MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}^n_k = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}^n_k$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}^n_k \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}^n_k$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}^2_{x,y}$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \ldots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \ldots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. □

Remark. The statement implies that if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then $V(f_1, \ldots, f_r) = V((f_1, \ldots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \ldots, x_n]$. By the Hilbert basis theorem that $k[x_1, \ldots, x_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $I = (f_1, \ldots, f_r)$. So $X = V(I) = V(f_1, \ldots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \ldots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. □

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \ldots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \dots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \dots, x_n]$;
2. $I(\mathbb{A}_k^n) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}_{\mathbb{R}}^n)$);
3. for $n = 1$, if $S \subseteq \mathbb{A}_k^1$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}_k^n$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \dots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. $\square$

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \dots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \dots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz). *If $J \leq k[x_1, \dots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ have no common zeros, then there exist $g_1, \dots, g_r \in k[x_1, \dots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. $\square$

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$ but $V(x^2 + 1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \dots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ with inverse $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \dots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \dots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. $\square$

Remark. This implies that maximal ideals in $k[x_1, \dots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. *$I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;*
2. *$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.*

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. $\square$

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0,0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x,y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization). *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \dots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \dots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \dots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \dots, z_n]$, which implies that $K[z_1, \dots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. □

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. □

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \dots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \dots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \dots, a_n \in k$, the ideal $(x_1 - a_1, \dots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \dots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \dots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \dots, a_n)\}$, so $J \neq \varnothing$.

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix $f \in I(V(J))$, and we want to show that $f^n \in J$ for some $n > 0$. Add a new variable y and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \dots, x_n, y].$$

Now $V(J_1) = \{(a, b) = (a_1, \dots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \varnothing$ since f vanishes on $V(J)$, so $f(a)b = 0$ for any b. Thus by the weak nullstellensatz, $J_1 = (1)$, so $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$ with

[1] The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

$f_1, \ldots, f_r \in J$ and $g_0, \ldots, g_r \in k[x_1, \ldots, x_n, y]$. Let N be the maximal power of y in the g_i. Multiplying by f^N, we get

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, fy) f_i + G_0(x_1, \ldots, x_n, fy)(fy - 1)$$

with $G_i \in k[x_1, \ldots, x_n, fy]$. So if we set $fy = 1$, then we have

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, 1) f_i + 0 \in J,$$

which gives $f \in \sqrt{J}$. To justify this substitution, we can consider the quotient $k[x_1, \ldots, x_n, y]/(fy - 1)$. We have a map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, y]/(fy - 1)$, which is injective since $(fy - 1)$ does not lie in $k[x_1, \ldots, x_n]$, so an equality in the quotient implies an equality in $k[x_1, \ldots, x_n]$. $\square$

Lecture 3

Aug. 26 — The Zariski Topology

3.1 Polynomial Functions and Subvarieties

Remark. Recall that a polynomial $f \in k[x_1, \dots, x_n]$ gives a function $\mathbb{A}^n_k \to k$ by $a \mapsto f(a)$.

Proposition 3.1. *If $f, g \in k[x_1, \dots, x_n]$ give the same function $\mathbb{A}^n_k \to k$, then $f = g$ in $k[x_1, \dots, x_n]$.*

Proof. Assume $f = g$ as polynomial functions. Then $V(f-g) = \mathbb{A}^n_k$, so $\sqrt{(f-g)} = I(\mathbb{A}^n_k) = (0)$ by Hilbert's nullstellensatz (note that we can also prove $I(\mathbb{A}^n_k) = (0)$ directly, it is enough to have k be an infinite field for this part). Thus $f - g = 0$, so $f = g$ in $k[x_1, \dots, x_n]$. $\square$

Remark. In the above proposition, we need k to be an infinite field (e.g. if $k = \overline{k}$): Otherwise, there are only finitely many functions $\mathbb{A}^n_k \to k$, but infinitely many polynomials in $k[x_1, \dots, x_n]$.

Remark. The set of polynomials functions $\mathbb{A}^n_k \to k$ form a ring, and the above proposition implies that this ring is isomorphic to $k[x_1, \dots, x_n]$.

Definition 3.1. A *polynomial function* on an affine variety $X \subseteq \mathbb{A}^n_k$ is a function $\varphi : X \to k$ such that there exists $f \in k[x_1, \dots, x_n]$ with $\varphi(a) = f(a)$ for every $a \in X$.

Definition 3.2. The *coordinate ring* of X is $A(X) = \{f : X \to k \mid f \text{ is a polynomial function}\}$, which is a ring under pointwise addition and multiplication.

Remark. Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel $I(X)$. Thus we have $A(X) \cong k[x_1, \dots, x_n]/I(X)$.

Remark. We can now replace $\mathbb{A}^n_k$ and $k[x_1, \dots, x_n]$ by X and $A(X)$ to study *subvarieties* of X.

Definition 3.3. Let $X \subseteq \mathbb{A}^n_k$ be an affine variety. If $S \subseteq A(X)$ is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of X of this form is called an *affine subvariety* of X. (Equivalently, these are the same as an affine variety $Y \subseteq \mathbb{A}^n_k$ such that $Y \subseteq X$.) For $Y \subseteq X$ a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

Proposition 3.2. *There is a bijective correspondence between radical ideals in $A(X)$ and affine subvarieties of X given by $J \mapsto V_X(J)$ and $Y \mapsto I_X(Y)$.*

Proof. See Homework 2. □

3.2 The Zariski Topology

Definition 3.4. The *Zariski topology* on $\mathbb{A}^n_k$ is the topology with closed sets $V(I) \subseteq \mathbb{A}^n_k$, where I is an ideal in $k[x_1, \ldots, x_n]$. (Equivalently, the closed sets are the affine varieties in $\mathbb{A}^n_k$.)

Remark. Note the following:

1. On $\mathbb{A}^1_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^1_k$, or finite collections of points.
2. When $k = \mathbb{C}$, then $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ being Zariski closed implies that X is closed in the analytic topology on $\mathbb{A}^n_{\mathbb{C}}$. In particular, the Zariski topology is coarser than the analytic topology.
3. On $\mathbb{A}^2_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^2_k$, finite collections of points, plane curves, and their finite unions.

Proposition 3.3. *The Zariski topology on $\mathbb{A}^n_k$ is indeed a topology.*

Proof. First note that $\varnothing = V((1))$ and $\mathbb{A}^n_k = V((0))$ are closed. For arbitrary intersections, note that $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$, and for finite unions, note that $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$. □

Example 3.4.1. The Zariski topology on $\mathbb{A}^{n+m}_k$ is in general *not* the product topology of the Zariski topologies on $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$. Consider $V(y - x^2) \subseteq \mathbb{A}^2_k$, which is a closed set in the Zariski topology, but the only closed sets in $\mathbb{A}^1_k$ are either $\varnothing$, $\mathbb{A}^1_k$, or finite.

Definition 3.5. If $X \subseteq \mathbb{A}^n_k$ is an affine variety, then we can define the *Zariski topology* on X in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on $\mathbb{A}^n_k$;
2. take the closed sets of X to be of the form $V_X(I)$ for some ideal $I \leq A(X)$.

This is because an affine subvariety of X is precisely the intersection of X with an affine variety in $\mathbb{A}^n_k$.

Remark. Our goal now is to relate properties of the Zariski topology on X to the ring $A(X)$, and then to the ideal $I(X) \leq k[x_1, \ldots, x_n]$.

Definition 3.6. A topological space X is *reducible* if we can write $X = X_1 \cup X_2$ for some closed sets $X_1, X_2 \subsetneq X$. Otherwise, X is called *irreducible*.

Example 3.6.1. The plane curve $X = V(y^2 - x^2y) = V(y) \cup V(y - x^2)$ is reducible.

Remark. Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g. $\mathbb{C}^n$, $\mathbb{R}^n$ with the analytic topology.
3. If X is irreducible and $U \subseteq X$ is a nonempty open set, then $\overline{U} = X$ (we have $\overline{U} \cup (X \setminus U) = X$).

Lecture 4

Aug. 28 — Irreducibility

4.1 Properties of Irreducibility

Proposition 4.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the following are equivalent:*

1. *X is irreducible;*
2. *$I(X) \leq k[x_1, \dots, x_n]$ is a prime ideal;*
3. *the coordinate ring $A(X)$ is an integral domain.*

Example 4.0.1. We have the following:

1. $\mathbb{A}^n_k$ is irreducible as $A(\mathbb{A}^n_k) = k[x_1, \dots, x_n]$, which is an integral domain.
2. A *hypersurface* $X \subseteq \mathbb{A}^n_k$ is an affine variety with $I(X) = (f)$ for some $f \in k[x_1, \dots, x_n]$. Then A is irreducible if and only if (f) is prime, if and only if f is irreducible.[1]

4.2 Dimension

Definition 4.1. Let X be a topological space.

- The *dimension* of X, denoted $\dim X$, is the supremum of the n such that there exists a chain of irreducible closed subspaces
$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \varnothing.$$
- For $Y \subseteq X$ closed and irreducible, the *codimension* of Y in X, denoted $\operatorname{codim}_X Y$, is the supremum of the n as above such that $X_n = Y$.

[1] Note that any prime ideal is radical.

Lecture 5

Sept. 2 — Dimension

5.1 More on Dimension

Remark. Recall the following correspondence from before: If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then there exists a bijection between the irreducible closed subsets $Y \subseteq X$ and the prime ideals $\mathfrak{p} \leq A(X)$.

Definition 5.1. For a ring A, the *(Krull) dimension* of A, denoted $\dim A$, is the supremum of the n such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

For a prime ideal $\mathfrak{q} \leq A$, the *height* of $\mathfrak{q}$, denoted $\operatorname{ht} \mathfrak{q}$, is the supremum of the n as above with $\mathfrak{p}_0 = \mathfrak{q}$.

Remark. If X is an affine variety, then we have the following:

1. $\dim X = \dim A(X)$;
2. for $Y \subseteq X$ a closed irreducible subset, $\operatorname{codim}_X Y = \operatorname{ht} I_X(Y)$.

These properties follow from the inclusion-reversing correspondence.

Definition 5.2. Let $K \subseteq L$ be a field extension.

1. A collection of elements $\{z_i : i \in I\} \subseteq L$ is a *transcendence basis* of $K \subseteq L$ if the z_i are algebraically independent (i.e. $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$ by $x_i \mapsto z_i$) and $K(z_i : i \in I) \subseteq L$ is algebraic.
2. The *transcendence degree* $\operatorname{tr.deg}_K L$ is the cardinality of a transcendence basis.

Theorem 5.1 (Dimension theory)**.** *Let A be a finitely generated k-algebra that is a domain. Then*

1. $\dim A = \operatorname{tr.deg}_k \operatorname{Frac}(A)$*;*
2. *for any prime ideal $\mathfrak{p} \leq A$, we have $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$;*
3. *all maximal chains of prime ideals $A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ are of the same length.*

Remark. The following are consequences of the above result from commutative algebra:

1. $\dim_k \mathbb{A}_k^n = \dim k[x_1, \ldots, x_n] = \operatorname{tr.deg}_k k(x_1, \ldots, x_n) = n$.
2. If X is irreducible, then $A(X)$ is a domain, so for $x \in X$, we have

   $$\operatorname{codim}_X \{x\} = \operatorname{ht} I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

   where we note that $A(X)/I(\{x\}) \cong k$ is a field.

3. If X is an irreducible affine variety and $U \subseteq X$ is a nonempty open subset, then
$$\dim U = \sup_{x \in U} \operatorname{codim}_U\{x\} = \sup_{x \in U} \operatorname{codim}_X\{x\} = \dim X.$$
This follows since we can pass from a chain in U to a chain in X by taking closures.

4. If X is an irreducible affine variety and $Z \subseteq X$ is an irreducible closed subset, then
$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if X is not irreducible. To contradict (4), let $X = V(x, y) \cup V(z) \subseteq \mathbb{A}_k^3$ with $Z = V(x, y)$. Then we have $\dim X = 2$, $\dim Z = 1$, $\operatorname{codim}_X Z = 0$.

5.2 Hypersurfaces

Remark. We now want to study hypersurfaces.

Theorem 5.2 (Krull's Hauptidealsatz)**.** *If A is a Noetherian ring and $f \in A$ is nonzero and a non-unit, then every minimal prime ideal containing f has height* 1*.*

Corollary 5.2.1. *If $X \subseteq \mathbb{A}_k^n$ is an irreducible affine variety and $f \in A(X)$ is a nonzero non-unit, then*
$$\dim Z = \dim X - 1$$
for every irreducible component Z of $V_X(f)$.

Proof. Since X is irreducible, $A(X)$ is a domain. So there is a correspondence between the minimal prime ideals $f \in \mathfrak{p} \subsetneq A(X)$ and the minimal irreducible closed subsets $Z \supseteq V_X(f)$, which corresponds to the irreducible components Z of $V_X(f)$. For such a component Z, we know
$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$
by Krull's Hauptidealsatz, which is the desired result. □

Example 5.2.1. Corollary 5.2.1 implies that if $f \in k[x_1, \ldots, x_n]$ is non-constant, then
$$\dim V(f) = \dim \mathbb{A}_k^n - 1 = n - 1.$$

Theorem 5.3. *An irreducible affine variety $Y \subseteq \mathbb{A}_k^n$ has $\dim Y = n - 1$ if and only if $Y = V(f)$ for some non-constant polynomial $f \in k[x_1, \ldots, x_n]$.*

Proof. ($\Leftarrow$) This was Corollary 5.2.1.

($\Rightarrow$) We will use that $A(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$ is a UFD. Since Y is irreducible and $\dim Y = n - 1$,
$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}_k^n} Y = \dim \mathbb{A}_k^n - \dim Y = 1.$$
Since $(0) \subsetneq I(Y) \subsetneq k[x_1, \ldots, x_n]$, there exists a non-constant $f \in k[x_1, \ldots, x_n]$ with $f \in I(Y)$. Write
$$f = f_1 \cdots f_r$$
with f_i irreducible by unique factorization, and note that the f_i are also prime since we are in a UFD. Since $I(Y)$ is prime, some f_i is in $I(Y)$, so we have the inclusions
$$(0) \subsetneq (f_i) \subseteq I(Y).$$
Since $\operatorname{ht} I(Y) = 1$, we must have $(f_i) = I(Y)$, so $Y = V(I(Y)) = V(f_i)$. □

5.3 Regular Functions

Definition 5.3. Let X be an affine variety and $U \subseteq X$ open. A function $\varphi : U \to k$ is *regular* if for each $a \in U$, there exists an open neighborhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define $\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is a regular function on } U\}$.

Exercise 5.1. Check that $\mathcal{O}_X(U)$ is a ring under pointwise addition and multiplication of outputs.

Remark. To patch open sets together, we will later need the notion of a *morphism*, and a morphism $U \to Y \subseteq \mathbb{A}_k^m$ should be given by

$$x \longmapsto (\varphi_1(x), \ldots, \varphi_m(x))$$

with φ_i regular functions on U.

Example 5.3.1. We have the following:

1. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then any $\varphi \in A(X)$ is regular. Furthermore, we get an injective ring homomorphism $A(X) \to \mathcal{O}_X(X)$. We will see that this is an isomorphism.
2. If $X = \mathbb{A}_k^1$ and $U = \mathbb{A}_k^1 \setminus \{0\}$, then for any $n \geq 0$ and $g \in k[x]$, the function g/x^n is regular on U. In general, if we fix $f, g \in A(X)$ and set $U = X \setminus V(f)$, then the map g/f^m is regular on U.
3. Let $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$ and $U = X \setminus V(x_2, x_4)$. Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

   Note that on $U \setminus V(x_2x_4)$, we have $x_1/x_2 = x_3/x_4$ since $x_1x_4 = x_2x_3$ on X.

Lecture 6

Sept. 4 — Regular Functions

6.1 Properties of Regular Functions

Proposition 6.1. *Let X be an affine variety and $U \subseteq X$ open. Then:*

1. *if $\varphi \in \mathcal{O}_X(U)$, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U;*
2. *(identity principle) If X is irreducible, $U \subseteq X$ is nonempty and open, and $\varphi, \psi \in \mathcal{O}_X(U)$ with $\varphi|_W = \psi|_W$ for some $W \subseteq U$ nonempty and open, then $\varphi = \psi$ in $\mathcal{O}_X(U)$.*

Proof. (1) It suffices to show that $U \setminus V(\varphi)$ is open in U. Fix $a \in U \setminus V(\varphi)$. Since φ is regular, there exists an open neighborhood $a \in U_a \subseteq U$ and $f_a, g_a \in A(X)$ such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$. This is an open set containing a in $U \setminus V(\varphi)$, so $U \setminus V(\varphi)$ is open.

(2) Since X is irreducible, U is also irreducible. The locus $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$ is closed in U by (1). It also contains W. Since W is dense (it is a nonempty open set in an irreducible topological space), we must have $V(\varphi - \psi) = U$. This proves the claim. $\square$

Example 6.0.1. In (2) of Proposition 6.1, the assumption that X is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}^2_k \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions $\varphi = x$ and $\psi = x + y$ agree on W but are not equal on U.

6.2 Distinguished Open Sets

Remark. We will see that an affine variety has a basis of open sets on which we can compute $\mathcal{O}_X(U)$.

Definition 6.1. A *distinguished open set* of an affine variety X is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function $f \in A(X)$.

Remark. We have the following:

1. The $D(f)$ are closed under (finite) intersection: $D(fg) = D(f) \cap D(g)$.

2. The $D(f)$ form a basis for the Zariski topology on X: If $U \subseteq X$ is open, then $U = X \setminus V(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in A(X)$ (since X is Noetherian). So $U = D(f_1) \cup \cdots \cup D(f_r)$.

Remark. We will view $D(f)$ as "small open sets" (under mild assumptions, $\operatorname{codim}_X(X \setminus D(f)) = 1$).

Theorem 6.1. *If X is an affine variety and $f \in A(X)$, then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

Proof. We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix $\varphi \in \mathcal{O}_X(D(f))$. For any $a \in D(f)$, there exists an open neighborhood $a \in U_a \subseteq D(f)$ and $f_a, g_a \in A(X)$ such that $\varphi|_{U_a} = g_a/f_a$. We may further assume that

1. $U_a = D(h_a)$ for some $h_a \in A(X)$ (by shrinking U_a if necessary, since the $D(h)$ form a basis);
2. $h_a = f_a$ (by rewriting $g_a/f_a = g_a h_a / f_a h_a$ and replacing h_a, f_a with $f_a h_a$).

Then for $a, b \in D(f)$, we have $f_a g_b = f_b g_a$ on $D(f_a) \cap D(f_b)$. Since both the left and right hand sides vanish on $X \setminus (D(f_a) \cap D(f_b))$, we have $f_a g_b = f_b g_a$ in $A(X)$. Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so $f \in I(V(f_a : a \in D(f)))$. By the Nullstellensatz, there exists $n \geq 0$ such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the k_a are nonzero. Set $g = \sum_{a \in D(f)} k_a g_a$, and we claim that $\varphi = g/f^n$. To see this, note that on U_b, we have $\varphi|_{U_b} = g_b/f_b$. Now since $f_a g_b = f_b g_a$, we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that $\varphi|_{U_b} = (g/f^n)|_{U_b}$. Since this holds for any U_b, we have $\varphi = g/f^n$ in $\mathcal{O}_X(D(f))$. $\square$

Remark. Theorem 6.1 has the following consequences:

1. The $f = 1$ case implies that the natural ring homomorphism $A(X) \to \mathcal{O}_X(X)$ is surjective and hence an isomorphism (note that $D(1) = X$).
2. We will see that $\mathcal{O}_X(D(f)) \cong A(X)_f$, the *localization* of $A(X)$ at f.

Example 6.1.1. How do we compute $\mathcal{O}_X(U)$ on non-distinguished open sets? Consider

$$X = \mathbb{A}^2_k \quad \text{and} \quad U = \mathbb{A}^2_k \setminus \{(0,0)\}.$$

Note that U is never a distinguished open set. We claim that the ring homomorphism

$$k[x,y] \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{(0,0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use $U = D(x) \cup D(y)$ (in general, cover U by basis elements). Fix $\varphi : U \to k$ regular, so

$$\begin{aligned} \varphi|_{D(x)} &= \frac{f}{x^m} \quad \text{for some } f \in k[x,y], m \geq 0 \\ \varphi|_{D(y)} &= \frac{g}{y^n} \quad \text{for some } g \in k[x,y], n \geq 0. \end{aligned}$$

Since we are in a UFD, we may assume that $x \nmid f$ and $y \nmid g$. Now $fy^n = gx^m$ on $D(y) \cap D(x)$, so by the identity principle, $fy^n = gx^m$ on $\mathbb{A}^2_k$, so $fy^n = gx^m$ in $k[x,y]$. Using that $y \nmid g$, $x \nmid f$, and that $k[x,y]$ is a UFD, we must have $n = m = 0$, hence $f = g$. In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map $k[x,y] \to \mathcal{O}_X(U)$ is surjective.

6.3 Localization

Remark. We want to invert a subset of a ring, in particular *multiplicative systems.*

Definition 6.2. A *multiplicative system* of a ring A is a subset such that

1. $1 \in S$;
2. S is closed under multiplication.

Example 6.2.1. The following examples of S are multiplicative systems:

1. $S = A$ or $S = \{1\}$;
2. if $\mathfrak{p} \leq A$ is a prime ideal, then $S = A \setminus \mathfrak{p}$;
3. if $f \in A$, then $S = \{f^m : m \geq 0\}$.

Definition 6.3. The *localization* of a ring A at a multiplicative system S is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the a/s are formal symbols with $a/s \sim a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$.[1] The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

Example 6.3.1. The following are examples of localization:

1. If A is a domain and $S = A \setminus \{0\}$, then $S^{-1}A = \operatorname{Frac} A$.
2. If $S = \langle f \rangle = \{1, f, f^2, \dots\}$, then we will write $A_f = S^{-1}A$.
3. If $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we will write $A_{\mathfrak{p}} = S^{-1}A$.

[1] Note that if A is a domain and $0 \notin S$, then this condition is equivalent to $as' = a's$.

Proposition 6.2. *We have the following properties of localization:*

1. *(Universal property of localization) For any ring homomorphism $\varphi : A \to B$ such that $\varphi(s)$ for all $s \in S$, then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi:a\mapsto a/1"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

2. *There is a bijection between the prime ideals $\mathfrak{p} \leq A$ with $\mathfrak{p} \cap S = \varnothing$ and the prime ideals $\mathfrak{q} \leq S^{-1}A$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$ with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : A \to S^{-1}A$ is the map $a \mapsto a/1$.*

Remark. In more generality, for an A-module M, we can define the localization $S^{-1}M$, which is an $S^{-1}A$-module. This gives a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which is exact.

Lecture 7

Sept. 9 — Germs and Sheaves

7.1 More on Localization

Proposition 7.1. *If X is an affine variety and $f \in A(X)$ is nonzero, then $\mathcal{O}_X(D(f)) \cong A(X)_f$.*

Proof. We define a ring homomorphism as follows:

$$\begin{aligned} A(X)_f &\longrightarrow \mathcal{O}_X(D(f)) \\ \frac{g}{f^m} &\longmapsto \left(x \mapsto \frac{g(x)}{f^m(x)} \right). \end{aligned}$$

To check that this is well-defined, assume $g/f^m \sim h/f^n$ in $A(X)_f$. So there exists $k \geq 0$ such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So $gf^n - hf^m = 0$ as functions $D(f) \to k$, so $g/f^m = h/f^n$ as functions $D(f) \to k$. Thus their images agree in $\mathcal{O}_X(D(f))$, so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume $g/f^m = 0$ as functions $D(f) \to k$ with $g \in A(X)$. Then $fg = 0$ in $A(X)$, so $g/f^m \sim 0/1$ in $A(X)_f$. □

7.2 Germs of Functions

Definition 7.1. Let $p \in X$ be a point on an affine variety.

1. A *germ* of a regular function of X at p is a pair (U, f) such that $x \in U \subseteq X$ is open and f is a regular function $U \to k$, up to the equivalence relation $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.
2. Define $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$.

Exercise 7.1. Check that $\mathcal{O}_{X,p}$ is a ring with operations

$$\begin{aligned} (U, \varphi) \cdot (V, \psi) &= (U \cap V, \varphi|_{U \cap V} \cdot \psi|_{U \cap V}), \\ (U, \varphi) + (V, \psi) &= (U \cap V, \varphi|_{U \cap V} + \psi|_{U \cap V}), \end{aligned}$$

with the zero function as the zero element and the constant 1 function as the unit element.

Lemma 7.1. *$\mathcal{O}_{X,p}$ is a local ring with unique maximal ideal $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$.*

Proof. It suffices to show that the units of $\mathcal{O}_{X,p}$ are precisely $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$. To see the reverse inclusion, fix $(U, \varphi) \in \mathcal{O}_{X,p}$ with $\varphi(p) \neq 0$. So there exists an open neighborhood $p \in W \subseteq U$ such that $\varphi|_W$ never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W)) = (W, 1),$$

so (U, φ) is a unit in $\mathcal{O}_{X,p}$. The forward inclusion is similar. □

Proposition 7.2. *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left(D(g), x \mapsto \frac{f(x)}{g(x)}\right) \end{aligned}$$

with $I(p) = \{f \in A(X) : f(p) = 0\}$.

Proof. To see that this is well-defined, let $f/g \sim f'/g' \in A(X)_{I(p)}$. Then $h(fg' - f'g) = 0$ for some $h \in A(X)$ with $h(p) \neq 0$. So $f/g = f'/g'$ as functions $D(h) \cap D(g) \to k$, which means that $f/g = f'/g'$ as elements in $\mathcal{O}_{X,p}$. Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose $(U, \varphi) \in \mathcal{O}_{X,p}$. Since $\varphi : U \to k$ is a regular function, there exists an open set $p \in U_p \subseteq U$ and $f, g \in A(X)$ such that g does not vanish on U_p and $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$. So $(U, \varphi) \sim (D(g), f/g)$ in $\mathcal{O}_{X,p}$, i.e. (U, φ) is in the image. □

Example 7.1.1. If $X = \mathbb{A}^n_k$ and $p = 0$, then

$$\mathcal{O}_{\mathbb{A}^n_k,0} \cong k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = \left\{\frac{f}{g} : f \in k[x_1, \ldots, x_n], g \in k[x_1, \ldots, x_n] \setminus (x_1, \ldots, x_n)\right\}.$$

Remark. We will relate the local properties of X at p to properties of $\mathcal{O}_{X,p}$. We will use the following statements from commutative algebra: Let A be a ring and $\mathfrak{p} \subseteq A$ a prime ideal. Then

1. $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$.
2. There is a bijection from the prime ideals of $A_\mathfrak{p}$ to the prime ideals of A contained in $\mathfrak{p}$.
3. $\operatorname{ht}_A \mathfrak{p} = \dim A_\mathfrak{p}$ (this follows from (2)).

This has the following consequence: If X is an affine variety and $p \in X$, then

$$\operatorname{codim}_X\{p\} = \operatorname{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

7.3 Sheaves

Remark. We will now formalize the structures $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ that we have seen before.

Definition 7.2. A *presheaf (of rings)* $\mathcal{F}$ on a topological space X is the data of

1. for every open set $U \subseteq X$, a ring $\mathcal{F}(U)$;
2. for every inclusion of open sets $U \subseteq V \subseteq X$, a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying the following properties:

1. $\mathcal{F}(\varnothing) = 0$;
2. $\rho_{U,U}$ is the identity map;
3. for inclusions of open sets $U \subseteq V \subseteq W \subseteq X$, we have $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$.

Example 7.2.1. If X is an affine variety, then $\mathcal{O}_X$ gives a presheaf of rings with

1. for $U \subseteq X$, the ring is $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \to k\}$;
2. for $U \subseteq V \subseteq X$, the map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is given by $\varphi \mapsto \varphi|_U$.

Remark. We often call $s \in \mathcal{F}(U)$ a *section*, and for $U \subseteq V$, we call $s|_U = \rho_{V,U}(s)$ the *restriction*.

Remark. A presheaf is the same thing as a functor $\mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Rings}$, where Open_X is the category with objects the nonempty open sets of X and morphisms corresponding to the inclusions $U \subseteq V$.

Definition 7.3. A presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the *gluing property*: For any $U \subseteq X$ open, an open cover $\{U_i\}_{i\in I}$ of U, and $\varphi_i \in \mathcal{F}(U_i)$ with $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$ for all $i, j \in I$, there exists a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Example 7.3.1. We have the following:

1. If X is an affine variety, then $\mathcal{O}_X$ is a sheaf (if we take $\varphi_i \in \mathcal{O}_X(U_i)$ that agree on the overlaps, then we get $\varphi : U \to k$, which is regular since regularity is a local property).
2. If M is a smooth manifold, then we can define a sheaf (on open subsets $U \subseteq M$) by
$$U \longmapsto \mathcal{F}^{\mathrm{sm}}(U) = \{\text{smooth functions } U \to \mathbb{R}\}.$$
We may also consider $\mathcal{F}^{\mathrm{cont}}$, $\mathcal{F}^{\mathrm{diff}}$, $\mathcal{F}^{\mathrm{loc,const}}$, etc. However, $\mathcal{F}^{\mathrm{const}}$ is a presheaf, but not a sheaf in general: We can take $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$, and we will only get a locally constant function. Similarly, $\mathcal{F}^{\mathrm{bounded}}$ is only a presheaf but not a sheaf.
3. If $\mathcal{F}$ is a sheaf on a topological space X and $U \subseteq X$ is open, then we get a sheaf $\mathcal{F}|_U$ on U defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 7.4. The *stalk* of a sheaf $\mathcal{F}$ on a topological space X at $x \in X$ is
$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\}/{\sim},$$
where $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.

Example 7.4.1. If X is an affine variety and $p \in X$, then $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$.

Remark. As before with $\mathcal{O}_{X,p}$, one can check that $\mathcal{F}_x$ naturally has the structure of a ring.

Remark. An alternative perspective is to define the stalk as a direct limit:
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where the limit is taken over all open $x \in U \subseteq X$ with respect to the ordering $U \le V$ if $V \subseteq U$.

Lecture 8

Sept. 11 — Morphisms

8.1 Morphisms of Open Sets

Remark. Recall that a continuous map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions $f_1, \dots, f_n : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = (f_1(x), \dots, f_n(x))$;
2. for each open set $U \subseteq \mathbb{R}^n$ and smooth $\varphi : U \to \mathbb{R}$, the function $f^*\varphi := \varphi \circ f : \mathbb{R}^m \to \mathbb{R}$ is smooth.

The implication ($1 \Rightarrow 2$) follows by the chain rule. To see ($2 \Rightarrow 1$), take $y_i : \mathbb{R}^n \to \mathbb{R}$ defined by $f_i := f^*y_i$. We want a similar definition in algebraic geometry.

Definition 8.1. Let X and Y be open sets of affine varieties. A *morphism* $f : X \to Y$ is a continuous map such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the map

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse (equivalently, f is a bijection and f^{-1} is a morphism).

Remark. We have the following properties of morphisms:

1. (Composition) If $f : X \to Y$ and $g : Y \to Z$ are morphisms of open sets of affine varieties, then so is $g \circ f : X \to Z$.
2. (Local on target) If $X \to Y$ is a map of open sets of affine varieties such that there exists an open cover $\{U_i\}_{i \in I}$ of Y with $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ a morphism for all $i \in I$, then f is a morphism.

Proposition 8.1. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Let $U \subseteq X$ and $V \subseteq Y$ be open sets. A map $f : U \to V$ is a morphism if and only if there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that*

$$f(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

Proof. ($\Rightarrow$) Let $U \subseteq \mathbb{A}^m_{x_i}$ and $V \subseteq \mathbb{A}^n_{y_i}$. By the definition of a morphism, $y_i : V \to k$ satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

($\Leftarrow$) Assume there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

We first show that f is continuous. Let $Z \subseteq V$ be a closed set. So we can write $Z = V(g_r, \ldots, g_r)$ for some $g_1, \ldots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \ldots, y_n]$. Now we have

$$\begin{aligned} f^{-1}(Z) = \{x \in U : f(x) \in Z\} &= \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \ldots, r\} \\ &= \{x \in U : (f^* g_i)(x) = 0 \text{ for } i = 1, \ldots, r\}. \end{aligned}$$

Note that $f^* g_i = g_i(\varphi_1, \ldots, \varphi_n)$, which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So $f^{-1}(Z)$ is closed in U.

Now to show that f is a morphism, it suffices to show that for any $W \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(W)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$. The proof of this is similar to before. □

Example 8.1.1. We have the following:

1. Morphisms $\mathbb{A}^m \to \mathbb{A}^n$ are of the form

$$x \longmapsto (f_1(x), \ldots, f_n(x))$$

   with $f_1, \ldots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_m]$.

2. Write $\mathbb{A}^1_t$ to mean $\mathbb{A}^1$ with variable t. Then we can define $\mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ by $t \mapsto (t, t^2)$. We can get an inverse $V(y - x^2) \to \mathbb{A}^1_t$ by $(x, y) \mapsto x$, so $\mathbb{A}^1_t$ and $V(y - x^2)$ are isomorphic.

3. Consider the map $g : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. This map is bijective, but it is not an isomorphism. To see this, we can show that $(g^{-1})^*\varphi$ is not regular for some regular function φ on $\mathbb{A}^t_1$. For instance, we can take $\varphi = t$, so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

   which we can see is not regular.

8.2 Relation to Coordinate Rings

Remark. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Then a morphism $f : X \to Y$ of affine varieties induces a k-algebra morphism (called the *pullback* of f)

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^* = \varphi \circ f \end{aligned}$$

with the properties $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{A(X)}$, i.e. $X \mapsto A(X)$ is a contravariant functor.

Proposition 8.2. *The following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_{\text{aff,var}}(X, Y) &\xrightarrow{\Phi} \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

Proof. Note that $A(X) \cong k[x_1, \ldots, x_m]/I(X)$ and $A(Y) \cong k[y_1, \ldots, y_n]/I(Y)$. Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \ldots, \varphi_n(x)), \end{aligned}$$

we can define $f^*\overline{y}_i = \varphi_i$. Conversely, given a k-algebra homomorphism $\phi : A(Y) \to A(X)$, we can set $\varphi_i = \phi(\overline{y}_i)$. Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}^n_{y_i} \\ x &\longmapsto (\varphi_1(x), \ldots, \varphi_n(x)). \end{aligned}$$

We claim that $f(X) \subseteq Y$. To see this, fix $x \in X$. If $h \in I(Y)$, then

$$h(f(x)) = h(\varphi_1(x), \ldots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so $f(X) \subseteq Y$. Thus we get a morphism $f : X \to Y$ by $x \mapsto (\varphi_1(x), \ldots, \varphi_n(x))$ with $f^*y_i = \varphi_i$. One can check that this gives a map $\Psi : \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \to \mathrm{Hom}_{\mathrm{aff,var}}(X, Y)$ which is inverse to Φ. □

Example 8.1.2. We have the following:

1. Recall the morphism $g : \mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t, t^2)$. The pullback is given by

$$\begin{aligned} g^* : \frac{k[x,y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

   Note that g^* is an isomorphism of k-algebras, so g is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to g.

2. Recall the morphism $h : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. The pullback is

$$\begin{aligned} h^* : \frac{k[x,y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

   Note that $t \notin \operatorname{Im} h^*$, so h^* is not an isomorphism, so h is not an isomorphism.

Remark. There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced k-algebras (up to isomorphism).

To see this, observe that if $X \subseteq \mathbb{A}^n$ is an affine variety, then $A(X) \cong k[x_1, \ldots, x_n]/I(X)$. This is finitely generated, and reduced since $I(X)$ is radical. Conversely, let A be a reduced finitely generated k-algebra. Then $A \cong k[y_1, \ldots, y_m]/I$ since A is finitely generated, and I is radical since A is reduced. Thus by Hilbert's nullstellensatz, $Y = V(I)$ satisfies $I(Y) = I(V(I)) = I$, so $A \cong A(Y)$.

In more abstract language, this means that there is an equivalence of categories

$$\mathrm{AffVar} \longleftrightarrow \mathrm{RedFGAlg}_k^{\mathrm{op}}.$$

Lecture 9

Sept. 16 — Morphisms, Part 2

9.1 An Example of Isomorphisms

Example 9.0.1. What of the following are isomorphic over $\mathbb{C}$?

1. $\mathbb{A}^1 \setminus \{1\}$;
2. $V(x^2 + y^2) \subseteq \mathbb{A}^2$;
3. $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$;
4. $V(xy) \subseteq \mathbb{A}^2$;
5. $V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$;
6. $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$.

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2 + y^2) = V((x + iy)(x - iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism $\mathbb{A}^1 \to Y = V(y - x^2, z - x^3)$ by $t \mapsto (t, t^2, t^3)$. We can also see this by noting that $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$. For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$, whereas $A(\mathbb{A}^1) \cong \mathbb{C}[x]$. So $A \setminus \{1\} \not\cong \mathbb{A}^1$. For (6), note that

$$V(x^2 - y^2 - 1) = V((x - y)(x + y) - 1) \cong V(uv - 1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map $V(uv - 1) \to \mathbb{A}^1 \setminus \{0\}$ given by $(u, v) \mapsto u$, with inverse $t \mapsto (t, 1/t)$. Finally, letting C be the curve in 6, one can show that there is a singularity at the origin with $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 2$, which is different than the other examples. So the isomorphism classes are $\{2, 4\}$, $\{1, 6\}$, $\{3\}$, and $\{5\}$.

9.2 Ringed Spaces and Morphisms

Definition 9.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X.

Example 9.1.1. If X is an affine variety and $\mathcal{O}_X$ is the sheaf of regular functions, then $(X, \mathcal{O}_X)$ is a ringed space. Similarly, if M is a complex manifold and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M, then $(M, \mathcal{O}_M)$ is a ringed space.

Remark. From now on, for a ringed space $(X, \mathcal{O}_X)$, we will always assume $\mathcal{O}_X$ is a sheaf of k-valued functions on X. With this assumption, we can make sense of pullbacks.

Definition 9.2. A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $f : X \to Y$ such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$,

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

is an element of $\mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse.

Remark. A one-sided inverse need not be two-sided: Consider $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$ and $g : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 0)$. Then $f \circ g = \mathrm{id}_{\mathbb{A}^1}$, but $g \circ f$ is not the identity on $\mathbb{A}^2$.

Remark. If $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then for $V \subseteq U \subseteq Y$ open, we get

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\text{res.}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\text{res.}"] \\ \mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \end{tikzcd}$$

which is a commutative diagram of ring homomorphisms.

Remark. If X and Y are open sets of affine varieties, then a map $f : X \to Y$ is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

Definition 9.3 (Redefinition of affine variety)**.** An *affine variety* $(X, \mathcal{O}_X)$ is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

Remark. We will often write just X for the affine variety instead of the pair $(X, \mathcal{O}_X)$.

Example 9.3.1. Recall that $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$ from Example 9.0.1. In particular, $\mathbb{A}^1 \setminus \{0\}$ is an affine variety in the new sense (but not in the old sense).

Proposition 9.1. *If X is an affine variety (in the old sense) and $f \in A(X)$, then $D(f)$ is an affine variety.*

Proof. Write $X = V(I) \subseteq \mathbb{A}^n_{x_i}$ and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse $V(I, fy - 1) \to D(f)$ given by $(x, y) \mapsto x$. So $D(f) \cong V(I, fy - 1)$ as ringed spaces. Thus $D(f)$ is an affine variety (in the new sense). □

9.3 Products of Affine Varieties

Remark. If $X \subseteq \mathbb{A}^m_{x_i}$ and $Y \subseteq \mathbb{A}^n_{y_i}$ are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing $I(X), I(Y)$ as ideals in $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. So $X \times Y$ is an affine variety with morphisms

$$\begin{tikzcd} & X \times Y \arrow[dl, "(x,y)\mapsto x"', "p_1"] \arrow[dr, "(x,y)\mapsto y", "p_2"'] & \\ X & & Y \end{tikzcd}$$

Proposition 9.2. *For every affine variety Z and diagram of morphisms*

$$\begin{tikzcd} Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\ & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\ & X & \end{tikzcd}$$

there is a unique morphism f which makes the diagram commute.

Proof. We already know that there is a unique set theoretic map which makes the diagram commute. Then since f_X and f_Y are given as regular functions, so is f. So f is a morphism. $\square$

Remark. We will now try to understand the isomorphism $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

9.4 Tensor Products

Definition 9.4. Let A be a (commutative) ring and M, N be A-modules. The *tensor product* $M \otimes_A N$ is the A-module generated by the symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

1. (distributive law): $(m + m') \otimes n = m \otimes n + m' \otimes n$,
2. (multiplication with scalars): $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.

To make this precise, $M \otimes_A N = A^{M \times N}/R$, where R is the submodule generated by these relations.

Example 9.4.1. We have $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$. We can compute

$$1 \otimes 1 = (3 - 2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if $\gcd(m, n) = 1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Proposition 9.3 (Universal property of the tensor product)**.** *For any bilinear map $\Phi : M \times N \to P$ to an A-module P (i.e. $n \mapsto \Phi(m, n)$ is A-linear for each $m \in M$ and the same for $m \mapsto \Phi(m, n)$),*

$$\begin{tikzcd} M \times N \arrow[r, "\Phi"] \arrow[d, "(m,n)\mapsto m\otimes n"'] & P \\ M \otimes N \arrow[ur, "\Psi"'] & \end{tikzcd}$$

there exists a unique A-module homomorphism $\Psi : M \otimes N \to P$ such that the above diagram commutes.

Remark. We have the following properties of the tensor product:

1. $A \otimes M \cong M$;

2. $M \otimes N \cong N \otimes M$;
3. $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$;
4. $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed A-module M and an exact sequence

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

the sequence (where F is defined by $m \otimes n' \mapsto m \otimes f(n')$ and G is defined by $m \otimes n \mapsto m \otimes g(n)$)

$$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

is also exact. In particular, $\otimes M$ induces a right exact functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$ by $N \mapsto M \otimes N$.

Example 9.4.2. The functor $\otimes M$ is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 9.1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

Lecture 10

Sept. 18 — Pre-varieties

10.1 More on Tensor Products

Proposition 10.1. *If B and C are A-algebras (i.e. there are ring homomorphisms $f : A \to B$ and $g : A \to C$ which give $a \cdot b := f(a)b$ and $a \cdot c = g(a)c$, then $B \otimes_A C$ is also an A-algebra with*

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc')$$

and ring homomorphism $A \to B \otimes_A C$ given by $a \mapsto a \otimes 1$ (equivalently, $1 \otimes a$).

Proposition 10.2. $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Proposition 10.3. $(k[x_1, \ldots, x_m]/I) \otimes_k (k[y_1, \ldots, y_n]/J) \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n)/\langle I, J \rangle$.

Proof. Set $R = k[x_1, \ldots, x_m]$ and $S = k[y_1, \ldots, y_n]$. We have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Applying the right exact functor $\otimes_k(S/J)$ (and vice versa with J and $\otimes R$) gives an exact sequence

$$\begin{tikzcd}
 & R \otimes_k J \arrow[d] & & \\
 & R \otimes_k S \arrow[d] & & \\
I \otimes_k (S/J) \arrow[r] & R \otimes_k (S/J) \arrow[r] \arrow[d] & (R/I) \otimes_k (S/J) \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

So we have

$$(R/I) \otimes_k (S/J) \cong \frac{R \otimes_k (S/J)}{\mathrm{Im}(I \otimes_k (S/J) \to R \otimes_k (S/J))} \cong \frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J},$$

which is the desired result since $I \otimes_k S + R \otimes_k J = \langle I, J \rangle$ in $R \otimes_k S$. $\square$

Proposition 10.4 (Milne)**.** *Let B and C be finitely generated k-algebras with $k = \overline{k}$.*

1. *If B and C are reduced, then so is $B \otimes_k C$.*

2. *If B and C are domains, then so is $B \otimes_k C$.*

Remark. We need $k = \overline{k}$ in Proposition 10.4. Consider the domains $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[y]/(y^2+1)$. Then

$$\frac{\mathbb{R}[x]}{(x^2+1)} \otimes_{\mathbb{R}} \frac{\mathbb{R}[y]}{(y^2+1)} \cong \frac{\mathbb{R}[x,y]}{(x^2+1, y^2+1)},$$

which is not a domain since $(\overline{x-y})(\overline{x+y}) = \overline{x^2-y^2} = \overline{-1-(-1)} = 0$.

Corollary 10.0.1. *If $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then*

1. *$I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.*
2. *$A(X \times Y) \cong A(X) \otimes_k A(Y)$.*
3. *If X and Y are irreducible, then $X \times Y$ is irreducible.*

Proof. Observe that $V(I(X), I(Y)) = X \times Y \subseteq \mathbb{A}^{m+n}$, so $I(X \times Y) = \sqrt{\langle I(X), I(Y) \rangle}$. Now we know that $I(X)$ and $I(Y)$ are radical in $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$, respectively, so

$$\frac{k[x_1, \ldots, x_m]}{I(X)} \quad \text{and} \quad \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

are reduced. By Proposition 10.4, we get that

$$\frac{k[x_1, \ldots, x_m, y_1, \ldots, y_n]}{\langle I(X), I(Y) \rangle} \cong \frac{k[x_1, \ldots, x_m]}{I(X)} \otimes_k \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

is reduced, so $\langle I(X), I(Y) \rangle$ is radical. Thus $I(X \times Y) = \langle I(X), I(Y) \rangle$, so (1) holds.

Now (1) implies (2), and (3) follows since X and Y being irreducible implies $A(X)$ and $A(Y)$ are domains, which implies $A(X \times Y)$ is a domain by Proposition 10.4 and (2), so $X \times Y$ is irreducible. $\square$

10.2 Pre-varieties

Remark. We will now head towards defining a *variety*, which is roughly finitely many affine varieties glued together (a *pre-variety*) with a separation condition (an algebraic version of Hausdorffness).

Definition 10.1. A *pre-variety* is a ringed space $(X, \mathcal{O}_X)$ such that there exists a finite open cover $X = \bigcup_{i=1}^{s} U_i$ with $(U_i, \mathcal{O}_X|_{U_i})$ being an affine variety for all $i = 1, \ldots, s$. A *morphism* of pre-varieties

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of the ringed spaces. We will often just write X for $(X, \mathcal{O}_X)$.

Remark. We call $\varphi \in \mathcal{O}_X(U)$ with $U \subseteq X$ open and $\varphi : U \to k$ a *regular function* on U.

Example 10.1.1. Consider the following:

1. An affine variety X is a pre-variety. However, we have multiple choices for the open cover: We can take $X = X$, or $X = \bigcup_{i=1}^{s} D(f_i)$ with $f_i \in \mathcal{O}_X(X)$ and $(f_1, \ldots, f_s) = (1)$ in $\mathcal{O}_X(X)$.
2. $\mathbb{P}_k^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$ is a pre-variety. We will see that $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\text{pt}\}$.

3. Let $X = V(I) \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ open. Set
$$\mathcal{O}_U(V) = \{\varphi : V \to k \mid \varphi \text{ is regular}\}.$$
Then $(U, \mathcal{O}_U)$ is a pre-variety. To see this, note that $U = \bigcup_{f \in I(X \setminus U)} D(f)$. Since U is Noetherian (hence is compact), we can find a finite subcover, so $U = \bigcup_{i=1}^s D(f_i)$ for some $f_i \in A(X)$.

4. (Gluing) Let X_1 and X_2 be affine varieties, and $U_{1,2} \subseteq X_1$, $U_{2,1} \subseteq X_2$ open, with an isomorphism
$$f : U_{1,2} \longrightarrow U_{2,1}.$$
Then we get a pre-variety by setting $X = (X_1 \sqcup X_2)/\sim$, where $a \sim f(a)$ for all $a \in U_{1,2}$, $f(a) \sim a$ for all $a \in U_{2,1}$, and $b \sim b$ for all $b \in X_1 \sqcup X_2$. We have quotient maps
$$j_1 : X_1 \longrightarrow X \quad \text{and} \quad j_2 : X_2 \longrightarrow X.$$
Now X is a topological space with the quotient topology, and j_1, j_2 are open embeddings (i.e. have open images and are homeomorphisms onto their images). Define a sheaf of rings $\mathcal{O}_X$ on X by
$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid j_1^*\varphi \in \mathcal{O}_{X_1}(j^{-1}(U)) \text{ and } j_2^*\varphi \in \mathcal{O}_{X_2}(j_2^{-1}(U))\}.$$
One can check $X = j_1(X_1) \cup j_2(X_2)$ and $(j(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Example 10.1.2. Consider $X_1 = \mathbb{A}^1_x$ and $X_2 = \mathbb{A}^1_y$, with $U_{1,2} = \mathbb{A}^1_x \setminus \{0\}$ and $U_{2,1} = \mathbb{A}^1_y \setminus \{0\}$. Define
$$\begin{aligned} f : U_{1,2} &\longrightarrow U_{2,1} \\ x &\longmapsto 1/x. \end{aligned}$$
Then we can take $\mathbb{P}^1_k = (X_1 \sqcup X_2)/\sim$. What are the regular functions $\mathbb{P}^1_k \to k$? We should get only the constant functions (When $k = \mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$ is compact, so a holomorphic function $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}$ is bounded. By restricting to X_1, we get a bounded map $f : \mathbb{C} \to \mathbb{C}$, so f is constant by Liouville's theorem).

In general, let $j_i : X_i \to \mathbb{P}^1_k$ be the quotient maps. Fix $\varphi \in \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$. Now
$$\varphi|_{X_1} := j_1^*\varphi = \sum_{i \geq 0} a_i x^i \quad \text{and} \quad \varphi|_{X_2} := j_2^*\varphi = \sum_{i \geq 0} b_i y^i$$
for some $a_i, b_i \in k$. They must agree on the overlap, so
$$\sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} b_i (1/x)^i$$
as functions on $\mathbb{A}^1 \setminus \{0\}$. Since $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$, we have $a_i = b_i = 0$ for $i > 0$ and $a_0 = b_0$ (since the powers of $x^{\pm 1}$ are k-linearly independent), so φ is a constant function.

If we instead took $f : U_{1,2} \to U_{2,1}$ to be $x \mapsto x$, then $X = (X_1 \sqcup X_2)/\sim$ is the "bug-eyed line" with two points $0, 0'$ at the origin (this is the *line with two origins* when $k = \mathbb{R}$, which is not Hausdorff.) Note that $X \setminus \{0, 0'\} \cong \mathbb{A}^1 \setminus \{0\}$. In our case, the bad property is that there exist two morphisms
$$g_1, g_2 : \mathbb{A}^1 \longrightarrow X$$
such that $g_1|_{\mathbb{A}^1 \setminus \{0\}} = g_2|_{\mathbb{A}^1 \setminus \{0\}}$ and $g_1 \neq g_2$, i.e. "limits are not unique" on X. Note that a similar computation shows $\mathcal{O}_X(X) \cong k[x]$, so in particular, $X \not\cong \mathbb{P}^1_k$.

Lecture 11

Sept. 23 — Pre-varieties, Part 2

11.1 More on Pre-varieties

Proposition 11.1. *Let $(X, \mathcal{O}_X)$ be a pre-variety.*

1. *X is Noetherian as a topological space.*
2. *X has a basis by affine varieties.*

Proof. (1) Note that X has a finite cover by affine varieties U_i, which are each Noetherian.

(2) If $(X, \mathcal{O}_X)$ is affine, then $\{D(f) : f \in \mathcal{O}_X(X)\}$ gives such a basis. Do this for each U_i. □

Example 11.0.1 (General gluing procedure)**.** Let I be a finite index set, $(X_i, \mathcal{O}_{X_i})$ affine varieties, $U_{i,j} \subseteq X_i$ open sets, and $f_{i,j} : U_{i,j} \to U_{j,i}$ isomorphisms for each $i, j \in I$, satisfying

1. $U_{i,i} = X_i$ and $f_{i,i} = \mathrm{id}$;
2. $f_{i,j}^{-1}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k}$;
3. the following diagram commutes:

$$\begin{tikzcd}
U_{i,j} \cap U_{i,k} \arrow[rr, "f_{i,k}"] \arrow[dr, "f_{i,j}"'] & & U_{k,i} \cap U_{k,j} \\
& U_{j,i} \cap U_{j,k} \arrow[ur, "f_{j,k}"'] &
\end{tikzcd}$$

We can define $X = (\bigsqcup_{i \in I} X_i)/\sim$ with the quotient topology, where $a \sim a$ for $a \in X_i$ and $a \sim f_{i,j}(a)$ for $a \in U_{i,j}$. The inclusions $j_i : X_i \hookrightarrow X$ are open embeddings, and we can set

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi|_{X_i} = j_i^* \varphi \text{ is regular for all } i\}.$$

Then $(X, \mathcal{O}_X)$ is a ringed space with $(j_i(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Remark. Any pre-variety X is a gluing of affine varieties. To see this, note that there exists a cover $X = \bigcup_{i=1}^s U_i$ by affine varieties. Then we can take $X_i = U_i$, $U_{i,j} = U_i \cap U_j \subseteq X_i$, and $f_{i,j} : U_{i,j} \to U_{j,i}$ to be the identity map.

Proposition 11.2. *Let X be a pre-variety.*

1. *If $U \subseteq X$ is an open set, then $(U, \mathcal{O}_X|_U)$ is again a pre-variety.*

2. *Let $Z \subseteq X$ be a closed set. For $U \subseteq Z$ open, set*

$$\mathcal{O}_Z(U) = \left\{ \varphi : U \to k \mid \begin{array}{c} \text{for each } a \in U, \text{ there exists open } a \in W \subseteq X \text{ and} \\ \psi : W \to k \text{ regular such that } \varphi|_{W\cap Z} = \psi|_{W \cap Z} \end{array} \right\}.$$

Then $(Z, \mathcal{O}_Z)$ is a pre-variety.

Proof. (1) Note that X has a basis by affine varieties, so we can cover U by affine varieties. This cover may be infinite, but we can pass to a finite subcover since X and hence U is Noetherian.

(2) The idea is to first reduce to the case $X = V(I) \subseteq \mathbb{A}^n$, so $Z \subseteq X$ is cut out by polynomials. Then observe that $\mathcal{O}_Z$ agrees with the previous definition. $\square$

Remark. Note that unions of intersections of closed and open sets are not necessarily pre-varieties. For instance, consider $(\mathbb{A}^2 \setminus V(xy)) \cup \{0\}$.

Proposition 11.3. *If X, Y are pre-varieties, then there exists a pre-variety with morphisms*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

with the property that for every diagram

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "f_X"'] \arrow[d, dashed, "f"] \arrow[ddr, bend left, "f_Y"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

there exists a unique morphism f such that the diagram commutes. We call P the product *of X and Y, and write $X \times Y := P$. Moreover, set theoretically $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.*

Proof. We know the result holds when X, Y, Z are affine or even open sets of affine varieties. In the general case, fix an open affine cover $X = \bigcup_{i=1}^s U_i$ and $Y = \bigcup_{j=1}^r V_j$. Then glue the products by:

1. $P_{(i,j)} := U_i \times V_j$,
2. along $P_{(i,j),(i',j')} : (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$,
3. via $f_{(i,j),(i',j')} : P_{(i,j),(i',j')} \xrightarrow{\cong} P_{(i',j'),(i,j)}$, the isomorphism from the universal property of products.

We get a pre-variety P, and the morphisms

$$\begin{tikzcd} & f_1^{-1}(U_2) \cap f_2^{-1}(V_j) \subseteq Z \arrow[ddl, bend right] \arrow[d, dashed, "f_{i,j}"'] \arrow[ddr, bend left] & \\ & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

glue to give a morphism

$$\begin{tikzcd} & Z \arrow[dl, bend right] \arrow[dr, bend left] \arrow[d, dashed, "f"] & \\ & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which is a morphism as the morphism condition can be checked locally. Furthermore, the diagram commutes (as can be checked locally). Last, f is unique: One can either this check locally or check set theoretically using $P = \{(x, y) : x \in X \text{ and } y \in Y\}$ as sets. $\square$

Remark. Note that $X \times Y$ is set theoretically the product of X and Y, but not the product of X and Y as topological spaces. Consider $X = Y = \mathbb{A}^1$ and $X \times Y = \mathbb{A}^2$.

11.2 Varieties

Remark. We want a version of Hausdorffness in algebraic geometry. However, an irreducible topological space (e.g. $\mathbb{A}^n$) is almost never Hausdorff (unless it is a single point). From a different perspective, note that X is Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is closed, where $X \times X$ is given the product topology.

Definition 11.1. A pre-variety is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$ (the product pre-variety). A *variety* is a pre-variety that is separated.

Example 11.1.1. $\mathbb{A}^n$ is separated. We have

$$V(x_1 - y_1, \ldots, x_n - y_n) = \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^n_{y_i} \cong \mathbb{A}^{2n},$$

so $\Delta_{\mathbb{A}^n}$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$.

Example 11.1.2. Any affine variety is separated. To see this, we may assume $X = V(I) \subseteq \mathbb{A}^n$. By the construction of the product, $X \times X \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is closed and

$$\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}.$$

Since $\Delta_{\mathbb{A}^n}$ is closed, we have Δ_X is closed in $X \times X$ since $X \times X$ has the subspace topology.

Proposition 11.4. *If X is a variety, then any closed or open set $Z \subseteq X$ is a variety.*

Proof. We have already seen that Z is a pre-variety, so it suffices to show that Z is separated. We note that $Z \times Z \hookrightarrow X \times X$ is an embedding of topological spaces, and $\Delta_Z = (Z \times Z) \cap \Delta_X$. Since Δ_X is closed and $Z \times Z$ has the subspace topology, Δ_Z is closed in $Z \times Z$. So Z is separated. $\square$

Example 11.1.3. Recall the bug-eyed line from Example 10.1.2. Let a, b be the two origins, and write $X = U_1 \cup U_2$, where $U_1 = X \setminus \{b\} \cong \mathbb{A}^1$ and $U_2 = X \setminus \{a\} \cong \mathbb{A}^1$. Then consider

$$\mathbb{A}^2 \cong U_1 \times U_2 \subseteq X \times X.$$

Note that $\Delta_X \cap (U_1 \times U_2) = \{(x, x) : x \in k \setminus \{0\}\} = \Delta_{\mathbb{A}^1} \setminus \{0\}$. So Δ_X is not closed in $X \times X$.

Exercise 11.1. Show that $\mathbb{P}^1_k$ is separated.

Proposition 11.5. *Let $f, g : X \to Y$ be morphisms of pre-varieties with Y a variety.*

1. *The graph $\Gamma_f := \{(x, f(x)) : x \in X\}$ of f is closed in $X \times Y$.*
2. *$\{x \in X : f(x) = g(x)\}$ is closed in X. This becomes a version of the identity principle in the case that X is irreducible: If X is irreducible and f, g agree on a nonempty open set, then $f = g$.*

Proof. (1) We can write $\Gamma_f = (f, \mathrm{id})^{-1}(\Delta_Y)$ where $(f, \mathrm{id}) : X \times Y \to Y \times Y$, and Δ_Y is closed.

(2) Consider the morphism

$$\begin{aligned} X &\xrightarrow{(f,g)} Y \times Y \\ x &\longmapsto (f(x), g(x)). \end{aligned}$$

Then $\{x \in X : f(x) = g(x)\} = (f, g)^{-1}(\Delta_Y)$, so it is closed. □

Lecture 12

Sept. 25 — Projective Varieties

12.1 Projective Space

Definition 12.1. Define *projective n-space* over k to be

$$\mathbb{P}^n_k = \mathbb{P}^n = \text{1-dimensional subspaces of } k^{n+1} = (k^{n+1} \setminus \{0\})/{\sim},$$

where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if there exists $\lambda \in k^\times$ such that $(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$. We write $[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k$ for the equivalence class of $(x_0, x_1, \dots, x_n)$.

Example 12.1.1. For $n = 2$, we have $[1 : 0 : 2] = [1/2 : 0 : 1] \in \mathbb{P}^2_k$ when $\operatorname{char} k \neq 2$.

Remark. For $0 \le i \le n$, define $U_i = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k : x_i \neq 0\}$. Then

$$\mathbb{P}^n_k = \bigcup_{i=0}^n U_i,$$

and there exist bijective maps $f_i : U_i \to \mathbb{A}^n$ given by

$$f_i([x_0 : \cdots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i),$$

where $\widehat{x_i/x_i}$ means we omit x_i/x_i. For $i = 0$, the inverse is $f_0^{-1}(x_1, \dots, x_n) = [1 : x_1 : \cdots : x_n]$.

Remark. Another way to think about $\mathbb{P}^n$ is via points at ∞. Observe that

$$\mathbb{P}^n \setminus U_0 = \{[0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k : (x_1, \dots, x_n) \in k^n \setminus \{0\}\} \cong \mathbb{P}^{n-1}.$$

So $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$.

Remark. Why work with $\mathbb{P}^n$? One motivation is analytic (e.g. for $k = \mathbb{C}$):

1. $\mathbb{P}^n_{\mathbb{C}}$ is compact with the analytic topology: There are surjective continuous maps

$$\begin{tikzcd} \mathbb{R}^{2n+2} \setminus \{0\} \arrow[r, two heads] & \mathbb{CP}^n \\ S^{2n+1} \arrow[u, hook] \arrow[ur, two heads] & \end{tikzcd}$$

2. *Chow's theorem*: Any closed complex submanifold of $\mathbb{CP}^n$ is a projective variety.

Another motivation is the extra data at ∞:

1. If ℓ_1, ℓ_2 are distinct lines in $\mathbb{A}^2$, then $\#(\ell_1 \cap \ell_2) = 0$ or 1. However, over $\mathbb{P}^2$, $\#(\ell_1 \cap \ell_2) = 1$ always.

2. *Bezout's theorem*: If $C_1, C_2 \subseteq \mathbb{A}^2$ are two distinct irreducible curves in $\mathbb{A}^2$, then
$$\#(C_1 \cap C_2) \leq (\deg C_1)(\deg C_2),$$
counting multiplicities. The version over $\mathbb{P}^2$ always gives equality.

12.2 Graded Rings

Remark. In projective space, for $f \in k[x_0, \ldots, x_n]$, we could try to define
$$V(f) = \{[a_0 : \cdots : a_n] : f(a_0, \ldots, a_n) = 0\}.$$
But this is bad notation as it is not well-defined ($f = 0$ depends on the representative in the equivalence class). Instead, if f is homogeneous of degree d, then
$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n),$$
so $V(f)$ is well-defined in this case, when f is homogeneous.

Definition 12.2. An $\mathbb{N}$-*graded ring* is a ring R with subgroups $R_d \subseteq R$ for $d \in \mathbb{N}$ such that
$$R = \bigoplus_{d \in \mathbb{N}} R_d \quad \text{and} \quad R_d R_e \subseteq R_{d+e}.$$
An element $f \in R$ is *homogeneous* if there exists d such that $f \in R_d$.

Example 12.2.1. For $S = k[x_0, \ldots, x_n]$, we can take $S_d = \bigoplus_{a_i \geq 0, \sum a_i = d} k x_0^{a_0} \cdots x_n^{a_n}$.

Definition 12.3. An ideal I in a graded ring is *homogeneous* if it is generated by homogeneous elements.

Example 12.3.1. We can write $(x, y^3 - 3x^2) \subseteq k[x, y]$ as (x, y^3), so it is homogeneous.

Proposition 12.1. *Let R be a graded ring with ideal I. The following are equivalent:*

1. *I is homogeneous;*
2. *for any $f = \sum_{d \in \mathbb{N}} f_d \in I$ with $f_d \in R_d$, then $f_d \in I$ for all d;*
3. *$I = \bigoplus_{d \in \mathbb{N}} (I \cap R_d)$.*

Proof. Left as an exercise. The interesting implication is ($1 \Rightarrow 2$). □

Proposition 12.2. *Let I, J be homogeneous ideals of a graded ring R. Then*

1. *$I + J$, IJ, $\sqrt{I}$, and $I \cap J$ are all homogeneous;*
2. *R/I is a graded ring with $R/I = \bigoplus_{d \in \mathbb{N}} R_d/I_d$, where $I_d = I \cap R_d$.*

Proof. (1) We prove that $\sqrt{I}$ is homogeneous. Assume $f \in \sqrt{I}$, and write $f = f_0 + f_1 + \cdots + f_d$ with $f_i \in R_i$ and $f_d \neq 0$. Now there exists $n > 0$ such that $f^n \in I$, and
$$f^n = f_d^n + \text{lower order terms}.$$
Since I is homogeneous, $f_d^n \in I$, so $f_d \in \sqrt{I}$. Then $f_0 + \cdots + f_{d-1} \in \sqrt{I}$, and we can repeat.

(2) We can write $R/I = (\bigoplus_{d \in \mathbb{N}} R_d)/(\bigoplus_{d \in \mathbb{N}} (I \cap R_d))$. As abelian groups, this is $R/I \cong \bigoplus_{d \in \mathbb{N}} R_d/I_d$. One can check that the multiplication also respects the grading, so this is an isomorphism of rings. □

12.3 Projective Varieties

Definition 12.4. For a set $T \subseteq k[x_0, \dots, x_n]$ of homogeneous elements, define its *vanishing locus*

$$V_p(T) := V(T) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\} \subseteq \mathbb{P}^n.$$

A *projective variety* is a subset of this form. For a homogeneous ideal $I \leq k[x_0, \dots, x_n]$, define

$$V(I) = \{x \in \mathbb{P}^n : f(x) \text{ for all } f \in I \text{ homogeneous}\}.$$

For a subset $X \subseteq \mathbb{P}^n$, define its *ideal*

$$I_p(X) := I(X) = (f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in X).$$

Note that we need to take the ideal generated by these elements, otherwise we may not get an ideal.

Remark. If $T \subseteq k[x_0, \dots, x_n]$ is a subset of homogeneous elements, then we have $V_p(T) = V_p((T))$. So projective varieties can equivalently be defined as vanishing sets of homogeneous ideals.

Example 12.4.1. Consider $X = V_p(x^2 - yz) \subseteq \mathbb{P}^2_{x:y:z}$. Set $H = V(x)$, then there is a bijection

$$\begin{aligned} U = \mathbb{P}^2 \setminus H &\xrightarrow{f} \mathbb{A}^2 \\ [1 : y : z] &\longmapsto (y, z). \end{aligned}$$

Then $f(X \cap U) = V(1 - yz)$. On the other hand, we can see that

$$X \cap H = \{[0 : 1 : 0], [0 : 0 : 1]\} = \{a, b\}.$$

If we were working with $\mathbb{C}$ with the analytic topology, then we can take limits on $V(1 - yz)$ and see

$$\lim_{t \to 0} [1 : t : 1/t] = \lim_{t \to 0} [t : t^2 : 1] = [0 : 0 : 1] = b.$$

Note that we essentially switched charts in order to take this limit. Similarly, we have

$$\lim_{t \to \infty} [1 : t : 1/t] = \lim_{t \to \infty} [1/t : 1 : 1/t^2] = [0 : 1 : 0] = a.$$

So we can see a, b as points at ∞ compactifying the curve $V(1 - yz)$.

Example 12.4.2. We have the following:

1. $V_p(0) = \mathbb{P}^n$;
2. $V_p(1) = \varnothing$;
3. if $p = [a_0 : \cdots : a_n]$ and $J = (a_i x_j - a_j x_i : 0 \leq i, j \leq n)$, then $V(J) = \{0\}$;
4. $I_0 = (x_0, \dots, x_n)$ is called the *irrelevant ideal*, which has $V_p(I_0) = \varnothing = V_p(1)$ but $I_0 = \sqrt{I_0} \subsetneq (1)$.

Lecture 13

Sept. 30 — Projective Varieties, Part 2

13.1 More on Projective Varieties

Example 13.0.1. Consider $X = V(y^2z - x^3 - zx^2 - z^3) \subseteq \mathbb{P}^2$ and $H_z = V(z)$. Let

$$\begin{aligned} U_z = \mathbb{P}^2 \setminus H_z &\xrightarrow[\text{bij}]{f} \mathbb{A}^2 \\ [x : y : 1] &\longmapsto (x, y). \end{aligned}$$

Then $f(X \cap U_z) = V(y^2 - x^3 - x^2 - 1)$, and

$$X \cap U_z = V(y^2z - x^3 - zx^2 - z^3, z) = V(x^3, z) = \{[0 : 1 : 0]\}.$$

Example 13.0.2. Let $I = (x_0, \dots, x_n)$ be the irrelevant ideal. Then I is radical, but

$$I_p(V_p(I)) = I_p(\varnothing) = (1) \neq \sqrt{I}.$$

13.2 Cones

Definition 13.1. A subset $C \subseteq \mathbb{A}^{n+1}$ is a *cone* if $0 \in C$ and $\lambda x \in C$ whenever $x \in C$ and $\lambda \in k$.

Example 13.1.1. If $X \subseteq \mathbb{P}^n$ is a projective variety, then we can set $C(X) = \pi^{-1}(X)\{0\}$, where

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [x]. \end{aligned}$$

Proposition 13.1. *If $C \subseteq \mathbb{A}^{n+1}$ is a cone, then $I_a(C) \leq k[x_0, \dots, x_n]$ is homogeneous.*

Proof. Fix $f \in I_a(C)$. Then we can write $f = \sum_{i=0}^d f_i$ with f_i homogeneous of degree i. We want to show that $f_i \in I_a(C)$ for each i. Fix $x \in C$. For any $\lambda \in k$,

$$0 = f(\lambda x) = \sum_{i=0}^{d} \lambda^i f_i(x).$$

Viewing this as a polynomial in λ (with x fixed), we must have each $f_i(x) = 0$. Thus $f_i \in I_a(C)$. $\square$

13.3 Projective Nullstellensatz

Theorem 13.1 (Projective Hilbert's Nullstellensatz)**.** *We have the following:*

1. *For a projective variety $X \subseteq \mathbb{P}^n$, $V_p(I_p(X)) = X$.*
2. *For a homogeneous ideal $J \leq k[x_0, \ldots, x_n]$ with $\sqrt{J} \neq (x_0, \ldots, x_n)$, $I_p(V_p(J)) = \sqrt{J}$.*

As a consequence, there is a bijection between projective varieties and radical homogeneous ideals of $k[x_0, \ldots, x_n]$ which are not equal to $(x_0, \ldots, x_n)$, given by $X \mapsto I_p(X)$ with inverse $J \mapsto V_p(J)$.

Proof. (1) This is similar to the affine case.

(2) Fix a homogeneous ideal $(1) \neq J \leq k[x_0, \ldots, x_n]$ such that $\sqrt{J} \neq (x_0, \ldots, x_n)$ (the theorem is clearly true for the unit ideal). Then observe that we can write

$$\begin{aligned} I_p(V_p(J)) &= (f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in V_p(J)) \\ &= (f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J) \setminus \{0\}) \\ &= (f \in k[x_0, \ldots, x_n] : f(x) = 0 \text{ for all } x \in \overline{V_a(J) \setminus \{0\}}) \\ &= \begin{cases} I_a(V_a(J)) & \text{if } V_a(J) \supsetneq \{0\}, \qquad \text{(A)} \\ I_a(\varnothing) & \text{if } V_a(J) = \{0\}, \qquad \text{(B)} \end{cases} \end{aligned}$$

In Case A, we get that $I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J}$ by the affine Nullstellensatz. In Case B, we have $V_a(J) = \{0\}$, so $\sqrt{J} = (x_0, \ldots, x_n)$, which we assumed was not the case. □

13.4 The Zariski Topology on $\mathbb{P}^n$

Remark. We have the following properties of I_p and V_p:

1. For homogeneous ideals $J_i \leq k[x_0, \ldots, x_n]$ for $i \in I$, we have $V_p(\sum_{i \in I} J_i) = \bigcap_{i \in I} V_p(J_i)$;

   If $I = \{1, 2\}$, then we have $V_p(J_1 J_2) = V_p(J_1) \cup V_p(J_2)$.
2. If $X_1, X_2 \subseteq \mathbb{P}^n$ are projective varieties, then

   $$I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2) \quad \text{and} \quad I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)},$$

   where we assume in the second equality that $X_1 \cap X_2 \neq \varnothing$.

The proofs are similar to the affine case.

Example 13.1.2. Let $X_1 = V(x) \subseteq \mathbb{P}^2$ and $X_2 = V(y, z) \subseteq \mathbb{P}^2$. Then $I(X_1 \cap X_1) = I(\varnothing) = (1)$, but we have $I(X_1) + I(X_2) = (x, y, z)$, which is already radical.

Definition 13.2. The *Zariski topology* on $\mathbb{P}^n$ is the topology whose closed sets are projective varieties $X \subseteq \mathbb{P}^n$ (equivalently, the vanishing loci of homogeneous ideals).

Remark. This is a topology by the above properties of I_p and V_p. We now want to relate this to the topology on our charts. Let $H_0 = V(x_0)$ and consider the bijection

$$\begin{aligned} \mathbb{A}^n &\xrightarrow{\rho_0} \mathbb{P}^n \setminus H_0 \\ (x_1, \ldots, x_n) &\longmapsto [1 : x_1 : \cdots : x_n]. \end{aligned}$$

We want to show that ρ_0 is a homeomorphism. Write $\mathbb{A}^n \subseteq \mathbb{P}^n$. Consider the ring homomorphism

$$\begin{aligned} k[x_0, \ldots, x_n] &\xrightarrow{\Phi} k[x_1, \ldots, x_n] \\ f(x_0, \ldots, x_n) &\longmapsto f(1, x_1, \ldots, x_n) =: f^i \end{aligned}$$

We call f^i the *dehomogenization* of f.

Example 13.2.1. Let $f(x) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3$, then $f^i(x) = x_2^2 - x_1^3 - x_1^2 - 1$.

Definition 13.3. If $J \leq k[x_0, \ldots, x_n]$ is homogeneous, then define its *dehomogenization* to be

$$J^i = (f^i : f \in J) = \Phi(J).$$

Proposition 13.2. *For $J \leq k[x_0, \ldots, x_n]$ homogeneous, $V_p(J) \cap \mathbb{A}^n = V_a(J^i)$.*

Proof. The idea is to use that for $[1 : x_1 : \cdots : x_n] \in \mathbb{P}^n$ and $f \in k[x_0, \ldots, x_n]$ homogeneous, we have $f([1 : x]) = 0$ if and only if $f^i(x) = 0$. Fill in the details as an exercise. □

Definition 13.4. If $f \in k[x_1, \ldots, x_n]$ with $\deg f = d$, then define its *homogenization* to be

$$f^h = x_0^d f(x_1/x_0, \ldots, x_n/x_0) \in k[x_0, x_1, \ldots, x_n],$$

which is homogeneous of degree d.

Example 13.4.1. Let $f = x_2^2 - x_1^3 - x_1^2 - 1$. Then we have

$$f^h = x_0^3((x_2/x_0)^2 - (x_1/x_0)^3 - (x_1/x_0)^2 - 1) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3.$$

Remark. While $f^h g^h = (fg)^h$, note that $(f+g)^h \neq f^h + g^h$ in general.

Definition 13.5. For $J \leq k[x_1, \ldots, x_n]$ an ideal, define its *homogenization* to be

$$J^h = (f^h : f \in J).$$

Proposition 13.3. *For $J \leq k[x_1, \ldots, x_n]$ an ideal, $V_a(J) = V_p(J^h) \cap \mathbb{A}^n$.*

Proof. Left as an exercise, use that $f(a_1, \ldots, a_n) = 0$ if and only if $f^h(1, a_1, \ldots, a_n) = 0$. □

Remark. The above results imply that $\rho_0 : \mathbb{A}^n \to \mathbb{P}^n \setminus H_0$ is a homeomorphism.

Lecture 14

Oct. 9 — Projective Space as Varieties

14.1 More on the Zariski Topology on $\mathbb{P}^n$

Proposition 14.1. *For each $0 \le i \le n$, the map*

$$\begin{aligned} U_i = \mathbb{P}^n \setminus V(x_i) &\xrightarrow{h_i} \mathbb{A}^n \\ [x_0 : \cdots : x_n] &\longmapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i) \end{aligned}$$

is a homeomorphism.

Proof. The main inputs to the proof are

- For $I \le k[x_0, \ldots, x_n]$ homogeneous, $h_0(V(I) \cap U_0) = V(I^i)$.
- For $J \le k[x_1, \ldots, x_n]$, $h_0^{-1}(V(J)) = V(J^h)$.

Fill in the remaining details as an exercise. □

Proposition 14.2 (Projective closure)**.** *For $J \le k[x_1, \ldots, x_n]$ and $X = V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$, we have*

$$\overline{X} = V_p(J^h).$$

Proof. See Gathmann. □

Proposition 14.3. *If $X = V_a(f) \subseteq \mathbb{A}^n$ with $f \in k[x_1, \ldots, x_n]$, then its projective closure in $\mathbb{P}^n$ is*

$$\overline{X} = V_p(f^h).$$

Proof. We know that $\overline{X} = V_p(\langle f \rangle^h)$ by Proposition 14.2. Now

$$\langle f \rangle^h = \langle (fg)^h : g \in k[x_1, \ldots, x_n] \rangle = \langle f^h g^h : g \in k[x_1, \ldots, x_n] \rangle = \langle f^h \rangle,$$

which implies the desired result. □

Example 14.0.1 (Twisted cubic)**.** Take $X = \operatorname{Im}(\mathbb{A}^1 \to \mathbb{A}^3 : t \mapsto (t, t^2, t^3))$. Note that $X \cong \mathbb{A}^1$, and

$$I_a(X) = (x^2 - y, x^3 - z) = (x^2 - y, x^3 - z, xy - z).$$

Then one can check that $\overline{X} \subseteq \mathbb{P}^3_{w:x:y:z}$ is given by $\overline{X} = V_p(x^2 - yw, x^3 - zw^2, xy - zw)$. However, one can also check that $\overline{X}$ cannot be cut out by 2 equations. For example,

$$V_p(x^2 - yw, x^3 - zw^2) = \overline{X} \cup V(w, x).$$

14.2 Projective Space as Varieties

Remark. Our goal now is to show that projective varieties are varieties. The first step is to define a sheaf of regular functions on $\mathbb{P}^n$.

Definition 14.1. Let U be an open set of a projective variety $X \subseteq \mathbb{P}^n$. A function $\varphi : U \to k$ is *regular* if for every $p \in U$, there exists $d \in \mathbb{N}$, $f, g \in k[x_0, \dots, x_n]$ homogeneous of degree d, and $U_p \subseteq U$ open such that

$$\varphi(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in U_p.$$

Remark. If $X \subseteq \mathbb{P}^n$ is a projective variety, then

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}$$

is a sheaf of rings on X. Again this is because the regular condition can be checked locally.

Proposition 14.4. *If $X \subseteq \mathbb{P}^n$ is a projective variety, then $(X, \mathcal{O}_X)$ is a pre-variety.*

Proof. Let $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$. It suffices to show $(X_i, \mathcal{O}_X|_{X_i})$ is an affine variety for each $0 \le i \le n$. For simplicity, assume $i = 0$. Let $J = I(X) \le k[x_0, \dots, x_n]$ and $Z_0 = V(J^i) \subseteq \mathbb{A}^n$. We have seen before that we have a homeomorphism

$$\begin{aligned} X_0 &\xrightarrow{F} Z_0 \\ [x_0 : \cdots : x_n] &\longmapsto (x_1/x_0, \dots, x_n/x_0). \end{aligned}$$

We claim that F induces an isomorphism of ringed spaces $(X_0, \mathcal{O}_X|_{X_0}) \cong (Z_0, \mathcal{O}_{Z_0})$. To see this, we need to check that regular functions pull back to regular functions via F and F^{-1}. A regular function on an open set of X_0 is locally of the form

$$\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)}$$

with f, g homogeneous of the same degree. Now

$$(F^{-1})^* \left(\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)} \right) = \frac{f(1, x_1, \dots, x_n)}{g(1, x_1, \dots, x_n)},$$

which is a fraction of polynomials and hence regular on Z_0. So F^{-1} pulls regular functions back to regular functions. Conversely, a regular function on Z_0 is locally given by

$$\frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)},$$

and its pullback via F is

$$F^* \left(\frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)} \right) = \frac{q(x_1/x_0, \dots, x_n/x_0)}{r(x_1/x_0, \dots, x_n/x_0)} = \frac{x_0^d q(x_1/x_0, \dots, x_n/x_0)}{x_0^d r(x_1/x_0, \dots, x_n/x_0)},$$

where $d = \max\{\deg q, \deg r\}$. This is regular on X_0, so F also pulls regular functions back to regular functions. So we get an isomorphism of ringed spaces, as desired. $\square$

Example 14.1.1. $\mathbb{P}^n$ is a pre-variety, and $\mathbb{P}^n \setminus V(x_i) =: U_i \cong \mathbb{A}^n$ as pre-varieties.

Definition 14.2. A *morphism* of projective varieties is a morphism of the underlying pre-varieties.

Remark. For a projective variety X, it will be convenient to work with "global coordinates," i.e.

$$S(X) := k[x_0, \dots, x_n]/I_p(X).$$

This is called the *homogeneous coordinate ring.* Note the following:

1. For $f \in S(X)$ homogeneous, f is not necessarily a well-defined function on X. But
$$V(f) = \{[x] \in X : f(x) = 0\}$$
is still well-defined.

2. A relative version of the projective Nullstellensatz holds: There is a bijection
$$\left\{\begin{matrix}\text{projective subvarieties}\\ Y \subseteq X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{radical homogeneous ideals in } S(X)\\ \text{not equal to } (\overline{x}_1, \dots, \overline{x}_n)\end{matrix}\right\}$$
$$Y \longmapsto I(Y)$$
$$V(J) \longleftarrow J$$

where $I(Y) = \langle f \in S(X) : f \text{ homogeneous and } f(y) = 0 \text{ for all } y \in Y\rangle$.

Lemma 14.1. *If $X \subseteq \mathbb{P}^n$ and $f_0, \dots, f_m \in S(X)$ are homogeneous of the same degree, then*

$$\begin{aligned} U = X \setminus V(f_0, \dots, f_m) &\xrightarrow{f} \mathbb{P}^m \\ [x_0 : \dots : x_n] &\longmapsto [f_0(x) : \dots : f_m(x)] \end{aligned}$$

is a morphism.

Proof. To see that f is well-defined, note that for $[a_0 : \dots : a_n] \in X \setminus V(f_0, \dots, f_m)$, we have

$$(f_0(\lambda a), \dots, f_m(\lambda a)) = \lambda^d (f_0(a), \dots, f_m(a))$$

with $d = \deg f_i$. So $[f_0(a) : \dots : f_m(a)] \in \mathbb{P}^m$ is well-defined. To see that f is a morphism, we check locally on $\mathbb{P}^m$. Let $V_i = \mathbb{P}^m \setminus V(x_i)$ and $U_i = f^{-1}(V_i)$. Then

$$\begin{aligned} U_i &\xrightarrow{f|_{U_i}} V_i \cong \mathbb{A}^m \\ a &\longmapsto \left(\frac{f_0(a)}{f_i(a)}, \dots, \frac{\widehat{f_i(a)}}{f_i(a)}, \dots, \frac{f_m(a)}{f_i(a)}\right). \end{aligned}$$

Since each f_j/f_i is regular, $f|_{U_i}$ is a morphism. So f is a morphism. $\square$

Example 14.2.1. Define a map

$$\begin{aligned} \mathbb{P}^1_{s:t} &\xrightarrow{f} \mathbb{P}^3_{x:y:z} \\ [s : t] &\longmapsto [s^3 : s^2t : st^2 : t^3]. \end{aligned}$$

Then $S(\mathbb{P}^1) = k[s, t]$ and $f(\mathbb{P}^1)$ is the projective twisted cubic in $\mathbb{P}^3$.

Example 14.2.2. Let $A \in \mathrm{GL}_{n+1}(k)$. Then

$$\begin{aligned} f_A : \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

is an isomorphism with inverse $f_{A^{-1}}$. We will see later that we have a surjective group homomorphism

$$\begin{aligned} \mathrm{GL}_{n+1}(k) &\longrightarrow \mathrm{Aut}(\mathbb{P}^n) \\ A &\longmapsto f_A \end{aligned}$$

with kernel $k^\times I$. So we get $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{GL}_{n+1}(k)/k^\times I =: \mathrm{PGL}_{n+1}(k)$.

Example 14.2.3 (Conics)**.** Let $f \in k[x, y, z]$ be homogeneous of degree 2, and write

$$f = (x, y, z)B(x, y, z)^T$$

with B a symmetric 3×3 matrix. We want to characterize $X = V(f)$. Choose $A \in \mathrm{GL}_3(k)$ such that

$$B' = ABA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $f' = (x, y, z)B'(x, y, z)^T$ has $f' = x^2 + y^2 + z^2$, $x^2 + y^2$, or x^2. Now A induces an isomorphism $h_{A^{-1}} : \mathbb{P}^2 \to \mathbb{P}^2$ and $g := h_{A^{-1}}|_X : X \to h_{A^{-1}}(X) = V(f')$, so any projective conic is isomorphic to

$$V(x^2 + y^2 + z^2), \quad V(x^2 + y^2), \quad \text{or} \quad V(x^2).$$

Example 14.2.4 (Projections)**.** Let $a = [1 : 0 : \cdots : 0]$ and define

$$\begin{aligned} \mathbb{P}^n \setminus \{a\} &\xrightarrow{f} \mathbb{P}^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_1 : \cdots : x_n]. \end{aligned}$$

Geometrically, if we fix $[b] \in \mathbb{P}^n \setminus \{a\}$ and set

$$\ell_{a,b} = \{[s : tb_1 : \cdots : tb_n] : (s, t) \in k^2 \setminus \{0\}\} = \text{the line through } a \text{ and } b,$$

then $\ell_{a,b} \cap V(x_0) = [0 : b_1 : \cdots : b_n] = [0 : f(b)]$.

Lecture 15

Oct. 14 — Projective Space as Varieties, Part 2

15.1 Example of Projective Morphism

Example 15.0.1 (Projections, continued)**.** Let $H \subseteq \mathbb{P}^n$ be a hyperplane and $p \notin H$. Then we can define

$$\begin{aligned} \mathbb{P}^n \setminus \{p\} &\xrightarrow{\pi} H \cong \mathbb{P}^{n-1} \\ q &\longmapsto \text{intersection point of } H \text{ and } \overline{pq}. \end{aligned}$$

For example, when $n = 2$, $p = [1 : 0 : 0] \in \mathbb{P}^2_{x_0:x_1:x_2}$, and $H = V(x_0)$, then we have

$$\begin{aligned} \mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2]. \end{aligned}$$

Note that this does not extend to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. But if we let $X = V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^2$, then the restriction of the above morphism to X:

$$\begin{aligned} X \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2]. \end{aligned}$$

does extend to a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto \begin{cases} [x_1 : x_2] & \text{if } [x] \neq [1 : 0 : 0], \\ [x_2 : x_0] & \text{if } [x] \neq [1 : 1 : 0]. \end{cases} \end{aligned}$$

15.2 The Segre Embedding

Remark. We now want to show that projective varieties are varieties, and understand and analogue of compactness in algebraic geometry. To do this, we will need to understand products.

Definition 15.1. Fix $m, n \geq 0$. The *Segre embedding* is the map $\Sigma : \mathbb{P}^m_{x_i} \times \mathbb{P}^n_{y_i} \to \mathbb{P}^N_{z_{i,j}}$ given by

$$\Sigma([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [x_iy_j : 0 \leq i \leq m, 0 \leq j \leq n],$$

where $N = (m + 1)(n + 1) - 1$.

Proposition 15.1. *Let Σ be the Segre embedding. Then*

1. $X = \Sigma(\mathbb{P}^m \times \mathbb{P}^n) = V(z_{i,j}z_{k,\ell} - z_{i,\ell}z_{k,j} : 0 \leq i, k \leq m,\ 0 \leq j, \ell \leq n)$.
2. *The map $\mathbb{P}^m \times \mathbb{P}^n \to X$ is an isomorphism, i.e. Σ is a closed embedding.*

Proof. (0) First one can check that Σ is a morphism. To do this, restrict to charts.

(1) Fix $[a_{i,j}] \in \mathbb{P}^N$. Then $[a_{i,j}] \in \operatorname{Im}\Sigma$ if and only if the matrix $(a_{i,j})$ has rank 1, which occurs if and only if all 2×2 minors of $(a_{i,j})$ vanish, which happens if and only if

$$a_{i,j}a_{k,\ell} - a_{i,\ell}a_{k,j} = 0$$

for all i, j, k, ℓ for which the above equation makes sense.

(2) We define a morphism $X \to \mathbb{P}^m \times \mathbb{P}^n$ that will be inverse to Σ. Set $U_{i,j} = X \cap \{z_{i,j} \neq 0\}$. Define

$$\begin{aligned} U_{i,j} &\xrightarrow{h_{i,j}} \mathbb{P}^m \times \mathbb{P}^n \\ [z_{i,j}] &\longmapsto ([z_{0,j} : \cdots : z_{m,j}], [z_{i,0} : \cdots : z_{i,n}]). \end{aligned}$$

Using the definition of X (as the set of rank 1 matrices up to scaling), these glue to give a morphism $X \to \mathbb{P}^m \times \mathbb{P}^n$ that is inverse to Σ. $\square$

Example 15.1.1. Let $m = n = 1$. Then the Segre embedding is given by

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3_{x:y:z:w} \\ ([a_0 : a_1], [b_0 : b_1]) &\longmapsto \begin{bmatrix} a_0b_0 & a_0b_1 \\ a_1b_0 & a_1b_1 \end{bmatrix}. \end{aligned}$$

Then $\operatorname{Im}\Sigma = V(xw - yz)$. Observe that the images of $\{a\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{b\}$ in $\Sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ are two families of lines, where the lines within each families do not intersect.

Remark. The following are consequences of the Segre embedding.

1. We can study products of projective varieties.

   Definition 15.2 (Redefinition of projective variety)**.** A *projective variety* is a (pre-)variety X such that there exists a closed embedding $X \hookrightarrow \mathbb{P}^n$ for some $n \geq 0$.

   Now using the Segre embedding, we get that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. Moreover, if $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ are projective varieties, then so is $X \times Y$.

2. We can show that $\mathbb{P}^n$ is separated.

   Lemma 15.1. *$\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.*

   Proof. Observe that

   $$\Delta_{\mathbb{P}^n} = \{([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \in \mathbb{P}^n \times \mathbb{P}^n : x_iy_j - x_jy_i = 0 \text{ for all } 0 \leq i, j \leq n\}.$$

   It suffices to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. There are two ways to see this. The first is to use the Segre embedding $\mathbb{P}^n_{x_i} \times \mathbb{P}^n_{y_i} \underset{\text{cl}}{\overset{\Sigma}{\hookrightarrow}} \mathbb{P}^N_{z_{i,j}}$ Then we can write

   $$\Sigma(\Delta_{\mathbb{P}^n}) = \Sigma(\mathbb{P}^n \times \mathbb{P}^n) \cap V(z_{i,j} - z_{j,i} : 0 \leq i, j \leq n),$$

   which is closed in $\mathbb{P}^N$, so $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. Alternatively, one can just compute $\Delta_{\mathbb{P}^n}$ directly on the affine charts. One can fill in the details of this method as an exercise. $\square$

Proposition 15.2. *Projective varieties are varieties.*

Proof. We have already seen that they are pre-varieties, so it suffices to show that they are separated. By Lemma 15.1, $\mathbb{P}^n$ is separated. Thus any closed sub-prevariety of $\mathbb{P}^n$ is also separated. $\square$

15.3 Completeness

Remark. We now want an analogue of compactness in algebraic geometry. One issue is that all varieties are compact to begin with, but $\mathbb{A}^n$ has points missing in some sense.

Example 15.2.1. Consider the projection map

$$\begin{aligned} \mathbb{A}^1 \times \mathbb{A}^1 &\xrightarrow{\mathrm{pr}_2} \mathbb{A}^1 \\ (x,t) &\longmapsto t. \end{aligned}$$

Then $X = V(xt-1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ is closed, but $\mathrm{pr}_2(X) = \mathbb{A}^1 \setminus \{0\}$. If we instead viewed this over $\mathbb{P}^1$:

$$\begin{aligned} \mathbb{P}^1_{[x:y]} \times \mathbb{A}^1_t &\xrightarrow{\mathrm{pr}_2} \mathbb{A}^1_t \\ ([x:y],t) &\longmapsto t \end{aligned}$$

with $\overline{X} = V(xt-y)$, then $\mathrm{pr}_2(\overline{X}) = \mathbb{A}^1$ as there is a point $([1:0],0)$ at infinity in $\overline{X}$. In other words, "compactifying" $\mathbb{A}^1$ to $\mathbb{P}^1$ gives the desired missing point.

Definition 15.3. A morphism $f : X \to Y$ is *closed* if $f(Z)$ is closed in Y for all closed sets $Z \subseteq X$.

Definition 15.4. A variety X is *complete* if the projection

$$\mathrm{pr}_2 : X \times Y \longrightarrow Y$$

is closed for all varieties Y.

Remark. The same definition for topological spaces gives the usual notion of compactness.

Example 15.4.1. Example 15.2.1 shows that $\mathbb{A}^1$ is not complete. Similar examples show that $\mathbb{A}^n$ is not complete for any $n \geq 1$.

Proposition 15.3. $\mathbb{P}^n$ *is complete.*

Proof. The steps to show this are the following:

1. For any $m, n \geq 0$, the projection $\mathrm{pr}_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

   See Gathmann for a proof of this fact.

2. If Y is an affine variety, then $\mathrm{pr}_2 : \mathbb{P}^n \times Y \to Y$ is closed.

   To see this, write $Y = V(I) \subseteq \mathbb{A}^m \subseteq \mathbb{P}^m$ and consider the diagram

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^m \arrow[r, "\mathrm{pr}_2"] & \mathbb{P}^m \\ \mathbb{P}^n \times \mathbb{A}^n \arrow[u, hook, "\mathrm{op}"] \arrow[r, "\mathrm{pr}_2"] & \mathbb{A}^n \arrow[u, hook] \\ \mathbb{P}^n \times Y \arrow[u, hook, "\mathrm{cl}"] \arrow[r, "\mathrm{pr}_2"] & Y \arrow[u, hook] \end{tikzcd}$$

Since the top row is closed, so is the bottom row.

Finally, we complete the proof. If Y is a variety, then it admits an open affine cover $Y = \bigcup_{i=1}^r U_i$. Now $\mathrm{pr}_2 : \mathbb{P}^n \times Y \to Y$ is closed when restricted to $\mathrm{pr}_2^{-1}(U_i)$. Since closedness of a map can be checked on an open cover of the target, we see that $\mathbb{P}^n \times Y \to Y$ is closed. $\square$

Remark. The same definition and arguments for completeness work if Y is replaced by a pre-variety.

Exercise 15.1. Show that if X is a complete variety, then so is any closed subvariety of X.

Corollary 15.0.1. *Any projective variety is complete.*

Lecture 16

Oct. 16 — Completeness and Embeddings

16.1 More on Completeness

Example 16.0.1. Recall from before that we have:

1. $\mathbb{A}^n$ is not complete for $n \geq 1$.
2. $\mathbb{P}^n$ is complete.
3. Any projective variety is complete.

Remark. Note the following:

1. If $k = \mathbb{C}$, then a variety is complete if and only if X^{an} is compact in the analytic topology.
2. *Nagata's compactification theorem*: Any variety X admits an open embedding $X \hookrightarrow \overline{X}$ with $\overline{X}$ complete.
3. In dimension 1, completeness is equivalent to being projective. In dimension ≥ 2, being projective implies completeness, but the converse may fail.

Proposition 16.1. *If $f : X \to Y$ is a morphism of varieties with X complete, then*

1. *$f(X)$ is closed in Y.*
2. *$f(X)$ is complete.*

Proof. (1) Consider the projection $\mathrm{pr}_2 : X \times Y \to Y$. As Y is separated, the graph Γ_f of f is closed in $X \times Y$. Now $f(X) = \mathrm{pr}_2(\Gamma_f)$, which is closed in Y as X is complete.

(2) Fix any variety Z. Now consider the projection $\pi' : f(X) \times Z \to Z$ and $W \subseteq f(X) \times Z$ closed. Now consider the projection $\pi : X \times Z \to Z$. Then $\pi'(W) = \pi((f, \mathrm{id})^{-1}(W))$. The set $(f, \mathrm{id})^{-1}(W)$ is closed in $X \times Z$ by continuity, and $\pi'(W)$ is closed in Z as X is complete. So $f(X)$ is complete. $\square$

Corollary 16.0.1. *If X is a complete variety that is connected, then any $\varphi \in \mathcal{O}_X(X)$ is constant. In particular, $\mathcal{O}_X(X) \cong k$.*

Proof. Any $\varphi \in \mathcal{O}_X(X)$ induces a morphism $f : X \to \mathbb{P}^1_{s:t}$ by $x \mapsto [1 : \varphi(x)]$. By construction, we have $f(X) \subseteq \{s \neq 0\} \cong \mathbb{A}^1$. As X is complete, $f(X)$ is closed, and as X is connected, $f(X)$ is connected. Since the only proper closed subsets of $\mathbb{P}^1$ are points, $f(X)$ be must a single point $[1 : a]$ since $f(X)$ is connected. So $\varphi(x) = a$ for every $x \in X$. $\square$

Corollary 16.0.2. *The only complete affine varieties are finite point sets.*

Proof. Assume $X = V(I) \subseteq \mathbb{A}^n$ is complete. Using the decomposition of X into connected components (there finitely many since X is Noetherian), we may reduce to the case that X is connected. Then $A(X) = \mathcal{O}_X(X) \cong k$, but this happens if and only if X is a single point (see Homework 1). $\square$

16.2 The Veronese Embedding

Remark. If $X \subseteq \mathbb{P}^n$ is a projective variety, which open sets in X are affine?

- $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$ is affine.
- $X \setminus (\mathbb{P}^n \setminus H)$ with $H \subseteq \mathbb{P}^n$ a hyperplane is affine.

We will see that $X \cap (\mathbb{P}^n \setminus V(g))$ is affine with $g \in k[x_0, \ldots, x_n]$ homogeneous of degree $d > 0$.

Definition 16.1 (Veronese embedding, d-tuple embedding)**.** Fix $n, d > 0$. Let $f_0, \ldots, f_N \in k[x_0, \ldots, x_n]$ denote the monomials of degree d, where $N = \binom{n+d}{d} - 1$. The *Veronese embedding* $\nu_{n,d}$ is the map

$$\begin{aligned} \nu_{n,d} : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ x &\longmapsto [f_0(x) : \cdots : f_N(x)]. \end{aligned}$$

Example 16.1.1. Let $n = 1$, $d = 3$. Then the degree-3 Veronese embedding is given by

$$\begin{aligned} f : \mathbb{P}^1 &\longmapsto \mathbb{P}^3_{s:t:u:v} \\ [x : y] &\longmapsto [x^3 : x^2y : xy^2 : y^3]. \end{aligned}$$

Then $X = f(\mathbb{P}^1) = V(sv - tu, t^2 - su, u^2 - vt)$. We can define an inverse

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1 \\ [s : t : u : v] &\longmapsto \begin{cases} [1 : t/s] & \text{if } s \neq 0, \\ [u/v : 1] & \text{if } v \neq 0. \end{cases} \end{aligned}$$

Note that $ut = sv$ on X when $sv \neq 0$, so this is well-defined.

Proposition 16.2. *$\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ is a closed embedding.*

Proof. As $\mathbb{P}^n$ is complete, $X = \nu_{n,d}(\mathbb{P}^n)$ is closed in $\mathbb{P}^N$. A similar computation as in Example 16.1.1 shows that $\nu_{n,d} : \mathbb{P}^n \to X$ has an inverse. So $\nu_{n,d}$ is a closed embedding. $\square$

Remark. Note the following:

1. With some work, one can show that $\nu_{n,d}(\mathbb{P}^n)$ can be cut out by quadratic equations

$$\{z_i z_j - z_k z_\ell : f_i f_j = f_k f_\ell \text{ as monomials}\}.$$

2. If $g \in k[x_0, \ldots, x_n]$ is homogeneous of degree $d > 0$, then

$$\mathbb{P}^n \supseteq V(g) = \nu_{n,d}^{-1}(H)$$

for some hyperplane $H \subseteq \mathbb{P}^N$.

Theorem 16.1. *If $X \subseteq \mathbb{P}^n$ is a projective variety, then for any $g \in k[x_0, \dots, x_n]$ homogeneous of degree d, the variety $X \setminus V(g)$ is affine.*

Proof. Consider the Veronese embedding $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$. Then $V(g) = \nu_{n,d}^{-1}(H)$ for some hyperplane $H \subseteq \mathbb{P}^N$. So $X \setminus V(g) \cong \nu_{n,d}(X) \setminus H$, which is affine. □

16.3 The Grassmannian

Definition 16.2. For $0 \le d \le n$, define the *Grassmannian*

$$G(d, n) := \{d\text{-dimensional subspaces of } k^n\}.$$

Example 16.2.1. We have the following:

- $G(0, n)$ and $G(n, n)$ are just points.
- $G(1, n) = \mathbb{P}^{n-1}$ set-theoretically.
- $G(d, n) \cong G(n - d, n)$ as dimension d subspaces of $V = k^n$ are in bijection with dimension $n - d$ subspaces of V^*, where $W \subseteq V$ corresponds to $\ker(V^* \to W^*)$.
- $G(2, n) = \{\text{lines in } \mathbb{P}^{n-1}\}$.

Theorem 16.2. *The Grassmannian $G(d, n)$ can be endowed with the structure of a (projective) variety of dimension $d(n - d)$.*

Proof. One strategy is to let $V = k^n = \operatorname{span}\{e_1, \dots, e_n\}$. We sketch this idea. Observe that

1. A d-dimensional subspace $W \subseteq V$ can be represented by a $d \times n$ matrix A of rank d (choose a basis of W and write the coordinates with respect to the basis $\{e_1, \dots, e_n\}$ of V). Note that A is unique up to the action by $\mathrm{GL}_d(k)$, so we get a point $[A] \in G(d, n)$.
2. For $I = \{1, \dots, n\}$ with $|I| = d$, define the set

$$U_I = \{[A] \in G(d, n) : \det(A_I) \neq 0\},$$

where A_I denotes the I-th $d \times d$ minor of A. One can check that the condition $\det(A_I) \neq 0$ is well-defined (i.e. $\det((BA)_I) = \det(B)\det(A_I)$ for $B \in \mathrm{GL}_d(k)$).

Then we have a bijection $\mathbb{A}^{d(n-d)} \to U_I$ given as follows. When $I = \{1, \dots, d\}$, define

$$\begin{aligned} \mathbb{A}^{d(n-d)} &\longrightarrow U_I \\ C &\longmapsto [I_d \mid C]. \end{aligned}$$

One can make a similar definition for other I. Note that $G(d, n) = \bigcup_I U_I$.

3. Show that the U_I glue to give $G(d, n)$ the structure of a variety.

The second strategy is to use the wedge product $\bigwedge^d V$. Recall that

- $\bigwedge^d V$ has basis given by $e_I := e_{i_1} \wedge \cdots \wedge e_{i_d}$ with $I = \{i_1 < \cdots < i_d\} \subseteq \{1, \dots, n\}$.
- For $v_1, \dots, v_d \in V$ with $v_i = \sum_{j=1}^d a_{i,j} e_j$, we have $v_1 \wedge \cdots \wedge v_d = \sum_{I \subseteq \{1,\dots,n\}} \det(a_I) e_I$.

The following are two linear algebra lemmas we will use:

Lemma 16.1. *Let $v_1, \dots, v_d \in V$. Then $v_1, \dots, v_d$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_d \neq 0$.*

Proof. Use the determinant formula. □

Lemma 16.2. *For linearly independent sets $\{v_1, \dots, v_d\}, \{w_1, \dots, w_d\} \subseteq V$, then $v_1 \wedge \cdots \wedge v_d$ and $w_1 \wedge \cdots \wedge w_d$ are linearly dependent in $\bigwedge^d V$ if and only if $\operatorname{span}\{v_1, \dots, v_d\} = \operatorname{span}\{w_1, \dots, w_d\}$.*

Proof. ($\Leftarrow$) It suffices to show that $k \cdot v_1 \wedge \cdots \wedge v_d \subseteq \bigwedge^d V$ is preserved under change of basis operations. Check this as an exercise.

($\Rightarrow$) We will prove this next class. □

We will complete the proof next class. □