

# MATH 6422: Algebraic Geometry II

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# Lecture 1

## Jan. 13 — Overview and Review

### 1.1 Course Overview

**Remark.** This course will cover the following topics:

- (i) vector bundles and line bundles in algebraic geometry;
- (ii) coherent sheaves;
- (iii) differentials;
- (iv) sheaf cohomology: in particular, we will see that  $H_{\mathrm{dR}}^k(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T_X^*)$ ;
- (v) the Riemann-Roch theorem: if  $\omega = f dz$  is a rational 1-form on a smooth projective curve  $C$ , then

$$(\# \text{ zeroes of } \omega) - (\# \text{ poles of } \omega) = 2 \operatorname{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example,  $\operatorname{Spec} \mathbb{Z}$  has points corresponding to the primes  $p$  and 0.

### 1.2 Review of Algebraic Geometry I

**Remark.** Let  $k = \bar{k}$  be an algebraically closed field.

**Remark** (Hilbert's Nullstellensatz). There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \dots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence,  $Z$  being irreducible (resp. a point) corresponds to  $I(Z)$  being prime (resp. maximal).

**Remark** (Zariski topology on  $\mathbb{A}^n$ ). The closed sets in  $\mathbb{A}^n$  are of the form  $V(J)$ , and this induces a Zariski topology on any subset of  $\mathbb{A}^n$ .

**Remark** (Embedded affine varieties). Let  $J \leq k[x_1, \dots, x_n]$ . Then we can associate to  $J$  a ringed space  $(X, \mathcal{O}_X)$  by setting  $X := V(J) \subseteq \mathbb{A}^n$  with the Zariski topology, and  $\mathcal{O}_X$  to be the sheaf of regular functions on  $X$ , i.e. for  $U \subseteq X$  open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \rightarrow k \mid \varphi \text{ is regular}\}.$$

Here,  $\varphi : U \rightarrow k$  is *regular* if for each  $p \in U$ , there exists an open set  $U_p \subseteq U$  and  $f, g \in k[x_1, \dots, x_n]$  such that  $\varphi(x) = f(x)/g(x)$  for all  $x \in U_p$ .

**Remark** (Coordinate ring). The *coordinate ring* of  $X$  is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \dots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for  $A(X)$ :

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

**Remark** (Distinguished open sets). The *distinguished open sets* of  $X$  are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These form a basis for  $X$  as we vary  $f \in A(X)$ .

**Definition 1.1.** An *affine variety* is a ringed space  $(X, \mathcal{O}_X)$  (here  $\mathcal{O}_X$  is a sheaf of  $k$ -valued functions) which is isomorphic to an embedded affine variety.

**Example 1.1.1.** If  $(X, \mathcal{O}_X)$  is an affine variety and  $f \in \mathcal{O}_X(X)$ , then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that  $X = V(J) \subseteq \mathbb{A}^n$  with  $J \subseteq k[x_1, \dots, x_n]$  a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_y^1 \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \dots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \dots, x_n]/J)[y]}{(fy - 1)} \cong A(X)_f.$$

**Theorem 1.1.** *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.-}k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

*This implies the following:*

1. *There is a bijection*

$$\begin{aligned} \text{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated  $k$ -algebra  $A$ , there exists an affine variety with  $A \cong A(X)$ .*

**Remark.** How can we explicitly define the inverse functor  $\text{Red-f.g.-}k\text{-alg}^{\text{op}} \rightarrow \text{Aff-var}$ ? We can define this as  $A \mapsto (X, \mathcal{O}_X)$ , where  $X$  is the set of maximal ideals of  $A$ . Think about what  $\mathcal{O}_X$  should be.

**Remark** (Varieties). A *variety*  $(X, \mathcal{O}_X)$  is a ringed space such that

- there exists a finite open cover of  $X$  by affine varieties,
- the diagonal  $\Delta_X$  is closed in  $X \times X$ .

**Example 1.1.2.** The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$ .

**Remark** (Projective spaces). Recall that  $\mathbb{P}^n$  has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for  $\mathbb{P}^n$  by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with  $f \in k[x_0, \dots, x_n]$  homogeneous.

## 1.3 Vector and Line Bundles

**Definition 1.2.** Let  $X$  be a variety. A *vector bundle* (of rank  $m$ ) on  $X$  is a variety  $\mathbb{E}$  with a morphism  $p : \mathbb{E} \rightarrow X$  such that

1.  $\mathbb{E}_x = p^{-1}(x)$  has the structure of a rank  $m$  vector space for every  $x \in X$  (i.e.  $k^m$ ),
2. for every  $x \in X$ , there exists an open neighborhood  $x \in U \subseteq X$  and an isomorphism  $p^{-1}(U) \rightarrow U \times \mathbb{A}^m$  such that for any  $y \in U$ , the map  $\mathbb{E}_y \rightarrow \{y\} \times \mathbb{A}^m$  is an isomorphism of vector spaces, i.e.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{A}^m \\ & \searrow p \quad \swarrow \text{proj} & \\ & U & \end{array}$$

commutes. We will call the map  $\phi_U$  a *trivialization*.

**Definition 1.3.** A *line bundle* on  $X$  is a rank 1 vector bundle.

**Remark.** A different way to think about this is the following:

1. Given two trivializations  $\phi_U : p^{-1}(U) \rightarrow U \times \mathbb{A}^m$  and  $\phi_V : p^{-1}(V) \rightarrow V \times \mathbb{A}^m$ , we get a morphism

$$\begin{array}{ccc} (U \cap V) \times \mathbb{A}^m & \xrightarrow{\phi_{U,V}} & (U \cap V) \times \mathbb{A}^m \\ & \searrow \phi_V^{-1} \quad \swarrow \phi_U & \\ & p^{-1}(U \cap V) & \end{array}$$

with  $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$ . Observe  $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$  for some  $g_{U,V}(x) \in \text{GL}(m, k)$ . Furthermore,  $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$  is a morphism. We will call the  $g_{U,V}$  *transition functions*.

In the special case where  $m = 1$  (so  $\mathbb{E}$  is a line bundle and  $\text{GL}(1, k) = k^\times$ ), the map  $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$  is equivalent to the data of a non-vanishing regular function  $g_{U,V} : U \cap V \rightarrow k$ .

2. The data of a vector bundle of rank  $m$  is equivalent to the data of

- an open cover  $X = \bigcup_{i \in I} U_i$ ,
- and morphisms  $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$

such that  $g_{i,k} = g_{i,j}g_{j,k}$ ,  $g_{i,j} = g_{j,i}^{-1}$ , and  $g_{i,i} = \mathrm{id}$ .

To recover the vector bundle, we can glue  $\mathbb{E}_i = U_i \times \mathbb{A}^m$  for  $i \in I$  via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x)v). \end{aligned}$$

One can check that this defines a vector bundle  $\mathbb{E}$  on  $X$ .

**Example 1.3.1** (Trivial vector bundle). Define the vector bundle  $\mathbb{E} : X \times \mathbb{A}^m \rightarrow X$  by  $(x, v) \rightarrow x$ . Given a cover  $X = \bigcup_{i \in I} U_i$ , we get  $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$  as  $x \mapsto I_m$ .

**Example 1.3.2** (Trivial line bundle). We will denote the trivial line bundle by  $\mathbb{1}_X : X \times \mathbb{A}^1 \rightarrow X$ .

# Lecture 2

## Jan. 15 — Vector and Line Bundles

### 2.1 Vector and Line Bundles, Continued

**Example 2.0.1** (Tautological bundle). Let  $X = \mathbb{P}^n$  and  $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$ . Consider

$$\begin{array}{ccc} & \mathbb{L} & \\ p \swarrow & & \searrow q \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{array}$$

The map  $q : \mathbb{L} \rightarrow \mathbb{A}^{n+1}$  is the blowup. We claim that  $p : \mathbb{L} \rightarrow \mathbb{P}^n$  is a line bundle. We have:

- $p^{-1}([x]) = \{([x], cx) : c \in k\} \cong kx$ , a 1-dimensional vector space;
- let  $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$ , then we can define

$$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

which we claim is a trivialization. To see this, observe that for fixed  $[x] \in \mathbb{P}^n$ , we have

$$\begin{aligned} \mathbb{L}_{[x]} &= \{([x], cx) : c \in k\} \longrightarrow \{[x]\} \times \mathbb{A}^1 \\ ([x], cx) &\longmapsto ([x], cx_i), \end{aligned}$$

which is a vector space isomorphism.

We can also compute the transitions functions. Let  $U_{i,j} = U_i \cap U_j$ . We have

$$\begin{array}{ccccc} & & \phi_{i,j} & & \\ & \searrow & & \nearrow & \\ U_{i,j} \times \mathbb{A}^1 & \xrightarrow{\phi_j^{-1}} & p^{-1}(U_{i,j}) & \xrightarrow{\phi_i} & U_{i,j} \times \mathbb{A}^1 \\ & & ([x], t) \longmapsto ([x], (tx_0/x_j, \dots, tx_n/x_j)) \longmapsto ([x], tx_i/x_j) & & \end{array}$$

Thus we see that  $g_{i,j} = x_i/x_j$ . This is called the *tautological bundle*, or  $\mathcal{O}_{\mathbb{P}^n}(-1)$ .

**Example 2.0.2** (Hyperplane bundle, or  $\mathcal{O}_{\mathbb{P}^n}(1)$ ). Consider

$$\begin{aligned} \mathbb{L} &:= \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \longrightarrow \mathbb{P}^n \\ [x_0 : \dots : x_n : x_{n+1}] &\longmapsto [x_0 : \dots : x_n]. \end{aligned}$$

Then  $\mathbb{L}$  is a line bundle with transition functions with respect to  $\{U_i\}$  given by  $g_{i,j} = x_j/x_i$  (HW).

## 2.2 Operations on Vector Bundles

**Remark.** The philosophy is: Every natural operation of vector spaces gives one for vector bundles.

**Example 2.0.3** (Direct sum). Let  $p : \mathbb{E} \rightarrow X$  and  $q : \mathbb{F} \rightarrow X$  be vector bundles of rank  $e$  and  $f$  on  $X$ , respectively. There exists trivializations with respect to a common open cover  $\{U_i\}$  (just take intersections) with transition functions  $g_{i,j}$  and  $h_{i,j}$  for  $\mathbb{E}$  and  $\mathbb{F}$ , respectively.

Then we define the vector bundle  $\mathbb{E} \oplus \mathbb{F} \rightarrow X$  as follows:

- As a set, it is  $r : \mathbb{E} \oplus \mathbb{F} = \{(x, u, v) : (x, u) \in \mathbb{E}, (x, v) \in \mathbb{F}\} \rightarrow X$ .
- We give  $\mathbb{E} \oplus \mathbb{F}$  the structure of a variety by requiring that

$$\begin{aligned} r^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^{e+f} \\ (x, u, v) &\longmapsto (x, \text{pr}_2(\phi_i^E(x, u)), \text{pr}_2(\phi_i^F(x, v))) \end{aligned}$$

be an isomorphism, where  $\phi_i^E$  and  $\phi_i^F$  are the trivializations of  $\mathbb{E}$  and  $\mathbb{F}$ , and  $\text{pr}_2$  is the second projection. This gives a variety structure on  $r^{-1}(U_i)$ , and one can show that these are consistent on  $U_{i,j}$ , so that this gives a variety structure on all of  $\mathbb{E} \oplus \mathbb{F}$ .

Note that the transition functions for  $\mathbb{E} \oplus \mathbb{F}$  with respect to  $\{U_i\}$  are given by the block matrix

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{bmatrix} : U_{i,j} \longrightarrow \text{GL}(e + f, k).$$

**Example 2.0.4.** Let  $\mathbb{E}$  and  $\mathbb{F}$  be vector bundles on  $X$  of ranks  $e$  and  $f$ , respectively. Then the following are also vector bundles on  $X$ :

1.  $\text{Hom}(\mathbb{E}, \mathbb{F})$ , of rank  $ef$ ;
2.  $\mathbb{E}^\vee = \text{Hom}(\mathbb{E}, \mathbb{1}_X)$ , of rank  $e$ ;
3.  $\mathbb{E} \otimes \mathbb{F}$ , of rank  $ef$ ;
4.  $\wedge^k \mathbb{E}$  and  $\text{Sym}^d \mathbb{E}$ .

**Remark.** Let  $\mathbb{L}, \mathbb{M}$  be line bundles on  $X$  with trivializations on  $\{U_i\}$  and transition functions  $g_{i,j}, h_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$ . In this case, we can describe operations on  $\mathbb{L}, \mathbb{M}$  more explicitly:

1.  $\mathbb{L} \otimes \mathbb{M}$  has transition functions  $g_{i,j}h_{i,j}$ ;
2.  $\text{Hom}(\mathbb{L}, \mathbb{M})$  has transition functions  $h_{i,j}/g_{i,j}$ ;
3.  $\mathbb{L}^\vee = \text{Hom}(\mathbb{L}, \mathbb{1}_X)$  has transition functions  $1/g_{i,j}$ ;
4.  $\mathbb{L}^{\otimes m} = \begin{cases} \mathbb{L}^{\otimes m}, & \text{if } m > 0, \\ \mathbb{1}_X, & \text{if } m = 0, \\ (\mathbb{L}^\vee)^{\otimes -m}, & \text{if } m < 0 \end{cases}$  has transition functions  $g_{i,j}^m$ .

**Example 2.0.5.** Define  $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$  with transition functions  $(x_j/x_i)^m$  with respect to the standard open cover for  $\mathbb{P}^n$ .

## 2.3 Morphisms of Vector Bundles

**Remark.** Let  $p : \mathbb{E} \rightarrow X$  and  $q : \mathbb{F} \rightarrow X$  be vector bundles on  $X$ , as before.

**Definition 2.1.** A *morphism of vector bundles*  $\mathbb{E} \rightarrow \mathbb{F}$  is a morphism of varieties

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{a} & \mathbb{F} \\ & \searrow p & \swarrow q \\ & X & \end{array}$$

such that the diagram commutes and  $a$  is linear on each fiber.

**Remark.** More concretely, given an open cover  $\{U_i\}$  which trivializes both vector bundles, we have

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{a} & q^{-1}(U_i) \\ \phi_i \downarrow \cong & & \cong \downarrow \psi_j \\ U_i \times \mathbb{A}^e & \longrightarrow & U_i \times \mathbb{A}^f \\ (x, v) & \longmapsto & (x, a_i(x)v) \end{array}$$

such that  $a_i : U_i \rightarrow \text{Hom}(k^e, k^f)$  is regular. On  $U_{i,j}$ , we have

$$\begin{array}{ccc} U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_j} & U_{i,j} \times \mathbb{A}^f \\ g_{i,j} \downarrow & & \downarrow h_{i,j} \\ U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_i} & U_{i,j} \times \mathbb{A}^f \end{array}$$

So  $h_{i,j}a_j = a_i g_{i,j}$ , or equivalently,  $a_i = h_{i,j}a_j g_{i,j}^{-1}$ .

As a special case when  $e = f$ ,  $a : \mathbb{E} \rightarrow \mathbb{F}$  is an isomorphism if and only if the  $a_i$  are isomorphisms.

**Remark.** When is a line bundle  $\mathbb{L}$  given by the trivialization data  $\{U_i, g_{i,j}\}$  isomorphic to  $\mathbb{1}_X$ ? We have

$$\begin{aligned} \mathbb{L} \cong \mathbb{1}_X &\iff \text{if and only if there exists an isomorphism } a : \mathbb{1}_X \rightarrow \mathbb{L} \\ &\iff \text{there exist } a_i \in \mathcal{O}_X(U_i)^\times \text{ such that } (a_j/a_i)|_{U_{i,j}} = g_{i,j}. \end{aligned}$$

**Definition 2.2.** Define the *Picard group* of  $X$  to be

$$\text{Pic } X := \{\text{line bundles on } X\} / \cong.$$

This is a group with respect to  $\otimes$  with  $\mathbb{1}_X$  as the identity and  $\mathbb{L}^\vee \otimes \mathbb{L} \cong \mathbb{1}_X$ .

## 2.4 Global Sections

**Definition 2.3.** A (*global*) *section* of a vector bundle  $p : \mathbb{E} \rightarrow X$  is a morphism  $s : X \rightarrow \mathbb{E}$  such that  $p \circ s = \text{id}_X$ . Note that for  $x \in X$ , we have  $s(x) \in \mathbb{E}_x$ .

**Example 2.3.1** (Zero section). Let  $s : X \rightarrow \mathbb{E}$  where  $s(x)$  is the zero element in  $\mathbb{E}_x$ .

**Example 2.3.2.** Let  $\mathbb{E} = \mathbb{1}_X$ . Then sections  $s : X \rightarrow X \times \mathbb{A}^1$  of  $\mathbb{E}$  correspond to morphisms  $X \rightarrow \mathbb{A}^1$ , which correspond to regular functions  $X \rightarrow k$ .

**Remark** (Local description of sections). Let  $\{U_i, g_{i,j}\}$  be the trivialization data for  $\mathbb{E} \rightarrow X$ , and let  $s : X \rightarrow \mathbb{E}$  be a section. On  $U_i$ , we have:

$$\begin{array}{ccc} & & U_i \times \mathbb{A}^e \\ & \nearrow x \mapsto (x, s_i(x)) & \uparrow \cong \phi_i \\ U_i & \xrightarrow{s|_{U_i}} & p^{-1}(U_i) \end{array}$$

Note that  $s_i : U_i \rightarrow k^e$  is a regular function (i.e. regular on each coordinate). These maps must satisfy the compatibility condition  $s_i = g_{i,j}s_j$ , since we have the diagram:

$$\begin{array}{ccccc} & & & & U_{i,j} \times \mathbb{A}^e \\ & & \nearrow (x, s_j(x)) & & \uparrow \phi_j \\ U_{i,j} & \xrightarrow{s|_{U_{i,j}}} & p^{-1}(U_{i,j}) & & \\ & \searrow (x, s_i(x)) & \downarrow \phi_i & & U_{i,j} \times \mathbb{A}^e \\ & & & & \downarrow (x,v) \mapsto (x, g_{i,j}(v)) \end{array}$$

**Example 2.3.3.** We can use the above compatibility condition to compute the global sections of  $\mathcal{O}_{\mathbb{P}^1}(1)$ . Write  $\mathbb{P}_{x_0:x_1}^1 = U_0 \cup U_1$ . Given a section  $s : \mathbb{P}^1 \rightarrow \mathcal{O}(1)$ , we get regular functions

$$\begin{aligned} s_0 : U_0 &\longrightarrow k \\ s_1 : U_1 &\longrightarrow k \end{aligned}$$

satisfying  $(x_1/x_0)s_1 = s_0 (*)$ . We can write

$$s_0 = \sum_{m \geq 0} a_m (x_1/x_0)^m \quad \text{and} \quad s_1 = \sum_{m \geq 0} b_m (x_0/x_1)^m$$

with  $a_m, b_m \in k$  (finitely many nonzero). Then  $(*)$  implies that

$$a_0 + a_1(x_1/x_0) + \cdots = (x_1/x_0)(b_0 + b_1(x_0/x_1) + \cdots),$$

so  $a_0 = b_1$ ,  $a_1 = b_0$ , and all other terms are 0. So we can relate  $s$  to a linear form

$$f = a_0 x_0 + a_1 x_1,$$

where  $s_0 = (1/x_0)f$  and  $s_1 = (1/x_1)f$ .

# Lecture 3

## Jan. 20 — Sections

### 3.1 Global Sections, Continued

**Definition 3.1.** Let  $\Gamma(X, \mathbb{E}) := \{\text{sections of } \mathbb{E} \rightarrow X\}$ , which has the structure of a  $k$ -vector space by

$$(s + t)(x) = s(x) + t(x) \quad \text{and} \quad (cs)(x) = cs(x)$$

for  $s, t \in \Gamma(X, \mathbb{E})$  and  $c \in k$ .

**Example 3.1.1.** One can check that  $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0, \dots, x_n]_d$  (HW). For example, for  $d < 0$ , we have  $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \{0\}$ , and for  $d = 0$ , we have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(0)) = \Gamma(\mathbb{P}^n, \mathbb{1}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k.$$

For  $d = 1$ , we can define an isomorphism

$$\begin{aligned} k[x_0, \dots, x_n]_1 &\xrightarrow{\cong} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \\ f &\longmapsto \text{section } s : \mathbb{P}^n \rightarrow \mathcal{O}(1) \text{ given by } s_i = f/x_i. \end{aligned}$$

Note that  $(x_j/x_i)s_j = s_i$  holds, so this is a section. An alternative perspective is that  $s$  corresponds to

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \\ x &\longmapsto [x_0 : \dots : x_n : f(x)]. \end{aligned}$$

### 3.2 Morphisms and Sections

**Definition 3.2.** Given a section  $s : X \rightarrow \mathbb{E}$ , its *vanishing locus* is

$$V(s) := \{s = 0\} = \{x \in X : s(x) = 0\}.$$

Using a trivializing cover, one can check that  $V(s)$  is closed in  $X$ .

**Example 3.2.1.** For a section  $s : \mathbb{P}^n \rightarrow \mathcal{O}(1)$  corresponding to  $f \in k[x_0, \dots, x_n]_1$ , we have  $V(s) = V_{\mathbb{P}^n}(f)$ .

**Remark.** Recall that there is a bijection

$$\begin{aligned} \{\text{morphisms } X \rightarrow \mathbb{A}^n\} &\longleftrightarrow \{f_1, \dots, f_n \in \mathcal{O}_X(X)\} \\ [f : X \rightarrow \mathbb{A}^n] &\longmapsto [f_1 = f^*x_1, \dots, f_n = f^*x_n \in \mathcal{O}_X(X)] \\ [x \mapsto (f_1(x), \dots, f_n(x))] &\longleftarrow [f_1, \dots, f_n \in \mathcal{O}_X(X)]. \end{aligned}$$

We want a similar statement for  $\mathbb{P}^n$ .

**Definition 3.3.** Given a line bundle  $\mathbb{L} \rightarrow X$  and  $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ , they are *nowhere vanishing* if

$$V(s_0) \cap \dots \cap V(s_n) = \emptyset.$$

**Example 3.3.1.** For  $\mathcal{O}(1)$ , the sections  $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$  are nowhere vanishing.

**Remark.** If  $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$  are nowhere vanishing, then we get a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

Note that  $(s_0(x), \dots, s_n(x))$  is a well-defined point in  $\mathbb{A}^{n+1}$  up to scaling. One can check that this map is a morphism by working locally.

**Example 3.3.2** (Linear maps). Let  $X = \mathbb{P}^n$  and  $\mathbb{L} = \mathcal{O}(1)$ .

(i)  $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$  gives  $\text{id} : \mathbb{P}^n \rightarrow \mathbb{P}^n$ .

(ii) For  $A \in \text{GL}_{n+1}(k)$ , we get a map

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

given by  $Ax_0, \dots, Ax_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ .

**Remark.** Now given a morphism  $X \rightarrow \mathbb{P}^n$ , we want to get a line bundle with sections.

**Definition 3.4** (Pullback). Let  $p : \mathbb{E} \rightarrow X$  be a vector bundle and  $f : Y \rightarrow X$  a morphism. Define

$$f^*\mathbb{E} = \{(e, y) : e \in \mathbb{E}, y \in Y \text{ with } p(e) = f(y)\} \longrightarrow Y.$$

One can show that this has the structure of a vector bundle in a natural way.

**Remark.** An alternative way to define the pullback is to choose trivialization data  $(U_i, g_{i,j})$  for  $\mathbb{E} \rightarrow X$ . Then we can define  $f^* : \mathbb{E} \rightarrow Y$  to be the vector bundle with trivialization data  $(f^{-1}(U_i), f^*g_{i,j})$ .

**Remark.** Now to go in reverse, given a morphism  $X \rightarrow \mathbb{P}^n$  and nowhere vanishing sections  $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ , we get nowhere vanishing sections

$$f^*x_0, \dots, f^*x_n \in \Gamma(X, f^*\mathcal{O}(1)).$$

We can define the pullback of a section in one of two ways: by  $f^*(x_i)(a) = (x_i(f(a)), a) \in f^*\mathcal{O}(1)$  for  $a \in X$  or by using trivializing covers.

**Remark.** Using the above, we get a bijection

$$\{\text{morphisms } X \rightarrow \mathbb{P}^n\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \rightarrow X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}.$$

Note that we should consider the right-hand side up to isomorphism of the line bundle. When do  $\mathbb{L} \rightarrow X$  and  $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$  give an injective morphism (or an embedding)?

**Definition 3.5.** Given a vector bundle  $\mathbb{E} \rightarrow X$ , we get a sheaf of abelian groups  $\mathcal{E}$  on  $X$  by

$$\mathcal{E}(U) := \{\text{sections of } p^{-1}(U) \rightarrow U\}$$

for  $U \subseteq X$  open. For  $V \subseteq U \subseteq X$  open, the restriction map is given by

$$\begin{aligned} \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto s|_V. \end{aligned}$$

We call  $\mathcal{E}$  the *sheaf of sections* of  $\mathbb{E}$ . Also note that  $\mathcal{E}(U)$  has the structure of an  $\mathcal{O}_X(U)$ -module. We

will see that this gives rise to the structure of an  $\mathcal{O}_X$ -module.

### 3.3 Review of Sheaves

**Definition 3.6.** A *presheaf* of abelian groups  $\mathcal{F}$  on a topological space  $X$  is the data of:

- for  $U \subseteq X$  open, an abelian group  $\mathcal{F}(U)$  (with  $\mathcal{F}(\emptyset) = 0$ ),
- for  $V \subseteq U \subseteq X$  open, a group homomorphism  $p_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ .

**Remark.** Note the following:

1. We may replace abelian groups in the above definition by rings, sets,  $R$ -modules, etc.
2. We denote  $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ , whose elements are called *sections*.
3.  $s|_V := p_{V,U}(s)$  is called the *restriction* for  $s \in \mathcal{F}(U)$  and  $V \subseteq U \subseteq X$  open.
4. We may view  $\mathcal{F}$  as a functor  $\text{Open}_X \rightarrow \text{Ab-gps}$  given by  $U \mapsto \mathcal{F}(U)$ .

**Definition 3.7.** For  $\mathcal{F}$  a presheaf on  $X$  and  $x \in X$ , the *stalk* of  $\mathcal{F}$  at  $x$  is

$$\mathcal{F}_x = \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U) = \{(s, U) : s \in \mathcal{F}(U)\} / \sim.$$

**Example 3.7.1.** The following are examples of presheaves:

1. Let  $M$  be a smooth manifold. Then
  - $\mathcal{O}_M$  = sheaf of smooth  $\mathbb{R}$ -valued functions on  $M$ ,
  - $\mathcal{E}$  = sheaf of sections of a vector bundle  $\mathbb{E} \rightarrow M$ .
2. Let  $X$  be an algebraic variety,  $\mathbb{E} \rightarrow X$  a vector bundle, and  $Z \subseteq X$  closed. Then
  - $\mathcal{O}_X$  and  $\mathcal{E}$  are sheaves,
  - $\mathcal{I}_Z$  = ideal sheaf of  $Z$ , given by  $\mathcal{I}_Z(U) = \{\varphi \in \mathcal{O}_X(U) : \varphi|_Z = 0\}$ .
3. Let  $X$  be a topological space and  $A$  an abelian group.
  - $\underline{A}^{\text{pre}}$  given by  $U \mapsto \{\text{constant functions } U \rightarrow A\}$ , i.e.  $\underline{A}^{\text{pre}}(U) \cong A$  for  $U \neq \emptyset$ ,
  - $\underline{A}$  given by  $U \mapsto \{\text{locally constant functions } U \rightarrow A\}$ ,
  - $i_p A$  = skyscraper sheaf, given by  $U \mapsto \begin{cases} A & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$

**Definition 3.8.** A presheaf  $\mathcal{F}$  is a *sheaf* if for any

- open set  $U \subseteq X$ ,
- open cover  $U = \bigcup_{i \in I} U_i$ ,
- and  $s_i \in \mathcal{F}(U_i)$  such that  $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$  for all  $i, j \in I$ ,

then there exists a unique  $s \in \mathcal{F}(U)$  such that  $s|_{U_i} = s_i$  for every  $i \in I$ .

**Remark.** The presheaf  $\underline{A}^{\text{pre}}$  is not a sheaf in general. All other examples above are sheaves.

**Definition 3.9.** A *morphism* of (pre)sheaves  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  on a topological space  $X$  is the data of group homomorphisms  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  for each  $U \subseteq X$  open such that for all  $V \subseteq U \subseteq X$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

**Example 3.9.1.** Let  $X$  be a variety.

1. If  $a : \mathbb{E} \rightarrow \mathbb{F}$  is a morphism of vector bundles on  $X$ , then we get a morphism of sheaves  $\mathcal{E} \rightarrow \mathcal{F}$  by  $s \mapsto a \circ s \in \mathcal{F}(U)$  for  $s \in \mathcal{E}(U)$ .
2. A closed subvariety  $Z \subseteq X$  induces a morphism  $\mathcal{I}_Z \rightarrow \mathcal{O}_X$  given by inclusion.

**Remark.** Given a morphism of (pre)sheaves and  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  and  $p \in X$ , we get an induced morphism

$$\begin{aligned} \mathcal{F}_p &\longrightarrow \mathcal{G}_p \\ (s, U) &\longmapsto (\varphi(s), U). \end{aligned}$$

# Lecture 4

## Jan. 22 — Sheaves

### 4.1 Sheafification

**Theorem 4.1** (Sheafification). *For a presheaf  $\mathcal{F}$  on a topological space  $X$ , there exists a morphism to a sheaf  $i : \mathcal{F} \rightarrow \mathcal{F}^+$  such that for any morphism to a sheaf  $g : \mathcal{F} \rightarrow \mathcal{G}$ , there exists a unique morphism  $g^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$  such that  $g = g^+ \circ i$ , i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathcal{G} \\ i \downarrow & \nearrow g^+ & \\ \mathcal{F}^+ & & \end{array}$$

In the above,  $\mathcal{F}^+$  is called the sheafification of  $\mathcal{F}$ , and the pair  $(i, \mathcal{F}^+)$  is unique up to isomorphism (as a consequence of the universal property).

*Proof.* We first define  $\mathcal{F}^+(U) = \{t : U \rightarrow \bigsqcup_{p \in X} \mathcal{F}_p : (1) \text{ and } (2) \text{ hold}\}$ , where

1.  $t(p) \in \mathcal{F}_p$ ;
2. for any  $x \in X$ , there is an open set  $x \in V_x \subseteq U$  with  $s \in \mathcal{F}(V_x)$  such that  $t(p) = s_p$  for all  $p \in V_x$ .

It is straightforward to see that  $\mathcal{F}^+$  is a sheaf and that

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto (X \ni p \mapsto s_p \in \mathcal{F}_p). \end{aligned}$$

gives a morphism  $i : \mathcal{F} \rightarrow \mathcal{F}^+$ . Now we check the universal property. Given a morphism  $g : \mathcal{F} \rightarrow \mathcal{G}$  with  $\mathcal{G}$  a sheaf, we need to define  $g^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ . Fix  $t \in \mathcal{F}^+(U)$ . By definition, there exists an open cover  $\{U_i\}$  of  $U$  and  $s_i \in \mathcal{F}(U_i)$  such that  $t(p) = (s_i)_p \in \mathcal{F}_p$  for all  $p \in U_i$ . Set  $t'_i := g(t_i)$ . Note that

$$(t'_i)_p = g_p(t_p) = (t'_j)_p \in \mathcal{G}_p$$

for every  $p \in U_i \cap U_j$ . Since  $\mathcal{G}$  is a sheaf, we get  $t'_i|_{U_i \cap U_j} = t'_j|_{U_i \cap U_j}$ . Thus there exists a unique  $t' \in \mathcal{G}(U)$  such that  $t'|_{U_i} = t'_i$  for every  $i \in I$ . Then we can set  $g^+(t) = t'$ . One can check as an exercise that this gives a morphism  $\mathcal{F}^+ \rightarrow \mathcal{G}$  satisfying the universal property.  $\square$

**Example 4.0.1.** We have the following:

1. If  $\mathcal{F}$  is a sheaf, then  $\mathcal{F} \rightarrow \mathcal{F}^+$  is an isomorphism.
2. For an abelian group  $A$  and topological space  $X$ , we have  $(\underline{A}^{\text{pre}})^+ \cong \underline{A}$ .

**Remark.** We have the following:

1. If  $p \in X$ , the induced morphism on stalks  $i_p : \mathcal{F}_p \rightarrow \mathcal{F}_p^+$  is an isomorphism for all  $p \in X$ .

To construct the inverse map, consider  $\mathcal{F}_p^+ \rightarrow \mathcal{F}_p$  defined by  $(t, U) \mapsto t_p$  for  $t \in \mathcal{F}^+(U)$ . Check as an exercise that this is well-defined and is inverse to  $i_p$ .

2. If  $\mathcal{F} \subseteq \mathcal{G}$  is a subpresheaf (i.e.  $\mathcal{F}(U) \subseteq \mathcal{G}(U)$  and  $\rho_{V,U}^{\mathcal{F}} = \rho_{V,U}^{\mathcal{G}}|_{\mathcal{F}(U)}$  for all  $V \subseteq U \subseteq X$ ) and  $\mathcal{G}$  is a sheaf, then we could alternatively define  $\mathcal{F}^+$  as

$$\mathcal{F}^+(U) = \{s \in \mathcal{G}(U) : \text{for all } x \in X, \text{ there exists } U_x \subseteq U \text{ open such that } s|_{U_x} \in \mathcal{F}(U_x)\}.$$

## 4.2 Kernel, Image, Cokernel for Sheaves

**Remark.** We want the following notions for sheaves:

- kernel, image, cokernel;
- short exact sequences;
- injectivity and surjectivity.

**Example 4.0.2.** We want the following to be short exact sequences of sheaves:

- for  $X$  a variety and  $Z \hookrightarrow X$  a closed subvariety,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

- for  $M$  a complex manifold (e.g.  $\mathbb{C}^n$ ) with  $\mathcal{O}_M$  the sheaf of  $\mathbb{C}$ -valued holomorphic functions,

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \times} \mathcal{O}_M \xrightarrow{\varphi \mapsto e^\varphi} \mathcal{O}_M^\times \longrightarrow 0$$

**Remark.** Let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves on a topological space  $X$ .

**Definition 4.1.** The *kernel* of  $\varphi$  is  $(\ker \varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$ .

**Remark** (Properties of the kernel). It is straightforward to check that  $\ker \varphi$  is a sheaf. Moreover:

1.  $\ker \varphi$  satisfies the following universal property: For any morphism to a sheaf  $\alpha$  such that  $\varphi \circ \alpha = 0$ , there exists a unique morphism  $\alpha'$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \text{---} & \searrow & \\ \mathcal{F}' & \xrightarrow{\alpha} & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ & \searrow \alpha' & \uparrow & & \\ & & \ker \varphi & & \end{array}$$

To see this, use the universal property of the kernel in the category of abelian groups.

2. Since filtered limits are exact, we have  $(\ker \varphi)_p = (\ker \varphi_p)_p$  for all  $p \in X$ .

**Lemma 4.1** (Injectivity for sheaves). *The following are equivalent:*

1.  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective for all  $U \subseteq X$  open;
2.  $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$  is injective for all  $x \in X$ .

We say that  $\varphi$  is injective if either of these equivalent conditions hold.

*Proof.*  $(1 \Rightarrow 2)$  This is clear.

$(2 \Rightarrow 1)$  Fix  $s \in \mathcal{F}(U)$  with  $\varphi(U)(s) = 0$ . Then

$$\varphi_p(s_p) = (\varphi(U)(s))_p = 0$$

for all  $p \in X$ , so  $s_p = 0$  for all  $p \in U$ , so  $s = 0$  by homework from Algebraic Geometry I.  $\square$

**Example 4.1.1** (Subtleties for the image). Consider the following:

1. Let  $\varphi : \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}^n}^\times$ . Then  $U \mapsto \text{im}(\mathcal{O}_{\mathbb{C}^n}(U) \rightarrow \mathcal{O}_{\mathbb{C}^n}^\times(U))$  is a presheaf but not a sheaf. This is because logarithms only exist locally.
2. Define  $\varphi : \mathcal{O}_{\mathbb{P}^n} \rightarrow i_{p_1}\underline{k} \oplus i_{p_2}\underline{k}$  by  $f \mapsto (f(p_1), f(p_2))$ . Again  $U \mapsto \text{im}(\varphi(U))$  is not a sheaf.

**Definition 4.2.** Let  $\widetilde{\text{im}} \varphi = (U \mapsto \text{im}(\varphi(U)))$ . This is a presheaf and  $\widetilde{\text{im}} \varphi \subseteq \mathcal{G}$ . Then the *image* of  $\varphi$  is  $\text{im} \varphi = (\widetilde{\text{im}} \varphi)^+$ . By Remark 4.1, we can equivalently define

$$(\text{im} \varphi)(U) = \{s \in \mathcal{G}(U) : \text{there exists cover } \{U_i\} \text{ of } U \text{ such that } s|_{U_i} \in \text{im}(\mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i))\}.$$

**Remark.** We have  $\text{im}(\varphi_x) \cong (\widetilde{\text{im}} \varphi)_x \cong (\text{im} \varphi)_x$ , where the first isomorphism is because filtered direct limits are exact and the second isomorphism is because sheafification preserves stalks.

**Definition 4.3.** Let  $\widetilde{\text{coker}}(\varphi) = (U \mapsto \text{coker}(\varphi(U)))$ . Then the *cokernel* of  $\varphi$  is  $\text{coker}(\varphi) = (\widetilde{\text{coker}}(\varphi))^+$ .

**Remark.** We have the following:

1.  $\text{coker}(\varphi)_x \cong \text{coker}(\varphi_x)$  (similar to above).
2.  $\text{coker}(\varphi)$  satisfies the universal property of the cokernel:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & & \searrow & \\
 \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{p_0} & \mathcal{G}' \\
 & & \downarrow & \nearrow & \uparrow \\
 & & \widetilde{\text{coker}}(\varphi) & \longrightarrow & \text{coker}(\varphi)
 \end{array}$$

3. For a subsheaf  $\mathcal{F}' \subseteq \mathcal{F}$ , we can define the *quotient sheaf*  $\mathcal{F}/\mathcal{F}' = \text{coker}(\mathcal{F}' \hookrightarrow \mathcal{F})$ .
4. By the universal property of the cokernel, we get natural maps

$$\begin{array}{ccccc}
 \ker \varphi & \longrightarrow & \mathcal{F} & \longrightarrow & \text{im } \mathcal{F} \\
 & & \downarrow & \nearrow & \\
 & & \mathcal{F}/\ker \varphi & & 
 \end{array}$$

As the following diagram commutes,

$$\begin{array}{ccc} (\mathcal{F}/\ker \varphi)_p & \xrightarrow{\alpha_p} & (\operatorname{im} \varphi)_p \\ \downarrow \cong & & \cong \downarrow \\ \mathcal{F}_p/(\ker \varphi)_p & \xrightarrow{\cong} & \operatorname{im}(\varphi_p) \end{array}$$

$\alpha_p$  is an isomorphism for all  $p \in X$ . So by HW,  $\alpha$  is an isomorphism. So  $\mathcal{F}/\ker \varphi \cong \operatorname{im} \varphi$ .

**Lemma 4.2** (Surjectivity for sheaves). *The following are equivalent:*

1.  $\operatorname{coker} \varphi = 0$ ;
2.  $\operatorname{im} \varphi = \mathcal{G}$ ;
3.  $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$  is surjective for all  $x \in X$ .

We say that  $\varphi$  is surjective if any of these equivalent conditions hold.

*Proof.* (3  $\Leftrightarrow$  1) We have (3) if and only if  $\operatorname{coker}(\varphi_x) = 0$  for all  $x \in X$ , if and only if  $(\operatorname{coker} \varphi)_x = 0$  for all  $x \in X$ , if and only if (1).

(3  $\Leftrightarrow$  2) We have (3) if and only if  $\operatorname{coker}(\varphi_x) = 0$  for all  $x \in X$ , if and only if  $\operatorname{im} \varphi_x = \mathcal{G}_x$  for all  $x \in X$ , if and only if  $(\operatorname{im} \varphi)_x \rightarrow \mathcal{G}_x$  is an isomorphism for all  $x \in X$ , if and only if (2) by HW.  $\square$

**Remark.** Note that if  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is surjective for all  $U \subseteq X$ , then  $\varphi$  is surjective. However, the converse is false in general.

**Definition 4.4.** A sequence of morphisms of sheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is *exact* at  $\mathcal{G}$  if  $\ker g = \operatorname{im} f$ .

**Lemma 4.3.** *The following are equivalent:*

1.  $\ker g = \operatorname{im} f$ ;
2.  $\ker g_x = \operatorname{im} f_x$  for all  $x \in X$ .

*Proof.* Similar to above.  $\square$

## 4.3 Constructions with Sheaves

**Definition 4.5.** Let  $\mathcal{F}_1, \mathcal{F}_2$  be sheaves on  $X$ . Then  $\mathcal{F}_1 \oplus \mathcal{F}_2$  is a sheaf defined by

$$U \mapsto \mathcal{F}_1(U) \oplus \mathcal{F}_2(U).$$

This is a *biproduct* in the category of sheaves.

**Example 4.5.1.** Let  $X$  be a variety with connected components  $X_1, \dots, X_n$ . Then

$$\mathcal{O}_X \cong \mathcal{O}_{X_1} \oplus \dots \oplus \mathcal{O}_{X_n}$$

as sheaves of abelian groups.

**Definition 4.6.** Let  $U \subseteq X$  be open. Then  $\mathcal{F}_i|_U$  is a sheaf on  $U$  given by  $V \mapsto \mathcal{F}_i(V)$

**Definition 4.7.**  $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$  is the sheaf  $U \mapsto \text{Hom}^{\text{sheaves}}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$ .

**Definition 4.8** (Gluing). Let  $\{U_i\}$  be an open cover of  $X$  with a sheaf  $\mathcal{F}_i$  on each  $U_i$  and isomorphisms  $\alpha_{i,j} : \mathcal{F}_j|_{U_{i,j}} \rightarrow \mathcal{F}_i|_{U_{i,j}}$  such that  $\alpha_{i,j} = \alpha_{j,i}^{-1}$ ,  $\alpha_{i,j} \circ \alpha_{j,k} = \alpha_{i,k}$ , and  $\alpha_{i,i} = \text{id}$ . Then there exists a sheaf  $\mathcal{F}$  with isomorphisms  $\beta_i : \mathcal{F}|_{U_i} \rightarrow \mathcal{F}_i$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}|_{U_{i,j}} & \xrightarrow{\text{id}} & \mathcal{F}|_{U_{i,j}} \\ \beta_j \downarrow & & \downarrow \beta_i \\ \mathcal{F}_i|_{U_{i,j}} & \xrightarrow{\alpha_{i,j}} & \mathcal{F}_j|_{U_{i,j}} \end{array}$$

One can define  $\mathcal{F}$  as follows and check that it satisfies the above properties:

$$\mathcal{F}(U) = \{(s_i)_{i \in I} : s_i \in \mathcal{F}(U_i) \text{ and } \alpha_{i,j}(s_j|_{U_{i,j}}) = s_i\}.$$

# Lecture 5

## Jan. 27 — $\mathcal{O}_X$ -Modules

### 5.1 Sheaves and Continuous Maps

**Remark.** The category  $\text{Sh}_X$  of sheaves of abelian groups on a topological space  $X$  is an abelian category, i.e. it has or satisfies the following:

- zero object (the  $\underline{0}$  sheaf);
- $\text{Hom}(\mathcal{F}, \mathcal{G})$  is an abelian group and composition is bilinear;
- finite biproducts exist;
- kernels and cokernels exist;
- the image coincides with the coimage.

**Remark.** For the rest of this section, let  $f : X \rightarrow Y$  be a continuous map of topological spaces,  $\mathcal{F}$  a sheaf on  $X$ , and  $\mathcal{G}$  a sheaf on  $Y$ .

**Definition 5.1.** The *pushforward*  $f_*\mathcal{F}$  is the sheaf on  $Y$  defined by

$$V \longmapsto \mathcal{F}(f^{-1}(V)).$$

**Example 5.1.1.** We have the following:

- If  $i : \{p\} \hookrightarrow X$  and  $A$  is an abelian group, then  $i_*\underline{A}$  is the skyscraper sheaf on  $X$  at  $p$ .
- If  $X$  is a variety and  $i : Z \hookrightarrow X$  with  $Z$  a closed subvariety, then

$$i_*\mathcal{O}_Z(U) = \{\varphi : U \cap Z \rightarrow k : \varphi \text{ is regular}\}.$$

**Definition 5.2.** Let  $\widetilde{f^{-1}\mathcal{G}}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ , which is a presheaf. The *pullback* is  $f^{-1}\mathcal{G} := (\widetilde{f^{-1}\mathcal{G}})^+$ .

**Example 5.2.1.** Let  $f : \{p\} \hookrightarrow X$ . Then  $\widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}}_p$ , which is a sheaf, so  $f^{-1}\mathcal{G} \cong \widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}}_p$ .

**Example 5.2.2.** If  $i : U \hookrightarrow X$  is the inclusion of an open set, then  $i^{-1}\mathcal{F} \cong \mathcal{F}|_U$ .

**Remark.** The pushforward  $f_*$  and pullback  $f^{-1}$  are functors:

$$\begin{array}{ccc} & \xrightarrow{f_*} & \\ \text{Sh}_X & & \text{Sh}_Y \\ & \xleftarrow{f^{-1}} & \end{array}$$

How are  $f_*$  and  $f^{-1}$  related? There are natural maps  $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$  and  $\mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$  induced by:

1. For  $U \subseteq X$ , define

$$\widetilde{f^{-1}f_*}\mathcal{F}(U) = \varinjlim_{V \supseteq f(U) \text{ open}} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$$

$$s \longmapsto s|_U.$$

2. For  $V \subseteq Y$ , define

$$\mathcal{G}(V) \longrightarrow \varinjlim_{V \supseteq V' \supseteq f(f^{-1}(V)) \text{ open}} \mathcal{G}(V') = (f_*\widetilde{f^{-1}\mathcal{G}})(V).$$

Note that  $V \supseteq f(f^{-1}(V))$ , so we can add the  $V \supseteq V' \supseteq f(f^{-1}(V))$  condition.

Another way to think about this is via adjoints.

**Proposition 5.1.** *For  $\mathcal{F} \in \text{Sh}_X$  and  $\mathcal{G} \in \text{Sh}_Y$ , there exist functorial bijections*

$$\text{Hom}_{\text{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{\text{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

i.e.  $(f^{-1}, f_*)$  is an adjoint pair.

*Proof.* Given  $\phi : f^{-1}\mathcal{G} \rightarrow \mathcal{F}$ , using map (2) from above we get

$$\mathcal{G} \xrightarrow{(2)} f_*f^{-1}\mathcal{G} \xrightarrow{f_*\phi} f_*\mathcal{F}.$$

Similarly, given  $\psi : \mathcal{G} \rightarrow f_*\mathcal{F}$ , using map (1) from above we get

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\psi} f^{-1}f_*\mathcal{F} \xrightarrow{(1)} \mathcal{F}.$$

One can (tediously) check that this gives a bijection. □

**Remark.** The following are consequences of adjointness:

- $f_*$  is left exact, i.e. given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

in  $\text{Sh}_X$ , we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

- $f^{-1}$  is right exact (defined similarly with the left 0 missing).

One can also directly check these properties from the definitions.

## 5.2 Sheaves of $\mathcal{O}_X$ -Modules

**Remark.** To use tools from commutative algebra, we want to consider modules. For the rest of this section, let  $X$  be a topological space with a sheaf of rings  $\mathcal{O}_X$  (e.g.  $X$  a variety with  $\mathcal{O}_X$  the sheaf of regular functions, or  $M$  a complex manifold with  $\mathcal{O}_M$  the sheaf of holomorphic functions).

**Definition 5.3.** A *(pre)sheaf of  $\mathcal{O}_X$ -modules* is a (pre)sheaf  $\mathcal{F}$  on  $X$  such that for each  $U \subseteq X$ ,  $\mathcal{F}(U)$  has the structure of an  $\mathcal{O}_X(U)$ -module compatible with restriction, i.e. such that

$$(a \cdot s)|_V = a|_V \cdot s|_V$$

for  $V \subseteq U \subseteq X$  open,  $a \in \mathcal{O}_X(U)$ , and  $s \in \mathcal{F}(U)$ .

**Remark.** We often say just “ $\mathcal{O}_X$ -module” to mean a “sheaf of  $\mathcal{O}_X$ -modules.”

**Definition 5.4.** A *morphism of (pre)sheaves of  $\mathcal{O}_X$ -modules*  $\phi : \mathcal{M} \rightarrow \mathcal{N}$  is a morphism of presheaves such that  $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$  is a morphism of  $\mathcal{O}_X(U)$ -modules for all  $U \subseteq X$ .

**Example 5.4.1.** We have the following:

1.  $\mathcal{O}_X$  has the structure of an  $\mathcal{O}_X$ -module (similar to how a ring  $A$  has the structure of an  $A$ -module).
2. Let  $X$  be a variety and  $p : \mathbb{E} \rightarrow X$  a vector bundle. Let  $\mathcal{E}$  be the sheaf of sections of  $p$ , with

$$[f \cdot s : U \rightarrow p^{-1}(U)] \in \mathcal{E}(U)$$

as the product of  $f \in \mathcal{O}_X(U)$  and  $[s : U \rightarrow p^{-1}(U)] \in \mathcal{E}(U)$ . Then  $\mathcal{E}$  is an  $\mathcal{O}_X$ -module.

- 2'. Let  $\mathbb{E} = \mathbb{1}_X$ , then  $\mathcal{E}(U) \cong \mathcal{O}_X(U)$  as  $\mathcal{O}_X(U)$ -modules. As the isomorphism is compatible with restriction, we have  $\mathcal{E} \cong \mathcal{O}_X$ .

3. Let  $\mathcal{F}_1, \mathcal{F}_2$ , be (pre)sheaves of  $\mathcal{O}_X$ -modules. Then so is  $\mathcal{F}_1 \oplus \mathcal{F}_2$ .

- 2''. If  $\mathbb{E}$  is a trivial vector bundle of rank  $e$ , then  $\mathcal{E} \cong \mathcal{O}_X^{\oplus e}$ .

4. If  $\mathcal{F}$  is a presheaf of  $\mathcal{O}_X$ -modules, then  $\mathcal{F}^+$  is naturally a sheaf of  $\mathcal{O}_X$ -modules (use the definition of  $\mathcal{F}^+$  in the proof).

5. If  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of sheaves of  $\mathcal{O}_X$ -modules, then  $\ker \varphi, \operatorname{im} \varphi, \operatorname{coker} \varphi$  are sheaves of  $\mathcal{O}_X$ -modules (use that  $\ker \varphi, \widetilde{\operatorname{im} \varphi}, \widetilde{\operatorname{coker} \varphi}$  are presheaves of  $\mathcal{O}_X$ -modules and then use (4)).

Furthermore, the category  $\operatorname{Mod}_{\mathcal{O}_X}$  of sheaves of  $\mathcal{O}_X$ -modules is an abelian.

6. We can define the usual constructions on  $\mathcal{O}_X$ -modules:  $\otimes, \operatorname{Sym}^d, \wedge^d$ , etc.

**Example 5.4.2.** Let  $\mathcal{F}, \mathcal{G}$  be  $\mathcal{O}_X$ -modules. Then their tensor product  $\mathcal{F} \otimes \mathcal{G}$  is the sheafification of

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

**Definition 5.5.** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is *locally free of rank  $e$*  if for any  $p \in X$ , there exists  $p \in U \subseteq X$  open such that  $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus e}$ . If  $e = 1$ , then we say that  $\mathcal{F}$

**Example 5.5.1.** Let  $X$  be a variety and  $p : \mathbb{E} \rightarrow X$  a vector bundle of rank  $e$ . For  $p \in X$ , there exists  $p \in U \subseteq X$  open such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \mathbb{A}^e \\ p \downarrow & \swarrow \operatorname{pr}_1 & \\ U & & \end{array}$$

Then  $\mathcal{E}|_U \cong$  sheaf of sections of  $U \times \mathbb{A}^e \cong \mathcal{O}_U^{\oplus e}$ .

**Remark** (Transition functions). Let  $e = 1$  for simplicity, and  $\mathcal{E}$  a locally free  $\mathcal{O}_X$ -module of rank  $e$ . Then there exists an open cover  $\{U_i\}$  of  $X$  with isomorphisms  $\alpha_i : \mathcal{E}|_{U_i} \rightarrow \mathcal{O}_{U_i}^{\oplus e}$ . So on  $U_{i,j} = U_i \cap U_j$ , we get isomorphisms  $\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}}^{\oplus e} \rightarrow \mathcal{O}_{U_{i,j}}^{\oplus e}$ .

# Lecture 6

## Jan. 29 — $\mathcal{O}_X$ -Modules, Part 2

### 6.1 More on $\mathcal{O}_X$ -Modules

**Remark.** Recall that we have operations  $\otimes$ ,  $\oplus$ ,  $\text{Sym}^d$ ,  $\wedge^d$ ,  $\mathcal{H}om(\cdot, \cdot)$  on  $\mathcal{O}_X$ -modules.

**Example 6.0.1.** The sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$  is the sheafification of

$$U \longmapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This is again an  $\mathcal{O}_X$ -module.

**Example 6.0.2.** We have  $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ .

**Exercise 6.1.** The invertible sheaves on  $X$  up to isomorphism forms a group with multiplication given by  $\otimes$ , identity  $\mathcal{O}_X$ , and inverse  $\mathcal{L}^{-1} = \mathcal{L}^\vee$ .

**Remark** (Transition data). Let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Then there exists an open cover  $\{U_i\}$  of  $X$  and isomorphisms  $\alpha_i : \mathcal{L}|_{U_i} \rightarrow \mathcal{O}_{U_i}$ , so we get isomorphisms

$$\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}} \rightarrow \mathcal{O}_{U_{i,j}}.$$

For this to be an isomorphism, we must have  $\alpha_{i,j}(U_{i,j})(1) = g_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$ .

**Proposition 6.1.** *If  $X$  is a variety, then there is a bijection*

$$\begin{aligned} \{\text{line bundles on } X\} / \cong &\longrightarrow \{\text{invertible sheaves on } X\} / \cong \\ \mathbb{L} &\longmapsto \mathcal{L} \end{aligned}$$

*Proof.* To get the reverse map, fix an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  with trivialization data  $(U_i, g_{i,j})$ . Send it to the line bundle with the same trivialization data. Check that this is well-defined as an exercise.

To show that this gives an inverse, it suffices to show that if  $\mathbb{L}$  is a line bundle with trivialization data  $(U_i, g_{i,j})$ , then the sheaf  $\mathcal{L}$  of sections of  $\mathbb{L}$  has the same trivialization data. The trivializations

$$\mathbb{L}|_{U_i} \xrightarrow[\cong]{\phi_i} U_i \times \mathbb{A}^1$$

give isomorphisms  $U_{i,j} \times \mathbb{A}^1 \rightarrow U_{i,j} \times \mathbb{A}^1$  by  $(x, v) \mapsto (x, g_{i,j}(x)v)$ . We get an isomorphism

$$\alpha_i : \mathcal{L}|_{U_i} \xrightarrow[\cong]{\phi_i} \mathcal{O}_{U_i}$$

where  $e_i := \alpha_i^{-1}(1) = [U_i \xrightarrow{x \mapsto (x,1)} U_i \times \mathbb{A}^1 \xrightarrow{\phi_i^{-1}} \mathbb{L}|_{U_i}]$ . Now we have

$$\begin{aligned} \mathcal{O}_{U_{i,j}} &\xrightarrow{\alpha_j^{-1}} \mathcal{L}|_{U_{i,j}} \xrightarrow{\alpha_i} \mathcal{O}_{U_{i,j}} \\ 1 &\longmapsto e_j = e_i g_{i,j} \longmapsto g_{i,j}. \end{aligned}$$

So we get the same transition functions  $(U_i, g_{i,j})$  for  $\mathcal{L}$ . □

**Remark.** Given a morphism of rings  $\phi : A \rightarrow B$ , we have functors

$$\begin{array}{ccc} & \xrightarrow{\Phi} & \\ \text{Mod}_B & & \text{Mod}_A \\ & \xleftarrow{\Psi} & \end{array}$$

given as follows:

1. *Extension of scalars:*  $\text{Mod}_A \ni M \mapsto M \otimes_A B \in \text{Mod}_B$ , where the multiplication by  $B$  is

$$c(m \otimes b) = m \otimes (cb).$$

For example, if  $M = A^{\oplus I}$ , then  $M \otimes_A B = B^{\oplus I}$ .

2. *Restriction of scalars:*  $\text{Mod}_B \ni N \mapsto N_A \in \text{Mod}_A$ , where  $N_A := N$  as abelian groups with

$$a \cdot n = \phi(a)n$$

as the multiplication by  $A$ .

**Proposition 6.2.** *There is a functorial bijection*

$$\begin{aligned} \text{Hom}_B(M \otimes_A B, N) &\longleftrightarrow \text{Hom}_A(M, N_A) \\ [f : M \otimes_A B \rightarrow N] &\longmapsto [m \mapsto f(m \otimes 1)] \\ [m \otimes b \mapsto b \cdot g(m)] &\longleftarrow [g : M \rightarrow N_A] \end{aligned}$$

for  $M \in \text{Mod}_A$  and  $N \in \text{Mod}_B$ .

**Remark.** Given the result for rings, we want a similar statement for  $\mathcal{O}_X$ -modules.

## 6.2 $\mathcal{O}_X$ -Modules and Continuous Maps

**Definition 6.1.** A morphism of ringed spaces  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is the data of

1. a continuous map  $f : X \rightarrow Y$ ,
2. a morphism of sheaves of rings  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ .

**Example 6.1.1.** If  $X \rightarrow Y$  is a morphism of varieties, then  $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  is given for  $U \subseteq Y$  open by

$$\begin{aligned} \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \varphi &\longmapsto f^* \varphi. \end{aligned}$$

**Remark.** Our goal will be to define functors

$$\begin{array}{ccc} & \xrightarrow{f_*} & \\ \text{Mod}_{\mathcal{O}_X} & & \text{Mod}_{\mathcal{O}_Y} \\ & \xleftarrow{f^*} & \end{array}$$

**Remark** (Pushforward). Given an  $\mathcal{O}_X$ -module  $\mathcal{F}$ , the sheaf pushforward  $f_*\mathcal{F}$  is naturally an  $f_*\mathcal{O}_X$ -module. Via the map  $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ , we get an  $\mathcal{O}_Y$ -module structure on  $f_*\mathcal{F}$ . More concretely, for  $U \subseteq Y$  open,  $s \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ , and  $a \in \mathcal{O}_Y(U)$ , we can define

$$a \cdot s = f^\#(U)(a) \cdot s.$$

**Remark** (Pullback). Given an  $\mathcal{O}_Y$ -module  $\mathcal{G}$ , we get an  $f^{-1}\mathcal{O}_Y$ -module  $f^{-1}\mathcal{G}$ . By the adjoint property for  $(f^{-1}, f_*)$ , the morphism  $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  corresponds to a morphism  $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ . So we get

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is an  $\mathcal{O}_X$ -module. Thus we get a functor  $f^* : \text{Mod}_{\mathcal{O}_Y} \rightarrow \text{Mod}_{\mathcal{O}_X}$ .

**Proposition 6.3.** *The pair  $(f^*, f_*)$  are adjoint functors.*

*Proof.* Similar to before. □

**Example 6.1.2.** Recall that if  $A \rightarrow B$  is a morphism of rings, then  $A \otimes_A B \cong B$ . In our setting, we get

$$f^*\mathcal{O}_Y = (f^{-1}\mathcal{O}_Y \widetilde{\otimes}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)^+ \cong \mathcal{O}_X^+ \cong \mathcal{O}_X.$$

Similarly, we have  $f^*(\mathcal{O}_Y^{\oplus I}) \cong (f^*\mathcal{O}_Y)^{\oplus I} \cong \mathcal{O}_X^{\oplus I}$  (as left-adjoint functors commute with coproducts).

**Remark.** If  $\mathcal{E}$  is a locally free rank  $m$   $\mathcal{O}_X$ -module, then  $f^*\mathcal{E}$  is a locally free rank  $m$   $\mathcal{O}_Y$ -module, (as  $f^*$  can be computed locally on  $Y$  using Example 6.1.2).

# Lecture 7

## Feb. 3 — Coherent Sheaves

### 7.1 Review of Localization

**Remark.** Let  $A$  be a ring and  $S \subseteq A$  a multiplicative system (i.e.  $1 \in S$ , and  $a, b \in S$  implies  $ab \in S$ ). For example, we could take  $S = \langle f \rangle = (1, f, f^2, \dots)$  for  $f \in A$  or  $S = A \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p} \leq A$ .

**Definition 7.1.** The *localization* of  $A$  at  $S$  is

$$S^{-1}A = \{a/s : a \in A, s \in S\},$$

where  $a/s = a'/s'$  if and only if  $t(as' - a's) = 0$  for some  $t \in S$ .

**Remark.** The localization satisfies the following universal property:

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto a/1} & S^{-1}A \\ & \searrow f & \downarrow \exists! \\ & & T \end{array}$$

whenever  $f(S)$  lands in the units of  $T$ .

**Definition 7.2.** For an  $A$ -module  $M$ , the *localization* of  $M$  at  $S$  is

$$S^{-1}M = \{m/s : m \in M, s \in S\},$$

where  $m/s = m'/s'$  if and only if  $t(s'm - sm') = 0$  for some  $t \in S$ .

**Remark.** For  $S = \langle f \rangle$ , we will write  $S^{-1}M = M_f$ . For  $S = A \setminus \mathfrak{p}$ , we will write  $S^{-1}M = M_{\mathfrak{p}}$ .

**Proposition 7.1.** *We have the following properties for localization:*

1. *There is an isomorphism*

$$\begin{aligned} M \otimes_A S^{-1}A &\xrightarrow{\cong} S^{-1}M \\ m \otimes (a/s) &\longmapsto (am)/s. \end{aligned}$$

2. *Localization gives an exact functor*

$$\begin{aligned} \text{Mod}_A &\longrightarrow \text{Mod}_{S^{-1}A} \\ M &\longmapsto S^{-1}M. \end{aligned}$$

3. A sequence in  $\text{Mod}_A$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow M'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow M''_\mathfrak{p} \longrightarrow 0$$

is exact for all maximal (equivalently, prime) ideals  $\mathfrak{p} \leq A$ .

**Example 7.2.1.** Recall that if  $X$  is an affine variety, then

$$\mathcal{O}_X(D(f)) \cong A(X)_f \quad \text{and} \quad \mathcal{O}_{X,x} \cong A(X)_{\mathfrak{m}_x},$$

where  $\mathfrak{m}_x = I(\{x\}) \leq A(X)$ .

## 7.2 Coherent Sheaves on Affine Varieties

**Remark.** For this section, let  $X$  be an affine variety and  $A = \mathcal{O}_X(X) = A(X)$ . We want a functor

$$\text{Mod}_A \longrightarrow \text{Mod}_{\mathcal{O}_X}.$$

**Theorem 7.1.** For  $M \in \text{Mod}_A$ , there exists a unique  $\mathcal{O}_X$ -module  $\widetilde{M}$  such that

1.  $\widetilde{M}(D(f)) \cong M_f$ ;
2. for  $D(g) \subseteq D(f)$ , we have

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{M}(D(g)) \\ \downarrow = & & \downarrow = \\ M_f & \xrightarrow{\text{natural map}} & M_g \end{array}$$

**Remark.** How is the natural map  $M_f \rightarrow M_g$  defined? If  $D(g) \subseteq D(f)$ , then we have  $V(g) \supseteq V(f)$ , so  $\sqrt{(g)} \subseteq \sqrt{(f)}$ . Thus  $g \in \sqrt{(f)}$ , so  $g^d = fh$  for some  $d > 0$  and  $h \in A$ . So we get a map

$$\begin{aligned} M_f &\longrightarrow M_g \\ m/f^i &\longmapsto mh^i/g^{di}. \end{aligned}$$

Alternatively, if  $D(g) \subseteq D(f)$ , then  $D(g) = D(gf)$ , so we could instead consider

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{M}(D(gf)) \\ \downarrow = & & \downarrow \\ M_f & \xrightarrow{m/f^i \mapsto mg^i/(fg)^i} & M_{fg} \end{array}$$

**Remark.** To construct  $\widetilde{M}$ , we need the notion of *sheaves on a basis*. Let  $(X, \mathcal{O}_X)$  be a ringed space and  $\mathcal{P}$  a collection of open sets in  $X$  such that

1.  $\mathcal{P}$  is a basis for  $X$ ;

2. if  $U, V \in \mathcal{P}$ , then  $U \cap V \in \mathcal{P}$ .

**Example 7.2.2.** If  $(X, \mathcal{O}_X)$  is an affine variety, we can take  $\mathcal{P} = \{D(f) : f \in A(X)\}$ . Also, if  $(X, \mathcal{O}_X)$  is a general algebraic variety, then we can take  $\mathcal{P}$  to be the affine open subsets of  $X$  (as  $X$  is separated, the intersection of two affine open subsets is again affine open).

**Definition 7.3.** A  $\mathcal{P}$ -sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  on  $X$  is the data of:

- an  $\mathcal{O}_X(U)$ -module  $\mathcal{F}(U)$  for each  $U \in \mathcal{P}$ ,
- homomorphisms of abelian groups  $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$  for all  $U, V \in \mathcal{P}$  with  $V \subseteq U$

satisfying the following properties:

- the multiplication is compatible with restriction,
- the sheaf axiom with respect to open sets in  $\mathcal{P}$ .

**Example 7.3.1.** If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_X$ -modules, then we get  $\mathcal{F}^\mathcal{P}$ , a  $\mathcal{P}$ -sheaf of  $\mathcal{O}_X$ -modules.

**Theorem 7.2.** *There is an equivalence of categories*

$$\begin{aligned} \text{Mod}_{\mathcal{O}_X} &\longrightarrow \text{Mod}_{\mathcal{O}_X}^\mathcal{P} \\ \mathcal{F} &\longmapsto \mathcal{F}^\mathcal{P}. \end{aligned}$$

*In particular,  $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}(\mathcal{F}^\mathcal{P}, \mathcal{G}^\mathcal{P})$  is a bijection, and for any  $\mathcal{H} \in \text{Mod}_{\mathcal{O}_X}^\mathcal{P}$ , there exists some  $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$  such that  $\mathcal{F}^\mathcal{P} \cong \mathcal{H}$ .*

*Proof.* We construct the inverse functor. Take  $\mathcal{H} \in \text{Mod}_{\mathcal{O}_X}^\mathcal{P}$ , and define  $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$  by setting

$$\mathcal{F}(U) = \{(s_V)_{V \in \mathcal{P}, V \subseteq U} : s_V|_{V \cap V'} = s'_{V'}|_{V \cap V'} \text{ for all } V, V' \in \mathcal{P} \text{ with } V, V' \subseteq U\}.$$

One can check that this defines a functor and an equivalence of categories. □

**Remark.** Returning to algebraic geometry, let  $X$  be an affine variety,  $A = A(X)$ , and  $M$  an  $A$ -module.

**Proposition 7.2.** *For  $\mathcal{P} = \{D(f) : f \in A\}$ , the assignment  $D(f) \mapsto M_f$  with the natural restriction maps defines a  $\mathcal{P}$ -sheaf of  $\mathcal{O}_X$ -modules.*

*Proof.* See Mustata Lemma 8.3.2. The hard part is to check the sheaf axiom for  $\mathcal{P}$ , which is similar to the computation that  $\mathcal{O}_X(D(f)) \cong A_f$  for an affine variety  $X$ . □

*Proof of Theorem 7.1.* Combining Theorem 7.2 and Proposition 7.2, we get an  $\mathcal{O}_X$ -module  $\widetilde{M}$  such that  $\widetilde{M}(D(f)) \cong M_f$ , and it is unique up to isomorphism. □

**Example 7.3.2.** We have  $\widetilde{A} \cong \mathcal{O}_X$ , since

$$\widetilde{A}(D(f)) \cong A_f \cong \mathcal{O}_X(D(f)).$$

Similarly, one can check that  $\widetilde{A^{\oplus I}} \cong \widetilde{\mathcal{O}_X^{\oplus I}}$ .

**Exercise 7.1.** For  $Z \subseteq X$  closed and  $I(Z) \leq A(X)$ , we have  $\widetilde{I(Z)} \cong \mathcal{I}_Z$  (the ideal sheaf of  $Z$ ).

**Remark.** We have the following:

1. For  $x \in X$  with  $\mathfrak{p} := I(\{x\}) \leq A$ , we have  $\widetilde{M}_x \cong \varinjlim_{D(f) \ni x} \widetilde{M}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} M_f \cong M_{\mathfrak{p}}$ .
2. For an  $A$ -module homomorphism  $\varphi : M \rightarrow N$ , we get homomorphisms

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{N}(D(f)) \\ \cong \downarrow & & \downarrow \cong \\ M_f & \xrightarrow{\text{natural map}} & N_f \end{array}$$

which are compatible with restriction. So we get a homomorphism of  $\mathcal{O}_X$ -modules  $\widetilde{M} \rightarrow \widetilde{N}$ . One can check that this gives a functor  $\Phi : \text{Mod}_A \rightarrow \text{Mod}_{\mathcal{O}_X}$ .

**Proposition 7.3.** *We have the following:*

1.  $\Phi$  is exact;
2.  $\Phi$  is fully faithful, i.e. the following map is a bijection:

$$\begin{array}{ccc} \text{Hom}_A(M, N) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \\ \varphi & \longmapsto & \widetilde{\varphi}. \end{array}$$

*Proof.* (1) If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact, then  $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$  is exact for every  $\mathfrak{p} \leq A$  maximal by Proposition 7.1(3). So the sequence

$$0 \longrightarrow \widetilde{M}'_x \longrightarrow \widetilde{M}_x \longrightarrow \widetilde{M}''_x \longrightarrow 0$$

is exact for all  $x \in X$ . So  $0 \rightarrow \widetilde{M}' \rightarrow \widetilde{M} \rightarrow \widetilde{M}'' \rightarrow 0$  is exact.

(2) We want a map  $\text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \rightarrow \text{Hom}_A(M, N)$ . Given an  $\mathcal{O}_X$ -module homomorphism  $\varphi : \widetilde{M} \rightarrow \widetilde{N}$ , we get  $f = \varphi(X) : \widetilde{M}(X) = M \rightarrow \widetilde{N}(X) = N$ . We want to show that  $\widetilde{f} = \varphi$ . We have

$$\begin{array}{ccc} \widetilde{M}(X) & \xrightarrow{\varphi(X)} & \widetilde{N}(X) \\ \downarrow & & \downarrow \\ \widetilde{M}(D(g)) & \xrightarrow{\varphi(D(g))} & \widetilde{N}(D(g)) \end{array}$$

and this diagram commutes. So we get  $\widetilde{f}(D(g)) = \varphi(D(g))$ . □

**Remark.** By Proposition 7.3(1), given an  $A$ -module homomorphism  $f : M \rightarrow N$ , we have

$$\ker(\widetilde{f}) \cong \widetilde{\ker(f)}, \quad \text{coker}(\widetilde{f}) \cong \widetilde{\text{coker}(f)}, \quad \text{im}(\widetilde{f}) \cong \widetilde{\text{im}(f)}.$$

**Proposition 7.4.** *For  $M, N \in \text{Mod}_A$ , we have*

1.  $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_A N}$ ;
2.  $\widetilde{\text{Hom}_A(M, N)} \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ ;
3.  $\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M_i}$ .

*Proof.* (1) We have a homomorphism

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{\cong} & \Gamma(X, \widetilde{M} \widetilde{\otimes}_{\mathcal{O}_X} \widetilde{N}) \\ & & \downarrow \\ & & \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \end{array}$$

This gives a homomorphism of  $\mathcal{O}_X$ -modules  $\widetilde{M \otimes_A N} \rightarrow \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ . Now at the stalks for  $x \in X$  with  $\mathfrak{m} = I(\{x\}) \leq A(X)$ , we can see that

$$(\widetilde{M \otimes_A N})_x \cong (M \otimes_A N) \otimes_A A_{\mathfrak{m}} \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Similarly, we have

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x \cong \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Thus we have isomorphisms at the stalks. □

**Remark.** The functor  $\Phi : \text{Mod}_A \rightarrow \text{Mod}_{\mathcal{O}_X}$  is left adjoint to  $\text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_A$  given by  $\mathcal{F} \mapsto \mathcal{F}(X)$ .

# Lecture 8

## Feb. 5 — Coherent Sheaves, Part 2

### 8.1 Coherent Sheaves on Affine Varieties, Continued

**Proposition 8.1.** *Let  $\varphi : X \rightarrow Y$  be a morphism of affine varieties, and let*

$$A = \mathcal{O}_Y(Y) \xrightarrow{\varphi^\#} \mathcal{O}_X(X) = B$$

*be the corresponding ring homomorphism on the coordinate rings. Then:*

1. *For  $N \in \text{Mod}_B$ , we have  $\varphi_* \widetilde{N} = \widetilde{N}_A$ .*
2. *For  $M \in \text{Mod}_A$ , we have  $\varphi^* \widetilde{M} = \widetilde{M \otimes_A B}$ .*

*Proof.* (1) For  $f \in A = \mathcal{O}_Y(Y)$ , the left-hand side is given on  $D(f)$  by

$$(\varphi_* \widetilde{N})(D(f)) = \widetilde{N}(\varphi^{-1}(D(f))) = \widetilde{N}(D(\varphi^\# f)) = N_{\varphi^\# f} = (N_A)_f.$$

So we get that  $\varphi_* \widetilde{N} = \widetilde{N}_A$ .

(2) (Hartshorne says this holds by definition lol.) First assume  $M = A^{\oplus I}$ . Then

$$\varphi^* \widetilde{M} = \varphi^*(\mathcal{O}_Y^{\oplus I}) = (\varphi^* \mathcal{O}_Y)^{\oplus I} = \mathcal{O}_X^{\oplus I} = \widetilde{B^{\oplus I}} = \widetilde{M \otimes_A B}.$$

For an arbitrary  $A$ -module  $M$ , we use the following:

**Claim:** There exists an exact sequence

$$A^{\oplus J} \xrightarrow{\alpha} A^{\oplus I} \xrightarrow{\beta} M \longrightarrow 0$$

*Proof of claim.* Choose generators  $(m_i)_{i \in I}$  for  $M$ , and set  $\beta(e_i) = m_i$ . Choose generators  $(n_j)_{j \in J}$  for  $\ker \beta$ . Then we can set  $\alpha(f_j) = n_j$ .  $\square$

Apply  $\cdot \otimes_A B$  (which is right exact) to the exact sequence from the claim to get

$$B^{\oplus J} \longrightarrow B^{\oplus I} \longrightarrow M \otimes_A B \longrightarrow 0.$$

Applying  $\varphi^*(\widetilde{\phantom{x}})$  (note that  $\widetilde{\phantom{x}}$  is exact and  $\varphi^*$  is right exact) to the original sequence to get

$$\begin{array}{ccccc} \varphi^*(\widetilde{A^{\oplus J}}) & \longrightarrow & \varphi^*(\widetilde{A^{\oplus I}}) & \longrightarrow & \varphi^* \widetilde{M} \longrightarrow 0 \\ \parallel & & \parallel & & \\ B^{\oplus J} & & B^{\oplus I} & & \end{array}$$

using the previous case. Thus  $\varphi^* \widetilde{M} = \text{coker}(\widetilde{B^{\oplus J}} \rightarrow \widetilde{B^{\oplus I}}) = (\text{coker}(B^{\oplus J} \rightarrow B^{\oplus I}))^\sim = \widetilde{M \otimes_A B}$ .  $\square$

## 8.2 Quasicoherent and Coherent Sheaves

**Remark.** For the rest of this lecture, assume  $(X, \mathcal{O}_X)$  is a variety (not necessarily affine).

**Definition 8.1.** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is *quasicoherent* if there exists an affine cover  $X = \bigcup_{i \in I} U_i$  such that  $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$  for some  $\mathcal{O}_X(U_i)$ -module  $M_i$ . It is *coherent* if the  $M_i$  are finitely generated  $\mathcal{O}_X(U_i)$ -modules.

**Remark.** We have  $M_i \cong \widetilde{M_i}(U_i) \cong \mathcal{F}(U_i)$ , so we may replace the  $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$  condition with one of the following:  $\mathcal{F}|_{U_i} \cong \widetilde{\mathcal{F}(U_i)}$ , or  $\mathcal{F}(U_i)_f \rightarrow \mathcal{F}(D_{U_i}(f))$  is an isomorphism for all  $f \in \mathcal{O}_X(U_i)$ .

**Example 8.1.1.** We have the following:

1. If  $\mathcal{E}$  is a locally free sheaf of rank  $r$  on  $X$ , then there exists an open cover  $X = \bigcup U_i$  such that  $\mathcal{E}|_{U_i} \cong \mathcal{O}_X^{\oplus r}|_{U_i}$ . Refining the cover, we may assume the  $U_i$  are affine, so

$$\mathcal{E}|_{U_i} = \widetilde{\mathcal{O}_X(U_i)^{\oplus r}}.$$

Thus we see that  $\mathcal{E}$  is coherent.

2. For  $Z \hookrightarrow X$  a closed embedding,  $\mathcal{I}_Z \subseteq \mathcal{O}_X$  is coherent.
3. For  $\mathcal{F}$  a (quasi)coherent  $\mathcal{O}_X$ -module and  $U \subseteq X$  open,  $\mathcal{F}|_U$  is (quasi)coherent.

To see this, use that if  $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$  for  $M_i$  an  $\mathcal{O}_X(U_i)$ -module and  $f \in \mathcal{O}_X(U_i)$ , then

$$\mathcal{F}|_{D_{U_i}(f)} \cong \widetilde{(M_i)_f}.$$

Furthermore, if  $M_i$  is finitely generated, then so is  $(M_i)_f$ . So by refining our open affine cover in with principal opens (which form a basis) we may assume  $U = \bigcup_{i, U_i \subseteq U} U_i$ . This gives the result.

**Proposition 8.2** (Key proposition). *Let  $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$ . The following are equivalent:*

1.  $\mathcal{F}$  is quasicoherent (resp. coherent).
2. For any affine open set  $U \subseteq X$ , we have  $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$  (with  $\mathcal{F}(U)$  finitely generated in the coherent case).

**Lemma 8.1** (Affine communication). *If  $X$  is a variety,  $U, V \subseteq X$  affine open subsets, and  $p \in U \cap V$ , then there exists an open set  $W \in U \cap V$  that is a principal open of both  $U$  and  $V$ .*

*Proof.* Choose a principal open of  $U$  with  $p \in W_1 = D_U(h) \subseteq U \cap V$  for some  $h \in \mathcal{O}_X(U)$ , and choose a principal open of  $V$  with  $p \in W = D_V(g) \subseteq W_1$  for some  $g \in \mathcal{O}_X(V)$ . Now

$$g|_{W_1} = \mathcal{O}_X(W_1) = \mathcal{O}_X(U)_h,$$

so  $g|_{W_1} = f/h^i$  for some  $f \in \mathcal{O}_X(U)$  and  $i \geq 0$ . Now  $D_V(g) = W = D_{W_1}(g|_{W_1}) = D_U(fh)$ .  $\square$

*Proof of Proposition 8.2.* (2  $\Rightarrow$  1) This is clear.

(1  $\Rightarrow$  2) Assume  $\mathcal{F} \in \text{QCoh}_X$ . So there exists an open affine cover  $\{U_i\}$  such that  $\mathcal{F}|_{U_i} = \widetilde{\mathcal{F}(U_i)}$ . Fix  $U \subseteq X$  affine open. By refining the cover, we may assume that  $U = \bigcup_{U_i \subseteq U} U_i$ . Now replacing  $X$  with  $U$ , we may assume that  $X = U$ . Using Lemma 8.1, we may assume  $U_i = D(f_i)$  for some  $f_i \in \mathcal{O}_X(X)$ .

So now  $X$  is affine,  $X = \bigcup_{i=1}^r D(f_i)$ , and  $\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))}$ . We want to show that for any  $f \in A$ , the natural map  $\mathcal{F}(X)_f \rightarrow \mathcal{F}(D(f))$  is an isomorphism (this would imply  $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$ ). Now

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_i))_f & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j))_f \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_i f)) & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j f)) \end{array}$$

where the first row is exact by the sheaf property and using that localization is exact, and the second row is exact by the sheaf property. Note that  $\beta$  and  $\gamma$  are isomorphisms as

$$\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))} \quad \text{and} \quad \mathcal{F}|_{D(f_i f_j)} \cong \widetilde{\mathcal{F}(D(f_i f_j))}.$$

So by the five lemma,  $\alpha$  is an isomorphism, and thus  $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$ . □

**Remark.** For the coherent case, use the fact that if  $M$  is an  $A$ -module,  $A = (f_1, \dots, f_r)$ , and  $M_{f_i}$  is finitely generated for  $i = 1, \dots, r$ , then  $M$  is finitely generated.

**Proposition 8.3.** *We have the following:*

1. If  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of (quasi)coherent sheaves, then  $\ker \varphi$ ,  $\operatorname{im} \varphi$ ,  $\operatorname{coker} \varphi$  are (quasi)coherent.
2. If  $\mathcal{F}$  and  $\mathcal{G}$  are (quasi)coherent, then so are  $\mathcal{F} \otimes \mathcal{G}$  and  $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ .
3. If  $\mathcal{F}_i$  is quasicoherent for  $i \in I$ , then  $\bigoplus_{i \in I} \mathcal{F}_i$  is quasicoherent. Furthermore, if  $\mathcal{F}_i$  is coherent and  $|I| < \infty$ , then  $\bigoplus_{i \in I} \mathcal{F}_i$  is coherent.

*Proof.* (1) Choose  $U \subseteq X$  affine open. So we can write  $\mathcal{F}|_U = \widetilde{M}$  and  $\mathcal{G}|_U = \widetilde{N}$ . Furthermore,  $\varphi|_U = \widetilde{\alpha}$  for some  $\mathcal{O}_X(U)$ -module homomorphism  $\alpha : M \rightarrow N$ . Now

$$(\ker \varphi)|_U = \ker \varphi|_U = \ker \widetilde{\alpha} = \widetilde{\ker \alpha}.$$

Furthermore, if  $\mathcal{F}$  and  $\mathcal{G}$  are coherent, then  $M$  and  $N$  are finitely generated, so  $\ker \alpha$  is also finitely generated. One can show the same for  $\operatorname{im} \varphi$  and  $\operatorname{coker} \varphi$  similarly. □

**Remark.** We have full subcategories  $\operatorname{Coh}_X \subseteq \operatorname{QCoh}_X \subseteq \operatorname{Mod}_{\mathcal{O}_X}$ , which are abelian.

# Lecture 9

## Feb. 10 — Coherent Sheaves, Part 3

### 9.1 Quasicoherent Sheaves, Continued

**Proposition 9.1.** *Let  $f : X \rightarrow Y$  be a morphism of varieties. Then*

1. *If  $\mathcal{G} \in \mathrm{QCoh}_Y$  (resp.  $\mathrm{Coh}_Y$ ), then  $f^*\mathcal{G} \in \mathrm{QCoh}_X$  (resp.  $\mathrm{Coh}_X$ ).*
2. *If  $\mathcal{F} \in \mathrm{QCoh}_X$ , then  $f_*\mathcal{F} \in \mathrm{QCoh}_Y$ .*

*Proof.* (1) Fix  $\mathcal{G} \in \mathrm{QCoh}_Y$  and  $x \in X$ . There exist affine opens  $x \in U \subseteq X$  and  $f(x) \in V \subseteq Y$  such that  $f(U) \subseteq V$ . Write  $g = f|_U : U \rightarrow V$ , then  $(f^*\mathcal{G})|_U = g^*(\mathcal{G}|_V)$ . Now letting  $A = \mathcal{O}_Y(V)$ ,  $B = \mathcal{O}_X(U)$ ,  $M = \mathcal{G}(V)$ , we see that

$$f^*\mathcal{G}|_U = g^*(\mathcal{G}|_V) = g^*\widetilde{M} = \widetilde{M \otimes_A B}.$$

So  $f^*\mathcal{G}$  is quasicoherent. The coherent version follows from the fact that if  $M$  is a finitely generated  $A$ -module, then  $M \otimes_A B$  is a finitely generated  $B$ -module.

(2) Fix  $\mathcal{F} \in \mathrm{QCoh}_X$ . We can check quasicoherence locally, so we can reduce to the case where  $Y$  is affine (cover  $Y$  by affine opens  $U_i$  and replace  $f$  with  $f^{-1}(U_i) \rightarrow U_i$ ). Choose an affine cover  $X = U_1 \cup \cdots \cup U_r$ , and note that  $U_i \cap U_j$  is again affine for a variety. Write  $\alpha_i : U_i \hookrightarrow X$  and  $\alpha_{i,j} : U_{i,j} \hookrightarrow X$ . As  $\mathcal{F}$  is a sheaf, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i (\alpha_i)_* \mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (\alpha_{i,j})_* \mathcal{F}|_{U_{i,j}}.$$

Applying  $f_*$ , which is left exact, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i (f \circ \alpha_i)_* \mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (f \circ \alpha_{i,j})_* \mathcal{F}|_{U_{i,j}}.$$

Note that the last two terms are both quasicoherent (e.g.  $\mathcal{F}|_{U_i}$  is quasicoherent and  $f \circ \alpha_i$  is a morphism of affine varieties, so  $(f \circ \alpha_i)_* \mathcal{F}|_{U_i}$  is quasicoherent).<sup>1</sup> □

**Remark.** The coherent version of Proposition 9.1(2) fails: For

$$i : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1,$$

the pushforward  $i_*\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$  is not coherent (but  $\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$  is coherent).

---

<sup>1</sup>If  $f : X \rightarrow Y$  is a morphism of affine varieties, and  $A := \mathcal{O}_Y(Y)$ ,  $B := \mathcal{O}_X(X)$ , then  $f_*\widetilde{N} = \widetilde{N_A}$  for  $N \in \mathrm{Mod}_B$  and  $f^*\widetilde{M}$  for  $M \in \mathrm{Mod}_A$ . This shows that pushforwards and pullbacks of quasicoherent sheaves on *affine* varieties are again quasicoherent.

**Remark.** If  $X \rightarrow Y$  is *projective*, i.e. there exists a factorization

$$\begin{array}{ccc} X & \hookrightarrow & Y \times \mathbb{P}^n \\ & \searrow & \downarrow \text{pr}_1 \\ & & Y \end{array}$$

then we do get a coherent version for the pushforward  $f_*\mathcal{F}$ . One can prove this via sheaf cohomology.

## 9.2 Morphisms to Projective Space

**Remark.** Recall that there is a bijection

$$\{\mathbb{L} \rightarrow X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\} / \cong \longleftrightarrow \{\text{morphisms } X \rightarrow \mathbb{P}^n\}.$$

We want to rephrase this using the bijection

$$\{\mathcal{L} \text{ invertible sheaves of } \mathcal{O}_X\text{-modules}\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \rightarrow X\}$$

and determine when  $X \rightarrow \mathbb{P}^n$  is injective or a closed embedding.

**Remark.** Let  $X$  be a variety and  $\mathcal{L}$  an invertible  $\mathcal{O}_X$ -module. For  $x \in X$ , we have

$$\mathcal{L}(x) := \mathcal{L}_x / \mathfrak{m}_x \mathcal{L}_x \cong \mathcal{O}_{X,x} / \mathfrak{m}_x \cong k,$$

where  $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$  is a maximal ideal.

**Definition 9.1.** We say  $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$  *generate*  $\mathcal{L}$  if

$$s_0(x), \dots, s_n(x) \in \mathcal{L}(x) \cong k$$

generate  $\mathcal{L}(x)$  as a  $k$ -vector space (i.e. at least one of  $s_0(x), \dots, s_n(x)$  is nonzero) for all  $x \in X$ .

**Proposition 9.2.** *The following are equivalent:*

1.  $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$  generate  $\mathcal{L}$ ;
2. the morphism  $\varphi : \mathcal{O}_X^{\oplus(n+1)} \rightarrow \mathcal{L}$  given by  $e_i \mapsto s_i$  is surjective;
3. for every  $U \subseteq X$  open affine,  $s_0|_U, \dots, s_n|_U$  generate  $\mathcal{L}(U)$  as an  $\mathcal{O}_X(U)$ -module.

*Proof.* By commutative algebra (in particular Nakayama's lemma),  $s_0(x), \dots, s_n(x)$  span  $\mathcal{L}(x)$  (1) if and only if  $(s_0)_x, \dots, (s_n)_x$  generate  $\mathcal{L}_x$  as an  $\mathcal{O}_{X,x}$ -module. This happens if and only if  $\varphi_x$  is surjective, which happens if and only if  $\varphi$  is surjective (2). This happens if and only if  $\varphi(U)$  is surjective for all  $U \subseteq X$  open affine (3) (note that  $\text{coker}(\widetilde{M} \rightarrow \widetilde{N}) = (\text{coker}(M \rightarrow N))^\sim$ ).  $\square$