MATH 6421: Algebraic Geometry II

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Contents

Lecture 1

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1.1 Course Overview

Remark. This course will cover the following topics:

(i) vector bundles and line bundles in algebraic geometry;

(ii) coherent sheaves;

(iii) differentials;

(iv) sheaf cohomology: in particular, we will see that $H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T^*_X))$;

(v) the Riemann-Roch theorem: if $\omega = f\,dz$ is a rational 1-form on a smooth projective curve C, then

$$(\#\text{ zeroes of }\omega) - (\#\text{ poles of }\omega) = 2\,\mathrm{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example, $\operatorname{Spec}\mathbb{Z}$ has points corresponding to the primes p and 0.

1.2 Review of Algebraic Geometry I

Remark. Let $k = \overline{k}$ be an algebraically closed field.

Remark (Hilbert's Nullstellensatz)**.** There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \ldots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence, Z being irreducible (resp. a point) corresponds to $I(Z)$ being prime (resp. maximal).

Remark (Zariski topology on $\mathbb{A}^n$)**.** The closed sets in $\mathbb{A}^n$ are of the form $V(J)$, and this induces a Zarisiki topology on any subset of $\mathbb{A}^n$.

Remark (Embedded affine varieties)**.** Let $J \leq k[x_1, \ldots, x_n]$. Then we can associate to J a ringed space $(X, \mathcal{O}_X)$ by setting $X := V(J) \subseteq \mathbb{A}^n$ with the Zariski topology, and $\mathcal{O}_X$ to be the sheaf of regular functions on X, i.e. for $U \subseteq X$ open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

Here, $\varphi : U \to k$ is *regular* if for each $p \in U$, there exists an open set $p \in U_p \subseteq U$ and $f, g \in k[x_1, \ldots, x_n]$ such that $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$.

Remark (Coordinate ring)**.** The *coordinate ring* of X is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \ldots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for $A(X)$:

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

Remark (Distinguished open sets)**.** The *distinguished open sets* of X are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These for a basis for X as we vary $f \in A(X)$.

Definition 1.1. An *affine variety* is a ringed space $(X, \mathcal{O}_X)$ (here $\mathcal{O}_X$ is a sheaf of k-valued functions) which is isomorphic to an embedded affine variety.

Example 1.1.1. If $(X, \mathcal{O}_X)$ is an affine variety and $f \in \mathcal{O}_X(X)$, then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that $X = V(J) \subseteq \mathbb{A}^n$ with $J \leq k[x_1, \ldots, x_n]$ a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}^n_{x_i} \times \mathbb{A}^1_y \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \ldots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \ldots, x_n]/J)[y]}{(\overline{f}y - 1)} \cong A(X)_f.$$

Theorem 1.1. *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.-}k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

This implies the following:

1. *There is a bijection*

$$\begin{aligned} \operatorname{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \operatorname{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated k-algebra A, there exists an affine variety with $A \cong A(X)$.*

Remark. How can we explicitly define the inverse functor Red-f.g.-k-alg$^{\text{op}} \to$ Aff-var? We can define this as $A \mapsto (X, \mathcal{O}_X)$, where X is the set of maximal ideals of A. Think about what $\mathcal{O}_X$ should be.

Remark (Varieties)**.** A *variety* $(X, \mathcal{O}_X)$ is a ringed space such that

- there exists a finite open cover of X by affine varieties,
- the diagonal Δ_X is closed in $X \times X$.

Example 1.1.2. The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$.

Remark (Projective spaces)**.** Recall that $\mathbb{P}^n$ has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for $\mathbb{P}^n$ by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with $f \in k[x_0, \ldots, x_n]$ homogeneous.

1.3 Vector and Line Bundles

Definition 1.2. Let X be a variety. A *vector bundle (of rank m)* on X is a variety $\mathbb{E}$ with a morphism $p : \mathbb{E} \to X$ such that

1. $\mathbb{E}_x = p^{-1}(X)$ has the structure of a rank m vector space for every $x \in X$ (i.e. k^m),
2. for every $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ and an isomorphism $p^{-1}(U) \to U \times \mathbb{A}^m$ such that for any $y \in U$, the map $\mathbb{E}_y \to \{y\} \times \mathbb{A}^m$ is an isomorphism of vector spaces, i.e.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr] & & U \times \mathbb{A}^m \arrow[dl] \\ & U & \end{tikzcd}$$

commutes. We will call the map ϕ_U a *trivialization.*

Definition 1.3. A *line bundle* on X is a rank 1 vector bundle.

Remark. A different way to think about this is the following:

1. Given two trivializations $\phi_U : p^{-1}(U) \to U \times \mathbb{A}^m$ and $\phi_V : p^{-1}(V) \to V \times \mathbb{A}^m$, we get a morphism

$$\begin{tikzcd} (U \cap V) \times \mathbb{A}^m \arrow[rr, "\phi_{U,V}"] \arrow[dr, "\phi_V^{-1}"'] & & (U \cap V) \times \mathbb{A}^m \\ & p^{-1}(U \cap V) \arrow[ur, "\phi_U"'] & \end{tikzcd}$$

with $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$. Observe $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$ for some $g_{U,V}(x) \in \mathrm{GL}(m, k)$. Furthermore, $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is a morphism. We will call the $g_{U,V}$ *transition functions.*

In the special case where $m = 1$ (so $\mathbb{E}$ is a line bundle and $\mathrm{GL}(1, k) = k^\times$), the map $g_{U,V} : U \cap V \to \mathrm{GL}(m, k)$ is equivalent to the data of a non-vanishing function $g_{U,V} : U \cap V \to k$.

2. The data of a vector bundle of rank m is equivalent to the data of
   - an open cover $X = \bigcup_{i \in I} U_i$,
   - and morphisms $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$

   such that $g_{i,k} = g_{i,j} g_{j,k}$, $g_{i,j} = g_{j,i}^{-1}$, and $g_{i,i} = \mathrm{id}$.

   To recover the vector bundle, we can glue $\mathbb{E}_i = U_i \times \mathbb{A}^m$ for $i \in I$ via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x)v). \end{aligned}$$

   One can check that this defines a vector bundle $\mathbb{E}$ on X.

Example 1.3.1 (Trivial vector bundle)**.** Define the vector bundle $\mathbb{E} : X \times \mathbb{A}^m \to X$ by $(x, v) \to x$. Given a cover $X = \bigcup_{i \in I} U_i$, we get $g_{i,j} : U_i \cap U_j \to \mathrm{GL}(m, k)$ as $x \mapsto I_m$.

Example 1.3.2 (Trivial line bundle)**.** We will denote the trivial line bundle by $\mathbb{1}_X : X \times \mathbb{A}^1 \to X$.

Example 1.3.3. Let $X = \mathbb{P}^n$ and $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$. Consider

$$\begin{tikzcd} & \mathbb{L} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{tikzcd}$$

The map $q : \mathbb{L} \to \mathbb{A}^{n+1}$ is the blowup. We claim that $p : \mathbb{L} \to \mathbb{P}^n$ is a line bundle. We have:

- $p^{-1}([x]) = kx$, a 1-dimensional vector space;
- let $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$, then we can define

$$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

  which one can check is a trivialization.