MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}^n_k = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}^n_k$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}^n_k \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}^n_k$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}^2_{x,y}$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \dots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \dots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. □

Remark. The statement implies that if $f_1, \dots, f_r \in k[x_1, \dots, x_n]$, then $V(f_1, \dots, f_r) = V((f_1, \dots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \dots, x_n]$. By the Hilbert basis theorem that $k[x_1, \dots, x_n]$ is Noetherian, there are finitely many $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ such that $I = (f_1, \dots, f_r)$. So $X = V(I) = V(f_1, \dots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \dots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. □

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \dots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \dots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \ldots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \ldots, x_n]$;
2. $I(\mathbb{A}^n_k) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}^n_{\mathbb{R}})$);
3. for $n = 1$, if $S \subseteq \mathbb{A}^1_k$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}^n_k$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \ldots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \ldots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \ldots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1] Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz). *If $J \leq k[x_1, \ldots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ have no common zeros, then there exist $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. □

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$ but $V(x^2 + 1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \ldots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ with inverse $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \ldots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \ldots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. □

Remark. This implies that maximal ideals in $k[x_1, \ldots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;
2. $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. □

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0, 0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x, y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization)**.** *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \ldots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \ldots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \ldots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \ldots, z_n]$, which implies that $K[z_1, \ldots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. □

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. □

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \ldots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \ldots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \ldots, a_n \in k$, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \ldots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \ldots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \ldots, a_n)\}$, so $J \neq \varnothing$.

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix $f \in I(V(J))$, and we want to show that $f^n \in J$ for some $n > 0$. Add a new variable y and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \ldots, x_n, y].$$

Now $V(J_1) = \{(a, b) = (a_1, \ldots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \varnothing$ since f vanishes on $V(J)$, so $f(a)b = 0$ for any b. Thus by the weak nullstellensatz, $J_1 = (1)$, so $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$ with

[1]The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

$f_1, \dots, f_r \in J$ and $g_0, \dots, g_r \in k[x_1, \dots, x_n, y]$. Let N be the maximal power of y in the g_i. Multiplying by f^N, we get

$$f^N = \sum_{i=1}^{r} G_i(x_1, \dots, x_n, fy) f_i + G_0(x_1, \dots, x_n, fy)(fy - 1)$$

with $G_i \in k[x_1, \dots, x_n, fy]$. So if we set $fy = 1$, then we have

$$f^N = \sum_{i=1}^{r} G_i(x_1, \dots, x_n, 1) f_i + 0 \in J,$$

which gives $f \in \sqrt{J}$. To justify this substitution, we can consider the quotient $k[x_1, \dots, x_n, y]/(fy - 1)$. We have a map $k[x_1, \dots, x_n] \to k[x_1, \dots, x_n, y]/(fy - 1)$, which is injective since $(fy - 1)$ does not lie in $k[x_1, \dots, x_n]$, so an equality in the quotient implies an equality in $k[x_1, \dots, x_n]$. $\square$

Lecture 3

Aug. 26 — The Zariski Topology

3.1 Polynomial Functions and Subvarieties

Remark. Recall that a polynomial $f \in k[x_1, \dots, x_n]$ gives a function $\mathbb{A}_k^n \to k$ by $a \mapsto f(a)$.

Proposition 3.1. *If $f, g \in k[x_1, \dots, x_n]$ give the same function $\mathbb{A}_k^n \to k$, then $f = g$ in $k[x_1, \dots, x_n]$.*

Proof. Assume $f = g$ as polynomial functions. Then $V(f - g) = \mathbb{A}_k^n$, so $\sqrt{(f-g)} = I(\mathbb{A}_k^n) = (0)$ by Hilbert's nullstellensatz (note that we can also prove $I(\mathbb{A}_k^n) = (0)$ directly, it is enough to have k be an infinite field for this part). Thus $f - g = 0$, so $f = g$ in $k[x_1, \dots, x_n]$. $\square$

Remark. In the above proposition, we need k to be an infinite field (e.g. if $k = \overline{k}$): Otherwise, there are only finitely many functions $\mathbb{A}_k^n \to k$, but infinitely many polynomials in $k[x_1, \dots, x_n]$.

Remark. The set of polynomials functions $\mathbb{A}_k^n \to k$ form a ring, and the above proposition implies that this ring is isomorphic to $k[x_1, \dots, x_n]$.

Definition 3.1. A *polynomial function* on an affine variety $X \subseteq \mathbb{A}_k^n$ is a function $\varphi : X \to k$ such that there exists $f \in k[x_1, \dots, x_n]$ with $\varphi(a) = f(a)$ for every $a \in X$.

Definition 3.2. The *coordinate ring* of X is $A(X) = \{f : X \to k \mid f \text{ is a polynomial function}\}$, which is a ring under pointwise addition and multiplication.

Remark. Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel $I(X)$. Thus we have $A(X) \cong k[x_1, \dots, x_n]/I(X)$.

Remark. We can now replace $\mathbb{A}_k^n$ and $k[x_1, \dots, x_n]$ by X and $A(X)$ to study *subvarieties* of X.

Definition 3.3. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety. If $S \subseteq A(X)$ is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of X of this form is called an *affine subvariety* of X. (Equivalently, these are the same as an affine variety $Y \subseteq \mathbb{A}_k^n$ such that $Y \subseteq X$.) For $Y \subseteq X$ a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

Proposition 3.2. *There is a bijective correspondence between radical ideals in $A(X)$ and affine subvarieties of X given by $J \mapsto V_X(J)$ and $Y \mapsto I_X(Y)$.*

Proof. See Homework 2. $\square$

3.2 The Zariski Topology

Definition 3.4. The *Zariski topology* on $\mathbb{A}_k^n$ is the topology with closed sets $V(I) \subseteq \mathbb{A}_k^n$, where I is an ideal in $k[x_1, \ldots, x_n]$. (Equivalently, the closed sets are the affine varieties in $\mathbb{A}_k^n$.)

Remark. Note the following:

1. On $\mathbb{A}_k^1$, the closed sets are of the form: $\varnothing$, $\mathbb{A}_k^1$, or finite collections of points.
2. When $k = \mathbb{C}$, then $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ being Zariski closed implies that X is closed in the analytic topology on $\mathbb{A}_{\mathbb{C}}^n$. In particular, the Zariski topology is coarser than the analytic topology.
3. On $\mathbb{A}_k^2$, the closed sets are of the form: $\varnothing$, $\mathbb{A}_k^2$, finite collections of points, plane curves, and their finite unions.

Proposition 3.3. *The Zariski topology on $\mathbb{A}_k^n$ is indeed a topology.*

Proof. First note that $\varnothing = V((1))$ and $\mathbb{A}_k^n = V((0))$ are closed. For arbitrary intersections, note that $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$, and for finite unions, note that $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$. $\square$

Example 3.4.1. The Zariski topology on $\mathbb{A}_k^{n+m}$ is in general *not* the product topology of the Zariski topologies on $\mathbb{A}_k^n$ and $\mathbb{A}_k^m$. Consider $V(y - x^2) \subseteq \mathbb{A}_k^2$, which is a closed set in the Zariski topology, but the only closed sets in $\mathbb{A}_k^1$ are either $\varnothing$, $\mathbb{A}_k^1$, or finite.

Definition 3.5. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then we can define the *Zariski topology* on X in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on $\mathbb{A}_k^n$;
2. take the closed sets of X to be of the form $V_X(I)$ for some ideal $I \leq A(X)$.

This is because an affine subvariety of X is precisely the intersection of X with an affine variety in $\mathbb{A}_k^n$.

Remark. Our goal now is to relate properties of the Zariski topology on X to the ring $A(X)$, and then to the ideal $I(X) \leq k[x_1, \ldots, x_n]$.

Definition 3.6. A topological space X is *reducible* if we can write $X = X_1 \cup X_2$ for some closed sets $X_1, X_2 \subsetneq X$. Otherwise, X is called *irreducible*.

Example 3.6.1. The plane curve $X = V(y^2 - x^2 y) = V(y) \cup V(y - x^2)$ is reducible.

Remark. Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g. $\mathbb{C}^n$, $\mathbb{R}^n$ with the analytic topology.
3. If X is irreducible and $U \subseteq X$ is a nonempty open set, then $\overline{U} = X$ (we have $\overline{U} \cup (X \setminus U) = X$).

Lecture 4

Aug. 28 — Irreducibility

4.1 Irreducibility

Definition 4.1. A topological space X is *reducible* if there exists a decomposition

$$X = X_1 \cup X_2$$

with $X_1, X_2 \subsetneq X$ closed. Otherwise X is *irreducible.*

Theorem 4.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the following are equivalent:*

1. *X is irreducible;*
2. *$I(X) \leq k[x_1, \ldots, x_n]$ is a prime ideal;*
3. *the coordinate ring $A(X)$ is an integral domain.*

Proof. ($2 \Leftrightarrow 3$) Use that $A(X) \cong k[x_1, \ldots, x_n]/I(X)$.

($1 \Leftrightarrow 2$) We have X is reducible if and only if $X = X_1 \cup X_2$ for some $X_1, X_2 \subsetneq X$ closed, if and only if

$$X = V_X(f) \cup V_X(g) = V_X(fg)$$

for some $f, g \in A(X)$ nonzero. This is equivalent to $fg = 0$ for some $f, g \in A(X)$ nonzero, i.e. $A(X)$ is not an integral domain. □

Example 4.1.1. We have the following:

1. $\mathbb{A}^n_k$ is irreducible as $A(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$, which is an integral domain.
2. A *hypersurface* $X \subseteq \mathbb{A}^n_k$ is an affine variety with $I(X) = (f)$ for some $f \in k[x_1, \ldots, x_n]$. Then A is irreducible if and only if (f) is prime, if and only if f is irreducible.[1]

Remark. Theorem 4.1 implies that there is a bijection

$$\{\text{irreducible subvarieties } Y \subseteq X\} \longleftrightarrow \{\text{prime ideals } \mathfrak{p} \leq A(X)\}.$$

Example 4.1.2. Let $f \in k[x_1, \ldots, x_n]$ be a nonzero nonunit. Write $f = f_1^{a_1} \cdots f_r^{a_r}$ with f_i irreducible and $a_i \in \mathbb{Z}_{>0}$. Then

$$V(f) = V(f_1^{a_1} \cdots f_r^{a_r}) = V(f_1) \cup \cdots \cup V(f_r).$$

Note f_i irreducible implies f_i is prime ($k[x_1, \ldots, x_n]$ is a UFD), so (f_i) is prime and $V(f_i)$ is irreducible.

[1] Note that any prime ideal is radical.

Remark. We want a unique decomposition into irreducibles for arbitrary varieties. For this, we need the notion of Noetherianity.

Definition 4.2. A topological space X is *Noetherian* if there is no infinite chain of closed subsets

$$X \supseteq X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots .$$

Example 4.2.1. The spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ are not Noetherian, e.g. take $X = \mathbb{R}^n$ and $X_n = [0, 1/n]$.

Lemma 4.1. *Any affine variety $X \subseteq \mathbb{A}^n_k$ is Noetherian.*

Proof. An infinite chain of closed subsets

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$$

gives an infinite chain of (radical) ideals

$$(0) = I_X(X) \subsetneq I_X(X_1) \subsetneq I_X(X_2) \subsetneq \cdots$$

of $A(X)$. As $A(X) \cong k[x_1, \ldots, x_n]/I(X)$ is a Noetherian ring, no such chain exists. □

Example 4.2.2. For $X = \mathbb{A}^n_k$, we have a chain

$$V(x_1) \supsetneq V(x_1, x_2) \supsetneq \cdots \supsetneq V(x_1, \ldots, x_n),$$

but such a chain stops.

Theorem 4.2. *If X is a Noetherian topological space, then there exists a decomposition*

$$X = X_1 \cup \cdots \cup X_r$$

with each X_i an irreducible closed subset and $X_i \not\subseteq X_j$ for $i \neq j$. Furthermore, this decomposition is unique up to reordering.

Proof. First we show existence. If X is not irreducible, then write

$$X = X_1 \cup X_1'$$

with $X_1, X_1' \subsetneq X$ closed. If X_1 or X_1' is not irreducible (say X_1), write

$$X_1 = X_2 \cup X_2'$$

with $X_2, X_2' \subsetneq X_1$ closed. If this process fails to terminate, then we get an infinite chain

$$X_1 \supsetneq X_2 \supsetneq \cdots ,$$

contradicting Noetherianity.

For uniqueness, say $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'$ with X_i, X_j' closed and irreducible. Fix $1 \leq i \leq r$. Since $X_i \subseteq \bigcup_{j=1}^s X_j'$, we have

$$X_i = (X_i \cap X_1') \cup \cdots \cup (X_i \cap X_s').$$

As X_i is irreducible, we have $X_i = X_i \cap X_j'$ for some j, so $X_i \subseteq X_j'$. By symmetry, $X_j' \subseteq X_k$ for some k. So $X_i \subseteq X_j' \subseteq X_k$, which implies $X_i = X_j' = X_k$. Hence X_i is an X_j' and vice versa. □

Remark. If X is an affine variety, then there exists a bijection

$$\{\text{irreducible components of } X\} \longleftrightarrow \{\text{minimal prime ideals in } A(X)\}.$$

Remark. The primary decomposition implies that an ideal $I \subseteq k[x_1, \dots, x_n]$ can be written as

$$I = Q_1 \cap \dots \cap Q_n$$

with Q_i primary (so that $P_i := \sqrt{Q_i}$ is prime).[2] Then

$$X := V(I) = V(Q_1 \cap \dots \cap Q_n) = V(Q_1) \cup \dots \cup V(Q_n) = V(P_1) \cup \dots \cup V(P_n),$$

though this decomposition is not necessarily minimal.

Example 4.2.3. Let $I = (x^2, xy) \le k[x, y]$. Then $I = (x) \cap (x^2, xy, y^2)$, and

$$V(I) = V(x) \cup V(x, y) = V(x).$$

4.2 Dimension

Example 4.2.4. The motivating example is the following: Consider

$$\mathbb{A}_k^n \supseteq V(0) \supsetneq V(x_1) \supsetneq V(x_1, x_2) \supsetneq \dots \supsetneq V(x_1, \dots, x_n).$$

This is a length n chain and there are no irreducible subspaces $V(x_1, \dots, x_i) \supsetneq Y \supsetneq V(x_1, \dots, x_{i+1})$.

Definition 4.3. Let X be a nonempty topological space.

- The *dimension* of X, denoted $\dim X$, is the supremum of the n such that there exists a chain of irreducible closed subspaces
$$X \supseteq X_0 \supsetneq X_1 \supsetneq \dots \supsetneq X_n \neq \varnothing.$$
- For $Y \subseteq X$ closed and irreducible, the *codimension* of Y in X, denoted $\operatorname{codim}_X Y$, is the supremum of the n as above such that $X_n = Y$.

Example 4.3.1. We have $\dim \mathbb{A}_k^1 = 1$, as all maximal chains are of the form $\mathbb{A}_k^1 \supsetneq \{p\}$.

Example 4.3.2. We clearly have $\dim \mathbb{A}_k^n \ge n$.

Remark (Properties of dimension)**.** We have the following:

1. $\dim_X = \sup\{\operatorname{codim}_X \{a\} : a \in X\}$.
2. If X is a Noetherian topological space with irreducible decomposition
$$X = X_1 \cup \dots \cup X_r,$$
then $\dim X = \max\{\dim X_1, \dots, \dim X_r\}$.

   Check $\ge$ as an exercise. To see $\le$, choose a chain of irreducible subspaces
$$X \supseteq Y_n \supsetneq \dots \supsetneq Y_0.$$
Then $Y_n = (Y_n \cap X_1) \cup \dots \cup (Y_n \cap X_r)$, so $Y_n \subseteq X_i$ for some i (as Y_n is irreducible).
$$X_i \supsetneq Y_n \supsetneq \dots \supsetneq Y_0,$$
hence $\dim X \le \max \dim X_i$.

[2]Recall that $\mathfrak{p}$ is *primary* if $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g^n \in \mathfrak{p}$ for some $n > 0$.

Lecture 5

Sept. 2 — Dimension

5.1 More on Dimension

Remark. Recall the following correspondence from before: If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then there exists a bijection between the irreducible closed subsets $Y \subseteq X$ and the prime ideals $\mathfrak{p} \leq A(X)$.

Definition 5.1. For a ring A, the *(Krull) dimension* of A, denoted $\dim A$, is the supremum of the n such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

For a prime ideal $\mathfrak{q} \leq A$, the *height* of $\mathfrak{q}$, denoted $\operatorname{ht} \mathfrak{q}$, is the supremum of the n as above with $\mathfrak{p}_0 = \mathfrak{q}$.

Remark. If X is an affine variety, then we have the following:

1. $\dim X = \dim A(X)$;
2. for $Y \subseteq X$ a closed irreducible subset, $\operatorname{codim}_X Y = \operatorname{ht} I_X(Y)$.

These properties follow from the inclusion-reversing correspondence.

Definition 5.2. Let $K \subseteq L$ be a field extension.

1. A collection of elements $\{z_i : i \in I\} \subseteq L$ is a *transcendence basis* of $K \subseteq L$ if the z_i are algebraically independent (i.e. $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$ by $x_i \mapsto z_i$) and $K(z_i : i \in I) \subseteq L$ is algebraic.
2. The *transcendence degree* $\operatorname{tr.deg}_K L$ is the cardinality of a transcendence basis.

Theorem 5.1 (Dimension theory)**.** *Let A be a finitely generated k-algebra that is a domain. Then*

1. $\dim A = \operatorname{tr.deg}_k \operatorname{Frac}(A)$*;*
2. *for any prime ideal* $\mathfrak{p} \leq A$*, we have* $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$*;*
3. *all maximal chains of prime ideals* $A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ *are of the same length.*

Remark. The following are consequences of the above result from commutative algebra:

1. $\dim_k \mathbb{A}_k^n = \dim k[x_1, \ldots, x_n] = \operatorname{tr.deg}_k k(x_1, \ldots, x_n) = n$.
2. If X is irreducible, then $A(X)$ is a domain, so for $x \in X$, we have

   $$\operatorname{codim}_X \{x\} = \operatorname{ht} I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

   where we note that $A(X)/I(\{x\}) \cong k$ is a field.

3. If X is an irreducible affine variety and $U \subseteq X$ is a nonempty open subset, then
$$\dim U = \sup_{x\in U} \operatorname{codim}_U\{x\} = \sup_{x\in U} \operatorname{codim}_X\{x\} = \dim X.$$
This follows since we can pass from a chain in U to a chain in X by taking closures.

4. If X is an irreducible affine variety and $Z \subseteq X$ is an irreducible closed subset, then
$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if X is not irreducible. To contradict (4), let $X = V(x,y) \cup V(z) \subseteq \mathbb{A}^3_k$ with $Z = V(x,y)$. Then we have $\dim X = 2$, $\dim Z = 1$, $\operatorname{codim}_X Z = 0$.

5.2 Hypersurfaces

Remark. We now want to study hypersurfaces.

Theorem 5.2 (Krull's Hauptidealsatz). *If A is a Noetherian ring and $f \in A$ is nonzero and a non-unit, then every minimal prime ideal containing f has height* 1*.*

Corollary 5.2.1. *If $X \subseteq \mathbb{A}^n_k$ is an irreducible affine variety and $f \in A(X)$ is a nonzero non-unit, then*
$$\dim Z = \dim X - 1$$
for every irreducible component Z of $V_X(f)$.

Proof. Since X is irreducible, $A(X)$ is a domain. So there is a correspondence between the minimal prime ideals $f \in \mathfrak{p} \subsetneq A(X)$ and the minimal irreducible closed subsets $Z \supseteq V_X(f)$, which corresponds to the irreducible components Z of $V_X(f)$. For such a component Z, we know
$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$
by Krull's Hauptidealsatz, which is the desired result. □

Example 5.2.1. Corollary 5.2.1 implies that if $f \in k[x_1, \dots, x_n]$ is non-constant, then
$$\dim V(f) = \dim \mathbb{A}^n_k - 1 = n - 1.$$

Theorem 5.3. *An irreducible affine variety $Y \subseteq \mathbb{A}^n_k$ has $\dim Y = n-1$ if and only if $Y = V(f)$ for some non-constant polynomial $f \in k[x_1, \dots, x_n]$.*

Proof. ($\Leftarrow$) This was Corollary 5.2.1.

($\Rightarrow$) We will use that $A(\mathbb{A}^n_k) = k[x_1, \dots, x_n]$ is a UFD. Since Y is irreducible and $\dim Y = n-1$,
$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}^n_k} Y = \dim \mathbb{A}^n_k - \dim Y = 1.$$
Since $(0) \subsetneq I(Y) \subsetneq k[x_1, \dots, x_n]$, there exists a non-constant $f \in k[x_1, \dots, x_n]$ with $f \in I(Y)$. Write
$$f = f_1 \cdots f_r$$
with f_i irreducible by unique factorization, and note that the f_i are also prime since we are in a UFD. Since $I(Y)$ is prime, some f_i is in $I(Y)$, so we have the inclusions
$$(0) \subsetneq (f_i) \subseteq I(Y).$$
Since $\operatorname{ht} I(Y) = 1$, we must have $(f_i) = I(Y)$, so $Y = V(I(Y)) = V(f_i)$. □

5.3 Regular Functions

Definition 5.3. Let X be an affine variety and $U \subseteq X$ open. A function $\varphi : U \to k$ is *regular* if for each $a \in U$, there exists an open neighborhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define $\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is a regular function on } U\}$.

Exercise 5.1. Check that $\mathcal{O}_X(U)$ is a ring under pointwise addition and multiplication of outputs.

Remark. To patch open sets together, we will later need the notion of a *morphism*, and a morphism $U \to Y \subseteq \mathbb{A}_k^m$ should be given by

$$x \longmapsto (\varphi_1(x), \ldots, \varphi_m(x))$$

with φ_i regular functions on U.

Example 5.3.1. We have the following:

1. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then any $\varphi \in A(X)$ is regular. Furthermore, we get an injective ring homomorphism $A(X) \to \mathcal{O}_X(X)$. We will see that this is an isomorphism.
2. If $X = \mathbb{A}_k^1$ and $U = \mathbb{A}_k^1 \setminus \{0\}$, then for any $n \geq 0$ and $g \in k[x]$, the function g/x^n is regular on U. In general, if we fix $f, g \in A(X)$ and set $U = X \setminus V(f)$, then the map g/f^m is regular on U.
3. Let $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$ and $U = X \setminus V(x_2, x_4)$. Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

   Note that on $U \setminus V(x_2x_4)$, we have $x_1/x_2 = x_3/x_4$ since $x_1x_4 = x_2x_3$ on X.

Lecture 6

Sept. 4 — Regular Functions

6.1 Properties of Regular Functions

Proposition 6.1. *Let X be an affine variety and $U \subseteq X$ open. Then:*

1. *if $\varphi \in \mathcal{O}_X(U)$, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U;*
2. *(identity principle) If X is irreducible, $U \subseteq X$ is nonempty and open, and $\varphi, \psi \in \mathcal{O}_X(U)$ with $\varphi|_W = \psi|_W$ for some $W \subseteq U$ nonempty and open, then $\varphi = \psi$ in $\mathcal{O}_X(U)$.*

Proof. (1) It suffices to show that $U \setminus V(\varphi)$ is open in U. Fix $a \in U \setminus V(\varphi)$. Since φ is regular, there exists an open neighborhood $a \in U_a \subseteq U$ and $f_a, g_a \in A(X)$ such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$. This is an open set containing a in $U \setminus V(\varphi)$, so $U \setminus V(\varphi)$ is open.

(2) Since X is irreducible, U is also irreducible. The locus $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$ is closed in U by (1). It also contains W. Since W is dense (it is a nonempty open set in an irreducible topological space), we must have $V(\varphi - \psi) = U$. This proves the claim. □

Example 6.0.1. In (2) of Proposition 6.1, the assumption that X is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}^2_k \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions $\varphi = x$ and $\psi = x + y$ agree on W but are not equal on U.

6.2 Distinguished Open Sets

Remark. We will see that an affine variety has a basis of open sets on which we can compute $\mathcal{O}_X(U)$.

Definition 6.1. A *distinguished open set* of an affine variety X is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function $f \in A(X)$.

Remark. We have the following:

1. The $D(f)$ are closed under (finite) intersection: $D(fg) = D(f) \cap D(g)$.

2. The $D(f)$ form a basis for the Zariski topology on X: If $U \subseteq X$ is open, then $U = X \setminus V(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in A(X)$ (since X is Noetherian). So $U = D(f_1) \cup \cdots \cup D(f_r)$.

Remark. We will view $D(f)$ as "small open sets" (under mild assumptions, $\operatorname{codim}_X(X \setminus D(f)) = 1$).

Theorem 6.1. *If X is an affine variety and $f \in A(X)$, then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

Proof. We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix $\varphi \in \mathcal{O}_X(D(f))$. For any $a \in D(f)$, there exists an open neighborhood $a \in U_a \subseteq D(f)$ and $f_a, g_a \in A(X)$ such that $\varphi|_{U_a} = g_a/f_a$. We may further assume that

1. $U_a = D(h_a)$ for some $h_a \in A(X)$ (by shrinking U_a if necessary, since the $D(h)$ form a basis);
2. $h_a = f_a$ (by rewriting $g_a/f_a = g_a h_a / f_a h_a$ and replacing h_a, f_a with $f_a h_a$).

Then for $a, b \in D(f)$, we have $f_a g_b = f_b g_a$ on $D(f_a) \cap D(f_b)$. Since both the left and right hand sides vanish on $X \setminus (D(f_a) \cap D(f_b))$, we have $f_a g_b = f_b g_a$ in $A(X)$. Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so $f \in I(V(f_a : a \in D(f)))$. By the Nullstellensatz, there exists $n \geq 0$ such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the k_a are nonzero. Set $g = \sum_{a \in D(f)} k_a g_a$, and we claim that $\varphi = g/f^n$. To see this, note that on U_b, we have $\varphi|_{U_b} = g_b/f_b$. Now since $f_a g_b = f_b g_a$, we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that $\varphi|_{U_b} = (g/f^n)|_{U_b}$. Since this holds for any U_b, we have $\varphi = g/f^n$ in $\mathcal{O}_X(D(f))$. $\square$

Remark. Theorem 6.1 has the following consequences:

1. The $f = 1$ case implies that the natural ring homomorphism $A(X) \to \mathcal{O}_X(X)$ is surjective and hence an isomorphism (note that $D(1) = X$).
2. We will see that $\mathcal{O}_X(D(f)) \cong A(X)_f$, the *localization* of $A(X)$ at f.

Example 6.1.1. How do we compute $\mathcal{O}_X(U)$ on non-distinguished open sets? Consider

$$X = \mathbb{A}^2_k \quad \text{and} \quad U = \mathbb{A}^2_k \setminus \{(0,0)\}.$$

Note that U is never a distinguished open set. We claim that the ring homomorphism

$$k[x,y] \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{(0,0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use $U = D(x) \cup D(y)$ (in general, cover U by basis elements). Fix $\varphi : U \to k$ regular, so

$$\varphi|_{D(x)} = \frac{f}{x^m} \quad \text{for some } f \in k[x,y], m \geq 0$$
$$\varphi|_{D(y)} = \frac{g}{y^n} \quad \text{for some } g \in k[x,y], n \geq 0.$$

Since we are in a UFD, we may assume that $x \nmid f$ and $y \nmid g$. Now $fy^n = gx^m$ on $D(y) \cap D(x)$, so by the identity principle, $fy^n = gx^m$ on $\mathbb{A}^2_k$, so $fy^n = gx^m$ in $k[x,y]$. Using that $y \nmid g$, $x \nmid f$, and that $k[x,y]$ is a UFD, we must have $n = m = 0$, hence $f = g$. In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map $k[x,y] \to \mathcal{O}_X(U)$ is surjective.

6.3 Localization

Remark. We want to invert a subset of a ring, in particular *multiplicative systems.*

Definition 6.2. A *multiplicative system* of a ring A is a subset such that

1. $1 \in S$;
2. S is closed under multiplication.

Example 6.2.1. The following examples of S are multiplicative systems:

1. $S = A$ or $S = \{1\}$;
2. if $\mathfrak{p} \leq A$ is a prime ideal, then $S = A \setminus \mathfrak{p}$;
3. if $f \in A$, then $S = \{f^m : m \geq 0\}$.

Definition 6.3. The *localization* of a ring A at a multiplicative system S is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the a/s are formal symbols with $a/s \sim a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$.[1] The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

Example 6.3.1. The following are examples of localization:

1. If A is a domain and $S = A \setminus \{0\}$, then $S^{-1}A = \operatorname{Frac} A$.
2. If $S = \langle f \rangle = \{1, f, f^2, \ldots\}$, then we will write $A_f = S^{-1}A$.
3. If $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we will write $A_{\mathfrak{p}} = S^{-1}A$.

[1] Note that if A is a domain and $0 \notin S$, then this condition is equivalent to $as' = a's$.

Proposition 6.2. *We have the following properties of localization:*

1. *(Universal property of localization) For any ring homomorphism $\varphi : A \to B$ such that $\varphi(s)$ for all $s \in S$, then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi: a \mapsto a/1"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

2. *There is a bijection between the prime ideals $\mathfrak{p} \leq A$ with $\mathfrak{p} \cap S = \varnothing$ and the prime ideals $\mathfrak{q} \leq S^{-1}A$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$ with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : A \to S^{-1}A$ is the map $a \mapsto a/1$.*

Remark. In more generality, for an A-module M, we can define the localization $S^{-1}M$, which is an $S^{-1}A$-module. This gives a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which is exact.

Lecture 7

Sept. 9 — Germs and Sheaves

7.1 More on Localization

Proposition 7.1. *If X is an affine variety and $f \in A(X)$ is nonzero, then $\mathcal{O}_X(D(f)) \cong A(X)_f$.*

Proof. We define a ring homomorphism as follows:

$$\begin{aligned} A(X)_f &\longrightarrow \mathcal{O}_X(D(f)) \\ \frac{g}{f^m} &\longmapsto \left(x \mapsto \frac{g(x)}{f^m(x)} \right). \end{aligned}$$

To check that this is well-defined, assume $g/f^m \sim h/f^n$ in $A(X)_f$. So there exists $k \geq 0$ such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So $gf^n - hf^m = 0$ as functions $D(f) \to k$, so $g/f^m = h/f^n$ as functions $D(f) \to k$. Thus their images agree in $\mathcal{O}_X(D(f))$, so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume $g/f^m = 0$ as functions $D(f) \to k$ with $g \in A(X)$. Then $fg = 0$ in $A(X)$, so $g/f^m \sim 0/1$ in $A(X)_f$. □

7.2 Germs of Functions

Definition 7.1. Let $p \in X$ be a point on an affine variety.

1. A *germ* of a regular function of X at p is a pair (U, f) such that $x \in U \subseteq X$ is open and f is a regular function $U \to k$, up to the equivalence relation $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.
2. Define $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$.

Exercise 7.1. Check that $\mathcal{O}_{X,p}$ is a ring with operations

$$\begin{aligned} (U, \varphi) \cdot (V, \psi) &= (U \cap V, \varphi|_{U \cap V} \cdot \psi|_{U \cap V}), \\ (U, \varphi) + (V, \psi) &= (U \cap V, \varphi|_{U \cap V} + \psi|_{U \cap V}), \end{aligned}$$

with the zero function as the zero element and the constant 1 function as the unit element.

Lemma 7.1. *$\mathcal{O}_{X,p}$ is a local ring with unique maximal ideal $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$.*

Proof. It suffices to show that the units of $\mathcal{O}_{X,p}$ are precisely $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$. To see the reverse inclusion, fix $(U, \varphi) \in \mathcal{O}_{X,p}$ with $\varphi(p) \neq 0$. So there exists an open neighborhood $p \in W \subseteq U$ such that $\varphi|_W$ never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W)) = (W, 1),$$

so (U, φ) is a unit in $\mathcal{O}_{X,p}$. The forward inclusion is similar. □

Proposition 7.2. *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left(D(g), x \mapsto \frac{f(x)}{g(x)}\right) \end{aligned}$$

with $I(p) = \{f \in A(X) : f(p) = 0\}$.

Proof. To see that this is well-defined, let $f/g \sim f'/g' \in A(X)_{I(p)}$. Then $h(fg' - f'g) = 0$ for some $h \in A(X)$ with $h(p) \neq 0$. So $f/g = f'/g'$ as functions $D(h) \cap D(g) \to k$, which means that $f/g = f'/g'$ as elements in $\mathcal{O}_{X,p}$. Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose $(U, \varphi) \in \mathcal{O}_{X,p}$. Since $\varphi : U \to k$ is a regular function, there exists an open set $p \in U_p \subseteq U$ and $f, g \in A(X)$ such that g does not vanish on U_p and $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$. So $(U, \varphi) \sim (D(g), f/g)$ in $\mathcal{O}_{X,p}$, i.e. (U, φ) is in the image. □

Example 7.1.1. If $X = \mathbb{A}^n_k$ and $p = 0$, then

$$\mathcal{O}_{\mathbb{A}^n_k,0} \cong k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = \left\{\frac{f}{g} : f \in k[x_1, \ldots, x_n], g \in k[x_1, \ldots, x_n] \setminus (x_1, \ldots, x_n)\right\}.$$

Remark. We will relate the local properties of X at p to properties of $\mathcal{O}_{X,p}$. We will use the following statements from commutative algebra: Let A be a ring and $\mathfrak{p} \subseteq A$ a prime ideal. Then

1. $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$.
2. There is a bijection from the prime ideals of $A_\mathfrak{p}$ to the prime ideals of A contained in $\mathfrak{p}$.
3. $\operatorname{ht}_A \mathfrak{p} = \dim A_\mathfrak{p}$ (this follows from (2)).

This has the following consequence: If X is an affine variety and $p \in X$, then

$$\operatorname{codim}_X\{p\} = \operatorname{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

7.3 Sheaves

Remark. We will now formalize the structures $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ that we have seen before.

Definition 7.2. A *presheaf (of rings)* $\mathcal{F}$ on a topological space X is the data of

1. for every open set $U \subseteq X$, a ring $\mathcal{F}(U)$;
2. for every inclusion of open sets $U \subseteq V \subseteq X$, a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying the following properties:

1. $\mathcal{F}(\varnothing) = 0$;
2. $\rho_{U,U}$ is the identity map;
3. for inclusions of open sets $U \subseteq V \subseteq W \subseteq X$, we have $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$.

Example 7.2.1. If X is an affine variety, then $\mathcal{O}_X$ gives a presheaf of rings with

1. for $U \subseteq X$, the ring is $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \to k\}$;
2. for $U \subseteq V \subseteq X$, the map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is given by $\varphi \mapsto \varphi|_U$.

Remark. We often call $s \in \mathcal{F}(U)$ a *section*, and for $U \subseteq V$, we call $s|_U = \rho_{V,U}(s)$ the *restriction*.

Remark. A presheaf is the same thing as a functor $\mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Rings}$, where Open_X is the category with objects the nonempty open sets of X and morphisms corresponding to the inclusions $U \subseteq V$.

Definition 7.3. A presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the *gluing property*: For any $U \subseteq X$ open, an open cover $\{U_i\}_{i \in I}$ of U, and $\varphi_i \in \mathcal{F}(U_i)$ with $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Example 7.3.1. We have the following:

1. If X is an affine variety, then $\mathcal{O}_X$ is a sheaf (if we take $\varphi_i \in \mathcal{O}_X(U_i)$ that agree on the overlaps, then we get $\varphi : U \to k$, which is regular since regularity is a local property).
2. If M is a smooth manifold, then we can define a sheaf (on open subsets $U \subseteq M$) by
$$U \longmapsto \mathcal{F}^{\mathrm{sm}}(U) = \{\text{smooth functions } U \to \mathbb{R}\}.$$
We may also consider $\mathcal{F}^{\mathrm{cont}}$, $\mathcal{F}^{\mathrm{diff}}$, $\mathcal{F}^{\mathrm{loc,const}}$, etc. However, $\mathcal{F}^{\mathrm{const}}$ is a presheaf, but not a sheaf in general: We can take $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$, and we will only get a locally constant function. Similarly, $\mathcal{F}^{\mathrm{bounded}}$ is only a presheaf but not a sheaf.
3. If $\mathcal{F}$ is a sheaf on a topological space X and $U \subseteq X$ is open, then we get a sheaf $\mathcal{F}|_U$ on U defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 7.4. The *stalk* of a sheaf $\mathcal{F}$ on a topological space X at $x \in X$ is
$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\}/\sim,$$
where $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.

Example 7.4.1. If X is an affine variety and $p \in X$, then $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$.

Remark. As before with $\mathcal{O}_{X,p}$, one can check that $\mathcal{F}_x$ naturally has the structure of a ring.

Remark. An alternative perspective is to define the stalk as a direct limit:
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where the limit is taken over all open $x \in U \subseteq X$ with respect to the ordering $U \leq V$ if $V \subseteq U$.

Lecture 8

Sept. 11 — Morphisms

8.1 Morphisms of Open Sets

Remark. Recall that a continuous map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions $f_1, \dots, f_n : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = (f_1(x), \dots, f_n(x))$;
2. for each open set $U \subseteq \mathbb{R}^n$ and smooth $\varphi : U \to \mathbb{R}$, the function $f^*\varphi := \varphi \circ f : \mathbb{R}^m \to \mathbb{R}$ is smooth.

The implication $(1 \Rightarrow 2)$ follows by the chain rule. To see $(2 \Rightarrow 1)$, take $y_i : \mathbb{R}^n \to \mathbb{R}$ defined by $f_i := f^*y_i$. We want a similar definition in algebraic geometry.

Definition 8.1. Let X and Y be open sets of affine varieties. A *morphism* $f : X \to Y$ is a continuous map such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the map

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse (equivalently, f is a bijection and f^{-1} is a morphism).

Remark. We have the following properties of morphisms:

1. (Composition) If $f : X \to Y$ and $g : Y \to Z$ are morphisms of open sets of affine varieties, then so is $g \circ f : X \to Z$.
2. (Local on target) If $X \to Y$ is a map of open sets of affine varieties such that there exists an open cover $\{U_i\}_{i \in I}$ of Y with $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ a morphism for all $i \in I$, then f is a morphism.

Proposition 8.1. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Let $U \subseteq X$ and $V \subseteq Y$ be open sets. A map $f : U \to V$ is a morphism if and only if there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that*

$$f(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

Proof. $(\Rightarrow)$ Let $U \subseteq \mathbb{A}^m_{x_i}$ and $V \subseteq \mathbb{A}^n_{y_i}$. By the definition of a morphism, $y_i : V \to k$ satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

$(\Leftarrow)$ Assume there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

We first show that f is continuous. Let $Z \subseteq V$ be a closed set. So we can write $Z = V(g_r, \ldots, g_r)$ for some $g_1, \ldots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \ldots, y_n]$. Now we have

$$\begin{aligned} f^{-1}(Z) = \{x \in U : f(x) \in Z\} &= \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \ldots, r\} \\ &= \{x \in U : (f^* g_i)(x) = 0 \text{ for } i = 1, \ldots, r\}. \end{aligned}$$

Note that $f^* g_i = g_i(\varphi_1, \ldots, \varphi_n)$, which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So $f^{-1}(Z)$ is closed in U.

Now to show that f is a morphism, it suffices to show that for any $W \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(W)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$. The proof of this is similar to before. □

Example 8.1.1. We have the following:

1. Morphisms $\mathbb{A}^m \to \mathbb{A}^n$ are of the form

$$x \longmapsto (f_1(x), \ldots, f_n(x))$$

   with $f_1, \ldots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_m]$.

2. Write $\mathbb{A}^1_t$ to mean $\mathbb{A}^1$ with variable t. Then we can define $\mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ by $t \mapsto (t, t^2)$. We can get an inverse $V(y - x^2) \to \mathbb{A}^1_t$ by $(x, y) \mapsto x$, so $\mathbb{A}^1_t$ and $V(y - x^2)$ are isomorphic.

3. Consider the map $g : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. This map is bijective, but it is not an isomorphism. To see this, we can show that $(g^{-1})^*\varphi$ is not regular for some regular function φ on $\mathbb{A}^t_1$. For instance, we can take $\varphi = t$, so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

   which we can see is not regular.

8.2 Relation to Coordinate Rings

Remark. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Then a morphism $f : X \to Y$ of affine varieties induces a k-algebra morphism (called the *pullback* of f)

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^* = \varphi \circ f \end{aligned}$$

with the properties $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{A(X)}$, i.e. $X \mapsto A(X)$ is a contravariant functor.

Proposition 8.2. *The following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{aff,var}}(X, Y) &\xrightarrow{\Phi} \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

Proof. Note that $A(X) \cong k[x_1, \ldots, x_m]/I(X)$ and $A(Y) \cong k[y_1, \ldots, y_n]/I(Y)$. Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \ldots, \varphi_n(x)), \end{aligned}$$

we can define $f^*\overline{y}_i = \varphi_i$. Conversely, given a k-algebra homomorphism $\phi : A(Y) \to A(X)$, we can set $\varphi_i = \phi(\overline{y}_i)$. Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}^n_{y_i} \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)). \end{aligned}$$

We claim that $f(X) \subseteq Y$. To see this, fix $x \in X$. If $h \in I(Y)$, then

$$h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so $f(X) \subseteq Y$. Thus we get a morphism $f : X \to Y$ by $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$ with $f^* y_i = \varphi_i$. One can check that this gives a map $\Psi : \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \to \mathrm{Hom}_{\mathrm{aff,var}}(X, Y)$ which is inverse to Φ. □

Example 8.1.2. We have the following:

1. Recall the morphism $g : \mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t, t^2)$. The pullback is given by

$$\begin{aligned} g^* : \frac{k[x,y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

   Note that g^* is an isomorphism of k-algebras, so g is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to g.

2. Recall the morphism $h : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. The pullback is

$$\begin{aligned} h^* : \frac{k[x,y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

   Note that $t \notin \operatorname{Im} h^*$, so h^* is not an isomorphism, so h is not an isomorphism.

Remark. There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced k-algebras (up to isomorphism).

To see this, observe that if $X \subseteq \mathbb{A}^n$ is an affine variety, then $A(X) \cong k[x_1, \dots, x_n]/I(X)$. This is finitely generated, and reduced since $I(X)$ is radical. Conversely, let A be a reduced finitely generated k-algebra. Then $A \cong k[y_1, \dots, y_m]/I$ since A is finitely generated, and I is radical since A is reduced. Thus by Hilbert's nullstellensatz, $Y = V(I)$ satisfies $I(Y) = I(V(I)) = I$, so $A \cong A(Y)$.

In more abstract language, this means that there is an equivalence of categories

$$\mathrm{AffVar} \longleftrightarrow \mathrm{RedFGAlg}_k^{\mathrm{op}}.$$

Lecture 9

Sept. 16 — Morphisms, Part 2

9.1 An Example of Isomorphisms

Example 9.0.1. What of the following are isomorphic over $\mathbb{C}$?

1. $\mathbb{A}^1 \setminus \{1\}$;
2. $V(x^2 + y^2) \subseteq \mathbb{A}^2$;
3. $V(y - x^2, z - x^3) \subseteq \mathbb{A}^3$;
4. $V(xy) \subseteq \mathbb{A}^2$;
5. $V(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$;
6. $V(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$.

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2 + y^2) = V((x + iy)(x - iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism $\mathbb{A}^1 \to Y = V(y - x^2, z - x^3)$ by $t \mapsto (t, t^2, t^3)$. We can also see this by noting that $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$. For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$, whereas $A(\mathbb{A}^1) \cong \mathbb{C}[x]$. So $A \setminus \{1\} \not\cong \mathbb{A}^1$. For (6), note that

$$V(x^2 - y^2 - 1) = V((x - y)(x + y) - 1) \cong V(uv - 1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map $V(uv - 1) \to \mathbb{A}^1 \setminus \{0\}$ given by $(u, v) \mapsto u$, with inverse $t \mapsto (t, 1/t)$. Finally, letting C be the curve in 6, one can show that there is a singularity at the origin with $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 2$, which is different than the other examples. So the isomorphism classes are $\{2, 4\}$, $\{1, 6\}$, $\{3\}$, and $\{5\}$.

9.2 Ringed Spaces and Morphisms

Definition 9.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X.

Example 9.1.1. If X is an affine variety and $\mathcal{O}_X$ is the sheaf of regular functions, then $(X, \mathcal{O}_X)$ is a ringed space. Similarly, if M is a complex manifold and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M, then $(M, \mathcal{O}_M)$ is a ringed space.

Remark. From now on, for a ringed space $(X, \mathcal{O}_X)$, we will always assume $\mathcal{O}_X$ is a sheaf of k-valued functions on X. With this assumption, we can make sense of pullbacks.

Definition 9.2. A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $f : X \to Y$ such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$,

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

is an element of $\mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse.

Remark. A one-sided inverse need not be two-sided: Consider $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$ and $g : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 0)$. Then $f \circ g = \mathrm{id}_{\mathbb{A}^1}$, but $g \circ f$ is not the identity on $\mathbb{A}^2$.

Remark. If $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then for $V \subseteq U \subseteq Y$ open, we get

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\text{res.}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\text{res.}"] \\ \mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \end{tikzcd}$$

which is a commutative diagram of ring homomorphisms.

Remark. If X and Y are open sets of affine varieties, then a map $f : X \to Y$ is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

Definition 9.3 (Redefinition of affine variety)**.** An *affine variety* $(X, \mathcal{O}_X)$ is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

Remark. We will often write just X for the affine variety instead of the pair $(X, \mathcal{O}_X)$.

Example 9.3.1. Recall that $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$ from Example 9.0.1. In particular, $\mathbb{A}^1 \setminus \{0\}$ is an affine variety in the new sense (but not in the old sense).

Proposition 9.1. *If X is an affine variety (in the old sense) and $f \in A(X)$, then $D(f)$ is an affine variety.*

Proof. Write $X = V(I) \subseteq \mathbb{A}^n_{x_i}$ and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse $V(I, fy - 1) \to D(f)$ given by $(x, y) \mapsto x$. So $D(f) \cong V(I, fy - 1)$ as ringed spaces. Thus $D(f)$ is an affine variety (in the new sense). $\square$

9.3 Products of Affine Varieties

Remark. If $X \subseteq \mathbb{A}^m_{x_i}$ and $Y \subseteq \mathbb{A}^n_{y_i}$ are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing $I(X), I(Y)$ as ideals in $k[x_1, \dots, x_m, y_1, \dots, y_n]$. So $X \times Y$ is an affine variety with morphisms

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1", "(x,y)\mapsto x"'] \arrow[dr, "p_2"', "(x,y)\mapsto y"] & \\ X & & Y \end{tikzcd}$$

Proposition 9.2. *For every affine variety Z and diagram of morphisms*

$$\begin{tikzcd} Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\ & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\ & X & \end{tikzcd}$$

there is a unique morphism f which makes the diagram commute.

Proof. We already know that there is a unique set theoretic map which makes the diagram commute. Then since f_X and f_Y are given as regular functions, so is f. So f is a morphism. □

Remark. We will now try to understand the isomorphism $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

9.4 Tensor Products

Definition 9.4. Let A be a (commutative) ring and M, N be A-modules. The *tensor product* $M \otimes_A N$ is the A-module generated by the symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

1. (distributive law): $(m + m') \otimes n = m \otimes n + m' \otimes n$,
2. (multiplication with scalars): $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.

To make this precise, $M \otimes_A N = A^{M \times N}/R$, where R is the submodule generated by these relations.

Example 9.4.1. We have $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$. We can compute

$$1 \otimes 1 = (3-2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if $\gcd(m, n) = 1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Proposition 9.3 (Universal property of the tensor product)**.** *For any bilinear map $\Phi : M \times N \to P$ to an A-module P (i.e. $n \mapsto \Phi(m, n)$ is A-linear for each $m \in M$ and the same for $m \mapsto \Phi(m, n)$),*

$$\begin{tikzcd} M \times N \arrow[r, "\Phi"] \arrow[d, "(m,n)\mapsto m\otimes n"'] & P \\ M \otimes N \arrow[ur, "\Psi"'] & \end{tikzcd}$$

there exists a unique A-module homomorphism $\Psi : M \otimes N \to P$ such that the above diagram commutes.

Remark. We have the following properties of the tensor product:

1. $A \otimes M \cong M$;

2. $M \otimes N \cong N \otimes M$;
3. $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$;
4. $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed A-module M and an exact sequence

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

the sequence (where F is defined by $m \otimes n' \mapsto m \otimes f(n')$ and G is defined by $m \otimes n \mapsto m \otimes g(n)$)

$$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

is also exact. In particular, $\otimes M$ induces a right exact functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$ by $N \mapsto M \otimes N$.

Example 9.4.2. The functor $\otimes M$ is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 9.1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

Lecture 10

Sept. 18 — Pre-varieties

10.1 More on Tensor Products

Proposition 10.1. *If B and C are A-algebras (i.e. there are ring homomorphisms $f : A \to B$ and $g : A \to C$ which give $a \cdot b := f(a)b$ and $a \cdot c = g(a)c$, then $B \otimes_A C$ is also an A-algebra with*

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc')$$

and ring homomorphism $A \to B \otimes_A C$ given by $a \mapsto a \otimes 1$ (equivalently, $1 \otimes a$).

Proposition 10.2. $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Proposition 10.3. $(k[x_1, \ldots, x_m]/I) \otimes_k (k[y_1, \ldots, y_n]/J) \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n)/\langle I, J\rangle$.

Proof. Set $R = k[x_1, \ldots, x_m]$ and $S = k[y_1, \ldots, y_n]$. We have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Applying the right exact functor $\otimes_k(S/J)$ (and vice versa with J and $\otimes R$) gives an exact sequence

$$\begin{tikzcd}
 & R \otimes_k J \arrow[d] & & \\
 & R \otimes_k S \arrow[d] & & \\
I \otimes_k (S/J) \arrow[r] & R \otimes_k (S/J) \arrow[r] \arrow[d] & (R/I) \otimes_k (S/J) \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

So we have

$$(R/I) \otimes_k (S/J) \cong \frac{R \otimes_k (S/J)}{\operatorname{Im}(I \otimes_k (S/J) \to R \otimes_k (S/J))} \cong \frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J},$$

which is the desired result since $I \otimes_k S + R \otimes_k J = \langle I, J\rangle$ in $R \otimes_k S$. $\square$

Proposition 10.4 (Milne)**.** *Let B and C be finitely generated k-algebras with $k = \overline{k}$.*

1. *If B and C are reduced, then so is $B \otimes_k C$.*

2. *If B and C are domains, then so is $B \otimes_k C$.*

Remark. We need $k = \overline{k}$ in Proposition 10.4. Consider the domains $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[y]/(y^2+1)$. Then

$$\frac{\mathbb{R}[x]}{(x^2+1)} \otimes_{\mathbb{R}} \frac{\mathbb{R}[y]}{(y^2+1)} \cong \frac{\mathbb{R}[x,y]}{(x^2+1, y^2+1)},$$

which is not a domain since $(\overline{x-y})(\overline{x+y}) = \overline{x^2-y^2} = \overline{-1-(-1)} = 0$.

Corollary 10.0.1. *If $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then*

1. *$I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.*
2. *$A(X \times Y) \cong A(X) \otimes_k A(Y)$.*
3. *If X and Y are irreducible, then $X \times Y$ is irreducible.*

Proof. Observe that $V(I(X), I(Y)) = X \times Y \subseteq \mathbb{A}^{m+n}$, so $I(X \times Y) = \sqrt{\langle I(X), I(Y) \rangle}$. Now we know that $I(X)$ and $I(Y)$ are radical in $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$, respectively, so

$$\frac{k[x_1, \ldots, x_m]}{I(X)} \quad \text{and} \quad \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

are reduced. By Proposition 10.4, we get that

$$\frac{k[x_1, \ldots, x_m, y_1, \ldots, y_n]}{\langle I(X), I(Y) \rangle} \cong \frac{k[x_1, \ldots, x_m]}{I(X)} \otimes_k \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

is reduced, so $\langle I(X), I(Y) \rangle$ is radical. Thus $I(X \times Y) = \langle I(X), I(Y) \rangle$, so (1) holds.

Now (1) implies (2), and (3) follows since X and Y being irreducible implies $A(X)$ and $A(Y)$ are domains, which implies $A(X \times Y)$ is a domain by Proposition 10.4 and (2), so $X \times Y$ is irreducible. $\square$

10.2 Pre-varieties

Remark. We will now head towards defining a *variety*, which is roughly finitely many affine varieties glued together (a *pre-variety*) with a separation condition (an algebraic version of Hausdorffness).

Definition 10.1. A *pre-variety* is a ringed space $(X, \mathcal{O}_X)$ such that there exists a finite open cover $X = \bigcup_{i=1}^{s} U_i$ with $(U_i, \mathcal{O}_X|_{U_i})$ being an affine variety for all $i = 1, \ldots, s$. A *morphism* of pre-varieties

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of the ringed spaces. We will often just write X for $(X, \mathcal{O}_X)$.

Remark. We call $\varphi \in \mathcal{O}_X(U)$ with $U \subseteq X$ open and $\varphi : U \to k$ a *regular function* on U.

Example 10.1.1. Consider the following:

1. An affine variety X is a pre-variety. However, we have multiple choices for the open cover: We can take $X = X$, or $X = \bigcup_{i=1}^{s} D(f_i)$ with $f_i \in \mathcal{O}_X(X)$ and $(f_1, \ldots, f_s) = (1)$ in $\mathcal{O}_X(X)$.
2. $\mathbb{P}_k^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$ is a pre-variety. We will see that $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\text{pt}\}$.

3. Let $X = V(I) \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ open. Set

$$\mathcal{O}_U(V) = \{\varphi : V \to k \mid \varphi \text{ is regular}\}.$$

Then $(U, \mathcal{O}_U)$ is a pre-variety. To see this, note that $U = \bigcup_{f \in I(X \setminus U)} D(f)$. Since U is Noetherian (hence is compact), we can find a finite subcover, so $U = \bigcup_{i=1}^{s} D(f_i)$ for some $f_i \in A(X)$.

4. (Gluing) Let X_1 and X_2 be affine varieties, and $U_{1,2} \subseteq X_1$, $U_{2,1} \subseteq X_2$ open, with an isomorphism

$$f : U_{1,2} \longrightarrow U_{2,1}.$$

Then we get a pre-variety by setting $X = (X_1 \sqcup X_2)/{\sim}$, where $a \sim f(a)$ for all $a \in U_{1,2}$, $f(a) \sim a$ for all $a \in U_{2,1}$, and $b \sim b$ for all $b \in X_1 \sqcup X_2$. We have quotient maps

$$j_1 : X_1 \longrightarrow X \quad \text{and} \quad j_2 : X_2 \longrightarrow X.$$

Now X is a topological space with the quotient topology, and j_1, j_2 are open embeddings (i.e. have open images and are homeomorphisms onto their images). Define a sheaf of rings $\mathcal{O}_X$ on X by

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid j_1^*\varphi \in \mathcal{O}_{X_1}(j^{-1}(U)) \text{ and } j_2^*\varphi \in \mathcal{O}_{X_2}(j_2^{-1}(U))\}.$$

One can check $X = j_1(X_1) \cup j_2(X_2)$ and $(j(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Example 10.1.2. Consider $X_1 = \mathbb{A}^1_x$ and $X_2 = \mathbb{A}^1_y$, with $U_{1,2} = \mathbb{A}^1_x \setminus \{0\}$ and $U_{2,1} = \mathbb{A}^1_y \setminus \{0\}$. Define

$$\begin{aligned} f : U_{1,2} &\longrightarrow U_{2,1} \\ x &\longmapsto 1/x. \end{aligned}$$

Then we can take $\mathbb{P}^1_k = (X_1 \sqcup X_2)/{\sim}$. What are the regular functions $\mathbb{P}^1_k \to k$? We should get only the constant functions (When $k = \mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$ is compact, so a holomorphic function $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}$ is bounded. By restricting to X_1, we get a bounded map $f : \mathbb{C} \to \mathbb{C}$, so f is constant by Liouville's theorem).

In general, let $j_i : X_i \to \mathbb{P}^1_k$ be the quotient maps. Fix $\varphi \in \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$. Now

$$\varphi|_{X_1} := j_1^*\varphi = \sum_{i \geq 0} a_i x^i \quad \text{and} \quad \varphi|_{X_2} := j_2^*\varphi = \sum_{i \geq 0} b_i y^i$$

for some $a_i, b_i \in k$. They must agree on the overlap, so

$$\sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} b_i (1/x)^i$$

as functions on $\mathbb{A}^1 \setminus \{0\}$. Since $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$, we have $a_i = b_i = 0$ for $i > 0$ and $a_0 = b_0$ (since the powers of $x^{\pm 1}$ are k-linearly independent), so φ is a constant function.

If we instead took $f : U_{1,2} \to U_{2,1}$ to be $x \mapsto x$, then $X = (X_1 \sqcup X_2)/{\sim}$ is the "bug-eyed line" with two points $0, 0'$ at the origin (this is the *line with two origins* when $k = \mathbb{R}$, which is not Hausdorff.) Note that $X \setminus \{0, 0'\} \cong \mathbb{A}^1 \setminus \{0\}$. In our case, the bad property is that there exist two morphisms

$$g_1, g_2 : \mathbb{A}^1 \longrightarrow X$$

such that $g_1|_{\mathbb{A}^1 \setminus \{0\}} = g_2|_{\mathbb{A}^1 \setminus \{0\}}$ and $g_1 \neq g_2$, i.e. "limits are not unique" on X. Note that a similar computation shows $\mathcal{O}_X(X) \cong k[x]$, so in particular, $X \ncong \mathbb{P}^1_k$.

Lecture 11

Sept. 23 — Pre-varieties, Part 2

11.1 More on Pre-varieties

Proposition 11.1. *Let $(X, \mathcal{O}_X)$ be a pre-variety.*

1. *X is Noetherian as a topological space.*
2. *X has a basis by affine varieties.*

Proof. (1) Note that X has a finite cover by affine varieties U_i, which are each Noetherian.

(2) If $(X, \mathcal{O}_X)$ is affine, then $\{D(f) : f \in \mathcal{O}_X(X)\}$ gives such a basis. Do this for each U_i. $\square$

Example 11.0.1 (General gluing procedure)**.** Let I be a finite index set, $(X_i, \mathcal{O}_{X_i})$ affine varieties, $U_{i,j} \subseteq X_i$ open sets, and $f_{i,j} : U_{i,j} \to U_{j,i}$ isomorphisms for each $i, j \in I$, satisfying

1. $U_{i,i} = X_i$ and $f_{i,i} = \mathrm{id}$;
2. $f_{i,j}^{-1}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k}$;
3. the following diagram commutes:

$$\begin{tikzcd} U_{i,j} \cap U_{i,k} \arrow[rr, "f_{i,k}"] \arrow[dr, "f_{i,j}"'] & & U_{k,i} \cap U_{k,j} \\ & U_{j,i} \cap U_{j,k} \arrow[ur, "f_{j,k}"'] & \end{tikzcd}$$

We can define $X = (\bigsqcup_{i \in I} X_i)/\sim$ with the quotient topology, where $a \sim a$ for $a \in X_i$ and $a \sim f_{i,j}(a)$ for $a \in U_{i,j}$. The inclusions $j_i : X_i \hookrightarrow X$ are open embeddings, and we can set

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi|_{X_i} = j_i^* \varphi \text{ is regular for all } i\}.$$

Then $(X, \mathcal{O}_X)$ is a ringed space with $(j_i(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Remark. Any pre-variety X is a gluing of affine varieties. To see this, note that there exists a cover $X = \bigcup_{i=1}^s U_i$ by affine varieties. Then we can take $X_i = U_i$, $U_{i,j} = U_i \cap U_j \subseteq X_i$, and $f_{i,j} : U_{i,j} \to U_{j,i}$ to be the identity map.

Proposition 11.2. *Let X be a pre-variety.*

1. *If $U \subseteq X$ is an open set, then $(U, \mathcal{O}_X|_U)$ is again a pre-variety.*

2. *Let $Z \subseteq X$ be a closed set. For $U \subseteq Z$ open, set*

$$\mathcal{O}_Z(U) = \left\{ \varphi : U \to k \mid \begin{array}{c} \text{for each } a \in U, \text{ there exists open } a \in W \subseteq X \text{ and} \\ \psi : W \to k \text{ regular such that } \varphi|_{W \cap Z} = \psi|_{W \cap Z} \end{array} \right\}.$$

Then $(Z, \mathcal{O}_Z)$ is a pre-variety.

Proof. (1) Note that X has a basis by affine varieties, so we can cover U by affine varieties. This cover may be infinite, but we can pass to a finite subcover since X and hence U is Noetherian.

(2) The idea is to first reduce to the case $X = V(I) \subseteq \mathbb{A}^n$, so $Z \subseteq X$ is cut out by polynomials. Then observe that $\mathcal{O}_Z$ agrees with the previous definition. □

Remark. Note that unions of intersections of closed and open sets are not necessarily pre-varieties. For instance, consider $(\mathbb{A}^2 \setminus V(xy)) \cup \{0\}$.

Proposition 11.3. *If X, Y are pre-varieties, then there exists a pre-variety with morphisms*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

with the property that for every diagram

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "f_X"'] \arrow[d, dashed, "f"] \arrow[ddr, bend left, "f_Y"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

there exists a unique morphism f such that the diagram commutes. We call P the product *of X and Y, and write $X \times Y := P$. Moreover, set theoretically $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.*

Proof. We know the result holds when X, Y, Z are affine or even open sets of affine varieties. In the general case, fix an open affine cover $X = \bigcup_{i=1}^{s} U_i$ and $Y = \bigcup_{j=1}^{r} V_j$. Then glue the products by:

1. $P_{(i,j)} := U_i \times V_j$,
2. along $P_{(i,j),(i',j')} : (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$,
3. via $f_{(i,j),(i',j')} : P_{(i,j),(i',j')} \xrightarrow{\cong} P_{(i',j'),(i,j)}$, the isomorphism from the universal property of products.

We get a pre-variety P, and the morphisms

$$\begin{tikzcd} & f_1^{-1}(U_2) \cap f_2^{-1}(V_j) \subseteq Z \arrow[ddl, bend right] \arrow[d, dashed, "f_{i,j}"] \arrow[ddr, bend left] & \\ & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

glue to give a morphism

$$\begin{tikzcd} & Z \arrow[d, dashed, "f"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which is a morphism as the morphism condition can be checked locally. Furthermore, the diagram commutes (as can be checked locally). Last, f is unique: One can either this check locally or check set theoretically using $P = \{(x, y) : x \in X \text{ and } y \in Y\}$ as sets. $\square$

Remark. Note that $X \times Y$ is set theoretically the product of X and Y, but not the product of X and Y as topological spaces. Consider $X = Y = \mathbb{A}^1$ and $X \times Y = \mathbb{A}^2$.

11.2 Varieties

Remark. We want a version of Hausdorffness in algebraic geometry. However, an irreducible topological space (e.g. $\mathbb{A}^n$) is almost never Hausdorff (unless it is a single point). From a different perspective, note that X is Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is closed, where $X \times X$ is given the product topology.

Definition 11.1. A pre-variety is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$ (the product pre-variety). A *variety* is a pre-variety that is separated.

Example 11.1.1. $\mathbb{A}^n$ is separated. We have

$$V(x_1 - y_1, \ldots, x_n - y_n) = \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^n_{y_i} \cong \mathbb{A}^{2n},$$

so $\Delta_{\mathbb{A}^n}$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$.

Example 11.1.2. Any affine variety is separated. To see this, we may assume $X = V(I) \subseteq \mathbb{A}^n$. By the construction of the product, $X \times X \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is closed and

$$\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}.$$

Since $\Delta_{\mathbb{A}^n}$ is closed, we have Δ_X is closed in $X \times X$ since $X \times X$ has the subspace topology.

Proposition 11.4. *If X is a variety, then any closed or open set $Z \subseteq X$ is a variety.*

Proof. We have already seen that Z is a pre-variety, so it suffices to show that Z is separated. We note that $Z \times Z \hookrightarrow X \times X$ is an embedding of topological spaces, and $\Delta_Z = (Z \times Z) \cap \Delta_X$. Since Δ_X is closed and $Z \times Z$ has the subspace topology, Δ_Z is closed in $Z \times Z$. So Z is separated. $\square$

Example 11.1.3. Recall the bug-eyed line from Example 10.1.2. Let a, b be the two origins, and write $X = U_1 \cup U_2$, where $U_1 = X \setminus \{b\} \cong \mathbb{A}^1$ and $U_2 = X \setminus \{a\} \cong \mathbb{A}^1$. Then consider

$$\mathbb{A}^2 \cong U_1 \times U_2 \subseteq X \times X.$$

Note that $\Delta_X \cap (U_1 \times U_2) = \{(x,x) : x \in k \setminus \{0\}\} = \Delta_{\mathbb{A}^1} \setminus \{0\}$. So Δ_X is not closed in $X \times X$.

Exercise 11.1. Show that $\mathbb{P}^1_k$ is separated.

Proposition 11.5. *Let $f, g : X \to Y$ be morphisms of pre-varieties with Y a variety.*

1. *The graph $\Gamma_f := \{(x, f(x)) : x \in X\}$ of f is closed in $X \times Y$.*
2. *$\{x \in X : f(x) = g(x)\}$ is closed in X. This becomes a version of the identity principle in the case that X is irreducible: If X is irreducible and f, g agree on a nonempty open set, then $f = g$.*

Proof. (1) We can write $\Gamma_f = (f, \mathrm{id})^{-1}(\Delta_Y)$ where $(f, \mathrm{id}) : X \times Y \to Y \times Y$, and Δ_Y is closed.

(2) Consider the morphism

$$\begin{aligned} X &\xrightarrow{(f,g)} Y \times Y \\ x &\longmapsto (f(x), g(x)). \end{aligned}$$

Then $\{x \in X : f(x) = g(x)\} = (f,g)^{-1}(\Delta_Y)$, so it is closed. $\square$

Lecture 12

Sept. 25 — Projective Varieties

12.1 Projective Space

Definition 12.1. Define *projective n-space* over k to be

$$\mathbb{P}^n_k = \mathbb{P}^n = \text{1-dimensional subspaces of } k^{n+1} = (k^{n+1} \setminus \{0\})/\sim,$$

where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if there exists $\lambda \in k^\times$ such that $(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$. We write $[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k$ for the equivalence class of $(x_0, x_1, \dots, x_n)$.

Example 12.1.1. For $n = 2$, we have $[1 : 0 : 2] = [1/2 : 0 : 1] \in \mathbb{P}^2_k$ when $\operatorname{char} k \neq 2$.

Remark. For $0 \leq i \leq n$, define $U_i = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k : x_i \neq 0\}$. Then

$$\mathbb{P}^n_k = \bigcup_{i=0}^n U_i,$$

and there exist bijective maps $f_i : U_i \to \mathbb{A}^n$ given by

$$f_i([x_0 : \cdots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i),$$

where $\widehat{x_i/x_i}$ means we omit x_i/x_i. For $i = 0$, the inverse is $f_0^{-1}(x_1, \dots, x_n) = [1 : x_1 : \cdots : x_n]$.

Remark. Another way to think about $\mathbb{P}^n$ is via points at ∞. Observe that

$$\mathbb{P}^n \setminus U_0 = \{[0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k : (x_1, \dots, x_n) \in k^n \setminus \{0\}\} \cong \mathbb{P}^{n-1}.$$

So $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$.

Remark. Why work with $\mathbb{P}^n$? One motivation is analytic (e.g. for $k = \mathbb{C}$):

1. $\mathbb{P}^n_\mathbb{C}$ is compact with the analytic topology: There are surjective continuous maps

$$\begin{tikzcd} \mathbb{R}^{2n+2} \setminus \{0\} \arrow[r, two heads] & \mathbb{CP}^n \\ S^{2n+1} \arrow[u, hook] \arrow[ur, two heads] & \end{tikzcd}$$

2. *Chow's theorem*: Any closed complex submanifold of $\mathbb{CP}^n$ is a projective variety.

Another motivation is the extra data at ∞:

1. If ℓ_1, ℓ_2 are distinct lines in $\mathbb{A}^2$, then $\#(\ell_1 \cap \ell_2) = 0$ or 1. However, over $\mathbb{P}^2$, $\#(\ell_1 \cap \ell_2) = 1$ always.

2. *Bezout's theorem*: If $C_1, C_2 \subseteq \mathbb{A}^2$ are two distinct irreducible curves in $\mathbb{A}^2$, then
$$\#(C_1 \cap C_2) \leq (\deg C_1)(\deg C_2),$$
counting multiplicities. The version over $\mathbb{P}^2$ always gives equality.

12.2 Graded Rings

Remark. In projective space, for $f \in k[x_0, \dots, x_n]$, we could try to define
$$V(f) = \{[a_0 : \cdots : a_n] : f(a_0, \dots, a_n) = 0\}.$$
But this is bad notation as it is not well-defined ($f = 0$ depends on the representative in the equivalence class). Instead, if f is homogeneous of degree d, then
$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n),$$
so $V(f)$ is well-defined in this case, when f is homogeneous.

Definition 12.2. An $\mathbb{N}$-*graded ring* is a ring R with subgroups $R_d \subseteq R$ for $d \in \mathbb{N}$ such that
$$R = \bigoplus_{d \in \mathbb{N}} R_d \quad \text{and} \quad R_d R_e \subseteq R_{d+e}.$$
An element $f \in R$ is *homogeneous* if there exists d such that $f \in R_d$.

Example 12.2.1. For $S = k[x_0, \dots, x_n]$, we can take $S_d = \bigoplus_{a_i \geq 0, \sum a_i = d} k x_0^{a_0} \cdots x_n^{a_n}$.

Definition 12.3. An ideal I in a graded ring is *homogeneous* if it is generated by homogeneous elements.

Example 12.3.1. We can write $(x, y^3 - 3x^2) \subseteq k[x, y]$ as (x, y^3), so it is homogeneous.

Proposition 12.1. *Let R be a graded ring with ideal I. The following are equivalent:*

1. *I is homogeneous;*
2. *for any $f = \sum_{d \in \mathbb{N}} f_d \in I$ with $f_d \in R_d$, then $f_d \in I$ for all d;*
3. *$I = \bigoplus_{d \in \mathbb{N}} (I \cap R_d)$.*

Proof. Left as an exercise. The interesting implication is $(1 \Rightarrow 2)$. $\square$

Proposition 12.2. *Let I, J be homogeneous ideals of a graded ring R. Then*

1. *$I + J$, IJ, $\sqrt{I}$, and $I \cap J$ are all homogeneous;*
2. *R/I is a graded ring with $R/I = \bigoplus_{d \in \mathbb{N}} R_d / I_d$, where $I_d = I \cap R_d$.*

Proof. (1) We prove that $\sqrt{I}$ is homogeneous. Assume $f \in \sqrt{I}$, and write $f = f_0 + f_1 + \cdots + f_d$ with $f_i \in R_i$ and $f_d \neq 0$. Now there exists $n > 0$ such that $f^n \in I$, and
$$f^n = f_d^n + \text{lower order terms}.$$
Since I is homogeneous, $f_d^n \in I$, so $f_d \in \sqrt{I}$. Then $f_0 + \cdots + f_{d-1} \in \sqrt{I}$, and we can repeat.

(2) We can write $R/I = (\bigoplus_{d \in \mathbb{N}} R_d)/(\bigoplus_{d \in \mathbb{N}} (I \cap R_d))$. As abelian groups, this is $R/I \cong \bigoplus_{d \in \mathbb{N}} R_d / I_d$. One can check that the multiplication also respects the grading, so this is an isomorphism of rings. $\square$

12.3 Projective Varieties

Definition 12.4. For a set $T \subseteq k[x_0, \dots, x_n]$ of homogeneous elements, define its *vanishing locus*

$$V_p(T) := V(T) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\} \subseteq \mathbb{P}^n.$$

A *projective variety* is a subset of this form. For a homogeneous ideal $I \leq k[x_0, \dots, x_n]$, define

$$V(I) = \{x \in \mathbb{P}^n : f(x) \text{ for all } f \in I \text{ homogeneous}\}.$$

For a subset $X \subseteq \mathbb{P}^n$, define its *ideal*

$$I_p(X) := I(X) = (f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in X).$$

Note that we need to take the ideal generated by these elements, otherwise we may not get an ideal.

Remark. If $T \subseteq k[x_0, \dots, x_n]$ is a subset of homogeneous elements, then we have $V_p(T) = V_p((T))$. So projective varieties can equivalently be defined as vanishing sets of homogeneous ideals.

Example 12.4.1. Consider $X = V_p(x^2 - yz) \subseteq \mathbb{P}^2_{x:y:z}$. Set $H = V(x)$, then there is a bijection

$$\begin{aligned} U = \mathbb{P}^2 \setminus H &\xrightarrow{f} \mathbb{A}^2 \\ [1 : y : z] &\longmapsto (y, z). \end{aligned}$$

Then $f(X \cap U) = V(1 - yz)$. On the other hand, we can see that

$$X \cap H = \{[0 : 1 : 0], [0 : 0 : 1]\} = \{a, b\}.$$

If we were working with $\mathbb{C}$ with the analytic topology, then we can take limits on $V(1 - yz)$ and see

$$\lim_{t \to 0} [1 : t : 1/t] = \lim_{t \to 0} [t : t^2 : 1] = [0 : 0 : 1] = b.$$

Note that we essentially switched charts in order to take this limit. Similarly, we have

$$\lim_{t \to \infty} [1 : t : 1/t] = \lim_{t \to \infty} [1/t : 1 : 1/t^2] = [0 : 1 : 0] = a.$$

So we can see a, b as points at ∞ compactifying the curve $V(1 - yz)$.

Example 12.4.2. We have the following:

1. $V_p(0) = \mathbb{P}^n$;
2. $V_p(1) = \varnothing$;
3. if $p = [a_0 : \cdots : a_n]$ and $J = (a_i x_j - a_j x_i : 0 \leq i, j \leq n)$, then $V(J) = \{0\}$;
4. $I_0 = (x_0, \dots, x_n)$ is called the *irrelevant ideal*, which has $V_p(I_0) = \varnothing = V_p(1)$ but $I_0 = \sqrt{I_0} \subsetneq (1)$.

Lecture 13

Sept. 30 — Projective Varieties, Part 2

13.1 More on Projective Varieties

Example 13.0.1. Consider $X = V(y^2z - x^3 - zx^2 - z^3) \subseteq \mathbb{P}^2$ and $H_z = V(z)$. Let

$$\begin{aligned} U_z = \mathbb{P}^2 \setminus H_z &\xrightarrow[\text{bij}]{f} \mathbb{A}^2 \\ [x : y : 1] &\longmapsto (x, y). \end{aligned}$$

Then $f(X \cap U_z) = V(y^2 - x^3 - x^2 - 1)$, and

$$X \cap U_z = V(y^2z - x^3 - zx^2 - z^3, z) = V(x^3, z) = \{[0 : 1 : 0]\}.$$

Example 13.0.2. Let $I = (x_0, \dots, x_n)$ be the irrelevant ideal. Then I is radical, but

$$I_p(V_p(I)) = I_p(\varnothing) = (1) \neq \sqrt{I}.$$

13.2 Cones

Definition 13.1. A subset $C \subseteq \mathbb{A}^{n+1}$ is a *cone* if $0 \in C$ and $\lambda x \in C$ whenever $x \in C$ and $\lambda \in k$.

Example 13.1.1. If $X \subseteq \mathbb{P}^n$ is a projective variety, then we can set $C(X) = \pi^{-1}(X)\{0\}$, where

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [x]. \end{aligned}$$

Proposition 13.1. *If $C \subseteq \mathbb{A}^{n+1}$ is a cone, then $I_a(C) \leq k[x_0, \dots, x_n]$ is homogeneous.*

Proof. Fix $f \in I_a(C)$. Then we can write $f = \sum_{i=0}^d f_i$ with f_i homogeneous of degree i. We want to show that $f_i \in I_a(C)$ for each i. Fix $x \in C$. For any $\lambda \in k$,

$$0 = f(\lambda x) = \sum_{i=0}^d \lambda^i f_i(x).$$

Viewing this as a polynomial in λ (with x fixed), we must have each $f_i(x) = 0$. Thus $f_i \in I_a(C)$. $\square$

13.3 Projective Nullstellensatz

Theorem 13.1 (Projective Hilbert's Nullstellensatz). *We have the following:*

1. *For a projective variety $X \subseteq \mathbb{P}^n$, $V_p(I_p(X)) = X$.*
2. *For a homogeneous ideal $J \leq k[x_0, \dots, x_n]$ with $\sqrt{J} \neq (x_0, \dots, x_n)$, $I_p(V_p(J)) = \sqrt{J}$.*

As a consequence, there is a bijection between projective varieties and radical homogeneous ideals of $k[x_0, \dots, x_n]$ which are not equal to $(x_0, \dots, x_n)$, given by $X \mapsto I_p(X)$ with inverse $J \mapsto V_p(J)$.

Proof. (1) This is similar to the affine case.

(2) Fix a homogeneous ideal $(1) \neq J \leq k[x_0, \dots, x_n]$ such that $\sqrt{J} \neq (x_0, \dots, x_n)$ (the theorem is clearly true for the unit ideal). Then observe that we can write

$$\begin{aligned}
I_p(V_p(J)) &= (f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in V_p(J)) \\
&= (f \in k[x_0, \dots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J) \setminus \{0\}) \\
&= (f \in k[x_0, \dots, x_n] : f(x) = 0 \text{ for all } x \in \overline{V_a(J) \setminus \{0\}}) \\
&= \begin{cases} I_a(V_a(J)) & \text{if } V_a(J) \supsetneq \{0\}, \qquad \text{(A)} \\ I_a(\varnothing) & \text{if } V_a(J) = \{0\}, \qquad \text{(B)} \end{cases}
\end{aligned}$$

In Case A, we get that $I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J}$ by the affine Nullstellensatz. In Case B, we have $V_a(J) = \{0\}$, so $\sqrt{J} = (x_0, \dots, x_n)$, which we assumed was not the case. □

13.4 The Zariski Topology on $\mathbb{P}^n$

Remark. We have the following properties of I_p and V_p:

1. For homogeneous ideals $J_i \leq k[x_0, \dots, x_n]$ for $i \in I$, we have $V_p(\sum_{i \in I} J_i) = \bigcap_{i \in I} V_p(J_i)$;

   If $I = \{1, 2\}$, then we have $V_p(J_1 J_2) = V_p(J_1) \cup V_p(J_2)$.
2. If $X_1, X_2 \subseteq \mathbb{P}^n$ are projective varieties, then

$$I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2) \quad \text{and} \quad I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)},$$

   where we assume in the second equality that $X_1 \cap X_2 \neq \varnothing$.

The proofs are similar to the affine case.

Example 13.1.2. Let $X_1 = V(x) \subseteq \mathbb{P}^2$ and $X_2 = V(y, z) \subseteq \mathbb{P}^2$. Then $I(X_1 \cap X_1) = I(\varnothing) = (1)$, but we have $I(X_1) + I(X_2) = (x, y, z)$, which is already radical.

Definition 13.2. The *Zariski topology* on $\mathbb{P}^n$ is the topology whose closed sets are projective varieties $X \subseteq \mathbb{P}^n$ (equivalently, the vanishing loci of homogeneous ideals).

Remark. This is a topology by the above properties of I_p and V_p. We now want to relate this to the topology on our charts. Let $H_0 = V(x_0)$ and consider the bijection

$$\begin{aligned}
\mathbb{A}^n &\xrightarrow{\rho_0} \mathbb{P}^n \setminus H_0 \\
(x_1, \dots, x_n) &\longmapsto [1 : x_1 : \cdots : x_n].
\end{aligned}$$

We want to show that ρ_0 is a homeomorphism. Write $\mathbb{A}^n \subseteq \mathbb{P}^n$. Consider the ring homomorphism

$$\begin{aligned} k[x_0,\dots,x_n] &\xrightarrow{\Phi} k[x_1,\dots,x_n] \\ f(x_0,\dots,x_n) &\longmapsto f(1,x_1,\dots,x_n) =: f^i \end{aligned}$$

We call f^i the *dehomogenization* of f.

Example 13.2.1. Let $f(x) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3$, then $f^i(x) = x_2^2 - x_1^3 - x_1^2 - 1$.

Definition 13.3. If $J \le k[x_0,\dots,x_n]$ is homogeneous, then define its *dehomogenization* to be

$$J^i = (f^i : f \in J) = \Phi(J).$$

Proposition 13.2. *For $J \le k[x_0,\dots,x_n]$ homogeneous, $V_p(J) \cap \mathbb{A}^n = V_a(J^i)$.*

Proof. The idea is to use that for $[1 : x_1 : \dots : x_n] \in \mathbb{P}^n$ and $f \in k[x_0,\dots,x_n]$ homogeneous, we have $f([1 : x]) = 0$ if and only if $f^i(x) = 0$. Fill in the details as an exercise. □

Definition 13.4. If $f \in k[x_1,\dots,x_n]$ with $\deg f = d$, then define its *homogenization* to be

$$f^h = x_0^d f(x_1/x_0,\dots,x_n/x_0) \in k[x_0,x_1,\dots,x_n],$$

which is homogeneous of degree d.

Example 13.4.1. Let $f = x_2^2 - x_1^3 - x_1^2 - 1$. Then we have

$$f^h = x_0^3((x_2/x_0)^2 - (x_1/x_0)^3 - (x_1/x_0)^2 - 1) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3.$$

Remark. While $f^hg^h = (fg)^h$, note that $(f+g)^h \neq f^h + g^h$ in general.

Definition 13.5. For $J \le k[x_1,\dots,x_n]$ an ideal, define its *homogenization* to be

$$J^h = (f^h : f \in J).$$

Proposition 13.3. *For $J \le k[x_1,\dots,x_n]$ an ideal, $V_a(J) = V_p(J^h) \cap \mathbb{A}^n$.*

Proof. Left as an exercise, use that $f(a_1,\dots,a_n) = 0$ if and only if $f^h(1,a_1,\dots,a_n) = 0$. □

Remark. The above results imply that $\rho_0 : \mathbb{A}^n \to \mathbb{P}^n \setminus H_0$ is a homeomorphism.

Lecture 14

Oct. 9 — Projective Space as Varieties

14.1 More on the Zariski Topology on $\mathbb{P}^n$

Proposition 14.1. *For each $0 \leq i \leq n$, the map*

$$\begin{aligned} U_i = \mathbb{P}^n \setminus V(x_i) &\xrightarrow{h_i} \mathbb{A}^n \\ [x_0 : \cdots : x_n] &\longmapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i) \end{aligned}$$

is a homeomorphism.

Proof. The main inputs to the proof are

- For $I \leq k[x_0, \ldots, x_n]$ homogeneous, $h_0(V(I) \cap U_0) = V(I^i)$.
- For $J \leq k[x_1, \ldots, x_n]$, $h_0^{-1}(V(J)) = V(J^h)$.

Fill in the remaining details as an exercise. □

Proposition 14.2 (Projective closure)**.** *For $J \leq k[x_1, \ldots, x_n]$ and $X = V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$, we have*

$$\overline{X} = V_p(J^h).$$

Proof. See Gathmann. □

Proposition 14.3. *If $X = V_a(f) \subseteq \mathbb{A}^n$ with $f \in k[x_1, \ldots, x_n]$, then its projective closure in $\mathbb{P}^n$ is*

$$\overline{X} = V_p(f^h).$$

Proof. We know that $\overline{X} = V_p(\langle f \rangle^h)$ by Proposition 14.2. Now

$$\langle f \rangle^h = \langle (fg)^h : g \in k[x_1, \ldots, x_n] \rangle = \langle f^h g^h : g \in k[x_1, \ldots, x_n] \rangle = \langle f^h \rangle,$$

which implies the desired result. □

Example 14.0.1 (Twisted cubic)**.** Take $X = \mathrm{Im}(\mathbb{A}^1 \to \mathbb{A}^3 : t \mapsto (t, t^2, t^3))$. Note that $X \cong \mathbb{A}^1$, and

$$I_a(X) = (x^2 - y, x^3 - z) = (x^2 - y, x^3 - z, xy - z).$$

Then one can check that $\overline{X} \subseteq \mathbb{P}^3_{w:x:y:z}$ is given by $\overline{X} = V_p(x^2 - yw, x^3 - zw^2, xy - zw)$. However, one can also check that $\overline{X}$ cannot be cut out by 2 equations. For example,

$$V_p(x^2 - yw, x^3 - zw^2) = \overline{X} \cup V(w, x).$$

14.2 Projective Space as Varieties

Remark. Our goal now is to show that projective varieties are varieties. The first step is to define a sheaf of regular functions on $\mathbb{P}^n$.

Definition 14.1. Let U be an open set of a projective variety $X \subseteq \mathbb{P}^n$. A function $\varphi : U \to k$ is *regular* if for every $p \in U$, there exists $d \in \mathbb{N}$, $f, g \in k[x_0, \dots, x_n]$ homogeneous of degree d, and $U_p \subseteq U$ open such that

$$\varphi(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in U_p.$$

Remark. If $X \subseteq \mathbb{P}^n$ is a projective variety, then

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}$$

is a sheaf of rings on X. Again this is because the regular condition can be checked locally.

Proposition 14.4. *If $X \subseteq \mathbb{P}^n$ is a projective variety, then $(X, \mathcal{O}_X)$ is a pre-variety.*

Proof. Let $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$. It suffices to show $(X_i, \mathcal{O}_X|_{X_i})$ is an affine variety for each $0 \le i \le n$. For simplicity, assume $i = 0$. Let $J = I(X) \le k[x_0, \dots, x_n]$ and $Z_0 = V(J^i) \subseteq \mathbb{A}^n$. We have seen before that we have a homeomorphism

$$\begin{aligned} X_0 &\xrightarrow{F} Z_0 \\ [x_0 : \cdots : x_n] &\longmapsto (x_1/x_0, \dots, x_n/x_0). \end{aligned}$$

We claim that F induces an isomorphism of ringed spaces $(X_0, \mathcal{O}_X|_{X_0}) \cong (Z_0, \mathcal{O}_{Z_0})$. To see this, we need to check that regular functions pull back to regular functions via F and F^{-1}. A regular function on an open set of X_0 is locally of the form

$$\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)}$$

with f, g homogeneous of the same degree. Now

$$(F^{-1})^* \left(\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)} \right) = \frac{f(1, x_1, \dots, x_n)}{g(1, x_1, \dots, x_n)},$$

which is a fraction of polynomials and hence regular on Z_0. So F^{-1} pulls regular functions back to regular functions. Conversely, a regular function on Z_0 is locally given by

$$\frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)},$$

and its pullback via F is

$$F^* \left(\frac{q(x_1, \dots, x_n)}{r(x_1, \dots, x_n)} \right) = \frac{q(x_1/x_0, \dots, x_n/x_0)}{r(x_1/x_0, \dots, x_n/x_0)} = \frac{x_0^d q(x_1/x_0, \dots, x_n/x_0)}{x_0^d r(x_1/x_0, \dots, x_n/x_0)},$$

where $d = \max\{\deg q, \deg r\}$. This is regular on X_0, so F also pulls regular functions back to regular functions. So we get an isomorphism of ringed spaces, as desired. $\square$

Example 14.1.1. $\mathbb{P}^n$ is a pre-variety, and $\mathbb{P}^n \setminus V(x_i) =: U_i \cong \mathbb{A}^n$ as pre-varieties.

Definition 14.2. A *morphism* of projective varieties is a morphism of the underlying pre-varieties.

Remark. For a projective variety X, it will be convenient to work with "global coordinates," i.e.

$$S(X) := k[x_0, \dots, x_n]/I_p(X).$$

This is called the *homogeneous coordinate ring.* Note the following:

1. For $f \in S(X)$ homogeneous, f is not necessarily a well-defined function on X. But
$$V(f) = \{[x] \in X : f(x) = 0\}$$
is still well-defined.

2. A relative version of the projective Nullstellensatz holds: There is a bijection
$$\left\{\begin{matrix}\text{projective subvarieties}\\ Y \subseteq X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{radical homogeneous ideals in } S(X)\\ \text{not equal to } (\overline{x}_1, \dots, \overline{x}_n)\end{matrix}\right\}$$
$$Y \longmapsto I(Y)$$
$$V(J) \longleftarrow J$$

where $I(Y) = \langle f \in S(X) : f \text{ homogeneous and } f(y) = 0 \text{ for all } y \in Y\rangle$.

Lemma 14.1. *If $X \subseteq \mathbb{P}^n$ and $f_0, \dots, f_m \in S(X)$ are homogeneous of the same degree, then*

$$\begin{aligned} U = X \setminus V(f_0, \dots, f_m) &\xrightarrow{f} \mathbb{P}^m \\ [x_0 : \dots : x_n] &\longmapsto [f_0(x) : \dots : f_m(x)] \end{aligned}$$

is a morphism.

Proof. To see that f is well-defined, note that for $[a_0 : \dots : a_n] \in X \setminus V(f_0, \dots, f_m)$, we have

$$(f_0(\lambda a), \dots, f_m(\lambda a)) = \lambda^d (f_0(a), \dots, f_m(a))$$

with $d = \deg f_i$. So $[f_0(a) : \dots : f_m(a)] \in \mathbb{P}^m$ is well-defined. To see that f is a morphism, we check locally on $\mathbb{P}^m$. Let $V_i = \mathbb{P}^m \setminus V(x_i)$ and $U_i = f^{-1}(V_i)$. Then

$$\begin{aligned} U_i &\xrightarrow{f|_{U_i}} V_i \cong \mathbb{A}^m \\ a &\longmapsto \left(\frac{f_0(a)}{f_i(a)}, \dots, \frac{\widehat{f_i(a)}}{f_i(a)}, \dots, \frac{f_m(a)}{f_i(a)}\right). \end{aligned}$$

Since each f_j/f_i is regular, $f|_{U_i}$ is a morphism. So f is a morphism. ☐

Example 14.2.1. Define a map

$$\begin{aligned} \mathbb{P}^1_{s:t} &\xrightarrow{f} \mathbb{P}^3_{x:y:z} \\ [s : t] &\longmapsto [s^3 : s^2 t : s t^2 : t^3]. \end{aligned}$$

Then $S(\mathbb{P}^1) = k[s, t]$ and $f(\mathbb{P}^1)$ is the projective twisted cubic in $\mathbb{P}^3$.

Example 14.2.2. Let $A \in \mathrm{GL}_{n+1}(k)$. Then

$$\begin{aligned} f_A : \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

is an isomorphism with inverse $f_{A^{-1}}$. We will see later that we have a surjective group homomorphism

$$\begin{aligned} \mathrm{GL}_{n+1}(k) &\longrightarrow \mathrm{Aut}(\mathbb{P}^n) \\ A &\longmapsto f_A \end{aligned}$$

with kernel $k^\times I$. So we get $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{GL}_{n+1}(k)/k^\times I =: \mathrm{PGL}_{n+1}(k)$.

Example 14.2.3 (Conics)**.** Let $f \in k[x,y,z]$ be homogeneous of degree 2, and write

$$f = (x,y,z)B(x,y,z)^T$$

with B a symmetric 3×3 matrix. We want to characterize $X = V(f)$. Choose $A \in \mathrm{GL}_3(k)$ such that

$$B' = ABA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $f' = (x,y,z)B'(x,y,z)^T$ has $f' = x^2 + y^2 + z^2$, $x^2 + y^2$, or x^2. Now A induces an isomorphism $h_{A^{-1}} : \mathbb{P}^2 \to \mathbb{P}^2$ and $g := h_{A^{-1}}|_X : X \to h_{A^{-1}}(X) = V(f')$, so any projective conic is isomorphic to

$$V(x^2 + y^2 + z^2), \quad V(x^2 + y^2), \quad \text{or} \quad V(x^2).$$

Example 14.2.4 (Projections)**.** Let $a = [1 : 0 : \cdots : 0]$ and define

$$\begin{aligned} \mathbb{P}^n \setminus \{a\} &\xrightarrow{f} \mathbb{P}^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_1 : \cdots : x_n]. \end{aligned}$$

Geometrically, if we fix $[b] \in \mathbb{P}^n \setminus \{a\}$ and set

$$\ell_{a,b} = \{[s : tb_1 : \cdots : tb_n] : (s,t) \in k^2 \setminus \{0\}\} = \text{the line through } a \text{ and } b,$$

then $\ell_{a,b} \cap V(x_0) = [0 : b_1 : \cdots : b_n] = [0 : f(b)]$.

Lecture 15

Oct. 14 — Projective Space as Varieties, Part 2

15.1 Example of Projective Morphism

Example 15.0.1 (Projections, continued)**.** Let $H \subseteq \mathbb{P}^n$ be a hyperplane and $p \notin H$. Then we can define

$$\begin{aligned} \mathbb{P}^n \setminus \{p\} &\xrightarrow{\pi} H \cong \mathbb{P}^{n-1} \\ q &\longmapsto \text{intersection point of } H \text{ and } \overline{pq}. \end{aligned}$$

For example, when $n = 2$, $p = [1 : 0 : 0] \in \mathbb{P}^2_{x_0:x_1:x_2}$, and $H = V(x_0)$, then we have

$$\begin{aligned} \mathbb{P}^2 \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2]. \end{aligned}$$

Note that this does not extend to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. But if we let $X = V(x_0x_1 - x_2^2) \subseteq \mathbb{P}^2$, then the restriction of the above morphism to X:

$$\begin{aligned} X \setminus \{p\} &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto [x_1 : x_2]. \end{aligned}$$

does extend to a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1 \\ [x_0 : x_1 : x_2] &\longmapsto \begin{cases} [x_1 : x_2] & \text{if } [x] \neq [1 : 0 : 0], \\ [x_2 : x_0] & \text{if } [x] \neq [1 : 1 : 0]. \end{cases} \end{aligned}$$

15.2 The Segre Embedding

Remark. We now want to show that projective varieties are varieties, and understand and analogue of compactness in algebraic geometry. To do this, we will need to understand products.

Definition 15.1. Fix $m, n \geq 0$. The *Segre embedding* is the map $\Sigma : \mathbb{P}^m_{x_i} \times \mathbb{P}^n_{y_i} \to \mathbb{P}^N_{z_{i,j}}$ given by

$$\Sigma([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [x_iy_j : 0 \leq i \leq m, 0 \leq j \leq n],$$

where $N = (m + 1)(n + 1) - 1$.

Proposition 15.1. *Let Σ be the Segre embedding. Then*

1. $X = \Sigma(\mathbb{P}^m \times \mathbb{P}^n) = V(z_{i,j}z_{k,\ell} - z_{i,\ell}z_{k,j} : 0 \le i,k \le m,\ 0 \le j,\ell \le n)$.
2. *The map $\mathbb{P}^m \times \mathbb{P}^n \to X$ is an isomorphism, i.e. Σ is a closed embedding.*

Proof. (0) First one can check that Σ is a morphism. To do this, restrict to charts.

(1) Fix $[a_{i,j}] \in \mathbb{P}^N$. Then $[a_{i,j}] \in \operatorname{Im}\Sigma$ if and only if the matrix $(a_{i,j})$ has rank 1, which occurs if and only if all 2×2 minors of $(a_{i,j})$ vanish, which happens if and only if

$$a_{i,j}a_{k,\ell} - a_{i,\ell}a_{k,j} = 0$$

for all i, j, k, ℓ for which the above equation makes sense.

(2) We define a morphism $X \to \mathbb{P}^m \times \mathbb{P}^n$ that will be inverse to Σ. Set $U_{i,j} = X \cap \{z_{i,j} \neq 0\}$. Define

$$\begin{aligned} U_{i,j} &\xrightarrow{h_{i,j}} \mathbb{P}^m \times \mathbb{P}^n \\ [z_{i,j}] &\longmapsto ([z_{0,j} : \cdots : z_{m,j}], [z_{i,0} : \cdots : z_{i,n}]). \end{aligned}$$

Using the definition of X (as the set of rank 1 matrices up to scaling), these glue to give a morphism $X \to \mathbb{P}^m \times \mathbb{P}^n$ that is inverse to Σ. $\square$

Example 15.1.1. Let $m = n = 1$. Then the Segre embedding is given by

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3_{x:y:z:w} \\ ([a_0 : a_1], [b_0 : b_1]) &\longmapsto \begin{bmatrix} a_0b_0 & a_0b_1 \\ a_1b_0 & a_1b_1 \end{bmatrix}. \end{aligned}$$

Then $\operatorname{Im}\Sigma = V(xw - yz)$. Observe that the images of $\{a\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{b\}$ in $\Sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ are two families of lines, where the lines within each families do not intersect.

Remark. The following are consequences of the Segre embedding.

1. We can study products of projective varieties.

   Definition 15.2 (Redefinition of projective variety)**.** A *projective variety* is a (pre-)variety X such that there exists a closed embedding $X \hookrightarrow \mathbb{P}^n$ for some $n \ge 0$.

   Now using the Segre embedding, we get that $\mathbb{P}^m \times \mathbb{P}^n$ is a projective variety. Moreover, if $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ are projective varieties, then so is $X \times Y$.

2. We can show that $\mathbb{P}^n$ is separated.

   Lemma 15.1. *$\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.*

   Proof. Observe that

   $$\Delta_{\mathbb{P}^n} = \{([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \in \mathbb{P}^n \times \mathbb{P}^n : x_iy_j - x_jy_i = 0 \text{ for all } 0 \le i, j \le n\}.$$

   It suffices to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. There are two ways to see this. The first is to use the Segre embedding $\mathbb{P}^n_{x_i} \times \mathbb{P}^n_{y_i} \underset{\text{cl}}{\overset{\Sigma}{\hookrightarrow}} \mathbb{P}^N_{z_{i,j}}$ Then we can write

   $$\Sigma(\Delta_{\mathbb{P}^n}) = \Sigma(\mathbb{P}^n \times \mathbb{P}^n) \cap V(z_{i,j} - z_{j,i} : 0 \le i, j \le n),$$

   which is closed in $\mathbb{P}^N$, so $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. Alternatively, one can just compute $\Delta_{\mathbb{P}^n}$ directly on the affine charts. One can fill in the details of this method as an exercise. $\square$

Proposition 15.2. *Projective varieties are varieties.*

Proof. We have already seen that they are pre-varieties, so it suffices to show that they are separated. By Lemma 15.1, $\mathbb{P}^n$ is separated. Thus any closed sub-prevariety of $\mathbb{P}^n$ is also separated. ☐

15.3 Completeness

Remark. We now want an analogue of compactness in algebraic geometry. One issue is that all varieties are compact to begin with, but $\mathbb{A}^n$ has points missing in some sense.

Example 15.2.1. Consider the projection map

$$\begin{aligned} \mathbb{A}^1 \times \mathbb{A}^1 &\xrightarrow{\mathrm{pr}_2} \mathbb{A}^1 \\ (x,t) &\longmapsto t. \end{aligned}$$

Then $X = V(xt-1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ is closed, but $\mathrm{pr}_2(X) = \mathbb{A}^1 \setminus \{0\}$. If we instead viewed this over $\mathbb{P}^1$:

$$\begin{aligned} \mathbb{P}^1_{[x:y]} \times \mathbb{A}^1_t &\xrightarrow{\mathrm{pr}_2} \mathbb{A}^1_t \\ ([x:y],t) &\longmapsto t \end{aligned}$$

with $\overline{X} = V(xt-y)$, then $\mathrm{pr}_2(\overline{X}) = \mathbb{A}^1$ as there is a point $([1:0],0)$ at infinity in $\overline{X}$. In other words, "compactifying" $\mathbb{A}^1$ to $\mathbb{P}^1$ gives the desired missing point.

Definition 15.3. A morphism $f : X \to Y$ is *closed* if $f(Z)$ is closed in Y for all closed sets $Z \subseteq X$.

Definition 15.4. A variety X is *complete* if the projection

$$\mathrm{pr}_2 : X \times Y \longrightarrow Y$$

is closed for all varieties Y.

Remark. The same definition for topological spaces gives the usual notion of compactness.

Example 15.4.1. Example 15.2.1 shows that $\mathbb{A}^1$ is not complete. Similar examples show that $\mathbb{A}^n$ is not complete for any $n \geq 1$.

Proposition 15.3. $\mathbb{P}^n$ *is complete.*

Proof. The steps to show this are the following:

1. For any $m, n \geq 0$, the projection $\mathrm{pr}_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

   See Gathmann for a proof of this fact.

2. If Y is an affine variety, then $\mathrm{pr}_2 : \mathbb{P}^n \times Y \to Y$ is closed.

   To see this, write $Y = V(I) \subseteq \mathbb{A}^m \subseteq \mathbb{P}^m$ and consider the diagram

$$\begin{tikzcd}
\mathbb{P}^n \times \mathbb{P}^m \arrow[r, "\mathrm{pr}_2"] & \mathbb{P}^m \\
\mathbb{P}^n \times \mathbb{A}^n \arrow[u, hook, "\mathrm{op}"] \arrow[r, "\mathrm{pr}_2"] & \mathbb{A}^n \arrow[u, hook] \\
\mathbb{P}^n \times Y \arrow[u, hook, "\mathrm{cl}"] \arrow[r, "\mathrm{pr}_2"] & Y \arrow[u, hook]
\end{tikzcd}$$

Since the top row is closed, so is the bottom row.

Finally, we complete the proof. If Y is a variety, then it admits an open affine cover $Y = \bigcup_{i=1}^r U_i$. Now $\mathrm{pr}_2 : \mathbb{P}^n \times Y \to Y$ is closed when restricted to $\mathrm{pr}_2^{-1}(U_i)$. Since closedness of a map can be checked on an open cover of the target, we see that $\mathbb{P}^n \times Y \to Y$ is closed. $\square$

Remark. The same definition and arguments for completeness work if Y is replaced by a pre-variety.

Exercise 15.1. Show that if X is a complete variety, then so is any closed subvariety of X.

Corollary 15.0.1. *Any projective variety is complete.*

Lecture 16

Oct. 16 — Completeness and Embeddings

16.1 More on Completeness

Example 16.0.1. Recall from before that we have:

1. $\mathbb{A}^n$ is not complete for $n \geq 1$.
2. $\mathbb{P}^n$ is complete.
3. Any projective variety is complete.

Remark. Note the following:

1. If $k = \mathbb{C}$, then a variety is complete if and only if X^{an} is compact in the analytic topology.
2. *Nagata's compactification theorem*: Any variety X admits an open embedding $X \hookrightarrow \overline{X}$ with $\overline{X}$ complete.
3. In dimension 1, completeness is equivalent to being projective. In dimension ≥ 2, being projective implies completeness, but the converse may fail.

Proposition 16.1. *If $f : X \to Y$ is a morphism of varieties with X complete, then*

1. *$f(X)$ is closed in Y.*
2. *$f(X)$ is complete.*

Proof. (1) Consider the projection $\text{pr}_2 : X \times Y \to Y$. As Y is separated, the graph Γ_f of f is closed in $X \times Y$. Now $f(X) = \text{pr}_2(\Gamma_f)$, which is closed in Y as X is complete.

(2) Fix any variety Z. Now consider the projection $\pi' : f(X) \times Z \to Z$ and $W \subseteq f(X) \times Z$ closed. Now consider the projection $\pi : X \times Z \to Z$. Then $\pi'(W) = \pi((f, \text{id})^{-1}(W))$. The set $(f, \text{id})^{-1}(W)$ is closed in $X \times Z$ by continuity, and $\pi'(W)$ is closed in Z as X is complete. So $f(X)$ is complete. $\square$

Corollary 16.0.1. *If X is a complete variety that is connected, then any $\varphi \in \mathcal{O}_X(X)$ is constant. In particular, $\mathcal{O}_X(X) \cong k$.*

Proof. Any $\varphi \in \mathcal{O}_X(X)$ induces a morphism $f : X \to \mathbb{P}^1_{s:t}$ by $x \mapsto [1 : \varphi(x)]$. By construction, we have $f(X) \subseteq \{s \neq 0\} \cong \mathbb{A}^1$. As X is complete, $f(X)$ is closed, and as X is connected, $f(X)$ is connected. Since the only proper closed subsets of $\mathbb{P}^1$ are points, $f(X)$ be must a single point $[1 : a]$ since $f(X)$ is connected. So $\varphi(x) = a$ for every $x \in X$. $\square$

Corollary 16.0.2. *The only complete affine varieties are finite point sets.*

Proof. Assume $X = V(I) \subseteq \mathbb{A}^n$ is complete. Using the decomposition of X into connected components (there finitely many since X is Noetherian), we may reduce to the case that X is connected. Then $A(X) = \mathcal{O}_X(X) \cong k$, but this happens if and only if X is a single point (see Homework 1). □

16.2 The Veronese Embedding

Remark. If $X \subseteq \mathbb{P}^n$ is a projective variety, which open sets in X are affine?

- $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$ is affine.
- $X \setminus (\mathbb{P}^n \setminus H)$ with $H \subseteq \mathbb{P}^n$ a hyperplane is affine.

We will see that $X \cap (\mathbb{P}^n \setminus V(g))$ is affine with $g \in k[x_0, \dots, x_n]$ homogeneous of degree $d > 0$.

Definition 16.1 (Veronese embedding, d-tuple embedding)**.** Fix $n, d > 0$. Let $f_0, \dots, f_N \in k[x_0, \dots, x_n]$ denote the monomials of degree d, where $N = \binom{n+d}{d} - 1$. The *Veronese embedding* $\nu_{n,d}$ is the map

$$\begin{aligned} \nu_{n,d} : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ x &\longmapsto [f_0(x) : \cdots : f_N(x)]. \end{aligned}$$

Example 16.1.1. Let $n = 1$, $d = 3$. Then the degree-3 Veronese embedding is given by

$$\begin{aligned} f : \mathbb{P}^1 &\longmapsto \mathbb{P}^3_{s:t:u:v} \\ [x : y] &\longmapsto [x^3 : x^2y : xy^2 : y^3]. \end{aligned}$$

Then $X = f(\mathbb{P}^1) = V(sv - tu, t^2 - su, u^2 - vt)$. We can define an inverse

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1 \\ [s : t : u : v] &\longmapsto \begin{cases} [1 : t/s] & \text{if } s \neq 0, \\ [u/v : 1] & \text{if } v \neq 0. \end{cases} \end{aligned}$$

Note that $ut = sv$ on X when $sv \neq 0$, so this is well-defined.

Proposition 16.2. $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ *is a closed embedding.*

Proof. As $\mathbb{P}^n$ is complete, $X = \nu_{n,d}(\mathbb{P}^n)$ is closed in $\mathbb{P}^N$. A similar computation as in Example 16.1.1 shows that $\nu_{n,d} : \mathbb{P}^n \to X$ has an inverse. So $\nu_{n,d}$ is a closed embedding. □

Remark. Note the following:

1. With some work, one can show that $\nu_{n,d}(\mathbb{P}^n)$ can be cut out by quadratic equations
$$\{z_i z_j - z_k z_\ell : f_i f_j = f_k f_\ell \text{ as monomials}\}.$$
2. If $g \in k[x_0, \dots, x_n]$ is homogeneous of degree $d > 0$, then
$$\mathbb{P}^n \supseteq V(g) = \nu_{n,d}^{-1}(H)$$
for some hyperplane $H \subseteq \mathbb{P}^N$.

Theorem 16.1. *If $X \subseteq \mathbb{P}^n$ is a projective variety, then for any $g \in k[x_0, \dots, x_n]$ homogeneous of degree d, the variety $X \setminus V(g)$ is affine.*

Proof. Consider the Veronese embedding $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$. Then $V(g) = \nu_{n,d}^{-1}(H)$ for some hyperplane $H \subseteq \mathbb{P}^N$. So $X \setminus V(g) \cong \nu_{n,d}(X) \setminus H$, which is affine. $\square$

16.3 The Grassmannian

Definition 16.2. For $0 \leq d \leq n$, define the *Grassmannian*

$$G(d, n) := \{d\text{-dimensional subspaces of } k^n\}.$$

Example 16.2.1. We have the following:

- $G(0, n)$ and $G(n, n)$ are just points.
- $G(1, n) = \mathbb{P}^{n-1}$ set-theoretically.
- $G(d, n) \cong G(n-d, n)$ as dimension d subspaces of $V = k^n$ are in bijection with dimension $n - d$ subspaces of V^*, where $W \subseteq V$ corresponds to $\ker(V^* \to W^*)$.
- $G(2, n) = \{\text{lines in } \mathbb{P}^{n-1}\}$.

Theorem 16.2. *The Grassmannian $G(d, n)$ can be endowed with the structure of a (projective) variety of dimension $d(n-d)$.*

Proof. One strategy is to let $V = k^n = \operatorname{span}\{e_1, \dots, e_n\}$. We sketch this idea. Observe that

1. A d-dimensional subspace $W \subseteq V$ can be represented by a $d \times n$ matrix A of rank d (choose a basis of W and write the coordinates with respect to the basis $\{e_1, \dots, e_n\}$ of V). Note that A is unique up to the action by $\mathrm{GL}_d(k)$, so we get a point $[A] \in G(d, n)$.
2. For $I = \{1, \dots, n\}$ with $|I| = d$, define the set

   $$U_I = \{[A] \in G(d, n) : \det(A_I) \neq 0\},$$

   where A_I denotes the I-th $d \times d$ minor of A. One can check that the condition $\det(A_I) \neq 0$ is well-defined (i.e. $\det((BA)_I) = \det(B)\det(A_I)$ for $B \in \mathrm{GL}_d(k)$).

   Then we have a bijection $\mathbb{A}^{d(n-d)} \to U_I$ given as follows. When $I = \{1, \dots, d\}$, define

   $$\begin{aligned} \mathbb{A}^{d(n-d)} &\longrightarrow U_I \\ C &\longmapsto [I_d \mid C]. \end{aligned}$$

   One can make a similar definition for other I. Note that $G(d, n) = \bigcup_I U_I$.
3. Show that the U_I glue to give $G(d, n)$ the structure of a variety.

The second strategy is to use the wedge product $\Lambda^d V$. Recall that

- $\Lambda^d V$ has basis given by $e_I := e_{i_1} \wedge \cdots \wedge e_{i_d}$ with $I = \{i_1 < \cdots < i_d\} \subseteq \{1, \dots, n\}$.
- For $v_1, \dots, v_d \in V$ with $v_i = \sum_{j=1}^d a_{i,j} e_j$, we have $v_1 \wedge \cdots \wedge v_d = \sum_{I \subseteq \{1,\dots,n\}} \det(a_I) e_I$.

The following are two linear algebra lemmas we will use:

Lemma 16.1. *Let $v_1, \ldots, v_d \in V$. Then $v_1, \ldots, v_d$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_d \neq 0$.*

Proof. Use the determinant formula. □

Lemma 16.2. *For linearly independent sets $\{v_1, \ldots, v_d\}, \{w_1, \ldots, w_d\} \subseteq V$, then $v_1 \wedge \cdots \wedge v_d$ and $w_1 \wedge \cdots \wedge w_d$ are linearly dependent in $\Lambda^d V$ if and only if* $\operatorname{span}\{v_1, \ldots, v_d\} = \operatorname{span}\{w_1, \ldots, w_d\}$.

Proof. ($\Leftarrow$) It suffices to show that $kv_1 \wedge \cdots \wedge v_d \subseteq \Lambda^d V$ is preserved under change of basis operations. Check this as an exercise.

($\Rightarrow$) Assume that $v_1 \wedge \cdots \wedge v_d = \lambda w_1 \wedge \cdots \wedge w_d$ for $\lambda \neq 0$. Then

$$w_i \wedge v_1 \wedge \cdots \wedge v_d = 0,$$

so $w_i \in \operatorname{span}\{v_1, \ldots, v_d\}$. So we have $\operatorname{span}\{w_1, \ldots, w_d\} \subseteq \operatorname{span}\{v_1, \ldots, v_d\}$, and by symmetry, the reverse inclusion holds as well. □

So given a d-dimensional vector subspace $\operatorname{span}\{v_1, \ldots, v_d\} = W \subseteq V$, we get a 1-dimensional subspace $k_1 v_1 \wedge \cdots \wedge v_d = \Lambda^d W \subseteq \Lambda^d V$. By Lemma 16.2, there is an injection

$$\begin{aligned} p_{n,d} : G(d,n) &\hookrightarrow \mathbb{P}^{\binom{n}{d}-1} \\ W = \operatorname{span}\{v_1, \ldots, v_d\} &\longmapsto kv_1 \wedge \cdots \wedge v_d \end{aligned}$$

This is called the *Plücker embedding*. It remains to show $p_{n,d}$ is closed, which is Corollary 17.0.1. □

Lecture 17

Oct. 21 — The Grassmannian

17.1 Plücker Coordinates

Remark. Consider coordinates on $\mathbb{P}^{\binom{n}{d}-1}$ with respect to

$$\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} : I = \{i_1 < \cdots < i_d\} \subseteq \{1, \ldots, n\}\}.$$

So when $d = 2$, $n = 3$, and $W = \operatorname{span}\{e_1 + e_2, e_1 + e_3\}$, then

$$p_{n,d}(W) = k(e_1 \wedge e_3 - e_1 \wedge e_2 + e_2 \wedge e_3) = [1 : -1 : 1] \in \mathbb{P}^2.$$

Note that $p_{n,d}(W)$ encodes the determinants of the 2×2 minors of

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Remark. We will now try to describe the image of $p_{n,d} : G(d, n) \to \mathbb{P}^{\binom{n}{d}-1}$. Note that $0 \neq \omega \in \Lambda^d V$ is in the image if and only if ω is *pure*, i.e. $\omega = v_1 \wedge \cdots \wedge v_d$ for some $v_1, \ldots, v_d \in V$.

Proposition 17.1. *For $0 \neq \omega \in \Lambda^d V$, the map*

$$\begin{aligned} g : V &\longrightarrow \Lambda^{d+1} V \\ v &\longmapsto w \wedge v \end{aligned}$$

has rank $\geq n - d$. Furthermore, ω is pure if and only if the rank is exactly $n - d$.

Proof. After a change of basis, we may assume that $\ker g = \operatorname{span}\{e_1, \ldots, e_r\}$ with $1 \leq r \leq n$. Write

$$\omega = \sum_{I \subseteq \{1, \ldots, n\}} a_I e_I.$$

Note that $e_i \wedge \omega = 0$ for $1 \leq i \leq r$, so $a_I = 0$ if there exists $1 \leq i \leq r$ with $i \notin I$. So

$$\omega = \sum_{\substack{\{1, \ldots, r\} \subseteq I \subseteq \{1, \ldots, n\} \\ |I| = d}} a_I e_I.$$

As $\omega \neq 0$, we have $r \leq d$. So $\operatorname{rk} g = n - r \geq n - d$. Furthermore, if $r = d$, then $\omega = a_{\{1, \ldots, d\}} e_1 \wedge \cdots \wedge e_d$. Conversely, if ω is pure, a similar computation shows that $\operatorname{rk} g = n - d$. $\square$

Example 17.0.1. Let $n = 3$, $d = 2$, $\omega \in \Lambda^2 V$. Consider a map

$$\begin{aligned} g : V &\longrightarrow \Lambda^3 V \\ v &\longmapsto \omega \wedge v. \end{aligned}$$

Then $\operatorname{rk} g \leq 1$. By Proposition 17.1, $\operatorname{rk} g = 1$ and ω is pure.

Example 17.0.2. Let $n = 4$, $d = 2$, and $\omega = e_1 \wedge e_2$. Then

$$g(a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4) = a_3 e_1 \wedge e_2 \wedge e_3 + a_4 e_1 \wedge e_2 \wedge e_4.$$

So we have $\operatorname{rk} g = 2 = 4 - 2$.

Corollary 17.0.1. *The image of $p_{n,d} : G(d,n) \to \mathbb{P}^{\binom{n}{d}-1}$ is a closed set. In particular, this endows $G(d,n)$ with the structure of an algebraic variety.*

Proof. Let $[\omega] \in \mathbb{P}^{\binom{n}{d}-1}$ where $0 \neq \omega = \sum_{|I|=d} b_I e_I$. Then $[\omega]$ is in $\operatorname{Im} p_{n,d}$ if and only if $\omega \in \Lambda^d V$ is pure, which happens if and only if the map

$$\begin{aligned} g_\omega : V &\longrightarrow \Lambda^{d+1} V \\ \sum a_i e_i &\longmapsto \sum_{|I|=d} b_I e_I \wedge \sum a_i e_i \end{aligned}$$

has rank $n - d$. Note that g_ω is given by some matrix in the b_I. This matrix has rank $n - d$ if and only if the $n - d + 1$ minors of this matrix vanish (we already know the rank is at least $n - d$ by Proposition 17.1). These give equations cutting out $\operatorname{Im} p_{n,d}$ in $\mathbb{P}^{\binom{n}{d}-1}$, so $\operatorname{Im} p_{n,d}$ is closed in $\mathbb{P}^{\binom{n}{d}-1}$. $\square$

Example 17.0.3. Let $n = 3$, $d = 2$. As all $\omega \in \Lambda^2 V$ are pure, $G(2,3) \to \mathbb{P}^{\binom{n}{d}-1}$ is an isomorphism.

17.2 Charts of the Grassmannian

Remark. For $I \subseteq \{1, \dots, n\}$ with $|I| = d$, let

$$U_I \subseteq G(d,n) \subseteq \mathbb{P}^{\binom{n}{d}-1}$$

denote the open set on which the e_I coordinate does not vanish.

Proposition 17.2. *$U_I \cong \mathbb{A}^{d(n-d)}$.*

Proof. After a change of coordinates, let $I = \{n-d+1, n-d+2, \dots, n\}$. Then define a map

$$\begin{aligned} h_I : \mathbb{A}^{d(n-d)} &\longrightarrow U_I \subseteq \mathbb{P}^{\binom{n}{d}-1} \\ A = (a_{i,j}) &\longmapsto p_{n,d}([A \mid I_d]), \end{aligned}$$

where A is a $d \times (n-d)$ matrix and I_d is the $d \times d$ identity matrix. This map is a morphism, and h_I has inverse given by

$$h_I^{-1}\Big(\Big[\sum_{|J|=d} b_J e_J\Big]\Big) = (\pm b_{(I \setminus \{i\}) \cup \{j\}} / b_I)_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n-d}}.$$

We can see that this is also a morphism, so h_I is an isomorphism $\mathbb{A}^{d(n-d)} \to U_I$. $\square$

Remark. Note that $G(d,n) = \bigcup_{|I|=d} U_I$. This gives us the following:

1. $\dim G(d,n) = d(n-d)$.
2. $G(d,n)$ is irreducible, as U_I is irreducible and $U_I \cap U_J \neq \varnothing$ (for the latter, it suffices to find some $d \times n$ matrix A whose I and J minors are both nonzero).

Example 17.0.4. We have $G(2,4) \hookrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$. Note that $\dim G(2,4) = 4$ and $\dim \mathbb{P}^5 = 5$. One can additionally show that $G(2,4)$ is cut out by a single equation via the *Plücker relations*.

17.3 Birational Maps

Remark. Many non-isomorphic varieties are isomorphic on an open set. Consider:

1. $\mathbb{P}^{n+m}$ and $\mathbb{P}^n \times \mathbb{P}^m$ both contain $\mathbb{A}^{n+m}$ as a dense open set.
2. $G(d,n)$ and $\mathbb{P}^{d(n-d)}$ both contain $\mathbb{A}^{d(n-d)}$ as a dense open set.
3. Consider the curve defined by

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow C = V(y^2 - x^3) \subseteq \mathbb{A}^2_{x,y} \\ t &\longmapsto (t^2, t^3). \end{aligned}$$

   Note that f is injective but not an isomorphism, as

$$f^{-1}(x,y) = \begin{cases} y/x & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

   which is not regular at the origin. But this does tell us that $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0,0)\}$.

So we want to study maps defined on an open set.

Definition 17.1. Let X and Y be irreducible varieties. Then a *rational map* $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ with $U \subseteq Y$ a nonempty open set, up to the equivalence $f_1 : U_1 \to Y \sim f_2 : U_2 \to Y$ if f_1, f_2 agree on a nonempty open set $W \subseteq U_1 \cap U_2$.

Remark. Note that if $f_1 : U_1 \to Y \sim f_2 : U_2 \to Y$, then they agree on $U_1 \cap U_2$ by the identity principle. So $f_1, \ldots, f_2$ are equivalent to a morphism $U_1 \cap U_2 \to Y$. So there exists a maximal nonempty open $U \subseteq X$ on which f is a morphism.

Remark. If $f, g \in \mathcal{O}_X(X)$, then we will see that the map $X \dashrightarrow \mathbb{A}^1$ given by $x \mapsto f(x)/g(x)$ is rational.

Lecture 18

Oct. 23 — Birational Maps

18.1 Birational Maps, Continued

Example 18.0.1. The following are examples of birational maps:

1. The *cuspidal curve* $C = V(x^2 - y^3) \dashrightarrow \mathbb{A}^1$ given by $(x, y) \mapsto x/y$ is a morphism on $C \setminus \{(0, 0)\}$.
2. The projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}]$ is a morphism on $\mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$.
3. Consider a map
$$\begin{aligned} \mathbb{A}^n &\dashrightarrow \mathbb{A}^n \\ x &\longmapsto \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_n(x)}{g_n(x)}\right) \end{aligned}$$
for some nonzero $f_i, g_j \in k[x_1, \ldots, x_n]$. This is a morphism on $\mathbb{A}^n \setminus V(g_1 \cdots g_n)$.

Definition 18.1. We define the following:

1. A map $f : X \dashrightarrow Y$ is *dominant* if the image of f contains a nonempty open set.
2. If $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are dominant rational maps, i.e.
$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y \arrow[r, dashed, "g"] & Z \\ U \arrow[u, hook] \arrow[ur, "f'"'] & V \arrow[u, hook] \arrow[ur, "g'"'] & \end{tikzcd}$$
then we can define the composition $g \circ f : X \dashrightarrow Z$ by
$$(f')^{-1}(V) \xrightarrow{f'} V \xrightarrow{g'} Z.$$

Definition 18.2. A rational map $f : X \dashrightarrow Y$ is *birational* if it is dominant and there exists a dominant rational map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ (as rational maps). In this case, we say that X and Y are are *birational*.

Example 18.2.1. We have the following:

1. Let X be an irreducible variety and $U \subseteq X$ a nonempty open set. Then the inclusion morphism $f : U \hookrightarrow X$ is birational with $g : X \dashrightarrow U$ the identity map on U.

2. Define a morphism

$$\begin{aligned} \mathbb{A}^n &\xrightarrow{f} \mathbb{A}^n \\ x &\longmapsto (x_1, x_1x_2, \ldots, x_1x_n). \end{aligned}$$

   Then we can define an inverse rational map

$$\begin{aligned} \mathbb{A}^n &\overset{g}{\dashrightarrow} \mathbb{A}^n \\ y &\longmapsto (x_1, x_2/x_1, \ldots, x_n/x_1). \end{aligned}$$

   So f is a birational map.

Proposition 18.1. *Two irreducible varieties are birational if and only if they contain isomorphic nonempty open sets.*

Proof. ($\Leftarrow$) Assume we have open sets $U \subseteq X$, $V \subseteq Y$ with an isomorphism $f : U \xrightarrow{\cong} V$. Then we get rational maps $f : X \dashrightarrow Y$ and $f^{-1} : Y \dashrightarrow X$ which compose to the identity.

($\Rightarrow$) Assume we have

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y \\ U \arrow[u, hook] \arrow[ur, "f'"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, dashed, "f"] & X \\ V \arrow[u, hook] \arrow[ur, "g'"'] & \end{tikzcd}$$

satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Then the composition

$$(g')^{-1}(U) \xrightarrow{g'} U \xrightarrow{f'} Y$$

is the inclusion map by the identity principle, so $g'((g')^{-1}(U)) = (f')^{-1}(g^{-1}(U)) \subseteq (f')^{-1}(V)$. Similarly, we can also get the inclusion $f'((f')^{-1}(V)) \subseteq (g')^{-1}(U)$. So we get morphisms

$$\begin{tikzcd} (f')^{-1}(U) \arrow[r, bend left, "f'"] & (g')^{-1}(V) \arrow[l, bend left, "g'"] \end{tikzcd}$$

that compose to the identity on some open sets, so they compose to id by the identity principle. ◻

Example 18.2.2. Consider the following:

1. $\mathbb{P}^{n+m}$ and $\mathbb{P}^n \times \mathbb{P}^m$ both contain $\mathbb{A}^{n+m}$, so they are birational.
2. $G(d, n)$ and $\mathbb{P}^{d(n-d)}$ both contain $\mathbb{A}^{d(n-d)}$, so they are birational.

18.2 Rational Functions

Definition 18.3. A *rational function* on a variety X is a rational map $X \dashrightarrow \mathbb{A}^1$. This is equivalent to the data (U, φ) for $U \subseteq X$ a nonempty open set and $\varphi \in \mathcal{O}_X(U)$, up to the equivalence $(U, \varphi) \sim (U', \varphi')$ if and only if $\varphi|_W = \varphi'|_W$ for some nonempty open $W \subseteq U \cap U'$.

Example 18.3.1. If $f, g \in k[x_1, \dots, x_n]$ with $g \neq 0$, then f/g gives a rational function on $\mathbb{A}^n$. The two ways of thinking about this are:

1. The rational map $\varphi : \mathbb{A}^n \dashrightarrow \mathbb{A}^1$ given by $x \mapsto f(x)/g(x)$.
2. The equivalence class of $(D(g), f(x)/g(x))$.

Definition 18.4. Let X be an irreducible variety X. Then the *function field* of X is

$$K(X) = \{\text{rational functions on } X\}$$

Remark. Note that $K(X)$ is in fact a field:

- Multiplication is defined by $(U, \varphi) \cdot (V, \psi) = (U \cap V, \varphi|_{U\cap V}\psi|_{U\cap V})$ and similarly for addition.
- One can check that the above operations turn $K(X)$ into a ring.
- For $0 \neq (U, \varphi) \in K(X)$, we have $(U, \varphi) \cdot (U \setminus \{\varphi = 0\}, 1/\varphi) = (X, 1)$.

Remark. We have the following properties of $K(X)$:

1. For a nonempty open set $U \subseteq X$, the restriction map $K(X) \to K(U)$ is an isomorphism.
2. For an irreducible affine variety X, the map

$$\begin{aligned} \operatorname{Frac}(A(X)) &\xrightarrow{\alpha} K(X) \\ \varphi/\psi &\longmapsto (D(\psi), \varphi/\psi) \end{aligned}$$

   is an isomorphism. One can check that it is a well-defined homomorphism, and it is injective since it is a homomorphism out of a field. To see surjectivity, let $(U, \varphi) \in K(X)$. By shrinking U, we may assume that $U = D(g)$ for some $0 \neq g \in A(X)$. Since

$$\mathcal{O}_X(D(g)) = A(X)_g,$$

   we can write $\varphi = f/g^n$ for some $n \geq 0$ and $f \in A(X)$. So $\alpha(f/g^n) = (U, \varphi)$.

Example 18.4.1. We have $K(\mathbb{A}^n) = \operatorname{Frac}(A(\mathbb{A}^n)) = \operatorname{Frac}(k[x_1, \dots, x_n]) = k(x_1, \dots, x_n)$.

Remark. If $f : X \dashrightarrow Y$ is a dominant morphism of irreducible varieties, then we get a k-algebra homomorphism on their function fields by the pullback:

$$\begin{aligned} f^* : K(Y) &\longrightarrow K(X) \\ (V, \varphi) &\longmapsto (f^{-1}(V), (f')^*(\varphi)). \end{aligned}$$

Exercise 18.1. Let X and Y be irreducible varieties. Then

1. If $f : X \dashrightarrow Y$ is birational, then $f^* : K(Y) \to K(X)$ is an isomorphism.
2. If $K(X) \cong K(Y)$ as k-algebras, then X is birational to Y.

Example 18.4.2. Since $G(d, n) \sim_{\text{bir}} \mathbb{P}^{d(n-d)}$, we have $K(G(d, n)) \cong K(\mathbb{P}^{d(n-d)})$.

Corollary 18.0.1. *Any irreducible variety X is birational to a hypersurface in $\mathbb{P}^n$.*

Proof. By Noether normalization, there exists a transcendence basis $x_1, \dots, x_n$ for $K(X)$ over k (this is finite as we may replace X by an affine open set). Assuming k is characteristic 0 (or working harder in nonzero characteristic), $k(x, \dots, x_n) \subseteq K(X)$ is separable. By the definition of a transcendence basis, this extension is finite. So by the primitive element theorem, there exists $y \in K(X)$ such that

$$K(X) = k(x_1, \dots, x_n, y).$$

Let $f \in k(x_1, \dots, x_n)[t]$ denote the minimal polynomial of y. Clearing denominators, we may assume $f \in k[x_1, \dots, x_n][t]$. By Gauss's lemma, f is still irreducible in $k[x_1, \dots, x_n, t]$. Now set

$$Y = V(f(x_1, \dots, x_n, x_{n+1})) \subseteq \mathbb{A}^{n+1},$$

which is irreducible. Now we have

$$K(Y) = \operatorname{Frac} \frac{(k[x_1, \dots, x_{n+1}])}{(f(x_1, \dots, x_{n+1}))} \cong \frac{k(x_1, \dots, x_n)[y]}{(f)} \cong K(X),$$

so $X \sim_{\mathrm{bir}} Y$. We get the desired statement after taking projective closures. $\square$

Lecture 19

Oct. 28 — The Blow-Up

19.1 Rational Varieties

Exercise 19.1. Show that two irreducible varieties X and Y are birational if and only if $K(X) \cong K(Y)$ as k-algebras.

Example 19.0.1. Let $C = V(x^2 - y^3)$, and define rational maps

$$f : \mathbb{A}^1 \dashrightarrow C,\ t \longmapsto (t^3, t^2) \quad \text{and} \quad g : C \dashrightarrow \mathbb{A}^1,\ (x, y) \longmapsto x/y.$$

For $t \neq 0$, we have $g(f(t)) = t^3/t^2$, so $g \circ f = \mathrm{id}_{\mathbb{A}^1}$. For $(a, b) \in C$,

$$(f \circ g)(a, b) = (a^3/b^3, a^2/b^2) = (a^3/a^2, b^3/b^2) = (a, b),$$

where the second equality follows since $a^2 = b^3$ on C. Thus $f \circ g = \mathrm{id}_C$, so f is birational.

Theorem 19.1. *Any irreducible algebraic variety is birational to a hypersurface in* $\mathbb{P}^n$.

Definition 19.1. An irreducible algebraic variety is *rational* if it is birational to $\mathbb{P}^n$.

Example 19.1.1. Rational varieties include $\mathbb{P}^n$, $\mathbb{A}^n$, $G(d, n)$, $\mathbb{P}^m \times \mathbb{P}^n$, etc.

Remark. Which hypersurfaces in $\mathbb{P}^n$ are rational? This is an open question. It is known that if $X \subseteq \mathbb{P}^n$ is a smooth hypersurface over a field of characteristic 0 and degree $d > n + 1$, then X is *not* rational.

19.2 The Blow-Up

Example 19.1.2. Consider the following birational map:

$$\begin{aligned} \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1 \\ (x_1, x_2) &\longmapsto [x_1 : x_2]. \end{aligned}$$

Why does this not extend to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$? The issue is where $(0, 0)$ would go. Let us try to make it extend. Consider $U = \mathbb{A}^2 \setminus \{0\}$ and the map

$$\begin{aligned} f : U &\longrightarrow \mathbb{P}^1 \\ (x_1, x_2) &\longmapsto [x_1 : x_2]. \end{aligned}$$

Consider the graph $\Gamma_f = \{(x, y) \in U \times \mathbb{P}^1 : x_1 y_2 = x_2 y_1\}$, and the *blow-up*

$$B_0\mathbb{A}^2 := \overline{\Gamma}_f \subseteq \mathbb{A}^2 \times \mathbb{P}^1,$$

with projection maps $\pi : B_0\mathbb{A}^2 \to \mathbb{A}^2$ and $B_0\mathbb{A}^2 \to \mathbb{P}^1$. Then

1. $B_0\mathbb{A}^2 = \{(x,y) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_1y_2 - x_2y_1 = 0\}$.

   Let Y be the right-hand side. Since $\Gamma_f \subseteq Y$, it is immediate that $B_0\mathbb{A}^2 = \overline{\Gamma}_f \subseteq \overline{Y} = Y$. For the reverse inclusion, it suffices to show Y is irreducible and Γ_f is open in Y. By the equation for Y,

   $$Y \setminus V(x_1, x_2) = \Gamma_f,$$

   which shows Γ_f is open in Y. For irreducibility, set $U_i = \{(x,y) \in Y : y_i \neq 0\}$. Then $U_i \cong \mathbb{A}^2$:

   $$\begin{aligned} U_1 &\longrightarrow \mathbb{A}^2 \\ (x,y) &\longmapsto (x_1, x_2/y_1) \\ ((x_1, x_1x_2), (1 : x_2)) &\longleftarrow\!\shortmid (x_1, x_2). \end{aligned}$$

   So each U_i is irreducible. As $U_1 \cap U_2 \neq \varnothing$ and $Y = U_1 \cap U_2$, Y is irreducible. Thus $B_0\mathbb{A}^2 \supseteq Y$.

2. $\pi^{-1}(\mathbb{A}^2 \setminus \{0\}) \to \mathbb{A}^2 \setminus \{0\}$ is an isomorphism.

   For this, we can give an inverse $\mathbb{A}^2 \setminus \{0\} \to \pi^{-1}(\mathbb{A}^2 \setminus \{0\})$ by $(x_1, x_2) \mapsto ((x_1, x_2), [x_1 : x_2])$.

3. $\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^1 \cong \mathbb{P}^1$.

   This follows from (1).

Remark. An alternative perspective is to think of $B_0\mathbb{A}^2$ as

$$B_0\mathbb{A}^2 = \{(p, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 : p \in \ell\}.$$

where we think of $\mathbb{P}^1$ as lines in $\mathbb{A}^2$ through 0.

Example 19.1.3. In higher dimensions, we define $B_0\mathbb{A}^n = \overline{\Gamma}_f \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$, where $f : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is given by $x \mapsto [x]$. Let $\pi : B_0\mathbb{A}^n \to \mathbb{A}^n$ be the projection. Then the same argument as before shows:

1. $B_0\mathbb{A}^n = \{(x,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x_iy_i - x_jy_j = 0 \text{ for all } 1 \leq i, j \leq n\}$.
2. $\pi^{-1}(\mathbb{A}^n \setminus \{0\}) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism.
3. $\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$. This is called the *exceptional locus*.

Remark. Define charts $U_i = \{(x,y) \in B_0\mathbb{A}^n : y_i \neq 0\}$. Then we have

$$\begin{tikzcd} \mathbb{A}^n \arrow[d, "\cong"'] & x \arrow[d, mapsto] \\ U_1 \arrow[d] & ((x_1, x_1x_2, \dots, x_1x_n), [1 : x_2 : \cdots : x_n]) \arrow[d, mapsto] \\ \mathbb{A}^n & (x_1, x_1x_2, \dots, x_1x_n) \end{tikzcd}$$

Check that $\mathbb{A}^n \to U_1$ is in fact an isomorphism.

Example 19.1.4. Let $n = 2$ and $\pi : B_0\mathbb{A}^2 \to \mathbb{A}^2$. What happens to $C := V(y^2 - x^3) \subseteq \mathbb{A}^2$ under the blow-up? We have $\pi^{-1}(C) = (\{0\} \times \mathbb{P}^1) \cup \overline{\pi^{-1}(C \setminus \{0\})}$. We can compute this explicitly on charts. Write

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r] \arrow[dr, "\pi_1"'] & U_1 \subseteq B_0\mathbb{A}^2 \arrow[d, "\pi"] \arrow[r, equal] & \{((x,y), [s:t]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt - ys = 0\} \\ & = \mathbb{A}^2 & \end{tikzcd}$$

where $\mathbb{A}^2 \to U_1$ is given by $(x, y) \mapsto ((x, xy), [1 : y])$ and π_1 is given by $(x, y) \mapsto (x, xy)$. Then

$$\pi_1^{-1}(C) = V((xy)^2 - x^3) = V(x^2(y^2 - x)) = V(x) \cup V(y^2 - x)$$

In particular, we see that the piece $\overline{\pi^{-1}(C \setminus \{0\})} = V(y^2 - x)$ no longer has a singularity at 0.

Definition 19.2. Let $X \subseteq \mathbb{A}^n$ be an affine variety, $f_1, \ldots, f_r \in A(X)$, and $U = X \setminus V(f_1, \ldots, f_r)$. We define the *blow-up* of X along $f_1, \ldots, f_r$ to be

$$B_{f_1,\ldots,f_r} X := \widetilde{X} := \text{closure of } \Gamma_f \text{ in } X \times \mathbb{P}^{n-1}$$

where $f : U \to \mathbb{P}^{r-1}$ is given by $x \mapsto [f_1(x) : \cdots : f_r(x)]$,

Remark. We have the following:

0. When $X = \mathbb{A}^n$ and $f_1, \ldots, f_r = x_1, \ldots, x_n$, then $B_{x_1,\ldots,x_n}\mathbb{A}^n = B_0\mathbb{A}^n$.
1. $B_{f_1,\ldots,f_r} X \subseteq \{(x, y) \in X \times \mathbb{P}^{n-1} : f_i(x)y_j - f_j(x)y_i = 0\}$. Equality does not necessarily hold.
2. π is an isomorphism over U.

   To see this, use that Γ_f is closed in $U \times \mathbb{P}^{r-1}$ and the projection $\mathrm{pr}_1 : \Gamma_f \to U$ is an isomorphism to get that $\overline{\Gamma}_f \cap (U \times \mathbb{P}^{r-1}) = \Gamma_f$. So $\pi^{-1}(U) = \Gamma_f \cong U$.
3. If X is irreducible and $U \neq \varnothing$, then π is birational.

   For this, use (1) to get $B_{f_1,\ldots,f_r}$ is irreducible and π being an isomorphism over U.
4. If $(f_1, \ldots, f_r) = (f_1', \ldots, f_s')$ as ideals in $A(X)$, then there is an isomorphism

   $$B_{f_1,\ldots,f_r} X \xrightarrow{\cong} B_{f_1',\ldots,f_s'} X.$$

   Hint: Write $f_j' = \sum_{i=1}^r g_{i,j} f_i$. Use the $g_{i,j}$ to get a morphism $\Gamma_f \to \Gamma_{f'}$, $(x, y) \mapsto (x, \big[\sum_{i=1}^r g_{i,1}(x)y_i : \cdots : \sum_{i=1}^r g_{i,s}(x)y_i\big])$, then use this to get a morphism $B_{f_1,\ldots,f_r} X \to B_{f_1',\ldots,f_s'} X$. We can obtain an inverse similarly by swapping the roles of f_i and f_j'. See Gathmann 9.16.
5. (Local) If $W \subseteq X$ is open and $W \setminus V(f_1, \ldots, f_r)$ is dense in W, then

   $$B_{f_1,\ldots,f_r} X|_{\pi^{-1}(W)} \cong B_{f_1|_W,\ldots,f_r|_W} W.$$

   (Use the properties of the graph.)

Remark. We can further generalize the blow-up as follows:

1. (Affine case) Let X be affine, $I \leq A(X)$, and $Z \subseteq X$ closed. Then we can define $B_I X := B_{f_1,\ldots,f_r} X$ with $f_1, \ldots, f_r$ generators of I (the isomorphism class is independent of the choice of generators), and $B_Z X = B_{I(Z)} X$ with $I(Z) \leq A(X)$ the ideal consisting of regular functions vanishing on Z.
2. Let X be a variety and $Z \subseteq X$ a closed subvariety. Choose an affine open cover $X = \bigcup_i U_i$, and define $B_Z X$ to be the $B_{Z \cap U_i} U_i$ glued together along $B_{Z \cap U_i \cap U_j}(U_i \cap U_j)$. (Use that the blow-up can be computed locally and glue these together to get $B_Z X$.)

Lecture 20

Oct. 30 — The Blow-Up, Part 2

20.1 The Blow-Up, Continued

Example 20.0.1. Let $\pi : B_{[1:0:0]}\mathbb{A}^2 \to \mathbb{A}^2$ be the blow-up. Then $\pi^{-1}(U_0) \cong B_0\mathbb{A}^2$ and $\pi^{-1}(U_i) \cong U_i$ for $i = 1, 2$. Note that π is an isomorphism over $\mathbb{A}^2 \setminus \{[1 : 0 : 0]\}$ and $\pi^{-1}([1 : 0 : 0]) = \mathbb{P}^1$.

Definition 20.1 (Blow-up for projective varieties)**.** Let $X \subseteq \mathbb{P}^n$ be a projective variety and $f_1, \ldots, f_r \in S(X)$ homogeneous of the same degree. Let $U = X \setminus V(f_1, \ldots, f_r)$ and define

$$\begin{aligned} f : U &\longrightarrow \mathbb{P}^{r-1} \\ x &\longmapsto [f_1(x) : \cdots : f_r(x)]. \end{aligned}$$

Then the *blow-up* of X along $f_1, \ldots, f_r$ is

$$B_{f_1,\ldots,f_r}X = \text{closure of } \Gamma_f \text{ in } X \times \mathbb{P}^{r-1}.$$

Example 20.1.1. We can compute the blow-up locally, e.g. for $U_0 = \{[x] \in X : x_0 \neq 0\}$, we have

$$B_{f_1,\ldots,f_r}X|_{\pi^{-1}(U_0)} \cong B_{f_1^i,\ldots,f_r^i}U_0.$$

Example 20.1.2. Consider the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $[x, y] \mapsto x$. We have

$$\begin{tikzcd} & B_{x,y}\mathbb{P}^2 \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \arrow[dl, "\pi"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, mapsto, "{[x,y,z]\mapsto[x:y]}"'] & & \mathbb{P}^1 \end{tikzcd}$$

Note that this resolves the indeterminacy.

What is $B_{x,y}\mathbb{P}^2$ isomorphic to? The answer is $B_{[0:0:1]}\mathbb{P}^2$. For example, on charts:

1. on $\mathbb{A}^2_{x,y} \hookrightarrow \mathbb{P}^2$ given by $(x, y) \mapsto [x : y : z]$, we have $(x^i, y^i) = (x, y) = I(0)$;
2. on $\mathbb{A}^2_{x,z} \hookrightarrow \mathbb{P}^2$, we have $(x^i, y^i) = (x, 1) = (1)$;
3. on $\mathbb{A}^2_{y,z} \hookrightarrow \mathbb{P}^2$, we have $(x^i, y^i) = (1, y) = (1)$.

Theorem 20.1 (Castelnuovo)**.** *If $X \dashrightarrow X'$ is a birational map between two smooth complex surfaces, then there exists a factorization*

$$\begin{tikzcd}
 & Y \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
\cdots \arrow[d, "f_m"'] & & \cdots \arrow[d, "g_n"] \\
X & & X'
\end{tikzcd}$$

with f_i, g_i point blow-ups.

20.2 Applications to Singularities

Example 20.1.3. Let $C = V(x^2 - y^3)$. Then we have $B_0C \subseteq B_0\mathbb{A}^2$ and

$$\begin{tikzcd}
B_0C \arrow[r, hook] \arrow[d] & B_0\mathbb{A}^2 \arrow[d] \\
C \arrow[r, hook] & \mathbb{A}^2
\end{tikzcd}$$

We claim that there is an isomorphism

$$\begin{aligned} g : \mathbb{A}^1 &\xrightarrow{\cong} B_0C \\ t &\longmapsto ((t^3, t^2), [t : 1]). \end{aligned}$$

To see this, note that $\operatorname{Im} f = B_0C$ (check this locally on charts), and g has inverse given by

$$\begin{aligned} B_0C &\longrightarrow \mathbb{A}^1 \\ ((x, y), [s : t]) &\longmapsto t/s. \end{aligned}$$

Theorem 20.2 (Hironaka, Fields Medal result)**.** *If X is an irreducible variety over a characteristic zero field, then there exists a sequence of blow-ups*

$$X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X$$

along subvarieties such that X_r is smooth.

Definition 20.2. The *tangent cone* of a variety X at a point a is

$$C_aX = \text{cone over } \pi^{-1}(a) \subseteq \mathbb{P}^{n-1}$$

with $\pi : B_aX \to X$ the blow-up.

Example 20.2.1. For $0 \in \mathbb{A}^n$, we have $\pi^{-1}(0) = \mathbb{P}^{n-1}$, so $C_0\mathbb{A}^n = \mathbb{A}^n$.

Example 20.2.2. The tangent cones for $X_1 = V(y - x^2)$ and $X_2 = V(y^2 - x^3)$ are both the x-axis, whereas the tangent cone for $X_3 = V(y^2 - x^2 - x^3)$ is a union of two lines $V(y - x)$ and $V(y + x)$.

Exercise 20.1. If $0 \in X \subseteq \mathbb{A}^n$ is a variety and $I(X) = (f)$, then show that $C_0(f) = V(f^{\text{init}})$, where $f^{\text{init}} := f_d$ if we write $f = f_d + f_{d+1} + \cdots$ with f_i homogeneous of degree i and $f_d \neq 0$.

Lecture 21

Nov. 11 — Tangent Spaces

21.1 Tangent Cones

Definition 21.1. Let $a \in X \subseteq \mathbb{A}^n$, where X is an affine variety. Define

$$\begin{aligned} f : X \setminus \{a\} &\longrightarrow \mathbb{P}^{n-1} \\ x &\longmapsto [x_1 - a_1 : \cdots : x_n - a_n], \end{aligned}$$

then the *blowup* of X at a is $B_a X = \overline{\Gamma}_f \subseteq X \times \mathbb{P}^{n-1}$. Let $\pi : B_a X \to X$ be the first projection. Then the *tangent cone* of X at a is $C_a X = C(\pi^{-1}(\{a\})) \subseteq \mathbb{A}^n$, where $\pi^{-1}(\{a\}) \subseteq \{a\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$.

Example 21.1.1. When $a = 0$ and $X = \mathbb{A}^n$, we have $\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}$ and $C_0 \mathbb{A}^n = \mathbb{A}^n$.

Example 21.1.2. Let $X = \mathbb{A}^2$ and $a = 0$. Then we have:

X	$V(y - x^2)$	$V(y^2 - x^3)$	$V(y^2 - x^2 - x^3)$
$C_0 X$	$V(y)$	$V(y^2)$	$V(y^2 - x^2)$

Theorem 21.1. *If $a \in X$ is a point on a variety, then* $\dim C_a X = \operatorname{codim}_X \{a\}$.[1]

Proof. Since both sides are local, we may assume X is affine. If $X = X_1 \cup \cdots \cup X_r$ are the irreducible components, then we can write $B_a X = \bigcup_{i=1}^r B_a X_i$ (where we take $B_a X_i = X_i$ if $a \notin X_i$). Thus we may assume that X is irreducible. Now let $\pi : B_a X \to X$ be the first projection, then

$$\dim C_a X = \dim C(\pi^{-1}(\{a\})) = \dim \pi^{-1}(\{a\}) + 1 = \dim X,$$

where the last equality is by Proposition 21.1. $\square$

Proposition 21.1. *Assume $X \subseteq \mathbb{A}^n$ is an irreducible affine variety and $f_1, \ldots, f_r \in A(X)$ such that $U = X \setminus V(f_1, \ldots, f_r) \neq \varnothing$. Then every irreducible component of $\pi^{-1}(V(f_1, \ldots, f_r))$ in $B_{f_1,\ldots,f_r} X$ is of codimension* 1.

Proof. Since X is irreducible, so is U. As $U \cong \Gamma_f$, we know Γ_f is irreducible, so $B_{f_1,\ldots,f_r} X$ is irreducible. Now observe that

$$B_{f_1,\ldots,f_r} X \subseteq \{(x, y) \in X \times \mathbb{P}^{r-1} : f_i(x) y_j - f_j(x) y_i = 0\}$$

as this containment holds on Γ_f and the right-hand side is closed. Consider

$$U_i = \{(x, y) \in B_{f_1,\ldots,f_r}(X) : y_i \neq 0\}.$$

[1] Note that $\operatorname{codim}_X \{a\} = \dim X$ when X is irreducible.

If $(x, y) \in U_i$ and $f_i(x) = 0$, then $f_j(x) = 0$ for all j. So $\pi^{-1}(V(f_1, \dots, f_r)) \cap U_i$ is cut out on U_i by $f_i = 0$, so $\pi^{-1}(V(f_1, \dots, f_r))$ is codimension 1 in $B_{f_1,\dots,f_r}X$ (it is codimension 1 on each chart). $\square$

Example 21.1.3. Let $X = \mathbb{A}^n$ and $f_1, \dots, f_r = x_1, \dots, x_n$. Then

$$\pi^{-1}(V(x_1, \dots, x_n)) = \pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}.$$

21.2 Tangent Spaces

Remark. Fix $f \in \mathbb{C}[x_1, \dots, x_n]$ with $0 \in X = V(f) \subseteq \mathbb{C}^n$. If

$$\left[\frac{\partial f}{\partial x_1}(0), \cdots, \frac{\partial f}{\partial x_n}(0)\right] \neq \vec{0},$$

then X is smooth at 0 with tangent plane $T_0X = V(f_1) \subseteq \mathbb{C}^n$, where f_1 is the linear part of f.

Definition 21.2. Let $X \subseteq \mathbb{A}^n$ be an affine variety with $0 \in X$. The *tangent space* of X at 0 is

$$T_0X = V(f_1 : f \in I(X)) \subseteq \mathbb{A}^n,$$

where f_1 is the homogeneous degree 1 part of f (note that T_0X is a linear subspace of k^n).

Remark. Note the following:

1. We will give an intrinsic definition of the tangent space later (i.e. without embedding X in $\mathbb{A}^n$).
2. If $f, g \in I(X)$ and $h \in k[x_1, \dots, x_n]$, then $[f + g]_1 = f_1 + g_1$ and $[fh]_1 = f_1h(0) + h_1f(0) = f_1h(0)$, where $f(0) = 0$ since $0 \in X$. So if $I(X) = (f^{(1)}, \dots, f^{(r)})$, then $T_0X = V(f_1^{(1)}, \dots, f_1^{(r)})$.
3. Recall that $C_0X = V(f^{\text{init}} : f \in I(X))$ and $T_0X = V(f_1 : f \in I(X))$. So $C_0X \subseteq T_0X$.

Example 21.2.1. $T_0\mathbb{A}^n = V(0) = \mathbb{A}^n$.

Example 21.2.2. We have the following table in $\mathbb{A}^2$:

X	$V(y - x^2)$	$V(y^2 - x^3)$	$V(y^2 - x^2 - x^3)$
T_0X	$V(y)$	$\mathbb{A}^2$	$\mathbb{A}^2$

Remark. To define a tangent space at $a \in X$, we can just translate a to 0.

Proposition 21.2. *Let $X \subseteq \mathbb{A}^n$ be an affine variety with $0 \in X$. Write*

$$I(0) = (\overline{x}_1, \dots, \overline{x}_n) \leq A(X) \cong k[x_1, \dots, x_n]/I(X).$$

Then there is a natural vector space isomorphism

$$I(0)/I(0)^2 \xrightarrow{\cong} \operatorname{Hom}_k(T_0X, k).$$

(Note that $I(0)/I(0)^2$ is a module over $A(X)/I(0) \cong k$, so it is a k-vector space.)

Proof. Consider the k-linear map

$$\begin{aligned} \varphi : I(0) &\longrightarrow \operatorname{Hom}_k(T_0X, k) \\ \overline{f} &\longmapsto f_1|_{T_0X} \end{aligned}$$

where we view $f \in (x_1, \ldots, x_n) \leq k[x_1, \ldots, x_n]$. We first check that this is well-defined. Assume we have $\overline{f} = \overline{g} \in I(0)$, so $f - g = I(X)$. So $[f_1 - g_1] = [f - g]_1 \in (h_1 : h \in I(X))$. This is the defining ideal for T_0X, so $(f_1 - g_1)|_{T_0X} = 0$. So $\varphi(\overline{f}) = \varphi(\overline{g})$, i.e. φ is well-defined.

Now we check that φ is surjective. For this, note that the right-hand side is generated by the coordinate functions, and we have $I(0) = (\overline{x}_1, \ldots, \overline{x}_n)$.

Finally, it suffices to check that $\ker \varphi = I(0)^2$. For the reverse containment, if $\overline{f}, \overline{g} \in I(0)$, then

$$(fg)_1 = f_1 g(0) + g_1 f(0) = 0$$

since $f(0) = g(0) = 0$. So $\varphi(\overline{f}\overline{g}) = 0$, i.e. $I(0)^2 \subseteq \varphi$. For the forward containment, if $f \in \ker \varphi$, then $f_1|_{T_0X} = 0$, so there exists $g \in I(X)$ such that $f_1 = g_1$. Then $f - g$ has no constant or degree 1 term, so $\overline{f} = \overline{f - g} \in I(0)^2$. Thus $\ker \varphi \subseteq I(0)^2$ as well, so we have $\ker \varphi = I(0)^2$. ◻

Example 21.2.3. If $X = \mathbb{A}^n$ and $a = 0$, then $I(0) = (x_1, \ldots, x_n) \leq k[x_1, \ldots, x_n]$, and

$$I(0)/I(0)^2 = k\overline{x}_1 \oplus \cdots \oplus k\overline{x}_n.$$

Remark. We now want a representation of T_0X, independent of its embedding in $\mathbb{A}^n$.

Proposition 21.3. *For an affine variety $X \subseteq \mathbb{A}^n$ with $a \in X$, let*

$$I(a) = (f \in A(X) : f(a) = 0) \quad \textit{and} \quad I_a = (f \in \mathcal{O}_{X,a} : f(a) = 0).$$

Then $I(a)/I(a)^2 \cong I_a/I_a^2$ as k-vector spaces.

Definition 21.3 (Redefinition of tangent space)**.** For a variety X and $a \in X$, the *tangent space* of X at a is the k-vector space

$$T_aX = \operatorname{Hom}_k(I_a/I_a^2, k).$$

Remark. Note that I_a/I_a^2 is a module over $\mathcal{O}_{X,a}/I_a \cong k$.

Lemma 21.1. *Given a ring A, $S \subseteq A$ a multiplicative system, and A-modules $N \subseteq M$, then*

$$S^{-1}(M/N) \cong S^{-1}M/S^{-1}N.$$

Proof. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0.$$

Using that localization is an exact functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$, we get a short exact sequence

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0.$$

So we get that $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$. ◻

Remark. Previously, we had $\mathcal{O}_{X,a} = S^{-1}A(X)$ where $S = A(X) \setminus I(a)$, and $I_a = S^{-1}I(a)$. So

$$\mathcal{O}_{X,a}/I_a = S^{-1}(A(X)/I(a)) = S^{-1}k \cong k$$

by Lemma 21.1 (note that any localization of k is still k).

Proof of Proposition 21.3. Observe that we have

$$I_a/I_a^2 = S^{-1}I(a)/S^{-1}I(a)^2 \cong S^{-1}(I(a)/I(a)^2).$$

Now we claim that the map

$$\begin{aligned} I(a)/I(a)^2 &\longrightarrow S^{-1}(I(a)/I(a)^2) \\ \overline{f} &\longmapsto \overline{f}/1 \end{aligned}$$

is an isomorphism. The hard part is surjectivity. The key idea is to note that for $g \in S$, we can write $\overline{f}/g = (1/g(a))\overline{f}/1$ (this is just a computation). $\square$

Lecture 22

Nov. 13 — Singularities

22.1 Tangent Spaces, Continued

Remark. Recall that the tangent space to a variety X at $a \in X$ is

$$T_aX = \mathrm{Hom}_k(\mathfrak{m}_a/\mathfrak{m}_a^2, k),$$

where $\mathcal{O}_{X,a} \supseteq \mathfrak{m}_a = \{\varphi \in \mathcal{O}_{X,a} : \varphi(a) = 0\}$. Note the following:

1. We have $\mathcal{O}_{X,a}/\mathfrak{m}_a \cong k$: We can define a map

$$\begin{aligned} \psi : \mathcal{O}_{X,a} &\longrightarrow k \\ \varphi &\longmapsto \varphi(a). \end{aligned}$$

   This is surjective and has kernel $\mathfrak{m}_a$, so $k \cong \mathcal{O}_{X,a}/\ker\psi = \mathcal{O}_{X,a}/\mathfrak{m}_a$.

2. Since $\mathfrak{m}_a \cdot (\mathfrak{m}_a/\mathfrak{m}_a^2) = 0$, we have that $\mathfrak{m}_a/\mathfrak{m}_a^2$ is a module over $\mathcal{O}_{X,a}/\mathfrak{m}_a \cong k$.

Remark. How do we compute the tangent space? We can do the following:

1. Shrink X so that it is affine: $a \in X \subseteq \mathbb{A}^n$ where X is closed.
2. Translate so that $a = 0$.
3. Use $T_0X = V(f_1 : f \in I(X))$.

Example 22.0.1. In $\mathbb{A}^2$, we have the following:

X	$V(y - x^2)$	$V(y^2 - x^2 - x^3)$
T_0X	$V(y)$	$V(0)$

Remark. Since $C_aX \subseteq T_aX$, we have

$$\dim T_aX \geq \dim C_aX = \mathrm{codim}_X\{a\}.$$

Note that this is equal to $\dim X$ when X is irreducible.

22.2 Singularities

Definition 22.1. Let X be a variety and $a \in X$.

1. X is *smooth* (or *non-singular*, *regular*) at a if

$$\dim T_aX = \mathrm{codim}_X\{a\}.$$

   Otherwise we say that X is singular at a.

2. X is *smooth* (or *non-singular*, *regular*) if it is smooth at every point.

Example 22.1.1. We have the following:

1. For $\mathbb{A}^n$, we have $T_0\mathbb{A}^n = \mathbb{A}^n$, so $\mathbb{A}^n$ is smooth at 0. By translation, $\mathbb{A}^n$ is smooth.
2. Let $C = V(y - x^2)$. Then $\dim T_0 C = 1 = \dim C$, so C is smooth at 0.
3. Let $0 \in X \subseteq \mathbb{A}^n$ be a hypersurface with $I(X) = (f)$. Then $T_0 X = V(f_1)$, so

$$\begin{aligned} X \text{ is singular at } 0 &\iff \dim T_0 X > n - 1 \\ &\iff \dim T_0 X = n \\ &\iff T_0 X = \mathbb{A}^n \\ &\iff f_1 = 0. \end{aligned}$$

4. $D = V(x^2 - y^2 - y^3)$ is singular at 0.

Remark (Relation to commutative algebra)**.** A local ring $(R, \mathfrak{m})$ is *regular* if it is Noetherian and

$$\dim R = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2.$$

We will use the following result from commutative algebra result without proof: Regular local rings are always integral domains.

In algebraic geometry, note that for a variety X and $a \in X$,

$$\begin{aligned} \dim T_a X &= \dim_k \mathfrak{m}_a/\mathfrak{m}_a^2 \\ \operatorname{codim}_X\{a\} &= \dim \mathcal{O}_{X,a}. \end{aligned}$$

So if X is non-singular at a, then $\mathcal{O}_{X,a}$ is regular, so $\mathcal{O}_{X,a}$ is a domain. Thus X is irreducible at a.

Proposition 22.1 (Jacobi criterion)**.** *Let $a \in X \subseteq \mathbb{A}^n$ be a point on an affine variety with $I(X) = (f_1, \ldots, f_r)$. Then X is smooth at a if and only if the Jacobian*

$$(\partial f_i/\partial x_j(a))_{i,j} \textit{ has rank } \geq n - \operatorname{codim}_X\{a\}.$$

Moreover, this happens if and only if the rank is equal to $n - \operatorname{codim}_X\{a\}$. This implies that the rank is always at most $n - \operatorname{codim}_X\{a\}$ regardless of smoothness.

Remark. Assume that X is irreducible. Then

$$\operatorname{Sing} X := \{a \in X : X \text{ is singular at } a\} = V_X((n - \dim X)\text{-dimensional minors of } \operatorname{Jac}(f))$$

where $I(X) = (f_1, \ldots, f_r)$.

Example 22.1.2. Let X be an irreducible hypersurface in $\mathbb{A}^n$ with $I(X) = (f)$. Then

$$\operatorname{Sing} X = V_X(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) = V(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

Example 22.1.3. Let $C = V(y^2 - x^3)$, then

$$\operatorname{Sing} C = V(y^2 - x^3, -3x^2, 2y) = \{0\}.$$

Example 22.1.4 (Whitney umbrella)**.** Let $X = V(x^2 - y^2z) \subseteq \mathbb{A}^3$. Then

$$\operatorname{Sing} X = V(x^2 - y^2z, 2x, -2yz, -y^2) = V(x, y),$$

which is a line in $\mathbb{A}^3$.

Proof of Proposition 22.1. By translating, we may assume $a = 0$. Now $I(X) = (f_1, \ldots, f_r)$, so

$$[f_i]_1 = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(0) x_j.$$

As $T_0X = V([f_1]_1, \ldots, [f_r]_1)$, we have $T_aX = \ker J$ with $J = (\partial f_i / \partial x_j(0))_{i,j}$. So

$$\dim T_aX = n - \operatorname{rk} J.$$

As $\dim T_aX \geq \operatorname{codim}_X\{a\}$, we get $\operatorname{codim}_X\{a\} \leq n - \operatorname{rk} J$, i.e. $\operatorname{rk} J \leq n - \operatorname{codim}_X\{a\}$. Furthermore, equality holds if and only if $\dim T_aX = \operatorname{codim}_X\{a\}$, which happens if and only if X is smooth at a. □

Corollary 22.0.1 (Generic smoothness)**.** *The smooth locus of a variety is a dense open set.*

Proof. Since the statement is local, we may assume X is affine and irreducible (it suffices to show that the smooth locus is nonempty and open in each irreducible component, away from the intersections with other irreducible components). Thus it suffices to show that the smooth locus is open and nonempty.

Openness follows from the Jacobi criterion, since

$$\operatorname{Sing} X = V_X((r \times r) - \text{minors of } \operatorname{Jac}(f))$$

with $I(X) = (f_1, \ldots, f_r)$ and $r = n - \dim X$. So $\operatorname{Smooth}(X)$ is open.

To see that it is nonempty, we first assume that $X \subseteq \mathbb{A}^n$ is a hypersurface with $I(X) = (f)$. Now

$$\operatorname{Sing} X = V(f, \partial f / \partial x_1, \ldots, \partial f / \partial x_n) \subsetneq X.$$

This is because f is irreducible (as X is irreducible) and $\deg \partial f / \partial x_i < \deg f$, so

$$\sqrt{(f)} = (f) \subsetneq (f, \partial f / \partial x_1, \ldots, \partial f / \partial x_n),$$

which implies that $X \supsetneq \operatorname{Sing} X$. So $\operatorname{Smooth}(X) \neq \varnothing$. In general, any irreducible variety X is birational to a hypersurface, so $\operatorname{Smooth}(X) \neq \varnothing$ in general as well.[1] □

Exercise 22.1 (Projective Jacobi criterion)**.** Let $X \subseteq \mathbb{P}^n$ be a projective variety and

$$I(X) = (f_1, \ldots, f_r) \leq k[x_0, \ldots, x_n], \quad f_i \text{ homogeneous}.$$

Then X is smooth at a if and only if $(\partial f_i / \partial x_j(a))_{i,j}$ has rank $\geq n - \operatorname{codim}_X\{a\}$. Moreover, the inequality $\leq n - \operatorname{codim}_X\{a\}$ always holds.

[1] Let H be the hypersurface. The birational map $\varphi : X \dashrightarrow H$ is defined on some open set $U \subseteq H$, so $\operatorname{Smooth}(X)$ being open dense implies $U \cap \operatorname{Smooth}(H) \neq \varnothing$, so $\operatorname{Smooth}(X) \supseteq \operatorname{Smooth}(X) \cap \varphi^{-1}(U) = \varphi^{-1}(U \cap \operatorname{Smooth}(H)) \neq \varnothing$.

Example 22.1.5 (Fermat hypersurface)**.** Let $F = x_0^d + \cdots + x_n^d$ with $n \geq 2$ and $\operatorname{char} k \nmid d$. Then for $X = V(F) \subseteq \mathbb{P}^n$, we have (using a relaxed version of the projective Jacobi criterion: if we only assume $X = V(f_1, \ldots, f_r)$, then we only get $\operatorname{rk}(\partial f_i/\partial x_j(a)) \geq n - \operatorname{codim}_X\{a\}$ implies smoothness at a)

$$\operatorname{Sing} X = V_X(dx_0^{d-1}, \ldots, dx_n^{d-1}) = V(F, dx_0^{d-1}, \ldots, dx_n^{d-1}) = \varnothing.$$

Exercise 22.2. Fix $n \geq 2$ and $d \geq 1$ ($\operatorname{char} k \nmid d$). Let $N = \dim k[x_0, \ldots, x_n]_d - 1 = \binom{n+d}{d} - 1$. View

$$\mathbb{P}^N_{a_I} = \text{parameter space of degree } d \text{ hypersurfaces in } \mathbb{P}^n,$$

where $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $\sum i_j = d$, i.e. $p \in [a_I] \in \mathbb{P}^N$ corresponds to

$$F_p = \sum a_I x^I \in k[x_0, \ldots, x_n]_d.$$

Then there exists some nonempty open set $U \subseteq \mathbb{P}^N$ such that $V(F_p) \subseteq \mathbb{P}^n$ is smooth for every $p \in U$. Colloquially, one says that a "general" hypersurface of degree d in $\mathbb{P}^n$ is smooth.

Hint: Consider the *incidence correspondence* (write $F = \sum c_I x^I$)

$$\Gamma = \{([F], [a]) : a \in V(F)\} = \left\{([c_I], [a]) : \sum c_I a^I = 0\right\} \subseteq \mathbb{P}^N \times \mathbb{P}^n$$

and the projection maps $p : \Gamma \to \mathbb{P}^N$, $q : \Gamma \to \mathbb{P}^n$. Then for $[F] \in \mathbb{P}^N$, the set $p^{-1}(\{F\}) = [F] \times V(F)$, and $q^{-1}([a])$ corresponds to the hypersurfaces containing $[a]$.

Lecture 23

Nov. 18 — Lines on Hypersurfaces

23.1 Lines on Hypersurfaces in $\mathbb{P}^3$

Remark. Let $X \subseteq \mathbb{P}^3_{x:y:z:w}$ be a smooth hypersurface of degree d. How many lines are on X?

Example 23.0.1. Let $d = 1$. After a linear change of coordinates, we can assume $X = V(x)$. This contains infinitely many lines, as any two points are connected by a line in X.

Example 23.0.2. Let $d = 2$. After a linear change of coordinates, $X = V(x^2 + y^2 + z^2 + w^2)$, which shows that all smooth degree 2 hypersurfaces are isomorphic. In particular, we may assume

$$X = V(xw - yz) \subseteq \mathbb{P}^3,$$

which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ which sends $([a : b], [c : d]) \mapsto [ac : ad : bc : bd]$. Note that $\operatorname{Im}([a : b] \times \mathbb{P}^1)$ and $\operatorname{Im}(\mathbb{P}^1 \times [c : d])$ are lines. So X contains infinitely many lines.

Example 23.0.3. Let $d = 3$. To understand the moduli of cubics, we roughly consider $\mathbb{P}/\mathrm{PGL}_4$, where $\mathbb{P}$ is the projective space of $k[x, y, z, w]_3$. Then the expected dimension is

$$\dim \mathbb{P} - \dim \mathrm{PGL}_4 = \left[\binom{3+3}{3} - 1\right] - [4^2 - 1] = 19 - 15 = 4.$$

Thus we expect that there is not just one smooth cubic in $\mathbb{P}^3$ up to a linear change of coordinates.

First consider the Fermat cubic $X = V(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3$, and assume $\operatorname{char} k \neq 3$. We compute the lines on X. A line $L \subseteq \mathbb{P}^3$ corresponds to a point $L \in G(2, 4)$. After permuting the coordinates,

$$L = \text{row span of } \begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}$$

with $a_i, b_i \in k$. Thus we can write

$$L = \{[s : t : sa_2 + tb_2 : sa_3 + tb_3] : [s : t] \in \mathbb{P}^1\}.$$

Now $L \subseteq X$ if and only if

$$s^3 + t^3 + (sa_2 + tb_2)^3 + (sa_3 + tb_3)^3 = 0$$

for all $s, t \in k$. This happens if and only if the coefficients of the above expression (as a polynomial in s, t) vanish, which happens if and only if the following equations are satisfied:

$$\begin{cases} A : a_2^3 + a_3^3 = -1, \\ B : b_2^3 + b_3^3 = -1, \\ C : a_2^2 b_2 = -a_3^2 b_3, \\ D : a_2 b_2^2 = -a_3 b_3^2. \end{cases}$$

Note that a_2, a_3, b_2, b_3 cannot all be nonzero, since otherwise C^2/D gives $a_2^3 = -a_3^3$, contradicting A. Assume $a_2 = 0$. Then we get

$$\begin{cases} a_3^3 = -1, \\ b_3 = 0, \\ b_2^3 = -1, \end{cases}$$

i.e. $a_2 = 0, a_3 = -\omega^j, b_2 = -\omega^k, b_3 = 0$ with ω a primitive 3rd root of unity and $0 \le j, k \le 2$. Thus

$$L = \begin{pmatrix} 1 & 0 & 0 & -\omega^j \\ 0 & 1 & -\omega^k & 0 \end{pmatrix},$$

which gives 9 lines by varying j, k. Permuting the coordinates gives 18 more:

$$\begin{pmatrix} 1 & 0 & -\omega^j & 0 \\ 0 & 1 & 0 & -\omega^k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -\omega^j & 0 & 0 \\ 0 & 0 & 1 & -\omega^k \end{pmatrix}.$$

Thus the Fermat cubic $V(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3$ contains exactly 27 lines.

Remark. A similar computation shows that $V(x^d + y^d + z^d + w^d) \subseteq \mathbb{P}^3$ contains $3d^2$ lines when $d \ge 3$ and $\operatorname{char} k \nmid d$.

Theorem 23.1. *We have the following:*

1. *Every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.*
2. *A general smooth hypersurface of degree $d > 3$ in $\mathbb{P}^3$ contains no lines.*

Remark. Let $N = \binom{3+d}{d} - 1$ and $\mathbb{P} = \mathbb{P}^N$ be the projective space of $k[x_0, x_1, x_2, x_3]_d$. We call this the "parameter space of hypersurfaces of degree d in $\mathbb{P}^3$." Define the *incidence correspondence*

$$\Gamma = \{([F], L) \in \mathbb{P} \times G(2,4) : L \in V(F)\} \subseteq \mathbb{P} \times G(2,4).$$

Let $p : \Gamma \to \mathbb{P}$ and $q : \Gamma \to G(2,4)$ be the projections. Then observe that:

1. $p^{-1}([F])$ is the set of lines contained in $V(F)$.
2. $p(\Gamma)$ is the set of hypersurfaces containing a line.

Lemma 23.1. Γ *is closed in* $\mathbb{P} \times G(2,4)$.

Proof. On the chart $\mathbb{A}^4 \cong U_{1,2} \subseteq G(2,4)$, the points of $U_{1,2}$ correspond to

$$L = \begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}.$$

Now $L \subseteq V(F)$ if and only if $F(s, t, sa_2 + tb_2, sa_3 + tb_3) = 0$ for all $s, t \in k$. As before, this corresponds to the condition that the coefficients of $F(s, t, sa_2 + tb_2, sa_3 + tb_3)$ as a polynomial in s, t vanish. Write $F = \sum c_I x^I$, then this is the vanishing locus of a polynomial in a_i, b_i, c_I. So $\Gamma \cap (\mathbb{P} \times U_{1,2})$ is closed in $\mathbb{P} \times U_{1,2}$. So Γ is closed in $\mathbb{P} \times G(2,4)$ as it is closed in every chart. $\square$

Remark. Which of the maps $p : \Gamma \to \mathbb{P}$ and $q : \Gamma \to G(2,4)$ is easier to analyze? Note that PGL_4 acts on all three of these spaces, and the action of PGL_4 is transitive on $G(2,4)$ but not on $\mathbb{P}$.

Lemma 23.2. *For any $L \in G(2,4)$, we have $q^{-1}(L) \cong \mathbb{P}^\nu$ where $\nu = \dim \mathbb{P} - (d+1)$.*

Proof. After a linear change of coordinates, we may assume

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in G(2,4),$$

i.e. $L = \{[s:t:0:0] : [s:t] \in \mathbb{P}^1\}$. Now if we write $F = \sum c_I x^I$, we have $([F], L) \in q^{-1}(L)$ if and only if $L \subseteq V(F)$, which happens if and only if the coefficients of F corresponding to the monomials $x_0^j x_1^{d-j}$ all vanish, i.e. that

$$c_{d,0,0,0} = c_{d-1,1,0,0} = \cdots = c_{0,d,0,0} = 0.$$

So $q^{-1}(L) \cong \mathbb{P}^\nu$ with $\nu = \dim \mathbb{P} - (d+1)$ (where $(d+1)$ is the number of zero coefficients above). □

Remark. The next steps are the following, which will follow from dimension theory:

1. $\dim \Gamma = \dim q^{-1}(L) + \dim G(2,4) = \dim \mathbb{P} - (d+1) + 4$.
2. $\dim p(\Gamma) \leq \dim \Gamma = \dim \mathbb{P} - (d+1) + 4$.

So once we show this, we get that $p(\Gamma) \subsetneq \mathbb{P}$ for $d > 3$ (i.e. when $(d+1) - 4 > 0$).

23.2 More Dimension Theory

Theorem 23.2. *Let $f : X \to Y$ be a surjective morphism between irreducible varieties with $n = \dim X$ and $m = \dim Y$. Then*

1. *$n \geq m$.*
2. *$\dim f^{-1}(y) \geq n - m$ for $y \in Y$.*
3. *There is an open set $\varnothing \neq U \subseteq Y$ such that equality holds in (2) for all $y \in U$.*

Proof. (1) Since $f : X \to Y$ is surjective, there is an inclusion $f^* : K(Y) \to K(X)$. Now

$$\dim Y = \text{tr.deg}_k K(Y) \leq \text{tr.deg}_k K(X) = \dim X.$$

(2) Fix $y \in Y$. By replacing Y with a neighborhood of y, we may assume that Y is affine. Using Krull's principal ideal theorem, we can argue that there exist $g_1, \ldots, g_r \in A(Y)$ such that $r = \dim Y$ and $\{y\}$ is an irreducible component of $V(g_1, \ldots, g_r)$. By shrinking Y we may assume that

$$\{y\} = V(g_1, \ldots, g_r).$$

Now $f^{-1}(y) = V(f^*g_1, \ldots, f^*g_r)$, so $\dim f^{-1}(y) \geq \dim X - r = \dim X - \dim Y$.

(3) See Milne. □

Proposition 23.1. *Let $f : X \to Y$ be a surjective morphism of varieties with Y irreducible and $f^{-1}(y)$ irreducible of constant dimension for all $y \in Y$. Then X is irreducible and* $\dim X = \dim Y + \dim f^{-1}(y)$.

Proof. See Milne. □

Lecture 24

Hilbert Functions

24.1 Lines on Hypersurfaces in $\mathbb{P}^3$, Continued

Proof of Theorem 23.1. Recall that we let $\mathbb{P} = \mathbb{P}^N$ be the projective space of $k[x_0, x_1, x_2, x_3]_d$, which is the "parameter space of degree d hypersurfaces in $\mathbb{P}^3$." Here $N = \binom{3+d}{d}$. We defined the incidence correspondence

$$\Gamma = \{([F], L) \in \mathbb{P} \times G(2,4) : L \subseteq V(F)\}$$

and the projections $p : \Gamma \to \mathbb{P}$ and $q : \Gamma \to G(2,4)$. Then we want to show:

1. For $d = 3$, that $p : \Gamma \to \mathbb{P}$ is surjective and has finite fibers over a dense open subset of $\mathbb{P}$.
2. For $d > 3$, $p(\Gamma)$ is closed with $p(\Gamma) \subsetneq \mathbb{P}$ (this is the locus of $\mathbb{P}$ which parametrizes hypersurfaces with lines).

Last time we showed $q^{-1}(L) \cong \mathbb{P}^\nu$ with $\nu = \dim \mathbb{P} - (d+1)$ for all $\Gamma \in G(2,4)$. So Γ is irreducible and

$$\dim \Gamma = \dim \mathbb{P} - (d+1) + \dim G(2,4) = \dim \mathbb{P} - (d+1) + 4.$$

Thus if $d > 3$, then $\dim \Gamma < \dim \mathbb{P}$. Thus $p : \Gamma \to \mathbb{P}$ is *not* surjective. So $p(\Gamma) \subsetneq \mathbb{P}$ is a strict closed subset. If $d = 3$, then $\dim \Gamma = \dim \mathbb{P}$. We have already computed that

$$\# p^{-1}(\text{Fermat cubic}) = 27,$$

so $\dim p^{-1}(\text{Fermat cubic}) = 0$. So we must have $p : \Gamma \to \mathbb{P}$ surjective (otherwise $\dim p(\Gamma) < \dim \mathbb{P} = \dim \Gamma$, so $\Gamma \to p(\Gamma)$ has all fibers of positive dimension). Now there exists an open $\varnothing \neq U \subseteq \mathbb{P}$ with $\dim p^{-1}(y) = \dim \Gamma - \dim \mathbb{P} = 0$ for all $y \in U$. This completes the proof. $\square$

Remark. The above proves that a general smooth cubic surface contains exactly 27 lines. To conclude that *all* smooth cubic surfaces contain 27 lines, one can take the following approaches:

1. Show that $p : \Gamma \to \mathbb{P}$ is "smooth" over the locus of smooth cubics, which implies that all fibers have the same dimension. This is done in Gathmann.
2. Show that any smooth cubic surface is isomorphic to

$$X = B_{p_1,\ldots,p_6}\mathbb{P}^2$$

with $p_1, \ldots, p_6 \in \mathbb{P}^2$ in general position (i.e. no 3 points lie on a line and no 6 points lie on a conic). Then we obtain 27 lines as follows:

- 6 from $\pi^{-1}(p_i)$ for $i = 1, \ldots, 6$.

- $15 = \binom{6}{2}$ from $\widetilde{L}_{i,j}$ (the strict transform of $L_{i,j}$), where $L_{i,j}$ is the line through p_i and p_j.
- $6 = \binom{6}{5}$ from $\widetilde{Q}_i$, where Q_i is the (unique) conic through $\{p_1, \ldots, p_6\} \setminus \{p_i\}$.

24.2 Hilbert Functions

Definition 24.1. For a hypersurface $X \subseteq \mathbb{P}^n$ with $I(X) = (f)$, we define the *degree* of X to be

$$\deg X = \deg f.$$

Theorem 24.1 (Bezout's theorem)**.** *If $C, D \subseteq \mathbb{P}^2$ are distinct irreducible curves, then*

$$\#(C \cap D) \leq (\deg C)(\deg D),$$

and equality holds when we count multiplicity.

Remark. We want a version of degree for any projective variety.

Definition 24.2. Let $I \leq k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then the *Hilbert function* of I is

$$\begin{aligned} h_I : \mathbb{N} &\longrightarrow \mathbb{N} \\ d &\longmapsto \dim[k[x_0, \ldots, x_n]/I]_d = \dim k[x_0, \ldots, x_n]_d - \dim I_d. \end{aligned}$$

Similarly, if $X \subseteq \mathbb{P}^n$ is a projective variety, we define its *Hilbert function* to be

$$\begin{aligned} h_X : \mathbb{N} &\longrightarrow \mathbb{N} \\ d &\longmapsto \dim S(X)_d. \end{aligned}$$

Remark. Since $S(X) = k[x_0, \ldots, x_n]/I(X)$, we have $h_X = h_{I(X)}$.

Example 24.2.1. For $\mathbb{P}^n$, we can compute that

$$\begin{aligned} h_{\mathbb{P}^n}(d) = \dim k[x_0, \ldots, x_n]_d &= \binom{n+d}{d} = \frac{(n+d)!}{n!d!} \\ &= \frac{(d+1)\ldots(d+n)}{n!} = \frac{1}{n!}d^n + \text{lower order terms.} \end{aligned}$$

Example 24.2.2. For $X = \{[1 : 0 : \cdots : 0]\} \subseteq \mathbb{P}^n$, we have $I(X) = (x_1, \ldots, x_n)$, so $S(X) \cong k[x_0]$ and

$$h_X(d) = \dim k[x_0]_d = \dim k x_0^d = 1.$$

Example 24.2.3. Let $I = (x^2) \subseteq k[x, y]$. View I as cutting out $[0 : 1]$ with multiplicity 2. For $d \geq 1$,

$$[k[x, y]/I]_d = k\overline{y^d} \oplus k\overline{xy^{d-1}}$$

since $\overline{x^2} = 0$ in the quotient. Thus we see that

$$h_I(d) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d \geq 1. \end{cases}$$

Example 24.2.4. Let $X \subseteq \mathbb{P}^n$ be a hypersurface of degree e, i.e. $I(X) = (f)$ with $\deg f = e$. Then

$$I(X)_d = \begin{cases} 0 & \text{if } d < e, \\ f \cdot k[x_0, \ldots, x_n]_{d-e} & \text{if } d \geq e. \end{cases}$$

So for $d \geq e$, we have

$$\begin{aligned} h_X(d) &= \dim k[x_0, \ldots, x_n]_d - \dim I(X)_d = \dim k[x_0, \ldots, x_n]_d - \dim k[x_0, \ldots, x_n]_{d-e} \\ &= \binom{n+d}{d} - \binom{n+d-e}{d-e} = \frac{d^n}{n!} - \frac{(d-e)^n}{n!} + O(d^{n-2}) = \frac{ed^{n-1}}{(n-1)!} + \text{lower order terms.} \end{aligned}$$

Note that $e = \deg X$ and $n - 1 = \dim X$.

Remark. We have seen the following:

1. $h_{\mathbb{P}^n}(d) = \dfrac{1}{n!} d^n + \text{lower order terms.}$
2. $h_{\text{hyp. of deg. } e}(d) = \dfrac{e}{(n-1)!} d^{n-1} + \text{lower order terms.}$
3. $h_{\text{pt}}(d) = 1$ for all $d \geq 1$.

In these cases, h_X agrees with a polynomial for $d \gg 0$, and the degree of the polynomial is $\dim X$.

Proposition 24.1. *For two homogeneous ideals $I, J \leq k[x_0, \ldots, x_n]$, we have*

$$h_{I \cap J} + h_{I+J} = h_I + h_J.$$

Proof. Use that $R = k[x_0, \ldots, x_n]$ satisfies the following short exact sequence of R-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/(I \cap J) & \longrightarrow & R/I \times R/J & \longrightarrow & R/(I+J) & \longrightarrow & 0 \\ & & \overline{f} & \longmapsto & (\overline{f}, \overline{f}) & & & & \\ & & & & (\overline{g}, \overline{h}) & \longmapsto & \overline{g-h} & & \end{array}$$

Check this as an exercise. Then restricting to degree d components gives the result. □

Remark. If $X, Y \subseteq \mathbb{P}^n$ are distinct projective varieties, then

$$I(X \cup Y) = I(X) \cap I(Y) \quad \text{and} \quad \sqrt{I(X) + I(Y)} \supseteq (x_0, \ldots, x_n)$$

So by Proposition 24.1,

$$h_{X \cup Y}(d) = h_{I(X) \cap I(Y)}(d) = h_{I(X)}(d) + h_{I(Y)}(d) - h_{I(X)+I(Y)}(d) = h_X(d) + h_Y(d)$$

for $d \gg 0$, since $h_{I(X)+I(Y)}(d) = 0$ for $d \gg 0$ by Lemma 24.1.

Example 24.2.5. Let $X = \{p_1, \ldots, p_r\} \subseteq \mathbb{P}^n$. Then

$$h_X(d) = \sum_{i=1}^{r} h_{\{p_i\}}(d) = r, \quad \text{for } d \gg 0.$$

Lemma 24.1. *If $I \le k[x_0, \ldots, x_n]$ is a homogeneous ideal with $\sqrt{I} \supseteq (x_0, \ldots, x_n)$, then $h_I(d) = 0$ for $d \gg 0$.*

Proof. Since $\sqrt{I} \supseteq (x_0, \ldots, x_n)$, there exists $k > 0$ such that $x_i^k \in I$ for every i. So if $d \geq k(n+1)$, then $I_d = k[x_0, \ldots, x_n]_d$. So $h_I(d) = \dim k[x_0, \ldots, x_n]_d - \dim I_d = 0$ for $d \geq k(n+1)$. □

Lecture 25

Nov. 25 — Hilbert Polynomials

25.1 Hilbert Polynomials

Remark. Let $R = k[x_0, \ldots, x_n]$. Geometrically, the functions $0 \neq f \in I(X)_d$ correspond to degree d hypersurfaces $X \subseteq V(f) \subseteq \mathbb{P}^n$ vanishing on X.

Theorem 25.1. *Let $I \leq R$ be a homogeneous ideal. Then there exists a unique polynomial $p_I(d) \in \mathbb{Q}[d]$ such that*

1. *$h_I(d) = p_I(d)$ for $d \gg 0$;*
2. *$m := \deg(p_I) = \dim V_p(I)$;*
3. *$p_I(d) = \frac{\text{integer}}{m!} d^{m-1} + \text{lower order terms}$.*

Definition 25.1. For $X \subseteq \mathbb{P}^n$ projective, set $p_X = p_{I(X)}$. Call p_X the *Hilbert polynomial* of X.

Definition 25.2. A *numerical polynomial* is a polynomial $P \in \mathbb{Q}[d]$ such that $P(d) \in \mathbb{Z}$ for all $d \gg 0$. (We will see that it is equivalent to ask $P(d) \in \mathbb{Z}$ for all $d \in \mathbb{Z}$.)

Example 25.2.1. The following are numerical polynomials:

- A constant polynomial which is integer-valued.
- $P(d) \in \mathbb{Z}[d]$.
- $P_n(d) = \binom{d}{n} = \frac{1}{n!} d(d-1)\ldots(d-n+1)$. Also observe that

$$P_n(d+1) - P_n(d) = \binom{d+1}{n} - \binom{d}{n} = \binom{d}{n-1} = P_{n-1}(d).$$

Lemma 25.1. *We have the following:*

1. *If P is a numerical polynomial, then there exist $c_i \in \mathbb{Z}$ such that*

$$P(d) = c_0 \binom{d}{n} + c_1 \binom{d}{n-1} + \cdots + c_n.$$

2. *If $f : \mathbb{N} \to \mathbb{N}$ is a function such that $\Delta f(d) = f(d+1) - f(d)$ agrees with a numerical polynomial for $d \gg 0$, then f agrees with a numerical polynomial for $d \gg 0$.*

Proof. (a) We induct on $\deg P$. The result is trivial when $\deg P = 0$. Thus assume by induction there exist $c_i \in \mathbb{Z}$ such that

$$\Delta P(d) = P(d+1) - P(d) = c_0\binom{d}{n-1} + c_1\binom{d}{n-2} + \cdots + c_{n-1}.$$

Set $Q(d) = c_0\binom{d}{n} + c_1\binom{d}{n-1} + \cdots + c_{n-1}\binom{d}{1}$. As $\Delta\binom{d}{i} = \binom{d}{i-1}$, we have $\Delta P = \Delta Q$, so $\Delta(P-Q) = 0$. So $P - Q$ is constant. As $P - Q$ is numerical, we have $c_n = P - Q \in \mathbb{Z}$. So $P = Q + c_n$.

(2) See Hartshorne. □

Remark. Now to prove Theorem 25.1, we want to show that $\Delta h_I(d) = h_I(d+1) - h_I(d)$ agrees with a numerical polynomial for $d \gg 0$.

Lemma 25.2. *Let $I \leq R$ be a homogeneous ideal and $0 \neq f \in R_e$. Assume there exists d_0 such that*

(∗) *If $g \in R_d$ with $d \geq d_0$ and $fg \in I$, then $g \in I$,*

then $h_{I+(f)}(d) = h_I(d) - h_I(d-e)$ for $d \gg 0$.

Proof. Consider the short exact sequence

$$R/I \xrightarrow{\cdot f} R/I \longrightarrow R/(I+(f)) \longrightarrow 0$$

For $d - e \geq d_0$, taking degree d parts, we get that

$$0 \longrightarrow [R/I]_{d-e} \xrightarrow{\cdot f} [R/I]_d \longrightarrow [R/(I+(f))]_d \longrightarrow 0$$

This gives the equality for the Hilbert functions when $d - e \geq d_0$. □

Remark. When does (∗) hold? There are essentially two cases:

1. $X \subseteq \mathbb{P}^n$ a projective variety with $I = I(X)$.

   We can write $X = X_1 \cup \cdots \cup X_r$ for the irreducible decomposition of X. So

   $$I = I(X) = I(X_1) \cap \cdots \cap I(X_r).$$

   If f does not vanish on any X_i, then $fg \in I$ implies that fg vanishes on each X_i, so g vanishes on each X_i. So in particular, $g \in I$.

2. $I \leq R$ an arbitrary homogeneous ideal.

   By commutative algebra there exists a *primary decomposition*

   $$I = I_1 \cap \cdots \cap I_r,$$

   which each I_j is *primary* (i.e. $gh \in I_j$ implies $g \in I_j$ or $h \in \sqrt{I_j}$). If we choose $V_p(I_j) \subsetneq V_p(f)$, then (∗) holds: If $gf \in I$, then $gf \in I_j$ for each j. By assumption $f \notin \sqrt{I_j}$ (by the Nullstellensatz), so $g \in I_j$ for every j.

In conclusion, we can always find $f \in R_1$ such that (∗) holds.

Proof of Theorem 25.1. Fix $I \leq R$ a homogeneous ideal. We want to show that $h_I(d)$ agrees with the monomial polynomial of $\deg \dim V_p(I)$ for $d \gg 0$. We induct on $\dim V_p(I)$. If $V_p(I) \neq \varnothing$, then

$$\sqrt{I} \supseteq (x_0, \ldots, x_n),$$

so $h_I(d) = 0$ for $d \gg 0$, which gives the result. Now assume $\dim V_p(I) \geq 0$. Choose $f \in R_1$ such that $(*)$ holds. Then Lemma 25.2 gives

$$h_I(d) - h_I(d-1) = h_{I+(f)}(d).$$

As $\dim V_p(I + (f)) < \dim V_p(I)$, by induction we get that $h_{I+(f)}(d)$ agrees with the numerical polynomial of degree $\dim V_p - 1$, so $h_I(d)$ agrees with a numerical polynomial of dimension $\deg \dim V_p(I)$. $\square$

25.2 Degree

Definition 25.3. For $I \leq R$ a homogeneous ideal with $m = \dim V_p(I)$, the *degree* of I is the integer $\deg I \in \mathbb{Z}$ such that

$$h_I(d) = \frac{\deg I}{m!} d^m + \text{lower order terms}.$$

For a projective variety $X \subseteq \mathbb{P}^n$, set $\deg X = \deg I(X)$, i.e. $h_X(d) = \frac{\deg X}{m!} d^m + \ldots$, where $m = \dim X$.

Example 25.3.1. Consider the following:

1. $X = \mathbb{P}^n$. Then $\deg \mathbb{P}^n = 1$ as $h_{\mathbb{P}^n}(d) = \frac{1}{n!} d^n + \ldots$.
2. If $X \subseteq \mathbb{P}^n$ is a hypersurface of degree e, then $\deg X = e$.
3. If $X = \{p_1, \ldots, p_r\} \subseteq \mathbb{P}^n$, then $\deg X = r$.

   Also, if $\dim V_p(I) = 0$, then $P_I \geq P_{\sqrt{I}} = \#V_p(I)$. In particular, $\deg I \geq \#V_p(I)$.[1]
4. Let $X \subseteq \mathbb{P}^n$ be a linear space. After a linear change of coordinates, we may assume

   $$X = V(x_{r+1}, \ldots, x_n)$$

   for some $0 \leq r \leq n$. So $S(X) = k[x_0, \ldots, x_n]/(x_{r+1}, \ldots, x_n) = k[x_0, \ldots, x_r] = S(\mathbb{P}^r)$.

Now we have that

$$h_X(d) = \binom{r+d}{r} + \frac{1}{r!} d^r + \text{lower order terms},$$

so we see that $\deg X = 1$.

Exercise 25.1. If a projective variety $X \subseteq \mathbb{P}^n$ has $\deg X = 1$, then X is a linear variety.

Proposition 25.1. *If $X, Y \subseteq \mathbb{P}^n$ are projective varieties of dimension m with no common irreducible components, then*

$$\deg(X \cup Y) = \deg X + \deg Y.$$

Proof. We know $h_{X \cup Y}(d) = h_{I(X)}(d) + h_{I(Y)} - h_{I(X)+I(Y)}(d).$, so

$$P_{X \cup Y} = P_X + P_Y - p_{I(X)+I(Y)}.$$

Now $\deg P_X = \deg P_Y = m$ and $\deg P_{I(X)+I(Y)} = \dim V_P(I(X) + I(Y)) = \dim(X \cap Y) < m$, so

$$p_{X \cup Y} = \frac{\deg X}{m!} d^m + \frac{\deg Y}{m!} d^m + \text{lower order terms}.$$

Thus we get that $\deg(X \cup Y) = \deg X + \deg Y$. $\square$

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[1] For an example where these are consider $I = (x^2) \leq k[x, y]$. But $\deg I = 2 > 1 = \#V_p(I)$.

Lecture 26

Dec. 2 — Bézout's Theorem

26.1 Bézout's Theorem

Proposition 26.1. *If $X \subseteq \mathbb{P}^n$ is a projective variety and $f \in k[x_0, \ldots, x_n]$ is a homogeneous polynomial not vanishing on any irreducible component of X, then*

$$\deg(I(X) + (f)) = (\deg X)(\deg f).$$

Proof. Set $e = \deg f$. We first show the $(*)$ condition from last lecture holds: If $g \in k[x_0, \ldots, x_n]$ is homogeneous with $fg \in I(X)$, then we want to show that $g \in I(X)$. As f does not vanish on any irreducible component of X, g must vanish on every irreducible component. So $g \in I(X)$ as desired.

Now we analyze the Hilbert functions. Write

$$h_X(d) = \frac{\deg X}{m!} d^m + ad^{m-1} + \text{lower order terms}$$

for some $a \in \mathbb{Q}$. As $(*)$ holds, for $d \gg 0$ we have

$$\begin{aligned} h_{I(X)+(f)}(d) &= h_X(d) - h_X(d-e) \\ &= \left(\frac{\deg X}{m!} d^m + ad^{m-1}\right) - \left(\frac{\deg X}{m!}(d-e)^m + a(d-e)^{m-1}\right) + \text{lower order terms} \\ &= \frac{e \deg X}{(m-1)!} d^{m-1} + \text{lower order terms}. \end{aligned}$$

Thus we see that $\deg(I(X) + (f)) = e(\deg X) = (\deg f)(\deg X)$. □

Example 26.0.1. If $X = \mathbb{P}^n$ and $f \in k[x_0, \ldots, x_n]_e$ is irreducible, then

$$\deg V_p(f) = \deg((f)) = \deg(I(X) + (f)) = (\deg X)(\deg f) = 1 \cdot \deg f = \deg f,$$

which reproves that the degree of a degree d hypersurface is d.

Theorem 26.1 (Bézout's theorem)**.** *For any two plane curves $X, Y \in \mathbb{P}^2$ (i.e. a closed projective variety in $\mathbb{P}^2$ of pure dimension 1) without common irreducible components,*

$$\#(X \cap Y) \leq (\deg X)(\deg Y).$$

Proof. Let $e = \deg Y$. As $Y \subseteq \mathbb{P}^2$ is a hypersurface, there exists $f \in k[x_0, \ldots, x_n]_e$ with $I(Y) = (f)$. So

$$(\deg X)(\deg Y) = (\deg X)(\deg f) = \deg(I(X) + (f)) = \deg(I(X) + I(Y)).$$

Note that $I(X \cap Y) = \sqrt{I(X) + I(Y)} \supseteq I(X) + I(Y)$, so we get

$$(\deg X)(\deg Y) = \deg(I(X) + I(Y)) \geq \deg(I(X \cap Y)).$$

Since $I(X \cap Y)$ is zero-dimensional, we have $\deg(I(X \cap Y)) = \deg(X \cap Y) = \#(X \cap Y)$. $\square$

Remark. We can make the following extensions to Bézout's theorem:

1. Multiplicities: We can define $\text{mult}_a(X, Y) \in \mathbb{Z}_{>0}$ for $a \in X \cap Y$. Then
$$\sum_{a \in X \cap Y} \text{mult}_a(X, Y) = (\deg X)(\deg Y).$$
2. For a general $Y \subseteq \mathbb{P}^2$ (relative to X), we have equality $(\deg X)(\deg Y) = \#(X \cap Y)$.
3. One can also get a similar statement in higher dimensions.

26.2 Divisors on Curves

Remark. Let C be a smooth projective curve, e.g. $C \cong \mathbb{P}^1$ or $C \subseteq \mathbb{P}^2$ a general degree d hypersurface.

Definition 26.1. A *divisor* on C is a formal $\mathbb{Z}$-linear combination of distinct points, i.e.

$$D = a_1 p_1 + \cdots + a_r p_r, \quad a_i \in \mathbb{Z},\ p_i \in C.$$

Let $\operatorname{Div} C = \bigoplus_{p \in C} \mathbb{Z}p = \{D : D \text{ is a divisor on } C\}$, which is a free abelian group. For $D = \sum a_i p_i \in \operatorname{Div} C$, its *degree* is $\deg D = \sum a_i$ (think of this as the number of points, counting multiplicity).

Example 26.1.1. Given $f \in K(C)$, we want to get a divisor $\operatorname{div} f \in \operatorname{Div}(C)$, called the *divisor of zeros and poles*. For example, if

$$f = \frac{x_0 x_1}{(x_0 - x_1)^2} \in K(\mathbb{P}^1),$$

then we want $\operatorname{div} f = [1:0] + [0:1] - 2[1:1]$.

To define this precisely, for $p \in C$ recall $\mathcal{O}_{C,p}$ is a regular local Noetherian ring of dimension 1. Thus commutative algebra implies that $\mathcal{O}_{C,p}$ is a discrete valuation ring (DVR).

Definition 26.2. A *discrete valuation ring (DVR)* is a UFD with unique irreducible element (up to multiplication by units).

Remark. Assume R is a DVR with $\pi \in R$ irreducible (such an element π is called a *uniformizer*). Then for $0 \neq r \in R$, we can write

$$r = u\pi^m, \quad u \in R^\times,\ m \geq \mathbb{N}.$$

Example 26.2.1. Consider $0 \in \mathbb{A}^1$, so $\mathcal{O}_{\mathbb{A}^1,0} = k[x]_{(x)}$. For $f \in \mathcal{O}_{\mathbb{A}^1,0}$, we can write

$$f = x^m \frac{q(x)}{r(x)}$$

with $m \in \mathbb{N}$ and $q, r \in k[x]$ such that $q(0), r(0) \neq 0$.

Definition 26.3. For $p \in C$, we get a function (in fact a *valuation*)

$$\operatorname{ord}_p : K(C)^\times \longrightarrow \mathbb{Z}$$

where $\operatorname{ord}_p(f)$ is the unique integer such that $f = u\pi^{\operatorname{ord}_p(f)}$ with $u \in \mathcal{O}_{C,p}^\times$ and $\pi \in \mathcal{O}_{C,p}$ a uniformizer.

Definition 26.4. For $0 \neq f \in K(C)$, the *divisor of zeros and poles* of f is

$$\operatorname{div}(f) := \sum_{p \in C} \operatorname{ord}_p(f) p.$$

Divisors of this form are called *principal*.

Exercise 26.1. Show that $\operatorname{div} f$ is a finite sum, so $\operatorname{div} f$ is well-defined.

Example 26.4.1. For $C = \mathbb{P}^1$ and $f = x_0x_1/(x_0 - x_1)^2$ as before, $\operatorname{div} f = [1 : 0] + [0 : 1] - 2[1 : 1]$.

Remark. One can check that $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$, i.e. div is a group homomorphism. So

$$\operatorname{PDiv}(C) = \{\operatorname{div} f : f \in K(C)^\times\}$$

is a subgroup of $\operatorname{Div} C$.

Definition 26.5. We say $D, D' \in \operatorname{Div} C$ are *linearly equivalent* if $D = D' + \operatorname{div} f$ for some $f \in K(C)^\times$, and write $D \sim D'$. The *Picard group* of C is

$$\operatorname{Pic} := (\operatorname{Div} C)/(\operatorname{PDiv} C) = \text{divisors on } C \text{ up to linear equivalence.}$$

Example 26.5.1. We have $\operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z}$.

We give a sketch of the proof. There is a surjective group homomorphism

$$\begin{aligned} \varphi : \operatorname{Div} \mathbb{P}^1 &\longrightarrow \mathbb{Z} \\ D &\longmapsto \deg D. \end{aligned}$$

Then we want to show that $\ker \varphi = \operatorname{PDiv}(\mathbb{P}^1)$. First fix $D \in \ker \varphi$, so $D = \sum_{i=1}^r a_i p_i$ with $\sum_{i=1}^r a_i = 0$. Write $p_i = [s_i : t_i]$, and consider $f = \prod_{i=1}^r (x_0 t_i - x_1 s_i)^{a_i}$. Since $\sum_{i=1}^r a_i = 0$, we have $f \in K(\mathbb{P}^1)$ and $\operatorname{div} f = D$, so $D \in \operatorname{PDiv}(\mathbb{P}^1)$. The reverse inclusion is similar.

Note that we always have $\operatorname{PDiv}(\mathbb{P}^1) \subseteq \ker \varphi$, but the other inclusion may fail for other curves.

Remark. Using Bézout's theorem (the one which counts multiplicities), one can show that

$$\deg(\operatorname{div} f) = 0$$

for any $f \in K(C)^\times$.

Example 26.5.2. Let $C \subseteq \mathbb{P}^2$ be a smooth cubic plane curve (i.e. an *elliptic curve*), and fix a point $p_0 \in C$. Up to a change of coordinates, we may assume that $p_0 = [1 : 0 : 0]$. Then

1. $\operatorname{Pic} C$ is *not* finitely generated.
2. The map $C \to \operatorname{Pic}^0(C)$ by $p \mapsto p - p_0$ is a bijection, where $\operatorname{Pic}^0(C) = \{D \in \operatorname{Pic} C : \deg D = 0\}$. As $\operatorname{Pic}^0(C)$ is a group, this bijection gives a group structure on C with identity p_0.

One can also describe this more geometrically. If $p = [1 : 0 : 0]$, then up to a change of coordinates, one can assume C is given by

$$y^2 = x^3 + ax + b$$

on a chart $\mathbb{A}^2_{x,y} \subseteq \mathbb{P}^2_{x:y:z}$. The usual group law on C is given as follows: For $p, q \in C$, let r be the unique third intersection point of the line through p, q with C. Then we define $p + q$ to be the reflection s of r across the x-axis.

To see this agrees with the group structure defined from $\mathrm{Pic}^0(C)$, we need to check

$$(p - p_0) + (q - p_0) \sim s - p_0$$

Let the line through p, q be $\ell = V(L(x, y, z))$ and the line through r, s be $\ell' = V(L'(x, y, z))$. Then

$$0 \sim \operatorname{div} \frac{L(x,y,z)}{z} = p + q + r - 3p_0 \quad \text{and} \quad 0 \sim \operatorname{div} \frac{L'(x,y,z)}{z} = r + s + p_0 - 3p_0,$$

which shows that the two definitions for the group law agree.