MATH 6421: Algebraic Geometry I

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Contents

Lecture 1

Aug. 19 — Affine Varieties

1.1 Motivation for Algebraic Geometry

Remark. Why study algebraic geometry? Algebraic geometry connects to many fields of math.

Example 1.0.1. Consider a plane curve $\{f(z_1, z_2) = 0\} \subseteq \mathbb{C}^2$, e.g. an elliptic curve $z_2^2 - z_1^3 + z_1 - 1 = 0$. Compactify and set C to be the closure of C^0 in $\mathbb{CP}^2$, and let $d = \deg f$. There are connections in

1. Topology: $H^1(C, \mathbb{C}) \cong \mathbb{C}^{2g}$, where $g = (d-1)(d-2)/12$;
2. Arithmetic: the number of $\mathbb{Q}$-points is finite if $d > 3$;
3. Complex geometry: We have $C \cong \mathbb{CP}^2$ for $d = 1, 2$, $C \cong \mathbb{C}/\Lambda$ for $d = 3$, and $C \cong \mathbb{H}/\Gamma$ for $d > 3$.

1.2 Affine Varieties

Fix an algebraically closed field k (e.g. $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}}_p$, etc.).

Definition 1.1. *Affine space* is the set $\mathbb{A}^n = \mathbb{A}_k^n = \{\vec{a} = (a_1, \ldots, a_n) : a_i \in k\}$.

Remark. Note the following:

1. $\mathbb{A}_k^n$ is the same set as k^n, but forgetting the vector space structure;
2. $f \in k[x_1, \ldots, x_n]$ gives a polynomial function $\mathbb{A}_k^n \to k$ by evaluation: $a \mapsto f(a)$.

Definition 1.2. For a subset $S \subseteq k[x_1, \ldots, x_n]$, its *vanishing set* is

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}.$$

An *affine variety* is a subset of $\mathbb{A}_k^n$ of this form.

Example 1.2.1. Consider the following:

1. $\mathbb{A}^n = V(\varnothing) = V(\{0\})$;
2. $\varnothing = V(1) = V(k[x_1, \ldots, x_n])$;
3. a point $a = (a_1, \ldots, a_n)$ is an affine variety: $V(\{x_1 - a_1, \ldots, x_n - a_n\}) = \{a\}$;
4. a linear space $L \subseteq \mathbb{A}^n$ (it is the kernel of some matrix);
5. plane curves $V(f(x, y)) \subseteq \mathbb{A}_{x,y}^2$;

6. $\mathrm{SL}_n(k) \subseteq \mathbb{A}^{n\times n}$ is an affine variety: $\mathrm{SL}_n(k) = V(\det([x_{i,j}]) - 1)$;
7. $\mathrm{GL}_n(k)$ (as a set) is an affine variety in $\mathbb{A}^{n\times n+1}$: $\mathrm{GL}_n(k) = V(\det([x_{i,j}])y - 1)$;
8. if $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety;
9. the affine varieties $X \subseteq \mathbb{A}^1_k$ are of the form: finite set of points, $\varnothing$, or $\mathbb{A}^1_k$.

Proposition 1.1 (Relation to ideals)**.** *If $S \subseteq k[x_1, \ldots, x_n]$, then $V(S) = V(\langle S\rangle)$, where $\langle S\rangle$ is the ideal generated by S.*

Proof. Since $S \subseteq \langle S\rangle$, we have $V(\langle S\rangle) \subseteq V(S)$. Conversely, if $f, g \in S$ and $h \in k[x_1, \ldots, x_n]$, then $f + g$ and hf vanish on $V(S)$, so we see that $V(S) \subseteq V(\langle S\rangle)$. □

Remark. The statement implies that if $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, then $V(f_1, \ldots, f_r) = V((f_1, \ldots, f_n))$. The following are some further applications of the statement:

1. affine varities are vanishing loci of ideals;
2. if $X \subseteq \mathbb{A}^n$ is an affine variety, then X is cut out by finitely many polynomial equations.

To see the second statement, note that $X = V(I)$ for some ideal $I \leq k[x_1, \ldots, x_n]$. By the Hilbert basis theorem that $k[x_1, \ldots, x_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $I = (f_1, \ldots, f_r)$. So $X = V(I) = V(f_1, \ldots, f_r)$.

Proposition 1.2 (Properties of the vanishing set)**.** *For ideals I, J of $k[x_1, \ldots, x_n]$,*

1. *if $I \subseteq J$, then $V(J) \subseteq V(I)$;*
2. *$V(I) \cap V(J) = V(I + J)$;*
3. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. (1) This follows from definitions and actually holds for general subsets.

(2) Note that $V(I) \cap V(J) = V(I \cap J) = V(\langle I \cup J\rangle) = V(I + J)$.

(3) We only prove the first equality, the second is similar. Recall that $IJ = \left\{\sum_{i=1}^r f_i g_i : f_i \in I, g_i \in J\right\}$. We have the forwards inclusion $V(I) \cup V(J) \subseteq V(IJ)$ from definitions. For the reverse inclusion, consider a point $x \notin V(I) \cup V(J)$. So there exists $f \in I$ and $g \in J$ such that $f(x), g(x) \neq 0$. So $f(x)g(x) \neq 0$, which implies that $x \notin V(IJ)$. Thus $V(IJ) \subseteq V(I) \cup V(J)$ as well. □

Remark. The above implies that if X and Y are affine varieties in $\mathbb{A}^n_k$, then so are $X \cup Y$ and $X \cap Y$.

Example 1.2.2. Consider $V(y^2 - x^2, y^2 + x^2) \subseteq \mathbb{A}^2$. Note that $(y^2 - x^2, y^2 + x^2) = (x^2, y^2)$, from which we can easily see that $V(y^2 - x^2, y^2 + x^2) = V(x^2, y^2) = \{0\}$.

1.3 Correspondence with Ideals

Remark. Our goal is to build a correspondence between affine varieties in $\mathbb{A}^n_k$ and ideals of $k[x_1, \ldots, x_n]$.

Definition 1.3. For a subset $X \subseteq \mathbb{A}^n_k$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

Remark. Note that $I(X)$ is in fact an ideal of $k[x_1, \dots, x_n]$.

Example 1.3.1. Consider the following:

1. $I(\varnothing) = k[x_1, \dots, x_n]$;
2. $I(\mathbb{A}_k^n) = \{0\}$, this will follow from the Hilbert nullstellensatz and relies on $k = \overline{k}$ (for $k = \mathbb{R}$, the polynomial $x^2 + y^2 + 1$ is always nonzero and thus lies in $I(\mathbb{A}_{\mathbb{R}}^n)$);
3. for $n = 1$, if $S \subseteq \mathbb{A}_k^1$ be an infinite set, then $I(S) = (0)$.
4. for $n = 1$, we have $I(V(x^2)) = I(\{0\}) = (x)$.

Remark. What properties does $I(X)$ satisfy?

Definition 1.4. Let R be a ring. The *radical* of an ideal $J \leq R$ is

$$\sqrt{J} = \{f \in R : f^n \in J \text{ for some } n > 0\}.$$

An ideal J is *radical* if $J = \sqrt{J}$.

Exercise 1.1. Check the following:

1. $\sqrt{J}$ is always an ideal.
2. $\sqrt{\sqrt{J}} = \sqrt{J}$.
3. An ideal $J \leq R$ is radical if and only if R/J is reduced.[1]

Proposition 1.3. *If $X \subseteq \mathbb{A}_k^n$ is a subset (not necessarily an affine variety), then $I(X)$ is radical.*

Proof. Fix $f \in k[x_1, \dots, x_n]$. If $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$. This implies $f(x) = 0$ for all $x \in X$, so $f \in I(X)$. Thus we see that $I(X) = \sqrt{I(X)}$. □

Theorem 1.1 (Hilbert's nullstellensatz)**.** *If $J \leq k[x_1, \dots, x_n]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

Example 1.4.1. Let $n = 1$, so that $k[x]$ is a PID. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. Then

$$I(V(f)) = I(\{a_1, \dots, a_r\}) = ((x - a_1) \cdots (x - a_r)).$$

[1]Recall that a ring R is *reduced* if for all nonzero $f \in R$ and positive integers n, we have $f^n \neq 0$. It is immediate that an integral domain is reduced.

Lecture 2

Aug. 21 — Hilbert's Nullstellensatz

2.1 Applications of Hilbert's Nullstellensatz

Corollary 2.0.1 (Weak nullstellensatz). *If $J \leq k[x_1, \dots, x_n]$ is an ideal with $J \neq (1)$, then $V(J) \neq \varnothing$. Equivalently, if $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ have no common zeros, then there exist $g_1, \dots, g_r \in k[x_1, \dots, x_n]$ such that $\sum_{i=1}^r f_i g_i = 1$.*

Proof. Assume otherwise that $V(J) = \varnothing$. Then $I(V(J)) = I(\varnothing) = (1)$, so by Hilbert's nullstellensatz, we have $\sqrt{J} = (1)$. Then $1^n \in J$ for some $n > 0$, so $1 \in J$, i.e. $J = (1)$. □

Remark. We need k to be algebraically closed. Note that $(1) \neq (x^2 + 1) \leq \mathbb{R}[x]$ but $V(x^2 + 1) = \varnothing$.

Corollary 2.0.2. *There is an inclusion-reversing bijection between radical ideals $J \leq k[x_1, \dots, x_n]$ and affine varieties $X \subseteq \mathbb{A}_k^n$ given by $J \mapsto V(J)$ with inverse $X \mapsto I(X)$.*

Proof. It suffices to show that these maps are inverses. For $J \leq k[x_1, \dots, x_n]$ a radical ideal, we have

$$I(V(J)) = \sqrt{J} = J$$

by Hilbert's nullstellensatz. For $X \subseteq \mathbb{A}_k^n$ an affine variety, we clearly have $X \subseteq V(I(X))X$. For the reverse inclusion, choose an ideal $J \leq k[x_1, \dots, x_n]$ such that $V(J) = X$. Then $J \subseteq I(X)$, so we have $V(I(X)) \subseteq V(J) = X$. Thus we also get $V(I(X)) = X$. □

Remark. This implies that maximal ideals in $k[x_1, \dots, x_n]$ correspond to points in $\mathbb{A}_k^n$, since maximal ideals correspond to minimal varieties under this bijection.

Corollary 2.0.3. *If X_1, X_2 are affine varieties in $\mathbb{A}_k^n$, then*

1. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;
2. $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (1) This follows from definitions.

(2) Write $I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$. □

Example 2.0.1. The radical in (2) is necessary. Consider $X_1 = V(y)$ and $X_2 = V(y - x^2)$ in $\mathbb{A}_k^2$. Then $X_1 \cap X_2 = \{(0,0)\} \subseteq \mathbb{A}_k^2$, so $I(X_1 \cap X_2) = (x, y)$. However, $I(X_1) + I(X_2) = (y) + (y - x^2) = (y, x^2)$.

Note that it is sometimes better to consider (y, x^2) anyway as it tracks multiplicities. In particular, we can see the multiplicity in the dimension of $k[x,y]/(x, y^2) \cong \overline{1}k \oplus \overline{y}k$ as a k-vector space.

2.2 Proof of Hilbert's Nullstellensatz

We will assume the following result from commutative algebra without proof:

Theorem 2.1 (Noether normalization)**.** *Let A be a finitely generated algebra over a field k with A a domain. Then there is an injective k-algebra homomorphism $k[z_1, \dots, z_n] \hookrightarrow A$ that is finite, i.e. A is a finitely generated $k[z_1, \dots, z_n]$-module.*

Corollary 2.1.1. *If $K \subseteq L$ is a field extension and L is a finitely generated K-algebra, then $K \subseteq L$ is a finite field extension. In particular, if in addition $K = \overline{K}$, then $K = L$.*

Proof. By Noether normalization, there exists a k-algebra homomorphism $K[z_1, \dots, z_n] \to L$ that is finite. Then by a result from commutative algebra, L is integral over $K[z_1, \dots, z_n]$, which implies that $K[z_1, \dots, z_n]$ must also be a field since L is. Thus $n = 0$, so $K \subseteq L$ is a finite extension. □

Proposition 2.1. *If $(1) \neq J \leq R$ is an ideal, then J is contained in some maximal ideal.*

Proof. Consider the set $P = \{I \leq R : J \subseteq I, I \neq (1)\}$ with the partial order given by inclusion. Note that $P \neq \varnothing$ since $J \in P$. Furthermore, every chain in P has an upper bound (for $\{I_\alpha : \alpha \in A\}$ a chain P, we can take $\bigcup_{\alpha \in A} I_\alpha$, which one can check is indeed an ideal that lies in P; note that $1 \notin I_\alpha$ implies $1 \notin \bigcup_{\alpha \in A} I_\alpha$). So Zorn's lemma implies there is a maximal element in P, which is a maximal ideal. □

Proof of Theorem 1.1. We will proceed in the following steps:

1. Show that the maximal ideals of $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for $a_i \in k$.
2. Prove the weak nullstellensatz: If $1 \neq J \leq k[x_1, \dots, x_n]$, is an ideal, then $V(J) \neq \varnothing$.
3. Prove the (strong) nullstellensatz: $I(V(J)) = \sqrt{J}$ for $J \leq k[x_1, \dots, x_n]$.

The most difficult part is the first step and is where we need k to be algebraically closed.[1]

(1) For $a_1, \dots, a_n \in k$, the ideal $(x_1 - a_1, \dots, x_n - a_n)$ is maximal (the quotient is k, which is a field). Conversely, fix a maximal ideal $\mathfrak{m} \in k[x_1, \dots, x_n]$. Since

$$k \xrightarrow{\phi} k[x_1, \dots, x_n]/\mathfrak{m} = L$$

is a finitely generated k-algebra and k is algebraically closed, ϕ is an isomorphism by Corollary 2.1.1. Choose $a_i \in k$ such that $\phi(a_i) = \overline{x}_i$, so $\overline{x_i - a_i} = 0$ in L Then $(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$, so they must be equal since both the left and right hand sides are maximal ideals.

(2) By Proposition 2.1, J is contained in some maximal ideal $\mathfrak{m}$. By (1), $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$. Since $J \subseteq \mathfrak{m}$, we have $V(J) \supseteq V(\mathfrak{m}) \supseteq \{(a_1, \dots, a_n)\}$, so $J \neq \varnothing$.

(3) The reverse inclusion follows from definitions. For the forward inclusion, fix $f \in I(V(J))$, and we want to show that $f^n \in J$ for some $n > 0$. Add a new variable y and consider

$$J_1 = (J, fy - 1) \leq k[x_1, \dots, x_n, y].$$

Now $V(J_1) = \{(a, b) = (a_1, \dots, a_n, b) \in \mathbb{A}_k^{n+1} : a \in V(J), f(a)b = 1\} = \varnothing$ since f vanishes on $V(J)$, so $f(a)b = 0$ for any b. Thus by the weak nullstellensatz, $J_1 = (1)$, so $1 = \sum_{i=1}^r g_i f_i + g_0(fy - 1)$ with

[1] The statement is false when k is not algebraically closed: $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

$f_1, \ldots, f_r \in J$ and $g_0, \ldots, g_r \in k[x_1, \ldots, x_n, y]$. Let N be the maximal power of y in the g_i. Multiplying by f^N, we get

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, fy) f_i + G_0(x_1, \ldots, x_n, fy)(fy - 1)$$

with $G_i \in k[x_1, \ldots, x_n, fy]$. So if we set $fy = 1$, then we have

$$f^N = \sum_{i=1}^{r} G_i(x_1, \ldots, x_n, 1) f_i + 0 \in J,$$

which gives $f \in \sqrt{J}$. To justify this substitution, we can consider the quotient $k[x_1, \ldots, x_n, y]/(fy - 1)$. We have a map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, y]/(fy - 1)$, which is injective since $(fy - 1)$ does not lie in $k[x_1, \ldots, x_n]$, so an equality in the quotient implies an equality in $k[x_1, \ldots, x_n]$. $\square$

Lecture 3

Aug. 26 — The Zariski Topology

3.1 Polynomial Functions and Subvarieties

Remark. Recall that a polynomial $f \in k[x_1, \ldots, x_n]$ gives a function $\mathbb{A}_k^n \to k$ by $a \mapsto f(a)$.

Proposition 3.1. *If $f, g \in k[x_1, \ldots, x_n]$ give the same function $\mathbb{A}_k^n \to k$, then $f = g$ in $k[x_1, \ldots, x_n]$.*

Proof. Assume $f = g$ as polynomial functions. Then $V(f-g) = \mathbb{A}_k^n$, so $\sqrt{(f-g)} = I(\mathbb{A}_k^n) = (0)$ by Hilbert's nullstellensatz (note that we can also prove $I(\mathbb{A}_k^n) = (0)$ directly, it is enough to have k be an infinite field for this part). Thus $f - g = 0$, so $f = g$ in $k[x_1, \ldots, x_n]$. $\square$

Remark. In the above proposition, we need k to be an infinite field (e.g. if $k = \overline{k}$): Otherwise, there are only finitely many functions $\mathbb{A}_k^n \to k$, but infinitely many polynomials in $k[x_1, \ldots, x_n]$.

Remark. The set of polynomials functions $\mathbb{A}_k^n \to k$ form a ring, and the above proposition implies that this ring is isomorphic to $k[x_1, \ldots, x_n]$.

Definition 3.1. A *polynomial function* on an affine variety $X \subseteq \mathbb{A}_k^n$ is a function $\varphi : X \to k$ such that there exists $f \in k[x_1, \ldots, x_n]$ with $\varphi(a) = f(a)$ for every $a \in X$.

Definition 3.2. The *coordinate ring* of X is $A(X) = \{f : X \to k \mid f \text{ is a polynomial function}\}$, which is a ring under pointwise addition and multiplication.

Remark. Observe that there exists a surjective ring homomorphism

$$\begin{aligned} k[x_1, \ldots, x_n] &\longrightarrow A(X) \\ f &\longmapsto (a \mapsto f(a)) \end{aligned}$$

with kernel $I(X)$. Thus we have $A(X) \cong k[x_1, \ldots, x_n]/I(X)$.

Remark. We can now replace $\mathbb{A}_k^n$ and $k[x_1, \ldots, x_n]$ by X and $A(X)$ to study *subvarieties* of X.

Definition 3.3. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety. If $S \subseteq A(X)$ is a subset, then define

$$V_X(S) = \{a \in X : f(a) = 0 \text{ for all } f \in S\}.$$

A subset of X of this form is called an *affine subvariety* of X. (Equivalently, these are the same as an affine variety $Y \subseteq \mathbb{A}_k^n$ such that $Y \subseteq X$.) For $Y \subseteq X$ a subvariety, define

$$I_X(Y) = \{f \in A(X) : f(a) = 0 \text{ for all } a \in Y\}.$$

Proposition 3.2. *There is a bijective correspondence between radical ideals in $A(X)$ and affine subvarieties of X given by $J \mapsto V_X(J)$ and $Y \mapsto I_X(Y)$.*

Proof. See Homework 2. □

3.2 The Zariski Topology

Definition 3.4. The *Zariski topology* on $\mathbb{A}^n_k$ is the topology with closed sets $V(I) \subseteq \mathbb{A}^n_k$, where I is an ideal in $k[x_1, \ldots, x_n]$. (Equivalently, the closed sets are the affine varieties in $\mathbb{A}^n_k$.)

Remark. Note the following:

1. On $\mathbb{A}^1_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^1_k$, or finite collections of points.
2. When $k = \mathbb{C}$, then $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ being Zariski closed implies that X is closed in the analytic topology on $\mathbb{A}^n_{\mathbb{C}}$. In particular, the Zariski topology is coarser than the analytic topology.
3. On $\mathbb{A}^2_k$, the closed sets are of the form: $\varnothing$, $\mathbb{A}^2_k$, finite collections of points, plane curves, and their finite unions.

Proposition 3.3. *The Zariski topology on $\mathbb{A}^n_k$ is indeed a topology.*

Proof. First note that $\varnothing = V((1))$ and $\mathbb{A}^n_k = V((0))$ are closed. For arbitrary intersections, note that $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$, and for finite unions, note that $\bigcup_{i=1}^r V(I_i) = V(I_1 \cdots I_r)$. □

Example 3.4.1. The Zariski topology on $\mathbb{A}^{n+m}_k$ is in general *not* the product topology of the Zariski topologies on $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$. Consider $V(y - x^2) \subseteq \mathbb{A}^2_k$, which is a closed set in the Zariski topology, but the only closed sets in $\mathbb{A}^1_k$ are either $\varnothing$, $\mathbb{A}^1_k$, or finite.

Definition 3.5. If $X \subseteq \mathbb{A}^n_k$ is an affine variety, then we can define the *Zariski topology* on X in the following two equivalent ways:

1. take the subspace topology from the Zariski topology on $\mathbb{A}^n_k$;
2. take the closed sets of X to be of the form $V_X(I)$ for some ideal $I \leq A(X)$.

This is because an affine subvariety of X is precisely the intersection of X with an affine variety in $\mathbb{A}^n_k$.

Remark. Our goal now is to relate properties of the Zariski topology on X to the ring $A(X)$, and then to the ideal $I(X) \leq k[x_1, \ldots, x_n]$.

Definition 3.6. A topological space X is *reducible* if we can write $X = X_1 \cup X_2$ for some closed sets $X_1, X_2 \subsetneq X$. Otherwise, X is called *irreducible*.

Example 3.6.1. The plane curve $X = V(y^2 - x^2y) = V(y) \cup V(y - x^2)$ is reducible.

Remark. Note the following:

1. A disconnected topological space is reducible.
2. Many topologies are reducible, e.g. $\mathbb{C}^n$, $\mathbb{R}^n$ with the analytic topology.
3. If X is irreducible and $U \subseteq X$ is a nonempty open set, then $\overline{U} = X$ (we have $\overline{U} \cup (X \setminus U) = X$).

Lecture 4

Aug. 28 — Irreducibility

4.1 Properties of Irreducibility

Proposition 4.1. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then the following are equivalent:*

1. *X is irreducible;*
2. *$I(X) \leq k[x_1, \ldots, x_n]$ is a prime ideal;*
3. *the coordinate ring $A(X)$ is an integral domain.*

Example 4.0.1. We have the following:

1. $\mathbb{A}_k^n$ is irreducible as $A(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$, which is an integral domain.
2. A *hypersurface* $X \subseteq \mathbb{A}_k^n$ is an affine variety with $I(X) = (f)$ for some $f \in k[x_1, \ldots, x_n]$. Then A is irreducible if and only if (f) is prime, if and only if f is irreducible.[1]

4.2 Dimension

Definition 4.1. Let X be a topological space.

- The *dimension* of X, denoted $\dim X$, is the supremum of the n such that there exists a chain of irreducible closed subspaces
$$X \supseteq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n \neq \varnothing.$$
- For $Y \subseteq X$ closed and irreducible, the *codimension* of Y in X, denoted $\operatorname{codim}_X Y$, is the supremum of the n as above such that $X_n = Y$.

[1] Note that any prime ideal is radical.

Lecture 5

Sept. 2 — Dimension

5.1 More on Dimension

Remark. Recall the following correspondence from before: If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then there exists a bijection between the irreducible closed subsets $Y \subseteq X$ and the prime ideals $\mathfrak{p} \leq A(X)$.

Definition 5.1. For a ring A, the *(Krull) dimension* of A, denoted $\dim A$, is the supremum of the n such that there exists a chain of prime ideals

$$A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

For a prime ideal $\mathfrak{q} \leq A$, the *height* of $\mathfrak{q}$, denoted $\operatorname{ht} \mathfrak{q}$, is the supremum of the n as above with $\mathfrak{p}_0 = \mathfrak{q}$.

Remark. If X is an affine variety, then we have the following:

1. $\dim X = \dim A(X)$;
2. for $Y \subseteq X$ a closed irreducible subset, $\operatorname{codim}_X Y = \operatorname{ht} I_X(Y)$.

These properties follow from the inclusion-reversing correspondence.

Definition 5.2. Let $K \subseteq L$ be a field extension.

1. A collection of elements $\{z_i : i \in I\} \subseteq L$ is a *transcendence basis* of $K \subseteq L$ if the z_i are algebraically independent (i.e. $K(x_i : i \in I) \xrightarrow{\cong} K(z_i : i \in I)$ by $x_i \mapsto z_i$) and $K(z_i : i \in I) \subseteq L$ is algebraic.
2. The *transcendence degree* $\operatorname{tr.deg}_K L$ is the cardinality of a transcendence basis.

Theorem 5.1 (Dimension theory)**.** *Let A be a finitely generated k-algebra that is a domain. Then*

1. $\dim A = \operatorname{tr.deg}_k \operatorname{Frac}(A)$*;*
2. *for any prime ideal $\mathfrak{p} \leq A$, we have* $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$*;*
3. *all maximal chains of prime ideals $A \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n$ are of the same length.*

Remark. The following are consequences of the above result from commutative algebra:

1. $\dim_k \mathbb{A}_k^n = \dim k[x_1, \dots, x_n] = \operatorname{tr.deg}_k k(x_1, \dots, x_n) = n$.
2. If X is irreducible, then $A(X)$ is a domain, so for $x \in X$, we have

   $$\operatorname{codim}_X \{x\} = \operatorname{ht} I(\{x\}) = \dim A(X) - \dim A(X)/I(\{x\}) = \dim A(X) = \dim X,$$

   where we note that $A(X)/I(\{x\}) \cong k$ is a field.

3. If X is an irreducible affine variety and $U \subseteq X$ is a nonempty open subset, then
$$\dim U = \sup_{x \in U} \operatorname{codim}_U \{x\} = \sup_{x \in U} \operatorname{codim}_X \{x\} = \dim X.$$
This follows since we can pass from a chain in U to a chain in X by taking closures.

4. If X is an irreducible affine variety and $Z \subseteq X$ is an irreducible closed subset, then
$$\dim Z = \dim X - \operatorname{codim}_X Z.$$

Note that (2)-(4) can be false if X is not irreducible. To contradict (4), let $X = V(x,y) \cup V(z) \subseteq \mathbb{A}^3_k$ with $Z = V(x,y)$. Then we have $\dim X = 2$, $\dim Z = 1$, $\operatorname{codim}_X Z = 0$.

5.2 Hypersurfaces

Remark. We now want to study hypersurfaces.

Theorem 5.2 (Krull's Hauptidealsatz)**.** *If A is a Noetherian ring and $f \in A$ is nonzero and a non-unit, then every minimal prime ideal containing f has height* 1*.*

Corollary 5.2.1. *If $X \subseteq \mathbb{A}^n_k$ is an irreducible affine variety and $f \in A(X)$ is a nonzero non-unit, then*
$$\dim Z = \dim X - 1$$
for every irreducible component Z of $V_X(f)$.

Proof. Since X is irreducible, $A(X)$ is a domain. So there is a correspondence between the minimal prime ideals $f \in \mathfrak{p} \subsetneq A(X)$ and the minimal irreducible closed subsets $Z \supseteq V_X(f)$, which corresponds to the irreducible components Z of $V_X(f)$. For such a component Z, we know
$$\dim Z = \dim Z - \operatorname{codim}_X Z = \dim X - \operatorname{ht} I(Z) = \dim X - 1$$
by Krull's Hauptidealsatz, which is the desired result. □

Example 5.2.1. Corollary 5.2.1 implies that if $f \in k[x_1, \ldots, x_n]$ is non-constant, then
$$\dim V(f) = \dim \mathbb{A}^n_k - 1 = n - 1.$$

Theorem 5.3. *An irreducible affine variety $Y \subseteq \mathbb{A}^n_k$ has $\dim Y = n-1$ if and only if $Y = V(f)$ for some non-constant polynomial $f \in k[x_1, \ldots, x_n]$.*

Proof. ($\Leftarrow$) This was Corollary 5.2.1.

($\Rightarrow$) We will use that $A(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$ is a UFD. Since Y is irreducible and $\dim Y = n-1$,
$$\operatorname{ht} I(Y) = \operatorname{codim}_{\mathbb{A}^n_k} Y = \dim \mathbb{A}^n_k - \dim Y = 1.$$
Since $(0) \subsetneq I(Y) \subsetneq k[x_1, \ldots, x_n]$, there exists a non-constant $f \in k[x_1, \ldots, x_n]$ with $f \in I(Y)$. Write
$$f = f_1 \cdots f_r$$
with f_i irreducible by unique factorization, and note that the f_i are also prime since we are in a UFD. Since $I(Y)$ is prime, some f_i is in $I(Y)$, so we have the inclusions
$$(0) \subsetneq (f_i) \subseteq I(Y).$$
Since $\operatorname{ht} I(Y) = 1$, we must have $(f_i) = I(Y)$, so $Y = V(I(Y)) = V(f_i)$. □

5.3 Regular Functions

Definition 5.3. Let X be an affine variety and $U \subseteq X$ open. A function $\varphi : U \to k$ is *regular* if for each $a \in U$, there exists an open neighborhood $a \in U_a \subseteq U$ and $f, g \in A(X)$ such that

$$\varphi(x) = \frac{g(x)}{f(x)}, \quad f(x) \neq 0, \quad \text{for all } x \in U_a.$$

Define $\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is a regular function on } U\}$.

Exercise 5.1. Check that $\mathcal{O}_X(U)$ is a ring under pointwise addition and multiplication of outputs.

Remark. To patch open sets together, we will later need the notion of a *morphism*, and a morphism $U \to Y \subseteq \mathbb{A}_k^m$ should be given by

$$x \longmapsto (\varphi_1(x), \ldots, \varphi_m(x))$$

with φ_i regular functions on U.

Example 5.3.1. We have the following:

1. If $X \subseteq \mathbb{A}_k^n$ is an affine variety, then any $\varphi \in A(X)$ is regular. Furthermore, we get an injective ring homomorphism $A(X) \to \mathcal{O}_X(X)$. We will see that this is an isomorphism.
2. If $X = \mathbb{A}_k^1$ and $U = \mathbb{A}_k^1 \setminus \{0\}$, then for any $n \geq 0$ and $g \in k[x]$, the function g/x^n is regular on U. In general, if we fix $f, g \in A(X)$ and set $U = X \setminus V(f)$, then the map g/f^m is regular on U.
3. Let $X = V(x_1x_4 - x_2x_3) \subseteq \mathbb{A}_k^4$ and $U = X \setminus V(x_2, x_4)$. Then the following map is regular:

$$\begin{aligned} \varphi : U &\longrightarrow k \\ (x_1, x_2, x_3, x_4) &\longmapsto \begin{cases} x_1/x_2, & \text{if } x_2 \neq 0, \\ x_3/x_4, & \text{if } x_4 \neq 0. \end{cases} \end{aligned}$$

   Note that on $U \setminus V(x_2x_4)$, we have $x_1/x_2 = x_3/x_4$ since $x_1x_4 = x_2x_3$ on X.

Lecture 6

Sept. 4 — Regular Functions

6.1 Properties of Regular Functions

Proposition 6.1. *Let X be an affine variety and $U \subseteq X$ open. Then:*

1. *if $\varphi \in \mathcal{O}_X(U)$, then $V(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U;*
2. *(identity principle) If X is irreducible, $U \subseteq X$ is nonempty and open, and $\varphi, \psi \in \mathcal{O}_X(U)$ with $\varphi|_W = \psi|_W$ for some $W \subseteq U$ nonempty and open, then $\varphi = \psi$ in $\mathcal{O}_X(U)$.*

Proof. (1) It suffices to show that $U \setminus V(\varphi)$ is open in U. Fix $a \in U \setminus V(\varphi)$. Since φ is regular, there exists an open neighborhood $a \in U_a \subseteq U$ and $f_a, g_a \in A(X)$ such that

$$\varphi|_{U_a} = \frac{g_a}{f_a}.$$

So $a \in \{g_a \neq 0\} \cap U_a \subseteq U \setminus V(\varphi)$. This is an open set containing a in $U \setminus V(\varphi)$, so $U \setminus V(\varphi)$ is open.

(2) Since X is irreducible, U is also irreducible. The locus $\{x \in U : \varphi(x) = \psi(x)\} = V(\varphi - \psi)$ is closed in U by (1). It also contains W. Since W is dense (it is a nonempty open set in an irreducible topological space), we must have $V(\varphi - \psi) = U$. This proves the claim. $\square$

Example 6.0.1. In (2) of Proposition 6.1, the assumption that X is irreducible is necessary. Consider

$$U = X = V(xy) \subseteq \mathbb{A}^2_k \quad \text{and} \quad W = V(xy) \setminus V(x).$$

Then the regular functions $\varphi = x$ and $\psi = x + y$ agree on W but are not equal on U.

6.2 Distinguished Open Sets

Remark. We will see that an affine variety has a basis of open sets on which we can compute $\mathcal{O}_X(U)$.

Definition 6.1. A *distinguished open set* of an affine variety X is a subset of the form

$$D(f) = X \setminus V(f)$$

for some polynomial function $f \in A(X)$.

Remark. We have the following:

1. The $D(f)$ are closed under (finite) intersection: $D(fg) = D(f) \cap D(g)$.

2. The $D(f)$ form a basis for the Zariski topology on X: If $U \subseteq X$ is open, then $U = X \setminus V(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in A(X)$ (since X is Noetherian). So $U = D(f_1) \cup \cdots \cup D(f_r)$.

Remark. We will view $D(f)$ as "small open sets" (under mild assumptions, $\operatorname{codim}_X(X \setminus D(f)) = 1$).

Theorem 6.1. *If X is an affine variety and $f \in A(X)$, then*

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\}.$$

Proof. We have an injective ring homomorphism

$$\left\{ \frac{g}{f^m} : g \in A(X), m \geq 0 \right\} \longrightarrow \mathcal{O}_X(D(f)),$$

it suffices to show this map is surjective. Fix $\varphi \in \mathcal{O}_X(D(f))$. For any $a \in D(f)$, there exists an open neighborhood $a \in U_a \subseteq D(f)$ and $f_a, g_a \in A(X)$ such that $\varphi|_{U_a} = g_a/f_a$. We may further assume that

1. $U_a = D(h_a)$ for some $h_a \in A(X)$ (by shrinking U_a if necessary, since the $D(h)$ form a basis);
2. $h_a = f_a$ (by rewriting $g_a/f_a = g_a h_a / f_a h_a$ and replacing h_a, f_a with $f_a h_a$).

Then for $a, b \in D(f)$, we have $f_a g_b = f_b g_a$ on $D(f_a) \cap D(f_b)$. Since both the left and right hand sides vanish on $X \setminus (D(f_a) \cap D(f_b))$, we have $f_a g_b = f_b g_a$ in $A(X)$. Now we can write

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(f_a : a \in D(f)),$$

so $f \in I(V(f_a : a \in D(f)))$. By the Nullstellensatz, there exists $n \geq 0$ such that

$$f^n = \sum_{a \in D(f)} k_a f_a, \quad k_a \in A(X),$$

where only finitely many of the k_a are nonzero. Set $g = \sum_{a \in D(f)} k_a g_a$, and we claim that $\varphi = g/f^n$. To see this, note that on U_b, we have $\varphi|_{U_b} = g_b/f_b$. Now since $f_a g_b = f_b g_a$, we have

$$g f_b = \sum_{a \in D(f)} k_a g_a f_b = \sum_{a \in D(f)} k_a f_a g_b = f^n g_b,$$

which shows that $\varphi|_{U_b} = (g/f^n)|_{U_b}$. Since this holds for any U_b, we have $\varphi = g/f^n$ in $\mathcal{O}_X(D(f))$. $\square$

Remark. Theorem 6.1 has the following consequences:

1. The $f = 1$ case implies that the natural ring homomorphism $A(X) \to \mathcal{O}_X(X)$ is surjective and hence an isomorphism (note that $D(1) = X$).
2. We will see that $\mathcal{O}_X(D(f)) \cong A(X)_f$, the *localization* of $A(X)$ at f.

Example 6.1.1. How do we compute $\mathcal{O}_X(U)$ on non-distinguished open sets? Consider

$$X = \mathbb{A}^2_k \quad \text{and} \quad U = \mathbb{A}^2_k \setminus \{(0,0)\}.$$

Note that U is never a distinguished open set. We claim that the ring homomorphism

$$k[x,y] \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k \setminus \{(0,0)\})$$

is an isomorphism. The map is injective by the identity principle, so it suffices to show surjectivity. The strategy is use $U = D(x) \cup D(y)$ (in general, cover U by basis elements). Fix $\varphi : U \to k$ regular, so

$$\begin{aligned} \varphi|_{D(x)} &= \frac{f}{x^m} \quad \text{for some } f \in k[x,y], m \geq 0 \\ \varphi|_{D(y)} &= \frac{g}{y^n} \quad \text{for some } g \in k[x,y], n \geq 0. \end{aligned}$$

Since we are in a UFD, we may assume that $x \nmid f$ and $y \nmid g$. Now $fy^n = gx^m$ on $D(y) \cap D(x)$, so by the identity principle, $fy^n = gx^m$ on $\mathbb{A}^2_k$, so $fy^n = gx^m$ in $k[x,y]$. Using that $y \nmid g$, $x \nmid f$, and that $k[x,y]$ is a UFD, we must have $n = m = 0$, hence $f = g$. In particular, we have

$$\varphi|_{D(x)} = \varphi|_{D(y)} = f,$$

so the map $k[x,y] \to \mathcal{O}_X(U)$ is surjective.

6.3 Localization

Remark. We want to invert a subset of a ring, in particular *multiplicative systems.*

Definition 6.2. A *multiplicative system* of a ring A is a subset such that

1. $1 \in S$;
2. S is closed under multiplication.

Example 6.2.1. The following examples of S are multiplicative systems:

1. $S = A$ or $S = \{1\}$;
2. if $\mathfrak{p} \leq A$ is a prime ideal, then $S = A \setminus \mathfrak{p}$;
3. if $f \in A$, then $S = \{f^m : m \geq 0\}$.

Definition 6.3. The *localization* of a ring A at a multiplicative system S is the ring

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / \sim$$

where the a/s are formal symbols with $a/s \sim a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$.[1] The operations are given by the usual addition and multiplication of fractions:

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'} \quad \text{and} \quad \frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}.$$

Check as an exercise that these operations respect the equivalence relation.

Example 6.3.1. The following are examples of localization:

1. If A is a domain and $S = A \setminus \{0\}$, then $S^{-1}A = \operatorname{Frac} A$.
2. If $S = \langle f \rangle = \{1, f, f^2, \ldots\}$, then we will write $A_f = S^{-1}A$.
3. If $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, then we will write $A_\mathfrak{p} = S^{-1}A$.

[1] Note that if A is a domain and $0 \notin S$, then this condition is equivalent to $as' = a's$.

Proposition 6.2. *We have the following properties of localization:*

1. *(Universal property of localization) For any ring homomorphism $\varphi : A \to B$ such that $\varphi(s)$ for all $s \in S$, then there exists a unique ring homomorphism which makes the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\pi:a\mapsto a/1"'] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

2. *There is a bijection between the prime ideals $\mathfrak{p} \leq A$ with $\mathfrak{p} \cap S = \varnothing$ and the prime ideals $\mathfrak{q} \leq S^{-1}A$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})S^{-1}A$ with inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$, where $\pi : A \to S^{-1}A$ is the map $a \mapsto a/1$.*

Remark. In more generality, for an A-module M, we can define the localization $S^{-1}M$, which is an $S^{-1}A$-module. This gives a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which is exact.

Lecture 7

Sept. 9 — Germs and Sheaves

7.1 More on Localization

Proposition 7.1. *If X is an affine variety and $f \in A(X)$ is nonzero, then $\mathcal{O}_X(D(f)) \cong A(X)_f$.*

Proof. We define a ring homomorphism as follows:

$$\begin{aligned} A(X)_f &\longrightarrow \mathcal{O}_X(D(f)) \\ \frac{g}{f^m} &\longmapsto \left(x \mapsto \frac{g(x)}{f^m(x)} \right). \end{aligned}$$

To check that this is well-defined, assume $g/f^m \sim h/f^n$ in $A(X)_f$. So there exists $k \geq 0$ such that

$$f^k(gf^n - hf^m) = 0 \quad \text{in } A(X).$$

So $gf^n - hf^m = 0$ as functions $D(f) \to k$, so $g/f^m = h/f^n$ as functions $D(f) \to k$. Thus their images agree in $\mathcal{O}_X(D(f))$, so the map is well-defined.

Surjectivity follows from the argument from last time. For injectivity, assume $g/f^m = 0$ as functions $D(f) \to k$ with $g \in A(X)$. Then $fg = 0$ in $A(X)$, so $g/f^m \sim 0/1$ in $A(X)_f$. $\square$

7.2 Germs of Functions

Definition 7.1. Let $p \in X$ be a point on an affine variety.

1. A *germ* of a regular function of X at p is a pair (U, f) such that $x \in U \subseteq X$ is open and f is a regular function $U \to k$, up to the equivalence relation $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.
2. Define $\mathcal{O}_{X,p} = \{\text{germs of regular functions of } X \text{ at } p\}$.

Exercise 7.1. Check that $\mathcal{O}_{X,p}$ is a ring with operations

$$\begin{aligned} (U, \varphi) \cdot (V, \psi) &= (U \cap V, \varphi|_{U\cap V} \cdot \psi|_{U \cap V}), \\ (U, \varphi) + (V, \psi) &= (U \cap V, \varphi|_{U\cap V} + \psi|_{U \cap V}), \end{aligned}$$

with the zero function as the zero element and the constant 1 function as the unit element.

Lemma 7.1. *$\mathcal{O}_{X,p}$ is a local ring with unique maximal ideal $\mathfrak{m}_p = \{(U, \varphi) \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$.*

Proof. It suffices to show that the units of $\mathcal{O}_{X,p}$ are precisely $\mathcal{O}_{X,p} \setminus \mathfrak{m}_p$. To see the reverse inclusion, fix $(U, \varphi) \in \mathcal{O}_{X,p}$ with $\varphi(p) \neq 0$. So there exists an open neighborhood $p \in W \subseteq U$ such that $\varphi|_W$ never vanishes. Then

$$(U, \varphi) \cdot (W, 1/\varphi|_W) = (W, \varphi|_W) \cdot (W, 1/\varphi|_W)) = (W, 1),$$

so (U, φ) is a unit in $\mathcal{O}_{X,p}$. The forward inclusion is similar. $\square$

Proposition 7.2. *With the above setup, there is an isomorphism*

$$\begin{aligned} A(X)_{I(p)} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto \left(D(g), x \mapsto \frac{f(x)}{g(x)}\right) \end{aligned}$$

with $I(p) = \{f \in A(X) : f(p) = 0\}$.

Proof. To see that this is well-defined, let $f/g \sim f'/g' \in A(X)_{I(p)}$. Then $h(fg' - f'g) = 0$ for some $h \in A(X)$ with $h(p) \neq 0$. So $f/g = f'/g'$ as functions $D(h) \cap D(g) \to k$, which means that $f/g = f'/g'$ as elements in $\mathcal{O}_{X,p}$. Thus the map is well-defined.

Injectivity is similar to before. For surjectivity, choose $(U, \varphi) \in \mathcal{O}_{X,p}$. Since $\varphi : U \to k$ is a regular function, there exists an open set $p \in U_p \subseteq U$ and $f, g \in A(X)$ such that g does not vanish on U_p and $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$. So $(U, \varphi) \sim (D(g), f/g)$ in $\mathcal{O}_{X,p}$, i.e. (U, φ) is in the image. $\square$

Example 7.1.1. If $X = \mathbb{A}^n_k$ and $p = 0$, then

$$\mathcal{O}_{\mathbb{A}^n_k,0} \cong k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = \left\{ \frac{f}{g} : f \in k[x_1, \ldots, x_n], g \in k[x_1, \ldots, x_n] \setminus (x_1, \ldots, x_n) \right\}.$$

Remark. We will relate the local properties of X at p to properties of $\mathcal{O}_{X,p}$. We will use the following statements from commutative algebra: Let A be a ring and $\mathfrak{p} \subseteq A$ a prime ideal. Then

1. $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$.
2. There is a bijection from the prime ideals of $A_\mathfrak{p}$ to the prime ideals of A contained in $\mathfrak{p}$.
3. $\operatorname{ht}_A \mathfrak{p} = \dim A_\mathfrak{p}$ (this follows from (2)).

This has the following consequence: If X is an affine variety and $p \in X$, then

$$\operatorname{codim}_X\{p\} = \operatorname{ht}_{A(X)} I(p) = \dim A(X)_{I(p)} = \dim \mathcal{O}_{X,p}.$$

7.3 Sheaves

Remark. We will now formalize the structures $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ that we have seen before.

Definition 7.2. A *presheaf (of rings)* $\mathcal{F}$ on a topological space X is the data of

1. for every open set $U \subseteq X$, a ring $\mathcal{F}(U)$;
2. for every inclusion of open sets $U \subseteq V \subseteq X$, a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$

satisfying the following properties:

1. $\mathcal{F}(\varnothing) = 0$;
2. $\rho_{U,U}$ is the identity map;
3. for inclusions of open sets $U \subseteq V \subseteq W \subseteq X$, we have $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$.

Example 7.2.1. If X is an affine variety, then $\mathcal{O}_X$ gives a presheaf of rings with

1. for $U \subseteq X$, the ring is $\mathcal{O}_X(U) = \{\text{regular functions } \varphi : U \to k\}$;
2. for $U \subseteq V \subseteq X$, the map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is given by $\varphi \mapsto \varphi|_U$.

Remark. We often call $s \in \mathcal{F}(U)$ a *section*, and for $U \subseteq V$, we call $s|_U = \rho_{V,U}(s)$ the *restriction*.

Remark. A presheaf is the same thing as a functor $\mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Rings}$, where Open_X is the category with objects the nonempty open sets of X and morphisms corresponding to the inclusions $U \subseteq V$.

Definition 7.3. A presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the *gluing property*: For any $U \subseteq X$ open, an open cover $\{U_i\}_{i \in I}$ of U, and $\varphi_i \in \mathcal{F}(U_i)$ with $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Example 7.3.1. We have the following:

1. If X is an affine variety, then $\mathcal{O}_X$ is a sheaf (if we take $\varphi_i \in \mathcal{O}_X(U_i)$ that agree on the overlaps, then we get $\varphi : U \to k$, which is regular since regularity is a local property).
2. If M is a smooth manifold, then we can define a sheaf (on open subsets $U \subseteq M$) by
$$U \longmapsto \mathcal{F}^{\mathrm{sm}}(U) = \{\text{smooth functions } U \to \mathbb{R}\}.$$
We may also consider $\mathcal{F}^{\mathrm{cont}}$, $\mathcal{F}^{\mathrm{diff}}$, $\mathcal{F}^{\mathrm{loc,const}}$, etc. However, $\mathcal{F}^{\mathrm{const}}$ is a presheaf, but not a sheaf in general: We can take $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$, and we will only get a locally constant function. Similarly, $\mathcal{F}^{\mathrm{bounded}}$ is only a presheaf but not a sheaf.
3. If $\mathcal{F}$ is a sheaf on a topological space X and $U \subseteq X$ is open, then we get a sheaf $\mathcal{F}|_U$ on U defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subseteq U$ open.

Definition 7.4. The *stalk* of a sheaf $\mathcal{F}$ on a topological space X at $x \in X$ is
$$\mathcal{F}_x = \{(U, \varphi) : U \subseteq X \text{ open and } \varphi \in \mathcal{F}(U)\}/\sim,$$
where $(U, \varphi) \sim (V, \psi)$ if there exists an open set $x \in W \subseteq U \cap V$ such that $\varphi|_W = \psi|_W$.

Example 7.4.1. If X is an affine variety and $p \in X$, then $\mathcal{O}_{X,p} \cong (\mathcal{O}_X)_p$.

Remark. As before with $\mathcal{O}_{X,p}$, one can check that $\mathcal{F}_x$ naturally has the structure of a ring.

Remark. An alternative perspective is to define the stalk as a direct limit:
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where the limit is taken over all open $x \in U \subseteq X$ with respect to the ordering $U \leq V$ if $V \subseteq U$.

Lecture 8

Sept. 11 — Morphisms

8.1 Morphisms of Open Sets

Remark. Recall that a continuous map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *smooth* if it satisfies either of the following equivalent conditions:

1. there exist smooth functions $f_1, \dots, f_n : \mathbb{R}^m \to \mathbb{R}$ such that $f(x) = (f_1(x), \dots, f_n(x))$;
2. for each open set $U \subseteq \mathbb{R}^n$ and smooth $\varphi : U \to \mathbb{R}$, the function $f^*\varphi := \varphi \circ f : \mathbb{R}^m \to \mathbb{R}$ is smooth.

The implication ($1 \Rightarrow 2$) follows by the chain rule. To see ($2 \Rightarrow 1$), take $y_i : \mathbb{R}^n \to \mathbb{R}$ defined by $f_i := f^*y_i$. We want a similar definition in algebraic geometry.

Definition 8.1. Let X and Y be open sets of affine varieties. A *morphism* $f : X \to Y$ is a continuous map such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the map

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse (equivalently, f is a bijection and f^{-1} is a morphism).

Remark. We have the following properties of morphisms:

1. (Composition) If $f : X \to Y$ and $g : Y \to Z$ are morphisms of open sets of affine varieties, then so is $g \circ f : X \to Z$.
2. (Local on target) If $X \to Y$ is a map of open sets of affine varieties such that there exists an open cover $\{U_i\}_{i \in I}$ of Y with $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ a morphism for all $i \in I$, then f is a morphism.

Proposition 8.1. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Let $U \subseteq X$ and $V \subseteq Y$ be open sets. A map $f : U \to V$ is a morphism if and only if there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that*

$$f(x) = (\varphi_1(x), \dots, \varphi_n(x)).$$

Proof. ($\Rightarrow$) Let $U \subseteq \mathbb{A}^m_{x_i}$ and $V \subseteq \mathbb{A}^n_{y_i}$. By the definition of a morphism, $y_i : V \to k$ satisfies

$$\varphi_i := f^*y_i \in \mathcal{O}_X(U),$$

so we can write $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

($\Leftarrow$) Assume there exist $\varphi_1, \dots, \varphi_n \in \mathcal{O}_X(U)$ such that $f(x) = (\varphi_1(x), \dots, \varphi_n(x))$.

We first show that f is continuous. Let $Z \subseteq V$ be a closed set. So we can write $Z = V(g_r, \ldots, g_r)$ for some $g_1, \ldots, g_r \in A(\mathbb{A}^n) \cong k[y_1, \ldots, y_n]$. Now we have

$$\begin{aligned} f^{-1}(Z) = \{x \in U : f(x) \in Z\} &= \{x \in U : g_i(f(x)) = 0 \text{ for } i = 1, \ldots, r\} \\ &= \{x \in U : (f^* g_i)(x) = 0 \text{ for } i = 1, \ldots, r\}. \end{aligned}$$

Note that $f^* g_i = g_i(\varphi_1, \ldots, \varphi_n)$, which is regular since a composition of a polynomial with fractions of polynomials is again a fraction of polynomials. So $f^{-1}(Z)$ is closed in U.

Now to show that f is a morphism, it suffices to show that for any $W \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(W)$, we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(W))$. The proof of this is similar to before. $\square$

Example 8.1.1. We have the following:

1. Morphisms $\mathbb{A}^m \to \mathbb{A}^n$ are of the form

$$x \longmapsto (f_1(x), \ldots, f_n(x))$$

   with $f_1, \ldots, f_n \in \mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_m]$.

2. Write $\mathbb{A}^1_t$ to mean $\mathbb{A}^1$ with variable t. Then we can define $\mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ by $t \mapsto (t, t^2)$. We can get an inverse $V(y - x^2) \to \mathbb{A}^1_t$ by $(x, y) \mapsto x$, so $\mathbb{A}^1_t$ and $V(y - x^2)$ are isomorphic.

3. Consider the map $g : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. This map is bijective, but it is not an isomorphism. To see this, we can show that $(g^{-1})^*\varphi$ is not regular for some regular function φ on $\mathbb{A}^t_1$. For instance, we can take $\varphi = t$, so that

$$(g^{-1})^*(t) = (x, y) \mapsto \begin{cases} x/y & \text{if } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

   which we can see is not regular.

8.2 Relation to Coordinate Rings

Remark. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Then a morphism $f : X \to Y$ of affine varieties induces a k-algebra morphism (called the *pullback* of f)

$$\begin{aligned} f^* : A(Y) &\longrightarrow A(X) \\ \varphi &\longmapsto f^* = \varphi \circ f \end{aligned}$$

with the properties $(g \circ f)^* = f^* \circ g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{A(X)}$, i.e. $X \mapsto A(X)$ is a contravariant functor.

Proposition 8.2. *The following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_{\text{aff,var}}(X, Y) &\xrightarrow{\Phi} \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^* \end{aligned}$$

Proof. Note that $A(X) \cong k[x_1, \ldots, x_m]/I(X)$ and $A(Y) \cong k[y_1, \ldots, y_n]/I(Y)$. Given a morphism

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto (\varphi_1(x), \ldots, \varphi_n(x)), \end{aligned}$$

we can define $f^*\overline{y}_i = \varphi_i$. Conversely, given a k-algebra homomorphism $\phi : A(Y) \to A(X)$, we can set $\varphi_i = \phi(\overline{y}_i)$. Now consider the morphism defined by

$$\begin{aligned} f : X &\longrightarrow \mathbb{A}^n_{y_i} \\ x &\longmapsto (\varphi_1(x), \dots, \varphi_n(x)). \end{aligned}$$

We claim that $f(X) \subseteq Y$. To see this, fix $x \in X$. If $h \in I(Y)$, then

$$h(f(x)) = h(\varphi_1(x), \dots, \varphi_n(x)) = \phi(h)(x) = 0(x) = 0,$$

so $f(X) \subseteq Y$. Thus we get a morphism $f : X \to Y$ by $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$ with $f^*y_i = \varphi_i$. One can check that this gives a map $\Psi : \mathrm{Hom}_{k\text{-alg}}(A(Y), A(X)) \to \mathrm{Hom}_{\mathrm{aff,var}}(X, Y)$ which is inverse to Φ. □

Example 8.1.2. We have the following:

1. Recall the morphism $g : \mathbb{A}^1_t \to V(y - x^2) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t, t^2)$. The pullback is given by

$$\begin{aligned} g^* : \frac{k[x,y]}{(y - x^2)} &\longmapsto k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2. \end{aligned}$$

   Note that g^* is an isomorphism of k-algebras, so g is an isomorphism of affine varieties. This gives an alternative way of seeing this without writing down an inverse to g.

2. Recall the morphism $h : \mathbb{A}^1_t \to V(x^2 - y^3) \subseteq \mathbb{A}^2_{x,y}$ given by $t \mapsto (t^3, t^2)$. The pullback is

$$\begin{aligned} h^* : \frac{k[x,y]}{(x^2 - y^3)} &\longmapsto k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^2. \end{aligned}$$

   Note that $t \notin \operatorname{Im} h^*$, so h^* is not an isomorphism, so h is not an isomorphism.

Remark. There is a one-to-one correspondence between affine varieties (up to isomorphism) and finitely generated reduced k-algebras (up to isomorphism).

To see this, observe that if $X \subseteq \mathbb{A}^n$ is an affine variety, then $A(X) \cong k[x_1, \dots, x_n]/I(X)$. This is finitely generated, and reduced since $I(X)$ is radical. Conversely, let A be a reduced finitely generated k-algebra. Then $A \cong k[y_1, \dots, y_m]/I$ since A is finitely generated, and I is radical since A is reduced. Thus by Hilbert's nullstellensatz, $Y = V(I)$ satisfies $I(Y) = I(V(I)) = I$, so $A \cong A(Y)$.

In more abstract language, this means that there is an equivalence of categories

$$\mathrm{AffVar} \longleftrightarrow \mathrm{RedFGAlg}_k^{\mathrm{op}}.$$

Lecture 9

Sept. 16 — Morphisms, Part 2

9.1 An Example of Isomorphisms

Example 9.0.1. What of the following are isomorphic over $\mathbb{C}$?

1. $\mathbb{A}^1 \setminus \{1\}$;
2. $V(x^2+y^2) \subseteq \mathbb{A}^2$;
3. $V(y-x^2, z-x^3) \subseteq \mathbb{A}^3$;
4. $V(xy) \subseteq \mathbb{A}^2$;
5. $V(y^2-x^2-x^3) \subseteq \mathbb{A}^2$;
6. $V(x^2-y^2-1) \subseteq \mathbb{A}^2$.

Note that (2) and (4) are not irreducible. In fact, they are isomorphic since we can write (2) as

$$V(x^2+y^2) = V((x+iy)(x-iy)) \cong V(xy).$$

We have seen (3) previously on homework, and we have an isomorphism $\mathbb{A}^1 \to Y = V(y-x^2, z-x^3)$ by $t \mapsto (t, t^2, t^3)$. We can also see this by noting that $A(Y) \cong \mathbb{C}[x] \cong A(\mathbb{A}^1)$. For (1), note that

$$\mathbb{A}^1 \setminus \{1\} \cong \mathbb{A}^1 \setminus \{0\}$$

and $A(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{C}[x^{\pm 1}]$, whereas $A(\mathbb{A}^1) \cong \mathbb{C}[x]$. So $A \setminus \{1\} \not\cong \mathbb{A}^1$. For (6), note that

$$V(x^2-y^2-1) = V((x-y)(x+y)-1) \cong V(uv-1) \cong \mathbb{A}^1 \setminus \{0\}$$

by the map $V(uv-1) \to \mathbb{A}^1 \setminus \{0\}$ given by $(u,v) \mapsto u$, with inverse $t \mapsto (t, 1/t)$. Finally, letting C be the curve in 6, one can show that there is a singularity at the origin with $\dim_{\mathbb{C}}(\mathcal{O}_{C,0}/\mathfrak{m}_0) = 2$, which is different than the other examples. So the isomorphism classes are $\{2,4\}$, $\{1,6\}$, $\{3\}$, and $\{5\}$.

9.2 Ringed Spaces and Morphisms

Definition 9.1. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$ on X.

Example 9.1.1. If X is an affine variety and $\mathcal{O}_X$ is the sheaf of regular functions, then $(X, \mathcal{O}_X)$ is a ringed space. Similarly, if M is a complex manifold and $\mathcal{O}_M$ is the sheaf of holomorphic functions on M, then $(M, \mathcal{O}_M)$ is a ringed space.

Remark. From now on, for a ringed space $(X, \mathcal{O}_X)$, we will always assume $\mathcal{O}_X$ is a sheaf of k-valued functions on X. With this assumption, we can make sense of pullbacks.

Definition 9.2. A *morphism* of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $f : X \to Y$ such that for every $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$,

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi"] & U \arrow[r, "\varphi"] & k \end{tikzcd}$$

is an element of $\mathcal{O}_X(f^{-1}(U))$. A morphism is an *isomorphism* if it has a two-sided inverse.

Remark. A one-sided inverse need not be two-sided: Consider $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$ and $g : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 0)$. Then $f \circ g = \mathrm{id}_{\mathbb{A}^1}$, but $g \circ f$ is not the identity on $\mathbb{A}^2$.

Remark. If $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then for $V \subseteq U \subseteq Y$ open, we get

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r] \arrow[d, "\text{res.}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\text{res.}"] \\ \mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(f^{-1}(V)) \end{tikzcd}$$

which is a commutative diagram of ring homomorphisms.

Remark. If X and Y are open sets of affine varieties, then a map $f : X \to Y$ is a morphism of open sets of affine varieties if and only if it is a morphism of ringed spaces.

Definition 9.3 (Redefinition of affine variety)**.** An *affine variety* $(X, \mathcal{O}_X)$ is a ringed space isomorphic to an affine variety in the original sense (as ringed spaces).

Remark. We will often write just X for the affine variety instead of the pair $(X, \mathcal{O}_X)$.

Example 9.3.1. Recall that $\mathbb{A}^1 \setminus \{0\} \cong V(xy - 1) \subseteq \mathbb{A}^2$ from Example 9.0.1. In particular, $\mathbb{A}^1 \setminus \{0\}$ is an affine variety in the new sense (but not in the old sense).

Proposition 9.1. *If X is an affine variety (in the old sense) and $f \in A(X)$, then $D(f)$ is an affine variety.*

Proof. Write $X = V(I) \subseteq \mathbb{A}^n_{x_i}$ and consider the map

$$\begin{aligned} D(f) &\longrightarrow V(I, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)). \end{aligned}$$

This has an inverse $V(I, fy - 1) \to D(f)$ given by $(x, y) \mapsto x$. So $D(f) \cong V(I, fy - 1)$ as ringed spaces. Thus $D(f)$ is an affine variety (in the new sense). □

9.3 Products of Affine Varieties

Remark. If $X \subseteq \mathbb{A}^m_{x_i}$ and $Y \subseteq \mathbb{A}^n_{y_i}$ are affine varieties, then

$$X \times Y = V(I(X), I(Y)) \subseteq \mathbb{A}^{m+n},$$

viewing $I(X), I(Y)$ as ideals in $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. So $X \times Y$ is an affine variety with morphisms

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1", "(x,y)\mapsto x"'] \arrow[dr, "p_2"', "(x,y)\mapsto y"] & \\ X & & Y \end{tikzcd}$$

Proposition 9.2. *For every affine variety Z and diagram of morphisms*

$$\begin{tikzcd} Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_Y"] \arrow[ddr, bend right, "f_X"'] & & \\ & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

there is a unique morphism f which makes the diagram commute.

Proof. We already know that there is a unique set theoretic map which makes the diagram commute. Then since f_X and f_Y are given as regular functions, so is f. So f is a morphism. $\square$

Remark. We will now try to understand the isomorphism $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

9.4 Tensor Products

Definition 9.4. Let A be a (commutative) ring and M, N be A-modules. The *tensor product* $M \otimes_A N$ is the A-module generated by the symbols $m \otimes n$ for $m \in M$ and $n \in N$, subject to the relations

1. (distributive law): $(m + m') \otimes n = m \otimes n + m' \otimes n$,
2. (multiplication with scalars): $a(m \otimes n) = (am) \otimes n = m \otimes (an)$.

To make this precise, $M \otimes_A N = A^{M \times N}/R$, where R is the submodule generated by these relations.

Example 9.4.1. We have $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$. We can compute

$$1 \otimes 1 = (3-2) \otimes 1 = 3 \otimes 1 - 2 \otimes 1 = 3 \otimes 1 + 1 \otimes (-2) = 0 \otimes 1 + 1 \otimes 0 = 0 \otimes 0,$$

and similarly for the other elements. In general, if $\gcd(m, n) = 1$, then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.

Proposition 9.3 (Universal property of the tensor product)**.** *For any bilinear map $\Phi : M \times N \to P$ to an A-module P (i.e. $n \mapsto \Phi(m, n)$ is A-linear for each $m \in M$ and the same for $m \mapsto \Phi(m, n)$),*

$$\begin{tikzcd} M \times N \arrow[r, "\Phi"] \arrow[d, "(m,n)\mapsto m\otimes n"'] & P \\ M \otimes N \arrow[ur, "\Psi"'] & \end{tikzcd}$$

there exists a unique A-module homomorphism $\Psi : M \otimes N \to P$ such that the above diagram commutes.

Remark. We have the following properties of the tensor product:

1. $A \otimes M \cong M$;

2. $M \otimes N \cong N \otimes M$;
3. $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$;
4. $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

The way to prove these is to use the universal property to construct maps in either direction and show that they compose to the identity.

5. For a fixed A-module M and an exact sequence

   $$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0,$$

   the sequence (where F is defined by $m \otimes n' \mapsto m \otimes f(n')$ and G is defined by $m \otimes n \mapsto m \otimes g(n)$)

   $$M \otimes N' \xrightarrow{F} M \otimes N \xrightarrow{G} M \otimes N'' \longrightarrow 0$$

   is also exact. In particular, $\otimes M$ induces a right exact functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$ by $N \mapsto M \otimes N$.

Example 9.4.2. The functor $\otimes M$ is in general not left exact. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

After tensoring with $\mathbb{Z}/2\mathbb{Z}$, we get the sequence

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where the first map is not injective. Note that right exactness gives $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 9.1. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

Lecture 10

Sept. 18 — Pre-varieties

10.1 More on Tensor Products

Proposition 10.1. *If B and C are A-algebras (i.e. there are ring homomorphisms $f : A \to B$ and $g : A \to C$ which give $a \cdot b := f(a)b$ and $a \cdot c = g(a)c$, then $B \otimes_A C$ is also an A-algebra with*

$$(b \otimes c) \cdot (b' \otimes c') := (bb') \otimes (cc')$$

and ring homomorphism $A \to B \otimes_A C$ given by $a \mapsto a \otimes 1$ (equivalently, $1 \otimes a$).

Proposition 10.2. $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Proposition 10.3. $(k[x_1, \ldots, x_m]/I) \otimes_k (k[y_1, \ldots, y_n]/J) \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n)/\langle I, J\rangle$.

Proof. Set $R = k[x_1, \ldots, x_m]$ and $S = k[y_1, \ldots, y_n]$. We have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Applying the right exact functor $\otimes_k(S/J)$ (and vice versa with J and $\otimes R$) gives an exact sequence

$$\begin{tikzcd}
& R \otimes_k J \arrow[d] & & \\
& R \otimes_k S \arrow[d] & & \\
I \otimes_k (S/J) \arrow[r] & R \otimes_k (S/J) \arrow[r] \arrow[d] & (R/I) \otimes_k (S/J) \arrow[r] & 0 \\
& 0 & &
\end{tikzcd}$$

So we have

$$(R/I) \otimes_k (S/J) \cong \frac{R \otimes_k (S/J)}{\mathrm{Im}(I \otimes_k (S/J) \to R \otimes_k (S/J))} \cong \frac{R \otimes_k S}{I \otimes_k S + R \otimes_k J},$$

which is the desired result since $I \otimes_k S + R \otimes_k J = \langle I, J\rangle$ in $R \otimes_k S$. $\square$

Proposition 10.4 (Milne)**.** *Let B and C be finitely generated k-algebras with $k = \overline{k}$.*

1. *If B and C are reduced, then so is $B \otimes_k C$.*

2. *If B and C are domains, then so is $B \otimes_k C$.*

Remark. We need $k = \overline{k}$ in Proposition 10.4. Consider the domains $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[y]/(y^2+1)$. Then

$$\frac{\mathbb{R}[x]}{(x^2+1)} \otimes_{\mathbb{R}} \frac{\mathbb{R}[y]}{(y^2+1)} \cong \frac{\mathbb{R}[x,y]}{(x^2+1, y^2+1)},$$

which is not a domain since $(\overline{x-y})(\overline{x+y}) = \overline{x^2-y^2} = \overline{-1-(-1)} = 0$.

Corollary 10.0.1. *If $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties, then*

1. *$I(X \times Y) = \langle I(X), I(Y) \rangle \subseteq k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.*
2. *$A(X \times Y) \cong A(X) \otimes_k A(Y)$.*
3. *If X and Y are irreducible, then $X \times Y$ is irreducible.*

Proof. Observe that $V(I(X), I(Y)) = X \times Y \subseteq \mathbb{A}^{m+n}$, so $I(X \times Y) = \sqrt{\langle I(X), I(Y) \rangle}$. Now we know that $I(X)$ and $I(Y)$ are radical in $k[x_1, \ldots, x_m]$ and $k[y_1, \ldots, y_n]$, respectively, so

$$\frac{k[x_1, \ldots, x_m]}{I(X)} \quad \text{and} \quad \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

are reduced. By Proposition 10.4, we get that

$$\frac{k[x_1, \ldots, x_m, y_1, \ldots, y_n]}{\langle I(X), I(Y) \rangle} \cong \frac{k[x_1, \ldots, x_m]}{I(X)} \otimes_k \frac{k[y_1, \ldots, y_n]}{I(Y)}$$

is reduced, so $\langle I(X), I(Y) \rangle$ is radical. Thus $I(X \times Y) = \langle I(X), I(Y) \rangle$, so (1) holds.

Now (1) implies (2), and (3) follows since X and Y being irreducible implies $A(X)$ and $A(Y)$ are domains, which implies $A(X \times Y)$ is a domain by Proposition 10.4 and (2), so $X \times Y$ is irreducible. □

10.2 Pre-varieties

Remark. We will now head towards defining a *variety*, which is roughly finitely many affine varieties glued together (a *pre-variety*) with a separation condition (an algebraic version of Hausdorffness).

Definition 10.1. A *pre-variety* is a ringed space $(X, \mathcal{O}_X)$ such that there exists a finite open cover $X = \bigcup_{i=1}^s U_i$ with $(U_i, \mathcal{O}_X|_{U_i})$ being an affine variety for all $i = 1, \ldots, s$. A *morphism* of pre-varieties

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a morphism of the ringed spaces. We will often just write X for $(X, \mathcal{O}_X)$.

Remark. We call $\varphi \in \mathcal{O}_X(U)$ with $U \subseteq X$ open and $\varphi : U \to k$ a *regular function* on U.

Example 10.1.1. Consider the following:

1. An affine variety X is a pre-variety. However, we have multiple choices for the open cover: We can take $X = X$, or $X = \bigcup_{i=1}^s D(f_i)$ with $f_i \in \mathcal{O}_X(X)$ and $(f_1, \ldots, f_s) = (1)$ in $\mathcal{O}_X(X)$.
2. $\mathbb{P}_k^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$ is a pre-variety. We will see that $\mathbb{P}_k^1 = \mathbb{A}_k^1 \cup \{\text{pt}\}$.

3. Let $X = V(I) \subseteq \mathbb{A}^n$ be an affine variety and $U \subseteq X$ open. Set
$$\mathcal{O}_U(V) = \{\varphi : V \to k \mid \varphi \text{ is regular}\}.$$
Then $(U, \mathcal{O}_U)$ is a pre-variety. To see this, note that $U = \bigcup_{f \in I(X \setminus U)} D(f)$. Since U is Noetherian (hence is compact), we can find a finite subcover, so $U = \bigcup_{i=1}^s D(f_i)$ for some $f_i \in A(X)$.

4. (Gluing) Let X_1 and X_2 be affine varieties, and $U_{1,2} \subseteq X_1$, $U_{2,1} \subseteq X_2$ open, with an isomorphism
$$f : U_{1,2} \longrightarrow U_{2,1}.$$
Then we get a pre-variety by setting $X = (X_1 \sqcup X_2)/\sim$, where $a \sim f(a)$ for all $a \in U_{1,2}$, $f(a) \sim a$ for all $a \in U_{2,1}$, and $b \sim b$ for all $b \in X_1 \sqcup X_2$. We have quotient maps
$$j_1 : X_1 \longrightarrow X \quad \text{and} \quad j_2 : X_2 \longrightarrow X.$$
Now X is a topological space with the quotient topology, and j_1, j_2 are open embeddings (i.e. have open images and are homeomorphisms onto their images). Define a sheaf of rings $\mathcal{O}_X$ on X by
$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid j_1^*\varphi \in \mathcal{O}_{X_1}(j^{-1}(U)) \text{ and } j_2^*\varphi \in \mathcal{O}_{X_2}(j_2^{-1}(U))\}.$$
One can check $X = j_1(X_1) \cup j_2(X_2)$ and $(j(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Example 10.1.2. Consider $X_1 = \mathbb{A}^1_x$ and $X_2 = \mathbb{A}^1_y$, with $U_{1,2} = \mathbb{A}^1_x \setminus \{0\}$ and $U_{2,1} = \mathbb{A}^1_y \setminus \{0\}$. Define
$$\begin{aligned} f : U_{1,2} &\longrightarrow U_{2,1} \\ x &\longmapsto 1/x. \end{aligned}$$
Then we can take $\mathbb{P}^1_k = (X_1 \sqcup X_2)/\sim$. What are the regular functions $\mathbb{P}^1_k \to k$? We should get only the constant functions (When $k = \mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$ is compact, so a holomorphic function $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{C}$ is bounded. By restricting to X_1, we get a bounded map $f : \mathbb{C} \to \mathbb{C}$, so f is constant by Liouville's theorem).

In general, let $j_i : X_i \to \mathbb{P}^1_k$ be the quotient maps. Fix $\varphi \in \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$. Now
$$\varphi|_{X_1} := j_1^*\varphi = \sum_{i \geq 0} a_i x^i \quad \text{and} \quad \varphi|_{X_2} := j_2^*\varphi = \sum_{i \geq 0} b_i y^i$$
for some $a_i, b_i \in k$. They must agree on the overlap, so
$$\sum_{i \geq 0} a_i x^i = \sum_{i \geq 0} b_i (1/x)^i$$
as functions on $\mathbb{A}^1 \setminus \{0\}$. Since $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1 \setminus \{0\}) = k[x^{\pm 1}]$, we have $a_i = b_i = 0$ for $i > 0$ and $a_0 = b_0$ (since the powers of $x^{\pm 1}$ are k-linearly independent), so φ is a constant function.

If we instead took $f : U_{1,2} \to U_{2,1}$ to be $x \mapsto x$, then $X = (X_1 \sqcup X_2)/\sim$ is the "bug-eyed line" with two points $0, 0'$ at the origin (this is the *line with two origins* when $k = \mathbb{R}$, which is not Hausdorff.) Note that $X \setminus \{0, 0'\} \cong \mathbb{A}^1 \setminus \{0\}$. In our case, the bad property is that there exist two morphisms
$$g_1, g_2 : \mathbb{A}^1 \longrightarrow X$$
such that $g_1|_{\mathbb{A}^1 \setminus \{0\}} = g_2|_{\mathbb{A}^1 \setminus \{0\}}$ and $g_1 \neq g_2$, i.e. "limits are not unique" on X. Note that a similar computation shows $\mathcal{O}_X(X) \cong k[x]$, so in particular, $X \ncong \mathbb{P}^1_k$.

Lecture 11

Sept. 23 — Pre-varieties, Part 2

11.1 More on Pre-varieties

Proposition 11.1. *Let $(X, \mathcal{O}_X)$ be a pre-variety.*

1. *X is Noetherian as a topological space.*
2. *X has a basis by affine varieties.*

Proof. (1) Note that X has a finite cover by affine varieties U_i, which are each Noetherian.

(2) If $(X, \mathcal{O}_X)$ is affine, then $\{D(f) : f \in \mathcal{O}_X(X)\}$ gives such a basis. Do this for each U_i. □

Example 11.0.1 (General gluing procedure)**.** Let I be a finite index set, $(X_i, \mathcal{O}_{X_i})$ affine varieties, $U_{i,j} \subseteq X_i$ open sets, and $f_{i,j} : U_{i,j} \to U_{j,i}$ isomorphisms for each $i, j \in I$, satisfying

1. $U_{i,i} = X_i$ and $f_{i,i} = \mathrm{id}$;
2. $f_{i,j}^{-1}(U_{j,i} \cap U_{j,k}) = U_{i,j} \cap U_{i,k}$;
3. the following diagram commutes:

$$\begin{tikzcd} U_{i,j} \cap U_{i,k} \arrow[rr, "f_{i,k}"] \arrow[dr, "f_{i,j}"'] & & U_{k,i} \cap U_{k,j} \\ & U_{j,i} \cap U_{j,k} \arrow[ur, "f_{j,k}"'] & \end{tikzcd}$$

We can define $X = (\bigsqcup_{i \in I} X_i)/\sim$ with the quotient topology, where $a \sim a$ for $a \in X_i$ and $a \sim f_{i,j}(a)$ for $a \in U_{i,j}$. The inclusions $j_i : X_i \hookrightarrow X$ are open embeddings, and we can set

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi|_{X_i} = j_i^* \varphi \text{ is regular for all } i\}.$$

Then $(X, \mathcal{O}_X)$ is a ringed space with $(j_i(X_i), \mathcal{O}_X|_{j_i(X_i)}) \cong (X_i, \mathcal{O}_{X_i})$, so $(X, \mathcal{O}_X)$ is a pre-variety.

Remark. Any pre-variety X is a gluing of affine varieties. To see this, note that there exists a cover $X = \bigcup_{i=1}^s U_i$ by affine varieties. Then we can take $X_i = U_i$, $U_{i,j} = U_i \cap U_j \subseteq X_i$, and $f_{i,j} : U_{i,j} \to U_{j,i}$ to be the identity map.

Proposition 11.2. *Let X be a pre-variety.*

1. *If $U \subseteq X$ is an open set, then $(U, \mathcal{O}_X|_U)$ is again a pre-variety.*

2. *Let $Z \subseteq X$ be a closed set. For $U \subseteq Z$ open, set*

$$\mathcal{O}_Z(U) = \left\{ \varphi : U \to k \mid \begin{array}{c} \text{for each } a \in U, \text{ there exists open } a \in W \subseteq X \text{ and} \\ \psi : W \to k \text{ regular such that } \varphi|_{W \cap Z} = \psi|_{W \cap Z} \end{array} \right\}.$$

Then $(Z, \mathcal{O}_Z)$ is a pre-variety.

Proof. (1) Note that X has a basis by affine varieties, so we can cover U by affine varieties. This cover may be infinite, but we can pass to a finite subcover since X and hence U is Noetherian.

(2) The idea is to first reduce to the case $X = V(I) \subseteq \mathbb{A}^n$, so $Z \subseteq X$ is cut out by polynomials. Then observe that $\mathcal{O}_Z$ agrees with the previous definition. □

Remark. Note that unions of intersections of closed and open sets are not necessarily pre-varieties. For instance, consider $(\mathbb{A}^2 \setminus V(xy)) \cup \{0\}$.

Proposition 11.3. *If X, Y are pre-varieties, then there exists a pre-variety with morphisms*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

with the property that for every diagram

$$\begin{tikzcd} & Z \arrow[d, dashed, "f"'] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\ & P \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ X & & Y \end{tikzcd}$$

there exists a unique morphism f such that the diagram commutes. We call P the product *of X and Y, and write $X \times Y := P$. Moreover, set theoretically $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.*

Proof. We know the result holds when X, Y, Z are affine or even open sets of affine varieties. In the general case, fix an open affine cover $X = \bigcup_{i=1}^s U_i$ and $Y = \bigcup_{j=1}^r V_j$. Then glue the products by:

1. $P_{(i,j)} := U_i \times V_j$,
2. along $P_{(i,j),(i',j')} : (U_i \cap U_{i'}) \times (V_j \cap V_{j'})$,
3. via $f_{(i,j),(i',j')} : P_{(i,j),(i',j')} \xrightarrow{\cong} P_{(i',j'),(i,j)}$, the isomorphism from the universal property of products.

We get a pre-variety P, and the morphisms

$$\begin{tikzcd} & f_1^{-1}(U_2) \cap f_2^{-1}(V_j) \subseteq Z \arrow[d, dashed, "f_{i,j}"'] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

glue to give a morphism

$$\begin{tikzcd}
& Z \arrow[d, dashed, "f"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& P \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

which is a morphism as the morphism condition can be checked locally. Furthermore, the diagram commutes (as can be checked locally). Last, f is unique: One can either this check locally or check set theoretically using $P = \{(x, y) : x \in X \text{ and } y \in Y\}$ as sets. $\square$

Remark. Note that $X \times Y$ is set theoretically the product of X and Y, but not the product of X and Y as topological spaces. Consider $X = Y = \mathbb{A}^1$ and $X \times Y = \mathbb{A}^2$.

11.2 Varieties

Remark. We want a version of Hausdorffness in algebraic geometry. However, an irreducible topological space (e.g. $\mathbb{A}^n$) is almost never Hausdorff (unless it is a single point). From a different perspective, note that X is Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is closed, where $X \times X$ is given the product topology.

Definition 11.1. A pre-variety is *separated* if the diagonal

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$ (the product pre-variety). A *variety* is a pre-variety that is separated.

Example 11.1.1. $\mathbb{A}^n$ is separated. We have

$$V(x_1 - y_1, \ldots, x_n - y_n) = \Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^n_{y_i} \cong \mathbb{A}^{2n},$$

so $\Delta_{\mathbb{A}^n}$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$.

Example 11.1.2. Any affine variety is separated. To see this, we may assume $X = V(I) \subseteq \mathbb{A}^n$. By the construction of the product, $X \times X \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is closed and

$$\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}.$$

Since $\Delta_{\mathbb{A}^n}$ is closed, we have Δ_X is closed in $X \times X$ since $X \times X$ has the subspace topology.

Proposition 11.4. *If X is a variety, then any closed or open set $Z \subseteq X$ is a variety.*

Proof. We have already seen that Z is a pre-variety, so it suffices to show that Z is separated. We note that $Z \times Z \hookrightarrow X \times X$ is an embedding of topological spaces, and $\Delta_Z = (Z \times Z) \cap \Delta_X$. Since Δ_X is closed and $Z \times Z$ has the subspace topology, Δ_Z is closed in $Z \times Z$. So Z is separated. $\square$

Example 11.1.3. Recall the bug-eyed line from Example 10.1.2. Let a, b be the two origins, and write $X = U_1 \cup U_2$, where $U_1 = X \setminus \{b\} \cong \mathbb{A}^1$ and $U_2 = X \setminus \{a\} \cong \mathbb{A}^1$. Then consider

$$\mathbb{A}^2 \cong U_1 \times U_2 \subseteq X \times X.$$

Note that $\Delta_X \cap (U_1 \times U_2) = \{(x, x) : x \in k \setminus \{0\}\} = \Delta_{\mathbb{A}^1} \setminus \{0\}$. So Δ_X is not closed in $X \times X$.

Exercise 11.1. Show that $\mathbb{P}^1_k$ is separated.

Proposition 11.5. *Let $f, g : X \to Y$ be morphisms of pre-varieties with Y a variety.*

1. *The graph $\Gamma_f := \{(x, f(x)) : x \in X\}$ of f is closed in $X \times Y$.*
2. *$\{x \in X : f(x) = g(x)\}$ is closed in X. This becomes a version of the identity principle in the case that X is irreducible: If X is irreducible and f, g agree on a nonempty open set, then $f = g$.*

Proof. (1) We can write $\Gamma_f = (f, \mathrm{id})^{-1}(\Delta_Y)$ where $(f, \mathrm{id}) : X \times Y \to Y \times Y$, and Δ_Y is closed.

(2) Consider the morphism

$$\begin{aligned} X &\xrightarrow{(f,g)} Y \times Y \\ x &\longmapsto (f(x), g(x)). \end{aligned}$$

Then $\{x \in X : f(x) = g(x)\} = (f, g)^{-1}(\Delta_Y)$, so it is closed. $\square$

Lecture 12

Sept. 25 — Projective Varieties

12.1 Projective Space

Definition 12.1. Define *projective n-space* over k to be

$$\mathbb{P}_k^n = \mathbb{P}^n = \text{1-dimensional subspaces of } k^{n+1} = (k^{n+1} \setminus \{0\})/\sim,$$

where $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$ if there exists $\lambda \in k^\times$ such that $(x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n)$. We write $[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}_k^n$ for the equivalence class of $(x_0, x_1, \ldots, x_n)$.

Example 12.1.1. For $n = 2$, we have $[1 : 0 : 2] = [1/2 : 0 : 1] \in \mathbb{P}_k^2$ when $\operatorname{char} k \neq 2$.

Remark. For $0 \leq i \leq n$, define $U_i = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}_k^n : x_i \neq 0\}$. Then

$$\mathbb{P}_k^n = \bigcup_{i=0}^n U_i,$$

and there exist bijective maps $f_i : U_i \to \mathbb{A}^n$ given by

$$f_i([x_0 : \cdots : x_n]) = (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i),$$

where $\widehat{x_i/x_i}$ means we omit x_i/x_i. For $i = 0$, the inverse is $f_0^{-1}(x_1, \ldots, x_n) = [1 : x_1 : \cdots : x_n]$.

Remark. Another way to think about $\mathbb{P}^n$ is via points at ∞. Observe that

$$\mathbb{P}^n \setminus U_0 = \{[0 : x_1 : \cdots : x_n] \in \mathbb{P}_k^n : (x_1, \ldots, x_n) \in k^n \setminus \{0\}\} \cong \mathbb{P}^{n-1}.$$

So $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \cup \mathbb{P}^{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0$.

Remark. Why work with $\mathbb{P}^n$? One motivation is analytic (e.g. for $k = \mathbb{C}$):

1. $\mathbb{P}_{\mathbb{C}}^n$ is compact with the analytic topology: There are surjective continuous maps

$$\begin{tikzcd}
\mathbb{R}^{2n+2} \setminus \{0\} \arrow[r, two heads] & \mathbb{CP}^n \\
S^{2n+1} \arrow[u, hook] \arrow[ur, two heads] &
\end{tikzcd}$$

2. *Chow's theorem*: Any closed complex submanifold of $\mathbb{CP}^n$ is a projective variety.

Another motivation is the extra data at ∞:

1. If ℓ_1, ℓ_2 are distinct lines in $\mathbb{A}^2$, then $\#(\ell_1 \cap \ell_2) = 0$ or 1. However, over $\mathbb{P}^2$, $\#(\ell_1 \cap \ell_2) = 1$ always.

2. *Bezout's theorem*: If $C_1, C_2 \subseteq \mathbb{A}^2$ are two distinct irreducible curves in $\mathbb{A}^2$, then
$$\#(C_1 \cap C_2) \leq (\deg C_1)(\deg C_2),$$
counting multiplicities. The version over $\mathbb{P}^2$ always gives equality.

12.2 Graded Rings

Remark. In projective space, for $f \in k[x_0, \ldots, x_n]$, we could try to define
$$V(f) = \{[a_0 : \cdots : a_n] : f(a_0, \ldots, a_n) = 0\}.$$
But this is bad notation as it is not well-defined ($f = 0$ depends on the representative in the equivalence class). Instead, if f is homogeneous of degree d, then
$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n),$$
so $V(f)$ is well-defined in this case, when f is homogeneous.

Definition 12.2. An $\mathbb{N}$-*graded ring* is a ring R with subgroups $R_d \subseteq R$ for $d \in \mathbb{N}$ such that
$$R = \bigoplus_{d \in \mathbb{N}} R_d \quad \text{and} \quad R_d R_e \subseteq R_{d+e}.$$
An element $f \in R$ is *homogeneous* if there exists d such that $f \in R_d$.

Example 12.2.1. For $S = k[x_0, \ldots, x_n]$, we can take $S_d = \bigoplus_{a_i \geq 0, \sum a_i = d} k x_0^{a_0} \cdots x_n^{a_n}$.

Definition 12.3. An ideal I in a graded ring is *homogeneous* if it is generated by homogeneous elements.

Example 12.3.1. We can write $(x, y^3 - 3x^2) \subseteq k[x, y]$ as (x, y^3), so it is homogeneous.

Proposition 12.1. *Let R be a graded ring with ideal I. The following are equivalent:*

1. *I is homogeneous;*
2. *for any $f = \sum_{d \in \mathbb{N}} f_d \in I$ with $f_d \in R_d$, then $f_d \in I$ for all d;*
3. *$I = \bigoplus_{d \in \mathbb{N}} (I \cap R_d)$.*

Proof. Left as an exercise. The interesting implication is $(1 \Rightarrow 2)$. □

Proposition 12.2. *Let I, J be homogeneous ideals of a graded ring R. Then*

1. *$I + J$, IJ, $\sqrt{I}$, and $I \cap J$ are all homogeneous;*
2. *R/I is a graded ring with $R/I = \bigoplus_{d \in \mathbb{N}} R_d/I_d$, where $I_d = I \cap R_d$.*

Proof. (1) We prove that $\sqrt{I}$ is homogeneous. Assume $f \in \sqrt{I}$, and write $f = f_0 + f_1 + \cdots + f_d$ with $f_i \in R_i$ and $f_d \neq 0$. Now there exists $n > 0$ such that $f^n \in I$, and
$$f^n = f_d^n + \text{lower order terms}.$$
Since I is homogeneous, $f_d^n \in I$, so $f_d \in \sqrt{I}$. Then $f_0 + \cdots + f_{d-1} \in \sqrt{I}$, and we can repeat.

(2) We can write $R/I = (\bigoplus_{d \in \mathbb{N}} R_d)/(\bigoplus_{d \in \mathbb{N}} (I \cap R_d))$. As abelian groups, this is $R/I \cong \bigoplus_{d \in \mathbb{N}} R_d/I_d$. One can check that the multiplication also respects the grading, so this is an isomorphism of rings. □

12.3 Projective Varieties

Definition 12.4. For a set $T \subseteq k[x_0, \ldots, x_n]$ of homogeneous elements, define its *vanishing locus*

$$V_p(T) := V(T) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in T\} \subseteq \mathbb{P}^n.$$

A *projective variety* is a subset of this form. For a homogeneous ideal $I \leq k[x_0, \ldots, x_n]$, define

$$V(I) = \{x \in \mathbb{P}^n : f(x) \text{ for all } f \in I \text{ homogeneous}\}.$$

For a subset $X \subseteq \mathbb{P}^n$, define its *ideal*

$$I_p(X) := I(X) = (f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in X).$$

Note that we need to take the ideal generated by these elements, otherwise we may not get an ideal.

Remark. If $T \subseteq k[x_0, \ldots, x_n]$ is a subset of homogeneous elements, then we have $V_p(T) = V_p((T))$. So projective varieties can equivalently be defined as vanishing sets of homogeneous ideals.

Example 12.4.1. Consider $X = V_p(x^2 - yz) \subseteq \mathbb{P}^2_{x:y:z}$. Set $H = V(x)$, then there is a bijection

$$\begin{aligned} U = \mathbb{P}^2 \setminus H &\xrightarrow{f} \mathbb{A}^2 \\ [1 : y : z] &\longmapsto (y, z). \end{aligned}$$

Then $f(X \cap U) = V(1 - yz)$. On the other hand, we can see that

$$X \cap H = \{[0 : 1 : 0], [0 : 0 : 1]\} = \{a, b\}.$$

If we were working with $\mathbb{C}$ with the analytic topology, then we can take limits on $V(1 - yz)$ and see

$$\lim_{t \to 0}[1 : t : 1/t] = \lim_{t \to 0}[t : t^2 : 1] = [0 : 0 : 1] = b.$$

Note that we essentially switched charts in order to take this limit. Similarly, we have

$$\lim_{t \to \infty}[1 : t : 1/t] = \lim_{t \to \infty}[1/t : 1 : 1/t^2] = [0 : 1 : 0] = a.$$

So we can see a, b as points at ∞ compactifying the curve $V(1 - yz)$.

Example 12.4.2. We have the following:

1. $V_p(0) = \mathbb{P}^n$;
2. $V_p(1) = \varnothing$;
3. if $p = [a_0 : \cdots : a_n]$ and $J = (a_i x_j - a_j x_i : 0 \leq i, j \leq n)$, then $V(J) = \{0\}$;
4. $I_0 = (x_0, \ldots, x_n)$ is called the *irrelevant ideal*, which has $V_p(I_0) = \varnothing = V_p(1)$ but $I_0 = \sqrt{I_0} \subsetneq (1)$.

Lecture 13

Sept. 30 — Projective Varieties, Part 2

13.1 More on Projective Varieties

Example 13.0.1. Consider $X = V(y^2z - x^3 - zx^2 - z^3) \subseteq \mathbb{P}^2$ and $H_z = V(z)$. Let

$$\begin{aligned} U_z = \mathbb{P}^2 \setminus H_z &\xrightarrow[\text{bij}]{f} \mathbb{A}^2 \\ [x : y : 1] &\longmapsto (x, y). \end{aligned}$$

Then $f(X \cap U_z) = V(y^2 - x^3 - x^2 - 1)$, and

$$X \cap U_z = V(y^2z - x^3 - zx^2 - z^3, z) = V(x^3, z) = \{[0 : 1 : 0]\}.$$

Example 13.0.2. Let $I = (x_0, \ldots, x_n)$ be the irrelevant ideal. Then I is radical, but

$$I_p(V_p(I)) = I_p(\varnothing) = (1) \neq \sqrt{I}.$$

13.2 Cones

Definition 13.1. A subset $C \subseteq \mathbb{A}^{n+1}$ is a *cone* if $0 \in C$ and $\lambda x \in C$ whenever $x \in C$ and $\lambda \in k$.

Example 13.1.1. If $X \subseteq \mathbb{P}^n$ is a projective variety, then we can set $C(X) = \pi^{-1}(X)\{0\}$, where

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [x]. \end{aligned}$$

Proposition 13.1. *If $C \subseteq \mathbb{A}^{n+1}$ is a cone, then $I_a(C) \leq k[x_0, \ldots, x_n]$ is homogeneous.*

Proof. Fix $f \in I_a(C)$. Then we can write $f = \sum_{i=0}^d f_i$ with f_i homogeneous of degree i. We want to show that $f_i \in I_a(C)$ for each i. Fix $x \in C$. For any $\lambda \in k$,

$$0 = f(\lambda x) = \sum_{i=0}^d \lambda^i f_i(x).$$

Viewing this as a polynomial in λ (with x fixed), we must have each $f_i(x) = 0$. Thus $f_i \in I_a(C)$. $\square$

13.3 Projective Nullstellensatz

Theorem 13.1 (Projective Hilbert's Nullstellensatz). *We have the following:*

1. *For a projective variety $X \subseteq \mathbb{P}^n$, $V_p(I_p(X)) = X$.*
2. *For a homogeneous ideal $J \leq k[x_0, \ldots, x_n]$ with $\sqrt{J} \neq (x_0, \ldots, x_n)$, $I_p(V_p(J)) = \sqrt{J}$.*

As a consequence, there is a bijection between projective varieties and radical homogeneous ideals of $k[x_0, \ldots, x_n]$ which are not equal to $(x_0, \ldots, x_n)$, given by $X \mapsto I_p(X)$ with inverse $J \mapsto V_p(J)$.

Proof. (1) This is similar to the affine case.

(2) Fix a homogeneous ideal $(1) \neq J \leq k[x_0, \ldots, x_n]$ such that $\sqrt{J} \neq (x_0, \ldots, x_n)$ (the theorem is clearly true for the unit ideal). Then observe that we can write

$$\begin{aligned} I_p(V_p(J)) &= (f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } [x] \in V_p(J)) \\ &= (f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \text{ for all } x \in V_a(J) \setminus \{0\}) \\ &= (f \in k[x_0, \ldots, x_n] : f(x) = 0 \text{ for all } x \in \overline{V_a(J) \setminus \{0\}}) \\ &= \begin{cases} I_a(V_a(J)) & \text{if } V_a(J) \supsetneq \{0\}, \qquad \text{(A)} \\ I_a(\varnothing) & \text{if } V_a(J) = \{0\}, \qquad \text{(B)} \end{cases} \end{aligned}$$

In Case A, we get that $I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J}$ by the affine Nullstellensatz. In Case B, we have $V_a(J) = \{0\}$, so $\sqrt{J} = (x_0, \ldots, x_n)$, which we assumed was not the case. □

13.4 The Zariski Topology on $\mathbb{P}^n$

Remark. We have the following properties of I_p and V_p:

1. For homogeneous ideals $J_i \leq k[x_0, \ldots, x_n]$ for $i \in I$, we have $V_p(\sum_{i\in I} J_i) = \bigcap_{i\in I} V_p(J_i)$;

   If $I = \{1, 2\}$, then we have $V_p(J_1J_2) = V_p(J_1) \cup V_p(J_2)$.
2. If $X_1, X_2 \subseteq \mathbb{P}^n$ are projective varieties, then

$$I_p(X_1 \cup X_2) = I_p(X_1) \cap I_p(X_2) \quad \text{and} \quad I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)},$$

   where we assume in the second equality that $X_1 \cap X_2 \neq \varnothing$.

The proofs are similar to the affine case.

Example 13.1.2. Let $X_1 = V(x) \subseteq \mathbb{P}^2$ and $X_2 = V(y, z) \subseteq \mathbb{P}^2$. Then $I(X_1 \cap X_1) = I(\varnothing) = (1)$, but we have $I(X_1) + I(X_2) = (x, y, z)$, which is already radical.

Definition 13.2. The *Zariski topology* on $\mathbb{P}^n$ is the topology whose closed sets are projective varieties $X \subseteq \mathbb{P}^n$ (equivalently, the vanishing loci of homogeneous ideals).

Remark. This is a topology by the above properties of I_p and V_p. We now want to relate this to the topology on our charts. Let $H_0 = V(x_0)$ and consider the bijection

$$\begin{aligned} \mathbb{A}^n &\xrightarrow{\rho_0} \mathbb{P}^n \setminus H_0 \\ (x_1, \ldots, x_n) &\longmapsto [1 : x_1 : \cdots : x_n]. \end{aligned}$$

We want to show that ρ_0 is a homeomorphism. Write $\mathbb{A}^n \subseteq \mathbb{P}^n$. Consider the ring homomorphism

$$\begin{aligned} k[x_0, \dots, x_n] &\xrightarrow{\Phi} k[x_1, \dots, x_n] \\ f(x_0, \dots, x_n) &\longmapsto f(1, x_1, \dots, x_n) =: f^i \end{aligned}$$

We call f^i the *dehomogenization* of f.

Example 13.2.1. Let $f(x) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3$, then $f^i(x) = x_2^2 - x_1^3 - x_1^2 - 1$.

Definition 13.3. If $J \leq k[x_0, \dots, x_n]$ is homogeneous, then define its *dehomogenization* to be

$$J^i = (f^i : f \in J) = \Phi(J).$$

Proposition 13.2. *For $J \leq k[x_0, \dots, x_n]$ homogeneous, $V_p(J) \cap \mathbb{A}^n = V_a(J^i)$.*

Proof. The idea is to use that for $[1 : x_1 : \cdots : x_n] \in \mathbb{P}^n$ and $f \in k[x_0, \dots, x_n]$ homogeneous, we have $f([1 : x]) = 0$ if and only if $f^i(x) = 0$. Fill in the details as an exercise. $\square$

Definition 13.4. If $f \in k[x_1, \dots, x_n]$ with $\deg f = d$, then define its *homogenization* to be

$$f^h = x_0^d f(x_1/x_0, \dots, x_n/x_0) \in k[x_0, x_1, \dots, x_n],$$

which is homogeneous of degree d.

Example 13.4.1. Let $f = x_2^2 - x_1^3 - x_1^2 - 1$. Then we have

$$f^h = x_0^3((x_2/x_0)^2 - (x_1/x_0)^3 - (x_1/x_0)^2 - 1) = x_0x_2^2 - x_1^3 - x_0x_1^2 - x_0^3.$$

Remark. While $f^hg^h = (fg)^h$, note that $(f+g)^h \neq f^h + g^h$ in general.

Definition 13.5. For $J \leq k[x_1, \dots, x_n]$ an ideal, define its *homogenization* to be

$$J^h = (f^h : f \in J).$$

Proposition 13.3. *For $J \leq k[x_1, \dots, x_n]$ an ideal, $V_a(J) = V_p(J^h) \cap \mathbb{A}^n$.*

Proof. Left as an exercise, use that $f(a_1, \dots, a_n) = 0$ if and only if $f^h(1, a_1, \dots, a_n) = 0$. $\square$

Remark. The above results imply that $\rho_0 : \mathbb{A}^n \to \mathbb{P}^n \setminus H_0$ is a homeomorphism.

Lecture 14

Oct. 9 — Projective Space as Varieties

14.1 More on the Zariski Topology on $\mathbb{P}^n$

Proposition 14.1. *For each $0 \le i \le n$, the map*

$$\begin{aligned} U_i = \mathbb{P}^n \setminus V(x_i) &\xrightarrow{h_i} \mathbb{A}^n \\ [x_0 : \dots : x_n] &\longmapsto (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i) \end{aligned}$$

is a homeomorphism.

Proof. The main inputs to the proof are

- For $I \le k[x_0, \dots, x_n]$ homogeneous, $h_0(V(I) \cap U_0) = V(I^i)$.
- For $J \le k[x_1, \dots, x_n]$, $h_0^{-1}(V(J)) = V(J^h)$.

Fill in the remaining details as an exercise. $\square$

Proposition 14.2 (Projective closure)**.** *For $J \le k[x_1, \dots, x_n]$ and $X = V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$, we have*

$$\overline{X} = V_p(J^h).$$

Proof. See Gathmann. $\square$

Proposition 14.3. *If $X = V_a(f) \subseteq \mathbb{A}^n$ with $f \in k[x_1, \dots, x_n]$, then its projective closure in $\mathbb{P}^n$ is*

$$\overline{X} = V_p(f^h).$$

Proof. We know that $\overline{X} = V_p(\langle f \rangle^h)$ by Proposition 14.2. Now

$$\langle f \rangle^h = \langle (fg)^h : g \in k[x_1, \dots, x_n] \rangle = \langle f^h g^h : g \in k[x_1, \dots, x_n] \rangle = \langle f^h \rangle,$$

which implies the desired result. $\square$

Example 14.0.1 (Twisted cubic)**.** Take $X = \operatorname{Im}(\mathbb{A}^1 \to \mathbb{A}^3 : t \mapsto (t, t^2, t^3))$. Note that $X \cong \mathbb{A}^1$, and

$$I_a(X) = (x^2 - y, x^3 - z) = (x^2 - y, x^3 - z, xy - z).$$

Then one can check that $\overline{X} \subseteq \mathbb{P}^3_{w:x:y:z}$ is given by $\overline{X} = V_p(x^2 - yw, x^3 - zw^2, xy - zw)$. However, one can also check that $\overline{X}$ cannot be cut out by 2 equations. For example,

$$V_p(x^2 - yw, x^3 - zw^2) = \overline{X} \cup V(w, x).$$

14.2 Projective Space as Varieties

Remark. Our goal now is to show that projective varieties are varieties. The first step is to define a sheaf of regular functions on $\mathbb{P}^n$.

Definition 14.1. Let U be an open set of a projective variety $X \subseteq \mathbb{P}^n$. A function $\varphi : U \to k$ is *regular* if for every $p \in U$, there exists $d \in \mathbb{N}$, $f, g \in k[x_0, \ldots, x_n]$ homogeneous of degree d, and $U_p \subseteq U$ open such that

$$\varphi(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in U_p.$$

Remark. If $X \subseteq \mathbb{P}^n$ is a projective variety, then

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}$$

is a sheaf of rings on X. Again this is because the regular condition can be checked locally.

Proposition 14.4. *If $X \subseteq \mathbb{P}^n$ is a projective variety, then $(X, \mathcal{O}_X)$ is a pre-variety.*

Proof. Let $X_i = X \cap (\mathbb{P}^n \setminus V(x_i))$. It suffices to show $(X_i, \mathcal{O}_X|_{X_i})$ is an affine variety for each $0 \leq i \leq n$. For simplicity, assume $i = 0$. Let $J = I(X) \leq k[x_0, \ldots, x_n]$ and $Z_0 = V(J^i) \subseteq \mathbb{A}^n$. We have seen before that we have a homeomorphism

$$\begin{aligned} X_0 &\xrightarrow{F} Z_0 \\ [x_0 : \cdots : x_n] &\longmapsto (x_1/x_0, \ldots, x_n/x_0). \end{aligned}$$

We claim that F induces an isomorphism of ringed spaces $(X_0, \mathcal{O}_X|_{X_0}) \cong (Z_0, \mathcal{O}_{Z_0})$. To see this, we need to check that regular functions pull back to regular functions via F and F^{-1}. A regular function on an open set of X_0 is locally of the form

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)}$$

with f, g homogeneous of the same degree. Now

$$(F^{-1})^* \left(\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} \right) = \frac{f(1, x_1, \ldots, x_n)}{g(1, x_1, \ldots, x_n)},$$

which is a fraction of polynomials and hence regular on Z_0. So F^{-1} pulls regular functions back to regular functions. Conversely, a regular function on Z_0 is locally given by

$$\frac{q(x_1, \ldots, x_n)}{r(x_1, \ldots, x_n)},$$

and its pullback via F is

$$F^* \left(\frac{q(x_1, \ldots, x_n)}{r(x_1, \ldots, x_n)} \right) = \frac{q(x_1/x_0, \ldots, x_n/x_0)}{r(x_1/x_0, \ldots, x_n/x_0)} = \frac{x_0^d q(x_1/x_0, \ldots, x_n/x_0)}{x_0^d r(x_1/x_0, \ldots, x_n/x_0)},$$

where $d = \max\{\deg q, \deg r\}$. This is regular on X_0, so F also pulls regular functions back to regular functions. So we get an isomorphism of ringed spaces, as desired. $\square$

Example 14.1.1. $\mathbb{P}^n$ is a pre-variety, and $\mathbb{P}^n \setminus V(x_i) =: U_i \cong \mathbb{A}^n$ as pre-varieties.

Definition 14.2. A *morphism* of projective varieties is a morphism of the underlying pre-varieties.

Remark. For a projective variety X, it will be convenient to work with "global coordinates," i.e.

$$S(X) := k[x_0, \dots, x_n]/I_p(X).$$

This is called the *homogeneous coordinate ring*. Note the following:

1. For $f \in S(X)$ homogeneous, f is not necessarily a well-defined function on X. But

$$V(f) = \{[x] \in X : f(x) = 0\}$$

   is still well-defined.

2. A relative version of the projective Nullstellensatz holds: There is a bijection

$$\left\{\begin{matrix}\text{projective subvarieties}\\ Y \subseteq X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{radical homogeneous ideals in } S(X)\\ \text{not equal to } (\overline{x}_1, \dots, \overline{x}_n)\end{matrix}\right\}$$

$$\begin{aligned} Y &\longmapsto I(Y) \\ V(J) &\longleftarrow\!\shortmid J \end{aligned}$$

where $I(Y) = \langle f \in S(X) : f \text{ homogeneous and } f(y) = 0 \text{ for all } y \in Y\rangle$.

Lemma 14.1. *If $X \subseteq \mathbb{P}^n$ and $f_0, \dots, f_m \in S(X)$ are homogeneous of the same degree, then*

$$\begin{aligned} U = X \setminus V(f_0, \dots, f_m) &\xrightarrow{f} \mathbb{P}^m \\ [x_0 : \cdots : x_n] &\longmapsto [f_0(x) : \cdots : f_m(x)] \end{aligned}$$

is a morphism.

Proof. To see that f is well-defined, note that for $[a_0 : \cdots : a_n] \in X \setminus V(f_0, \dots, f_m)$, we have

$$(f_0(\lambda a), \dots, f_m(\lambda a)) = \lambda^d (f_0(a), \dots, f_m(a))$$

with $d = \deg f_i$. So $[f_0(a) : \cdots : f_m(a)] \in \mathbb{P}^m$ is well-defined. To see that f is a morphism, we check locally on $\mathbb{P}^m$. Let $V_i = \mathbb{P}^m \setminus V(x_i)$ and $U_i = f^{-1}(V_i)$. Then

$$\begin{aligned} U_i &\xrightarrow{f|_{U_i}} V_i \cong \mathbb{A}^m \\ a &\longmapsto \left(\frac{f_0(a)}{f_i(a)}, \dots, \frac{\widehat{f_i(a)}}{f_i(a)}, \dots, \frac{f_m(a)}{f_i(a)} \right). \end{aligned}$$

Since each f_j/f_i is regular, $f|_{U_i}$ is a morphism. So f is a morphism. □

Example 14.2.1. Define a map

$$\begin{aligned} \mathbb{P}^1_{s:t} &\xrightarrow{f} \mathbb{P}^3_{x:y:z} \\ [s : t] &\longmapsto [s^3 : s^2 t : s t^2 : t^3]. \end{aligned}$$

Then $S(\mathbb{P}^1) = k[s, t]$ and $f(\mathbb{P}^1)$ is the projective twisted cubic in $\mathbb{P}^3$.

Example 14.2.2. Let $A \in \mathrm{GL}_{n+1}(k)$. Then

$$\begin{aligned} f_A : \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

is an isomorphism with inverse $f_{A^{-1}}$. We will see later that we have a surjective group homomorphism

$$\begin{aligned} \mathrm{GL}_{n+1}(k) &\longrightarrow \mathrm{Aut}(\mathbb{P}^n) \\ A &\longmapsto f_A \end{aligned}$$

with kernel $k^\times I$. So we get $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{GL}_{n+1}(k)/k^\times I =: \mathrm{PGL}_{n+1}(k)$.

Example 14.2.3 (Conics)**.** Let $f \in k[x,y,z]$ be homogeneous of degree 2, and write

$$f = (x,y,z)B(x,y,z)^T$$

with B a symmetric 3×3 matrix. We want to characterize $X = V(f)$. Choose $A \in \mathrm{GL}_3(k)$ such that

$$B' = ABA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $f' = (x,y,z)B'(x,y,z)^T$ has $f' = x^2 + y^2 + z^2$, $x^2 + y^2$, or x^2. Now A induces an isomorphism $h_{A^{-1}} : \mathbb{P}^2 \to \mathbb{P}^2$ and $g := h_{A^{-1}}|_X : X \to h_{A^{-1}}(X) = V(f')$, so any projective conic is isomorphic to

$$V(x^2 + y^2 + z^2), \quad V(x^2 + y^2), \quad \text{or} \quad V(x^2).$$

Example 14.2.4 (Projections)**.** Let $a = [1 : 0 : \cdots : 0]$ and define

$$\begin{aligned} \mathbb{P}^n \setminus \{a\} &\xrightarrow{f} \mathbb{P}^{n-1} \\ [x_0 : \cdots : x_n] &\longmapsto [x_1 : \cdots : x_n]. \end{aligned}$$

Geometrically, if we fix $[b] \in \mathbb{P}^n \setminus \{a\}$ and set

$$\ell_{a,b} = \{[s : tb_1 : \cdots : tb_n] : (s,t) \in k^2 \setminus \{0\}\} = \text{the line through } a \text{ and } b,$$

then $\ell_{a,b} \cap V(x_0) = [0 : b_1 : \cdots : b_n] = [0 : f(b)]$.