

# MATH 6421: Algebraic Geometry I

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# Lecture 1

## Aug. 19 — Affine Varieties

### 1.1 Motivation for Algebraic Geometry

**Remark.** Why study algebraic geometry? Algebraic geometry connects to many fields of math.

**Example 1.0.1.** Consider a plane curve  $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$ , e.g. an elliptic curve  $z_2^2 - z_1^3 + z_1 - 1 = 0$ . Compactify and set  $C$  to be the closure of  $C^0$  in  $\mathbb{CP}^2$ , and let  $d = \deg f$ . There are connections in

1. Topology:  $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$ , where  $g = (d-1)(d-2)/12$ ;
2. Arithmetic: the number of  $\mathbb{Q}$ -points is finite if  $d > 3$ ;
3. Complex geometry: We have  $C \cong \mathbb{CP}^2$  for  $d = 1, 2$ ,  $C \cong \mathbb{C}/\Lambda$  for  $d = 3$ , and  $C \cong \mathbb{H}/\Gamma$  for  $d > 3$ .

### 1.2 Affine Varieties

Fix an algebraically closed field  $k$  (e.g.  $\mathbb{C}$ ,  $\overline{\mathbb{Q}}$ ,  $\overline{\mathbb{F}}_p$ , etc.).

**Definition 1.1.** *Affine space* is the set  $\mathbb{A}^n = \mathbb{A}_k^n = \{\vec{a} = (a_1, \dots, a_n) : a_i \in k\}$ .

**Remark.** Note the following:

1.  $\mathbb{A}_k^n$  is the same set as  $k^n$ , but forgetting the vector space structure;
2.  $f \in k[x_1, \dots, x_n]$  gives a polynomial function  $\mathbb{A}_k^n \rightarrow k$  by evaluation:  $a \mapsto f(a)$ .

**Definition 1.2.** For a subset  $S \subseteq k[x_1, \dots, x_n]$ , its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of  $\mathbb{A}_k^n$  of this form.

**Example 1.2.1.** Consider the following:

1.  $\mathbb{A}^n = V(\emptyset) = V(\{0\})$ ;
2.  $\emptyset = V(1) = V(k[x_1, \dots, x_n])$ ;
3. a point  $a = (a_1, \dots, a_n)$  is an affine variety:  $V(\{x_1 - a_1, \dots, x_n - a_n\}) = \{a\}$ ;
4. a linear space  $L \subseteq \mathbb{A}^n$  (it is the kernel of some matrix);
5. plane curves  $V(f(x, y)) \subseteq \mathbb{A}_{x,y}^2$ ;

6.  $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n \times n}$  is an affine variety:  $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$ ;
7.  $\mathrm{GL}_n(k)$  (as a set) is an affine variety in  $\mathbb{A}^{n \times n+1}$ :  $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$ ;
8. if  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  are affine varieties, then  $X \times Y \subseteq \mathbb{A}^{m+n}$  is an affine variety;
9. the affine varieties  $X \subseteq \mathbb{A}_k^1$  are of the form: finite set of points,  $\emptyset$ , or  $\mathbb{A}_k^1$ .

**Proposition 1.1** (Relation to ideals). *If  $S \subseteq k[x_1, \dots, x_n]$ , then  $V(S) = V(\langle S \rangle)$ , where  $\langle S \rangle$  is the ideal generated by  $S$ .*

*Proof.* Since  $S \subseteq \langle S \rangle$ , we have  $V(\langle S \rangle) \subseteq V(S)$ . Conversely, if  $f, g \in S$  and  $h \in k[x_1, \dots, x_n]$ , then  $f + g$  and  $hf$  vanish on  $V(S)$ , so we see that  $V(S) \subseteq V(\langle S \rangle)$ .  $\square$

**Remark.** The statement implies that if  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ , then  $V(f_1, \dots, f_r) = V((f_1, \dots, f_r))$ . The following are some further applications of the statement:

1. affine varieties are vanishing loci of ideals;
2. if  $X \subseteq \mathbb{A}^n$  is an affine variety, then  $X$  is cut out by finitely many polynomial equations.

To see the second statement, note that  $X = V(I)$  for some ideal  $I \leq k[x_1, \dots, x_n]$ . By the Hilbert basis theorem that  $k[x_1, \dots, x_n]$  is Noetherian, there are finitely many  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$  such that  $I = (f_1, \dots, f_r)$ . So  $X = V(I) = V(f_1, \dots, f_r)$ .

**Proposition 1.2** (Properties of the vanishing set). *For ideals  $I, J$  of  $k[x_1, \dots, x_n]$ ,*

1. *if  $I \subseteq J$ , then  $V(J) \subseteq V(I)$ ;*
2.  *$V(I) \cap V(J) = V(I + J)$ ;*
3.  *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$ .*

*Proof.* (1) This follows from definitions and actually holds for general subsets.

(2) Note that  $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J \rangle) = V(I + J)$ .

(3) We only prove the first equality, the second is similar. Recall that  $IJ = \{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\}$ . We have the forwards inclusion  $V(I) \cup V(J) \subseteq V(IJ)$  from definitions. For the reverse inclusion, consider a point  $x \notin V(I) \cup V(J)$ . So there exists  $f \in I$  and  $g \in J$  such that  $f(x), g(x) \neq 0$ . So  $f(x)g(x) \neq 0$ , which implies that  $x \notin V(IJ)$ . Thus  $V(IJ) \subseteq V(I) \cup V(J)$  as well.  $\square$

**Remark.** The above implies that if  $X$  and  $Y$  are affine varieties in  $\mathbb{A}_k^n$ , then so are  $X \cup Y$  and  $X \cap Y$ .

**Example 1.2.2.** Consider  $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$ . Note that  $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$ , from which we can easily see that  $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$ .

## 1.3 Correspondence with Ideals

**Remark.** Our goal is to build a correspondence between affine varieties in  $\mathbb{A}_k^n$  and ideals of  $k[x_1, \dots, x_n]$ .

**Definition 1.3.** For a subset  $X \subseteq \mathbb{A}_k^n$ , define

$$I(X) = \{f \in k[x_1, \dots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

**Remark.** Note that  $I(X)$  is in fact an ideal of  $k[x_1, \dots, x_n]$ .

**Example 1.3.1.** Consider the following:

1.  $I(\emptyset) = k[x_1, \dots, x_n]$ ;
2.  $I(\mathbb{A}_k^n) = \{0\}$ , this will follow from the Hilbert nullstellensatz and relies on  $k = \bar{k}$  (for  $k = \mathbb{R}$ , the polynomial  $x^2 + y^2 + 1$  is always nonzero and thus lies in  $I(\mathbb{A}_{\mathbb{R}}^n)$ );
3. for  $n = 1$ , if  $S \subseteq \mathbb{A}_k^1$  be an infinite set, then  $I(S) = (0)$ .
4. for  $n = 1$ , we have  $I(V(x^2)) = I(\{0\}) = (x)$ .

**Remark.** What properties does  $I(X)$  satisfy?

**Definition 1.4.** Let  $R$  be a ring. The *radical* of an ideal  $J \leq R$  is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal  $J$  is *radical* if  $J = \sqrt{J}$ .

**Exercise 1.1.** Check the following:

1.  $\sqrt{J}$  is always an ideal.
2.  $\sqrt{\sqrt{J}} = \sqrt{J}$ .
3. An ideal  $J \leq R$  is radical if and only if  $R/J$  is reduced.<sup>1</sup>

**Proposition 1.3.** If  $X \subseteq \mathbb{A}_k^n$  is a subset (not necessarily an affine variety), then  $I(X)$  is radical.

*Proof.* Fix  $f \in k[x_1, \dots, x_n]$ . If  $f^n \in I(X)$ , then  $f^n(x) = 0$  for all  $x \in X$ . This implies  $f(x) = 0$  for all  $x \in X$ , so  $f \in I(X)$ . Thus we see that  $I(X) = \sqrt{I(X)}$ .  $\square$

**Theorem 1.1** (Hilbert's nullstellensatz). If  $J \leq k[x_1, \dots, x_n]$  is an ideal, then  $I(V(J)) = \sqrt{J}$ .

**Example 1.4.1.** Let  $n = 1$ , so that  $k[x]$  is a PID. Let  $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$ . Then

$$I(V(f)) = I(\{a_1, \dots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

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<sup>1</sup>Recall that a ring  $R$  is *reduced* if for all nonzero  $f \in R$  and positive integers  $n$ , we have  $f^n \neq 0$ . It is immediate that an integral domain is reduced.

# Lecture 2

## Aug. 21 — Hilbert's Nullstellensatz

### 2.1 Applications of Hilbert's Nullstellensatz

**Corollary 2.0.1** (Weak nullstellensatz). *If  $J \leq k[x_1, \dots, x_n]$  is an ideal with  $J \neq (1)$ , then  $V(J) \neq \emptyset$ . Equivalently, if  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$  have no common zeros, then there exist  $g_1, \dots, g_r \in k[x_1, \dots, x_n]$  such that  $\sum_{i=1}^r f_i g_i = 1$ .*

*Proof.* Assume otherwise that  $V(J) = \emptyset$ . Then  $I(V(J)) = I(\emptyset) = (1)$ , so by Hilbert's nullstellensatz, we have  $\sqrt{J} = (1)$ . Then  $1^n \in J$  for some  $n > 0$ , so  $1 \in J$ , i.e.  $J = (1)$ .  $\square$

**Remark.** We need  $k$  to be algebraically closed. Note that  $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$  but  $V(x^2 + 1) = \emptyset$ .

**Corollary 2.0.2.** *There is an inclusion-reversing bijection between radical ideals  $J \leq k[x_1, \dots, x_n]$  and affine varieties  $X \subseteq \mathbb{A}_k^n$  given by  $J \mapsto V(J)$  with inverse  $X \mapsto I(X)$ .*

*Proof.* It suffices to show that these maps are inverses. For  $J \leq k[x_1, \dots, x_n]$  a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For  $X \subseteq \mathbb{A}_k^n$  an affine variety, we clearly have  $X \subseteq V(I(X))$ . For the reverse inclusion, choose an ideal  $J \leq k[x_1, \dots, x_n]$  such that  $V(J) = X$ . Then  $J \subseteq I(X)$ , so we have  $V(I(X)) \subseteq V(J) = X$ . Thus we also get  $V(I(X)) = X$ .  $\square$

**Remark.** This implies that maximal ideals in  $k[x_1, \dots, x_n]$  correspond to points in  $\mathbb{A}_k^n$ , since maximal ideals correspond to minimal varieties under this bijection.

**Corollary 2.0.3.** *If  $X_1, X_2$  are affine varieties in  $\mathbb{A}_k^n$ , then*

1.  $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$ ;
2.  $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$ .

*Proof.* (1) This follows from definitions.

(2) Write  $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$ .  $\square$

**Example 2.0.1.** The radical in (2) is necessary. Consider  $X_1 = V(y)$  and  $X_2 = V(y - x^2)$  in  $\mathbb{A}_k^2$ . Then  $X_1 \cap X_2 = \{(0, 0)\} \subseteq \mathbb{A}_k^2$ , so  $I(X_1 \cap X_2) = (x, y)$ . However,  $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$ .

Note that it is sometimes better to consider  $(y, x^2)$  anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of  $k[x, y]/(x, y^2) \cong \bar{1}k \oplus \bar{y}k$  as a  $k$ -vector space.

## 2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

**Theorem 2.1** (Noether normalization). *Let  $A$  be a finitely generated algebra over a field  $k$  with  $A$  a domain. Then there is an injective  $k$ -algebra homomorphism  $k[z_1, \dots, z_n] \hookrightarrow A$  that is finite, i.e.  $A$  is a finitely generated  $k[z_1, \dots, z_n]$ -module.*

**Corollary 2.1.1.** *If  $K \subseteq L$  is a field extension and  $L$  is a finitely generated  $K$ -algebra, then  $K \subseteq L$  is a finite field extension. In particular, if in addition  $K = \overline{K}$ , then  $K = L$ .*

*Proof.* By Noether normalization, there exists a  $k$ -algebra homomorphism  $K[z_1, \dots, z_n] \rightarrow L$  that is finite. Then by a result from commutative algebra,  $L$  is integral over  $K[z_1, \dots, z_n]$ , which implies that  $K[z_1, \dots, z_n]$  must also be a field since  $L$  is. Thus  $n = 0$ , so  $K \subseteq L$  is a finite extension.  $\square$

**Proposition 2.1.** *If  $(1) \neq J \leq R$  is an ideal, then  $J$  is contained in some maximal ideal.*

*Proof.* Consider the set  $P = \{I \leq R : J \subseteq I, I \neq (1)\}$  with the partial order given by inclusion. Note that  $P \neq \emptyset$  since  $J \in P$ . Furthermore, every chain in  $P$  has an upper bound (for  $\{I_\alpha : \alpha \in A\}$  a chain  $P$ , we can take  $\bigcup_{\alpha \in A} I_\alpha$ , which one can check is indeed an ideal that lies in  $P$ ; note that  $1 \notin I_\alpha$  implies  $1 \notin \bigcup_{\alpha \in A} I_\alpha$ ). So Zorn's lemma implies there is a maximal element in  $P$ , which is a maximal ideal.  $\square$

*Proof of Theorem 1.1.* We will proceed in the following steps:

1. Show that the maximal ideals of  $k[x_1, \dots, x_n]$  are of the form  $(x_1 - a_1, \dots, x_n - a_n)$  for  $a_i \in k$ .
2. Prove the weak nullstellensatz: If  $(1) \neq J \leq k[x_1, \dots, x_n]$ , is an ideal, then  $V(J) \neq \emptyset$ .
3. Prove the (strong) nullstellensatz:  $I(V(J)) = \sqrt{J}$  for  $J \leq k[x_1, \dots, x_n]$ .

The most difficult part is the first step and is where we need  $k$  to be algebraically closed.<sup>1</sup>

(1) For  $a_1, \dots, a_n \in k$ , the ideal  $(x_1 - a_1, \dots, x_n - a_n)$  is maximal (the quotient is  $k$ , which is a field). Conversely, fix a maximal ideal  $\mathfrak{m} \in k[x_1, \dots, x_n]$ . Since

$$k \xrightarrow{\phi} k[x_1, \dots, x_n]/\mathfrak{m} = L$$

is a finitely generated  $k$ -algebra and  $k$  is algebraically closed,  $\phi$  is an isomorphism by Corollary 2.1.1. Choose  $a_i \in k$  such that  $\phi(a_i) = \overline{x_i}$ , so  $\overline{x_i - a_i} = 0$  in  $L$ . Then  $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$ , so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1,  $J$  is contained in some maximal ideal  $\mathfrak{m}$ . By (1),  $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$  for some  $a_1, \dots, a_n \in k$ . Since  $J \subseteq \mathfrak{m}$ , we have  $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \dots, a_n)\}$ , so  $J \neq \emptyset$ .

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix  $f \in I(V(J))$ , and we want to show that  $f^n \in J$  for some  $n > 0$ . Add a new variable  $y$  and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \dots, x_n, y].$$

Now  $V(J_1) = \{(a, b) = (a_1, \dots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \emptyset$  since  $f$  vanishes on  $V(J)$ , so  $f(a)b = 0$  for any  $b$ . Thus by the weak nullstellensatz,  $J_1 = (1)$ , so  $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$  with

<sup>1</sup>The statement is false when  $k$  is not algebraically closed:  $(x^2 + 1)$  is maximal in  $\mathbb{R}[x]$ .



$f_1, \dots, f_r \in J$  and  $g_0, \dots, g_r \in k[x_1, \dots, x_n, y]$ . Let  $N$  be the maximal power of  $y$  in the  $g_i$ . Multiplying by  $f^N$ , we get

$$f^N = \sum_{i=1}^r G_i(x_1, \dots, x_n, fy) f_i + G_0(x_1, \dots, x_n, fy)(fy - 1)$$

with  $G_i \in k[x_1, \dots, x_n, fy]$ . So if we set  $fy = 1$ , then we have

$$f^N = \sum_{i=1}^r G_i(x_1, \dots, x_n, 1) f_i + 0 \in J,$$

which gives  $f \in \sqrt{J}$ . To justify this substitution, we can consider the quotient  $k[x_1, \dots, x_n, y]/(fy - 1)$ . We have a map  $k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n, y]/(fy - 1)$ , which is injective since  $(fy - 1)$  does not lie in  $k[x_1, \dots, x_n]$ , so an equality in the quotient implies an equality in  $k[x_1, \dots, x_n]$ .  $\square$

# Lecture 3

## Aug. 26 — The Zariski Topology

### 3.1 Polynomial Functions and Subvarieties

**Remark.** Recall that a polynomial  $f \in k[x_1, \dots, x_n]$  gives a function  $\mathbb{A}_k^n \rightarrow k$  by  $a \mapsto f(a)$ .

**Proposition 3.1.** *If  $f, g \in k[x_1, \dots, x_n]$  give the same function  $\mathbb{A}_k^n \rightarrow k$ , then  $f = g$  in  $k[x_1, \dots, x_n]$ .*

*Proof.* Assume  $f = g$  as polynomial functions. Then  $V(f - g) = \mathbb{A}_k^n$ , so  $\sqrt{(f - g)} = I(\mathbb{A}_k^n) = (0)$  by Hilbert's nullstellensatz (note that we can also prove  $I(\mathbb{A}_k^n) = (0)$  directly, it is enough to have  $k$  be an infinite field for this part). Thus  $f - g = 0$ , so  $f = g$  in  $k[x_1, \dots, x_n]$ .  $\square$

**Remark.** In the above proposition, we need  $k$  to be an infinite field (e.g. if  $k = \bar{k}$ ): Otherwise, there are only finitely many functions  $\mathbb{A}_k^n \rightarrow k$ , but infinitely many polynomials in  $k[x_1, \dots, x_n]$ .

**Remark.** The set of polynomial functions  $\mathbb{A}_k^n \rightarrow k$  form a ring, and the above proposition implies that this ring is isomorphic to  $k[x_1, \dots, x_n]$ .

**Definition 3.1.** A *polynomial function* on an affine variety  $X \subseteq \mathbb{A}_k^n$  is a function  $\varphi : X \rightarrow k$  such that there exists  $f \in k[x_1, \dots, x_n]$  with  $\varphi(a) = f(a)$  for every  $a \in X$ .

**Definition 3.2.** The *coordinate ring* of  $X$  is  $A(X) = \{f : X \rightarrow k \mid f \text{ is a polynomial function}\}$ , which is a ring under pointwise addition and multiplication.

**Remark.** Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel  $I(X)$ . Thus we have  $A(X) \cong k[x_1, \dots, x_n]/I(X)$ .

**Remark.** We can now replace  $\mathbb{A}_k^n$  and  $k[x_1, \dots, x_n]$  by  $X$  and  $A(X)$  to study *subvarieties* of  $X$ .

**Definition 3.3.** Let  $X \subseteq \mathbb{A}_k^n$  be an affine variety. If  $S \subseteq A(X)$  is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of  $X$  of this form is called an *affine subvariety* of  $X$ . (Equivalently, these are the same as an affine variety  $Y \subseteq \mathbb{A}_k^n$  such that  $Y \subseteq X$ .) For  $Y \subseteq X$  a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

**Proposition 3.2.** *There is a bijective correspondence between radical ideals in  $A(X)$  and affine subvarieties of  $X$  given by  $J \mapsto V_X(J)$  and  $Y \mapsto I_X(Y)$ .*

*Proof.* See Homework 2. □

## 3.2 The Zariski Topology

**Definition 3.4.** The *Zariski topology* on  $\mathbb{A}_k^n$  is the topology with closed sets  $V(I) \subseteq \mathbb{A}_k^n$ , where  $I$  is an ideal in  $k[x_1, \dots, x_n]$ . (Equivalently, the closed sets are the affine varieties in  $\mathbb{A}_k^n$ .)

**Remark.** Note the following:

1. On  $\mathbb{A}_k^1$ , the closed sets are of the form:  $\emptyset$ ,  $\mathbb{A}_k^1$ , or finite collections of points.
2. When  $k = \mathbb{C}$ , then  $X \subseteq \mathbb{A}_{\mathbb{C}}^n$  being Zariski closed implies that  $X$  is closed in the analytic topology on  $\mathbb{A}_{\mathbb{C}}^n$ . In particular, the Zariski topology is coarser than the analytic topology.
3. On  $\mathbb{A}_k^2$ , the closed sets are of the form:  $\emptyset$ ,  $\mathbb{A}_k^2$ , finite collections of points, plane curves, and their finite unions.

**Proposition 3.3.** *The Zariski topology on  $\mathbb{A}_k^n$  is indeed a topology.*

*Proof.* First note that  $\emptyset = V((1))$  and  $\mathbb{A}_k^n = V((0))$  are closed. For arbitrary intersections, note that  $\bigcap_{\alpha} V(I_{\alpha}) = V(\sum_{\alpha} I_{\alpha})$ , and for finite unions, note that  $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$ . □

**Example 3.4.1.** The Zariski topology on  $\mathbb{A}_k^{n+m}$  is in general *not* the product topology of the Zariski topologies on  $\mathbb{A}_k^n$  and  $\mathbb{A}_k^m$ . Consider  $V(y - x^2) \subseteq \mathbb{A}_k^2$ , which is a closed set in the Zariski topology, but the only closed sets in  $\mathbb{A}_k^1$  are either  $\emptyset$ ,  $\mathbb{A}_k^1$ , or finite.

**Definition 3.5.** If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then we can define the *Zariski topology* on  $X$  in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on  $\mathbb{A}_k^n$ ;
2. take the closed sets of  $X$  to be of the form  $V_X(I)$  for some ideal  $I \leq A(X)$ .

This is because an affine subvariety of  $X$  is precisely the intersection of  $X$  with an affine variety in  $\mathbb{A}_k^n$ .

**Remark.** Our goal now is to relate properties of the Zariski topology on  $X$  to the ring  $A(X)$ , and then to the ideal  $I(X) \leq k[x_1, \dots, x_n]$ .

**Definition 3.6.** A topological space  $X$  is *reducible* if we can write  $X = X_1 \cup X_2$  for some closed sets  $X_1, X_2 \subsetneq X$ . Otherwise,  $X$  is called *irreducible*.

**Example 3.6.1.** The plane curve  $X = V(y^2 - x^2y) = V(y) \cup V(y - x^2)$  is reducible.

**Remark.** Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g.  $\mathbb{C}^n$ ,  $\mathbb{R}^n$  with the analytic topology.
3. If  $X$  is irreducible and  $U \subseteq X$  is a nonempty open set, then  $\overline{U} = X$  (we have  $\overline{U} \cup (X \setminus U) = X$ ).

# Lecture 4

## Aug. 28 — Irreducibility

### 4.1 Properties of Irreducibility

**Proposition 4.1.** *Let  $X \subseteq \mathbb{A}^n$  be an affine variety. Then the following are equivalent:*

1.  $X$  is irreducible;
2.  $I(X) \leq k[x_1, \dots, x_n]$  is a prime ideal;
3. the coordinate ring  $A(X)$  is an integral domain.

**Example 4.0.1.** We have the following:

1.  $\mathbb{A}_k^n$  is irreducible as  $A(\mathbb{A}_k^n) = k[x_1, \dots, x_n]$ , which is an integral domain.
2. A *hypersurface*  $X \subseteq \mathbb{A}_k^n$  is an affine variety with  $I(X) = (f)$  for some  $f \in k[x_1, \dots, x_n]$ . Then  $A$  is irreducible if and only if  $(f)$  is prime, if and only if  $f$  is irreducible.<sup>1</sup>

### 4.2 Dimension

**Definition 4.1.** Let  $X$  be a topological space.

- The *dimension* of  $X$ , denoted  $\dim X$ , is the supremum of the  $n$  such that there exists a chain of irreducible closed subspaces

$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \emptyset.$$

- For  $Y \subseteq X$  closed and irreducible, the *codimension* of  $Y$  in  $X$ , denoted  $\operatorname{codim}_X Y$ , is the supremum of the  $n$  as above such that  $X_n = Y$ .

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<sup>1</sup>Note that any prime ideal is radical.

# Lecture 5

## Sept. 2 — Dimension

### 5.1 More on Dimension

**Remark.** Recall the following correspondence from before: If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then there exists a bijection between the irreducible closed subsets  $Y \subseteq X$  and the prime ideals  $\mathfrak{p} \leq A(X)$ .

**Definition 5.1.** For a ring  $A$ , the *(Krull) dimension* of  $A$ , denoted  $\dim A$ , is the supremum of the  $n$  such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

For a prime ideal  $\mathfrak{q} \leq A$ , the *height* of  $\mathfrak{q}$ , denoted  $\text{ht } \mathfrak{q}$ , is the supremum of the  $n$  as above with  $\mathfrak{p}_0 = \mathfrak{q}$ .

**Remark.** If  $X$  is an affine variety, then we have the following:

1.  $\dim X = \dim A(X)$ ;
2. for  $Y \subseteq X$  a closed irreducible subset,  $\text{codim}_X Y = \text{ht } I_X(Y)$ .

These properties follow from the inclusion-reversing correspondence.

**Definition 5.2.** Let  $K \subseteq L$  be a field extension.

1. A collection of elements  $\{z_i : i \in I\} \subseteq L$  is a *transcendence basis* of  $K \subseteq L$  if the  $z_i$  are algebraically independent (i.e.  $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$  by  $x_i \mapsto z_i$ ) and  $K(z_i : i \in I) \subseteq L$  is algebraic.
2. The *transcendence degree*  $\text{tr.deg}_K L$  is the cardinality of a transcendence basis.

**Theorem 5.1** (Dimension theory). *Let  $A$  be a finitely generated  $k$ -algebra that is a domain. Then*

1.  $\dim A = \text{tr.deg}_k \text{Frac}(A)$ ;
2. for any prime ideal  $\mathfrak{p} \leq A$ , we have  $\text{ht } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$ ;
3. all maximal chains of prime ideals  $A \supseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$  are of the same length.

**Remark.** The following are consequences of the above result from commutative algebra:

1.  $\dim_k \mathbb{A}_k^n = \dim k[x_1, \dots, x_n] = \text{tr.deg}_k k(x_1, \dots, x_n) = n$ .
2. If  $X$  is irreducible, then  $A(X)$  is a domain, so for  $x \in X$ , we have

$$\text{codim}_X \{x\} = \text{ht } I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

where we note that  $A(X)/I(\{x\}) \cong k$  is a field.

3. If  $X$  is an irreducible affine variety and  $U \subseteq X$  is a nonempty open subset, then

$$\dim U = \sup_{x \in U} \operatorname{codim}_U \{x\} = \sup_{x \in U} \operatorname{codim}_X \{x\} = \dim X.$$

This follows since we can pass from a chain in  $U$  to a chain in  $X$  by taking closures.

4. If  $X$  is an irreducible affine variety and  $Z \subseteq X$  is an irreducible closed subset, then

$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if  $X$  is not irreducible. To contradict (4), let  $X = V(x, y) \cup V(z) \subseteq \mathbb{A}_k^3$  with  $Z = V(x, y)$ . Then we have  $\dim X = 2$ ,  $\dim Z = 1$ ,  $\operatorname{codim}_X Z = 0$ .

## 5.2 Hypersurfaces

**Remark.** We now want to study hypersurfaces.

**Theorem 5.2** (Krull's Hauptidealsatz). *If  $A$  is a Noetherian ring and  $f \in A$  is nonzero and a non-unit, then every minimal prime ideal containing  $f$  has height 1.*

**Corollary 5.2.1.** *If  $X \subseteq \mathbb{A}_k^n$  is an irreducible affine variety and  $f \in A(X)$  is a nonzero non-unit, then*

$$\dim Z = \dim X - 1$$

*for every irreducible component  $Z$  of  $V_X(f)$ .*

*Proof.* Since  $X$  is irreducible,  $A(X)$  is a domain. So there is a correspondence between the minimal prime ideals  $\mathfrak{p} \subsetneq A(X)$  and the minimal irreducible closed subsets  $Z \supseteq V_X(f)$ , which corresponds to the irreducible components  $Z$  of  $V_X(f)$ . For such a component  $Z$ , we know

$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$

by Krull's Hauptidealsatz, which is the desired result.  $\square$

**Example 5.2.1.** Corollary 5.2.1 implies that if  $f \in k[x_1, \dots, x_n]$  is non-constant, then

$$\dim V(f) = \dim \mathbb{A}_k^n - 1 = n - 1.$$

**Theorem 5.3.** *An irreducible affine variety  $Y \subseteq \mathbb{A}_k^n$  has  $\dim Y = n - 1$  if and only if  $Y = V(f)$  for some non-constant polynomial  $f \in k[x_1, \dots, x_n]$ .*

*Proof.* ( $\Leftarrow$ ) This was Corollary 5.2.1.

( $\Rightarrow$ ) We will use that  $A(\mathbb{A}_k^n) = k[x_1, \dots, x_n]$  is a UFD. Since  $Y$  is irreducible and  $\dim Y = n - 1$ ,

$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}_k^n} Y = \dim \mathbb{A}_k^n - \dim Y = 1.$$

Since  $(0) \subsetneq I(Y) \subsetneq k[x_1, \dots, x_n]$ , there exists a non-constant  $f \in k[x_1, \dots, x_n]$  with  $f \in I(Y)$ . Write

$$f = f_1 \cdots f_r$$

with  $f_i$  irreducible by unique factorization, and note that the  $f_i$  are also prime since we are in a UFD. Since  $I(Y)$  is prime, some  $f_i$  is in  $I(Y)$ , so we have the inclusions

$$(0) \subsetneq (f_i) \subseteq I(Y).$$

Since  $\operatorname{ht} I(Y) = 1$ , we must have  $(f_i) = I(Y)$ , so  $Y = V(I(Y)) = V(f_i)$ .  $\square$

## 5.3 Regular Functions

**Definition 5.3.** Let  $X$  be an affine variety and  $U \subseteq X$  open. A function  $\varphi : U \rightarrow k$  is *regular* if for each  $a \in U$ , there exists an open neighborhood  $a \in U_a \subseteq U$  and  $f, g \in A(X)$  such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define  $\mathcal{O}_X(U) = \{\varphi : U \rightarrow k \mid \varphi \text{ is a regular function on } U\}$ .

**Exercise 5.1.** Check that  $\mathcal{O}_X(U)$  is a ring under pointwise addition and multiplication of outputs.

**Remark.** To patch open sets together, we will later need the notion of a *morphism*, and a morphism  $U \rightarrow Y \subseteq \mathbb{A}_k^m$  should be given by

$$x \longmapsto (\varphi_1(x), \dots, \varphi_m(x))$$

with  $\varphi_i$  regular functions on  $U$ .

**Example 5.3.1.** We have the following:

1. If  $X \subseteq \mathbb{A}_k^n$  is an affine variety, then any  $\varphi \in A(X)$  is regular. Furthermore, we get an injective ring homomorphism  $A(X) \rightarrow \mathcal{O}_X(X)$ . We will see that this is an isomorphism.
2. If  $X = \mathbb{A}_k^1$  and  $U = \mathbb{A}_k^1 \setminus \{0\}$ , then for any  $n \geq 0$  and  $g \in k[x]$ , the function  $g/x^n$  is regular on  $U$ . In general, if we fix  $f, g \in A(X)$  and set  $U = X \setminus V(f)$ , then the map  $g/f^m$  is regular on  $U$ .
3. Let  $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$  and  $U = X \setminus V(x_2, x_4)$ . Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

Note that on  $U \setminus V(x_2x_4)$ , we have  $x_1/x_2 = x_3/x_4$  since  $x_1x_4 = x_2x_3$  on  $X$ .

# Lecture 6

## Sept. 4 — Regular Functions

### 6.1 Properties of Regular Functions

**Proposition 6.1.** *Let  $X$  be an affine variety and  $U \subseteq X$  open. Then:*

1. *if  $\varphi \in \mathcal{O}_X(U)$ , then  $V(\varphi) = \{x \in U : \varphi(x) = 0\}$  is closed in  $U$ ;*
2. *(identity principle) If  $X$  is irreducible,  $U \subseteq X$  is nonempty and open, and  $\varphi, \psi \in \mathcal{O}_X(U)$  with  $\varphi|_W = \psi|_W$  for some  $W \subseteq U$  nonempty and open, then  $\varphi = \psi$  in  $\mathcal{O}_X(U)$ .*

*Proof.* (1) It suffices to show that  $U \setminus V(\varphi)$  is open in  $U$ . Fix  $a \in U \setminus V(\varphi)$ . Since  $\varphi$  is regular, there exists an open neighborhood  $a \in U_a \subseteq U$  and  $f_a, g_a \in A(X)$  such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So  $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$ . This is an open set containing  $a$  in  $U \setminus V(\varphi)$ , so  $U \setminus V(\varphi)$  is open.

(2) Since  $X$  is irreducible,  $U$  is also irreducible. The locus  $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$  is closed in  $U$  by (1). It also contains  $W$ . Since  $W$  is dense (it is a nonempty open set in an irreducible topological space), we must have  $V(\varphi - \psi) = U$ . This proves the claim.  $\square$

**Example 6.0.1.** In (2) of Proposition 6.1, the assumption that  $X$  is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}_k^2 \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions  $\varphi = x$  and  $\psi = x + y$  agree on  $W$  but are not equal on  $U$ .

### 6.2 Distinguished Open Sets

**Remark.** We will see that an affine variety has a basis of open sets on which we can compute  $\mathcal{O}_X(U)$ .

**Definition 6.1.** A *distinguished open set* of an affine variety  $X$  is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function  $f \in A(X)$ .

**Remark.** We have the following:

1. The  $D(f)$  are closed under (finite) intersection:  $D(fg) = D(f) \cap D(g)$ .



2. The  $D(f)$  form a basis for the Zariski topology on  $X$ : If  $U \subseteq X$  is open, then  $U = X \setminus V(f_1, \dots, f_r)$  for some  $f_1, \dots, f_r \in A(X)$  (since  $X$  is Noetherian). So  $U = D(f_1) \cup \dots \cup D(f_r)$ .

**Remark.** We will view  $D(f)$  as “small open sets” (under mild assumptions,  $\text{codim}_X(X \setminus D(f)) = 1$ ).

**Theorem 6.1.** *If  $X$  is an affine variety and  $f \in A(X)$ , then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

*Proof.* We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix  $\varphi \in \mathcal{O}_X(D(f))$ . For any  $a \in D(f)$ , there exists an open neighborhood  $a \in U_a \subseteq D(f)$  and  $f_a, g_a \in A(X)$  such that  $\varphi|_{U_a} = g_a/f_a$ . We may further assume that

1.  $U_a = D(h_a)$  for some  $h_a \in A(X)$  (by shrinking  $U_a$  if necessary, since the  $D(h)$  form a basis);
2.  $h_a = f_a$  (by rewriting  $g_a/f_a = g_a h_a / f_a h_a$  and replacing  $h_a, f_a$  with  $f_a h_a$ ).

Then for  $a, b \in D(f)$ , we have  $f_a g_b = f_b g_a$  on  $D(f_a) \cap D(f_b)$ . Since both the left and right hand sides vanish on  $X \setminus (D(f_a) \cap D(f_b))$ , we have  $f_a g_b = f_b g_a$  in  $A(X)$ . Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so  $f \in I(V(f_a : a \in D(f)))$ . By the Nullstellensatz, there exists  $n \geq 0$  such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the  $k_a$  are nonzero. Set  $g = \sum_{a \in D(f)} k_a g_a$ , and we claim that  $\varphi = g/f^n$ . To see this, note that on  $U_b$ , we have  $\varphi|_{U_b} = g_b/f_b$ . Now since  $f_a g_b = f_b g_a$ , we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that  $\varphi|_{U_b} = (g/f^n)|_{U_b}$ . Since this holds for any  $U_b$ , we have  $\varphi = g/f^n$  in  $\mathcal{O}_X(D(f))$ .  $\square$

**Remark.** Theorem 6.1 has the following consequences:

1. The  $f = 1$  case implies that the natural ring homomorphism  $A(X) \rightarrow \mathcal{O}_X(X)$  is surjective and hence an isomorphism (note that  $D(1) = X$ ).
2. We will see that  $\mathcal{O}_X(D(f)) \cong A(X)_f$ , the *localization* of  $A(X)$  at  $f$ .

**Example 6.1.1.** How do we compute  $\mathcal{O}_X(U)$  on non-distinguished open sets? Consider

$$X = \mathbb{A}_k^2 \quad \text{and} \quad U = \mathbb{A}_k^2 \setminus \{(0, 0)\}.$$

Note that  $U$  is never a distinguished open set. We claim that the ring homomorphism

$$k[x, y] \longrightarrow \mathcal{O}_{\mathbb{A}_k^2}(\mathbb{A}_k^2 \setminus \{(0, 0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use  $U = D(x) \cup D(y)$  (in general, cover  $U$  by basis elements). Fix  $\varphi : U \rightarrow k$  regular, so

$$\begin{aligned}\varphi|_{D(x)} &= \frac{f}{x^m} \quad \text{for some } f \in k[x, y], m \geq 0 \\ \varphi|_{D(y)} &= \frac{g}{y^n} \quad \text{for some } g \in k[x, y], n \geq 0.\end{aligned}$$

Since we are in a UFD, we may assume that  $x \nmid f$  and  $y \nmid g$ . Now  $fy^n = gx^m$  on  $D(y) \cap D(x)$ , so by the identity principle,  $fy^n = gx^m$  on  $\mathbb{A}_k^2$ , so  $fy^n = gx^m$  in  $k[x, y]$ . Using that  $y \nmid g$ ,  $x \nmid f$ , and that  $k[x, y]$  is a UFD, we must have  $n = m = 0$ , hence  $f = g$ . In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map  $k[x, y] \rightarrow \mathcal{O}_X(U)$  is surjective.

## 6.3 Localization

**Remark.** We want to invert a subset of a ring, in particular *multiplicative systems*.

**Definition 6.2.** A *multiplicative system* of a ring  $A$  is a subset such that

1.  $1 \in S$ ;
2.  $S$  is closed under multiplication.

**Example 6.2.1.** The following examples of  $S$  are multiplicative systems:

1.  $S = A$  or  $S = \{1\}$ ;
2. if  $\mathfrak{p} \leq A$  is a prime ideal, then  $S = A \setminus \mathfrak{p}$ ;
3. if  $f \in A$ , then  $S = \{f^m : m \geq 0\}$ .

**Definition 6.3.** The *localization* of a ring  $A$  at a multiplicative system  $S$  is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the  $a/s$  are formal symbols with  $a/s \sim a'/s'$  if  $t(as' - a's) = 0$  for some  $t \in S$ .<sup>1</sup> The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

**Example 6.3.1.** The following are examples of localization:

1. If  $A$  is a domain and  $S = A \setminus \{0\}$ , then  $S^{-1}A = \text{Frac } A$ .
2. If  $S = \langle f \rangle = \{1, f, f^2, \dots\}$ , then we will write  $A_f = S^{-1}A$ .
3. If  $S = A \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p}$ , then we will write  $A_{\mathfrak{p}} = S^{-1}A$ .

---

<sup>1</sup>Note that if  $A$  is a domain and  $0 \notin S$ , then this condition is equivalent to  $as' = a's$ .

**Proposition 6.2.** *We have the following properties of localization:*

1. (Universal property of localization) *For any ring homomorphism  $\varphi : A \rightarrow B$  such that  $\varphi(s)$  for all  $s \in S$ , then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \pi: a \mapsto a/1 \searrow & & \nearrow \exists! \\ & S^{-1}A & \end{array}$$

2. *There is a bijection between the prime ideals  $\mathfrak{p} \leq A$  with  $\mathfrak{p} \cap S = \emptyset$  and the prime ideals  $\mathfrak{q} \leq S^{-1}A$  given by  $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$  with inverse  $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ , where  $\pi : A \rightarrow S^{-1}A$  is the map  $a \mapsto a/1$ .*

**Remark.** In more generality, for an  $A$ -module  $M$ , we can define the localization  $S^{-1}M$ , which is an  $S^{-1}A$ -module. This gives a functor  $\text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$  which is exact.

# Lecture 7

## Sept. 9 — Germs and Sheaves

### 7.1 More on Localization

**Proposition 7.1.** *If  $X$  is an affine variety and  $f \in A(X)$  is nonzero, then  $\mathcal{O}_X(D(f)) \cong A(X)_f$ .*

*Proof.* We define a ring homomorphism as follows:

$$\begin{aligned} A(X)_f &\longrightarrow \mathcal{O}_X(D(f)) \\ \frac{g}{f^m} &\longmapsto \left( x \mapsto \frac{g(x)}{f^m(x)} \right). \end{aligned}$$

To check that this is well-defined, assume  $g/f^m \sim h/f^n$  in  $A(X)_f$ . So there exists  $k \geq 0$  such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So  $gf^n - hf^m = 0$  as functions  $D(f) \rightarrow k$ , so  $g/f^m = h/f^n$  as functions  $D(f) \rightarrow k$ . Thus their images agree in  $\mathcal{O}_X(D(f))$ , so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume  $g/f^m = 0$  as functions  $D(f) \rightarrow k$  with  $g \in A(X)$ . Then  $fg = 0$  in  $A(X)$ , so  $g/f^m \sim 0/1$  in  $A(X)_f$ .  $\square$

### 7.2 Germs of Functions

**Definition 7.1.** Let  $p \in X$  be a point on an affine variety.

1. A *germ* of a regular function of  $X$  at  $p$  is a pair  $(U, f)$  such that  $x \in U \subseteq X$  is open and  $f$  is a regular function  $U \rightarrow k$ , up to the equivalence relation  $(U, \varphi) \sim (V, \psi)$  if there exists an open set  $x \in W \subseteq U \cap V$  such that  $\varphi|_W = \psi|_W$ .
2. Define  $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$ .

**Exercise 7.1.** Check that  $\mathcal{O}_{X,p}$  is a ring with operations

$$\begin{aligned} (U, \varphi) \cdot (V, \psi) &= (U \cap V, \varphi|_{U \cap V} \cdot \psi|_{U \cap V}), \\ (U, \varphi) + (V, \psi) &= (U \cap V, \varphi|_{U \cap V} + \psi|_{U \cap V}), \end{aligned}$$

with the zero function as the zero element and the constant 1 function as the unit element.

**Lemma 7.1.**  $\mathcal{O}_{X,p}$  is a local ring with unique maximal ideal  $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$ .

*Proof.* It suffices to show that the units of  $\mathcal{O}_{X,p}$  are precisely  $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$ . To see the reverse inclusion, fix  $(U, \varphi) \in \mathcal{O}_{X,p}$  with  $\varphi(p) \neq 0$ . So there exists an open neighborhood  $p \in W \subseteq U$  such that  $\varphi|_W$  never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W) = (W, 1),$$

so  $(U, \varphi)$  is a unit in  $\mathcal{O}_{X,p}$ . The forward inclusion is similar.  $\square$

**Proposition 7.2.** *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left( D(g), x \mapsto \frac{f(x)}{g(x)} \right) \end{aligned}$$

with  $I(p) = \{f \in A(X) : f(p) = 0\}$ .

*Proof.* To see that this is well-defined, let  $f/g \sim f'/g' \in A(X)_{I(p)}$ . Then  $h(fg' - f'g) = 0$  for some  $h \in A(X)$  with  $h(p) \neq 0$ . So  $f/g = f'/g'$  as functions  $D(h) \cap D(g) \rightarrow k$ , which means that  $f/g = f'/g'$  as elements in  $\mathcal{O}_{X,p}$ . Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose  $(U, \varphi) \in \mathcal{O}_{X,p}$ . Since  $\varphi : U \rightarrow k$  is a regular function, there exists an open set  $p \in U_p \subseteq U$  and  $f, g \in A(X)$  such that  $g$  does not vanish on  $U_p$  and  $\varphi(x) = f(x)/g(x)$  for all  $x \in U_p$ . So  $(U, \varphi) \sim (D(g), f/g)$  in  $\mathcal{O}_{X,p}$ , i.e.  $(U, \varphi)$  is in the image.  $\square$

**Example 7.1.1.** If  $X = \mathbb{A}_k^n$  and  $p = 0$ , then

$$\mathcal{O}_{\mathbb{A}_k^n, 0} \cong k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = \left\{ \frac{f}{g} : f \in k[x_1, \dots, x_n], g \in k[x_1, \dots, x_n] \setminus (x_1, \dots, x_n) \right\}.$$

**Remark.** We will relate the local properties of  $X$  at  $p$  to properties of  $\mathcal{O}_{X,p}$ . We will use the following statements from commutative algebra: Let  $A$  be a ring and  $\mathfrak{p} \subseteq A$  a prime ideal. Then

1.  $A_{\mathfrak{p}}$  is a local ring with unique maximal ideal  $\mathfrak{p}A_{\mathfrak{p}}$ .
2. There is a bijection from the prime ideals of  $A_{\mathfrak{p}}$  to the prime ideals of  $A$  contained in  $\mathfrak{p}$ .
3.  $\text{ht}_A \mathfrak{p} = \dim A_{\mathfrak{p}}$  (this follows from (2)).

This has the following consequence: If  $X$  is an affine variety and  $p \in X$ , then

$$\text{codim}_X \{p\} = \text{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

## 7.3 Sheaves

**Remark.** We will now formalize the structures  $\mathcal{O}_X(U)$  and  $\mathcal{O}_{X,p}$  that we have seen before.

**Definition 7.2.** A *presheaf (of rings)*  $\mathcal{F}$  on a topological space  $X$  is the data of

1. for every open set  $U \subseteq X$ , a ring  $\mathcal{F}(U)$ ;
2. for every inclusion of open sets  $U \subseteq V \subseteq X$ , a ring homomorphism  $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$

satisfying the following properties:

1.  $\mathcal{F}(\emptyset) = 0$ ;
2.  $\rho_{U,U}$  is the identity map;
3. for inclusions of open sets  $U \subseteq V \subseteq W \subseteq X$ , we have  $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$ .

**Example 7.2.1.** If  $X$  is an affine variety, then  $\mathcal{O}_X$  gives a presheaf of rings with

1. for  $U \subseteq X$ , the ring is  $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \rightarrow k\}$ ;
2. for  $U \subseteq V \subseteq X$ , the map  $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$  is given by  $\varphi \mapsto \varphi|_U$ .

**Remark.** We often call  $s \in \mathcal{F}(U)$  a *section*, and for  $U \subseteq V$ , we call  $s|_U = \rho_{V,U}(s)$  the *restriction*.

**Remark.** A presheaf is the same thing as a functor  $\text{Open}_X^{\text{op}} \rightarrow \text{Rings}$ , where  $\text{Open}_X$  is the category with objects the nonempty open sets of  $X$  and morphisms corresponding to the inclusions  $U \subseteq V$ .

**Definition 7.3.** A presheaf  $\mathcal{F}$  on  $X$  is a *sheaf* if it satisfies the *gluing property*: For any  $U \subseteq X$  open, an open cover  $\{U_i\}_{i \in I}$  of  $U$ , and  $\varphi_i \in \mathcal{F}(U_i)$  with  $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$  for all  $i, j \in I$ , there exists a unique  $\varphi \in \mathcal{F}(U)$  such that  $\varphi|_{U_i} = \varphi_i$  for all  $i \in I$ .

**Example 7.3.1.** We have the following:

1. If  $X$  is an affine variety, then  $\mathcal{O}_X$  is a sheaf (if we take  $\varphi_i \in \mathcal{O}_X(U_i)$  that agree on the overlaps, then we get  $\varphi : U \rightarrow k$ , which is regular since regularity is a local property).
2. If  $M$  is a smooth manifold, then we can define a sheaf (on open subsets  $U \subseteq M$ ) by

$$U \mapsto \mathcal{F}^{\text{sm}}(U) = \{\text{smooth functions } U \rightarrow \mathbb{R}\}.$$

We may also consider  $\mathcal{F}^{\text{cont}}$ ,  $\mathcal{F}^{\text{diff}}$ ,  $\mathcal{F}^{\text{loc, const}}$ , etc. However,  $\mathcal{F}^{\text{const}}$  is a presheaf, but not a sheaf in general: We can take  $U = U_1 \cup U_2$  with  $U_1 \cap U_2 = \emptyset$ , and we will only get a locally constant function. Similarly,  $\mathcal{F}^{\text{bounded}}$  is only a presheaf but not a sheaf.

3. If  $\mathcal{F}$  is a sheaf on a topological space  $X$  and  $U \subseteq X$  is open, then we get a sheaf  $\mathcal{F}|_U$  on  $U$  defined by  $\mathcal{F}|_U(V) = \mathcal{F}(V)$  for  $V \subseteq U$  open.

**Definition 7.4.** The *stalk* of a sheaf  $\mathcal{F}$  on a topological space  $X$  at  $x \in X$  is

$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\} / \sim,$$

where  $(U, \varphi) \sim (V, \psi)$  if there exists an open set  $x \in W \subseteq U \cap V$  such that  $\varphi|_W = \psi|_W$ .

**Example 7.4.1.** If  $X$  is an affine variety and  $p \in X$ , then  $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$ .

**Remark.** As before with  $\mathcal{O}_{X,p}$ , one can check that  $\mathcal{F}_x$  naturally has the structure of a ring.

**Remark.** An alternative perspective is to define the stalk as a direct limit:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

where the limit is taken over all open  $x \in U \subseteq X$  with respect to the ordering  $U \leq V$  if  $V \subseteq U$ .

# Lecture 8

## Sept. 11 — Morphisms

### 8.1 Morphisms of Open Sets

**Remark.** Recall that a continuous map  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions  $f_1, \dots, f_n : \mathbb{R}^m \rightarrow \mathbb{R}$  such that  $f(x) = (f_1(x), \dots, f_n(x))$ ;
2. for each open set  $U \subseteq \mathbb{R}^n$  and smooth  $\varphi : U \rightarrow \mathbb{R}$ , the function  $f^*\varphi := \varphi \circ f : \mathbb{R}^m \rightarrow \mathbb{R}$  is smooth.

The implication  $(1 \Rightarrow 2)$  follows by the chain rule. To see  $(2 \Rightarrow 1)$ , take  $y_i : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f_i := f^*y_i$ . We want a similar definition in algebraic geometry.

**Definition 8.1.** Let  $X$  and  $Y$  be open sets of affine varieties. A *morphism*  $f : X \rightarrow Y$  is a continuous map such that for every  $U \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(U)$ , the map

$$\begin{array}{ccccc} & & f^*\varphi & & \\ & \nearrow & & \searrow & \\ f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{\varphi} & k \end{array}$$

satisfies  $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$ . A morphism is an *isomorphism* if it has a two-sided inverse (equivalently,  $f$  is a bijection and  $f^{-1}$  is a morphism).

**Remark.** We have the following properties of morphisms:

1. (Composition) If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are morphisms of open sets of affine varieties, then so is  $g \circ f : X \rightarrow Z$ .
2. (Local on target) If  $X \rightarrow Y$  is a map of open sets of affine varieties such that there exists an open cover  $\{U_i\}_{i \in I}$  of  $Y$  with  $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \rightarrow U_i$  a morphism for all  $i \in I$ , then  $f$  is a morphism.

**Proposition 8.1.** Let  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  be affine varieties. Let  $U \subseteq X$  and  $V \subseteq Y$  be open sets. A map  $f : U \rightarrow V$  is a morphism if and only if there exist  $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$  such that

$$f(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

*Proof.*  $(\Rightarrow)$  Let  $U \subseteq \mathbb{A}^m_{x_i}$  and  $V \subseteq \mathbb{A}^n_{y_i}$ . By the definition of a morphism,  $y_i : V \rightarrow k$  satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write  $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$ .

$(\Leftarrow)$  Assume there exist  $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$  such that  $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$ .

We first show that  $f$  is continuous. Let  $Z \subseteq V$  be a closed set. So we can write  $Z = V(g_1, \dots, g_r)$  for some  $g_1, \dots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \dots, y_n]$ . Now we have

$$\begin{aligned} f^{-1}(Z) &= \{x \in U : f(x) \in Z\} = \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \dots, r\} \\ &= \{x \in U : (f^*g_i)(x) = 0 \text{ for } i = 1, \dots, r\}. \end{aligned}$$

Note that  $f^*g_i = g_i(\varphi_1, \dots, \varphi_n)$ , which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So  $f^{-1}(Z)$  is closed in  $U$ .

Now to show that  $f$  is a morphism, it suffices to show that for any  $W \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(W)$ , we have  $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$ . The proof of this is similar to before.  $\square$

**Example 8.1.1.** We have the following:

1. Morphisms  $\mathbb{A}^m \rightarrow \mathbb{A}^n$  are of the form

$$x \mapsto (f_1(x), \dots, f_n(x))$$

with  $f_1, \dots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \dots, x_m]$ .

2. Write  $\mathbb{A}_t^1$  to mean  $\mathbb{A}^1$  with variable  $t$ . Then we can define  $\mathbb{A}_t^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}_{x,y}^2$  by  $t \mapsto (t, t^2)$ . We can get an inverse  $V(y - x^2) \rightarrow \mathbb{A}_t^1$  by  $(x, y) \mapsto x$ , so  $\mathbb{A}_t^1$  and  $V(y - x^2)$  are isomorphic.
3. Consider the map  $g : \mathbb{A}_t^1 \rightarrow V(x^2 - y^3) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t^3, t^2)$ . This map is bijective, but it is not an isomorphism. To see this, we can show that  $(g^{-1})^*\varphi$  is not regular for some regular function  $\varphi$  on  $\mathbb{A}_1^1$ . For instance, we can take  $\varphi = t$ , so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which we can see is not regular.

## 8.2 Relation to Coordinate Rings

**Remark.** Let  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  be affine varieties. Then a morphism  $f : X \rightarrow Y$  of affine varieties induces a  $k$ -algebra morphism (called the *pullback* of  $f$ )

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^*\varphi = \varphi \circ f \end{aligned}$$

with the properties  $(g \circ f)^* = f^* \circ g^*$  and  $(\text{id}_X)^* = \text{id}_{A(X)}$ , i.e.  $X \mapsto A(X)$  is a contravariant functor.

**Proposition 8.2.** *The following map is a bijection:*

$$\begin{aligned} \text{Hom}_{\text{aff, var}}(X, Y) &\xrightarrow{\Phi} \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

*Proof.* Note that  $A(X) \cong k[x_1, \dots, x_m]/I(X)$  and  $A(Y) \cong k[y_1, \dots, y_n]/I(Y)$ . Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)), \end{aligned}$$



we can define  $f^*\bar{y}_i = \varphi_i$ . Conversely, given a  $k$ -algebra homomorphism  $\phi : A(Y) \rightarrow A(X)$ , we can set  $\varphi_i = \phi(\bar{y}_i)$ . Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}_{y_i}^n \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)). \end{aligned}$$

We claim that  $f(X) \subseteq Y$ . To see this, fix  $x \in X$ . If  $h \in I(Y)$ , then

$$h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so  $f(X) \subseteq Y$ . Thus we get a morphism  $f : X \rightarrow Y$  by  $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$  with  $f^*y_i = \varphi_i$ . One can check that this gives a map  $\Psi : \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \rightarrow \text{Hom}_{\text{aff, var}}(X, Y)$  which is inverse to  $\Phi$ .  $\square$

**Example 8.1.2.** We have the following:

1. Recall the morphism  $g : \mathbb{A}_t^1 \rightarrow V(y - x^2) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t, t^2)$ . The pullback is given by

$$\begin{aligned} g^* : \frac{k[x, y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

Note that  $g^*$  is an isomorphism of  $k$ -algebras, so  $g$  is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to  $g$ .

2. Recall the morphism  $h : \mathbb{A}_t^1 \rightarrow V(x^2 - y^3) \subseteq \mathbb{A}_{x,y}^2$  given by  $t \mapsto (t^3, t^2)$ . The pullback is

$$\begin{aligned} h^* : \frac{k[x, y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

Note that  $t \notin \text{Im } h^*$ , so  $h^*$  is not an isomorphism, so  $h$  is not an isomorphism.

**Remark.** There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced  $k$ -algebras (up to isomorphism).

To see this, observe that if  $X \subseteq \mathbb{A}^n$  is an affine variety, then  $A(X) \cong k[x_1, \dots, x_n]/I(X)$ . This is finitely generated, and reduced since  $I(X)$  is radical. Conversely, let  $A$  be a reduced finitely generated  $k$ -algebra. Then  $A \cong k[y_1, \dots, y_m]/I$  since  $A$  is finitely generated, and  $I$  is radical since  $A$  is reduced. Thus by Hilbert's nullstellensatz,  $Y = V(I)$  satisfies  $I(Y) = I(V(I)) = I$ , so  $A \cong A(Y)$ .

In more abstract language, this means that there is an equivalence of categories

$$\text{AffVar} \longleftrightarrow \text{RedFGAlg}_k^{\text{op}}.$$

# Lecture 9

## Sept. 16 — Morphisms, Part 2

### 9.1 An Example of Isomorphisms

**Example 9.0.1.** What of the following are isomorphic over  $\mathbb{C}$ ?

1.  $\mathbb{A}^1 \setminus \{1\}$ ;
2.  $V(x^2 + y^2) \subseteq \mathbb{A}^2$ ;
3.  $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$ ;
4.  $V(xy) \subseteq \mathbb{A}^2$ ;
5.  $V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$ ;
6.  $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$ .

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2 + y^2) = V((x + iy)(x - iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism  $\mathbb{A}^1 \rightarrow Y = V(y - x^2, z - x^3)$  by  $t \mapsto (t, t^2, t^3)$ . We can also see this by noting that  $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$ . For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and  $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$ , whereas  $A(\mathbb{A}^1) \cong \mathbb{C}[x]$ . So  $\mathbb{A}^1 \setminus \{1\} \not\cong \mathbb{A}^1$ . For (6), note that

$$V(x^2 - y^2 - 1) = V((x - y)(x + y) - 1) \cong V(uv - 1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map  $V(uv - 1) \rightarrow \mathbb{A}^1 \setminus \{0\}$  given by  $(u, v) \mapsto u$ , with inverse  $t \mapsto (t, 1/t)$ . Finally, letting  $C$  be the curve in 6, one can show that there is a singularity at the origin with  $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 2$ , which is different than the other examples. So the isomorphism classes are  $\{2, 4\}$ ,  $\{1, 6\}$ ,  $\{3\}$ , and  $\{5\}$ .

### 9.2 Ringed Spaces and Morphisms

**Definition 9.1.** A *ringed space*  $(X, \mathcal{O}_X)$  is a topological space  $X$  with a sheaf of rings  $\mathcal{O}_X$  on  $X$ .

**Example 9.1.1.** If  $X$  is an affine variety and  $\mathcal{O}_X$  is the sheaf of regular functions, then  $(X, \mathcal{O}_X)$  is a ringed space. Similarly, if  $M$  is a complex manifold and  $\mathcal{O}_M$  is the sheaf of holomorphic functions on  $M$ , then  $(M, \mathcal{O}_M)$  is a ringed space.

**Remark.** From now on, for a ringed space  $(X, \mathcal{O}_X)$ , we will always assume  $\mathcal{O}_X$  is a sheaf of  $k$ -valued functions on  $X$ . With this assumption, we can make sense of pullbacks.

**Definition 9.2.** A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map  $f : X \rightarrow Y$  such that for every  $U \subseteq Y$  open and  $\varphi \in \mathcal{O}_Y(U)$ ,

$$\begin{array}{ccccc} & & f^*\varphi & & \\ & \nearrow & & \searrow & \\ f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{\varphi} & k \end{array}$$

is an element of  $\mathcal{O}_X(f^{-1}(U))$ . A morphism is an *isomorphism* if it has a two-sided inverse.

**Remark.** A one-sided inverse need not be two-sided: Consider  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^1$  given by  $(x, y) \mapsto x$  and  $g : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  given by  $x \mapsto (x, 0)$ . Then  $f \circ g = \text{id}_{\mathbb{A}^1}$ , but  $g \circ f$  is not the identity on  $\mathbb{A}^2$ .

**Remark.** If  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism of ringed spaces, then for  $V \subseteq U \subseteq Y$  open, we get

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X(f^{-1}(U)) \\ \downarrow \text{res.} & & \downarrow \text{res.} \\ \mathcal{O}_Y(V) & \longrightarrow & \mathcal{O}_X(f^{-1}(V)) \end{array}$$

which is a commutative diagram of ring homomorphisms.

**Remark.** If  $X$  and  $Y$  are open sets of affine varieties, then a map  $f : X \rightarrow Y$  is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

**Definition 9.3** (Redefinition of affine variety). An *affine variety*  $(X, \mathcal{O}_X)$  is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

**Remark.** We will often write just  $X$  for the affine variety instead of the pair  $(X, \mathcal{O}_X)$ .

**Example 9.3.1.** Recall that  $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$  from Example 9.0.1. In particular,  $\mathbb{A}^1 \setminus \{0\}$  is an affine variety in the new sense (but not in the old sense).

**Proposition 9.1.** If  $X$  is an affine variety (in the old sense) and  $f \in A(X)$ , then  $D(f)$  is an affine variety.

*Proof.* Write  $X = V(I) \subseteq \mathbb{A}_{x_i}^n$  and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_y^1 \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse  $V(I, fy - 1) \rightarrow D(f)$  given by  $(x, y) \mapsto x$ . So  $D(f) \cong V(I, fy - 1)$  as ringed spaces. Thus  $D(f)$  is an affine variety (in the new sense).  $\square$

## 9.3 Products of Affine Varieties

**Remark.** If  $X \subseteq \mathbb{A}_{x_i}^m$  and  $Y \subseteq \mathbb{A}_{y_i}^n$  are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing  $I(X), I(Y)$  as ideals in  $k[x_1, \dots, x_m, y_1, \dots, y_n]$ . So  $X \times Y$  is an affine variety with morphisms

$$\begin{array}{ccc} & X \times Y & \\ (x,y) \mapsto x \swarrow p_1 & & \searrow p_2 (x,y) \mapsto y \\ X & & Y \end{array}$$

**Proposition 9.2.** *For every affine variety  $Z$  and diagram of morphisms*

$$\begin{array}{ccccc} Z & & & & \\ & \searrow f & & \nearrow f_Y & \\ & & X \times Y & \xrightarrow{p_2} & Y \\ & \nearrow f_X & \downarrow p_1 & & \\ & & X & & \end{array}$$

*there is a unique morphism  $f$  which makes the diagram commute.*

*Proof.* We already know that there is a unique set theoretic map which makes the diagram commute. Then since  $f_X$  and  $f_Y$  are given as regular functions, so is  $f$ . So  $f$  is a morphism.  $\square$

**Remark.** We will now try to understand the isomorphism  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .

## 9.4 Tensor Products

**Definition 9.4.** Let  $A$  be a (commutative) ring and  $M, N$  be  $A$ -modules. The *tensor product*  $M \otimes_A N$  is the  $A$ -module generated by the symbols  $m \otimes n$  for  $m \in M$  and  $n \in N$ , subject to the relations

1. (distributive law):  $(m + m') \otimes n = m \otimes n + m' \otimes n$ ,
2. (multiplication with scalars):  $a(m \otimes n) = (am) \otimes n = m \otimes (an)$ .

To make this precise,  $M \otimes_A N = A^{M \times N} / R$ , where  $R$  is the submodule generated by these relations.

**Example 9.4.1.** We have  $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$ . We can compute

$$1 \otimes 1 = (3 - 2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if  $\gcd(m, n) = 1$ , then  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ .

**Proposition 9.3** (Universal property of the tensor product). *For any bilinear map  $\Phi : M \times N \rightarrow P$  to an  $A$ -module  $P$  (i.e.  $n \mapsto \Phi(m, n)$  is  $A$ -linear for each  $m \in M$  and the same for  $m \mapsto \Phi(m, n)$ ),*

$$\begin{array}{ccc} M \times N & \xrightarrow{\Phi} & P \\ (m,n) \mapsto m \otimes n \downarrow & \nearrow \Psi & \\ M \otimes N & & \end{array}$$

*there exists a unique  $A$ -module homomorphism  $\Psi : M \otimes N \rightarrow P$  such that the above diagram commutes.*

**Remark.** We have the following properties of the tensor product:

1.  $A \otimes M \cong M$ ;

2.  $M \otimes N \cong N \otimes M$ ;
3.  $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ ;
4.  $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$ .

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed  $A$ -module  $M$  and an exact sequence

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

the sequence (where  $F$  is defined by  $m \otimes n' \mapsto m \otimes f(n')$  and  $G$  is defined by  $m \otimes n \mapsto m \otimes g(n)$ )

$$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

is also exact. In particular,  $\otimes M$  induces a right exact functor  $\text{Mod}_A \rightarrow \text{Mod}_A$  by  $N \mapsto M \otimes N$ .

**Example 9.4.2.** The functor  $\otimes M$  is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with  $\mathbb{Z}/2\mathbb{Z}$ , we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives  $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ .

**Exercise 9.1.** Show that  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m, n)\mathbb{Z}$ .

# Lecture 10

## Sept. 18 — Pre-varieties

### 10.1 More on Tensor Products

**Proposition 10.1.** *If  $B$  and  $C$  are  $A$ -algebras (i.e. there are ring homomorphisms  $f : A \rightarrow B$  and  $g : A \rightarrow C$  which give  $a \cdot b := f(a)b$  and  $a \cdot c = g(a)c$ , then  $B \otimes_A C$  is also an  $A$ -algebra with*

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc')$$

*and ring homomorphism  $A \rightarrow B \otimes_A C$  given by  $a \mapsto a \otimes 1$  (equivalently,  $1 \otimes a$ ).*

**Proposition 10.2.**  $k[x_1, \dots, x_m] \otimes_k k[y_1, \dots, y_n] \cong k[x_1, \dots, x_m, y_1, \dots, y_n]$ .

**Proposition 10.3.**  $(k[x_1, \dots, x_m]/I) \otimes_k (k[y_1, \dots, y_n]/J) \cong k[x_1, \dots, x_m, y_1, \dots, y_n]/\langle I, J \rangle$ .

*Proof.* Set  $R = k[x_1, \dots, x_m]$  and  $S = k[y_1, \dots, y_n]$ . We have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Applying the right exact functor  $\otimes_k (S/J)$  (and vice versa with  $J$  and  $\otimes R$ ) gives an exact sequence

$$\begin{array}{ccccccc} & & R \otimes_k J & & & & \\ & & \downarrow & & & & \\ & & R \otimes_k S & & & & \\ & & \downarrow & & & & \\ I \otimes_k (S/J) & \longrightarrow & R \otimes_k (S/J) & \longrightarrow & (R/I) \otimes_k (S/J) & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

So we have

$$(R/I) \otimes_k (S/J) \cong \frac{R \otimes_k (S/J)}{\text{Im}(I \otimes_k (S/J) \rightarrow R \otimes_k (S/J))} \cong \frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J},$$

which is the desired result since  $I \otimes_k S + R \otimes_k J = \langle I, J \rangle$  in  $R \otimes_k S$ . □

**Proposition 10.4** (Milne). *Let  $B$  and  $C$  be finitely generated  $k$ -algebras with  $k = \bar{k}$ .*

1. *If  $B$  and  $C$  are reduced, then so is  $B \otimes_k C$ .*

2. If  $B$  and  $C$  are domains, then so is  $B \otimes_k C$ .

**Remark.** We need  $k = \bar{k}$  in Proposition 10.4. Consider the domains  $\mathbb{R}[x]/(x^2 + 1)$ ,  $\mathbb{R}[y]/(y^2 + 1)$ . Then

$$\frac{\mathbb{R}[x]}{(x^2 + 1)} \otimes_{\mathbb{R}} \frac{\mathbb{R}[y]}{(y^2 + 1)} \cong \frac{\mathbb{R}[x, y]}{(x^2 + 1, y^2 + 1)},$$

which is not a domain since  $(\overline{x - y})(\overline{x + y}) = \overline{x^2 - y^2} = \overline{-1 - (-1)} = 0$ .

**Corollary 10.0.1.** If  $X \subseteq \mathbb{A}^m$  and  $Y \subseteq \mathbb{A}^n$  are affine varieties, then

1.  $I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \dots, x_m, y_1, \dots, y_n]$ .
2.  $A(X \times Y) \cong A(X) \otimes_k A(Y)$ .
3. If  $X$  and  $Y$  are irreducible, then  $X \times Y$  is irreducible.

*Proof.* Observe that  $V(I(X), I(Y)) = X \times Y \subseteq \mathbb{A}^{m+n}$ , so  $I(X \times Y) = \sqrt{\langle I(X), I(Y) \rangle}$ . Now we know that  $I(X)$  and  $I(Y)$  are radical in  $k[x_1, \dots, x_m]$  and  $k[y_1, \dots, y_n]$ , respectively, so

$$\frac{k[x_1, \dots, x_m]}{I(X)} \quad \text{and} \quad \frac{k[y_1, \dots, y_n]}{I(Y)}$$

are reduced. By Proposition 10.4, we get that

$$\frac{k[x_1, \dots, x_m, y_1, \dots, y_n]}{\langle I(X), I(Y) \rangle} \cong \frac{k[x_1, \dots, x_m]}{I(X)} \otimes_k \frac{k[y_1, \dots, y_n]}{I(Y)}$$

is reduced, so  $\langle I(X), I(Y) \rangle$  is radical. Thus  $I(X \times Y) = \langle I(X), I(Y) \rangle$ , so (1) holds.

Now (1) implies (2), and (3) follows since  $X$  and  $Y$  being irreducible implies  $A(X)$  and  $A(Y)$  are domains, which implies  $A(X \times Y)$  is a domain by Proposition 10.4 and (2), so  $X \times Y$  is irreducible.  $\square$

## 10.2 Pre-varieties

**Remark.** We will now head towards defining a *variety*, which is roughly finitely many affine varieties glued together (a *pre-variety*) with a separation condition (an algebraic version of Hausdorffness).

**Definition 10.1.** A *pre-variety* is a ringed space  $(X, \mathcal{O}_X)$  such that there exists a finite open cover  $X = \bigcup_{i=1}^s U_i$  with  $(U_i, \mathcal{O}_X|_{U_i})$  being an affine variety for all  $i = 1, \dots, s$ . A *morphism* of pre-varieties

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of the ringed spaces. We will often just write  $X$  for  $(X, \mathcal{O}_X)$ .

**Remark.** We call  $\varphi \in \mathcal{O}_X(U)$  with  $U \subseteq X$  open and  $\varphi : U \rightarrow k$  a *regular function* on  $U$ .

**Example 10.1.1.** Consider the following:

1. An affine variety  $X$  is a pre-variety. However, we have multiple choices for the open cover: We can take  $X = X$ , or  $X = \bigcup_{i=1}^s D(f_i)$  with  $f_i \in \mathcal{O}_X(X)$  and  $(f_1, \dots, f_s) = (1)$  in  $\mathcal{O}_X(X)$ .
2.  $\mathbb{P}_k^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$  is a pre-variety. We will see that  $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\text{pt}\}$ .

3. Let  $X = V(I) \subseteq \mathbb{A}^n$  be an affine variety and  $U \subseteq X$  open. Set

$$\mathcal{O}_U(V) = \{\varphi : V \rightarrow k \mid \varphi \text{ is regular}\}.$$

Then  $(U, \mathcal{O}_U)$  is a pre-variety. To see this, note that  $U = \bigcup_{f \in I(X \setminus U)} D(f)$ . Since  $U$  is Noetherian (hence is compact), we can find a finite subcover, so  $U = \bigcup_{i=1}^s D(f_i)$  for some  $f_i \in A(X)$ .

4. (Gluing) Let  $X_1$  and  $X_2$  be affine varieties, and  $U_{1,2} \subseteq X_1$ ,  $U_{2,1} \subseteq X_2$  open, with an isomorphism

$$f : U_{1,2} \longrightarrow U_{2,1}.$$

Then we get a pre-variety by setting  $X = (X_1 \sqcup X_2)/\sim$ , where  $a \sim f(a)$  for all  $a \in U_{1,2}$ ,  $f(a) \sim a$  for all  $a \in U_{2,1}$ , and  $b \sim b$  for all  $b \in X_1 \sqcup X_2$ . We have quotient maps

$$j_1 : X_1 \longrightarrow X \quad \text{and} \quad j_2 : X_2 \longrightarrow X.$$

Now  $X$  is a topological space with the quotient topology, and  $j_1, j_2$  are open embeddings (i.e. have open images and are homeomorphisms onto their images). Define a sheaf of rings  $\mathcal{O}_X$  on  $X$  by

$$\mathcal{O}_X(U) = \{\varphi : U \rightarrow k \mid j_1^* \varphi \in \mathcal{O}_{X_1}(j_1^{-1}(U)) \text{ and } j_2^* \varphi \in \mathcal{O}_{X_2}(j_2^{-1}(U))\}.$$

One can check  $X = j_1(X_1) \cup j_2(X_2)$  and  $(j(X_i), \mathcal{O}_X|_{j(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$ , so  $(X, \mathcal{O}_X)$  is a pre-variety.

**Example 10.1.2.** Consider  $X_1 = \mathbb{A}_x^1$  and  $X_2 = \mathbb{A}_y^1$ , with  $U_{1,2} = \mathbb{A}_x^1 \setminus \{0\}$  and  $U_{2,1} = \mathbb{A}_y^1 \setminus \{0\}$ . Define

$$\begin{aligned} f : U_{1,2} &\longrightarrow U_{2,1} \\ x &\longmapsto 1/x. \end{aligned}$$

Then we can take  $\mathbb{P}_k^1 = (X_1 \sqcup X_2)/\sim$ . What are the regular functions  $\mathbb{P}_k^1 \rightarrow k$ ? We should get only the constant functions (When  $k = \mathbb{C}$ ,  $\mathbb{P}_{\mathbb{C}}^1$  is compact, so a holomorphic function  $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{C}$  is bounded. By restricting to  $X_1$ , we get a bounded map  $f : \mathbb{C} \rightarrow \mathbb{C}$ , so  $f$  is constant by Liouville's theorem).

In general, let  $j_i : X_i \rightarrow \mathbb{P}_k^1$  be the quotient maps. Fix  $\varphi \in \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ . Now

$$\varphi|_{X_1} := j_1^* \varphi = \sum_{i \geq 0} a_i x^i \quad \text{and} \quad \varphi|_{X_2} := j_2^* \varphi = \sum_{i \geq 0} b_i y^i$$

for some  $a_i, b_i \in k$ . They must agree on the overlap, so

$$\sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} b_i (1/x)^i$$

as functions on  $\mathbb{A}^1 \setminus \{0\}$ . Since  $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$ , we have  $a_i = b_i = 0$  for  $i > 0$  and  $a_0 = b_0$  (since the powers of  $x^{\pm 1}$  are  $k$ -linearly independent), so  $\varphi$  is a constant function.

If we instead took  $f : U_{1,2} \rightarrow U_{2,1}$  to be  $x \mapsto x$ , then  $X = (X_1 \sqcup X_2)/\sim$  is the “bug-eyed line” with two points  $0, 0'$  at the origin (this is the *line with two origins* when  $k = \mathbb{R}$ , which is not Hausdorff.) Note that  $X \setminus \{0, 0'\} \cong \mathbb{A}^1 \setminus \{0\}$ . In our case, the bad property is that there exist two morphisms

$$g_1, g_2 : \mathbb{A}^1 \longrightarrow X$$

such that  $g_1|_{\mathbb{A}^1 \setminus \{0\}} = g_2|_{\mathbb{A}^1 \setminus \{0\}}$  and  $g_1 \neq g_2$ , i.e. “limits are not unique” on  $X$ . Note that a similar computation shows  $\mathcal{O}_X(X) \cong k[x]$ , so in particular,  $X \not\cong \mathbb{P}_k^1$ .



# Lecture 11

## Sept. 23 — Pre-varieties, Part 2

### 11.1 More on Pre-varieties

**Proposition 11.1.** *Let  $(X, \mathcal{O}_X)$  be a pre-variety.*

1.  $X$  is Noetherian as a topological space.
2.  $X$  has a basis by affine varieties.

*Proof.* (1) Note that  $X$  has a finite cover by affine varieties  $U_i$ , which are each Noetherian.

(2) If  $(X, \mathcal{O}_X)$  is affine, then  $\{D(f) : f \in \mathcal{O}_X(X)\}$  gives such a basis. Do this for each  $U_i$ .  $\square$

**Example 11.0.1** (General gluing procedure). Let  $I$  be a finite index set,  $(X_i, \mathcal{O}_{X_i})$  affine varieties,  $U_{i,j} \subseteq X_i$  open sets, and  $f_{i,j} : U_{i,j} \rightarrow U_{j,i}$  isomorphisms for each  $i, j \in I$ , satisfying

1.  $U_{i,i} = X_i$  and  $f_{i,i} = \text{id}$ ;
2.  $f_{i,j}^{-1}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k}$ ;
3. the following diagram commutes:

$$\begin{array}{ccc} U_{i,j} \cap U_{i,k} & \xrightarrow{f_{i,k}} & U_{k,i} \cap U_{k,j} \\ & \searrow f_{i,j} \quad \nearrow f_{j,k} & \\ & U_{j,i} \cap U_{j,k} & \end{array}$$

We can define  $X = (\bigsqcup_{i \in I} X_i) / \sim$  with the quotient topology, where  $a \sim a$  for  $a \in X_i$  and  $a \sim f_{i,j}(a)$  for  $a \in U_{i,j}$ . The inclusions  $j_i : X_i \hookrightarrow X$  are open embeddings, and we can set

$$\mathcal{O}_X(U) := \{\varphi : U \rightarrow k \mid \varphi|_{X_i} = j_i^* \varphi \text{ is regular for all } i\}.$$

Then  $(X, \mathcal{O}_X)$  is a ringed space with  $(j_i(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$ , so  $(X, \mathcal{O}_X)$  is a pre-variety.

**Remark.** Any pre-variety  $X$  is a gluing of affine varieties. To see this, note that there exists a cover  $X = \bigcup_{i=1}^s U_i$  by affine varieties. Then we can take  $X_i = U_i$ ,  $U_{i,j} = U_i \cap U_j \subseteq X_i$ , and  $f_{i,j} : U_{i,j} \rightarrow U_{j,i}$  to be the identity map.

**Proposition 11.2.** *Let  $X$  be a pre-variety.*

1. If  $U \subseteq X$  is an open set, then  $(U, \mathcal{O}_X|_U)$  is again a pre-variety.

2. Let  $Z \subseteq X$  be a closed set. For  $U \subseteq Z$  open, set

$$\mathcal{O}_Z(U) = \left\{ \varphi : U \rightarrow k \mid \begin{array}{l} \text{for each } a \in U, \text{ there exists open } a \in W \subseteq X \text{ and} \\ \psi : W \rightarrow k \text{ regular such that } \varphi|_{W \cap Z} = \psi|_{W \cap Z} \end{array} \right\}.$$

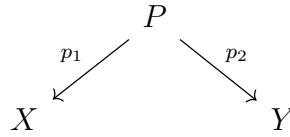
Then  $(Z, \mathcal{O}_Z)$  is a pre-variety.

*Proof.* (1) Note that  $X$  has a basis by affine varieties, so we can cover  $U$  by affine varieties. This cover may be infinite, but we can pass to a finite subcover since  $X$  and hence  $U$  is Noetherian.

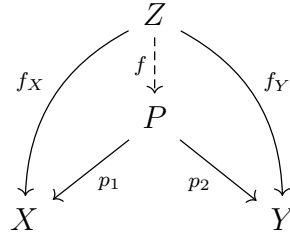
(2) The idea is to first reduce to the case  $X = V(I) \subseteq \mathbb{A}^n$ , so  $Z \subseteq X$  is cut out by polynomials. Then observe that  $\mathcal{O}_Z$  agrees with the previous definition.  $\square$

**Remark.** Note that unions of intersections of closed and open sets are not necessarily pre-varieties. For instance, consider  $(\mathbb{A}^2 \setminus V(xy)) \cup \{0\}$ .

**Proposition 11.3.** *If  $X, Y$  are pre-varieties, then there exists a pre-variety with morphisms*



with the property that for every diagram

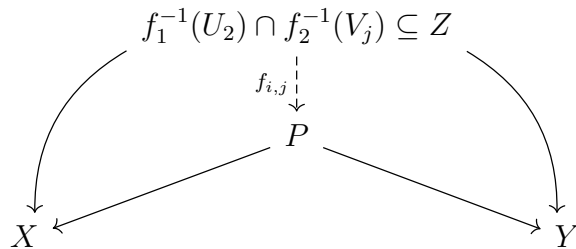


there exists a unique morphism  $f$  such that the diagram commutes. We call  $P$  the product of  $X$  and  $Y$ , and write  $X \times Y := P$ . Moreover, set theoretically  $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$ .

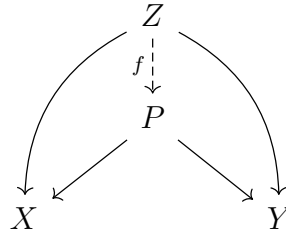
*Proof.* We know the result holds when  $X, Y, Z$  are affine or even open sets of affine varieties. In the general case, fix an open affine cover  $X = \bigcup_{i=1}^s U_i$  and  $Y = \bigcup_{j=1}^r V_j$ . Then glue the products by:

1.  $P_{(i,j)} := U_i \times V_j$ ,
2. along  $P_{(i,j),(i',j')} : (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$ ,
3. via  $f_{(i,j),(i',j')} : P_{(i,j),(i',j')} \xrightarrow{\cong} P_{(i',j'),(i,j)}$ , the isomorphism from the universal property of products.

We get a pre-variety  $P$ , and the morphisms



glue to give a morphism



which is a morphism as the morphism condition can be checked locally. Furthermore, the diagram commutes (as can be checked locally). Last,  $f$  is unique: One can either check this locally or check set theoretically using  $P = \{(x, y) : x \in X \text{ and } y \in Y\}$  as sets.  $\square$

**Remark.** Note that  $X \times Y$  is set theoretically the product of  $X$  and  $Y$ , but not the product of  $X$  and  $Y$  as topological spaces. Consider  $X = Y = \mathbb{A}^1$  and  $X \times Y = \mathbb{A}^2$ .

## 11.2 Varieties

**Remark.** We want a version of Hausdorffness in algebraic geometry. However, an irreducible topological space (e.g.  $\mathbb{A}^n$ ) is almost never Hausdorff (unless it is a single point). From a different perspective, note that  $X$  is Hausdorff if and only if the diagonal  $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$  is closed, where  $X \times X$  is given the product topology.

**Definition 11.1.** A pre-variety is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in  $X \times X$  (the product pre-variety). A *variety* is a pre-variety that is separated.

**Example 11.1.1.**  $\mathbb{A}^n$  is separated. We have

$$V(x_1 - y_1, \dots, x_n - y_n) = \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_{y_i}^n \cong \mathbb{A}^{2n},$$

so  $\Delta_{\mathbb{A}^n}$  is closed in  $\mathbb{A}^n \times \mathbb{A}^n$ .

**Example 11.1.2.** Any affine variety is separated. To see this, we may assume  $X = V(I) \subseteq \mathbb{A}^n$ . By the construction of the product,  $X \times X \subseteq \mathbb{A}^n \times \mathbb{A}^n$  is closed and

$$\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}.$$

Since  $\Delta_{\mathbb{A}^n}$  is closed, we have  $\Delta_X$  is closed in  $X \times X$  since  $X \times X$  has the subspace topology.

**Proposition 11.4.** *If  $X$  is a variety, then any closed or open set  $Z \subseteq X$  is a variety.*

*Proof.* We have already seen that  $Z$  is a pre-variety, so it suffices to show that  $Z$  is separated. We note that  $Z \times Z \hookrightarrow X \times X$  is an embedding of topological spaces, and  $\Delta_Z = (Z \times Z) \cap \Delta_X$ . Since  $\Delta_X$  is closed and  $Z \times Z$  has the subspace topology,  $\Delta_Z$  is closed in  $Z \times Z$ . So  $Z$  is separated.  $\square$

**Example 11.1.3.** Recall the bug-eyed line from Example 10.1.2. Let  $a, b$  be the two origins, and write  $X = U_1 \cup U_2$ , where  $U_1 = X \setminus \{b\} \cong \mathbb{A}^1$  and  $U_2 = X \setminus \{a\} \cong \mathbb{A}^1$ . Then consider

$$\mathbb{A}^2 \cong U_1 \times U_2 \subseteq X \times X.$$

Note that  $\Delta_X \cap (U_1 \times U_2) = \{(x, x) : x \in k \setminus \{0\}\} = \Delta_{\mathbb{A}^1} \setminus \{0\}$ . So  $\Delta_X$  is not closed in  $X \times X$ .

**Exercise 11.1.** Show that  $\mathbb{P}_k^1$  is separated.

**Proposition 11.5.** *Let  $f, g : X \rightarrow Y$  be morphisms of pre-varieties with  $Y$  a variety.*

1. *The graph  $\Gamma_f := \{(x, f(x)) : x \in X\}$  of  $f$  is closed in  $X \times Y$ .*
2.  *$\{x \in X : f(x) = g(x)\}$  is closed in  $X$ . This becomes a version of the identity principle in the case that  $X$  is irreducible: If  $X$  is irreducible and  $f, g$  agree on a nonempty open set, then  $f = g$ .*

*Proof.* (1) We can write  $\Gamma_f = (f, \text{id})^{-1}(\Delta_Y)$  where  $(f, \text{id}) : X \times Y \rightarrow Y \times Y$ , and  $\Delta_Y$  is closed.

(2) Consider the morphism

$$\begin{aligned} X &\xrightarrow{(f, g)} Y \times Y \\ x &\longmapsto (f(x), g(x)). \end{aligned}$$

Then  $\{x \in X : f(x) = g(x)\} = (f, g)^{-1}(\Delta_Y)$ , so it is closed. □

# Lecture 12

## Sept. 25 — Projective Varieties

### 12.1 Projective Space

**Definition 12.1.** Define *projective  $n$ -space* over  $k$  to be

$$\mathbb{P}_k^n = \mathbb{P}^n = \text{1-dimensional subspaces of } k^{n+1} = (k^{n+1} \setminus \{0\})/\sim,$$

where  $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$  if there exists  $\lambda \in k^\times$  such that  $(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$ . We write  $[x_0 : x_1 : \dots : x_n] \in \mathbb{P}_k^n$  for the equivalence class of  $(x_0, x_1, \dots, x_n)$ .

**Example 12.1.1.** For  $n = 2$ , we have  $[1 : 0 : 2] = [1/2 : 0 : 1] \in \mathbb{P}_k^2$  when  $\text{char } k \neq 2$ .

**Remark.** For  $0 \leq i \leq n$ , define  $U_i = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{P}_k^n : x_i \neq 0\}$ . Then

$$\mathbb{P}_k^n = \bigcup_{i=0}^n U_i,$$

and there exist bijective maps  $f_i : U_i \rightarrow \mathbb{A}^n$  given by

$$f_i([x_0 : \dots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i),$$

where  $\widehat{x_i/x_i}$  means we omit  $x_i/x_i$ . For  $i = 0$ , the inverse is  $f_0^{-1}(x_1, \dots, x_n) = [1 : x_1 : \dots : x_n]$ .

**Remark.** Another way to think about  $\mathbb{P}^n$  is via points at  $\infty$ . Observe that

$$\mathbb{P}^n \setminus U_0 = \{[0 : x_1 : \dots : x_n] \in \mathbb{P}_k^n : (x_1, \dots, x_n) \in k^n \setminus \{0\}\} \cong \mathbb{P}^{n-1}.$$

So  $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} = \dots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$ .

**Remark.** Why work with  $\mathbb{P}^n$ ? One motivation is analytic (e.g. for  $k = \mathbb{C}$ ):

1.  $\mathbb{P}_{\mathbb{C}}^n$  is compact with the analytic topology: There are surjective continuous maps

$$\begin{array}{ccc} \mathbb{R}^{2n+2} \setminus \{0\} & \twoheadrightarrow & \mathbb{CP}^n \\ \uparrow & \nearrow & \\ S^{2n+1} & & \end{array}$$

2. *Chow's theorem:* Any closed complex submanifold of  $\mathbb{CP}^n$  is a projective variety.

Another motivation is the extra data at  $\infty$ :

1. If  $\ell_1, \ell_2$  are distinct lines in  $\mathbb{A}^2$ , then  $\#(\ell_1 \cap \ell_2) = 0$  or 1. However, over  $\mathbb{P}^2$ ,  $\#(\ell_1 \cap \ell_2) = 1$  always.

2. *Bezout's theorem*: If  $C_1, C_2 \subseteq \mathbb{A}^2$  are two distinct irreducible curves in  $\mathbb{A}^2$ , then

$$\#(C_1 \cap C_2) \leq (\deg C_1)(\deg C_2),$$

counting multiplicities. The version over  $\mathbb{P}^2$  always gives equality.

## 12.2 Graded Rings

**Remark.** In projective space, for  $f \in k[x_0, \dots, x_n]$ , we could try to define

$$V(f) = \{[a_0 : \dots : a_n] : f(a_0, \dots, a_n) = 0\}.$$

But this is bad notation as it is not well-defined ( $f = 0$  depends on the representative in the equivalence class). Instead, if  $f$  is homogeneous of degree  $d$ , then

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n),$$

so  $V(f)$  is well-defined in this case, when  $f$  is homogeneous.

**Definition 12.2.** An  $\mathbb{N}$ -graded ring is a ring  $R$  with subgroups  $R_d \subseteq R$  for  $d \in \mathbb{N}$  such that

$$R = \bigoplus_{d \in \mathbb{N}} R_d \quad \text{and} \quad R_d R_e \subseteq R_{d+e}.$$

An element  $f \in R$  is *homogeneous* if there exists  $d$  such that  $f \in R_d$ .

**Example 12.2.1.** For  $S = k[x_0, \dots, x_n]$ , we can take  $S_d = \bigoplus_{a_i \geq 0, \sum a_i = d} kx_0^{a_0} \cdots x_n^{a_n}$ .

**Definition 12.3.** An ideal  $I$  in a graded ring is *homogeneous* if it is generated by homogeneous elements.

**Example 12.3.1.** We can write  $(x, y^3 - 3x^2) \subseteq k[x, y]$  as  $(x, y^3)$ , so it is homogeneous.

**Proposition 12.1.** Let  $R$  be a graded ring with ideal  $I$ . The following are equivalent:

1.  $I$  is homogeneous;
2. for any  $f = \sum_{d \in \mathbb{N}} f_d \in I$  with  $f_d \in R_d$ , then  $f_d \in I$  for all  $d$ ;
3.  $I = \bigoplus_{d \in \mathbb{N}} (I \cap R_d)$ .

*Proof.* Left as an exercise. The interesting implication is  $(1 \Rightarrow 2)$ . □

**Proposition 12.2.** Let  $I, J$  be homogeneous ideals of a graded ring  $R$ . Then

1.  $I + J$ ,  $IJ$ ,  $\sqrt{I}$ , and  $I \cap J$  are all homogeneous;
2.  $R/I$  is a graded ring with  $R/I = \bigoplus_{d \in \mathbb{N}} R_d/I_d$ , where  $I_d = I \cap R_d$ .

*Proof.* (1) We prove that  $\sqrt{I}$  is homogeneous. Assume  $f \in \sqrt{I}$ , and write  $f = f_0 + f_1 + \dots + f_d$  with  $f_i \in R_i$  and  $f_d \neq 0$ . Now there exists  $n > 0$  such that  $f^n \in I$ , and

$$f^n = f_d^n + \text{lower order terms}.$$

Since  $I$  is homogeneous,  $f_d^n \in I$ , so  $f_d \in \sqrt{I}$ . Then  $f_0 + \dots + f_{d-1} \in \sqrt{I}$ , and we can repeat.

(2) We can write  $R/I = (\bigoplus_{d \in \mathbb{N}} R_d) / (\bigoplus_{d \in \mathbb{N}} (I \cap R_d))$ . As abelian groups, this is  $R/I \cong \bigoplus_{d \in \mathbb{N}} R_d/I_d$ . One can check that the multiplication also respects the grading, so this is an isomorphism of rings. □

## 12.3 Projective Varieties

**Definition 12.4.** For a set  $T \subseteq k[x_0, \dots, x_n]$  of homogeneous elements, define its *vanishing locus*

$$V_p(T) := V(T) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\} \subseteq \mathbb{P}^n.$$

A *projective variety* is a subset of this form. For a homogeneous ideal  $I \leq k[x_0, \dots, x_n]$ , define

$$V(I) = \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in I \text{ homogeneous}\}.$$

For a subset  $X \subseteq \mathbb{P}^n$ , define its *ideal*

$$I_p(X) := I(X) = (f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in X).$$

Note that we need to take the ideal generated by these elements, otherwise we may not get an ideal.

**Remark.** If  $T \subseteq k[x_0, \dots, x_n]$  is a subset of homogeneous elements, then we have  $V_p(T) = V_p((T))$ . So projective varieties can equivalently be defined as vanishing sets of homogeneous ideals.

**Example 12.4.1.** Consider  $X = V_p(x^2 - yz) \subseteq \mathbb{P}^2_{x,y,z}$ . Set  $H = V(x)$ , then there is a bijection

$$\begin{aligned} U = \mathbb{P}^2 \setminus H &\xrightarrow{f} \mathbb{A}^2 \\ [1 : y : z] &\longmapsto (y, z). \end{aligned}$$

Then  $f(X \cap U) = V(1 - yz)$ . On the other hand, we can see that

$$X \cap H = \{[0 : 1 : 0], [0 : 0 : 1]\} = \{a, b\}.$$

If we were working with  $\mathbb{C}$  with the analytic topology, then we can take limits on  $V(1 - yz)$  and see

$$\lim_{t \rightarrow 0} [1 : t : 1/t] = \lim_{t \rightarrow 0} [t : t^2 : 1] = [0 : 0 : 1] = b.$$

Note that we essentially switched charts in order to take this limit. Similarly, we have

$$\lim_{t \rightarrow \infty} [1 : t : 1/t] = \lim_{t \rightarrow \infty} [1/t : 1 : 1/t^2] = [0 : 1 : 0] = a.$$

So we can see  $a, b$  as points at  $\infty$  compactifying the curve  $V(1 - yz)$ .

**Example 12.4.2.** We have the following:

1.  $V_p(0) = \mathbb{P}^n$ ;
2.  $V_p(1) = \emptyset$ ;
3. if  $p = [a_0 : \dots : a_n]$  and  $J = (a_i x_j - a_j x_i : 0 \leq i, j \leq n)$ , then  $V(J) = \{0\}$ ;
4.  $I_0 = (x_0, \dots, x_n)$  is called the *irrelevant ideal*, which has  $V_p(I_0) = \emptyset = V_p(1)$  but  $I_0 = \sqrt{I_0} \subsetneq (1)$ .

# Lecture 13

## Sept. 30 — Projective Varieties, Part 2

### 13.1 More on Projective Varieties

**Example 13.0.1.** Consider  $X = V(y^2z - x^3 - zx^2 - z^3) \subseteq \mathbb{P}^2$  and  $H_z = V(z)$ . Let

$$U_z = \mathbb{P}^2 \setminus H_z \xrightarrow[\text{bij}]{f} \mathbb{A}^2$$
$$[x : y : 1] \mapsto (x, y).$$

Then  $f(X \cap U_z) = V(y^2 - x^3 - x^2 - 1)$ , and

$$X \cap U_z = V(y^2z - x^3 - zx^2 - z^3, z) = V(x^3, z) = \{[0 : 1 : 0]\}.$$

**Example 13.0.2.** Let  $I = (x_0, \dots, x_n)$  be the irrelevant ideal. Then  $I$  is radical, but

$$I_p(V_p(I)) = I_p(\emptyset) = (1) \neq \sqrt{I}.$$

### 13.2 Cones

**Definition 13.1.** A subset  $C \subseteq \mathbb{A}^{n+1}$  is a *cone* if  $0 \in C$  and  $\lambda x \in C$  whenever  $x \in C$  and  $\lambda \in k$ .

**Example 13.1.1.** If  $X \subseteq \mathbb{P}^n$  is a projective variety, then we can set  $C(X) = \pi^{-1}(X)\{0\}$ , where

$$\pi : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$$
$$x \mapsto [x].$$

**Proposition 13.1.** If  $C \subseteq \mathbb{A}^{n+1}$  is a cone, then  $I_a(C) \leq k[x_0, \dots, x_n]$  is homogeneous.

*Proof.* Fix  $f \in I_a(C)$ . Then we can write  $f = \sum_{i=0}^d f_i$  with  $f_i$  homogeneous of degree  $i$ . We want to show that  $f_i \in I_a(C)$  for each  $i$ . Fix  $x \in C$ . For any  $\lambda \in k$ ,

$$0 = f(\lambda x) = \sum_{i=0}^d \lambda^i f_i(x).$$

Viewing this as a polynomial in  $\lambda$  (with  $x$  fixed), we must have each  $f_i(x) = 0$ . Thus  $f_i \in I_a(C)$ .  $\square$



### 13.3 Projective Nullstellensatz

**Theorem 13.1** (Projective Hilbert's Nullstellensatz). *We have the following:*

1. For a projective variety  $X \subseteq \mathbb{P}^n$ ,  $V_p(I_p(X)) = X$ .
2. For a homogeneous ideal  $J \leq k[x_0, \dots, x_n]$  with  $\sqrt{J} \neq (x_0, \dots, x_n)$ ,  $I_p(V_p(J)) = \sqrt{J}$ .

As a consequence, there is a bijection between projective varieties and radical homogeneous ideals of  $k[x_0, \dots, x_n]$  which are not equal to  $(x_0, \dots, x_n)$ , given by  $X \mapsto I_p(X)$  with inverse  $J \mapsto V_p(J)$ .

*Proof.* (1) This is similar to the affine case.

(2) Fix a homogeneous ideal  $(1) \neq J \leq k[x_0, \dots, x_n]$  such that  $\sqrt{J} \neq (x_0, \dots, x_n)$  (the theorem is clearly true for the unit ideal). Then observe that we can write

$$\begin{aligned} I_p(V_p(J)) &= \{f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in V_p(J)\} \\ &= \{f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J) \setminus \{0\}\} \\ &= \{f \in k[x_0, \dots, x_n] : f(x) = 0 \text{ for all } x \in \overline{V_a(J) \setminus \{0\}}\} \\ &= \begin{cases} I_a(V_a(J)) & \text{if } V_a(J) \supsetneq \{0\}, \\ I_a(\emptyset) & \text{if } V_a(J) = \{0\}, \end{cases} \end{aligned} \quad \begin{array}{l} \text{(A)} \\ \text{(B)} \end{array}$$

In Case A, we get that  $I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J}$  by the affine Nullstellensatz. In Case B, we have  $V_a(J) = \{0\}$ , so  $\sqrt{J} = (x_0, \dots, x_n)$ , which we assumed was not the case.  $\square$

### 13.4 The Zariski Topology on $\mathbb{P}^n$

**Remark.** We have the following properties of  $I_p$  and  $V_p$ :

1. For homogeneous ideals  $J_i \leq k[x_0, \dots, x_n]$  for  $i \in I$ , we have  $V_p(\sum_{i \in I} J_i) = \bigcap_{i \in I} V_p(J_i)$ ;  
If  $I = \{1, 2\}$ , then we have  $V_p(J_1 J_2) = V_p(J_1) \cup V_p(J_2)$ .
2. If  $X_1, X_2 \subseteq \mathbb{P}^n$  are projective varieties, then

$$I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2) \quad \text{and} \quad I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)},$$

where we assume in the second equality that  $X_1 \cap X_2 \neq \emptyset$ .

The proofs are similar to the affine case.

**Example 13.1.2.** Let  $X_1 = V(x) \subseteq \mathbb{P}^2$  and  $X_2 = V(y, z) \subseteq \mathbb{P}^2$ . Then  $I(X_1 \cap X_2) = I(\emptyset) = (1)$ , but we have  $I(X_1) + I(X_2) = (x, y, z)$ , which is already radical.

**Definition 13.2.** The *Zariski topology* on  $\mathbb{P}^n$  is the topology whose closed sets are projective varieties  $X \subseteq \mathbb{P}^n$  (equivalently, the vanishing loci of homogeneous ideals).

**Remark.** This is a topology by the above properties of  $I_p$  and  $V_p$ . We now want to relate this to the topology on our charts. Let  $H_0 = V(x_0)$  and consider the bijection

$$\begin{aligned} \mathbb{A}^n &\xrightarrow{\rho_0} \mathbb{P}^n \setminus H_0 \\ (x_1, \dots, x_n) &\longmapsto [1 : x_1 : \dots : x_n]. \end{aligned}$$

We want to show that  $\rho_0$  is a homeomorphism. Write  $\mathbb{A}^n \subseteq \mathbb{P}^n$ . Consider the ring homomorphism

$$\begin{aligned} k[x_0, \dots, x_n] &\xrightarrow{\Phi} k[x_1, \dots, x_n] \\ f(x_0, \dots, x_n) &\mapsto f(1, x_1, \dots, x_n) =: f^i \end{aligned}$$

We call  $f^i$  the *dehomogenization* of  $f$ .

**Example 13.2.1.** Let  $f(x) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3$ , then  $f^i(x) = x_2^2 - x_1^3 - x_1^2 - 1$ .

**Definition 13.3.** If  $J \leq k[x_0, \dots, x_n]$  is homogeneous, then define its *dehomogenization* to be

$$J^i = (f^i : f \in J) = \Phi(J).$$

**Proposition 13.2.** For  $J \leq k[x_0, \dots, x_n]$  homogeneous,  $V_p(J) \cap \mathbb{A}^n = V_a(J^i)$ .

*Proof.* The idea is to use that for  $[1 : x_1 : \dots : x_n] \in \mathbb{P}^n$  and  $f \in k[x_0, \dots, x_n]$  homogeneous, we have  $f([1 : x]) = 0$  if and only if  $f^i(x) = 0$ . Fill in the details as an exercise.  $\square$

**Definition 13.4.** If  $f \in k[x_1, \dots, x_n]$  with  $\deg f = d$ , then define its *homogenization* to be

$$f^h = x_0^d f(x_1/x_0, \dots, x_n/x_0) \in k[x_0, x_1, \dots, x_n],$$

which is homogeneous of degree  $d$ .

**Example 13.4.1.** Let  $f = x_2^2 - x_1^3 - x_1^2 - 1$ . Then we have

$$f^h = x_0^3((x_2/x_0)^2 - (x_1/x_0)^3 - (x_1/x_0)^2 - 1) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3.$$

**Remark.** While  $f^h g^h = (fg)^h$ , note that  $(f + g)^h \neq f^h + g^h$  in general.

**Definition 13.5.** For  $J \leq k[x_1, \dots, x_n]$  an ideal, define its *homogenization* to be

$$J^h = (f^h : f \in J).$$

**Proposition 13.3.** For  $J \leq k[x_1, \dots, x_n]$  an ideal,  $V_a(J) = V_p(J^h) \cap \mathbb{A}^n$ .

*Proof.* Left as an exercise, use that  $f(a_1, \dots, a_n) = 0$  if and only if  $f^h(1, a_1, \dots, a_n) = 0$ .  $\square$

**Remark.** The above results imply that  $\rho_0 : \mathbb{A}^n \rightarrow \mathbb{P}^n \setminus H_0$  is a homeomorphism.

# Lecture 14

## Oct. 9 — Projective Space as Varieties

### 14.1 More on the Zariski Topology on $\mathbb{P}^n$

**Proposition 14.1.** *For each  $0 \leq i \leq n$ , the map*

$$U_i = \mathbb{P}^n \setminus V(x_i) \xrightarrow{h_i} \mathbb{A}^n$$

$$[x_0 : \cdots : x_n] \mapsto (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$$

*is a homeomorphism.*

*Proof.* The main inputs to the proof are

- For  $I \leq k[x_0, \dots, x_n]$  homogeneous,  $h_0(V(I) \cap U_0) = V(I^i)$ .
- For  $J \leq k[x_1, \dots, x_n]$ ,  $h_0^{-1}(V(J)) = V(J^h)$ .

Fill in the remaining details as an exercise. □

**Proposition 14.2** (Projective closure). *For  $J \leq k[x_1, \dots, x_n]$  and  $X = V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ , we have*

$$\overline{X} = V_p(J^h).$$

*Proof.* See Gathmann. □

**Proposition 14.3.** *If  $X = V_a(f) \subseteq \mathbb{A}^n$  with  $f \in k[x_1, \dots, x_n]$ , then its projective closure in  $\mathbb{P}^n$  is*

$$\overline{X} = V_p(f^h).$$

*Proof.* We know that  $\overline{X} = V_p(\langle f \rangle^h)$  by Proposition 14.2. Now

$$\langle f \rangle^h = \langle (fg)^h : g \in k[x_1, \dots, x_n] \rangle = \langle f^h g^h : g \in k[x_1, \dots, x_n] \rangle = \langle f^h \rangle,$$

which implies the desired result. □

**Example 14.0.1** (Twisted cubic). Take  $X = \text{Im}(\mathbb{A}^1 \rightarrow \mathbb{A}^3 : t \mapsto (t, t^2, t^3))$ . Note that  $X \cong \mathbb{A}^1$ , and

$$I_a(X) = (x^2 - y, x^3 - z) = (x^2 - y, x^3 - z, xy - zw).$$

Then one can check that  $\overline{X} \subseteq \mathbb{P}^3_{w:x:y:z}$  is given by  $\overline{X} = V_p(x^2 - yw, x^3 - zw^2, xy - zw)$ . However, one can also check that  $\overline{X}$  cannot be cut out by 2 equations. For example,

$$V_p(x^2 - yw, x^3 - zw^2) = \overline{X} \cup V(w, x).$$

## 14.2 Projective Space as Varieties

**Remark.** Our goal now is to show that projective varieties are varieties. The first step is to define a sheaf of regular functions on  $\mathbb{P}^n$ .

**Definition 14.1.** Let  $U$  be an open set of a projective variety  $X \subseteq \mathbb{P}^n$ . A function  $\varphi : U \rightarrow k$  is *regular* if for every  $p \in U$ , there exists  $d \in \mathbb{N}$ ,  $f, g \in k[x_0, \dots, x_n]$  homogeneous of degree  $d$ , and  $U_p \subseteq U$  open such that

$$\varphi(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in U_p.$$

**Remark.** If  $X \subseteq \mathbb{P}^n$  is a projective variety, then

$$\mathcal{O}_X(U) = \{\varphi : U \rightarrow k \mid \varphi \text{ is regular}\}$$

is a sheaf of rings on  $X$ . Again this is because the regular condition can be checked locally.

**Proposition 14.4.** *If  $X \subseteq \mathbb{P}^n$  is a projective variety, then  $(X, \mathcal{O}_X)$  is a pre-variety.*

*Proof.* Let  $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$ . It suffices to show  $(X_i, \mathcal{O}_X|_{X_i})$  is an affine variety for each  $0 \leq i \leq n$ . For simplicity, assume  $i = 0$ . Let  $J = I(X) \leq k[x_0, \dots, x_n]$  and  $Z_0 = V(J^i) \subseteq \mathbb{A}^n$ . We have seen before that we have a homeomorphism

$$\begin{aligned} X_0 &\xrightarrow{F} Z_0 \\ [x_0 : \dots : x_n] &\longmapsto (x_1/x_0, \dots, x_n/x_0). \end{aligned}$$

We claim that  $F$  induces an isomorphism of ringed spaces  $(X_0, \mathcal{O}_X|_{X_0}) \cong (Z_0, \mathcal{O}_{Z_0})$ . To see this, we need to check that regular functions pull back to regular functions via  $F$  and  $F^{-1}$ . A regular function on an open set of  $X_0$  is locally of the form

$$\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)}$$

with  $f, g$  homogeneous of the same degree. Now

$$(F^{-1})^* \left( \frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)} \right) = \frac{f(1, x_1, \dots, x_n)}{g(1, x_1, \dots, x_n)},$$

which is a fraction of polynomials and hence regular on  $Z_0$ . So  $F^{-1}$  pulls regular functions back to regular functions. Conversely, a regular function on  $Z_0$  is locally given by

$$\frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)},$$

and its pullback via  $F$  is

$$F^* \left( \frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)} \right) = \frac{q(x_1/x_0, \dots, x_n/x_0)}{r(x_1/x_0, \dots, x_n/x_0)} = \frac{x_0^d q(x_1/x_0, \dots, x_n/x_0)}{x_0^d r(x_1/x_0, \dots, x_n/x_0)},$$

where  $d = \max\{\deg q, \deg r\}$ . This is regular on  $X_0$ , so  $F$  also pulls regular functions back to regular functions. So we get an isomorphism of ringed spaces, as desired.  $\square$

**Example 14.1.1.**  $\mathbb{P}^n$  is a pre-variety, and  $\mathbb{P}^n \setminus V(x_i) =: U_i \cong \mathbb{A}^n$  as pre-varieties.

**Definition 14.2.** A *morphism* of projective varieties is a morphism of the underlying pre-varieties.

**Remark.** For a projective variety  $X$ , it will be convenient to work with “global coordinates,” i.e.

$$S(X) := k[x_0, \dots, x_n]/I_p(X).$$

This is called the *homogeneous coordinate ring*. Note the following:

1. For  $f \in S(X)$  homogeneous,  $f$  is not necessarily a well-defined function on  $X$ . But

$$V(f) = \{[x] \in X : f(x) = 0\}$$

is still well-defined.

2. A relative version of the projective Nullstellensatz holds: There is a bijection

$$\begin{aligned} \left\{ \begin{array}{c} \text{projective subvarieties} \\ Y \subseteq X \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{c} \text{radical homogeneous ideals in } S(X) \\ \text{not equal to } (\bar{x}_1, \dots, \bar{x}_n) \end{array} \right\} \\ Y &\longmapsto I(Y) \\ V(J) &\longleftarrow J \end{aligned}$$

where  $I(Y) = \langle f \in S(X) : f \text{ homogeneous and } f(y) = 0 \text{ for all } y \in Y \rangle$ .

**Lemma 14.1.** If  $X \subseteq \mathbb{P}^n$  and  $f_0, \dots, f_m \in S(X)$  are homogeneous of the same degree, then

$$\begin{aligned} U = X \setminus V(f_0, \dots, f_m) &\xrightarrow{f} \mathbb{P}^m \\ [x_0 : \dots : x_n] &\longmapsto [f_0(x) : \dots : f_m(x)] \end{aligned}$$

is a morphism.

*Proof.* To see that  $f$  is well-defined, note that for  $[a_0 : \dots : a_n] \in X \setminus V(f_0, \dots, f_m)$ , we have

$$(f_0(\lambda a), \dots, f_m(\lambda a)) = \lambda^d (f_0(a), \dots, f_m(a))$$

with  $d = \deg f_i$ . So  $[f_0(a) : \dots : f_m(a)] \in \mathbb{P}^m$  is well-defined. To see that  $f$  is a morphism, we check locally on  $\mathbb{P}^m$ . Let  $V_i = \mathbb{P}^m \setminus V(x_i)$  and  $U_i = f^{-1}(V_i)$ . Then

$$\begin{aligned} U_i &\xrightarrow{f|_{U_i}} V_i \cong \mathbb{A}^m \\ a &\longmapsto \left( \frac{f_0(a)}{f_i(a)}, \dots, \frac{\widehat{f_i(a)}}{f_i(a)}, \dots, \frac{f_m(a)}{f_i(a)} \right). \end{aligned}$$

Since each  $f_j/f_i$  is regular,  $f|_{U_i}$  is a morphism. So  $f$  is a morphism. □

**Example 14.2.1.** Define a map

$$\begin{aligned} \mathbb{P}_{s:t}^1 &\xrightarrow{f} \mathbb{P}_{x:y:z}^3 \\ [s : t] &\longmapsto [s^3 : s^2t : st^2 : t^3]. \end{aligned}$$

Then  $S(\mathbb{P}^1) = k[s, t]$  and  $f(\mathbb{P}^1)$  is the projective twisted cubic in  $\mathbb{P}^3$ .

**Example 14.2.2.** Let  $A \in \mathrm{GL}_{n+1}(k)$ . Then

$$\begin{aligned} f_A : \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

is an isomorphism with inverse  $f_{A^{-1}}$ . We will see later that we have a surjective group homomorphism

$$\begin{aligned} \mathrm{GL}_{n+1}(k) &\longrightarrow \mathrm{Aut}(\mathbb{P}^n) \\ A &\longmapsto f_A \end{aligned}$$

with kernel  $k^\times I$ . So we get  $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{GL}_{n+1}(k)/k^\times I =: \mathrm{PGL}_{n+1}(k)$ .

**Example 14.2.3** (Conics). Let  $f \in k[x, y, z]$  be homogeneous of degree 2, and write

$$f = (x, y, z)B(x, y, z)^T$$

with  $B$  a symmetric  $3 \times 3$  matrix. We want to characterize  $X = V(f)$ . Choose  $A \in \mathrm{GL}_3(k)$  such that

$$B' = ABA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then  $f' = (x, y, z)B'(x, y, z)^T$  has  $f' = x^2 + y^2 + z^2$ ,  $x^2 + y^2$ , or  $x^2$ . Now  $A$  induces an isomorphism  $h_{A^{-1}} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  and  $g := h_{A^{-1}}|_X : X \rightarrow h_{A^{-1}}(X) = V(f')$ , so any projective conic is isomorphic to

$$V(x^2 + y^2 + z^2), \quad V(x^2 + y^2), \quad \text{or} \quad V(x^2).$$

**Example 14.2.4** (Projections). Let  $a = [1 : 0 : \cdots : 0]$  and define

$$\begin{aligned} \mathbb{P}^n \setminus \{a\} &\xrightarrow{f} \mathbb{P}^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_1 : \cdots : x_n]. \end{aligned}$$

Geometrically, if we fix  $[b] \in \mathbb{P}^n \setminus \{a\}$  and set

$$\ell_{a,b} = \{[s : tb_1 : \cdots : tb_n] : (s, t) \in k^2 \setminus \{0\}\} = \text{the line through } a \text{ and } b,$$

then  $\ell_{a,b} \cap V(x_0) = [0 : b_1 : \cdots : b_n] = [0 : f(b)]$ .

# Lecture 15

## Oct. 14 — Projective Space as Varieties, Part 2

### 15.1 Example of Projective Morphism

**Example 15.0.1** (Projections, continued). Let  $H \subseteq \mathbb{P}^n$  be a hyperplane and  $p \notin H$ . Then we can define

$$\begin{aligned}\mathbb{P}^n \setminus \{p\} &\xrightarrow{\pi} H \cong \mathbb{P}^{n-1} \\ q &\longmapsto \text{intersection point of } H \text{ and } \overline{pq}.\end{aligned}$$

For example, when  $n = 2$ ,  $p = [1 : 0 : 0] \in \mathbb{P}_{x_0:x_1:x_2}^2$ , and  $H = V(x_0)$ , then we have

$$\begin{aligned}\mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2].\end{aligned}$$

Note that this does not extend to a morphism  $\mathbb{P}^2 \rightarrow \mathbb{P}^1$ . But if we let  $X = V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^2$ , then the restriction of the above morphism to  $X$ :

$$\begin{aligned}X \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2].\end{aligned}$$

does extend to a morphism

$$\begin{aligned}X &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto \begin{cases} [x_1 : x_2] & \text{if } [x] \neq [1 : 0 : 0], \\ [x_2 : x_0] & \text{if } [x] \neq [1 : 1 : 0]. \end{cases}\end{aligned}$$

### 15.2 The Segre Embedding

**Remark.** We now want to show that projective varieties are varieties, and understand an analogue of compactness in algebraic geometry. To do this, we will need to understand products.

**Definition 15.1.** Fix  $m, n \geq 0$ . The *Segre embedding* is the map  $\Sigma : \mathbb{P}_{x_i}^m \times \mathbb{P}_{y_i}^n \rightarrow \mathbb{P}_{z_{i,j}}^N$  given by

$$\Sigma([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [x_i y_j : 0 \leq i \leq m, 0 \leq j \leq n],$$

where  $N = (m+1)(n+1) - 1$ .

**Proposition 15.1.** *Let  $\Sigma$  be the Segre embedding. Then*

1.  $X = \Sigma(\mathbb{P}^m \times \mathbb{P}^n) = V(z_{i,j}z_{k,\ell} - z_{i,\ell}z_{k,j} : 0 \leq i, k \leq m, 0 \leq j, \ell \leq n)$ .
2. *The map  $\mathbb{P}^m \times \mathbb{P}^n \rightarrow X$  is an isomorphism, i.e.  $\Sigma$  is a closed embedding.*

*Proof.* (0) First one can check that  $\Sigma$  is a morphism. To do this, restrict to charts.

(1) Fix  $[a_{i,j}] \in \mathbb{P}^N$ . Then  $[a_{i,j}] \in \text{Im } \Sigma$  if and only if the matrix  $(a_{i,j})$  has rank 1, which occurs if and only if all  $2 \times 2$  minors of  $(a_{i,j})$  vanish, which happens if and only if

$$a_{i,j}a_{k,\ell} - a_{i,\ell}a_{k,j} = 0$$

for all  $i, j, k, \ell$  for which the above equation makes sense.

(2) We define a morphism  $X \rightarrow \mathbb{P}^m \times \mathbb{P}^n$  that will be inverse to  $\Sigma$ . Set  $U_{i,j} = X \cap \{z_{i,j} \neq 0\}$ . Define

$$\begin{aligned} U_{i,j} &\xrightarrow{h_{i,j}} \mathbb{P}^m \times \mathbb{P}^n \\ [z_{i,j}] &\longmapsto ([z_{0,j} : \cdots : z_{m,j}], [z_{i,0} : \cdots : z_{i,n}]). \end{aligned}$$

Using the definition of  $X$  (as the set of rank 1 matrices up to scaling), these glue to give a morphism  $X \rightarrow \mathbb{P}^m \times \mathbb{P}^n$  that is inverse to  $\Sigma$ .  $\square$

**Example 15.1.1.** Let  $m = n = 1$ . Then the Segre embedding is given by

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3_{x:y:z:w} \\ ([a_0 : a_1], [b_0 : b_1]) &\longmapsto \begin{bmatrix} a_0b_0 & a_0b_1 \\ a_1b_0 & a_1b_1 \end{bmatrix}. \end{aligned}$$

Then  $\text{Im } \Sigma = V(xw - yz)$ . Observe that the images of  $\{a\} \times \mathbb{P}^1$  and  $\mathbb{P}^1 \times \{b\}$  in  $\Sigma(\mathbb{P}^1 \times \mathbb{P}^1)$  are two families of lines, where the lines within each families do not intersect.

**Remark.** The following are consequences of the Segre embedding.

1. We can study products of projective varieties.

**Definition 15.2** (Redefinition of projective variety). A *projective variety* is a (pre-)variety  $X$  such that there exists a closed embedding  $X \hookrightarrow \mathbb{P}^n$  for some  $n \geq 0$ .

Now using the Segre embedding, we get that  $\mathbb{P}^m \times \mathbb{P}^n$  is a projective variety. Moreover, if  $X \subseteq \mathbb{P}^m$  and  $Y \subseteq \mathbb{P}^n$  are projective varieties, then so is  $X \times Y$ .

2. We can show that  $\mathbb{P}^n$  is separated.

**Lemma 15.1.**  $\Delta_{\mathbb{P}^n}$  is closed in  $\mathbb{P}^n \times \mathbb{P}^n$ .

*Proof.* Observe that

$$\Delta_{\mathbb{P}^n} = \{([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \in \mathbb{P}^n \times \mathbb{P}^n : x_i y_j - x_j y_i = 0 \text{ for all } 0 \leq i, j \leq n\}.$$

It suffices to show that  $\Delta_{\mathbb{P}^n}$  is closed in  $\mathbb{P}^n \times \mathbb{P}^n$ . There are two ways to see this. The first is to use the Segre embedding  $\mathbb{P}^n \times \mathbb{P}^n \xrightarrow[\text{cl}]{\Sigma} \mathbb{P}^N_{z_{i,j}}$ . Then we can write

$$\Sigma(\Delta_{\mathbb{P}^n}) = \Sigma(\mathbb{P}^n \times \mathbb{P}^n) \cap V(z_{i,j} - z_{j,i} : 0 \leq i, j \leq n),$$

which is closed in  $\mathbb{P}^N$ , so  $\Delta_{\mathbb{P}^n}$  is closed in  $\mathbb{P}^n \times \mathbb{P}^n$ . Alternatively, one can just compute  $\Delta_{\mathbb{P}^n}$  directly on the affine charts. One can fill in the details of this method as an exercise.  $\square$



**Proposition 15.2.** *Projective varieties are varieties.*

*Proof.* We have already seen that they are pre-varieties, so it suffices to show that they are separated. By Lemma 15.1,  $\mathbb{P}^n$  is separated. Thus any closed sub-prevariety of  $\mathbb{P}^n$  is also separated.  $\square$

## 15.3 Completeness

**Remark.** We now want an analogue of compactness in algebraic geometry. One issue is that all varieties are compact to begin with, but  $\mathbb{A}^n$  has points missing in some sense.

**Example 15.2.1.** Consider the projection map

$$\begin{aligned} \mathbb{A}^1 \times \mathbb{A}^1 &\xrightarrow{\text{pr}_2} \mathbb{A}^1 \\ (x, t) &\longmapsto t. \end{aligned}$$

Then  $X = V(xt - 1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$  is closed, but  $\text{pr}_2(X) = \mathbb{A}^1 \setminus \{0\}$ . If we instead viewed this over  $\mathbb{P}^1$ :

$$\begin{aligned} \mathbb{P}_{[x:y]}^1 \times \mathbb{A}_t^1 &\xrightarrow{\text{pr}_2} \mathbb{A}_t^1 \\ ([x : y], t) &\longmapsto t \end{aligned}$$

with  $\overline{X} = V(xt - y)$ , then  $\text{pr}_2(\overline{X}) = \mathbb{A}^1$  as there is a point  $([1 : 0], 0)$  at infinity in  $\overline{X}$ . In other words, “compactifying”  $\mathbb{A}^1$  to  $\mathbb{P}^1$  gives the desired missing point.

**Definition 15.3.** A morphism  $f : X \rightarrow Y$  is *closed* if  $f(Z)$  is closed in  $Y$  for all closed sets  $Z \subseteq X$ .

**Definition 15.4.** A variety  $X$  is *complete* if the projection

$$\text{pr}_2 : X \times Y \longrightarrow Y$$

is closed for all varieties  $Y$ .

**Remark.** The same definition for topological spaces gives the usual notion of compactness.

**Example 15.4.1.** Example 15.2.1 shows that  $\mathbb{A}^1$  is not complete. Similar examples show that  $\mathbb{A}^n$  is not complete for any  $n \geq 1$ .

**Proposition 15.3.**  $\mathbb{P}^n$  is complete.

*Proof.* The steps to show this are the following:

1. For any  $m, n \geq 0$ , the projection  $\text{pr}_2 : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$  is closed.

See Gathmann for a proof of this fact.

2. If  $Y$  is an affine variety, then  $\text{pr}_2 : \mathbb{P}^n \times Y \rightarrow Y$  is closed.

To see this, write  $Y = V(I) \subseteq \mathbb{A}^m \subseteq \mathbb{P}^m$  and consider the diagram

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{\text{pr}_2} & \mathbb{P}^m \\ \uparrow \text{op} & & \uparrow \\ \mathbb{P}^n \times \mathbb{A}^n & \xrightarrow{\text{pr}_2} & \mathbb{A}^n \\ \uparrow \text{cl} & & \uparrow \\ \mathbb{P}^n \times Y & \xrightarrow{\text{pr}_2} & Y \end{array}$$

Since the top row is closed, so is the bottom row.

Finally, we complete the proof. If  $Y$  is a variety, then it admits an open affine cover  $Y = \bigcup_{i=1}^r U_i$ . Now  $\text{pr}_2 : \mathbb{P}^n \times Y \rightarrow Y$  is closed when restricted to  $\text{pr}_2^{-1}(U_i)$ . Since closedness of a map can be checked on an open cover of the target, we see that  $\mathbb{P}^n \times Y \rightarrow Y$  is closed.  $\square$

**Remark.** The same definition and arguments for completeness work if  $Y$  is replaced by a pre-variety.

**Exercise 15.1.** Show that if  $X$  is a complete variety, then so is any closed subvariety of  $X$ .

**Corollary 15.0.1.** *Any projective variety is complete.*

# Lecture 16

## Oct. 16 — Completeness and Embeddings

### 16.1 More on Completeness

**Example 16.0.1.** Recall from before that we have:

1.  $\mathbb{A}^n$  is not complete for  $n \geq 1$ .
2.  $\mathbb{P}^n$  is complete.
3. Any projective variety is complete.

**Remark.** Note the following:

1. If  $k = \mathbb{C}$ , then a variety is complete if and only if  $X^{\text{an}}$  is compact in the analytic topology.
2. *Nagata's compactification theorem:* Any variety  $X$  admits an open embedding  $X \hookrightarrow \overline{X}$  with  $\overline{X}$  complete.
3. In dimension 1, completeness is equivalent to being projective. In dimension  $\geq 2$ , being projective implies completeness, but the converse may fail.

**Proposition 16.1.** *If  $f : X \rightarrow Y$  is a morphism of varieties with  $X$  complete, then*

1.  $f(X)$  is closed in  $Y$ .
2.  $f(X)$  is complete.

*Proof.* (1) Consider the projection  $\text{pr}_2 : X \times Y \rightarrow Y$ . As  $Y$  is separated, the graph  $\Gamma_f$  of  $f$  is closed in  $X \times Y$ . Now  $f(X) = \text{pr}_2(\Gamma_f)$ , which is closed in  $Y$  as  $X$  is complete.

(2) Fix any variety  $Z$ . Now consider the projection  $\pi' : f(X) \times Z \rightarrow Z$  and  $W \subseteq f(X) \times Z$  closed. Now consider the projection  $\pi : X \times Z \rightarrow Z$ . Then  $\pi'(W) = \pi((f, \text{id})^{-1}(W))$ . The set  $(f, \text{id})^{-1}(W)$  is closed in  $X \times Z$  by continuity, and  $\pi'(W)$  is closed in  $Z$  as  $X$  is complete. So  $f(X)$  is complete.  $\square$

**Corollary 16.0.1.** *If  $X$  is a complete variety that is connected, then any  $\varphi \in \mathcal{O}_X(X)$  is constant. In particular,  $\mathcal{O}_X(X) \cong k$ .*

*Proof.* Any  $\varphi \in \mathcal{O}_X(X)$  induces a morphism  $f : X \rightarrow \mathbb{P}_{s:t}^1$  by  $x \mapsto [1 : \varphi(x)]$ . By construction, we have  $f(X) \subseteq \{s \neq 0\} \cong \mathbb{A}^1$ . As  $X$  is complete,  $f(X)$  is closed, and as  $X$  is connected,  $f(X)$  is connected. Since the only proper closed subsets of  $\mathbb{P}^1$  are points,  $f(X)$  must be a single point  $[1 : a]$  since  $f(X)$  is connected. So  $\varphi(x) = a$  for every  $x \in X$ .  $\square$

**Corollary 16.0.2.** *The only complete affine varieties are finite point sets.*

*Proof.* Assume  $X = V(I) \subseteq \mathbb{A}^n$  is complete. Using the decomposition of  $X$  into connected components (there finitely many since  $X$  is Noetherian), we may reduce to the case that  $X$  is connected. Then  $A(X) = \mathcal{O}_X(X) \cong k$ , but this happens if and only if  $X$  is a single point (see Homework 1).  $\square$

## 16.2 The Veronese Embedding

**Remark.** If  $X \subseteq \mathbb{P}^n$  is a projective variety, which open sets in  $X$  are affine?

- $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$  is affine.
- $X \setminus (\mathbb{P}^n \setminus H)$  with  $H \subseteq \mathbb{P}^n$  a hyperplane is affine.

We will see that  $X \cap (\mathbb{P}^n \setminus V(g))$  is affine with  $g \in k[x_0, \dots, x_n]$  homogeneous of degree  $d > 0$ .

**Definition 16.1** (Veronese embedding,  $d$ -tuple embedding). Fix  $n, d > 0$ . Let  $f_0, \dots, f_N \in k[x_0, \dots, x_n]$  denote the monomials of degree  $d$ , where  $N = \binom{n+d}{d} - 1$ . The *Veronese embedding*  $\nu_{n,d}$  is the map

$$\begin{aligned} \nu_{n,d} : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ x &\longmapsto [f_0(x) : \dots : f_N(x)]. \end{aligned}$$

**Example 16.1.1.** Let  $n = 1, d = 3$ . Then the degree-3 Veronese embedding is given by

$$\begin{aligned} f : \mathbb{P}^1 &\longmapsto \mathbb{P}_{s:t:u:v}^3 \\ [x : y] &\longmapsto [x^3 : x^2y : xy^2 : y^3]. \end{aligned}$$

Then  $X = f(\mathbb{P}^1) = V(sv - tu, t^2 - su, u^2 - vt)$ . We can define an inverse

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1 \\ [s : t : u : v] &\longmapsto \begin{cases} [1 : t/s] & \text{if } s \neq 0, \\ [u/v : 1] & \text{if } v \neq 0. \end{cases} \end{aligned}$$

Note that  $ut = sv$  on  $X$  when  $sv \neq 0$ , so this is well-defined.

**Proposition 16.2.**  $\nu_{n,d} : \mathbb{P}^n \rightarrow \mathbb{P}^N$  is a closed embedding.

*Proof.* As  $\mathbb{P}^n$  is complete,  $X = \nu_{n,d}(\mathbb{P}^n)$  is closed in  $\mathbb{P}^N$ . A similar computation as in Example 16.1.1 shows that  $\nu_{n,d} : \mathbb{P}^n \rightarrow X$  has an inverse. So  $\nu_{n,d}$  is a closed embedding.  $\square$

**Remark.** Note the following:

1. With some work, one can show that  $\nu_{n,d}(\mathbb{P}^n)$  can be cut out by quadratic equations

$$\{z_i z_j - z_k z_\ell : f_i f_j = f_k f_\ell \text{ as monomials}\}.$$

2. If  $g \in k[x_0, \dots, x_n]$  is homogeneous of degree  $d > 0$ , then

$$\mathbb{P}^n \supseteq V(g) = \nu_{n,d}^{-1}(H)$$

for some hyperplane  $H \subseteq \mathbb{P}^N$ .

**Theorem 16.1.** *If  $X \subseteq \mathbb{P}^n$  is a projective variety, then for any  $g \in k[x_0, \dots, x_n]$  homogeneous of degree  $d$ , the variety  $X \setminus V(g)$  is affine.*

*Proof.* Consider the Veronese embedding  $\nu_{n,d} : \mathbb{P}^n \rightarrow \mathbb{P}^N$ . Then  $V(g) = \nu_{n,d}^{-1}(H)$  for some hyperplane  $H \subseteq \mathbb{P}^N$ . So  $X \setminus V(g) \cong \nu_{n,d}(X) \setminus H$ , which is affine.  $\square$

## 16.3 The Grassmannian

**Definition 16.2.** For  $0 \leq d \leq n$ , define the *Grassmannian*

$$G(d, n) := \{d\text{-dimensional subspaces of } k^n\}.$$

**Example 16.2.1.** We have the following:

- $G(0, n)$  and  $G(n, n)$  are just points.
- $G(1, n) = \mathbb{P}^{n-1}$  set-theoretically.
- $G(d, n) \cong G(n-d, n)$  as dimension  $d$  subspaces of  $V = k^n$  are in bijection with dimension  $n-d$  subspaces of  $V^*$ , where  $W \subseteq V$  corresponds to  $\ker(V^* \rightarrow W^*)$ .
- $G(2, n) = \{\text{lines in } \mathbb{P}^{n-1}\}$ .

**Theorem 16.2.** *The Grassmannian  $G(d, n)$  can be endowed with the structure of a (projective) variety of dimension  $d(n-d)$ .*

*Proof.* One strategy is to let  $V = k^n = \text{span}\{e_1, \dots, e_n\}$ . We sketch this idea. Observe that

1. A  $d$ -dimensional subspace  $W \subseteq V$  can be represented by a  $d \times n$  matrix  $A$  of rank  $d$  (choose a basis of  $W$  and write the coordinates with respect to the basis  $\{e_1, \dots, e_n\}$  of  $V$ ). Note that  $A$  is unique up to the action by  $\text{GL}_d(k)$ , so we get a point  $[A] \in G(d, n)$ .
2. For  $I = \{1, \dots, n\}$  with  $|I| = d$ , define the set

$$U_I = \{[A] \in G(d, n) : \det(A_I) \neq 0\},$$

where  $A_I$  denotes the  $I$ -th  $d \times d$  minor of  $A$ . One can check that the condition  $\det(A_I) \neq 0$  is well-defined (i.e.  $\det((BA)_I) = \det(B) \det(A_I)$  for  $B \in \text{GL}_d(k)$ ).

Then we have a bijection  $\mathbb{A}^{d(n-d)} \rightarrow U_I$  given as follows. When  $I = \{1, \dots, d\}$ , define

$$\begin{aligned} \mathbb{A}^{d(n-d)} &\longrightarrow U_I \\ C &\longmapsto [I_d \mid C]. \end{aligned}$$

One can make a similar definition for other  $I$ . Note that  $G(d, n) = \bigcup_I U_I$ .

3. Show that the  $U_I$  glue to give  $G(d, n)$  the structure of a variety.

The second strategy is to use the wedge product  $\Lambda^d V$ . Recall that

- $\Lambda^d V$  has basis given by  $e_I := e_{i_1} \wedge \dots \wedge e_{i_d}$  with  $I = \{i_1 < \dots < i_d\} \subseteq \{1, \dots, n\}$ .
- For  $v_1, \dots, v_d \in V$  with  $v_i = \sum_{j=1}^d a_{i,j} e_j$ , we have  $v_1 \wedge \dots \wedge v_d = \sum_{I \subseteq \{1, \dots, n\}} \det(a_I) e_I$ .

The following are two linear algebra lemmas we will use:

**Lemma 16.1.** *Let  $v_1, \dots, v_d \in V$ . Then  $v_1, \dots, v_d$  are linearly independent if and only if  $v_1 \wedge \dots \wedge v_d \neq 0$ .*

*Proof.* Use the determinant formula.  $\square$

**Lemma 16.2.** *For linearly independent sets  $\{v_1, \dots, v_d\}, \{w_1, \dots, w_d\} \subseteq V$ , then  $v_1 \wedge \dots \wedge v_d$  and  $w_1 \wedge \dots \wedge w_d$  are linearly dependent in  $\Lambda^d V$  if and only if  $\text{span}\{v_1, \dots, v_d\} = \text{span}\{w_1, \dots, w_d\}$ .*

*Proof.* ( $\Leftarrow$ ) It suffices to show that  $kv_1 \wedge \dots \wedge v_d \subseteq \Lambda^d V$  is preserved under change of basis operations. Check this as an exercise.

( $\Rightarrow$ ) Assume that  $v_1 \wedge \dots \wedge v_d = \lambda w_1 \wedge \dots \wedge w_d$  for  $\lambda \neq 0$ . Then

$$w_i \wedge v_1 \wedge \dots \wedge v_d = 0,$$

so  $w_i \in \text{span}\{v_1, \dots, v_d\}$ . So we have  $\text{span}\{w_1, \dots, w_d\} \subseteq \text{span}\{v_1, \dots, v_d\}$ , and by symmetry, the reverse inclusion holds as well.  $\square$

So given a  $d$ -dimensional vector subspace  $\text{span}\{v_1, \dots, v_d\} = W \subseteq V$ , we get a 1-dimensional subspace  $kv_1 \wedge \dots \wedge v_d = \Lambda^d W \subseteq \Lambda^d V$ . By Lemma 16.2, there is an injection

$$\begin{aligned} p_{n,d} : G(d, n) &\hookrightarrow \mathbb{P}^{\binom{n}{d}-1} \\ W = \text{span}\{v_1, \dots, v_d\} &\longmapsto kv_1 \wedge \dots \wedge v_d \end{aligned}$$

This is called the *Plücker embedding*. It remains to show  $p_{n,d}$  is closed, which is Corollary 17.0.1.  $\square$

# Lecture 17

## Oct. 21 — The Grassmannian

### 17.1 Plücker Coordinates

**Remark.** Consider coordinates on  $\mathbb{P}^{\binom{n}{d}-1}$  with respect to

$$\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} : I = \{i_1 < \cdots < i_d\} \subseteq \{1, \dots, n\}\}.$$

So when  $d = 2$ ,  $n = 3$ , and  $W = \text{span}\{e_1 + e_2, e_1 + e_3\}$ , then

$$p_{n,d}(W) = k(e_1 \wedge e_3 - e_1 \wedge e_2 + e_2 \wedge e_3) = [1 : -1 : 1] \in \mathbb{P}^2.$$

Note that  $p_{n,d}(W)$  encodes the determinants of the  $2 \times 2$  minors of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**Remark.** We will now try to describe the image of  $p_{n,d} : G(d, n) \rightarrow \mathbb{P}^{\binom{n}{d}-1}$ . Note that  $0 \neq \omega \in \Lambda^d V$  is in the image if and only if  $\omega$  is *pure*, i.e.  $\omega = v_1 \wedge \cdots \wedge v_d$  for some  $v_1, \dots, v_d \in V$ .

**Proposition 17.1.** *For  $0 \neq \omega \in \Lambda^d V$ , the map*

$$\begin{aligned} g : V &\longrightarrow \Lambda^{d+1} V \\ v &\longmapsto \omega \wedge v \end{aligned}$$

*has rank  $\geq n - d$ . Furthermore,  $\omega$  is pure if and only if the rank is exactly  $n - d$ .*

*Proof.* After a change of basis, we may assume that  $\ker g = \text{span}\{e_1, \dots, e_r\}$  with  $1 \leq r \leq n$ . Write

$$\omega = \sum_{I \subseteq \{1, \dots, n\}} a_I e_I.$$

Note that  $e_i \wedge \omega = 0$  for  $1 \leq i \leq r$ , so  $a_I = 0$  if there exists  $1 \leq i \leq r$  with  $i \notin I$ . So

$$\omega = \sum_{\substack{\{1, \dots, r\} \subseteq I \subseteq \{1, \dots, n\} \\ |I|=d}} a_I e_I.$$

As  $\omega \neq 0$ , we have  $r \leq d$ . So  $\text{rk } g = n - r \geq n - d$ . Furthermore, if  $r = d$ , then  $\omega = a_{\{1, \dots, d\}} e_1 \wedge \cdots \wedge e_d$ . Conversely, if  $\omega$  is pure, a similar computation shows that  $\text{rk } g = n - d$ .  $\square$

**Example 17.0.1.** Let  $n = 3$ ,  $d = 2$ ,  $\omega \in \Lambda^2 V$ . Consider a map

$$\begin{aligned} g : V &\longrightarrow \Lambda^3 V \\ v &\longmapsto \omega \wedge v. \end{aligned}$$

Then  $\text{rk } g \leq 1$ . By Proposition 17.1,  $\text{rk } g = 1$  and  $\omega$  is pure.

**Example 17.0.2.** Let  $n = 4$ ,  $d = 2$ , and  $\omega = e_1 \wedge e_2$ . Then

$$g(a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4) = a_3 e_1 \wedge e_2 \wedge e_3 + a_4 e_1 \wedge e_2 \wedge e_4.$$

So we have  $\text{rk } g = 2 = 4 - 2$ .

**Corollary 17.0.1.** *The image of  $p_{n,d} : G(d, n) \rightarrow \mathbb{P}^{\binom{n}{d}-1}$  is a closed set. In particular, this endows  $G(d, n)$  with the structure of an algebraic variety.*

*Proof.* Let  $[\omega] \in \mathbb{P}^{\binom{n}{d}-1}$  where  $0 \neq \omega = \sum_{|I|=d} b_I e_I$ . Then  $[\omega]$  is in  $\text{Im } p_{n,d}$  if and only if  $\omega \in \Lambda^d V$  is pure, which happens if and only if the map

$$\begin{aligned} g_\omega : V &\longrightarrow \Lambda^{d+1} V \\ \sum a_i e_i &\longmapsto \sum_{|I|=d} b_I e_I \wedge \sum a_i e_i \end{aligned}$$

has rank  $n - d$ . Note that  $g_\omega$  is given by some matrix in the  $b_I$ . This matrix has rank  $n - d$  if and only if the  $n - d + 1$  minors of this matrix vanish (we already know the rank is at least  $n - d$  by Proposition 17.1). These give equations cutting out  $\text{Im } p_{n,d}$  in  $\mathbb{P}^{\binom{n}{d}-1}$ , so  $\text{Im } p_{n,d}$  is closed in  $\mathbb{P}^{\binom{n}{d}-1}$ .  $\square$

**Example 17.0.3.** Let  $n = 3$ ,  $d = 2$ . As all  $\omega \in \Lambda^2 V$  are pure,  $G(2, 3) \rightarrow \mathbb{P}^{\binom{3}{2}-1}$  is an isomorphism.

## 17.2 Charts of the Grassmannian

**Remark.** For  $I \subseteq \{1, \dots, n\}$  with  $|I| = d$ , let

$$U_I \subseteq G(d, n) \subseteq \mathbb{P}^{\binom{n}{d}-1}$$

denote the open set on which the  $e_I$  coordinate does not vanish.

**Proposition 17.2.**  $U_I \cong \mathbb{A}^{d(n-d)}$ .

*Proof.* After a change of coordinates, let  $I = \{n - d + 1, n - d + 2, \dots, n\}$ . Then define a map

$$\begin{aligned} h_I : \mathbb{A}^{d(n-d)} &\longrightarrow U_I \subseteq \mathbb{P}^{\binom{n}{d}-1} \\ A = (a_{i,j}) &\longmapsto p_{n,d}([A \mid I_d]), \end{aligned}$$

where  $A$  is a  $d \times (n - d)$  matrix and  $I_d$  is the  $d \times d$  identity matrix. This map is a morphism, and  $h_I$  has inverse given by

$$h_I^{-1} \left( \left[ \sum_{|J|=d} b_J e_J \right] \right) = (\pm b_{(I \setminus \{i\}) \cup \{j\}} / b_I)_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n-d}}.$$

We can see that this is also a morphism, so  $h_I$  is an isomorphism  $\mathbb{A}^{d(n-d)} \rightarrow U_I$ .  $\square$



**Remark.** Note that  $G(d, n) = \bigcup_{|I|=d} U_I$ . This gives us the following:

1.  $\dim G(d, n) = d(n - d)$ .
2.  $G(d, n)$  is irreducible, as  $U_I$  is irreducible and  $U_I \cap U_J \neq \emptyset$  (for the latter, it suffices to find some  $d \times n$  matrix  $A$  whose  $I$  and  $J$  minors are both nonzero).

**Example 17.0.4.** We have  $G(2, 4) \hookrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ . Note that  $\dim G(2, 4) = 4$  and  $\dim \mathbb{P}^5 = 5$ . One can additionally show that  $G(2, 4)$  is cut out by a single equation via the *Plücker relations*.

## 17.3 Birational Maps

**Remark.** Many non-isomorphic varieties are isomorphic on an open set. Consider:

1.  $\mathbb{P}^{n+m}$  and  $\mathbb{P}^n \times \mathbb{P}^m$  both contain  $\mathbb{A}^{n+m}$  as a dense open set.
2.  $G(d, n)$  and  $\mathbb{P}^{d(n-d)}$  both contain  $\mathbb{A}^{d(n-d)}$  as a dense open set.
3. Consider the curve defined by

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow C = V(y^2 - x^3) \subseteq \mathbb{A}_{x,y}^2 \\ t &\longmapsto (t^2, t^3). \end{aligned}$$

Note that  $f$  is injective but not an isomorphism, as

$$f^{-1}(x, y) = \begin{cases} y/x & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is not regular at the origin. But this does tell us that  $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0, 0)\}$ .

So we want to study maps defined on an open set.

**Definition 17.1.** Let  $X$  and  $Y$  be irreducible varieties. Then a *rational map*  $f : X \dashrightarrow Y$  is a morphism  $f : U \rightarrow Y$  with  $U \subseteq X$  a nonempty open set, up to the equivalence  $f_1 : U_1 \rightarrow Y \sim f_2 : U_2 \rightarrow Y$  if  $f_1, f_2$  agree on a nonempty open set  $W \subseteq U_1 \cap U_2$ .

**Remark.** Note that if  $f_1 : U_1 \rightarrow Y \sim f_2 : U_2 \rightarrow Y$ , then they agree on  $U_1 \cap U_2$  by the identity principle. So  $f_1, \dots, f_2$  are equivalent to a morphism  $U_1 \cap U_2 \rightarrow Y$ . So there exists a maximal nonempty open  $U \subseteq X$  on which  $f$  is a morphism.

**Remark.** If  $f, g \in \mathcal{O}_X(X)$ , then we will see that the map  $X \dashrightarrow \mathbb{A}^1$  given by  $x \mapsto f(x)/g(x)$  is rational.

# Lecture 18

## Oct. 23 — Birational Maps

### 18.1 Birational Maps, Continued

**Example 18.0.1.** The following are examples of birational maps:

1. The *cuspidal curve*  $C = V(x^2 - y^3) \dashrightarrow \mathbb{A}^1$  given by  $(x, y) \mapsto x/y$  is a morphism on  $C \setminus \{(0, 0)\}$ .
2. The projection  $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ ,  $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}]$  is a morphism on  $\mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$ .
3. Consider a map

$$\begin{aligned} \mathbb{A}^n &\dashrightarrow \mathbb{A}^n \\ x &\mapsto \left( \frac{f_1(x)}{g_1(x)}, \dots, \frac{f_n(x)}{g_n(x)} \right) \end{aligned}$$

for some nonzero  $f_i, g_j \in k[x_1, \dots, x_n]$ . This is a morphism on  $\mathbb{A}^n \setminus V(g_1 \cdots g_n)$ .

**Definition 18.1.** We define the following:

1. A map  $f : X \dashrightarrow Y$  is *dominant* if the image of  $f$  contains a nonempty open set.
2. If  $f : X \dashrightarrow Y$  and  $g : Y \dashrightarrow Z$  are dominant rational maps, i.e.

$$\begin{array}{ccccc} X & \dashrightarrow^f & Y & \dashrightarrow^g & Z \\ \uparrow & \nearrow f' & \uparrow & \nearrow g' & \\ U & & V & & \end{array}$$

then we can define the composition  $g \circ f : X \dashrightarrow Z$  by

$$(f')^{-1}(V) \xrightarrow{f'} V \xrightarrow{g'} Z.$$

**Definition 18.2.** A rational map  $f : X \dashrightarrow Y$  is *birational* if it is dominant and there exists a dominant rational map  $g : Y \dashrightarrow X$  such that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$  (as rational maps). In this case, we say that  $X$  and  $Y$  are *birational*.

**Example 18.2.1.** We have the following:

1. Let  $X$  be an irreducible variety and  $U \subseteq X$  a nonempty open set. Then the inclusion morphism  $f : U \hookrightarrow X$  is birational with  $g : X \dashrightarrow U$  the identity map on  $U$ .

2. Define a morphism

$$\begin{aligned}\mathbb{A}^n &\xrightarrow{f} \mathbb{A}^n \\ x &\longmapsto (x_1, x_1x_2, \dots, x_1x_n).\end{aligned}$$

Then we can define an inverse rational map

$$\begin{aligned}\mathbb{A}^n &\xrightarrow{g} \mathbb{A}^n \\ y &\longmapsto (x_1, x_2/x_1, \dots, x_n/x_1).\end{aligned}$$

So  $f$  is a birational map.

**Proposition 18.1.** *Two irreducible varieties are birational if and only if they contain isomorphic nonempty open sets.*

*Proof.* ( $\Leftarrow$ ) Assume we have open sets  $U \subseteq X$ ,  $V \subseteq Y$  with an isomorphism  $f : U \xrightarrow{\cong} V$ . Then we get rational maps  $f : X \dashrightarrow Y$  and  $f^{-1} : Y \dashrightarrow X$  which compose to the identity.

( $\Rightarrow$ ) Assume we have

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & \nearrow f' & \\ U & & \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow & \nearrow g' & \\ V & & \end{array}$$

satisfying  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ . Then the composition

$$(g')^{-1}(U) \xrightarrow{g'} U \xrightarrow{f'} Y$$

is the inclusion map by the identity principle, so  $g'((g')^{-1}(U)) = (f')^{-1}(g^{-1}(U)) \subseteq (f')^{-1}(V)$ . Similarly, we can also get the inclusion  $f'((f')^{-1}(V)) \subseteq (g')^{-1}(U)$ . So we get morphisms

$$\begin{array}{ccc} & \xrightarrow{f'} & \\ (f')^{-1}(U) & & (g')^{-1}(V) \\ & \xleftarrow{g'} & \end{array}$$

that compose to the identity on some open sets, so they compose to  $\text{id}$  by the identity principle.  $\square$

**Example 18.2.2.** Consider the following:

1.  $\mathbb{P}^{n+m}$  and  $\mathbb{P}^n \times \mathbb{P}^m$  both contain  $\mathbb{A}^{n+m}$ , so they are birational.
2.  $G(d, n)$  and  $\mathbb{P}^{d(n-d)}$  both contain  $\mathbb{A}^{d(n-d)}$ , so they are birational.

## 18.2 Rational Functions

**Definition 18.3.** A *rational function* on a variety  $X$  is a rational map  $X \dashrightarrow \mathbb{A}^1$ . This is equivalent to the data  $(U, \varphi)$  for  $U \subseteq X$  a nonempty open set and  $\varphi \in \mathcal{O}_X(U)$ , up to the equivalence  $(U, \varphi) \sim (U', \varphi')$  if and only if  $\varphi|_W = \varphi'|_W$  for some nonempty open  $W \subseteq U \cap U'$ .

**Example 18.3.1.** If  $f, g \in k[x_1, \dots, x_n]$  with  $g \neq 0$ , then  $f/g$  gives a rational function on  $\mathbb{A}^n$ . The two ways of thinking about this are:

1. The rational map  $\varphi : \mathbb{A}^n \dashrightarrow \mathbb{A}^1$  given by  $x \mapsto f(x)/g(x)$ .
2. The equivalence class of  $(D(g), f(x)/g(x))$ .

**Definition 18.4.** Let  $X$  be an irreducible variety  $X$ . Then the *function field* of  $X$  is

$$K(X) = \{\text{rational functions on } X\}$$

**Remark.** Note that  $K(X)$  is in fact a field:

- Multiplication is defined by  $(U, \varphi) \cdot (V, \psi) = (U \cap V, \varphi|_{U \cap V} \psi|_{U \cap V})$  and similarly for addition.
- One can check that the above operations turn  $K(X)$  into a ring.
- For  $0 \neq (U, \varphi) \in K(X)$ , we have  $(U, \varphi) \cdot (U \setminus \{\varphi = 0\}, 1/\varphi) = (X, 1)$ .

**Remark.** We have the following properties of  $K(X)$ :

1. For a nonempty open set  $U \subseteq X$ , the restriction map  $K(X) \rightarrow K(U)$  is an isomorphism.
2. For an irreducible affine variety  $X$ , the map

$$\begin{aligned} \text{Frac}(A(X)) &\xrightarrow{\alpha} K(X) \\ \varphi/\psi &\longmapsto (D(\psi), \varphi/\psi) \end{aligned}$$

is an isomorphism. One can check that it is a well-defined homomorphism, and it is injective since it is a homomorphism out of a field. To see surjectivity, let  $(U, \varphi) \in K(X)$ . By shrinking  $U$ , we may assume that  $U = D(g)$  for some  $0 \neq g \in A(X)$ . Since

$$\mathcal{O}_X(D(g)) = A(X)_g,$$

we can write  $\varphi = f/g^n$  for some  $n \geq 0$  and  $f \in A(X)$ . So  $\alpha(f/g^n) = (U, \varphi)$ .

**Example 18.4.1.** We have  $K(\mathbb{A}^n) = \text{Frac}(A(\mathbb{A}^n)) = \text{Frac}(k[x_1, \dots, x_n]) = k(x_1, \dots, x_n)$ .

**Remark.** If  $f : X \dashrightarrow Y$  is a dominant morphism of irreducible varieties, then we get a  $k$ -algebra homomorphism on their function fields by the pullback:

$$\begin{aligned} f^* : K(Y) &\longrightarrow K(X) \\ (V, \varphi) &\longmapsto (f^{-1}(V), (f')^*(\varphi)). \end{aligned}$$

**Exercise 18.1.** Let  $X$  and  $Y$  be irreducible varieties. Then

1. If  $f : X \dashrightarrow Y$  is birational, then  $f^* : K(Y) \rightarrow K(X)$  is an isomorphism.
2. If  $K(X) \cong K(Y)$  as  $k$ -algebras, then  $X$  is birational to  $Y$ .

**Example 18.4.2.** Since  $G(d, n) \sim_{\text{bir}} \mathbb{P}^{d(n-d)}$ , we have  $K(G(d, n)) \cong K(\mathbb{P}^{d(n-d)})$ .

**Corollary 18.0.1.** Any irreducible variety  $X$  is birational to a hypersurface in  $\mathbb{P}^n$ .

*Proof.* By Noether normalization, there exists a transcendence basis  $x_1, \dots, x_n$  for  $K(X)$  over  $k$  (this is finite as we may replace  $X$  by an affine open set). Assuming  $k$  is characteristic 0 (or working harder in nonzero characteristic),  $k(x_1, \dots, x_n) \subseteq K(X)$  is separable. By the definition of a transcendence basis, this extension is finite. So by the primitive element theorem, there exists  $y \in K(X)$  such that

$$K(X) = k(x_1, \dots, x_n, y).$$

Let  $f \in k(x_1, \dots, x_n)[t]$  denote the minimal polynomial of  $y$ . Clearing denominators, we may assume  $f \in k[x_1, \dots, x_n][t]$ . By Gauss's lemma,  $f$  is still irreducible in  $k[x_1, \dots, x_n, t]$ . Now set

$$Y = V(f(x_1, \dots, x_n, x_{n+1})) \subseteq \mathbb{A}^{n+1},$$

which is irreducible. Now we have

$$K(Y) = \text{Frac} \frac{(k[x_1, \dots, x_{n+1}])}{(f(x_1, \dots, x_{n+1}))} \cong \frac{k(x_1, \dots, x_n)[y]}{(f)} \cong K(X),$$

so  $X \sim_{\text{bir}} Y$ . We get the desired statement after taking projective closures. □

# Lecture 19

## Oct. 28 — The Blow-Up

### 19.1 Rational Varieties

TODO: fill in notes from beginning of class.

**Theorem 19.1.** *Any irreducible algebraic variety is birational to a hypersurface in  $\mathbb{P}^n$ .*

**Definition 19.1.** An irreducible algebraic variety is *rational* if it is birational to  $\mathbb{P}^n$ .

**Example 19.1.1.** Rational varieties include  $\mathbb{P}^n$ ,  $\mathbb{A}^n$ ,  $G(d, n)$ ,  $\mathbb{P}^m \times \mathbb{P}^n$ , etc.

**Remark.** Which hypersurfaces in  $\mathbb{P}^n$  are rational? This is an open question. It is known that if  $X \subseteq \mathbb{P}^n$  is a smooth hypersurface over a field of characteristic 0 and degree  $d > n + 1$ , then  $X$  is *not* rational.

### 19.2 The Blow-Up

**Example 19.1.2.** Consider the following birational map:

$$\begin{aligned}\mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x_1, x_2) &\longmapsto [x_1 : x_2].\end{aligned}$$

Why does this not extend to a morphism  $\mathbb{A}^2 \rightarrow \mathbb{P}^1$ ? The issue is where  $(0, 0)$  would go. Let us try to make it extend. Consider  $U = \mathbb{A}^2 \setminus \{0\}$  and the map

$$\begin{aligned}f : U &\longrightarrow \mathbb{P}^1 \\ (x_1, x_2) &\longmapsto [x_1 : x_2].\end{aligned}$$

Consider the graph  $\Gamma_f = \{(x, y) \in U \times \mathbb{P}^1 : x_1 y_2 = x_2 y_1\}$ , and the *blow-up*

$$B_0 \mathbb{A}^2 := \overline{\Gamma}_f \subseteq \mathbb{A}^2 \times \mathbb{P}^1,$$

with projection maps  $\pi : B_0 \mathbb{A}^2 \rightarrow \mathbb{A}^2$  and  $B_0 \mathbb{A}^2 \rightarrow \mathbb{P}^1$ . Then

1.  $B_0 \mathbb{A}^2 = \{(x, y) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_1 y_2 - x_2 y_1 = 0\}$ .

Let  $Y$  be the right-hand side. Since  $\Gamma_f \subseteq Y$ , it is immediate that  $B_0 \mathbb{A}^2 = \overline{\Gamma}_f \subseteq \overline{Y} = Y$ . For the reverse inclusion, it suffices to show  $Y$  is irreducible and  $\Gamma_f$  is open in  $Y$ . By the equation for  $Y$ ,

$$Y \setminus V(x_1, x_2) = \Gamma_f,$$

which shows  $\Gamma_f$  is open in  $Y$ . For irreducibility, set  $U_i = \{(x, y) \in Y : y_i \neq 0\}$ . Then  $U_i \cong \mathbb{A}^2$ :

$$\begin{aligned} U_1 &\longrightarrow \mathbb{A}^2 \\ (x, y) &\longmapsto (x_1, x_2/y_1) \\ ((x_1, x_1x_2), (1 : x_2)) &\longleftarrow (x_1, x_2). \end{aligned}$$

So each  $U_i$  is irreducible. As  $U_1 \cap U_2 \neq \emptyset$  and  $Y = U_1 \cap U_2$ ,  $Y$  is irreducible. Thus  $B_0\mathbb{A}^2 \supseteq Y$ .

2.  $\pi^{-1}(\mathbb{A}^2 \setminus \{0\}) \rightarrow \mathbb{A}^2 \setminus \{0\}$  is an isomorphism.

For this, we can give an inverse  $\mathbb{A}^2 \setminus \{0\} \rightarrow \pi^{-1}(\mathbb{A}^2 \setminus \{0\})$  by  $(x_1, x_2) \mapsto ((x_1, x_2), [x_1 : x_2])$ .

3.  $\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^1 \cong \mathbb{P}^1$ .

This follows from (1).

**Remark.** An alternative perspective is to think of  $B_0\mathbb{A}^2$  as

$$B_0\mathbb{A}^2 = \{(p, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 : p \in \ell\}.$$

where we think of  $\mathbb{P}^1$  as lines in  $\mathbb{A}^2$  through 0.

**Example 19.1.3.** In higher dimensions, we define  $B_0\mathbb{A}^n = \bar{\Gamma}_f \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ , where  $f : \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$  is given by  $x \mapsto [x]$ . Let  $\pi : B_0\mathbb{A}^n \rightarrow \mathbb{A}^n$  be the projection. Then the same argument as before shows:

1.  $B_0\mathbb{A}^n = \{(x, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x_i y_i - x_j y_j = 0 \text{ for all } 1 \leq i, j \leq n\}$ .
2.  $\pi^{-1}(\mathbb{A}^n \setminus \{0\}) \rightarrow \mathbb{A}^n \setminus \{0\}$  is an isomorphism.
3.  $\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$ .

**Remark.** Define charts  $U_i = \{(x, y) \in B_0\mathbb{A}^n : y_i \neq 0\}$ . Then we have

$$\begin{array}{ccc} \mathbb{A}^n & & x \\ \cong \downarrow & & \downarrow \\ U_1 & ((x_1, x_1x_2, \dots, x_1x_n), [1 : x_2 : \dots : x_n]) & \\ \downarrow & & \downarrow \\ \mathbb{A}^n & (x_1, x_1x_2, \dots, x_1x_n) & \end{array}$$

Check that  $\mathbb{A}^n \rightarrow U_1$  is in fact an isomorphism.

**Example 19.1.4.** Let  $n = 2$  and  $\pi : B_0\mathbb{A}^2 \rightarrow \mathbb{A}^2$ . What happens to  $C := V(y^2 - x^3) \subseteq \mathbb{A}^2$  under the blow-up? We have  $\pi^{-1}(C) = (\{0\} \times \mathbb{P}^1) \cup \overline{\pi^{-1}(C \setminus \{0\})}$ . We call  $\{0\} \times \mathbb{P}^1$  the *exceptional locus* of  $\pi$ . We can compute this explicitly on charts. Write

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & U_1 \subseteq B_0\mathbb{A}^2 = \{((x, y), [s : t]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt - ys = 0\} \\ & \searrow \pi_1 & \downarrow \pi \\ & & \mathbb{A}^2 \end{array}$$

where  $\mathbb{A}^2 \rightarrow U_1$  is given by  $(x, y) \mapsto ((x, xy), [1 : y])$  and  $\pi_1$  is given by  $(x, y) \mapsto (x, xy)$ . Then

$$\pi_1^{-1}(C) = V((xy)^2 - x^3) = V(x^2(y^2 - x)) = V(x) \cup V(y^2 - x)$$

In particular, we see that the piece  $\overline{\pi^{-1}(C \setminus \{0\})} = V(y^2 - x)$  no longer has a singularity at 0.

**Definition 19.2.** Let  $X \subseteq \mathbb{A}^n$  be an affine variety,  $f_1, \dots, f_r \in A(X)$ , and  $U = X \setminus V(f_1, \dots, f_r)$ . We define the *blow-up* of  $X$  along  $f_1, \dots, f_r$  to be

$$B_{f_1, \dots, f_r} X := \tilde{X} := \text{closure of } \Gamma_f \text{ in } X \times \mathbb{P}^{r-1}$$

where  $f : U \rightarrow \mathbb{P}^{r-1}$  is given by  $x \mapsto [f_1(x) : \dots : f_r(x)]$ ,

**Remark.** We have the following:

0. When  $X = \mathbb{A}^n$  and  $f_1, \dots, f_r = x_1, \dots, x_n$ , then  $B_{x_1, \dots, x_n} \mathbb{A}^n = B_0 \mathbb{A}^n$ .
1.  $\pi$  is an isomorphism over  $U$ .

To see this, use that  $\Gamma_f$  is closed in  $U \times \mathbb{P}^{r-1}$  and the projection  $\text{pr}_1 : \Gamma_f \rightarrow U$  is an isomorphism to get that  $\bar{\Gamma}_f \cap (U \times \mathbb{P}^{r-1}) = \Gamma_f$ . So  $\pi^{-1}(U) = \Gamma_f \cong U$ .

2. If  $X$  is irreducible and  $U \neq \emptyset$ , then  $\pi$  is birational.

For this, use (1) to get  $B_{f_1, \dots, f_r}$  is irreducible and  $\pi$  being an isomorphism over  $U$ .

3. If  $(f_1, \dots, f_r) = (f'_1, \dots, f'_s)$  as ideals in  $A(X)$ , then there is an isomorphism

$$B_{f_1, \dots, f_r} X \xrightarrow{\cong} B_{f'_1, \dots, f'_s} X.$$

Hint: Write  $f'_j = \sum_{i=1}^r g_{i,j} f_i$ . Use the  $g_{i,j}$  to get a morphism  $\Gamma_f \rightarrow \Gamma_{f'}$ ,  $(x, y) \mapsto (x, [\sum_{i=1}^r g_{i,1}(x) y_i : \dots : \sum_{i=1}^r g_{i,s}(x) y_i])$ , then use this to get a morphism  $B_{f_1, \dots, f_r} X \rightarrow B_{f'_1, \dots, f'_s} X$ . We can obtain an inverse similarly by swapping the roles of  $f_i$  and  $f'_j$ . See Gathmann 9.16.

4. (Local) If  $W \subseteq X$  is open and  $W \setminus V(f_1, \dots, f_r)$  is dense in  $W$ , then

$$B_{f_1, \dots, f_r} X|_{\pi^{-1}(W)} \cong B_{f_1|_W, \dots, f_r|_W} W.$$

(Use the properties of the graph.)

**Remark.** We can further generalize the blow-up as follows:

1. (Affine case) Let  $X$  be affine,  $I \leq A(X)$ , and  $Z \subseteq X$  closed. Then we can define  $B_I X := B_{f_1, \dots, f_r} X$  with  $f_1, \dots, f_r$  generators of  $I$  (the isomorphism class is independent of the choice of generators), and  $B_Z X = B_{I(Z)} X$  with  $I(Z) \leq A(X)$  the ideal consisting of regular functions vanishing on  $Z$ .
2. Let  $X$  be a variety and  $Z \subseteq X$  a closed subvariety. Choose an affine open cover  $X = \bigcup_i U_i$ , and define  $B_Z X$  to be the  $B_{Z \cap U_i} U_i$  glued together along  $B_{Z \cap U_i \cap U_j} (U_i \cap U_j)$ . (Use that the blow-up can be computed locally and glue these together to get  $B_Z X$ .)