

MATH 6422: Algebraic Geometry II

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Lecture 1

Jan. 13 — Overview and Review

1.1 Course Overview

Remark. This course will cover the following topics:

- (i) vector bundles and line bundles in algebraic geometry;
- (ii) coherent sheaves;
- (iii) differentials;
- (iv) sheaf cohomology: in particular, we will see that $H_{\mathrm{dR}}^k(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T_X^*)$;
- (v) the Riemann-Roch theorem: if $\omega = f dz$ is a rational 1-form on a smooth projective curve C , then

$$(\# \text{ zeroes of } \omega) - (\# \text{ poles of } \omega) = 2 \operatorname{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example, $\operatorname{Spec} \mathbb{Z}$ has points corresponding to the primes p and 0.

1.2 Review of Algebraic Geometry I

Remark. Let $k = \bar{k}$ be an algebraically closed field.

Remark (Hilbert's Nullstellensatz). There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \dots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence, Z being irreducible (resp. a point) corresponds to $I(Z)$ being prime (resp. maximal).

Remark (Zariski topology on \mathbb{A}^n). The closed sets in \mathbb{A}^n are of the form $V(J)$, and this induces a Zariski topology on any subset of \mathbb{A}^n .

Remark (Embedded affine varieties). Let $J \leq k[x_1, \dots, x_n]$. Then we can associate to J a ringed space (X, \mathcal{O}_X) by setting $X := V(J) \subseteq \mathbb{A}^n$ with the Zariski topology, and \mathcal{O}_X to be the sheaf of regular functions on X , i.e. for $U \subseteq X$ open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \rightarrow k \mid \varphi \text{ is regular}\}.$$

Here, $\varphi : U \rightarrow k$ is *regular* if for each $p \in U$, there exists an open set $p \in U_p \subseteq U$ and $f, g \in k[x_1, \dots, x_n]$ such that $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$.

Remark (Coordinate ring). The *coordinate ring* of X is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \dots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for $A(X)$:

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

Remark (Distinguished open sets). The *distinguished open sets* of X are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These form a basis for X as we vary $f \in A(X)$.

Definition 1.1. An *affine variety* is a ringed space (X, \mathcal{O}_X) (here \mathcal{O}_X is a sheaf of k -valued functions) which is isomorphic to an embedded affine variety.

Example 1.1.1. If (X, \mathcal{O}_X) is an affine variety and $f \in \mathcal{O}_X(X)$, then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that $X = V(J) \subseteq \mathbb{A}^n$ with $J \subseteq k[x_1, \dots, x_n]$ a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_y^1 \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \dots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \dots, x_n]/J)[y]}{(fy - 1)} \cong A(X)_f.$$

Theorem 1.1. *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.-}k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

This implies the following:

1. *There is a bijection*

$$\begin{aligned} \text{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated k -algebra A , there exists an affine variety with $A \cong A(X)$.*

Remark. How can we explicitly define the inverse functor $\text{Red-f.g.-}k\text{-alg}^{\text{op}} \rightarrow \text{Aff-var}$? We can define this as $A \mapsto (X, \mathcal{O}_X)$, where X is the set of maximal ideals of A . Think about what \mathcal{O}_X should be.

Remark (Varieties). A *variety* (X, \mathcal{O}_X) is a ringed space such that

- there exists a finite open cover of X by affine varieties,
- the diagonal Δ_X is closed in $X \times X$.

Example 1.1.2. The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$.

Remark (Projective spaces). Recall that \mathbb{P}^n has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for \mathbb{P}^n by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with $f \in k[x_0, \dots, x_n]$ homogeneous.

1.3 Vector and Line Bundles

Definition 1.2. Let X be a variety. A *vector bundle (of rank m)* on X is a variety \mathbb{E} with a morphism $p : \mathbb{E} \rightarrow X$ such that

1. $\mathbb{E}_x = p^{-1}(x)$ has the structure of a rank m vector space for every $x \in X$ (i.e. k^m),
2. for every $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ and an isomorphism $p^{-1}(U) \rightarrow U \times \mathbb{A}^m$ such that for any $y \in U$, the map $\mathbb{E}_y \rightarrow \{y\} \times \mathbb{A}^m$ is an isomorphism of vector spaces, i.e.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{A}^m \\ & \searrow p \quad \swarrow \text{proj} & \\ & U & \end{array}$$

commutes. We will call the map ϕ_U a *trivialization*.

Definition 1.3. A *line bundle* on X is a rank 1 vector bundle.

Remark. A different way to think about this is the following:

1. Given two trivializations $\phi_U : p^{-1}(U) \rightarrow U \times \mathbb{A}^m$ and $\phi_V : p^{-1}(V) \rightarrow V \times \mathbb{A}^m$, we get a morphism

$$\begin{array}{ccc} (U \cap V) \times \mathbb{A}^m & \xrightarrow{\phi_{U,V}} & (U \cap V) \times \mathbb{A}^m \\ & \searrow \phi_V^{-1} \quad \swarrow \phi_U & \\ & p^{-1}(U \cap V) & \end{array}$$

with $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$. Observe $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$ for some $g_{U,V}(x) \in \text{GL}(m, k)$. Furthermore, $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$ is a morphism. We will call the $g_{U,V}$ *transition functions*.

In the special case where $m = 1$ (so \mathbb{E} is a line bundle and $\text{GL}(1, k) = k^\times$), the map $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$ is equivalent to the data of a non-vanishing regular function $g_{U,V} : U \cap V \rightarrow k$.

2. The data of a vector bundle of rank m is equivalent to the data of

- an open cover $X = \bigcup_{i \in I} U_i$,
- and morphisms $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$

such that $g_{i,k} = g_{i,j}g_{j,k}$, $g_{i,j} = g_{j,i}^{-1}$, and $g_{i,i} = \mathrm{id}$.

To recover the vector bundle, we can glue $\mathbb{E}_i = U_i \times \mathbb{A}^m$ for $i \in I$ via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x)v). \end{aligned}$$

One can check that this defines a vector bundle \mathbb{E} on X .

Example 1.3.1 (Trivial vector bundle). Define the vector bundle $\mathbb{E} : X \times \mathbb{A}^m \rightarrow X$ by $(x, v) \rightarrow x$. Given a cover $X = \bigcup_{i \in I} U_i$, we get $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$ as $x \mapsto I_m$.

Example 1.3.2 (Trivial line bundle). We will denote the trivial line bundle by $\mathbb{1}_X : X \times \mathbb{A}^1 \rightarrow X$.

Lecture 2

Jan. 15 — Vector and Line Bundles

2.1 Vector and Line Bundles, Continued

Example 2.0.1 (Tautological bundle). Let $X = \mathbb{P}^n$ and $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$. Consider

$$\begin{array}{ccc} & \mathbb{L} & \\ p \swarrow & & \searrow q \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{array}$$

The map $q : \mathbb{L} \rightarrow \mathbb{A}^{n+1}$ is the blowup. We claim that $p : \mathbb{L} \rightarrow \mathbb{P}^n$ is a line bundle. We have:

- $p^{-1}([x]) = \{([x], cx) : c \in k\} \cong kx$, a 1-dimensional vector space;
- let $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$, then we can define

$$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

which we claim is a trivialization. To see this, observe that for fixed $[x] \in \mathbb{P}^n$, we have

$$\begin{aligned} \mathbb{L}_{[x]} &= \{([x], cx) : c \in k\} \longrightarrow \{[x]\} \times \mathbb{A}^1 \\ ([x], cx) &\longmapsto ([x], cx_i), \end{aligned}$$

which is a vector space isomorphism.

We can also compute the transitions functions. Let $U_{i,j} = U_i \cap U_j$. We have

$$\begin{array}{ccccc} & & \phi_{i,j} & & \\ & \searrow & & \swarrow & \\ U_{i,j} \times \mathbb{A}^1 & \xrightarrow{\phi_j^{-1}} & p^{-1}(U_{i,j}) & \xrightarrow{\phi_i} & U_{i,j} \times \mathbb{A}^1 \end{array}$$

$$([x], t) \longmapsto ([x], (tx_0/x_j, \dots, tx_n/x_j)) \longmapsto ([x], tx_i/x_j).$$

Thus we see that $g_{i,j} = x_i/x_j$. This is called the *tautological bundle*, or $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 2.0.2 (Hyperplane bundle, or $\mathcal{O}_{\mathbb{P}^n}(1)$). Consider

$$\begin{aligned} \mathbb{L} &:= \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \longrightarrow \mathbb{P}^n \\ [x_0 : \dots : x_n : x_{n+1}] &\longmapsto [x_0 : \dots : x_n]. \end{aligned}$$

Then \mathbb{L} is a line bundle with transition functions with respect to $\{U_i\}$ given by $g_{i,j} = x_j/x_i$ (HW).

2.2 Operations on Vector Bundles

Remark. The philosophy is: Every natural operation of vector spaces gives one for vector bundles.

Example 2.0.3 (Direct sum). Let $p : \mathbb{E} \rightarrow X$ and $q : \mathbb{F} \rightarrow X$ be vector bundles of rank e and f on X , respectively. There exists trivializations with respect to a common open cover $\{U_i\}$ (just take intersections) with transition functions $g_{i,j}$ and $h_{i,j}$ for \mathbb{E} and \mathbb{F} , respectively.

Then we define the vector bundle $\mathbb{E} \oplus \mathbb{F} \rightarrow X$ as follows:

- As a set, it is $r : \mathbb{E} \oplus \mathbb{F} = \{(x, u, v) : (x, u) \in \mathbb{E}, (x, v) \in \mathbb{F}\} \rightarrow X$.
- We give $\mathbb{E} \oplus \mathbb{F}$ the structure of a variety by requiring that

$$\begin{aligned} r^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^{e+f} \\ (x, u, v) &\longmapsto (x, \text{pr}_2(\phi_i^E(x, u)), \text{pr}_2(\phi_i^F(x, v))) \end{aligned}$$

be an isomorphism, where ϕ_i^E and ϕ_i^F are the trivializations of \mathbb{E} and \mathbb{F} , and pr_2 is the second projection. This gives a variety structure on $r^{-1}(U_i)$, and one can show that these are consistent on $U_{i,j}$, so that this gives a variety structure on all of $\mathbb{E} \oplus \mathbb{F}$.

Note that the transition functions for $\mathbb{E} \oplus \mathbb{F}$ with respect to $\{U_i\}$ are given by the block matrix

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{bmatrix} : U_{i,j} \longrightarrow \text{GL}(e + f, k).$$

Example 2.0.4. Let \mathbb{E} and \mathbb{F} be vector bundles on X of ranks e and f , respectively. Then the following are also vector bundles on X :

1. $\text{Hom}(\mathbb{E}, \mathbb{F})$, of rank ef ;
2. $\mathbb{E}^\vee = \text{Hom}(\mathbb{E}, \mathbb{1}_X)$, of rank e ;
3. $\mathbb{E} \otimes \mathbb{F}$, of rank ef ;
4. $\wedge^k \mathbb{E}$ and $\text{Sym}^d \mathbb{E}$.

Remark. Let \mathbb{L}, \mathbb{M} be line bundles on X with trivializations on $\{U_i\}$ and transition functions $g_{i,j}, h_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$. In this case, we can describe operations on \mathbb{L}, \mathbb{M} more explicitly:

1. $\mathbb{L} \otimes \mathbb{M}$ has transition functions $g_{i,j}h_{i,j}$;
2. $\text{Hom}(\mathbb{L}, \mathbb{M})$ has transition functions $h_{i,j}/g_{i,j}$;
3. $\mathbb{L}^\vee = \text{Hom}(\mathbb{L}, \mathbb{1}_X)$ has transition functions $1/g_{i,j}$;
4. $\mathbb{L}^{\otimes m} = \begin{cases} \mathbb{L}^{\otimes m}, & \text{if } m > 0, \\ \mathbb{1}_X, & \text{if } m = 0, \text{ has transition functions } g_{i,j}^m. \\ (\mathbb{L}^\vee)^{\otimes -m}, & \text{if } m < 0 \end{cases}$

Example 2.0.5. Define $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$ with transition functions $(x_j/x_i)^m$ with respect to the standard open cover for \mathbb{P}^n .

2.3 Morphisms of Vector Bundles

Remark. Let $p : \mathbb{E} \rightarrow X$ and $q : \mathbb{F} \rightarrow X$ be vector bundles on X , as before.

Definition 2.1. A *morphism of vector bundles* $\mathbb{E} \rightarrow \mathbb{F}$ is a morphism of varieties

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{a} & \mathbb{F} \\ & \searrow p & \swarrow q \\ & X & \end{array}$$

such that the diagram commutes and a is linear on each fiber.

Remark. More concretely, given an open cover $\{U_i\}$ which trivializes both vector bundles, we have

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{a} & q^{-1}(U_i) \\ \phi_i \downarrow \cong & & \cong \downarrow \psi_j \\ U_i \times \mathbb{A}^e & \longrightarrow & U_i \times \mathbb{A}^f \\ (x, v) & \longmapsto & (x, a_i(x)v) \end{array}$$

such that $a_i : U_i \rightarrow \text{Hom}(k^e, k^f)$ is regular. On $U_{i,j}$, we have

$$\begin{array}{ccc} U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_j} & U_{i,j} \times \mathbb{A}^f \\ g_{i,j} \downarrow & & \downarrow h_{i,j} \\ U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_i} & U_{i,j} \times \mathbb{A}^f \end{array}$$

So $h_{i,j}a_j = a_i g_{i,j}$, or equivalently, $a_i = h_{i,j}a_j g_{i,j}^{-1}$.

As a special case when $e = f$, $a : \mathbb{E} \rightarrow \mathbb{F}$ is an isomorphism if and only if the a_i are isomorphisms.

Remark. When is a line bundle \mathbb{L} given by the trivialization data $\{U_i, g_{i,j}\}$ isomorphic to $\mathbb{1}_X$? We have

$$\begin{aligned} \mathbb{L} \cong \mathbb{1}_X &\iff \text{if and only if there exists an isomorphism } a : \mathbb{1}_X \rightarrow \mathbb{L} \\ &\iff \text{there exist } a_i \in \mathcal{O}_X(U_i)^\times \text{ such that } (a_j/a_i)|_{U_{i,j}} = g_{i,j}. \end{aligned}$$

Definition 2.2. Define the *Picard group* of X to be

$$\text{Pic } X := \{\text{line bundles on } X\} / \cong.$$

This is a group with respect to \otimes with $\mathbb{1}_X$ as the identity and $\mathbb{L}^\vee \otimes \mathbb{L} \cong \mathbb{1}_X$.

2.4 Global Sections

Definition 2.3. A (*global*) *section* of a vector bundle $p : \mathbb{E} \rightarrow X$ is a morphism $s : X \rightarrow \mathbb{E}$ such that $p \circ s = \text{id}_X$. Note that for $x \in X$, we have $s(x) \in \mathbb{E}_x$.

Example 2.3.1 (Zero section). Let $s : X \rightarrow \mathbb{E}$ where $s(x)$ is the zero element in \mathbb{E}_x .

Example 2.3.2. Let $\mathbb{E} = \mathbb{1}_X$. Then sections $s : X \rightarrow X \times \mathbb{A}^1$ of \mathbb{E} correspond to morphisms $X \rightarrow \mathbb{A}^1$, which correspond to regular functions $X \rightarrow k$.

Remark (Local description of sections). Let $\{U_i, g_{i,j}\}$ be the trivialization data for $\mathbb{E} \rightarrow X$, and let $s : X \rightarrow \mathbb{E}$ be a section. On U_i , we have:

$$\begin{array}{ccc} & & U_i \times \mathbb{A}^e \\ & \nearrow x \mapsto (x, s_i(x)) & \uparrow \cong \phi_i \\ U_i & \xrightarrow{s|_{U_i}} & p^{-1}(U_i) \end{array}$$

Note that $s_i : U_i \rightarrow k^e$ is a regular function (i.e. regular on each coordinate). These maps must satisfy the compatibility condition $s_i = g_{i,j}s_j$, since we have the diagram:

$$\begin{array}{ccc} & & U_{i,j} \times \mathbb{A}^e \\ & \nearrow (x, s_j(x)) & \uparrow \phi_j \\ U_{i,j} & \xrightarrow{s|_{U_{i,j}}} & p^{-1}(U_{i,j}) \\ & \searrow \phi_i & \downarrow (x,v) \mapsto (x, g_{i,j}(v)) \\ & & U_{i,j} \times \mathbb{A}^e \\ & \nwarrow (x, s_i(x)) & \end{array}$$

Example 2.3.3. We can use the above compatibility condition to compute the global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Write $\mathbb{P}_{x_0:x_1}^1 = U_0 \cup U_1$. Given a section $s : \mathbb{P}^1 \rightarrow \mathcal{O}(1)$, we get regular functions

$$\begin{aligned} s_0 : U_0 &\longrightarrow k \\ s_1 : U_1 &\longrightarrow k \end{aligned}$$

satisfying $(x_1/x_0)s_1 = s_0 (*)$. We can write

$$s_0 = \sum_{m \geq 0} a_m (x_1/x_0)^m \quad \text{and} \quad s_1 = \sum_{m \geq 0} b_m (x_0/x_1)^m$$

with $a_m, b_m \in k$ (finitely many nonzero). Then $(*)$ implies that

$$a_0 + a_1(x_1/x_0) + \cdots = (x_1/x_0)(b_0 + b_1(x_0/x_1) + \cdots),$$

so $a_0 = b_1$, $a_1 = b_0$, and all other terms are 0. So we can relate s to a linear form

$$f = a_0 x_0 + a_1 x_1,$$

where $s_0 = (1/x_0)f$ and $s_1 = (1/x_1)f$.

Lecture 3

Jan. 20 — Sections

3.1 Global Sections, Continued

Definition 3.1. Let $\Gamma(X, \mathbb{E}) := \{\text{sections of } \mathbb{E} \rightarrow X\}$, which has the structure of a k -vector space by

$$(s + t)(x) = s(x) + t(x) \quad \text{and} \quad (cs)(x) = cs(x)$$

for $s, t \in \Gamma(X, \mathbb{E})$ and $c \in k$.

Example 3.1.1. One can check that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0, \dots, x_n]_d$ (HW). For example, for $d < 0$, we have $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \{0\}$, and for $d = 0$, we have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(0)) = \Gamma(\mathbb{P}^n, \mathbb{1}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k.$$

For $d = 1$, we can define an isomorphism

$$\begin{aligned} k[x_0, \dots, x_n]_1 &\xrightarrow{\cong} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \\ f &\longmapsto \text{section } s : \mathbb{P}^n \rightarrow \mathcal{O}(1) \text{ given by } s_i = f/x_i. \end{aligned}$$

Note that $(x_j/x_i)s_j = s_i$ holds, so this is a section. An alternative perspective is that s corresponds to

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \\ x &\longmapsto [x_0 : \dots : x_n : f(x)]. \end{aligned}$$

3.2 Morphisms and Sections

Definition 3.2. Given a section $s : X \rightarrow \mathbb{E}$, its *vanishing locus* is

$$V(s) := \{s = 0\} = \{x \in X : s(x) = 0\}.$$

Using a trivializing cover, one can check that $V(s)$ is closed in X .

Example 3.2.1. For a section $s : \mathbb{P}^n \rightarrow \mathcal{O}(1)$ corresponding to $f \in k[x_0, \dots, x_n]_1$, we have $V(s) = V_{\mathbb{P}^n}(f)$.

Remark. Recall that there is a bijection

$$\begin{aligned} \{\text{morphisms } X \rightarrow \mathbb{A}^n\} &\longleftrightarrow \{f_1, \dots, f_n \in \mathcal{O}_X(X)\} \\ [f : X \rightarrow \mathbb{A}^n] &\longmapsto [f_1 = f^*x_1, \dots, f_n = f^*x_n \in \mathcal{O}_X(X)] \\ [x \mapsto (f_1(x), \dots, f_n(x))] &\longleftarrow [f_1, \dots, f_n \in \mathcal{O}_X(X)]. \end{aligned}$$

We want a similar statement for \mathbb{P}^n .

Definition 3.3. Given a line bundle $\mathbb{L} \rightarrow X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$, they are *nowhere vanishing* if

$$V(s_0) \cap \dots \cap V(s_n) = \emptyset.$$

Example 3.3.1. For $\mathcal{O}(1)$, the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ are nowhere vanishing.

Remark. If $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ are nowhere vanishing, then we get a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

Note that $(s_0(x), \dots, s_n(x))$ is a well-defined point in \mathbb{A}^{n+1} up to scaling. One can check that this map is a morphism by working locally.

Example 3.3.2 (Linear maps). Let $X = \mathbb{P}^n$ and $\mathbb{L} = \mathcal{O}(1)$.

(i) $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ gives $\text{id} : \mathbb{P}^n \rightarrow \mathbb{P}^n$.

(ii) For $A \in \text{GL}_{n+1}(k)$, we get a map

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

given by $Ax_0, \dots, Ax_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Remark. Now given a morphism $X \rightarrow \mathbb{P}^n$, we want to get a line bundle with sections.

Definition 3.4 (Pullback). Let $p : \mathbb{E} \rightarrow X$ be a vector bundle and $f : Y \rightarrow X$ a morphism. Define

$$f^*\mathbb{E} = \{(e, y) : e \in \mathbb{E}, y \in Y \text{ with } p(e) = f(y)\} \longrightarrow Y.$$

One can show that this has the structure of a vector bundle in a natural way.

Remark. An alternative way to define the pullback is to choose trivialization data $(U_i, g_{i,j})$ for $\mathbb{E} \rightarrow X$. Then we can define $f^* : \mathbb{E} \rightarrow Y$ to be the vector bundle with trivialization data $(f^{-1}(U_i), f^*g_{i,j})$.

Remark. Now to go in reverse, given a morphism $X \rightarrow \mathbb{P}^n$ and nowhere vanishing sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we get nowhere vanishing sections

$$f^*x_0, \dots, f^*x_n \in \Gamma(X, f^*\mathcal{O}(1)).$$

We can define the pullback of a section in one of two ways: by $f^*(x_i)(a) = (x_i(f(a)), a) \in f^*\mathcal{O}(1)$ for $a \in X$ or by using trivializing covers.

Remark. Using the above, we get a bijection

$$\{\text{morphisms } X \rightarrow \mathbb{P}^n\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \rightarrow X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}.$$

Note that we should consider the right-hand side up to isomorphism of the line bundle. When do $\mathbb{L} \rightarrow X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ give an injective morphism (or an embedding)?

Definition 3.5. Given a vector bundle $\mathbb{E} \rightarrow X$, we get a sheaf of abelian groups \mathcal{E} on X by

$$\mathcal{E}(U) := \{\text{sections of } p^{-1}(U) \rightarrow U\}$$

for $U \subseteq X$ open. For $V \subseteq U \subseteq X$ open, the restriction map is given by

$$\begin{aligned} \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto s|_V. \end{aligned}$$

We call \mathcal{E} the *sheaf of sections* of \mathbb{E} . Also note that $\mathcal{E}(U)$ has the structure of an $\mathcal{O}_X(U)$ -module. We

will see that this gives rise to the structure of an \mathcal{O}_X -module.

3.3 Review of Sheaves

Definition 3.6. A *presheaf* of abelian groups \mathcal{F} on a topological space X is the data of:

- for $U \subseteq X$ open, an abelian group $\mathcal{F}(U)$ (with $\mathcal{F}(\emptyset) = 0$),
- for $V \subseteq U \subseteq X$ open, a group homomorphism $p_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$.

Remark. Note the following:

1. We may replace abelian groups in the above definition by rings, sets, R -modules, etc.
2. We denote $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, whose elements are called *sections*.
3. $s|_V := p_{V,U}(s)$ is called the *restriction* for $s \in \mathcal{F}(U)$ and $V \subseteq U \subseteq X$ open.
4. We may view \mathcal{F} as a functor $\text{Open}_X \rightarrow \text{Ab-gps}$ given by $U \mapsto \mathcal{F}(U)$.

Definition 3.7. For \mathcal{F} a presheaf on X and $x \in X$, the *stalk* of \mathcal{F} at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U) = \{(s, U) : s \in \mathcal{F}(U)\} / \sim.$$

Example 3.7.1. The following are examples of presheaves:

1. Let M be a smooth manifold. Then
 - \mathcal{O}_M = sheaf of smooth \mathbb{R} -valued functions on M ,
 - \mathcal{E} = sheaf of sections of a vector bundle $\mathbb{E} \rightarrow M$.
2. Let X be an algebraic variety, $\mathbb{E} \rightarrow X$ a vector bundle, and $Z \subseteq X$ closed. Then
 - \mathcal{O}_X and \mathcal{E} are sheaves,
 - \mathcal{I}_Z = ideal sheaf of Z , given by $\mathcal{I}_Z(U) = \{\varphi \in \mathcal{O}_X(U) : \varphi|_Z = 0\}$.
3. Let X be a topological space and A an abelian group.
 - $\underline{A}^{\text{pre}}$ given by $U \mapsto \{\text{constant functions } U \rightarrow A\}$, i.e. $\underline{A}^{\text{pre}}(U) \cong A$ for $U \neq \emptyset$,
 - \underline{A} given by $U \mapsto \{\text{locally constant functions } U \rightarrow A\}$,
 - $i_p A$ = skyscraper sheaf, given by $U \mapsto \begin{cases} A & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$

Definition 3.8. A presheaf \mathcal{F} is a *sheaf* if for any

- open set $U \subseteq X$,
- open cover $U = \bigcup_{i \in I} U_i$,
- and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$ for all $i, j \in I$,

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Remark. The presheaf $\underline{A}^{\text{pre}}$ is not a sheaf in general. All other examples above are sheaves.

Definition 3.9. A *morphism* of (pre)sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ on a topological space X is the data of group homomorphisms $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each $U \subseteq X$ open such that for all $V \subseteq U \subseteq X$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Example 3.9.1. Let X be a variety.

1. If $a : \mathbb{E} \rightarrow \mathbb{F}$ is a morphism of vector bundles on X , then we get a morphism of sheaves $\mathcal{E} \rightarrow \mathcal{F}$ by $s \mapsto a \circ s \in \mathcal{F}(U)$ for $s \in \mathcal{E}(U)$.
2. A closed subvariety $Z \subseteq X$ induces a morphism $\mathcal{I}_Z \rightarrow \mathcal{O}_X$ given by inclusion.

Remark. Given a morphism of (pre)sheaves and $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $p \in X$, we get an induced morphism

$$\begin{aligned} \mathcal{F}_p &\longrightarrow \mathcal{G}_p \\ (s, U) &\longmapsto (\varphi(s), U). \end{aligned}$$

Lecture 4

Jan. 22 — Sheaves

4.1 Sheafification

Theorem 4.1 (Sheafification). *For a presheaf \mathcal{F} on a topological space X , there exists a morphism to a sheaf $i : \mathcal{F} \rightarrow \mathcal{F}^+$ such that for any morphism to a sheaf $g : \mathcal{F} \rightarrow \mathcal{G}$, there exists a unique morphism $g^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $g = g^+ \circ i$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathcal{G} \\ i \downarrow & \nearrow g^+ & \\ \mathcal{F}^+ & & \end{array}$$

In the above, \mathcal{F}^+ is called the sheafification of \mathcal{F} , and the pair (i, \mathcal{F}^+) is unique up to isomorphism (as a consequence of the universal property).

Proof. We first define $\mathcal{F}^+(U) = \{t : U \rightarrow \bigsqcup_{p \in X} \mathcal{F}_p : (1) \text{ and } (2) \text{ hold}\}$, where

1. $t(p) \in \mathcal{F}_p$;
2. for any $x \in X$, there is an open set $x \in V_x \subseteq U$ with $s \in \mathcal{F}(V_x)$ such that $t(p) = s_p$ for all $p \in V_x$.

It is straightforward to see that \mathcal{F}^+ is a sheaf and that

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto (X \ni p \mapsto s_p \in \mathcal{F}_p). \end{aligned}$$

gives a morphism $i : \mathcal{F} \rightarrow \mathcal{F}^+$. Now we check the universal property. Given a morphism $g : \mathcal{F} \rightarrow \mathcal{G}$ with \mathcal{G} a sheaf, we need to define $g^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$. Fix $t \in \mathcal{F}^+(U)$. By definition, there exists an open cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $t(p) = (s_i)_p \in \mathcal{F}_p$ for all $p \in U_i$. Set $t'_i := g(t_i)$. Note that

$$(t'_i)_p = g_p(t_p) = (t'_j)_p \in \mathcal{G}_p$$

for every $p \in U_i \cap U_j$. Since \mathcal{G} is a sheaf, we get $t'_i|_{U_i \cap U_j} = t'_j|_{U_i \cap U_j}$. Thus there exists a unique $t' \in \mathcal{G}(U)$ such that $t'|_{U_i} = t'_i$ for every $i \in I$. Then we can set $g^+(t) = t'$. One can check as an exercise that this gives a morphism $\mathcal{F}^+ \rightarrow \mathcal{G}$ satisfying the universal property. \square

Example 4.0.1. We have the following:

1. If \mathcal{F} is a sheaf, then $\mathcal{F} \rightarrow \mathcal{F}^+$ is an isomorphism.
2. For an abelian group A and topological space X , we have $(\underline{A}^{\text{pre}})^+ \cong \underline{A}$.

Remark. We have the following:

1. If $p \in X$, the induced morphism on stalks $i_p : \mathcal{F}_p \rightarrow \mathcal{F}_p^+$ is an isomorphism for all $p \in X$.

To construct the inverse map, consider $\mathcal{F}_p^+ \rightarrow \mathcal{F}_p$ defined by $(t, U) \mapsto t_p$ for $t \in \mathcal{F}^+(U)$. Check as an exercise that this is well-defined and is inverse to i_p .

2. If $\mathcal{F} \subseteq \mathcal{G}$ is a subpresheaf (i.e. $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and $\rho_{V,U}^{\mathcal{F}} = \rho_{V,U}^{\mathcal{G}}|_{\mathcal{F}(U)}$ for all $V \subseteq U \subseteq X$) and \mathcal{G} is a sheaf, then we could alternatively define \mathcal{F}^+ as

$$\mathcal{F}^+(U) = \{s \in \mathcal{G}(U) : \text{for all } x \in X, \text{ there exists } U_x \subseteq U \text{ open such that } s|_{U_x} \in \mathcal{F}(U_x)\}.$$

4.2 Kernel, Image, Cokernel for Sheaves

Remark. We want the following notions for sheaves:

- kernel, image, cokernel;
- short exact sequences;
- injectivity and surjectivity.

Example 4.0.2. We want the following to be short exact sequences of sheaves:

- for X a variety and $Z \hookrightarrow X$ a closed subvariety,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

- for M a complex manifold (e.g. \mathbb{C}^n) with \mathcal{O}_M the sheaf of \mathbb{C} -valued holomorphic functions,

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \times} \mathcal{O}_M \xrightarrow{\varphi \mapsto e^\varphi} \mathcal{O}_M^\times \longrightarrow 0$$

Remark. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on a topological space X .

Definition 4.1. The *kernel* of φ is $(\ker \varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U))$.

Remark (Properties of the kernel). It is straightforward to check that $\ker \varphi$ is a sheaf. Moreover:

1. $\ker \varphi$ satisfies the following universal property: For any morphism to a sheaf α such that $\varphi \circ \alpha = 0$, there exists a unique morphism α' such that the following diagram commutes:

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \text{---} & \nearrow & \\ \mathcal{F}' & \xrightarrow{\alpha} & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\ & \searrow \alpha' & \uparrow & & \\ & & \ker \varphi & & \end{array}$$

To see this, use the universal property of the kernel in the category of abelian groups.

2. Since filtered limits are exact, we have $(\ker \varphi)_p = (\ker \varphi_p)_p$ for all $p \in X$.

Lemma 4.1 (Injectivity for sheaves). *The following are equivalent:*

1. $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for all $U \subseteq X$ open;
2. $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective for all $x \in X$.

We say that φ is injective if either of these equivalent conditions hold.

Proof. $(1 \Rightarrow 2)$ This is clear.

$(2 \Rightarrow 1)$ Fix $s \in \mathcal{F}(U)$ with $\varphi(U)(s) = 0$. Then

$$\varphi_p(s_p) = (\varphi(U)(s))_p = 0$$

for all $p \in X$, so $s_p = 0$ for all $p \in U$, so $s = 0$ by homework from Algebraic Geometry I. \square

Example 4.1.1 (Subtleties for the image). Consider the following:

1. Let $\varphi : \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}^n}^\times$. Then $U \mapsto \text{im}(\mathcal{O}_{\mathbb{C}^n}(U) \rightarrow \mathcal{O}_{\mathbb{C}^n}^\times(U))$ is a presheaf but not a sheaf. This is because logarithms only exist locally.
2. Define $\varphi : \mathcal{O}_{\mathbb{P}^n} \rightarrow i_{p_1}\underline{k} \oplus i_{p_2}\underline{k}$ by $f \mapsto (f(p_1), f(p_2))$. Again $U \mapsto \text{im}(\varphi(U))$ is not a sheaf.

Definition 4.2. Let $\widetilde{\text{im}} \varphi = (U \mapsto \text{im}(\varphi(U)))$. This is a presheaf and $\widetilde{\text{im}} \varphi \subseteq \mathcal{G}$. Then the *image* of φ is $\text{im} \varphi = (\widetilde{\text{im}} \varphi)^+$. By Remark 4.1, we can equivalently define

$$(\text{im} \varphi)(U) = \{s \in \mathcal{G}(U) : \text{there exists cover } \{U_i\} \text{ of } U \text{ such that } s|_{U_i} \in \text{im}(\mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i))\}.$$

Remark. We have $\text{im}(\varphi_x) \cong (\widetilde{\text{im}} \varphi)_x \cong (\text{im} \varphi)_x$, where the first isomorphism is because filtered direct limits are exact and the second isomorphism is because sheafification preserves stalks.

Definition 4.3. Let $\widetilde{\text{coker}}(\varphi) = (U \mapsto \text{coker}(\varphi(U)))$. Then the *cokernel* of φ is $\text{coker}(\varphi) = (\widetilde{\text{coker}}(\varphi))^+$.

Remark. We have the following:

1. $\text{coker}(\varphi)_x \cong \text{coker}(\varphi_x)$ (similar to above).
2. $\text{coker}(\varphi)$ satisfies the universal property of the cokernel:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & & \searrow & \\
 \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \xrightarrow{p_0} & \mathcal{G}' \\
 & & \downarrow & \nearrow & \uparrow \\
 & & \widetilde{\text{coker}}(\varphi) & \longrightarrow & \text{coker}(\varphi)
 \end{array}$$

3. For a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, we can define the *quotient sheaf* $\mathcal{F}/\mathcal{F}' = \text{coker}(\mathcal{F}' \hookrightarrow \mathcal{F})$.
4. By the universal property of the cokernel, we get natural maps

$$\begin{array}{ccccc}
 \ker \varphi & \longrightarrow & \mathcal{F} & \longrightarrow & \text{im } \mathcal{F} \\
 & & \downarrow & \nearrow \alpha & \\
 & & \mathcal{F}/\ker \varphi & &
 \end{array}$$

As the following diagram commutes,

$$\begin{array}{ccc} (\mathcal{F}/\ker \varphi)_p & \xrightarrow{\alpha_p} & (\operatorname{im} \varphi)_p \\ \downarrow \cong & & \cong \downarrow \\ \mathcal{F}_p/(\ker \varphi)_p & \xrightarrow{\cong} & \operatorname{im}(\varphi_p) \end{array}$$

α_p is an isomorphism for all $p \in X$. So by HW, α is an isomorphism. So $\mathcal{F}/\ker \varphi \cong \operatorname{im} \varphi$.

Lemma 4.2 (Surjectivity for sheaves). *The following are equivalent:*

1. $\operatorname{coker} \varphi = 0$;
2. $\operatorname{im} \varphi = \mathcal{G}$;
3. $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is surjective for all $x \in X$.

We say that φ is surjective if any of these equivalent conditions hold.

Proof. $(3 \Leftrightarrow 1)$ We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $(\operatorname{coker} \varphi)_x = 0$ for all $x \in X$, if and only if (1).

$(3 \Leftrightarrow 2)$ We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $\operatorname{im} \varphi_x = \mathcal{G}_x$ for all $x \in X$, if and only if $(\operatorname{im} \varphi)_x \rightarrow \mathcal{G}_x$ is an isomorphism for all $x \in X$, if and only if (2) by HW. \square

Remark. Note that if $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is surjective for all $U \subseteq X$, then φ is surjective. However, the converse is false in general.

Definition 4.4. A sequence of morphisms of sheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is *exact* at \mathcal{G} if $\ker g = \operatorname{im} f$.

Lemma 4.3. *The following are equivalent:*

1. $\ker g = \operatorname{im} f$;
2. $\ker g_x = \operatorname{im} f_x$ for all $x \in X$.

Proof. Similar to above. \square

4.3 Constructions with Sheaves

Definition 4.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves on X . Then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a sheaf defined by

$$U \mapsto \mathcal{F}_1(U) \oplus \mathcal{F}_2(U).$$

This is a *biproduct* in the category of sheaves.

Example 4.5.1. Let X be a variety with connected components X_1, \dots, X_n . Then

$$\mathcal{O}_X \cong \mathcal{O}_{X_1} \oplus \dots \oplus \mathcal{O}_{X_n}$$

as sheaves of abelian groups.

Definition 4.6. Let $U \subseteq X$ be open. Then $\mathcal{F}_i|_U$ is a sheaf on U given by $V \mapsto \mathcal{F}_i(V)$

Definition 4.7. $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ is the sheaf $U \mapsto \text{Hom}^{\text{sheaves}}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$.

Definition 4.8 (Gluing). Let $\{U_i\}$ be an open cover of X with a sheaf \mathcal{F}_i on each U_i and isomorphisms $\alpha_{i,j} : \mathcal{F}_j|_{U_{i,j}} \rightarrow \mathcal{F}_i|_{U_{i,j}}$ such that $\alpha_{i,j} = \alpha_{j,i}^{-1}$, $\alpha_{i,j} \circ \alpha_{j,k} = \alpha_{i,k}$, and $\alpha_{i,i} = \text{id}$. Then there exists a sheaf \mathcal{F} with isomorphisms $\beta_i : \mathcal{F}|_{U_i} \rightarrow \mathcal{F}_i$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}|_{U_{i,j}} & \xrightarrow{\text{id}} & \mathcal{F}|_{U_{i,j}} \\ \beta_j \downarrow & & \downarrow \beta_i \\ \mathcal{F}_i|_{U_{i,j}} & \xrightarrow{\alpha_{i,j}} & \mathcal{F}_j|_{U_{i,j}} \end{array}$$

One can define \mathcal{F} as follows and check that it satisfies the above properties:

$$\mathcal{F}(U) = \{(s_i)_{i \in I} : s_i \in \mathcal{F}(U_i) \text{ and } \alpha_{i,j}(s_j|_{U_{i,j}}) = s_i\}.$$

Lecture 5

Jan. 27 — \mathcal{O}_X -Modules

5.1 Sheaves and Continuous Maps

Remark. The category Sh_X of sheaves of abelian groups on a topological space X is an abelian category, i.e. it has or satisfies the following:

- zero object (the $\underline{0}$ sheaf);
- $\text{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group and composition is bilinear;
- finite biproducts exist;
- kernels and cokernels exist;
- the image coincides with the coimage.

Remark. For the rest of this section, let $f : X \rightarrow Y$ be a continuous map of topological spaces, \mathcal{F} a sheaf on X , and \mathcal{G} a sheaf on Y .

Definition 5.1. The *pushforward* $f_*\mathcal{F}$ is the sheaf on Y defined by

$$V \longmapsto \mathcal{F}(f^{-1}(V)).$$

Example 5.1.1. We have the following:

- If $i : \{p\} \hookrightarrow X$ and A is an abelian group, then $i_*\underline{A}$ is the skyscraper sheaf on X at p .
- If X is a variety and $i : Z \hookrightarrow X$ with Z a closed subvariety, then

$$i_*\mathcal{O}_Z(U) = \{\varphi : U \cap Z \rightarrow k : \varphi \text{ is regular}\}.$$

Definition 5.2. Let $\widetilde{f^{-1}\mathcal{G}}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, which is a presheaf. The *pullback* is $f^{-1}\mathcal{G} := (\widetilde{f^{-1}\mathcal{G}})^+$.

Example 5.2.1. Let $f : \{p\} \hookrightarrow X$. Then $\widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$, which is a sheaf, so $f^{-1}\mathcal{G} \cong \widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$.

Example 5.2.2. If $i : U \hookrightarrow X$ is the inclusion of an open set, then $i^{-1}\mathcal{F} \cong \mathcal{F}|_U$.

Remark. The pushforward f_* and pullback f^{-1} are functors:

$$\begin{array}{ccc} \text{Sh}_X & \xrightarrow{f_*} & \text{Sh}_Y \\ & \xleftarrow{f^{-1}} & \end{array}$$

How are f_* and f^{-1} related? There are natural maps $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$ and $\mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$ induced by:

1. For $U \subseteq X$, define

$$\widetilde{f^{-1}f_*}\mathcal{F}(U) = \varinjlim_{V \supseteq f(U) \text{ open}} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$$

$$s \longmapsto s|_U.$$

2. For $V \subseteq Y$, define

$$\mathcal{G}(V) \longrightarrow \varinjlim_{V \supseteq V' \supseteq f(f^{-1}(V)) \text{ open}} \mathcal{G}(V') = (f_*\widetilde{f^{-1}}\mathcal{G})(V).$$

Note that $V \supseteq f(f^{-1}(V))$, so we can add the $V \supseteq V' \supseteq f(f^{-1}(V))$ condition.

Another way to think about this is via adjoints.

Proposition 5.1. *For $\mathcal{F} \in \text{Sh}_X$ and $\mathcal{G} \in \text{Sh}_Y$, there exist functorial bijections*

$$\text{Hom}_{\text{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \text{Hom}_{\text{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

i.e. (f^{-1}, f_*) is an adjoint pair.

Proof. Given $\phi : f^{-1}\mathcal{G} \rightarrow \mathcal{F}$, using map (2) from above we get

$$\mathcal{G} \xrightarrow{(2)} f_*f^{-1}\mathcal{G} \xrightarrow{f_*\phi} f_*\mathcal{F}.$$

Similarly, given $\psi : \mathcal{G} \rightarrow f_*\mathcal{F}$, using map (1) from above we get

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\psi} f^{-1}f_*\mathcal{F} \xrightarrow{(1)} \mathcal{F}.$$

One can (tediously) check that this gives a bijection. □

Remark. The following are consequences of adjointness:

- f_* is left exact, i.e. given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

in Sh_X , we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

- f^{-1} is right exact (defined similarly with the left 0 missing).

One can also directly check these properties from the definitions.

5.2 Sheaves of \mathcal{O}_X -Modules

Remark. To use tools from commutative algebra, we want to consider modules. For the rest of this section, let X be a topological space with a sheaf of rings \mathcal{O}_X (e.g. X a variety with \mathcal{O}_X the sheaf of regular functions, or M a complex manifold with \mathcal{O}_M the sheaf of holomorphic functions).

Definition 5.3. A (pre)sheaf of \mathcal{O}_X -modules is a (pre)sheaf \mathcal{F} on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$ -module compatible with restriction, i.e. such that

$$(a \cdot s)|_V = a|_V \cdot s|_V$$

for $V \subseteq U \subseteq X$ open, $a \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}(U)$.

Remark. We often say just “ \mathcal{O}_X -module” to mean a “sheaf of \mathcal{O}_X -modules.”

Definition 5.4. A morphism of (pre)sheaves of \mathcal{O}_X -modules $\phi : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of presheaves such that $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$ is a morphism of $\mathcal{O}_X(U)$ -modules for all $U \subseteq X$.

Example 5.4.1. We have the following:

1. \mathcal{O}_X has the structure of an \mathcal{O}_X -module (similar to how a ring A has the structure of an A -module).
2. Let X be a variety and $p : \mathbb{E} \rightarrow X$ a vector bundle. Let \mathcal{E} be the sheaf of sections of p , with

$$[f \cdot s : U \rightarrow p^{-1}(U)] \in \mathcal{E}(U)$$

as the product of $f \in \mathcal{O}_X(U)$ and $[s : U \rightarrow p^{-1}(U)] \in \mathcal{E}(U)$. Then \mathcal{E} is an \mathcal{O}_X -module.

- 2'. Let $\mathbb{E} = \mathbb{1}_X$, then $\mathcal{E}(U) \cong \mathcal{O}_X(U)$ as $\mathcal{O}_X(U)$ -modules. As the isomorphism is compatible with restriction, we have $\mathcal{E} \cong \mathcal{O}_X$.
3. Let $\mathcal{F}_1, \mathcal{F}_2$, be (pre)sheaves of \mathcal{O}_X -modules. Then so is $\mathcal{F}_1 \oplus \mathcal{F}_2$.
- 2''. If \mathbb{E} is a trivial vector bundle of rank e , then $\mathcal{E} \cong \mathcal{O}_X^{\oplus e}$.
4. If \mathcal{F} is a presheaf of \mathcal{O}_X -modules, then \mathcal{F}^+ is naturally a sheaf of \mathcal{O}_X -modules (use the definition of \mathcal{F}^+ in the proof).
5. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of \mathcal{O}_X -modules, then $\ker \varphi, \operatorname{im} \varphi, \operatorname{coker} \varphi$ are sheaves of \mathcal{O}_X -modules (use that $\ker \varphi, \widetilde{\operatorname{im} \varphi}, \widetilde{\operatorname{coker} \varphi}$ are presheaves of \mathcal{O}_X -modules and then use (4)).

Furthermore, the category $\operatorname{Mod}_{\mathcal{O}_X}$ of sheaves of \mathcal{O}_X -modules is an abelian.

6. We can define the usual constructions on \mathcal{O}_X -modules: $\otimes, \operatorname{Sym}^d, \wedge^d$, etc.

Example 5.4.2. Let \mathcal{F}, \mathcal{G} be \mathcal{O}_X -modules. Then their tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Definition 5.5. An \mathcal{O}_X -module \mathcal{F} is *locally free* of rank e if for any $p \in X$, there exists $p \in U \subseteq X$ open such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus e}$. If $e = 1$, then we say that \mathcal{F} is *invertible*.

Example 5.5.1. Let X be a variety and $p : \mathbb{E} \rightarrow X$ a vector bundle of rank e . For $p \in X$, there exists $p \in U \subseteq X$ open such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times \mathbb{A}^e \\ p \downarrow & \swarrow \operatorname{pr}_1 & \\ U & & \end{array}$$

Then $\mathcal{E}|_U \cong$ sheaf of sections of $U \times \mathbb{A}^e \cong \mathcal{O}_U^{\oplus e}$.

Remark (Transition functions). Let $e = 1$ for simplicity, and \mathcal{E} a locally free \mathcal{O}_X -module of rank e . Then there exists an open cover $\{U_i\}$ of X with isomorphisms $\alpha_i : \mathcal{E}|_{U_i} \rightarrow \mathcal{O}_{U_i}^{\oplus e}$. So on $U_{i,j} = U_i \cap U_j$, we get isomorphisms $\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}}^{\oplus e} \rightarrow \mathcal{O}_{U_{i,j}}^{\oplus e}$.

Lecture 6

Jan. 29 — \mathcal{O}_X -Modules, Part 2

6.1 More on \mathcal{O}_X -Modules

Remark. Recall that we have operations \otimes , \oplus , Sym^d , \wedge^d , $\mathcal{H}om(\cdot, \cdot)$ on \mathcal{O}_X -modules.

Example 6.0.1. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheafification of

$$U \longmapsto \text{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This is again an \mathcal{O}_X -module.

Example 6.0.2. We have $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Exercise 6.1. The invertible sheaves on X up to isomorphism forms a group with multiplication given by \otimes , identity \mathcal{O}_X , and inverse $\mathcal{L}^{-1} = \mathcal{L}^\vee$.

Remark (Transition data). Let \mathcal{L} be an invertible \mathcal{O}_X -module. Then there exists an open cover $\{U_i\}$ of X and isomorphisms $\alpha_i : \mathcal{L}|_{U_i} \rightarrow \mathcal{O}_{U_i}$, so we get isomorphisms

$$\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}} \rightarrow \mathcal{O}_{U_{i,j}}.$$

For this to be an isomorphism, we must have $\alpha_{i,j}(U_{i,j})(1) = g_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$.

Proposition 6.1. *If X is a variety, then there is a bijection*

$$\begin{aligned} \{\text{line bundles on } X\} / \cong &\longrightarrow \{\text{invertible sheaves on } X\} / \cong \\ \mathbb{L} &\longmapsto \mathcal{L} \end{aligned}$$

Proof. To get the reverse map, fix an invertible \mathcal{O}_X -module \mathcal{L} with trivialization data $(U_i, g_{i,j})$. Send it to the line bundle with the same trivialization data. Check that this is well-defined as an exercise.

To show that this gives an inverse, it suffices to show that if \mathbb{L} is a line bundle with trivialization data $(U_i, g_{i,j})$, then the sheaf \mathcal{L} of sections of \mathbb{L} has the same trivialization data. The trivializations

$$\mathbb{L}|_{U_i} \xrightarrow[\cong]{\phi_i} U_i \times \mathbb{A}^1$$

give isomorphisms $U_{i,j} \times \mathbb{A}^1 \rightarrow U_{i,j} \times \mathbb{A}^1$ by $(x, v) \mapsto (x, g_{i,j}(x)v)$. We get an isomorphism

$$\alpha_i : \mathcal{L}|_{U_i} \xrightarrow[\cong]{\phi_i} \mathcal{O}_{U_i}$$

where $e_i := \alpha_i^{-1}(1) = [U_i \xrightarrow{x \mapsto (x,1)} U_i \times \mathbb{A}^1 \xrightarrow{\phi_i^{-1}} \mathbb{L}|_{U_i}]$. Now we have

$$\begin{aligned} \mathcal{O}_{U_{i,j}} &\xrightarrow{\alpha_j^{-1}} \mathcal{L}|_{U_{i,j}} \xrightarrow{\alpha_i} \mathcal{O}_{U_{i,j}} \\ 1 &\longmapsto e_j = e_i g_{i,j} \longmapsto g_{i,j}. \end{aligned}$$

So we get the same transition functions $(U_i, g_{i,j})$ for \mathcal{L} . □

Remark. Given a morphism of rings $\phi : A \rightarrow B$, we have functors

$$\begin{array}{ccc} & \xrightarrow{\Phi} & \\ \text{Mod}_B & & \text{Mod}_A \\ & \xleftarrow{\Psi} & \end{array}$$

given as follows:

1. *Extension of scalars:* $\text{Mod}_A \ni M \mapsto M \otimes_A B \in \text{Mod}_B$, where the multiplication by B is

$$c(m \otimes b) = m \otimes (cb).$$

For example, if $M = A^{\oplus I}$, then $M \otimes_A B = B^{\oplus I}$.

2. *Restriction of scalars:* $\text{Mod}_B \ni N \mapsto N_A \in \text{Mod}_A$, where $N_A := N$ as abelian groups with

$$a \cdot n = \phi(a)n$$

as the multiplication by A .

Proposition 6.2. *There is a functorial bijection*

$$\begin{aligned} \text{Hom}_B(M \otimes_A B, N) &\longleftrightarrow \text{Hom}_A(M, N_A) \\ [f : M \otimes_A B \rightarrow N] &\longmapsto [m \mapsto f(m \otimes 1)] \\ [m \otimes b \mapsto b \cdot g(m)] &\longleftarrow [g : M \rightarrow N_A] \end{aligned}$$

for $M \in \text{Mod}_A$ and $N \in \text{Mod}_B$.

Remark. Given the result for rings, we want a similar statement for \mathcal{O}_X -modules.

6.2 \mathcal{O}_X -Modules and Continuous Maps

Definition 6.1. A morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is the data of

1. a continuous map $f : X \rightarrow Y$,
2. a morphism of sheaves of rings $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$.

Example 6.1.1. If $X \rightarrow Y$ is a morphism of varieties, then $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is given for $U \subseteq Y$ open by

$$\begin{aligned} \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \varphi &\longmapsto f^* \varphi. \end{aligned}$$

Remark. Our goal will be to define functors

$$\begin{array}{ccc} & \xrightarrow{f_*} & \\ \text{Mod}_{\mathcal{O}_X} & & \text{Mod}_{\mathcal{O}_Y} \\ & \xleftarrow{f^*} & \end{array}$$

Remark (Pushforward). Given an \mathcal{O}_X -module \mathcal{F} , the sheaf pushforward $f_*\mathcal{F}$ is naturally an $f_*\mathcal{O}_X$ -module. Via the map $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$, we get an \mathcal{O}_Y -module structure on $f_*\mathcal{F}$. More concretely, for $U \subseteq Y$ open, $s \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and $a \in \mathcal{O}_Y(U)$, we can define

$$a \cdot s = f^\#(U)(a) \cdot s.$$

Remark (Pullback). Given an \mathcal{O}_Y -module \mathcal{G} , we get an $f^{-1}\mathcal{O}_Y$ -module $f^{-1}\mathcal{G}$. By the adjoint property for (f^{-1}, f_*) , the morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ corresponds to a morphism $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$. So we get

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is an \mathcal{O}_X -module. Thus we get a functor $f^* : \text{Mod}_{\mathcal{O}_Y} \rightarrow \text{Mod}_{\mathcal{O}_X}$.

Proposition 6.3. *The pair (f^*, f_*) are adjoint functors.*

Proof. Similar to before. □

Example 6.1.2. Recall that if $A \rightarrow B$ is a morphism of rings, then $A \otimes_A B \cong B$. In our setting, we get

$$f^*\mathcal{O}_Y = (f^{-1}\mathcal{O}_Y \widetilde{\otimes}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)^+ \cong \mathcal{O}_X^+ \cong \mathcal{O}_X.$$

Similarly, we have $f^*(\mathcal{O}_Y^{\oplus I}) \cong (f^*\mathcal{O}_Y)^{\oplus I} \cong \mathcal{O}_X^{\oplus I}$ (as left-adjoint functors commute with coproducts).

Remark. If \mathcal{E} is a locally free rank m \mathcal{O}_X -module, then $f^*\mathcal{E}$ is a locally free rank m \mathcal{O}_Y -module, (as f^* can be computed locally on Y using Example 6.1.2).

Lecture 7

Feb. 3 — Coherent Sheaves

7.1 Review of Localization

Remark. Let A be a ring and $S \subseteq A$ a multiplicative system (i.e. $1 \in S$, and $a, b \in S$ implies $ab \in S$). For example, we could take $S = \langle f \rangle = (1, f, f^2, \dots)$ for $f \in A$ or $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \leq A$.

Definition 7.1. The *localization* of A at S is

$$S^{-1}A = \{a/s : a \in A, s \in S\},$$

where $a/s = a'/s'$ if and only if $t(as' - a's) = 0$ for some $t \in S$.

Remark. The localization satisfies the following universal property:

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto a/1} & S^{-1}A \\ & \searrow f & \downarrow \exists! \\ & & T \end{array}$$

whenever $f(S)$ lands in the units of T .

Definition 7.2. For an A -module M , the *localization* of M at S is

$$S^{-1}M = \{m/s : m \in M, s \in S\},$$

where $m/s = m'/s'$ if and only if $t(s'm - sm') = 0$ for some $t \in S$.

Remark. For $S = \langle f \rangle$, we will write $S^{-1}M = M_f$. For $S = A \setminus \mathfrak{p}$, we will write $S^{-1}M = M_{\mathfrak{p}}$.

Proposition 7.1. *We have the following properties for localization:*

1. *There is an isomorphism*

$$\begin{aligned} M \otimes_A S^{-1}A &\xrightarrow{\cong} S^{-1}M \\ m \otimes (a/s) &\longmapsto (am)/s. \end{aligned}$$

2. *Localization gives an exact functor*

$$\begin{aligned} \text{Mod}_A &\longrightarrow \text{Mod}_{S^{-1}A} \\ M &\longmapsto S^{-1}M. \end{aligned}$$

3. A sequence in Mod_A

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow M'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow M''_\mathfrak{p} \longrightarrow 0$$

is exact for all maximal (equivalently, prime) ideals $\mathfrak{p} \leq A$.

Example 7.2.1. Recall that if X is an affine variety, then

$$\mathcal{O}_X(D(f)) \cong A(X)_f \quad \text{and} \quad \mathcal{O}_{X,x} \cong A(X)_{\mathfrak{m}_x},$$

where $\mathfrak{m}_x = I(\{x\}) \leq A(X)$.

7.2 Coherent Sheaves on Affine Varieties

Remark. For this section, let X be an affine variety and $A = \mathcal{O}_X(X) = A(X)$. We want a functor

$$\text{Mod}_A \longrightarrow \text{Mod}_{\mathcal{O}_X}.$$

Theorem 7.1. For $M \in \text{Mod}_A$, there exists a unique \mathcal{O}_X -module \widetilde{M} such that

1. $\widetilde{M}(D(f)) \cong M_f$;
2. for $D(g) \subseteq D(f)$, we have

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{M}(D(g)) \\ \downarrow = & & \downarrow = \\ M_f & \xrightarrow{\text{natural map}} & M_g \end{array}$$

Remark. How is the natural map $M_f \rightarrow M_g$ defined? If $D(g) \subseteq D(f)$, then we have $V(g) \supseteq V(f)$, so $\sqrt{(g)} \subseteq \sqrt{(f)}$. Thus $g \in \sqrt{(f)}$, so $g^d = fh$ for some $d > 0$ and $h \in A$. So we get a map

$$\begin{aligned} M_f &\longrightarrow M_g \\ m/f^i &\longmapsto mh^i/g^{di}. \end{aligned}$$

Alternatively, if $D(g) \subseteq D(f)$, then $D(g) = D(gf)$, so we could instead consider

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{M}(D(gf)) \\ \downarrow = & & \downarrow \\ M_f & \xrightarrow{m/f^i \mapsto mg^i/(fg)^i} & M_{fg} \end{array}$$

Remark. To construct \widetilde{M} , we need the notion of *sheaves on a basis*. Let (X, \mathcal{O}_X) be a ringed space and \mathcal{P} a collection of open sets in X such that

1. \mathcal{P} is a basis for X ;

2. if $U, V \in \mathcal{P}$, then $U \cap V \in \mathcal{P}$.

Example 7.2.2. If (X, \mathcal{O}_X) is an affine variety, we can take $\mathcal{P} = \{D(f) : f \in A(X)\}$. Also, if (X, \mathcal{O}_X) is a general algebraic variety, then we can take \mathcal{P} to be the affine open subsets of X (as X is separated, the intersection of two affine open subsets is again affine open).

Definition 7.3. A \mathcal{P} -sheaf of \mathcal{O}_X -modules \mathcal{F} on X is the data of:

- an $\mathcal{O}_X(U)$ -module $\mathcal{F}(U)$ for each $U \in \mathcal{P}$,
- homomorphisms of abelian groups $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for all $U, V \in \mathcal{P}$ with $V \subseteq U$

satisfying the following properties:

- the multiplication is compatible with restriction,
- the sheaf axiom with respect to open sets in \mathcal{P} .

Example 7.3.1. If \mathcal{F} is a sheaf of \mathcal{O}_X -modules, then we get $\mathcal{F}^\mathcal{P}$, a \mathcal{P} -sheaf of \mathcal{O}_X -modules.

Theorem 7.2. *There is an equivalence of categories*

$$\begin{aligned} \text{Mod}_{\mathcal{O}_X} &\longrightarrow \text{Mod}_{\mathcal{O}_X}^\mathcal{P} \\ \mathcal{F} &\longmapsto \mathcal{F}^\mathcal{P}. \end{aligned}$$

In particular, $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}(\mathcal{F}^\mathcal{P}, \mathcal{G}^\mathcal{P})$ is a bijection, and for any $\mathcal{H} \in \text{Mod}_{\mathcal{O}_X}^\mathcal{P}$, there exists some $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$ such that $\mathcal{F}^\mathcal{P} \cong \mathcal{H}$.

Proof. We construct the inverse functor. Take $\mathcal{H} \in \text{Mod}_{\mathcal{O}_X}^\mathcal{P}$, and define $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$ by setting

$$\mathcal{F}(U) = \{(s_V)_{V \in \mathcal{P}, V \subseteq U} : s_V|_{V \cap V'} = s_{V'}|_{V \cap V'} \text{ for all } V, V' \in \mathcal{P} \text{ with } V, V' \subseteq U\}.$$

One can check that this defines a functor and an equivalence of categories. \square

Remark. Returning to algebraic geometry, let X be an affine variety, $A = A(X)$, and M an A -module.

Proposition 7.2. *For $\mathcal{P} = \{D(f) : f \in A\}$, the assignment $D(f) \mapsto M_f$ with the natural restriction maps defines a \mathcal{P} -sheaf of \mathcal{O}_X -modules.*

Proof. See Mustata Lemma 8.3.2. The hard part is to check the sheaf axiom for \mathcal{P} , which is similar to the computation that $\mathcal{O}_X(D(f)) \cong A_f$ for an affine variety X . \square

Proof of Theorem 7.1. Combining Theorem 7.2 and Proposition 7.2, we get an \mathcal{O}_X -module \widetilde{M} such that $\widetilde{M}(D(f)) \cong M_f$, and it is unique up to isomorphism. \square

Example 7.3.2. We have $\widetilde{A} \cong \mathcal{O}_X$, since

$$\widetilde{A}(D(f)) \cong A_f \cong \mathcal{O}_X(D(f)).$$

Similarly, one can check that $\widetilde{A^{\oplus I}} \cong \widetilde{\mathcal{O}_X^{\oplus I}}$.

Exercise 7.1. For $Z \subseteq X$ closed and $I(Z) \leq A(X)$, we have $\widetilde{I(Z)} \cong \mathcal{I}_Z$ (the ideal sheaf of Z).

Remark. We have the following:

1. For $x \in X$ with $\mathfrak{p} := I(\{x\}) \leq A$, we have $\widetilde{M}_x \cong \varinjlim_{D(f) \ni x} \widetilde{M}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} M_f \cong M_{\mathfrak{p}}$.
2. For an A -module homomorphism $\varphi : M \rightarrow N$, we get homomorphisms

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{N}(D(f)) \\ \cong \downarrow & & \downarrow \cong \\ M_f & \xrightarrow{\text{natural map}} & N_f \end{array}$$

which are compatible with restriction. So we get a homomorphism of \mathcal{O}_X -modules $\widetilde{M} \rightarrow \widetilde{N}$. One can check that this gives a functor $\Phi : \text{Mod}_A \rightarrow \text{Mod}_{\mathcal{O}_X}$.

Proposition 7.3. *We have the following:*

1. Φ is exact;
2. Φ is fully faithful, i.e. the following map is a bijection:

$$\begin{array}{ccc} \text{Hom}_A(M, N) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \\ \varphi & \longmapsto & \widetilde{\varphi}. \end{array}$$

Proof. (1) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$ is exact for every $\mathfrak{p} \leq A$ maximal by Proposition 7.1(3). So the sequence

$$0 \longrightarrow \widetilde{M}'_x \longrightarrow \widetilde{M}_x \longrightarrow \widetilde{M}''_x \longrightarrow 0$$

is exact for all $x \in X$. So $0 \rightarrow \widetilde{M}' \rightarrow \widetilde{M} \rightarrow \widetilde{M}'' \rightarrow 0$ is exact.

(2) We want a map $\text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \rightarrow \text{Hom}_A(M, N)$. Given an \mathcal{O}_X -module homomorphism $\varphi : \widetilde{M} \rightarrow \widetilde{N}$, we get $f = \varphi(X) : \widetilde{M}(X) = M \rightarrow \widetilde{N}(X) = N$. We want to show that $\widetilde{f} = \varphi$. We have

$$\begin{array}{ccc} \widetilde{M}(X) & \xrightarrow{\varphi(X)} & \widetilde{N}(X) \\ \downarrow & & \downarrow \\ \widetilde{M}(D(g)) & \xrightarrow{\varphi(D(g))} & \widetilde{N}(D(g)) \end{array}$$

and this diagram commutes. So we get $\widetilde{f}(D(g)) = \varphi(D(g))$. □

Remark. By Proposition 7.3(1), given an A -module homomorphism $f : M \rightarrow N$, we have

$$\ker(\widetilde{f}) \cong \widetilde{\ker(f)}, \quad \text{coker}(\widetilde{f}) \cong \widetilde{\text{coker}(f)}, \quad \text{im}(\widetilde{f}) \cong \widetilde{\text{im}(f)}.$$

Proposition 7.4. *For $M, N \in \text{Mod}_A$, we have*

1. $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_A N}$;
2. $\widetilde{\text{Hom}_A(M, N)} \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$;
3. $\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M_i}$.

Proof. (1) We have a homomorphism

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{\cong} & \Gamma(X, \widetilde{M} \widetilde{\otimes}_{\mathcal{O}_X} \widetilde{N}) \\ & & \downarrow \\ & & \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \end{array}$$

This gives a homomorphism of \mathcal{O}_X -modules $\widetilde{M \otimes_A N} \rightarrow \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Now at the stalks for $x \in X$ with $\mathfrak{m} = I(\{x\}) \leq A(X)$, we can see that

$$(\widetilde{M \otimes_A N})_x \cong (M \otimes_A N) \otimes_A A_{\mathfrak{m}} \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Similarly, we have

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x \cong \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Thus we have isomorphisms at the stalks. □

Remark. The functor $\Phi : \text{Mod}_A \rightarrow \text{Mod}_{\mathcal{O}_X}$ is left adjoint to $\text{Mod}_{\mathcal{O}_X} \rightarrow \text{Mod}_A$ given by $\mathcal{F} \mapsto \mathcal{F}(X)$.

Lecture 8

Feb. 5 — Coherent Sheaves, Part 2

8.1 Coherent Sheaves on Affine Varieties, Continued

Proposition 8.1. *Let $\varphi : X \rightarrow Y$ be a morphism of affine varieties, and let*

$$A = \mathcal{O}_Y(Y) \xrightarrow{\varphi^\#} \mathcal{O}_X(X) = B$$

be the corresponding ring homomorphism on the coordinate rings. Then:

1. *For $N \in \text{Mod}_B$, we have $\varphi_* \widetilde{N} = \widetilde{N}_A$.*
2. *For $M \in \text{Mod}_A$, we have $\varphi^* \widetilde{M} = \widetilde{M \otimes_A B}$.*

Proof. (1) For $f \in A = \mathcal{O}_Y(Y)$, the left-hand side is given on $D(f)$ by

$$(\varphi_* \widetilde{N})(D(f)) = \widetilde{N}(\varphi^{-1}(D(f))) = \widetilde{N}(D(\varphi^\# f)) = N_{\varphi^\# f} = (N_A)_f.$$

So we get that $\varphi_* \widetilde{N} = \widetilde{N}_A$.

(2) (Hartshorne says this holds by definition lol.) First assume $M = A^{\oplus I}$. Then

$$\varphi^* \widetilde{M} = \varphi^*(\mathcal{O}_Y^{\oplus I}) = (\varphi^* \mathcal{O}_Y)^{\oplus I} = \mathcal{O}_X^{\oplus I} = \widetilde{B^{\oplus I}} = \widetilde{M \otimes_A B}.$$

For an arbitrary A -module M , we use the following:

Claim: There exists an exact sequence

$$A^{\oplus J} \xrightarrow{\alpha} A^{\oplus I} \xrightarrow{\beta} M \longrightarrow 0$$

Proof of claim. Choose generators $(m_i)_{i \in I}$ for M , and set $\beta(e_i) = m_i$. Choose generators $(n_j)_{j \in J}$ for $\ker \beta$. Then we can set $\alpha(f_j) = n_j$. \square

Apply $\cdot \otimes_A B$ (which is right exact) to the exact sequence from the claim to get

$$B^{\oplus J} \longrightarrow B^{\oplus I} \longrightarrow M \otimes_A B \longrightarrow 0.$$

Applying $\varphi^*(\widetilde{})$ (note that $\widetilde{}$ is exact and φ^* is right exact) to the original sequence to get

$$\begin{array}{ccccccc} \varphi^*(\widetilde{A^{\oplus J}}) & \longrightarrow & \varphi^*(\widetilde{A^{\oplus I}}) & \longrightarrow & \varphi^* \widetilde{M} & \longrightarrow & 0 \\ \parallel & & \parallel & & & & \\ B^{\oplus J} & & B^{\oplus I} & & & & \end{array}$$

using the previous case. Thus $\varphi^* \widetilde{M} = \text{coker}(\widetilde{B^{\oplus J}} \rightarrow \widetilde{B^{\oplus I}}) = (\text{coker}(B^{\oplus J} \rightarrow B^{\oplus I}))^\sim = \widetilde{M \otimes_A B}$. \square

8.2 Quasicoherent and Coherent Sheaves

Remark. For the rest of this lecture, assume (X, \mathcal{O}_X) is a variety (not necessarily affine).

Definition 8.1. An \mathcal{O}_X -module \mathcal{F} is *quasicoherent* if there exists an affine cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $\mathcal{O}_X(U_i)$ -module M_i . It is *coherent* if the M_i are finitely generated $\mathcal{O}_X(U_i)$ -modules.

Remark. We have $M_i \cong \widetilde{M_i}(U_i) \cong \mathcal{F}(U_i)$, so we may replace the $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ condition with one of the following: $\mathcal{F}|_{U_i} \cong \widetilde{\mathcal{F}(U_i)}$, or $\mathcal{F}(U_i)_f \rightarrow \mathcal{F}(D_{U_i}(f))$ is an isomorphism for all $f \in \mathcal{O}_X(U_i)$.

Example 8.1.1. We have the following:

1. If \mathcal{E} is a locally free sheaf of rank r on X , then there exists an open cover $X = \bigcup U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_X^{\oplus r}|_{U_i}$. Refining the cover, we may assume the U_i are affine, so

$$\mathcal{E}|_{U_i} = \widetilde{\mathcal{O}_X(U_i)^{\oplus r}}.$$

Thus we see that \mathcal{E} is coherent.

2. For $Z \hookrightarrow X$ a closed embedding, $\mathcal{I}_Z \subseteq \mathcal{O}_X$ is coherent.
3. For \mathcal{F} a (quasi)coherent \mathcal{O}_X -module and $U \subseteq X$ open, $\mathcal{F}|_U$ is (quasi)coherent.

To see this, use that if $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for M_i an $\mathcal{O}_X(U_i)$ -module and $f \in \mathcal{O}_X(U_i)$, then

$$\mathcal{F}|_{D_{U_i}(f)} \cong \widetilde{(M_i)_f}.$$

Furthermore, if M_i is finitely generated, then so is $(M_i)_f$. So by refining our open affine cover in with principal opens (which form a basis) we may assume $U = \bigcup_{i, U_i \subseteq U} U_i$. This gives the result.

Proposition 8.2 (Key proposition). *Let $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$. The following are equivalent:*

1. \mathcal{F} is quasicoherent (resp. coherent).
2. For any affine open set $U \subseteq X$, we have $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$ (with $\mathcal{F}(U)$ finitely generated in the coherent case).

Lemma 8.1 (Affine communication). *If X is a variety, $U, V \subseteq X$ affine open subsets, and $p \in U \cap V$, then there exists an open set $W \in U \cap V$ that is a principal open of both U and V .*

Proof. Choose a principal open of U with $p \in W_1 = D_U(h) \subseteq U \cap V$ for some $h \in \mathcal{O}_X(U)$, and choose a principal open of V with $p \in W = D_V(g) \subseteq W_1$ for some $g \in \mathcal{O}_X(V)$. Now

$$g|_{W_1} = \mathcal{O}_X(W_1) = \mathcal{O}_X(U)_h,$$

so $g|_{W_1} = f/h^i$ for some $f \in \mathcal{O}_X(U)$ and $i \geq 0$. Now $D_V(g) = W = D_{W_1}(g|_{W_1}) = D_U(fh)$. \square

Proof of Proposition 8.2. (2 \Rightarrow 1) This is clear.

(1 \Rightarrow 2) Assume $\mathcal{F} \in \text{QCoh}_X$. So there exists an open affine cover $\{U_i\}$ such that $\mathcal{F}|_{U_i} = \widetilde{\mathcal{F}(U_i)}$. Fix $U \subseteq X$ affine open. By refining the cover, we may assume that $U = \bigcup_{U_i \subseteq U} U_i$. Now replacing X with U , we may assume that $X = U$. Using Lemma 8.1, we may assume $U_i = D(f_i)$ for some $f_i \in \mathcal{O}_X(X)$.

So now X is affine, $X = \bigcup_{i=1}^r D(f_i)$, and $\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))}$. We want to show that for any $f \in A$, the natural map $\mathcal{F}(X)_f \rightarrow \mathcal{F}(D(f))$ is an isomorphism (this would imply $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$). Now

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_i))_f & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j))_f \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_i f)) & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_i f_j f)) \end{array}$$

where the first row is exact by the sheaf property and using that localization is exact, and the second row is exact by the sheaf property. Note that β and γ are isomorphisms as

$$\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))} \quad \text{and} \quad \mathcal{F}|_{D(f_i f_j)} \cong \widetilde{\mathcal{F}(D(f_i f_j))}.$$

So by the five lemma, α is an isomorphism, and thus $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$. □

Remark. For the coherent case, use the fact that if M is an A -module, $A = (f_1, \dots, f_r)$, and M_{f_i} is finitely generated for $i = 1, \dots, r$, then M is finitely generated.

Proposition 8.3. *We have the following:*

1. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of (quasi)coherent sheaves, then $\ker \varphi$, $\operatorname{im} \varphi$, $\operatorname{coker} \varphi$ are (quasi)coherent.
2. If \mathcal{F} and \mathcal{G} are (quasi)coherent, then so are $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.
3. If \mathcal{F}_i is quasicoherent for $i \in I$, then $\bigoplus_{i \in I} \mathcal{F}_i$ is quasicoherent. Furthermore, if \mathcal{F}_i is coherent and $|I| < \infty$, then $\bigoplus_{i \in I} \mathcal{F}_i$ is coherent.

Proof. (1) Choose $U \subseteq X$ affine open. So we can write $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Furthermore, $\varphi|_U = \widetilde{\alpha}$ for some $\mathcal{O}_X(U)$ -module homomorphism $\alpha : M \rightarrow N$. Now

$$(\ker \varphi)|_U = \ker \varphi|_U = \ker \widetilde{\alpha} = \widetilde{\ker \alpha}.$$

Furthermore, if \mathcal{F} and \mathcal{G} are coherent, then M and N are finitely generated, so $\ker \alpha$ is also finitely generated. One can show the same for $\operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ similarly. □

Remark. We have full subcategories $\operatorname{Coh}_X \subseteq \operatorname{QCoh}_X \subseteq \operatorname{Mod}_{\mathcal{O}_X}$, which are abelian.

Lecture 9

Feb. 10 — Coherent Sheaves, Part 3

9.1 Quasicoherent Sheaves, Continued

Proposition 9.1. *Let $f : X \rightarrow Y$ be a morphism of varieties. Then*

1. *If $\mathcal{G} \in \mathrm{QCoh}_Y$ (resp. Coh_Y), then $f^*\mathcal{G} \in \mathrm{QCoh}_X$ (resp. Coh_X).*
2. *If $\mathcal{F} \in \mathrm{QCoh}_X$, then $f_*\mathcal{F} \in \mathrm{QCoh}_Y$.*

Proof. (1) Fix $\mathcal{G} \in \mathrm{QCoh}_Y$ and $x \in X$. There exist affine opens $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that $f(U) \subseteq V$. Write $g = f|_U : U \rightarrow V$, then $(f^*\mathcal{G})|_U = g^*(\mathcal{G}|_V)$. Now letting $A = \mathcal{O}_Y(V)$, $B = \mathcal{O}_X(U)$, $M = \mathcal{G}(V)$, we see that

$$f^*\mathcal{G}|_U = g^*(\mathcal{G}|_V) = g^*\widetilde{M} = \widetilde{M \otimes_A B}.$$

So $f^*\mathcal{G}$ is quasicoherent. The coherent version follows from the fact that if M is a finitely generated A -module, then $M \otimes_A B$ is a finitely generated B -module.

(2) Fix $\mathcal{F} \in \mathrm{QCoh}_X$. We can check quasicoherence locally, so we can reduce to the case where Y is affine (cover Y by affine opens U_i and replace f with $f^{-1}(U_i) \rightarrow U_i$). Choose an affine cover $X = U_1 \cup \cdots \cup U_r$, and note that $U_i \cap U_j$ is again affine for a variety. Write $\alpha_i : U_i \hookrightarrow X$ and $\alpha_{i,j} : U_{i,j} \hookrightarrow X$. As \mathcal{F} is a sheaf, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i (\alpha_i)_* \mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (\alpha_{i,j})_* \mathcal{F}|_{U_{i,j}}.$$

Applying f_* , which is left exact, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i (f \circ \alpha_i)_* \mathcal{F}|_{U_i} \longrightarrow \bigoplus_{i,j} (f \circ \alpha_{i,j})_* \mathcal{F}|_{U_{i,j}}.$$

Note that the last two terms are both quasicoherent (e.g. $\mathcal{F}|_{U_i}$ is quasicoherent and $f \circ \alpha_i$ is a morphism of affine varieties, so $(f \circ \alpha_i)_* \mathcal{F}|_{U_i}$ is quasicoherent).¹ □

Remark. The coherent version of Proposition 9.1(2) fails: For

$$i : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1,$$

the pushforward $i_*\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$ is not coherent (but $\mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$ is coherent).

¹If $f : X \rightarrow Y$ is a morphism of affine varieties, and $A := \mathcal{O}_Y(Y)$, $B := \mathcal{O}_X(X)$, then $f_*\widetilde{N} = \widetilde{N_A}$ for $N \in \mathrm{Mod}_B$ and $f^*\widetilde{M}$ for $M \in \mathrm{Mod}_A$. This shows that pushforwards and pullbacks of quasicoherent sheaves on *affine* varieties are again quasicoherent.

Remark. If $X \rightarrow Y$ is *projective*, i.e. there exists a factorization

$$\begin{array}{ccc} X & \hookrightarrow & Y \times \mathbb{P}^n \\ & \searrow & \downarrow \text{pr}_1 \\ & & Y \end{array}$$

then we do get a coherent version for the pushforward $f_*\mathcal{F}$. One can prove this via sheaf cohomology.

9.2 Morphisms to Projective Space

Remark. Recall that there is a bijection

$$\{\mathbb{L} \rightarrow X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\} / \cong \longleftrightarrow \{\text{morphisms } X \rightarrow \mathbb{P}^n\}.$$

We want to rephrase this using the bijection

$$\{\mathcal{L} \text{ invertible sheaves of } \mathcal{O}_X\text{-modules}\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \rightarrow X\}$$

and determine when $X \rightarrow \mathbb{P}^n$ is injective or a closed embedding.

Remark. Let X be a variety and \mathcal{L} an invertible \mathcal{O}_X -module. For $x \in X$, we have

$$\mathcal{L}(x) := \mathcal{L}_x / \mathfrak{m}_x \mathcal{L}_x \cong \mathcal{O}_{X,x} / \mathfrak{m}_x \cong k,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is a maximal ideal.

Definition 9.1. We say $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ *generate* \mathcal{L} if

$$s_0(x), \dots, s_n(x) \in \mathcal{L}(x) \cong k$$

generate $\mathcal{L}(x)$ as a k -vector space (i.e. at least one of $s_0(x), \dots, s_n(x)$ is nonzero) for all $x \in X$.

Proposition 9.2. *The following are equivalent:*

1. $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generate \mathcal{L} ;
2. $(s_0)_x, \dots, (s_n)_x$ generate \mathcal{L}_x as an $\mathcal{O}_{X,x}$ -module for all $x \in X$;
3. the morphism $\varphi : \mathcal{O}_X^{\oplus(n+1)} \rightarrow \mathcal{L}$ given by $e_i \mapsto s_i$ is surjective;
4. for every $U \subseteq X$ open affine, $s_0|_U, \dots, s_n|_U$ generate $\mathcal{L}(U)$ as an $\mathcal{O}_X(U)$ -module.

Proof. By commutative algebra (in particular Nakayama's lemma), $s_0(x), \dots, s_n(x)$ span $\mathcal{L}(x)$ (1) if and only if $(s_0)_x, \dots, (s_n)_x$ generate \mathcal{L}_x as an $\mathcal{O}_{X,x}$ -module (2). This happens if and only if φ_x is surjective, which happens if and only if φ is surjective (3). This happens if and only if $\varphi(U)$ is surjective for all $U \subseteq X$ open affine (4) (note that $\text{coker}(\widetilde{M} \rightarrow \widetilde{N} = (\text{coker}(M \rightarrow N))^\sim)$). \square

Lecture 10

Feb. 12 — Morphisms to Projective Space

10.1 Morphisms to Projective Space, Continued

Definition 10.1. When any of the conditions in Proposition 9.2 hold, we say that \mathcal{L} is *generated* by s_0, \dots, s_n . Furthermore, we say \mathcal{L} is *globally generated* if there exist s_0, \dots, s_n generating \mathcal{L} .

Example 10.1.1. \mathcal{O}_X is globally generated, e.g. by $1 \in \Gamma(X, \mathcal{O}_X)$.

Example 10.1.2. For $X = \mathbb{P}^n$, the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ on \mathbb{P}^n defined by transition data $(U_i, x_j/x_i)$ is globally generated. There are isomorphisms $\alpha_i : \mathcal{O}_{\mathbb{P}^n}(1)|_{U_i} \rightarrow \mathcal{O}_{U_i}$ such that $\alpha_i \circ \alpha_j^{-1} =$ multiplication by x_j/x_i . Note that we have an isomorphism

$$\begin{aligned} k[x_0, \dots, x_n]_1 &\longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \\ f &\longmapsto s_f, \end{aligned}$$

where $\alpha_i(s_f|_{U_i}) = f/x_i$ (check that this is well-defined).

Then $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$. To see this, note that on U_i , we have

$$\alpha_i(x_i) = 1 \in \mathcal{O}_{U_i},$$

so $x_i|_{U_i}$ generates $\mathcal{O}_{\mathbb{P}^n}(1)(U_i)$ as an $\mathcal{O}_{\mathbb{P}^n}(U_i)$ -module by Proposition 9.2(4). So the x_0, \dots, x_n generate $\mathcal{O}_{\mathbb{P}^n}(1)$. Alternatively, one can note that for $a = [a_0 : \dots : a_n] \in \mathbb{P}^n$, we have $[x_0(a) : \dots : x_n(a)] = [a_0 : \dots : a_n]$, so the x_i generate $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 10.1.3. Let $f : X \rightarrow \mathbb{P}^n$ be a morphism. Then $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by

$$f^*x_0, \dots, f^*x_n \in \Gamma(X, \mathcal{L}).$$

Remark. If \mathcal{L} is generated by s_0, \dots, s_n , then we get a map

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

This is a morphism: If we fix $a \in X$ and pick $U \subseteq X$ open such that

$$\alpha : \mathcal{L}|_U \xrightarrow{\cong} \mathcal{O}_U,$$

then $f|_U(x) = [\alpha(s_0|_U)(x) : \dots : \alpha(s_n|_U)(x)]$. Since each $\alpha(s_i)$ is a regular function, $f|_U$ is a morphism. Thus we see that f is a morphism.

Remark. Similar to before, we have a bijection

$$\{\text{morphisms } X \rightarrow \mathbb{P}^n\} \longleftrightarrow \{\text{invertible sheaves } \mathcal{L} \text{ on } X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \text{ generators}\} / \cong$$

The maps are given by $f \mapsto \mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n}(1)$ with $s_i = f^* x_i$ with inverse $(\mathcal{L}, s_i) \mapsto [x \mapsto [s(x)]]$.

Example 10.1.4. Recall the *Veronese embedding* given by

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{Nd} \\ a &\longmapsto [x^I(a) : x^I \text{ monomial of degree } d \text{ in } x_0, \dots, x_n]. \end{aligned}$$

This map corresponds to $\mathcal{O}_{\mathbb{P}^n}(d)$ with sections $x^I \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$.

10.2 Injectivity and Closed Embeddings

Remark. When is $f : X \rightarrow \mathbb{P}^n$ injective or a closed embedding?

Definition 10.2. Let $V = \text{span}\{s_0, \dots, s_n\} \subseteq \Gamma(X, \mathcal{L})$ such that s_0, \dots, s_n generate \mathcal{L} . We say that

1. s_0, \dots, s_n *separate points* of X if for all $x \neq y$, there exists $s \in V$ such that $s(x) = 0 \neq s(y)$.¹
2. s_0, \dots, s_n *separate tangent vectors* if for each $p \in X$,

$$\{s_p : s \in V \text{ and } s(p) = 0\}$$

generates $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$ as a k -vector space.²

Proposition 10.1. Assume that X is complete and \mathcal{L} an invertible sheaf on X generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Write

$$\begin{aligned} f : X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

Then we have the following:

1. f is injective if and only if s_0, \dots, s_n separate points;
2. f is a closed embedding if and only if s_0, \dots, s_n separate points and tangent vectors.

Proof. (1) For $s = \sum_{i=0}^n a_i s_i$ with $a_i \in k$, we have

$$\{x \in X : s(x) = 0\} = f^{-1}(H_a)$$

where $H_a = V(\sum a_i x_i)$. Now f is injective if and only if for all $x \neq y \in X$, there exists a hyperplane $H \subseteq \mathbb{P}^n$ such that $f(x) \in H$ and $f(y) \notin H$. This happens if and only if s_0, \dots, s_n separate points.

(2) Assume that f is injective (equivalently, s_0, \dots, s_n separate points by (1)). For $p \in X$, we get a local ring homomorphism $\mathcal{O}_{\mathbb{P}^n, f(p)} \rightarrow \mathcal{O}_{X, p}$, which induces a map on cotangent spaces

$$\begin{array}{ccc} \mathfrak{m}_{f(p)} / \mathfrak{m}_{f(p)}^2 & \longrightarrow & \mathfrak{m}_p / \mathfrak{m}_p^2 \\ \cong \downarrow & & \downarrow \cong \\ \mathfrak{m}_{f(p)} \mathcal{O}_{\mathbb{P}^n}(1)_{f(p)} / \mathfrak{m}_{f(p)}^2 \mathcal{O}_{\mathbb{P}^n}(1)_{f(p)} & & \mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p \end{array}$$

¹Note that $s(x) = 0$ if and only if $s \in \mathfrak{m}_p \mathcal{L}_p$.

²Here $\mathfrak{m}_p \leq \mathcal{O}_{X, p}$ is its unique maximal ideal. Recall that $(T_p X)^* \cong \mathfrak{m}_p / \mathfrak{m}_p^2 \cong \mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$.

(as \mathcal{L} is invertible, it is locally trivial), where the bottom map is given by

$$\left[\sum a_i x_i \right] \mapsto \left[\sum a_i s_i \right].$$

So f separates tangent vectors at p if and only if $\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \rightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$ is surjective. Since $f : X \hookrightarrow \mathbb{P}^n$ is injective and a closed map (as X is complete), we get $X \hookrightarrow f(X) \subseteq \mathbb{P}^n$ is a homeomorphism (we know it is a bijection, preimages of open sets are open, and images of closed sets are closed), and $f(X) \subseteq \mathbb{P}^n$ is closed. Using this, f is a closed embedding if and only if $\mathcal{O}_{\mathbb{P}^n} \rightarrow f_*\mathcal{O}_X$ is surjective (HW) which happens if and only if $\mathcal{O}_{\mathbb{P}^n, f(p)} \rightarrow \mathcal{O}_{X,p}$ is surjective for all $p \in X$. This happens if and only if

$$\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$$

is surjective for all $p \in X$ (see Mustata's notes, uses that $f_*\mathcal{O}_X$ is coherent). \square

Definition 10.3. Let \mathcal{L} be an invertible sheaf on a complete variety X .

1. \mathcal{L} is *very ample* if there exists a closed embedding $f : X \hookrightarrow \mathbb{P}^n$ and $\mathcal{L} \cong \mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}^n}(1)$.
2. \mathcal{L} is *ample* if there exists $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.

Example 10.3.1. Let $X = \mathbb{P}^n$ (for $n \geq 1$), then $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ if and only if $d > 0$, if and only if $\mathcal{O}_{\mathbb{P}^n}(d)$ is ample (though this is not true in general).

Exercise 10.1. If \mathcal{F} is a coherent sheaf on a complete variety X and $\mathcal{O}_X(1)$ is then show that there exists an exact sequence:

$$\bigoplus_{i=1}^s \mathcal{O}_X(n_i) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_X(m_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

10.3 Divisors

Remark. For the rest of the lecture, let X be an irreducible variety.

Definition 10.4. A *prime divisor* on X is a closed irreducible subvariety $D \subseteq X$ of codimension 1.

Example 10.4.1. We have the following:

1. Prime divisors on \mathbb{P}^n correspond to irreducible hypersurfaces.
2. Prime divisors on a curve C correspond to points.

Definition 10.5. A (*Weil*) *divisor* on X is a formal sum

$$D = \sum_{i=1}^r a_i D_i,$$

such $a_i \in \mathbb{Z}$ and the D_i are distinct prime divisors. Write

$$\text{Div } X = \{\text{divisors on } X\} = \bigoplus_{\substack{E \subseteq X \\ \text{prime divisor}}} \mathbb{Z}[E].$$

Remark. We will see that when X is smooth, there is a surjective group homomorphism

$$\begin{aligned} \Phi : \text{Div } X &\longrightarrow \text{Pic } X = \{\text{invertible sheaves on } X\}/\cong \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

Furthermore, $\ker \Phi = \{\text{div } \varphi : \varphi \in K(X)^\times\}$, where $\text{div } \varphi$ is the divisor of zeros and poles of φ .

Definition 10.6. For a closed irreducible subset $Z \subseteq X$, the *local ring at Z* is

$$\begin{aligned}\mathcal{O}_{X,Z} &= \{(\varphi, U) : \varphi \in \mathcal{O}_X(U), U \subseteq X \text{ open}, Z \cap U \neq \emptyset\} / \sim \\ &= \varinjlim_{\substack{U \subseteq X \text{ open} \\ Z \cap U \neq \emptyset}} \mathcal{O}_X(U),\end{aligned}$$

where $(\varphi, U) \sim (\psi, V)$ if there exists $W \subseteq U \cap V$ open with $Z \cap W \neq \emptyset$ such that $\varphi|_W = \psi|_W$.

Remark. We consider open sets intersecting Z (and not those containing Z) because otherwise there not be many such U . For example, if $X = \mathbb{P}^n$ and Z is a hyperplane, then we can only have $U = X$.

Remark. This construction works for any sheaf on X , not just \mathcal{O}_X .

Lecture 11

Feb. 17 — Divisors

11.1 Divisors, Continued

Proposition 11.1. *We have the following:*

1. *If $U \subseteq X$ is affine open, then we have an isomorphism*

$$\mathcal{O}_X(U)_{\mathcal{I}_Z(U)} \xrightarrow{\cong} \mathcal{O}_{X,Z}$$

when $Z \cap U \neq \emptyset$.

2. $\dim \mathcal{O}_{X,Z} = \text{codim}_X Z$.
3. $\mathcal{O}_{X,Z}$ is a local ring.
4. Let \mathfrak{m}_Z be the maximal ideal of $\mathcal{O}_{X,Z}$. If X is smooth, then $\mathcal{O}_{X,Z}$ is regular, i.e.

$$\dim \mathcal{O}_{X,Z} = \dim \mathfrak{m}_Z / \mathfrak{m}_Z^2,$$

where the right-hand side is the dimension as a vector space over $\mathcal{O}_{X,Z} / \mathfrak{m}_Z$.

Proof. (1) The proof is the same as when $Z = \{x\}$.

(2) We can reduce to the case where X is affine. Then we have a bijection

$$\begin{array}{c} \{\text{prime ideals } \mathfrak{p} \leq A \text{ such that } \mathfrak{p} \cap (A \setminus I_Z) = \emptyset\} \xleftrightarrow{\mathfrak{q} \mapsto \mathfrak{q}\mathcal{O}_{X,Z}} \{\text{prime ideals of } \mathcal{O}_{X,Z}\} \\ \updownarrow \\ \{\text{closed irreducible subvarieties } Z \subseteq Y \subseteq X\} \end{array}$$

where $A = \mathcal{O}_X(X)$. This proves the claim.

(3) We use (1). Since $\mathcal{I}_Z(U) \subseteq \mathcal{O}_X(U)$ is prime, $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal

$$\mathfrak{m}_Z := \mathcal{I}_Z(U)\mathcal{O}_{X,Z} = \{(\varphi, U) \in \mathcal{O}_{X,Z} : \varphi|_{Z \cap U} = 0\} \subseteq \mathcal{O}_{X,Z}.$$

(4) In Algebraic Geometry I, we have seen that this holds when Z is a point. Choose $x \in Z$, then

$$\mathcal{O}_{X,Z} \cong (\mathcal{O}_{X,x})_{\{\varphi \in \mathcal{O}_{X,x} : \varphi|_Z = 0\}}.$$

As $\mathcal{O}_{X,x}$ is regular, so is $\mathcal{O}_{X,Z}$ by commutative algebra. □

Remark. For the rest of this lecture, assume that X is a smooth irreducible variety (so that the local rings are regular and we can talk about the function field of X).

Remark. Let $E \subseteq X$ be a prime divisor. Then $\mathcal{O}_{X,E}$ is a regular local ring of dimension 1. By results from commutative algebra, $\mathcal{O}_{X,E}$ is a *discrete valuation ring* (DVR). In particular, this implies that $\mathcal{O}_{X,E}$ is an integral domain which is a UFD with a unique irreducible element up to multiplication by units.

So there exist an irreducible element $\pi \in \mathcal{O}_{X,E}$ such that for any $0 \neq f \in \text{Frac}(\mathcal{O}_{X,E}) = K(X)$,

$$f = u\pi^d$$

for some $u \in \mathcal{O}_{X,E}^\times$ and $d \in \mathbb{Z}$ (with $d \geq 0$ if and only if $f \in \mathcal{O}_{X,E}$). Such a π is called a *uniformizer*.

We will write $\text{ord}_E(f) = d$ when $f = u\pi^d$, which we view as the multiplicity of vanishing of f along E .

Remark. Note the following:

- $\text{ord}_E(fg) = \text{ord}_E(f) + \text{ord}_E(g)$.
- $\text{ord}_E(f + g) \geq \min\{\text{ord}_E(f), \text{ord}_E(g)\}$.

Along with some other properties, the above implies that ord_E is a *valuation* on $K(X)$.

Example 11.0.1. Consider $0 \in \mathbb{A}^1$. Then we have

$$\mathcal{O}_{\mathbb{A}^1,0} \cong \left\{ \frac{f}{g} : f, g \in k[x], g(0) \neq 0 \right\}.$$

For $\varphi \in \mathcal{O}_{\mathbb{A}^1,0}$, we can write

$$\varphi = ux^m$$

such that $u \in \mathcal{O}_{\mathbb{A}^1,0}^\times = \{f/g : f, g \in k[x], f(0), g(0) \neq 0\}$. So $\text{ord}_0 \varphi = m$.

Remark. We can also think of $\mathcal{O}_{X,E} = \{\varphi \in K(X) : \text{ord}_E(\varphi) \geq 0\}$.

Definition 11.1. For $0 \neq \varphi \in K(X)$, its *divisor of zeros and poles* is

$$\text{div}(\varphi) = \text{div}_X(\varphi) := \sum_{\substack{E \subseteq X \\ \text{prime}}} \text{ord}_E(\varphi) E.$$

Proposition 11.2. *The divisor of zeros and poles $\text{div}(\varphi)$ is a divisor, i.e. we have $\text{ord}_E(\varphi) = 0$ for all but finitely many prime divisors $E \subseteq X$.*

Proof. For an open affine $\emptyset \neq U \subseteq X$. Since $X \setminus U$ contains at most finitely many prime divisors, it suffices to show that $\text{div}_U(\varphi)$ is a divisor. Write $\varphi = f/g$ with $f, g \in \mathcal{O}_X(U) = A$. Since

$$\text{div}_U(\varphi) = \text{div}_U(f) - \text{div}_U(g),$$

it suffices to consider the case when $\varphi \in \mathcal{O}_X(U)$. Now for $E \subseteq X$ prime with $E \cap U \neq \emptyset$, we have

$$\begin{aligned} \text{ord}_E(\varphi) > 0 &\iff \varphi \in \mathfrak{m}_E \subseteq \mathcal{O}_{X,E} \\ &\iff \varphi \text{ vanishes on } E \cap U \\ &\iff E \subseteq V(\varphi). \end{aligned}$$

As $V(\varphi)$ contains finitely many prime divisors, we get that $\text{div}_U(\varphi)$ is a divisor. □

Example 11.1.1. Let $f \in k[x_1, \dots, x_n] \in \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$. We can write

$$f = c f_1^{a_1} \cdots f_r^{a_r}$$

with f_i irreducible and $a_i \geq 0$. Then we have

$$\operatorname{div}_{\mathbb{A}^n}(f) = \sum_{i=1}^r a_i V(f_i).$$

This follows since $\operatorname{ord}_{V(f_i)} f_i = 1$, which in turns follows from $\mathcal{O}_{\mathbb{A}^n, V(f_i)} \cong k[x_1, \dots, x_n]_{(f_i)}$, which has maximal ideal (f_i) and hence uniformizer f_i .

Proposition 11.3 (Algebraic Hartog's lemma). *For $0 \neq \varphi \in K(X)$, we have $\operatorname{div} \varphi \geq 0$ (i.e. $\operatorname{ord}_E \varphi \geq 0$ for all prime divisors $E \subseteq X$) if and only if $\varphi \in \mathcal{O}_X(X)$.*

Proof. It suffices to show the statement on an affine cover. So we may assume that X is affine. Write $A = \mathcal{O}_X(X)$. Hartog's lemma in commutative algebra implies that

$$A = \bigcap_{\substack{\mathfrak{p} \leq A \\ \text{prime} \\ \text{height } 1}} A_{\mathfrak{p}} \subseteq \operatorname{Frac}(A).$$

Thus for $\varphi \in \operatorname{Frac}(A) \cong K(X)$, we have $\varphi \in A$ if and only if $\varphi \in A_{\mathfrak{p}}$ for all height 1 prime ideals $\mathfrak{p} \leq A$, which happens if and only if $\operatorname{ord}_E(\varphi) \geq 0$ for all prime divisors $E \subseteq X$, i.e. $\operatorname{div}_X(\varphi) \geq 0$. \square

11.2 Class Groups

Definition 11.2. A divisor $D \in \operatorname{Div} X$ is *principal* if $D = \operatorname{div}_X(\varphi)$ for some $0 \neq \varphi \in K(X)$.

Example 11.2.1. Any divisor on \mathbb{A}^n is principal by the previous examples (this is also true when \mathbb{A}^n is replaced by an affine variety X such that $\mathcal{O}_X(X)$ is a UFD).

Remark. Note that $\operatorname{PDiv} X = \{\operatorname{div} \varphi : 0 \neq \varphi \in K(X)\} \subseteq \operatorname{Div} X$ is a subgroup.

Definition 11.3. We say that $D_1, D_2 \in \operatorname{Div} X$ are *linearly equivalent* if $D_1 - D_2$ is principal. The *class group* of X is

$$\operatorname{Cl}(X) = \operatorname{Div}(X) / \operatorname{PDiv}(X) = \text{group of divisors up to } \sim.$$

Example 11.3.1. We have the following:

1. $\operatorname{Cl}(\mathbb{A}^n) = 0$.
2. $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}[H]$, where $H \subseteq \mathbb{P}^n$ is a hyperplane.

To see this, recall that we have a bijection

$$\{\text{prime divisors in } \mathbb{P}^n\} \longleftrightarrow \{\text{irreducible hypersurfaces}\}.$$

Consider $\deg : \operatorname{Div}(\mathbb{P}^n) \rightarrow \mathbb{Z}$ which sends $\sum a_i E_i = \sum a_i \deg E_i$, which is clearly surjective. For $\varphi \in K(\mathbb{P}^n)$, write $\varphi = f/g$ where $f, g \in k[x_0, \dots, x_n]$ are homogeneous of the same degree. So

$$f = f_1^{a_1} \cdots f_r^{a_r} \quad \text{and} \quad g = g_1^{b_1} \cdots g_s^{b_s}$$

with f_i, g_i homogeneous and $a_i, b_i \in \mathbb{Z}_{\geq 0}$, and $\sum a_i \deg f_i = \sum b_i \deg g_i$. Now

$$\operatorname{div} \varphi = \sum_{i=1}^r a_i V(f_i) - \sum_{i=1}^s b_i V(g_i).$$

So $\deg \varphi = 0$. This shows that $\operatorname{PDiv}(\mathbb{P}^n) \subseteq \ker \deg$. One can show the reverse inclusion as well, so $\operatorname{PDiv}(\mathbb{P}^n) = \ker \deg$. Thus we have $\operatorname{Cl}(\mathbb{P}^n) = \operatorname{Div}(\mathbb{P}^n) / \operatorname{PDiv}(\mathbb{P}^n) \cong \mathbb{Z}$.

11.3 Cartier Divisors

Definition 11.4. A divisor D on X is *Cartier* if it is locally principal, i.e. at each $x \in X$, there exists an open neighborhood $x \in U_x \subseteq X$ and $0 \neq \varphi_x \in K(X)$ such that $\operatorname{div}_{U_x}(\varphi_x) = D|_{U_x}$.

Example 11.4.1. Principal divisors are Cartier (in fact, they are globally principal).

Example 11.4.2. Let $H = V(x_0) \subseteq \mathbb{P}^n$. Note $H \approx 0$, so H is not principal. But on $U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n$,

$$H|_{U_i} = \operatorname{div}_{U_i}(x_0/x_i),$$

so H is Cartier (alternatively, \mathbb{P}^n has an open cover by the U_i , and each $\operatorname{Cl}(U_i) = 0$).

Proposition 11.4. *Every divisor D on X is Cartier.*

Proof. If D_1, D_2 are Cartier, then so is $D_1 + D_2$. So it suffices to consider the case when $D = E$ for some prime divisor E on X . Now fix $x \in X$. If $x \notin E$, then set $U_x = X \setminus E$ and $\varphi_x = 1$. Now assume $x \in E$. Since $\mathcal{O}_{X,x}$ is a regular local ring, the Auslander-Buchsbaum theorem implies that $\mathcal{O}_{X,x}$ is a UFD. Now we have a bijection

$$\{\text{prime ideals of } \mathcal{O}_{X,x}\} \longleftrightarrow \{\text{irreducible closed subvarieties } x \in Y \subseteq X\}.$$

Write $J \subseteq \mathcal{O}_{X,x}$ for the prime ideal corresponding to E . As J is a height 1 prime ideal and $\mathcal{O}_{X,x}$ is a UFD, there exists $f \in \mathcal{O}_{X,x}$ such that $J = (f)$. So there exists an isomorphism

$$\varphi : (\mathcal{O}_{X,x})_{(f)} \xrightarrow{\cong} \mathcal{O}_{X,E}$$

such that $\varphi(f)\mathcal{O}_{X,E} = \mathfrak{m}_E$. So $\operatorname{ord}_E(f) = 1$. One then checks that $\operatorname{div}_{U_x}(f) = E|_{U_x}$ for some U_x . \square

Remark. Proposition 11.4 crucially uses our assumption that X is smooth. It may fail in general.