MATH 6422: Algebraic Geometry II

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Contents

Lecture 1

Jan. 13 — Overview and Review

1.1 Course Overview

Remark. This course will cover the following topics:

(i) vector bundles and line bundles in algebraic geometry;

(ii) coherent sheaves;

(iii) differentials;

(iv) sheaf cohomology: in particular, we will see that $H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T_X^*))$;

(v) the Riemann-Roch theorem: if $\omega = f\,dz$ is a rational 1-form on a smooth projective curve C, then

$$(\#\text{ zeroes of }\omega) - (\#\text{ poles of }\omega) = 2\,\mathrm{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example, $\operatorname{Spec}\mathbb{Z}$ has points corresponding to the primes p and 0.

1.2 Review of Algebraic Geometry I

Remark. Let $k = \overline{k}$ be an algebraically closed field.

Remark (Hilbert's Nullstellensatz)**.** There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \dots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence, Z being irreducible (resp. a point) corresponds to $I(Z)$ being prime (resp. maximal).

Remark (Zariski topology on $\mathbb{A}^n$)**.** The closed sets in $\mathbb{A}^n$ are of the form $V(J)$, and this induces a Zarisiki topology on any subset of $\mathbb{A}^n$.

Remark (Embedded affine varieties)**.** Let $J \leq k[x_1, \dots, x_n]$. Then we can associate to J a ringed space $(X, \mathcal{O}_X)$ by setting $X := V(J) \subseteq \mathbb{A}^n$ with the Zariski topology, and $\mathcal{O}_X$ to be the sheaf of regular functions on X, i.e. for $U \subseteq X$ open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

Here, $\varphi : U \to k$ is *regular* if for each $p \in U$, there exists an open set $p \in U_p \subseteq U$ and $f, g \in k[x_1, \dots, x_n]$ such that $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$.

Remark (Coordinate ring)**.** The *coordinate ring* of X is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \dots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for $A(X)$:

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

Remark (Distinguished open sets)**.** The *distinguished open sets* of X are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These for a basis for X as we vary $f \in A(X)$.

Definition 1.1. An *affine variety* is a ringed space $(X, \mathcal{O}_X)$ (here $\mathcal{O}_X$ is a sheaf of k-valued functions) which is isomorphic to an embedded affine variety.

Example 1.1.1. If $(X, \mathcal{O}_X)$ is an affine variety and $f \in \mathcal{O}_X(X)$, then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that $X = V(J) \subseteq \mathbb{A}^n$ with $J \leq k[x_1, \dots, x_n]$ a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \dots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \dots, x_n]/J)[y]}{(\overline{f}y - 1)} \cong A(X)_f.$$

Theorem 1.1. *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.-}k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

This implies the following:

1. *There is a bijection*

$$\begin{aligned} \text{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated k-algebra A, there exists an affine variety with $A \cong A(X)$.*

Remark. How can we explicitly define the inverse functor Red-f.g.-k-alg$^{\text{op}} \to$ Aff-var? We can define this as $A \mapsto (X, \mathcal{O}_X)$, where X is the set of maximal ideals of A. Think about what $\mathcal{O}_X$ should be.

Remark (Varieties)**.** A *variety* $(X, \mathcal{O}_X)$ is a ringed space such that

- there exists a finite open cover of X by affine varieties,
- the diagonal Δ_X is closed in $X \times X$.

Example 1.1.2. The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$.

Remark (Projective spaces)**.** Recall that $\mathbb{P}^n$ has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for $\mathbb{P}^n$ by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with $f \in k[x_0, \dots, x_n]$ homogeneous.

1.3 Vector and Line Bundles

Definition 1.2. Let X be a variety. A *vector bundle (of rank m)* on X is a variety $\mathbb{E}$ with a morphism $p : \mathbb{E} \to X$ such that

1. $\mathbb{E}_x = p^{-1}(x)$ has the structure of a rank m vector space for every $x \in X$ (i.e. k^m),
2. for every $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ and an isomorphism $p^{-1}(U) \to U \times \mathbb{A}^m$ such that for any $y \in U$, the map $\mathbb{E}_y \to \{y\} \times \mathbb{A}^m$ is an isomorphism of vector spaces, i.e.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "p"'] & & U \times \mathbb{A}^m \arrow[dl, "\text{proj}"] \\ & U & \end{tikzcd}$$

commutes. We will call the map ϕ_U a *trivialization.*

Definition 1.3. A *line bundle* on X is a rank 1 vector bundle.

Remark. A different way to think about this is the following:

1. Given two trivializations $\phi_U : p^{-1}(U) \to U \times \mathbb{A}^m$ and $\phi_V : p^{-1}(V) \to V \times \mathbb{A}^m$, we get a morphism

$$\begin{tikzcd} (U \cap V) \times \mathbb{A}^m \arrow[rr, "\phi_{U,V}"] \arrow[dr, "\phi_V^{-1}"'] & & (U \cap V) \times \mathbb{A}^m \\ & p^{-1}(U \cap V) \arrow[ur, "\phi_U"'] & \end{tikzcd}$$

with $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$. Observe $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$ for some $g_{U,V}(x) \in \mathrm{GL}(m, k)$. Furthermore, $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is a morphism. We will call the $g_{U,V}$ *transition functions.*

In the special case where $m = 1$ (so $\mathbb{E}$ is a line bundle and $\mathrm{GL}(1, k) = k^\times$), the map $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is equivalent to the data of a non-vanishing regular function $g_{U,V} : U \cap V \to k$.

2. The data of a vector bundle of rank m is equivalent to the data of
   - an open cover $X = \bigcup_{i \in I} U_i$,
   - and morphisms $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$

   such that $g_{i,k} = g_{i,j} g_{j,k}$, $g_{i,j} = g_{j,i}^{-1}$, and $g_{i,i} = \mathrm{id}$.

   To recover the vector bundle, we can glue $\mathbb{E}_i = U_i \times \mathbb{A}^m$ for $i \in I$ via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x) v). \end{aligned}$$

   One can check that this defines a vector bundle $\mathbb{E}$ on X.

Example 1.3.1 (Trivial vector bundle)**.** Define the vector bundle $\mathbb{E} : X \times \mathbb{A}^m \to X$ by $(x, v) \to x$. Given a cover $X = \bigcup_{i \in I} U_i$, we get $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$ as $x \mapsto I_m$.

Example 1.3.2 (Trivial line bundle)**.** We will denote the trivial line bundle by $\mathbb{1}_X : X \times \mathbb{A}^1 \to X$.

Lecture 2

Jan. 15 — Vector and Line Bundles

2.1 Vector and Line Bundles, Continued

Example 2.0.1 (Tautological bundle)**.** Let $X = \mathbb{P}^n$ and $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$. Consider

$$\begin{tikzcd} & \mathbb{L} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{tikzcd}$$

The map $q : \mathbb{L} \to \mathbb{A}^{n+1}$ is the blowup. We claim that $p : \mathbb{L} \to \mathbb{P}^n$ is a line bundle. We have:

- $p^{-1}([x]) = \{([x], cx) : c \in k\} \cong kx$, a 1-dimensional vector space;
- let $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$, then we can define

  $$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

  which we claim is a trivialization. To see this, observe that for fixed $[x] \in \mathbb{P}^n$, we have

  $$\begin{aligned} \mathbb{L}_{[x]} = \{([x], cx) : c \in k\} &\longrightarrow \{[x]\} \times \mathbb{A}^1 \\ ([x], cx) &\longmapsto ([x], cx_i), \end{aligned}$$

  which is a vector space isomorphism.

We can also compute the transitions functions. Let $U_{i,j} = U_i \cap U_j$. We have

$$\begin{tikzcd} U_{i,j} \times \mathbb{A}^1 \arrow[r, "\phi_j^{-1}"'] \arrow[rr, bend left, "\phi_{i,j}"] & p^{-1}(U_{i,j}) \arrow[r, "\phi_i"'] & U_{i,j} \times \mathbb{A}^1 \end{tikzcd}$$

$$([x], t) \longmapsto ([x], (tx_0/x_j, \ldots, tx_n/x_j)) \longmapsto ([x], tx_i/x_j).$$

Thus we see that $g_{i,j} = x_i/x_j$. This is called the *tautological bundle*, or $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 2.0.2 (Hyperplane bundle, or $\mathcal{O}_{\mathbb{P}^n}(1)$)**.** Consider

$$\begin{aligned} \mathbb{L} := \mathbb{P}^{n+1} \setminus \{[0 : \cdots : 0 : 1]\} &\longrightarrow \mathbb{P}^n \\ [x_0 : \cdots : x_n : x_{n+1}] &\longmapsto [x_0 : \cdots : x_n]. \end{aligned}$$

Then $\mathbb{L}$ is a line bundle with transition functions with respect to $\{U_i\}$ given by $g_{i,j} = x_j/x_i$ (HW).

2.2 Operations on Vector Bundles

Remark. The philosophy is: Every natural operation of vector spaces gives one for vector bundles.

Example 2.0.3 (Direct sum)**.** Let $p : \mathbb{E} \to X$ and $q : \mathbb{F} \to X$ be vector bundles of rank e and f on X, respectively. There exists trivializations with respect to a common open cover $\{U_i\}$ (just take intersections) with transition functions $g_{i,j}$ and $h_{i,j}$ for $\mathbb{E}$ and $\mathbb{F}$, respectively.

Then we define the vector bundle $\mathbb{E} \oplus \mathbb{F} \longrightarrow X$ as follows:

- As a set, it is $r : \mathbb{E} \oplus \mathbb{F} = \{(x, u, v) : (x, u) \in \mathbb{E}, (x, v) \in \mathbb{F}\} \to X$.
- We give $\mathbb{E} \oplus \mathbb{F}$ the structure of a variety by requiring that

$$\begin{aligned} r^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^{e+f} \\ (x, u, v) &\longmapsto (x, \mathrm{pr}_2(\phi_i^E(x, u)), \mathrm{pr}_2(\phi_i^F(x, v))) \end{aligned}$$

  be an isomorphism, where ϕ_i^E and ϕ_i^F are the trivializations of $\mathbb{E}$ and $\mathbb{F}$, and pr_2 is the second projection. This gives a variety structure on $r^{-1}(U_i)$, and one can show that these are consistent on $U_{i,j}$, so that this gives a variety structure on all of $\mathbb{E} \oplus \mathbb{F}$.

Note that the transition functions for $\mathbb{E} \oplus \mathbb{F}$ with respect to $\{U_i\}$ are given by the block matrix

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{bmatrix} : U_{i,j} \longrightarrow \mathrm{GL}(e + f, k).$$

Example 2.0.4. Let $\mathbb{E}$ and $\mathbb{F}$ be vector bundles on X of ranks e and f, respectively. Then the following are also vector bundles on X:

1. $\mathrm{Hom}(\mathbb{E}, \mathbb{F})$, of rank ef;
2. $\mathbb{E}^\vee = \mathrm{Hom}(\mathbb{E}, \mathbb{1}_X)$, of rank e;
3. $\mathbb{E} \otimes \mathbb{F}$, of rank ef;
4. $\wedge^k \mathbb{E}$ and $\mathrm{Sym}^d \mathbb{E}$.

Remark. Let $\mathbb{L}, \mathbb{M}$ be line bundles on X with trivializations on $\{U_i\}$ and transition functions $g_{i,j}, h_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$. In this case, we can describe operations on $\mathbb{L}, \mathbb{M}$ more explicitly:

1. $\mathbb{L} \otimes \mathbb{M}$ has transition functions $g_{i,j} h_{i,j}$;
2. $\mathrm{Hom}(\mathbb{L}, \mathbb{M})$ has transition functions $h_{i,j}/g_{i,j}$;
3. $\mathbb{L}^\vee = \mathrm{Hom}(\mathbb{L}, \mathbb{1}_X)$ has transition functions $1/g_{i,j}$;
4. $\mathbb{L}^{\otimes m} = \begin{cases} \mathbb{L}^{\otimes m}, & \text{if } m > 0, \\ \mathbb{1}_X, & \text{if } m = 0, \\ (\mathbb{L}^\vee)^{\otimes -m}, & \text{if } m < 0 \end{cases}$ has transition functions $g_{i,j}^m$.

Example 2.0.5. Define $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$ with transition functions $(x_j/x_i)^m$ with respect to the standard open cover for $\mathbb{P}^n$.

2.3 Morphisms of Vector Bundles

Remark. Let $p : \mathbb{E} \to X$ and $q : \mathbb{F} \to X$ be vector bundles on X, as before.

Definition 2.1. A *morphism of vector bundles* $\mathbb{E} \to \mathbb{F}$ is a morphism of varieties

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "a"] \arrow[dr, "p"'] & & \mathbb{F} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

such that the diagram commutes and a is linear on each fiber.

Remark. More concretely, given an open cover $\{U_i\}$ which trivializes both vector bundles, we have

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "a"] \arrow[d, "\phi_i"', "\cong"] & q^{-1}(U_i) \arrow[d, "\cong"', "\psi_j"] \\ U_i \times \mathbb{A}^e \arrow[r] & U_i \times \mathbb{A}^f \end{tikzcd}$$

$$(x, v) \longmapsto (x, a_i(x)v)$$

such that $a_i : U_i \to \mathrm{Hom}(k^e, k^f)$ is regular. On $U_{i,j}$, we have

$$\begin{tikzcd} U_{i,j} \times \mathbb{A}^e \arrow[r, "a_j"] \arrow[d, "g_{i,j}"'] & U_{i,j} \times \mathbb{A}^f \arrow[d, "h_{i,j}"] \\ U_{i,j} \times \mathbb{A}^e \arrow[r, "a_i"'] & U_{i,j} \times \mathbb{A}^f \end{tikzcd}$$

So $h_{i,j}a_j = a_i g_{i,j}$, or equivalently, $a_i = h_{i,j}a_j g_{i,j}^{-1}$.

As a special case when $e = f$, $a : \mathbb{E} \to \mathbb{F}$ is an isomorphism if and only if the a_i are isomorphisms.

Remark. When is a line bundle $\mathbb{L}$ given by the trivialization data $\{U_i, g_{i,j}\}$ isomorphic to $\mathbb{1}_X$? We have

$$\begin{aligned} \mathbb{L} \cong \mathbb{1}_X &\iff \text{if and only if there exists an isomorphism } a : \mathbb{1}_X \to \mathbb{L} \\ &\iff \text{there exist } a_i \in \mathcal{O}_X(U_i)^\times \text{ such that } (a_j/a_i)|_{U_{i,j}} = g_{i,j}. \end{aligned}$$

Definition 2.2. Define the *Picard group* of X to be

$$\operatorname{Pic} X := \{\text{line bundles on } X\}/\cong.$$

This is a group with respect to $\otimes$ with $\mathbb{1}_X$ as the identity and $\mathbb{L}^\vee \otimes \mathbb{L} \cong \mathbb{1}_X$.

2.4 Global Sections

Definition 2.3. A *(global) section* of a vector bundle $p : \mathbb{E} \to X$ is a morphism $s : X \to \mathbb{E}$ such that $p \circ s = \mathrm{id}_X$. Note that for $x \in X$, we have $s(x) \in \mathbb{E}_x$.

Example 2.3.1 (Zero section)**.** Let $s : X \to \mathbb{E}$ where $s(x)$ is the zero element in $\mathbb{E}_x$.

Example 2.3.2. Let $\mathbb{E} = \mathbb{1}_X$. Then sections $s : X \to X \times \mathbb{A}^1$ of $\mathbb{E}$ correspond to morphisms $X \to \mathbb{A}^1$, which correspond to regular functions $X \to k$.

Remark (Local description of sections)**.** Let $\{U_i, g_{i,j}\}$ be the trivialization data for $\mathbb{E} \to X$, and let $s : X \to \mathbb{E}$ be a section. On U_i, we have:

$$\begin{tikzcd} & U_i \times \mathbb{A}^e \\ U_i \arrow[ur, "x \mapsto (x, s_i(x))"] \arrow[r, "s|_{U_i}"'] & p^{-1}(U_i) \arrow[u, "\cong"', "\phi_i"] \end{tikzcd}$$

Note that $s_i : U_i \to k^e$ is a regular function (i.e. regular on each coordinate). These maps must satisfy the compatibility condition $s_i = g_{i,j} s_j$, since we have the diagram:

$$\begin{tikzcd} & & U_{i,j} \times \mathbb{A}^e \arrow[dd, "(x,v) \mapsto (x, g_{i,j}(v))"] \\ U_{i,j} \arrow[urr, bend left, "(x, s_j(x))"] \arrow[drr, bend right, "(x, s_i(x))"'] \arrow[r, "s|_{U_{i,j}}"] & p^{-1}(U_{i,j}) \arrow[ur, "\phi_j"] \arrow[dr, "\phi_i"'] & \\ & & U_{i,j} \times \mathbb{A}^e \end{tikzcd}$$

Example 2.3.3. We can use the above compatibility condition to compute the global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Write $\mathbb{P}^1_{x_0:x_1} = U_0 \cup U_1$. Given a section $s : \mathbb{P}^1 \to \mathcal{O}(1)$, we get regular functions

$$s_0 : U_0 \longrightarrow k$$
$$s_1 : U_1 \longrightarrow k$$

satisfying $(x_1/x_0)s_1 = s_0$ $(*)$. We can write

$$s_0 = \sum_{m \geq 0} a_m (x_1/x_0)^m \quad \text{and} \quad s_1 = \sum_{m \geq 0} b_m (x_0/x_1)^m$$

with $a_m, b_m \in k$ (finitely many nonzero). Then $(*)$ implies that

$$a_0 + a_1(x_1/x_0) + \cdots = (x_1/x_0)(b_0 + b_1(x_0/x_1) + \cdots),$$

so $a_0 = b_1$, $a_1 = b_0$, and all other terms are 0. So we can relate s to a linear form

$$f = a_0 x_0 + a_1 x_1,$$

where $s_0 = (1/x_0)f$ and $s_1 = (1/x_1)f$.

Lecture 3

Jan. 20 — Sections

3.1 Global Sections, Continued

Definition 3.1. Let $\Gamma(X, \mathbb{E}) := \{\text{sections of } \mathbb{E} \to X\}$, which has the structure of a k-vector space by

$$(s+t)(x) = s(x) + t(x) \quad \text{and} \quad (cs)(x) = cs(x)$$

for $s, t \in \Gamma(X, \mathbb{E})$ and $c \in k$.

Example 3.1.1. One can check that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0, \ldots, x_n]_d$ (HW). For example, for $d < 0$, we have $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \{0\}$, and for $d = 0$, we have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(0)) = \Gamma(\mathbb{P}^n, \mathbb{1}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k.$$

For $d = 1$, we can define an isomorphism

$$\begin{aligned} k[x_0, \ldots, x_n]_1 &\xrightarrow{\cong} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \\ f &\longmapsto \text{section } s : \mathbb{P}^n \to \mathcal{O}(1) \text{ given by } s_i = f/x_i. \end{aligned}$$

Note that $(x_j/x_i)s_j = s_i$ holds, so this is a section. An alternative perspective is that s corresponds to

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{n+1} \setminus \{[0 : \cdots : 0 : 1]\} \\ x &\longmapsto [x_0 : \cdots : x_n : f(x)]. \end{aligned}$$

3.2 Morphisms and Sections

Definition 3.2. Given a section $s : X \to \mathbb{E}$, its *vanishing locus* is

$$V(s) := \{s = 0\} = \{x \in X : s(x) = 0\}.$$

Using a trivializing cover, one can check that $V(s)$ is closed in X.

Example 3.2.1. For a section $s : \mathbb{P}^n \to \mathcal{O}(1)$ corresponding to $f \in k[x_0, \ldots, x_n]_1$, we have $V(s) = V_{\mathbb{P}^n}(f)$.

Remark. Recall that there is a bijection

$$\begin{aligned} \{\text{morphisms } X \to \mathbb{A}^n\} &\longleftrightarrow \{f_1, \ldots, f_n \in \mathcal{O}_X(X)\} \\ [f : X \to \mathbb{A}^n] &\longmapsto [f_1 = f^* x_1, \ldots, f_n = f^* x_i \in \mathcal{O}_X(X)] \\ [x \mapsto (f_1(x), \ldots, f_n(x))] &\longleftarrow [f_1, \ldots, f_n \in \mathcal{O}_X(X)]. \end{aligned}$$

We want a similar statement for $\mathbb{P}^n$.

Definition 3.3. Given a line bundle $\mathbb{L} \to X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$, they are *nowhere vanishing* if

$$V(s_0) \cap \cdots \cap V(s_n) = \varnothing.$$

Example 3.3.1. For $\mathcal{O}(1)$, the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ are nowhere vanishing.

Remark. If $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ are nowhere vanishing, then we get a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \cdots : s_n(x)]. \end{aligned}$$

Note that $(s_0(x), \dots, s_n(x))$ is a well-defined point in $\mathbb{A}^{n+1}$ up to scaling. One can check that this map is a morphism by working locally.

Example 3.3.2 (Linear maps)**.** Let $X = \mathbb{P}^n$ and $\mathbb{L} = \mathcal{O}(1)$.

(i) $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ gives $\mathrm{id} : \mathbb{P}^n \to \mathbb{P}^n$.

(ii) For $A \in \mathrm{GL}_{n+1}(k)$, we get a map

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

given by $Ax_0, \dots, Ax_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Remark. Now given a morphism $X \to \mathbb{P}^n$, we want to get a line bundle with sections.

Definition 3.4 (Pullback)**.** Let $p : \mathbb{E} \to X$ be a vector bundle and $f : Y \to X$ a morphism. Define

$$f^*\mathbb{E} = \{(e, y) : e \in \mathbb{E}, y \in Y \text{ with } p(e) = f(y)\} \longrightarrow Y.$$

One can show that this has the structure of a vector bundle in a natural way.

Remark. An alternative way to define the pullback is to choose trivialization data $(U_i, g_{i,j})$ for $\mathbb{E} \to X$. Then we can define $f^* : \mathbb{E} \to Y$ to be the vector bundle with trivialization data $(f^{-1}(U_i), f^*g_{i,j})$.

Remark. Now to go in reverse, given a morphism $X \to \mathbb{P}^n$ and nowhere vanishing sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we get nowhere vanishing sections

$$f^*x_0, \dots, f^*x_n \in \Gamma(X, f^*\mathcal{O}(1)).$$

We can define the pullback of a section in one of two ways: by $f^*(x_i)(a) = (x_i(f(a)), a) \in f^*\mathcal{O}(1))$ for $a \in X$ or by using trivializing covers.

Remark. Using the above, we get a bijection

$$\{\text{morphisms } X \to \mathbb{P}^n\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \to X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}.$$

Note that we should consider the right-hand side up to isomorphism of the line bundle. When do $\mathbb{L} \to X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ give an injective morphism (or an embedding)?

Definition 3.5. Given a vector bundle $\mathbb{E} \to X$, we get a sheaf of abelian groups $\mathcal{E}$ on X by

$$\mathcal{E}(U) := \{\text{sections of } p^{-1}(U) \to U\}$$

for $U \subseteq X$ open. For $V \subseteq U \subseteq X$ open, the restriction map is given by

$$\begin{aligned} \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto s|_V. \end{aligned}$$

We call $\mathcal{E}$ the *sheaf of sections* of $\mathbb{E}$. Also note that $\mathcal{E}(U)$ has the structure of an $\mathcal{O}_X(U)$-module. We will see that this gives rise to the structure of an $\mathcal{O}_X$-module.

3.3 Review of Sheaves

Definition 3.6. A *presheaf* of abelian groups $\mathcal{F}$ on a topological space X is the data of:

- for $U \subseteq X$ open, an abelian group $\mathcal{F}(U)$ (with $\mathcal{F}(\varnothing) = 0$),
- for $V \subseteq U \subseteq X$ open, a group homomorphism $p_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$.

Remark. Note the following:

1. We may replace abelian groups in the above definition by rings, sets, R-modules, etc.
2. We denote $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, whose elements are called *sections*.
3. $s|_V := p_{V,U}(s)$ is called the *restriction* for $s \in \mathcal{F}(U)$ and $V \subseteq U \subseteq X$ open.
4. We may view $\mathcal{F}$ as a functor $\mathrm{Open}_X \to$ Ab-gps given by $U \mapsto \mathcal{F}(U)$.

Definition 3.7. For $\mathcal{F}$ a presheaf on X and $x \in X$, the *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U) = \{(s, U) : s \in \mathcal{F}(U)\}/\sim.$$

Example 3.7.1. The following are examples of presheaves:

1. Let M be a smooth manifold. Then
   - $\mathcal{O}_M$ = sheaf of smooth $\mathbb{R}$-valued functions on M,
   - $\mathcal{E}$ = sheaf of sections of a vector bundle $\mathbb{E} \to M$.
2. Let X be an algebraic variety, $\mathbb{E} \to X$ a vector bundle, and $Z \subseteq X$ closed. Then
   - $\mathcal{O}_X$ and $\mathcal{E}$ are sheaves,
   - $\mathcal{I}_Z$ = ideal sheaf of Z, given by $\mathcal{I}_Z(U) = \{\varphi \in \mathcal{O}_X(U) : \varphi|_Z = 0\}$.
3. Let X be a topological space and A an abelian group.
   - $\underline{A}^{\text{pre}}$ given by $U \mapsto \{\text{constant functions } U \to A\}$, i.e. $\underline{A}^{\text{pre}}(U) \cong A$ for $U \neq \varnothing$,
   - $\underline{A}$ given by $U \mapsto \{\text{locally constant functions } U \to A\}$,
   - $i_p A$ = skyscraper sheaf, given by $U \mapsto \begin{cases} A & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$

Definition 3.8. A presheaf $\mathcal{F}$ is a *sheaf* if for any

- open set $U \subseteq X$,
- open cover $U = \bigcup_{i \in I} U_i$,
- and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$ for all $i, j \in I$,

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Remark. The presheaf $\underline{A}^{\text{pre}}$ is not a sheaf in general. All other examples above are sheaves.

Definition 3.9. A *morphism* of (pre)sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on a topological space X is the data of group homomorphisms $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each $U \subseteq X$ open such that for all $V \subseteq U \subseteq X$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

Example 3.9.1. Let X be a variety.

1. If $a : \mathbb{E} \to \mathbb{F}$ is a morphism of vector bundles on X, then we get a morphism of sheaves $\mathcal{E} \to \mathcal{F}$ by $s \mapsto a \circ s \in \mathcal{F}(U)$ for $s \in \mathcal{E}(U)$.
2. A closed subvariety $Z \subseteq X$ induces a morphism $\mathcal{I}_Z \to \mathcal{O}_X$ given by inclusion.

Remark. Given a morphism of (pre)sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ and $p \in X$, we get an induced morphism

$$\begin{aligned} \mathcal{F}_p &\longrightarrow \mathcal{G}_p \\ (s, U) &\longmapsto (\varphi(s), U). \end{aligned}$$

Lecture 4

Jan. 22 — Sheaves

4.1 Sheafification

Theorem 4.1 (Sheafification). *For a presheaf $\mathcal{F}$ on a topological space X, there exists a morphism to a sheaf $i : \mathcal{F} \to \mathcal{F}^+$ such that for any morphism to a sheaf $g : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $g^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $g = g^+ \circ i$, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g"] \arrow[d, "i"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, dashed, "g^+"'] & \end{tikzcd}$$

In the above, $\mathcal{F}^+$ is called the sheafification *of $\mathcal{F}$, and the pair $(i, \mathcal{F}^+)$ is unique up to isomorphism (as a consequence of the universal property).*

Proof. We first define $\mathcal{F}^+(U) = \{t : U \to \bigsqcup_{p \in X} \mathcal{F}_p : (1) \text{ and } (2) \text{ hold}\}$, where

1. $t(p) \in \mathcal{F}_p$;
2. for any $x \in X$, there is an open set $x \in V_x \subseteq U$ with $s \in \mathcal{F}(V_x)$ such that $t(p) = s_p$ for all $p \in V_x$.

It is straightforward to see that $\mathcal{F}^+$ is a sheaf and that

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto (X \ni p \mapsto s_p \in \mathcal{F}_p). \end{aligned}$$

gives a morphism $i : \mathcal{F} \to \mathcal{F}^+$. Now we check the universal property. Given a morphism $g : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf, we need to define $g^+ : \mathcal{F}^+ \to \mathcal{G}$. Fix $t \in \mathcal{F}^+(U)$. By definition, there exists an open cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $t(p) = (s_i)_p \in \mathcal{F}_p$ for all $p \in U_i$. Set $t_i' := g(t_i)$. Note that

$$(t_i')_p = g_p(t_p) = (t_j')_p \in \mathcal{G}_p$$

for every $p \in U_i \cap U_j$. Since $\mathcal{G}$ is a sheaf, we get $t_i'|_{U_i \cap U_j} = t_j'|_{U_i \cap U_j}$. Thus there exists a unique $t' \in \mathcal{G}(U)$ such that $t'|_{U_i} = t_i$ for every $i \in I$. Then we can set $g^+(t) = t'$. One can check as an exercise that this gives a morphism $\mathcal{F}^+ \to \mathcal{G}$ satisfying the universal property. $\square$

Example 4.0.1. We have the following:

1. If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
2. For an abelian group A and topological space X, we have $(\underline{A}^{\text{pre}})^+ \cong \underline{A}$.

Remark. We have the following:

1. If $p \in X$, the induced morphism on stalks $i_p : \mathcal{F}_p \to \mathcal{F}_p^+$ is an isomorphism for all $p \in X$.

   To construct the inverse map, consider $\mathcal{F}_p^+ \to \mathcal{F}_p$ defined by $(t, U) \mapsto t_p$ for $t \in \mathcal{F}^+(U)$. Check as an exercise that this is well-defined and is inverse to i_p.

2. If $\mathcal{F} \subseteq \mathcal{G}$ is a subpresheaf (i.e. $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and $\rho^{\mathcal{F}}_{V,U} = \rho^{\mathcal{G}}_{V,U}|_{\mathcal{F}(U)}$ for all $V \subseteq U \subseteq X$) and $\mathcal{G}$ is a sheaf, then we could alternatively define $\mathcal{F}^+$ as

$$\mathcal{F}^+(U) = \{s \in \mathcal{G}(U) : \text{for all } x \in X, \text{ there exists } x \in U_x \subseteq U \text{ open such that } s|_{U_x} \in \mathcal{F}(U_x)\}.$$

4.2 Kernel, Image, Cokernel for Sheaves

Remark. We want the following notions for sheaves:

- kernel, image, cokernel;
- short exact sequences;
- injectivity and surjectivity.

Example 4.0.2. We want the following to be short exact sequences of sheaves:

- for X a variety and $Z \hookrightarrow X$ a closed subvariety,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

- for M a complex manifold (e.g. $\mathbb{C}^n$) with $\mathcal{O}_M$ the sheaf of $\mathbb{C}$-valued holomorphic functions,

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i \times} \mathcal{O}_M \xrightarrow{\varphi \mapsto e^{\varphi}} \mathcal{O}_M^\times \longrightarrow 0$$

Remark. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X.

Definition 4.1. The *kernel* of φ is $(\ker \varphi)(U) = \ker(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$.

Remark (Properties of the kernel)**.** It is straightforward to check that $\ker \varphi$ is a sheaf. Moreover:

1. $\ker \varphi$ satisfies the following universal property: For any morphism to a sheaf α such that $\varphi \circ \alpha = 0$, there exists a unique morphism α' such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\alpha"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\alpha'"'] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \\ & \ker \varphi \arrow[u] & \end{tikzcd}$$

   To see this, use the universal property of the kernel in the category of abelian groups.

2. Since filtered limits are exact, we have $(\ker \varphi_p) = (\ker \varphi)_p$ for all $p \in X$.

Lemma 4.1 (Injectivity for sheaves)**.** *The following are equivalent:*

1. *$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subseteq X$ open;*
2. *$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.*

We say that φ is injective *if either of these equivalent conditions hold.*

Proof. ($1 \Rightarrow 2$) This is clear.

($2 \Rightarrow 1$) Fix $s \in \mathcal{F}(U)$ with $\varphi(U)(s) = 0$. Then

$$\varphi_p(s_p) = (\varphi(U)(s))_p = 0$$

for all $p \in X$, so $s_p = 0$ for all $p \in U$, so $s = 0$ by homework from Algebraic Geometry I. $\square$

Example 4.1.1 (Subtleties for the image)**.** Consider the following:

1. Let $\varphi : \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}^n}^\times$. Then $U \mapsto \operatorname{im}(\mathcal{O}_{\mathbb{C}^n}(U) \to \mathcal{O}_{\mathbb{C}^n}^\times(U))$ is a presheaf but not a sheaf. This is because logarithms only exist locally.
2. Define $\varphi : \mathcal{O}_{\mathbb{P}^n} \to i_{p_1}\underline{k} \oplus i_{p_2}\underline{k}$ by $f \mapsto (f(p_1), f(p_2))$. Again $U \mapsto \operatorname{im}(\varphi(U))$ is not a sheaf.

Definition 4.2. Let $\widetilde{\operatorname{im}}\,\varphi = (U \mapsto \operatorname{im}(\varphi(U))$. This is a presheaf and $\widetilde{\operatorname{im}}\,\varphi \subseteq \mathcal{G}$. Then the *image* of φ is $\operatorname{im}\varphi = (\widetilde{\operatorname{im}}\,\varphi)^+$. By Remark 4.1, we can equivalently define

$$(\operatorname{im}\varphi)(U) = \{s \in \mathcal{G}(U) : \text{there exists cover } \{U_i\} \text{ of } U \text{ such that } s|_{U_i} \in \operatorname{im}(\mathcal{F}(U_i) \to \mathcal{G}(U_i))\}.$$

Remark. We have $\operatorname{im}(\varphi_x) \cong (\widetilde{\operatorname{im}}\,\varphi)_x \cong (\operatorname{im}\varphi)_x$, where the first isomorphism is because filtered direct limits are exact and the second isomorphism is because sheafification preserves stalks.

Definition 4.3. Let $\widetilde{\operatorname{coker}}(\varphi) = (U \mapsto \operatorname{coker}(\varphi(U)))$. Then the *cokernel* of φ is $\operatorname{coker}(\varphi) = (\widetilde{\operatorname{coker}}(\varphi))^+$.

Remark. We have the following:

1. $\operatorname{coker}(\varphi)_x \cong \operatorname{coker}(\varphi_x)$ (similar to above).
2. $\operatorname{coker}(\varphi)$ satisfies the universal property of the cokernel:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[rr, bend left, "0"] & \mathcal{G} \arrow[r, "p_0"] \arrow[d] & \mathcal{G}' \\
& \widetilde{\operatorname{coker}}(\varphi) \arrow[r] \arrow[ur, dashed] & \operatorname{coker}(\varphi) \arrow[u, dashed]
\end{tikzcd}$$

3. For a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, we can define the *quotient sheaf* $\mathcal{F}/\mathcal{F}' = \operatorname{coker}(\mathcal{F}' \hookrightarrow \mathcal{F})$.
4. By the universal property of the cokernel, we get natural maps

$$\begin{tikzcd}
\ker\varphi \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \operatorname{im}\mathcal{F} \\
& \mathcal{F}/\ker\varphi \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

As the following diagram commutes,

$$\begin{tikzcd}
(\mathcal{F}/\ker\varphi)_p \arrow[r, "\alpha_p"] \arrow[d, "\cong"] & (\operatorname{im}\varphi)_p \arrow[d, "\cong"'] \\
\mathcal{F}_p/(\ker\varphi)_p \arrow[r, "\cong"'] & \operatorname{im}(\varphi_p)
\end{tikzcd}$$

α_p is an isomorphism for all $p \in X$. So by HW, α is an isomorphism. So $\mathcal{F}/\ker\varphi \cong \operatorname{im}\varphi$.

Lemma 4.2 (Surjectivity for sheaves)**.** *The following are equivalent:*

1. $\operatorname{coker}\varphi = 0$*;*
2. $\operatorname{im}\varphi = \mathcal{G}$*;*
3. $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ *is surjective for all* $x \in X$*.*

We say that φ *is* surjective *if any of these equivalent conditions hold.*

Proof. ($3 \Leftrightarrow 1$) We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $(\operatorname{coker}\varphi)_x = 0$ for all $x \in X$, if and only if (1).

($3 \Leftrightarrow 2$) We have (3) if and only if $\operatorname{coker}(\varphi_x) = 0$ for all $x \in X$, if and only if $\operatorname{im}\varphi_x = \mathcal{G}_x$ for all $x \in X$, if and only if $(\operatorname{im}\varphi)_x \to \mathcal{G}_x$ is an isomorphism for all $x \in X$, if and only if (2) by HW. □

Remark. Note that if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all $U \subseteq X$, then φ is surjective. However, the converse is false in general.

Definition 4.4. A sequence of morphisms of sheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is *exact* at $\mathcal{G}$ if $\ker g = \operatorname{im} f$.

Lemma 4.3. *The following are equivalent:*

1. $\ker g = \operatorname{im} f$*;*
2. $\ker g_x = \operatorname{im} f_x$ *for all* $x \in X$*.*

Proof. Similar to above. □

4.3 Constructions with Sheaves

Definition 4.5. Let $\mathcal{F}_1, \mathcal{F}_2$ be sheaves on X. Then $\mathcal{F}_1 \oplus \mathcal{F}_2$ is a sheaf defined by

$$U \longmapsto \mathcal{F}_1(U) \oplus \mathcal{F}_2(U).$$

This is a *biproduct* in the category of sheaves.

Example 4.5.1. Let X be a variety with connected components $X_1, \ldots, X_n$. Then

$$\mathcal{O}_X \cong \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_n}$$

as sheaves of abelian groups.

Definition 4.6. Let $U \subseteq X$ be open. Then $\mathcal{F}_i|_U$ is a sheaf on U given by $V \mapsto \mathcal{F}_i(V)$

Definition 4.7. $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ is the sheaf $U \mapsto \mathrm{Hom}^{\text{sheaves}}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$.

Definition 4.8 (Gluing)**.** Let $\{U_i\}$ be an open cover of X with a sheaf $\mathcal{F}_i$ on each U_i and isomorphisms $\alpha_{i,j} : \mathcal{F}_j|_{U_{i,j}} \to \mathcal{F}_i|_{U_{i,j}}$ such that $\alpha_{i,j} = \alpha_{j,i}^{-1}$, $\alpha_{i,j} \circ \alpha_{j,k} = \alpha_{i,k}$, and $\alpha_{i,i} = \mathrm{id}$. Then there exists a sheaf $\mathcal{F}$ with isomorphisms $\beta_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}|_{U_{i,j}} \arrow[r, "\mathrm{id}"] \arrow[d, "\beta_j"'] & \mathcal{F}|_{U_{i,j}} \arrow[d, "\beta_i"] \\
\mathcal{F}_i|_{U_{i,j}} \arrow[r, "\alpha_{i,j}"'] & \mathcal{F}_j|_{U_{i,j}}
\end{tikzcd}$$

One can define $\mathcal{F}$ as follows and check that it satisfies the above properties:

$$\mathcal{F}(U) = \{(s_i)_{i \in I} : s_i \in \mathcal{F}(U_i) \text{ and } \alpha_{i,j}(s_j|_{U_{i,j}}) = s_i\}.$$

Lecture 5

Jan. 27 — $\mathcal{O}_X$-Modules

5.1 Sheaves and Continuous Maps

Remark. The category Sh_X of sheaves of abelian groups on a topological space X is an abelian category, i.e. it has or satisfies the following:

- zero object (the $\underline{0}$ sheaf);
- $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is an abelian group and composition is bilinear;
- finite biproducts exist;
- kernels and cokernels exist;
- the image coincides with the coimage.

Remark. For the rest of this section, let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y.

Definition 5.1. The *pushforward* $f_*\mathcal{F}$ is the sheaf on Y defined by

$$V \longmapsto \mathcal{F}(f^{-1}(V)).$$

Example 5.1.1. We have the following:

- If $i : \{p\} \hookrightarrow X$ and A is an abelian group, then $i_*\underline{A}$ is the skyscraper sheaf on X at p.
- If X is a variety and $i : Z \hookrightarrow X$ with Z a closed subvariety, then
$$i_*\mathcal{O}_Z(U) = \{\varphi : U \cap Z \to k : \varphi \text{ is regular}\}.$$

Definition 5.2. Let $\widetilde{f^{-1}\mathcal{G}}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, which is a presheaf. The *pullback* is $f^{-1}\mathcal{G} := (\widetilde{f^{-1}\mathcal{G}})^+$.

Example 5.2.1. Let $f : \{p\} \hookrightarrow X$. Then $\widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$, which is a sheaf, so $f^{-1}\mathcal{G} \cong \widetilde{f^{-1}\mathcal{G}} \cong \underline{\mathcal{G}_p}$.

Example 5.2.2. If $i : U \hookrightarrow X$ is the inclusion of an open set, then $i^{-1}\mathcal{F} \cong \mathcal{F}|_U$.

Remark. The pushforward f_* and pullback f^{-1} are functors:

$$\begin{tikzcd} \mathrm{Sh}_X \arrow[r, bend left, "f_*"] & \mathrm{Sh}_Y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

How are f_* and f^{-1} related? There are natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ induced by:

1. For $U \subseteq X$, define

$$\widetilde{f^{-1}} f_* \mathcal{F}(U) = \varinjlim_{V \supseteq f(U) \text{ open}} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$$
$$s \longmapsto s|_U.$$

2. For $V \subseteq Y$, define

$$\mathcal{G}(V) \longrightarrow \varinjlim_{V \supseteq V' \supseteq f(f^{-1}(V)) \text{ open}} \mathcal{G}(V') = (f_* \widetilde{f^{-1}} \mathcal{G})(V).$$

   Note that $V \supseteq f(f^{-1}(V))$, so we can add the $V \supseteq V' \supseteq f(f^{-1}(V))$ condition.

Another way to think about this is via adjoints.

Proposition 5.1. *For $\mathcal{F} \in \mathrm{Sh}_X$ and $\mathcal{G} \in \mathrm{Sh}_Y$, there exist functorial bijections*

$$\mathrm{Hom}_{\mathrm{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

i.e. (f^{-1}, f_*) is an adjoint pair.

Proof. Given $\phi : f^{-1}\mathcal{G} \to \mathcal{F}$, using map (2) from above we get

$$\mathcal{G} \xrightarrow{(2)} f_* f^{-1} \mathcal{G} \xrightarrow{f_*\phi} f_* \mathcal{F}.$$

Similarly, given $\psi : \mathcal{G} \to f_*\mathcal{F}$, using map (1) from above we get

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\psi} f^{-1} f_* \mathcal{F} \xrightarrow{(1)} \mathcal{F}.$$

One can (tediously) check that this gives a bijection. □

Remark. The following are consequences of adjointness:

- f_* is left exact, i.e. given an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

  in Sh_X, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

- f^{-1} is right exact (defined similarly with the left 0 missing).

One can also directly check these properties from the definitions.

5.2 Sheaves of $\mathcal{O}_X$-Modules

Remark. To use tools from commutative algebra, we want to consider modules. For the rest of this section, let X be a topological space with a sheaf of rings $\mathcal{O}_X$ (e.g. X a variety with $\mathcal{O}_X$ the sheaf of regular functions, or M a complex manifold with $\mathcal{O}_M$ the sheaf of holomorphic functions).

Definition 5.3. A *(pre)sheaf of $\mathcal{O}_X$-modules* is a (pre)sheaf $\mathcal{F}$ on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module compatible with restriction, i.e. such that

$$(a \cdot s)|_V = a|_V \cdot s|_V$$

for $V \subseteq U \subseteq X$ open, $a \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}(U)$.

Remark. We often say just "$\mathcal{O}_X$-module" to mean a "sheaf of $\mathcal{O}_X$-modules."

Definition 5.4. A *morphism* of (pre)sheaves of $\mathcal{O}_X$-modules $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of presheaves such that $\mathcal{M}(U) \to \mathcal{N}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules for all $U \subseteq X$.

Example 5.4.1. We have the following:

1. $\mathcal{O}_X$ has the structure of an $\mathcal{O}_X$-module (similar to how a ring A has the structure of an A-module).
2. Let X be a variety and $p : \mathbb{E} \to X$ a vector bundle. Let $\mathcal{E}$ be the sheaf of sections of p, with
$$[f \cdot s : U \to p^{-1}(U)] \in \mathcal{E}(U)$$
as the product of $f \in \mathcal{O}_X(U)$ and $[s : U \to p^{-1}(U)] \in \mathcal{E}(U)$. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module.

2'. Let $\mathbb{E} = \mathbb{1}_X$, then $\mathcal{E}(U) \cong \mathcal{O}_X(U)$ as $\mathcal{O}_X(U)$-modules. As the isomorphism is compatible with restriction, we have $\mathcal{E} \cong \mathcal{O}_X$.

3. Let $\mathcal{F}_1, \mathcal{F}_2$, be (pre)sheaves of $\mathcal{O}_X$-modules. Then so is $\mathcal{F}_1 \oplus \mathcal{F}_2$.

2". If $\mathbb{E}$ is a trivial vector bundle of rank e, then $\mathcal{E} \cong \mathcal{O}_X^{\oplus e}$.

4. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, then $\mathcal{F}^+$ is naturally a sheaf of $\mathcal{O}_X$-modules (use the definition of $\mathcal{F}^+$ in the proof).
5. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of $\mathcal{O}_X$-modules, then $\ker\varphi, \operatorname{im}\varphi, \operatorname{coker}\varphi$ are sheaves of $\mathcal{O}_X$-modules (use that $\ker\varphi, \widetilde{\operatorname{im}}\,\varphi, \widetilde{\operatorname{coker}}\,\varphi$ are presheaves of $\mathcal{O}_X$-modules and then use (4)).

   Furthermore, the category $\mathrm{Mod}_{\mathcal{O}_X}$ of sheaves of $\mathcal{O}_X$-modules is an abelian.
6. We can define the usual constructions on $\mathcal{O}_X$-modules: $\otimes$, Sym^d, $\wedge^d$, etc.

Example 5.4.2. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then their tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Definition 5.5. An $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* of rank e if for any $p \in X$, there exists $p \in U \subseteq X$ open such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus e}$. If $e = 1$, then we say that $\mathcal{F}$ is *invertible*.

Example 5.5.1. Let X be a variety and $p : \mathbb{E} \to X$ a vector bundle of rank e. For $p \in X$, there exists $p \in U \subseteq X$ open such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[d, "p"'] & U \times \mathbb{A}^e \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

Then $\mathcal{E}|_U \cong$ sheaf of sections of $U \times \mathbb{A}^e \cong \mathcal{O}_U^{\oplus e}$.

Remark (Transition functions)**.** Let $e = 1$ for simplicity, and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank e. Then there exists an open cover $\{U_i\}$ of X with isomorphisms $\alpha_i : \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus e}$. So on $U_{i,j} = U_i \cap U_j$, we get isomorphisms $\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}}^{\oplus e} \to \mathcal{O}_{U_{i,j}}^{\oplus e}$.

Lecture 6

Jan. 29 — $\mathcal{O}_X$-Modules, Part 2

6.1 More on $\mathcal{O}_X$-Modules

Remark. Recall that we have operations $\otimes$, $\oplus$, Sym^d, $\wedge^d$, $\mathcal{H}om(\cdot,\cdot)$ on $\mathcal{O}_X$-modules.

Example 6.0.1. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is the sheafification of

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This is again an $\mathcal{O}_X$-module.

Example 6.0.2. We have $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Exercise 6.1. The invertible sheaves on X up to isomorphism forms a group with multiplication given by $\otimes$, identity $\mathcal{O}_X$, and inverse $\mathcal{L}^{-1} = \mathcal{L}^\vee$.

Remark (Transition data)**.** Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then there exists an open cover $\{U_i\}$ of X and isomorphisms $\alpha_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$, so we get isomorphisms

$$\alpha_{i,j} = \alpha_i \circ \alpha_j^{-1} : \mathcal{O}_{U_{i,j}} \to \mathcal{O}_{U_{i,j}}.$$

For this to be an isomorphism, we must have $\alpha_{i,j}(U_{i,j})(1) = g_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$.

Proposition 6.1. *If X is a variety, then there is a bijection*

$$\begin{aligned} \{\textit{line bundles on } X\}/\cong &\longrightarrow \{\textit{invertible sheaves on } X\}/\cong \\ \mathbb{L} &\longmapsto \mathcal{L} \end{aligned}$$

Proof. To get the reverse map, fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with trivialization data $(U_i, g_{i,j})$. Send it to the line bundle with the same trivialization data. Check that this is well-defined as an exercise.

To show that this gives an inverse, it suffices to show that if $\mathbb{L}$ is a line bundle with trivialization data $(U_i, g_{i,j})$, then the sheaf $\mathcal{L}$ of sections of $\mathbb{L}$ has the same trivialization data. The trivializations

$$\mathbb{L}|_{U_i} \xrightarrow[\cong]{\phi_i} U_i \times \mathbb{A}^1$$

give isomorphisms $U_{i,j} \times \mathbb{A}^1 \to U_{i,j} \times \mathbb{A}^1$ by $(x, v) \mapsto (x, g_{i,j}(x)v)$. We get an isomorphism

$$\alpha_i : \mathcal{L}|_{U_i} \xrightarrow[\cong]{\phi_i} \mathcal{O}_{U_i}$$

where $e_i := \alpha_i^{-1}(1) = [U_i \xrightarrow{x \mapsto (x,1)} U_i \times \mathbb{A}^1 \xrightarrow{\phi_i^{-1}} \mathbb{L}|_{U_i}]$. Now we have

$$\begin{array}{ccccc} \mathcal{O}_{U_{i,j}} & \xrightarrow{\alpha_j^{-1}} & \mathcal{L}|_{U_{i,j}} & \xrightarrow{\alpha_i} & \mathcal{O}_{U_{i,j}} \\ 1 & \longmapsto & e_j = e_i g_{i,j} & \longmapsto & g_{i,j}. \end{array}$$

So we get the same transition functions $(U_i, g_{i,j})$ for $\mathcal{L}$. $\square$

Remark. Given a morphism of rings $\phi : A \to B$, we have functors

$$\begin{tikzcd} \mathrm{Mod}_B \arrow[r, bend left, "\Phi"] & \mathrm{Mod}_A \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

given as follows:

1. *Extension of scalars*: $\mathrm{Mod}_A \ni M \longmapsto M \otimes_A B \in \mathrm{Mod}_B$, where the multiplication by B is
$$c(m \otimes b) = m \otimes (cb).$$
For example, if $M = A^{\oplus I}$, then $M \otimes_A B = B^{\oplus I}$.
2. *Restriction of scalars*: $\mathrm{Mod}_B \ni N \longmapsto N_A \in \mathrm{Mod}_A$, where $N_A := N$ as abelian groups with
$$a \cdot n = \phi(a)n$$
as the multiplication by A.

Proposition 6.2. *There is a functorial bijection*

$$\begin{aligned} \mathrm{Hom}_B(M \otimes_A B, N) &\longleftrightarrow \mathrm{Hom}_A(M, N_A) \\ [f : M \otimes_A B \to N] &\longmapsto [m \mapsto f(m \otimes 1)] \\ [m \otimes b \mapsto b \cdot g(m)] &\longleftarrow [g : M \to N_A] \end{aligned}$$

for $M \in \mathrm{Mod}_A$ and $N \in \mathrm{Mod}_B$.

Remark. Given the result for rings, we want a similar statement for $\mathcal{O}_X$-modules.

6.2 $\mathcal{O}_X$-Modules and Continuous Maps

Definition 6.1. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of

1. a continuous map $f : X \to Y$,
2. a morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$.

Example 6.1.1. If $X \to Y$ is a morphism of varieties, then $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is given for $U \subseteq Y$ open by

$$\begin{aligned} \mathcal{O}_Y(U) &\longrightarrow \mathcal{O}_X(f^{-1}(U)) \\ \varphi &\longmapsto f^* \varphi. \end{aligned}$$

Remark. Our goal will be to define functors

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}_X} \arrow[r, bend left, "f_*"] & \mathrm{Mod}_{\mathcal{O}_Y} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Remark (Pushforward)**.** Given an $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf pushforward $f_*\mathcal{F}$ is naturally an $f_*\mathcal{O}_X$-module. Via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, we get an $\mathcal{O}_Y$-module structure on $f_*\mathcal{F}$. More concretely, for $U \subseteq Y$ open, $s \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and $a \in \mathcal{O}_Y(U)$, we can define

$$a \cdot s = f^\#(U)(a) \cdot s.$$

Remark (Pullback)**.** Given an $\mathcal{O}_Y$-module $\mathcal{G}$, we get an $f^{-1}\mathcal{O}_Y$-module $f^{-1}\mathcal{G}$. By the adjoint property for (f^{-1}, f_*), the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ corresponds to a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. So we get

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is an $\mathcal{O}_X$-module. Thus we get a functor $f^* : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Proposition 6.3. *The pair (f^*, f_*) are adjoint functors.*

Proof. Similar to before. □

Example 6.1.2. Recall that if $A \to B$ is a morphism of rings, then $A \otimes_A B \cong B$. In our setting, we get

$$f^*\mathcal{O}_Y = (f^{-1}\mathcal{O}_Y \mathbin{\widetilde{\otimes}}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)^+ \cong \mathcal{O}_X^+ \cong \mathcal{O}_X.$$

Similarly, we have $f^*(\mathcal{O}_Y^{\oplus I}) \cong (f^*\mathcal{O}_Y)^{\oplus I} \cong \mathcal{O}_X^{\oplus I}$ (as left-adjoint functors commute with coproducts).

Remark. If $\mathcal{E}$ is a locally free rank m $\mathcal{O}_X$-module, then $f^*\mathcal{E}$ is a locally free rank m $\mathcal{O}_Y$-module, (as f^* can be computed locally on Y using Example 6.1.2).

Lecture 7

Feb. 3 — Coherent Sheaves

7.1 Review of Localization

Remark. Let A be a ring and $S \subseteq A$ a multiplicative system (i.e. $1 \in S$, and $a, b \in S$ implies $ab \in S$). For example, we could take $S = \langle f \rangle = (1, f, f^2, \dots)$ for $f \in A$ or $S = A \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \leq A$.

Definition 7.1. The *localization* of A at S is

$$S^{-1}A = \{a/s : a \in A, s \in S\},$$

where $a/s = a'/s'$ if and only if $t(as' - a's) = 0$ for some $t \in S$.

Remark. The localization satisfies the following universal property:

$$\begin{tikzcd} A \arrow[r, "a \mapsto a/1"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, dashed, "\exists!"] \\ & T \end{tikzcd}$$

whenever $f(S)$ lands in the units of T.

Definition 7.2. For an A-module M, the *localization* of M at S is

$$S^{-1}M = \{m/s : m \in M, s \in S\},$$

where $m/s = m'/s'$ if and only if $t(s'm - sm') = 0$ for some $t \in S$.

Remark. For $S = \langle f \rangle$, we will write $S^{-1}M = M_f$. For $S = A \setminus \mathfrak{p}$, we will write $S^{-1}M = M_{\mathfrak{p}}$.

Proposition 7.1. *We have the following properties for localization:*

1. *There is an isomorphism*

$$\begin{aligned} M \otimes_A S^{-1}A &\xrightarrow{\cong} S^{-1}M \\ m \otimes (a/s) &\longmapsto (am)/s. \end{aligned}$$

2. *Localization gives an exact functor*

$$\begin{aligned} \mathrm{Mod}_A &\longrightarrow \mathrm{Mod}_{S^{-1}A} \\ M &\longmapsto S^{-1}M. \end{aligned}$$

3. *A sequence in* Mod_A

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$$

is exact for all maximal (equivalently, prime) ideals $\mathfrak{p} \leq A$.

Example 7.2.1. Recall that if X is an affine variety, then

$$\mathcal{O}_X(D(f)) \cong A(X)_f \quad \text{and} \quad \mathcal{O}_{X,x} \cong A(X)_{\mathfrak{m}_x},$$

where $\mathfrak{m}_x = I(\{x\}) \leq A(X)$.

7.2 Coherent Sheaves on Affine Varieties

Remark. For this section, let X be an affine variety and $A = \mathcal{O}_X(X) = A(X)$. We want a functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathcal{O}_X}.$$

Theorem 7.1. *For* $M \in \mathrm{Mod}_A$*, there exists a unique* $\mathcal{O}_X$*-module* $\widetilde{M}$ *such that*

1. $\widetilde{M}(D(f)) \cong M_f$*;*
2. *for* $D(g) \subseteq D(f)$*, we have*

$$\begin{tikzcd} \widetilde{M}(D(f)) \arrow[r] \arrow[d, "="'] & \widetilde{M}(D(g)) \arrow[d, "="] \\ M_f \arrow[r, "\text{natural map}"'] & M_g \end{tikzcd}$$

Remark. How is the natural map $M_f \to M_g$ defined? If $D(g) \subseteq D(f)$, then we have $V(g) \supseteq V(f)$, so $\sqrt{(g)} \subseteq \sqrt{(f)}$. Thus $g \in \sqrt{(f)}$, so $g^d = fh$ for some $d > 0$ and $h \in A$. So we get a map

$$\begin{aligned} M_f &\longrightarrow M_g \\ m/f^i &\longmapsto mh^i/g^{di}. \end{aligned}$$

Alternatively, if $D(g) \subseteq D(f)$, then $D(g) = D(gf)$, so we could instead consider

$$\begin{tikzcd} \widetilde{M}(D(f)) \arrow[r] \arrow[d, "="'] & \widetilde{M}(D(fg)) \arrow[d] \\ M_f \arrow[r, "m/f^i \mapsto mg^i/(fg)^i"'] & M_{fg} \end{tikzcd}$$

Remark. To construct $\widetilde{M}$, we need the notion of *sheaves on a basis*. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{P}$ a collection of open sets in X such that

1. $\mathcal{P}$ is a basis for X;

2. if $U, V \in \mathcal{P}$, then $U \cap V \in \mathcal{P}$.

Example 7.2.2. If $(X, \mathcal{O}_X)$ is an affine variety, we can take $\mathcal{P} = \{D(f) : f \in A(X)\}$. Also, if $(X, \mathcal{O}_X)$ is a general algebraic variety, then we can take $\mathcal{P}$ to be the affine open subsets of X (as X is separated, the intersection of two affine open subsets is again affine open).

Definition 7.3. A *$\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules* $\mathcal{F}$ on X is the data of:

- an $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ for each $U \in \mathcal{P}$,
- homomorphisms of abelian groups $\mathcal{F}(U) \to \mathcal{F}(V)$ for all $U, V \in \mathcal{P}$ with $V \subseteq U$

satisfying the following properties:

- the multiplication is compatible with restriction,
- the sheaf axiom with respect to open sets in $\mathcal{P}$.

Example 7.3.1. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we get $\mathcal{F}^{\mathcal{P}}$, a $\mathcal{P}$-sheaf of $\mathcal{O}_X$-modules.

Theorem 7.2. *There is an equivalence of categories*

$$\begin{aligned} \mathrm{Mod}_{\mathcal{O}_X} &\longrightarrow \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}} \\ \mathcal{F} &\longmapsto \mathcal{F}^{\mathcal{P}}. \end{aligned}$$

In particular, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^{\mathcal{P}}, \mathcal{G}^{\mathcal{P}})$ *is a bijection, and for any* $\mathcal{H} \in \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}}$*, there exists some* $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ *such that* $\mathcal{F}^{\mathcal{P}} \cong \mathcal{H}$*.*

Proof. We construct the inverse functor. Take $\mathcal{H} \in \mathrm{Mod}_{\mathcal{O}_X}^{\mathcal{P}}$, and define $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ by setting

$$\mathcal{F}(U) = \{(s_V)_{V \in \mathcal{P}, V \subseteq U} : s_V|_{V \cap V'} = s'_V|_{V \cap V'} \text{ for all } V, V' \in \mathcal{P} \text{ with } V, V' \subseteq U\}.$$

One can check that this defines a functor and an equivalence of categories. □

Remark. Returning to algebraic geometry, let X be an affine variety, $A = A(X)$, and M an A-module.

Proposition 7.2. *For* $\mathcal{P} = \{D(f) : f \in A\}$*, the assignment* $D(f) \mapsto M_f$ *with the natural restriction maps defines a* $\mathcal{P}$*-sheaf of* $\mathcal{O}_X$*-modules.*

Proof. See Mustata Lemma 8.3.2. The hard part is to check the sheaf axiom for $\mathcal{P}$, which is similar to the computation that $\mathcal{O}_X(D(f)) \cong A_f$ for an affine variety X. □

Proof of Theorem 7.1. Combining Theorem 7.2 and Proposition 7.2, we get an $\mathcal{O}_X$-module $\widetilde{M}$ such that $\widetilde{M}(D(f)) \cong M_f$, and it is unique up to isomorphism. □

Example 7.3.2. We have $\widetilde{A} \cong \mathcal{O}_X$, since

$$\widetilde{A}(D(f)) \cong A_f \cong \mathcal{O}_X(D(f)).$$

Similarly, one can check that $\widetilde{A^{\oplus I}} \cong \widetilde{\mathcal{O}_X^{\oplus I}}$.

Exercise 7.1. For $Z \subseteq X$ closed and $I(Z) \leq A(X)$, we have $\widetilde{I(Z)} \cong \mathcal{I}_Z$ (the ideal sheaf of Z).

Remark. We have the following:

1. For $x \in X$ with $\mathfrak{p} := I(\{x\}) \leq A$, we have $\widetilde{M}_x \cong \varinjlim_{D(f) \ni x} \widetilde{M}(D(f)) = \varinjlim_{f \in A \setminus \mathfrak{p}} M_f \cong M_{\mathfrak{p}}$.
2. For an A-module homomorphism $\varphi : M \to N$, we get homomorphisms

$$\begin{array}{ccc} \widetilde{M}(D(f)) & \longrightarrow & \widetilde{N}(D(f)) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ M_f & \xrightarrow[\text{natural map}]{} & N_f \end{array}$$

which are compatible with restriction. So we get a homomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$. One can check that this gives a functor $\Phi : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$.

Proposition 7.3. *We have the following:*

1. *Φ is exact;*
2. *Φ is fully faithful, i.e. the following map is a bijection:*

$$\begin{aligned} \mathrm{Hom}_A(M, N) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \\ \varphi &\longmapsto \widetilde{\varphi}. \end{aligned}$$

Proof. (1) If $0 \to M' \to M \to M'' \to 0$ is exact, then $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$ is exact for every $\mathfrak{p} \leq A$ maximal by Proposition 7.1(3). So the sequence

$$0 \longrightarrow \widetilde{M}'_x \longrightarrow \widetilde{M}_x \longrightarrow \widetilde{M}''_x \longrightarrow 0$$

is exact for all $x \in X$. So $0 \to \widetilde{M}' \to \widetilde{M} \to \widetilde{M}'' \to 0$ is exact.

(2) We want a map $\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_A(M, N)$. Given an $\mathcal{O}_X$-module homomorphism $\varphi : \widetilde{M} \to \widetilde{N}$, we get $f = \varphi(X) : \widetilde{M}(X) = M \to \widetilde{N}(X) = N$. We want to show that $\widetilde{f} = \varphi$. We have

$$\begin{array}{ccc} \widetilde{M}(X) & \xrightarrow{\varphi(X)} & \widetilde{N}(X) \\ \big\downarrow & & \big\downarrow \\ \widetilde{M}(D(g)) & \xrightarrow[\varphi(D(f))]{} & \widetilde{N}(D(g)) \end{array}$$

and this diagram commutes. So we get $\widetilde{f}(D(g)) = \varphi(D(g))$. $\square$

Remark. By Proposition 7.3(1), given an A-module homomorphism $f : M \to N$, we have

$$\ker(\widetilde{f}) \cong \widetilde{\ker(f)}, \quad \mathrm{coker}(\widetilde{f}) \cong \widetilde{\mathrm{coker}(f)}, \quad \mathrm{im}(\widetilde{f}) \cong \widetilde{\mathrm{im}(f)}.$$

Proposition 7.4. *For $M, N \in \mathrm{Mod}_A$, we have*

1. *$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_A N}$;*
2. *$\widetilde{\mathrm{Hom}_A(M, N)} \cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$;*
3. *$\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M}_i$.*

Proof. (1) We have a homomorphism

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{\cong} & \Gamma(X, \widetilde{M} \, \widetilde{\otimes}_{\mathcal{O}_X} \widetilde{N}) \\ & & \downarrow \\ & & \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) \end{array}$$

This gives a homomorphism of $\mathcal{O}_X$-modules $\widetilde{M \otimes_A N} \to \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Now at the stalks for $x \in X$ with $\mathfrak{m} = I(\{x\}) \leq A(X)$, we can see that

$$(\widetilde{M \otimes_A N})_x \cong (M \otimes_A N) \otimes_A A_{\mathfrak{m}} \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Similarly, we have

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x \cong \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Thus we have isomorphisms at the stalks. □

Remark. The functor $\Phi : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}$ is left adjoint to $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_A$ given by $\mathcal{F} \mapsto \mathcal{F}(X)$.

Lecture 8

Feb. 5 — Coherent Sheaves, Part 2

8.1 Coherent Sheaves on Affine Varieties, Continued

Proposition 8.1. *Let $\varphi : X \to Y$ be a morphism of affine varieties, and let*

$$A = \mathcal{O}_Y(Y) \xrightarrow{\varphi^\#} \mathcal{O}_X(X) = B$$

be the corresponding ring homomorphism on the coordinate rings. Then:

1. *For $N \in \mathrm{Mod}_B$, we have $\varphi_* \widetilde{N} = \widetilde{N_A}$.*
2. *For $M \in \mathrm{Mod}_A$, we have $\varphi^* \widetilde{M} = \widetilde{M \otimes_A B}$.*

Proof. (1) For $f \in A = \mathcal{O}_Y(Y)$, the left-hand side is given on $D(f)$ by

$$(\varphi_* \widetilde{N})(D(f)) = \widetilde{N}(\varphi^{-1}(D(f))) = \widetilde{N}(D(\varphi^\# f)) = N_{\varphi^\# f} = (N_A)_f.$$

So we get that $\varphi_* \widetilde{N} = \widetilde{N_A}$.

(2) (Hartshorne says this holds by definition lol.) First assume $M = A^{\oplus I}$. Then

$$\varphi^* \widetilde{M} = \varphi^*(\mathcal{O}_Y^{\oplus I}) = (\varphi^* \mathcal{O}_Y)^{\oplus I} = \mathcal{O}_X^{\oplus I} = \widetilde{B^{\oplus I}} = \widetilde{M \otimes_A B}.$$

For an arbitrary A-module M, we use the following:

> **Claim:** There exists an exact sequence
>
> $$A^{\oplus J} \xrightarrow{\alpha} A^{\oplus I} \xrightarrow{\beta} M \longrightarrow 0$$
>
> *Proof of claim.* Choose generators $(m_i)_{i \in I}$ for M, and set $\beta(e_i) = m_i$. Choose generators $(n_j)_{j \in J}$ for $\ker \beta$. Then we can set $\alpha(f_j) = n_j$. □

Apply $\cdot \otimes_A B$ (which is right exact) to the exact sequence from the claim to get

$$B^{\oplus J} \longrightarrow B^{\oplus I} \longrightarrow M \otimes_A B \longrightarrow 0.$$

Applying $\varphi^*(\widetilde{\cdot})$ (note that $\widetilde{\cdot}$ is exact and φ^* is right exact) to the original sequence to get

$$\begin{tikzcd} \varphi^*(\widetilde{A^{\oplus J}}) \arrow[r] \arrow[d, equal] & \varphi^*(\widetilde{A^{\oplus I}}) \arrow[r] \arrow[d, equal] & \varphi^* \widetilde{M} \arrow[r] & 0 \\ B^{\oplus J} & B^{\oplus I} & & \end{tikzcd}$$

using the previous case. Thus $\varphi^* \widetilde{M} = \mathrm{coker}(\widetilde{B^{\oplus J}} \to \widetilde{B^{\oplus I}}) = (\mathrm{coker}(B^{\oplus J} \to B^{\oplus I}))^\sim = \widetilde{M \otimes_A B}$. □

8.2 Quasicoherent and Coherent Sheaves

Remark. For the rest of this lecture, assume $(X, \mathcal{O}_X)$ is a variety (not necessarily affine).

Definition 8.1. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* if there exists an affine cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $\mathcal{O}_X(U_i)$-module M_i. It is *coherent* if the M_i are finitely generated $\mathcal{O}_X(U_i)$-modules.

Remark. We have $M_i \cong \widetilde{M_i}(U_i) \cong \mathcal{F}(U_i)$, so we may replace the $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ condition with one of the following: $\mathcal{F}|_{U_i} \cong \widetilde{\mathcal{F}(U_i)}$, or $\mathcal{F}(U_i)_f \to \mathcal{F}(D_{U_i}(f))$ is an isomorphism for all $f \in \mathcal{O}_X(U_i)$.

Example 8.1.1. We have the following:

1. If $\mathcal{E}$ is a locally free sheaf of rank r on X, then there exists an open cover $X = \bigcup U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_X^{\oplus r}|_{U_i}$. Refining the cover, we may assume the U_i are affine, so
$$\mathcal{E}|_{U_i} = \widetilde{\mathcal{O}_X(U_i)^{\oplus r}}.$$
Thus we see that $\mathcal{E}$ is coherent.

2. For $Z \hookrightarrow X$ a closed embedding, $\mathcal{I}_Z \subseteq \mathcal{O}_X$ is coherent.

3. For $\mathcal{F}$ a (quasi)coherent $\mathcal{O}_X$-module and $U \subseteq X$ open, $\mathcal{F}|_U$ is (quasi)coherent.

   To see this, use that if $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for M_i an $\mathcal{O}_X(U_i)$-module and $f \in \mathcal{O}_X(U_i)$, then
$$\mathcal{F}|_{D_{U_i}(f)} \cong \widetilde{(M_i)_f}.$$
   Furthermore, if M_i is finitely generated, then so is $(M_i)_f$. So by refining our open affine cover in with principal opens (which form a basis) we may assume $U = \bigcup_{i, U_i \subseteq U} U_i$. This gives the result.

Proposition 8.2 (Key proposition)**.** *Let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$. The following are equivalent:*

1. *$\mathcal{F}$ is quasicoherent (resp. coherent).*

2. *For any affine open set $U \subseteq X$, we have $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$ (with $\mathcal{F}(U)$ finitely generated in the coherent case).*

Lemma 8.1 (Affine communication)**.** *If X is a variety, $U, V \subseteq X$ affine open subsets, and $p \in U \cap V$, then there exists an open set $p \in W \in U \cap V$ that is a principal open of both U and V.*

Proof. Choose a principal open of U with $p \in W_1 = D_U(h) \subseteq U \cap V$ for some $h \in \mathcal{O}_X(U)$, and choose a principal open of V with $p \in W = D_V(g) \subseteq W_1$ for some $g \in \mathcal{O}_X(V)$. Now
$$g|_{W_1} = \mathcal{O}_X(W_1) = \mathcal{O}_X(U)_h,$$
so $g|_{W_1} = f/h^i$ for some $f \in \mathcal{O}_X(U)$ and $i \geq 0$. Now $D_V(g) = W = D_{W_1}(g|_{W_1}) = D_U(fh)$. □

Proof of Proposition 8.2. $(2 \Rightarrow 1)$ This is clear.

$(1 \Rightarrow 2)$ Assume $\mathcal{F} \in \mathrm{QCoh}_X$. So there exists an open affine cover $\{U_i\}$ such that $\mathcal{F}|_{U_i} = \widetilde{\mathcal{F}(U_i)}$. Fix $U \subseteq X$ affine open. By refining the cover, we may assume that $U = \bigcup_{U_i \subseteq U} U_i$. Now replacing X with U, we may assume that $X = U$. Using Lemma 8.1, we may assume $U_i = D(f_i)$ for some $f_i \in \mathcal{O}_X(X)$.

So now X is affine, $X = \bigcup_{i=1}^r D(f_i)$, and $\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))}$. We want to show that for any $f \in A$, the natural map $\mathcal{F}(X)_f \to \mathcal{F}(D(f))$ is an isomorphism (this would imply $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$). Now

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_i))_f & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_if_j))_f \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \bigoplus_i \mathcal{F}(D(f_if)) & \longrightarrow & \bigoplus_{i,j} \mathcal{F}(D(f_if_jf)) \end{array}$$

where the first row is exact by the sheaf property and using that localization is exact, and the second row is exact by the sheaf property. Note that β and γ are isomorphisms as

$$\mathcal{F}|_{D(f_i)} \cong \widetilde{\mathcal{F}(D(f_i))} \quad \text{and} \quad \mathcal{F}|_{D(f_if_j)} \cong \widetilde{\mathcal{F}(D(f_if_j))}.$$

So by the five lemma, α is an isomorphism, and thus $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$. $\square$

Remark. For the coherent case, use the fact that if M is an A-module, $A = (f_1, \ldots, f_r)$, and M_{f_i} is finitely generated for $i = 1, \ldots, r$, then M is finitely generated.

Proposition 8.3. *We have the following:*

1. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of (quasi)coherent sheaves, then* $\ker\varphi, \operatorname{im}\varphi, \operatorname{coker}\varphi$ *are (quasi)coherent.*
2. *If $\mathcal{F}$ and $\mathcal{G}$ are (quasi)coherent, then so are $\mathcal{F} \otimes \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.*
3. *If $\mathcal{F}_i$ is quasicoherent for $i \in I$, then $\bigoplus_{i\in I} \mathcal{F}_i$ is quasicoherent. Furthermore, if $\mathcal{F}_i$ is coherent and $|I| < \infty$, then $\bigoplus_{i\in I} \mathcal{F}_i$ is coherent.*

Proof. (1) Choose $U \subseteq X$ affine open. So we can write $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Furthermore, $\varphi|_U = \widetilde{\alpha}$ for some $\mathcal{O}_X(U)$-module homomorphism $\alpha : M \to N$. Now

$$(\ker\varphi)|_U = \ker\varphi|_U = \ker\widetilde{\alpha} = \widetilde{\ker\alpha}.$$

Furthermore, if $\mathcal{F}$ and $\mathcal{G}$ are coherent, then M and N are finitely generated, so $\ker\alpha$ is also finitely generated. One can show the same for $\operatorname{im}\varphi$ and $\operatorname{coker}\varphi$ similarly. $\square$

Remark. We have full subcategories $\mathrm{Coh}_X \subseteq \mathrm{QCoh}_X \subseteq \mathrm{Mod}_{\mathcal{O}_X}$, which are abelian.

Lecture 9

Feb. 10 — Coherent Sheaves, Part 3

9.1 Quasicoherent Sheaves, Continued

Proposition 9.1. *Let $f : X \to Y$ be a morphism of varieties. Then*

1. *If $\mathcal{G} \in \mathrm{QCoh}_Y$ (resp. Coh_Y), then $f^*\mathcal{G} \in \mathrm{QCoh}_X$ (resp. Coh_X).*
2. *If $\mathcal{F} \in \mathrm{QCoh}_X$, then $f_*\mathcal{F} \in \mathrm{QCoh}_Y$.*

Proof. (1) Fix $\mathcal{G} \in \mathrm{QCoh}_Y$ and $x \in X$. There exist affine opens $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ such that $f(U) \subseteq V$. Write $g = f|_U : U \to V$, then $(f^*\mathcal{G})|_U = g^*(\mathcal{G}|_V)$. Now letting $A = \mathcal{O}_Y(V)$, $B = \mathcal{O}_X(U)$, $M = \mathcal{G}(V)$, we see that

$$f^*\mathcal{G}|_U = g^*(\mathcal{G}|_V) = g^*\widetilde{M} = \widetilde{M \otimes_A B}.$$

So $f^*\mathcal{G}$ is quasicoherent. The coherent version follows from the fact that if M is a finitely generated A-module, then $M \otimes_A B$ is a finitely generated B-module.

(2) Fix $\mathcal{F} \in \mathrm{QCoh}_X$. We can check quasicoherence locally, so we can reduce to the case where Y is affine (cover Y by affine opens U_i and replace f with $f^{-1}(U_i) \to U_i$). Choose an affine cover $X = U_1 \cup \cdots \cup U_r$, and note that $U_i \cap U_j$ is again affine for a variety. Write $\alpha_i : U_i \hookrightarrow X$ and $\alpha_{i,j} : U_{i,j} \hookrightarrow X$. As $\mathcal{F}$ is a sheaf, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \textstyle\bigoplus_i (\alpha_i)_*\mathcal{F}|_{U_i} \longrightarrow \textstyle\bigoplus_{i,j} (\alpha_{i,j})_*\mathcal{F}|_{U_{i,j}}.$$

Applying f_*, which is left exact, we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \textstyle\bigoplus_i (f \circ \alpha_i)_*\mathcal{F}|_{U_i} \longrightarrow \textstyle\bigoplus_{i,j} (f \circ \alpha_{i,j})_*\mathcal{F}|_{U_{i,j}}.$$

Note that the last two terms are both quasicoherent (e.g. $\mathcal{F}|_{U_i}$ is quasicoherent and $f \circ \alpha_i$ is a morphism of affine varieties, so $(f \circ \alpha_i)_*\mathcal{F}|_{U_i}$ is quasicoherent).[1] □

Remark. The coherent version of Proposition 9.1(2) fails: For

$$i : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1,$$

the pushforward $i_*\mathcal{O}_{\mathbb{A}^1\setminus\{0\}}$ is not coherent (but $\mathcal{O}_{\mathbb{A}^1\setminus\{0\}}$ is coherent).

[1] If $f : X \to Y$ is a morphism of affine varieties, and $A := \mathcal{O}_Y(Y)$, $B := \mathcal{O}_X(X)$, then $f_*\widetilde{N} = \widetilde{N_A}$ for $N \in \mathrm{Mod}_B$ and $f^*\widetilde{M}$ for $M \in \mathrm{Mod}_A$. This shows that pushforwards and pullbacks of quasicoherent sheaves on *affine* varieties are again quasicoherent.

Remark. If $X \to Y$ is *projective*, i.e. there exists a factorization

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & Y \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\ & Y \end{tikzcd}$$

then we do get a coherent version for the pushfoward $f_*\mathcal{F}$. One can prove this via sheaf cohomology.

9.2 Morphisms to Projective Space

Remark. Recall that there is a bijection

$$\{\mathbb{L} \to X \text{ with } s_0, \ldots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}/\cong \longleftrightarrow \{\text{morphisms } X \to \mathbb{P}^n\}.$$

We want to rephrase this using the bijection

$$\{\mathcal{L} \text{ invertible sheaves of } \mathcal{O}_X\text{-modules}\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \to X\}$$

and determine when $X \to \mathbb{P}^n$ is injective or a closed embedding.

Remark. Let X be a variety and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. For $x \in X$, we have

$$\mathcal{L}(x) := \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x \cong \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is a maximal ideal.

Definition 9.1. We say $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$ if

$$s_0(x), \ldots, s_n(x) \in \mathcal{L}(x) \cong k$$

generate $\mathcal{L}(x)$ as a k-vector space (i.e. at least one of $s_0(x), \ldots, s_n(x)$ is nonzero) for all $x \in X$.

Proposition 9.2. *The following are equivalent:*

1. $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$*;*
2. *the morphism* $\varphi : \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ *given by* $e_i \mapsto s_i$ *is surjective;*
3. *for every* $U \subseteq X$ *open affine,* $s_0|_U, \ldots, s_n|_U$ *generate* $\mathcal{L}(U)$ *as an* $\mathcal{O}_X(U)$*-module.*

Proof. By commutative algebra (in particular Nakayama's lemma), $s_0(x), \ldots, s_n(x)$ span $\mathcal{L}(x)$ (1) if and only if $(s_0)_x, \ldots, (s_n)_x$ generate $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. This happens if and only if φ_x is surjective, which happens if and only if φ is surjective (2). This happens if and only if $\varphi(U)$ is surjective for all $U \subseteq X$ open affine (3) (note that $\operatorname{coker}(\widetilde{M} \to \widetilde{N} = (\operatorname{coker}(M \to N))^\sim)$. $\square$