

MATH 6422: Algebraic Geometry II

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Lecture 1

Jan. 13 — Overview and Review

1.1 Course Overview

Remark. This course will cover the following topics:

- (i) vector bundles and line bundles in algebraic geometry;
- (ii) coherent sheaves;
- (iii) differentials;
- (iv) sheaf cohomology: in particular, we will see that $H_{\mathrm{dR}}^k(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{i+j=k} H^i(X, \wedge^j T_X^*)$;
- (v) the Riemann-Roch theorem: if $\omega = f dz$ is a rational 1-form on a smooth projective curve C , then

$$(\# \text{ zeroes of } \omega) - (\# \text{ poles of } \omega) = 2 \operatorname{genus}(C) - 2;$$

(vi) surfaces and toric varieties;

(vii) schemes: for example, $\operatorname{Spec} \mathbb{Z}$ has points corresponding to the primes p and 0.

1.2 Review of Algebraic Geometry I

Remark. Let $k = \bar{k}$ be an algebraically closed field.

Remark (Hilbert's Nullstellensatz). There is a correspondence

$$\begin{aligned} \text{closed subvarieties of } \mathbb{A}^n &\longleftrightarrow \text{radical ideals in } k[x_1, \dots, x_n] \\ Z &\longmapsto I(Z) \\ V(J) &\longleftarrow J. \end{aligned}$$

Under this correspondence, Z being irreducible (resp. a point) corresponds to $I(Z)$ being prime (resp. maximal).

Remark (Zariski topology on \mathbb{A}^n). The closed sets in \mathbb{A}^n are of the form $V(J)$, and this induces a Zariski topology on any subset of \mathbb{A}^n .

Remark (Embedded affine varieties). Let $J \leq k[x_1, \dots, x_n]$. Then we can associate to J a ringed space (X, \mathcal{O}_X) by setting $X := V(J) \subseteq \mathbb{A}^n$ with the Zariski topology, and \mathcal{O}_X to be the sheaf of regular functions on X , i.e. for $U \subseteq X$ open, we have

$$\mathcal{O}_X(U) := \{\varphi : U \rightarrow k \mid \varphi \text{ is regular}\}.$$

Here, $\varphi : U \rightarrow k$ is *regular* if for each $p \in U$, there exists an open set $p \in U_p \subseteq U$ and $f, g \in k[x_1, \dots, x_n]$ such that $\varphi(x) = f(x)/g(x)$ for all $x \in U_p$.

Remark (Coordinate ring). The *coordinate ring* of X is

$$A(X) := \mathcal{O}_X(X) \cong k[x_1, \dots, x_n]/I(X).$$

We also get a version of Hilbert's Nullstellensatz for $A(X)$:

$$\text{closed subsets of } X \longleftrightarrow \text{radical ideals of } A(X).$$

Remark (Distinguished open sets). The *distinguished open sets* of X are

$$D(f) := \{x \in X : f(x) \neq 0\} = X \setminus V(f).$$

These form a basis for X as we vary $f \in A(X)$.

Definition 1.1. An *affine variety* is a ringed space (X, \mathcal{O}_X) (here \mathcal{O}_X is a sheaf of k -valued functions) which is isomorphic to an embedded affine variety.

Example 1.1.1. If (X, \mathcal{O}_X) is an affine variety and $f \in \mathcal{O}_X(X)$, then

$$(D(f), \mathcal{O}_X|_{D(f)})$$

is again an affine variety. To see this, we may assume that $X = V(J) \subseteq \mathbb{A}^n$ with $J \subseteq k[x_1, \dots, x_n]$ a radical ideal. Now we can define a map

$$\begin{aligned} D(f) &\longrightarrow V(J, fy - 1) \subseteq \mathbb{A}_{x_i}^n \times \mathbb{A}_y^1 \\ x &\longmapsto (x, 1/f(x)), \end{aligned}$$

which one can check is an isomorphism. Now that this also shows

$$\mathcal{O}_X(D(f)) = A(D(f)) \cong \frac{k[x_1, \dots, x_n, y]}{(J, fy - 1)} \cong \frac{(k[x_1, \dots, x_n]/J)[y]}{(fy - 1)} \cong A(X)_f.$$

Theorem 1.1. *There is an equivalence of categories*

$$\begin{aligned} \Phi : \text{Aff-var} &\longrightarrow \text{Red-f.g.}-k\text{-alg}^{\text{op}} \\ (X, \mathcal{O}_X) &\longmapsto A(X). \end{aligned}$$

This implies the following:

1. *There is a bijection*

$$\begin{aligned} \text{Hom}_{\text{aff-var}}(X, Y) &\longrightarrow \text{Hom}_{k\text{-alg}}(A(Y), A(X)) \\ f &\longmapsto f^*. \end{aligned}$$

2. *For any reduced finitely generated k -algebra A , there exists an affine variety with $A \cong A(X)$.*

Remark. How can we explicitly define the inverse functor $\text{Red-f.g.}-k\text{-alg}^{\text{op}} \rightarrow \text{Aff-var}$? We can define this as $A \mapsto (X, \mathcal{O}_X)$, where X is the set of maximal ideals of A . Think about what \mathcal{O}_X should be.

Remark (Varieties). A *variety* (X, \mathcal{O}_X) is a ringed space such that

- there exists a finite open cover of X by affine varieties,
- the diagonal Δ_X is closed in $X \times X$.

Example 1.1.2. The following are examples of varieties:

- affine varieties,
- open or closed subsets of varieties,
- $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^\times$.

Remark (Projective spaces). Recall that \mathbb{P}^n has an open cover by

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \cong \mathbb{A}^n.$$

A basis for \mathbb{P}^n by distinguished open sets is given by

$$D(f) = \{[x] \in \mathbb{P}^n : f(x) \neq 0\}$$

with $f \in k[x_0, \dots, x_n]$ homogeneous.

1.3 Vector and Line Bundles

Definition 1.2. Let X be a variety. A *vector bundle* (of rank m) on X is a variety \mathbb{E} with a morphism $p : \mathbb{E} \rightarrow X$ such that

1. $\mathbb{E}_x = p^{-1}(x)$ has the structure of a rank m vector space for every $x \in X$ (i.e. k^m),
2. for every $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ and an isomorphism $p^{-1}(U) \rightarrow U \times \mathbb{A}^m$ such that for any $y \in U$, the map $\mathbb{E}_y \rightarrow \{y\} \times \mathbb{A}^m$ is an isomorphism of vector spaces, i.e.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{A}^m \\ & \searrow p \quad \swarrow \text{proj} & \\ & U & \end{array}$$

commutes. We will call the map ϕ_U a *trivialization*.

Definition 1.3. A *line bundle* on X is a rank 1 vector bundle.

Remark. A different way to think about this is the following:

1. Given two trivializations $\phi_U : p^{-1}(U) \rightarrow U \times \mathbb{A}^m$ and $\phi_V : p^{-1}(V) \rightarrow V \times \mathbb{A}^m$, we get a morphism

$$\begin{array}{ccc} (U \cap V) \times \mathbb{A}^m & \xrightarrow{\phi_{U,V}} & (U \cap V) \times \mathbb{A}^m \\ & \searrow \phi_V^{-1} \quad \swarrow \phi_U & \\ & p^{-1}(U \cap V) & \end{array}$$

with $\phi_{U,V} = \phi_U \circ \phi_V^{-1}$. Observe $\phi_{U,V}(x, v) = (x, g_{U,V}(x)v)$ for some $g_{U,V}(x) \in \text{GL}(m, k)$. Furthermore, $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$ is a morphism. We will call the $g_{U,V}$ *transition functions*.

In the special case where $m = 1$ (so \mathbb{E} is a line bundle and $\text{GL}(1, k) = k^\times$), the map $g_{U,V} : U \cap V \rightarrow \text{GL}(m, k)$ is equivalent to the data of a non-vanishing regular function $g_{U,V} : U \cap V \rightarrow k$.

2. The data of a vector bundle of rank m is equivalent to the data of

- an open cover $X = \bigcup_{i \in I} U_i$,
- and morphisms $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$

such that $g_{i,k} = g_{i,j}g_{j,k}$, $g_{i,j} = g_{j,i}^{-1}$, and $g_{i,i} = \mathrm{id}$.

To recover the vector bundle, we can glue $\mathbb{E}_i = U_i \times \mathbb{A}^m$ for $i \in I$ via

$$\begin{aligned} \mathbb{E}_{i,j} = (U_i \cap U_j) \times \mathbb{A}^m &\longrightarrow \mathbb{E}_{j,i} = (U_j \cap U_i) \times \mathbb{A}^m \\ (x, v) &\longmapsto (x, g_{i,j}(x)v). \end{aligned}$$

One can check that this defines a vector bundle \mathbb{E} on X .

Example 1.3.1 (Trivial vector bundle). Define the vector bundle $\mathbb{E} : X \times \mathbb{A}^m \rightarrow X$ by $(x, v) \rightarrow x$. Given a cover $X = \bigcup_{i \in I} U_i$, we get $g_{i,j} : U_i \cap U_j \rightarrow \mathrm{GL}(m, k)$ as $x \mapsto I_m$.

Example 1.3.2 (Trivial line bundle). We will denote the trivial line bundle by $\mathbb{1}_X : X \times \mathbb{A}^1 \rightarrow X$.

Lecture 2

Jan. 15 — Vector and Line Bundles

2.1 Vector and Line Bundles, Continued

Example 2.0.1 (Tautological bundle). Let $X = \mathbb{P}^n$ and $\mathbb{L} = \{(\ell, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in \ell\}$. Consider

$$\begin{array}{ccc} & \mathbb{L} & \\ p \swarrow & & \searrow q \\ \mathbb{P}^n & & \mathbb{A}^{n+1} \end{array}$$

The map $q : \mathbb{L} \rightarrow \mathbb{A}^{n+1}$ is the blowup. We claim that $p : \mathbb{L} \rightarrow \mathbb{P}^n$ is a line bundle. We have:

- $p^{-1}([x]) = \{([x], cx) : c \in k\} \cong kx$, a 1-dimensional vector space;
- let $U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$, then we can define

$$\begin{aligned} p^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^1 \\ ([x], y) &\longmapsto ([x], y_i), \end{aligned}$$

which we claim is a trivialization. To see this, observe that for fixed $[x] \in \mathbb{P}^n$, we have

$$\begin{aligned} \mathbb{L}_{[x]} &= \{([x], cx) : c \in k\} \longrightarrow \{[x]\} \times \mathbb{A}^1 \\ ([x], cx) &\longmapsto ([x], cx_i), \end{aligned}$$

which is a vector space isomorphism.

We can also compute the transitions functions. Let $U_{i,j} = U_i \cap U_j$. We have

$$\begin{array}{ccccc} & & \phi_{i,j} & & \\ & \searrow & & \nearrow & \\ U_{i,j} \times \mathbb{A}^1 & \xrightarrow{\phi_j^{-1}} & p^{-1}(U_{i,j}) & \xrightarrow{\phi_i} & U_{i,j} \times \mathbb{A}^1 \\ & & ([x], t) \longmapsto ([x], (tx_0/x_j, \dots, tx_n/x_j)) \longmapsto ([x], tx_i/x_j) & & \end{array}$$

Thus we see that $g_{i,j} = x_i/x_j$. This is called the *tautological bundle*, or $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 2.0.2 (Hyperplane bundle, or $\mathcal{O}_{\mathbb{P}^n}(1)$). Consider

$$\begin{aligned} \mathbb{L} &:= \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \longrightarrow \mathbb{P}^n \\ [x_0 : \dots : x_n : x_{n+1}] &\longmapsto [x_0 : \dots : x_n]. \end{aligned}$$

Then \mathbb{L} is a line bundle with transition functions with respect to $\{U_i\}$ given by $g_{i,j} = x_j/x_i$ (HW).

2.2 Operations on Vector Bundles

Remark. The philosophy is: Every natural operation of vector spaces gives one for vector bundles.

Example 2.0.3 (Direct sum). Let $p : \mathbb{E} \rightarrow X$ and $q : \mathbb{F} \rightarrow X$ be vector bundles of rank e and f on X , respectively. There exists trivializations with respect to a common open cover $\{U_i\}$ (just take intersections) with transition functions $g_{i,j}$ and $h_{i,j}$ for \mathbb{E} and \mathbb{F} , respectively.

Then we define the vector bundle $\mathbb{E} \oplus \mathbb{F} \rightarrow X$ as follows:

- As a set, it is $r : \mathbb{E} \oplus \mathbb{F} = \{(x, u, v) : (x, u) \in \mathbb{E}, (x, v) \in \mathbb{F}\} \rightarrow X$.
- We give $\mathbb{E} \oplus \mathbb{F}$ the structure of a variety by requiring that

$$\begin{aligned} r^{-1}(U_i) &\longrightarrow U_i \times \mathbb{A}^{e+f} \\ (x, u, v) &\longmapsto (x, \text{pr}_2(\phi_i^E(x, u)), \text{pr}_2(\phi_i^F(x, v))) \end{aligned}$$

be an isomorphism, where ϕ_i^E and ϕ_i^F are the trivializations of \mathbb{E} and \mathbb{F} , and pr_2 is the second projection. This gives a variety structure on $r^{-1}(U_i)$, and one can show that these are consistent on $U_{i,j}$, so that this gives a variety structure on all of $\mathbb{E} \oplus \mathbb{F}$.

Note that the transition functions for $\mathbb{E} \oplus \mathbb{F}$ with respect to $\{U_i\}$ are given by the block matrix

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & h_{i,j} \end{bmatrix} : U_{i,j} \longrightarrow \text{GL}(e + f, k).$$

Example 2.0.4. Let \mathbb{E} and \mathbb{F} be vector bundles on X of ranks e and f , respectively. Then the following are also vector bundles on X :

1. $\text{Hom}(\mathbb{E}, \mathbb{F})$, of rank ef ;
2. $\mathbb{E}^\vee = \text{Hom}(\mathbb{E}, \mathbb{1}_X)$, of rank e ;
3. $\mathbb{E} \otimes \mathbb{F}$, of rank ef ;
4. $\wedge^k \mathbb{E}$ and $\text{Sym}^d \mathbb{E}$.

Remark. Let \mathbb{L}, \mathbb{M} be line bundles on X with trivializations on $\{U_i\}$ and transition functions $g_{i,j}, h_{i,j} \in \mathcal{O}_X(U_{i,j})^\times$. In this case, we can describe operations on \mathbb{L}, \mathbb{M} more explicitly:

1. $\mathbb{L} \otimes \mathbb{M}$ has transition functions $g_{i,j}h_{i,j}$;
2. $\text{Hom}(\mathbb{L}, \mathbb{M})$ has transition functions $h_{i,j}/g_{i,j}$;
3. $\mathbb{L}^\vee = \text{Hom}(\mathbb{L}, \mathbb{1}_X)$ has transition functions $1/g_{i,j}$;
4. $\mathbb{L}^{\otimes m} = \begin{cases} \mathbb{L}^{\otimes m}, & \text{if } m > 0, \\ \mathbb{1}_X, & \text{if } m = 0, \\ (\mathbb{L}^\vee)^{\otimes -m}, & \text{if } m < 0 \end{cases}$ has transition functions $g_{i,j}^m$.

Example 2.0.5. Define $\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m}$ with transition functions $(x_j/x_i)^m$ with respect to the standard open cover for \mathbb{P}^n .

2.3 Morphisms of Vector Bundles

Remark. Let $p : \mathbb{E} \rightarrow X$ and $q : \mathbb{F} \rightarrow X$ be vector bundles on X , as before.

Definition 2.1. A *morphism of vector bundles* $\mathbb{E} \rightarrow \mathbb{F}$ is a morphism of varieties

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{a} & \mathbb{F} \\ & \searrow p & \swarrow q \\ & X & \end{array}$$

such that the diagram commutes and a is linear on each fiber.

Remark. More concretely, given an open cover $\{U_i\}$ which trivializes both vector bundles, we have

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{a} & q^{-1}(U_i) \\ \phi_i \downarrow \cong & & \cong \downarrow \psi_j \\ U_i \times \mathbb{A}^e & \longrightarrow & U_i \times \mathbb{A}^f \\ (x, v) & \longmapsto & (x, a_i(x)v) \end{array}$$

such that $a_i : U_i \rightarrow \text{Hom}(k^e, k^f)$ is regular. On $U_{i,j}$, we have

$$\begin{array}{ccc} U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_j} & U_{i,j} \times \mathbb{A}^f \\ g_{i,j} \downarrow & & \downarrow h_{i,j} \\ U_{i,j} \times \mathbb{A}^e & \xrightarrow{a_i} & U_{i,j} \times \mathbb{A}^f \end{array}$$

So $h_{i,j}a_j = a_i g_{i,j}$, or equivalently, $a_i = h_{i,j}a_j g_{i,j}^{-1}$.

As a special case when $e = f$, $a : \mathbb{E} \rightarrow \mathbb{F}$ is an isomorphism if and only if the a_i are isomorphisms.

Remark. When is a line bundle \mathbb{L} given by the trivialization data $\{U_i, g_{i,j}\}$ isomorphic to $\mathbb{1}_X$? We have

$$\begin{aligned} \mathbb{L} \cong \mathbb{1}_X &\iff \text{if and only if there exists an isomorphism } a : \mathbb{1}_X \rightarrow \mathbb{L} \\ &\iff \text{there exist } a_i \in \mathcal{O}_X(U_i)^\times \text{ such that } (a_j/a_i)|_{U_{i,j}} = g_{i,j}. \end{aligned}$$

Definition 2.2. Define the *Picard group* of X to be

$$\text{Pic } X := \{\text{line bundles on } X\} / \cong.$$

This is a group with respect to \otimes with $\mathbb{1}_X$ as the identity and $\mathbb{L}^\vee \otimes \mathbb{L} \cong \mathbb{1}_X$.

2.4 Global Sections

Definition 2.3. A (*global*) *section* of a vector bundle $p : \mathbb{E} \rightarrow X$ is a morphism $s : X \rightarrow \mathbb{E}$ such that $p \circ s = \text{id}_X$. Note that for $x \in X$, we have $s(x) \in \mathbb{E}_x$.

Example 2.3.1 (Zero section). Let $s : X \rightarrow \mathbb{E}$ where $s(x)$ is the zero element in \mathbb{E}_x .

Example 2.3.2. Let $\mathbb{E} = \mathbb{1}_X$. Then sections $s : X \rightarrow X \times \mathbb{A}^1$ of \mathbb{E} correspond to morphisms $X \rightarrow \mathbb{A}^1$, which correspond to regular functions $X \rightarrow k$.

Remark (Local description of sections). Let $\{U_i, g_{i,j}\}$ be the trivialization data for $\mathbb{E} \rightarrow X$, and let $s : X \rightarrow \mathbb{E}$ be a section. On U_i , we have:

$$\begin{array}{ccc} & & U_i \times \mathbb{A}^e \\ & \nearrow x \mapsto (x, s_i(x)) & \uparrow \cong \phi_i \\ U_i & \xrightarrow{s|_{U_i}} & p^{-1}(U_i) \end{array}$$

Note that $s_i : U_i \rightarrow k^e$ is a regular function (i.e. regular on each coordinate). These maps must satisfy the compatibility condition $s_i = g_{i,j}s_j$, since we have the diagram:

$$\begin{array}{ccc} & & U_{i,j} \times \mathbb{A}^e \\ & \nearrow (x, s_j(x)) & \uparrow \phi_j \\ U_{i,j} & \xrightarrow{s|_{U_{i,j}}} & p^{-1}(U_{i,j}) \\ & \searrow \phi_i & \downarrow (x,v) \mapsto (x, g_{i,j}(v)) \\ & & U_{i,j} \times \mathbb{A}^e \\ & \nwarrow (x, s_i(x)) & \end{array}$$

Example 2.3.3. We can use the above compatibility condition to compute the global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Write $\mathbb{P}_{x_0:x_1}^1 = U_0 \cup U_1$. Given a section $s : \mathbb{P}^1 \rightarrow \mathcal{O}(1)$, we get regular functions

$$\begin{aligned} s_0 : U_0 &\longrightarrow k \\ s_1 : U_1 &\longrightarrow k \end{aligned}$$

satisfying $(x_1/x_0)s_1 = s_0 (*)$. We can write

$$s_0 = \sum_{m \geq 0} a_m (x_1/x_0)^m \quad \text{and} \quad s_1 = \sum_{m \geq 0} b_m (x_0/x_1)^m$$

with $a_m, b_m \in k$ (finitely many nonzero). Then $(*)$ implies that

$$a_0 + a_1(x_1/x_0) + \cdots = (x_1/x_0)(b_0 + b_1(x_0/x_1) + \cdots),$$

so $a_0 = b_1$, $a_1 = b_0$, and all other terms are 0. So we can relate s to a linear form

$$f = c_0 x_0 + c_1 x_1,$$

where $s_0 = (x_1/x_0)f$ and $s_1 = (x_0/x_1)f$.

Lecture 3

Jan. 20 — Sections

3.1 Global Sections, Continued

Definition 3.1. Let $\Gamma(X, \mathbb{E}) := \{\text{sections of } \mathbb{E} \rightarrow X\}$, which has the structure of a k -vector space by

$$(s + t)(x) = s(x) + t(x) \quad \text{and} \quad (cs)(x) = cs(x)$$

for $s, t \in \Gamma(X, \mathbb{E})$ and $c \in k$.

Example 3.1.1. One can check that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong k[x_0, \dots, x_n]_d$ (HW). For example, for $d < 0$, we have $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) = \{0\}$, and for $d = 0$, we have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(0)) = \Gamma(\mathbb{P}^n, \mathbb{1}_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k.$$

For $d = 1$, we can define an isomorphism

$$\begin{aligned} k[x_0, \dots, x_n]_1 &\xrightarrow{\cong} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \\ f &\longmapsto \text{section } s : \mathbb{P}^n \rightarrow \mathcal{O}(1) \text{ given by } s_i = f/x_i. \end{aligned}$$

Note that $(x_j/x_i)s_j = s_i$ holds, so this is a section. An alternative perspective is that s corresponds to

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{n+1} \setminus \{[0 : \dots : 0 : 1]\} \\ x &\longmapsto [x_0 : \dots : x_n : f(x)]. \end{aligned}$$

3.2 Morphisms and Sections

Definition 3.2. Given a section $s : X \rightarrow \mathbb{E}$, its *vanishing locus* is

$$V(s) := \{s = 0\} = \{x \in X : s(x) = 0\}.$$

Using a trivializing cover, one can check that $V(s)$ is closed in X .

Example 3.2.1. For a section $s : \mathbb{P}^n \rightarrow \mathcal{O}(1)$ corresponding to $f \in k[x_0, \dots, x_n]_1$, we have $V(s) = V_{\mathbb{P}^n}(f)$.

Remark. Recall that there is a bijection

$$\begin{aligned} \{\text{morphisms } X \rightarrow \mathbb{A}^n\} &\longleftrightarrow \{f_1, \dots, f_n \in \mathcal{O}_X(X)\} \\ [f : X \rightarrow \mathbb{A}^n] &\longmapsto [f_1 = f^*x_1, \dots, f_n = f^*x_n \in \mathcal{O}_X(X)] \\ [x \mapsto (f_1(x), \dots, f_n(x))] &\longleftarrow [f_1, \dots, f_n \in \mathcal{O}_X(X)]. \end{aligned}$$

We want a similar statement for \mathbb{P}^n .

Definition 3.3. Given a line bundle $\mathbb{L} \rightarrow X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$, they are *nowhere vanishing* if

$$V(s_0) \cap \dots \cap V(s_n) = \emptyset.$$

Example 3.3.1. For $\mathcal{O}(1)$, the sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ are nowhere vanishing.

Remark. If $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ are nowhere vanishing, then we get a morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto [s_0(x) : \dots : s_n(x)]. \end{aligned}$$

Note that $(s_0(x), \dots, s_n(x))$ is a well-defined point in \mathbb{A}^{n+1} up to scaling. One can check that this map is a morphism by working locally.

Example 3.3.2 (Linear maps). Let $X = \mathbb{P}^n$ and $\mathbb{L} = \mathcal{O}(1)$.

- (i) $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ gives $\text{id} : \mathbb{P}^n \rightarrow \mathbb{P}^n$.
- (ii) For $A \in \text{GL}_{n+1}(k)$, we get a map

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto [Ax] \end{aligned}$$

given by $Ax_0, \dots, Ax_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Remark. Now given a morphism $X \rightarrow \mathbb{P}^n$, we want to get a line bundle with sections.

Definition 3.4 (Pullback). Let $p : \mathbb{E} \rightarrow X$ be a vector bundle and $f : Y \rightarrow X$ a morphism. Define

$$f^*\mathbb{E} = \{(e, y) : e \in \mathbb{E}, y \in Y \text{ with } p(e) = f(y)\} \longrightarrow Y.$$

One can show that this has the structure of a vector bundle in a natural way.

Remark. An alternative way to define the pullback is to choose trivialization data $(U_i, g_{i,j})$ for $\mathbb{E} \rightarrow X$. Then we can define $f^* : \mathbb{E} \rightarrow Y$ to be the vector bundle with trivialization data $(f^{-1}(U_i), f^*g_{i,j})$.

Remark. Now to go in reverse, given a morphism $X \rightarrow \mathbb{P}^n$ and nowhere vanishing sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, we get nowhere vanishing sections

$$f^*x_0, \dots, f^*x_n \in \Gamma(X, f^*\mathcal{O}(1)).$$

We can define the pullback of a section in one of two ways: by $f^*(x_i)(a) = (x_i(f(a)), a) \in f^*\mathcal{O}(1)$ for $a \in X$ or by using trivializing covers.

Remark. Using the above, we get a bijection

$$\{\text{morphisms } X \rightarrow \mathbb{P}^n\} \longleftrightarrow \{\text{line bundles } \mathbb{L} \rightarrow X \text{ with } s_0, \dots, s_n \in \Gamma(X, \mathbb{L}) \text{ nowhere vanishing}\}.$$

Note that we should consider the right-hand side up to isomorphism of the line bundle. When do $\mathbb{L} \rightarrow X$ and $s_0, \dots, s_n \in \Gamma(X, \mathbb{L})$ give an injective morphism (or an embedding)?

Definition 3.5. Given a vector bundle $\mathbb{E} \rightarrow X$, we get a sheaf of abelian groups \mathcal{E} on X by

$$\mathcal{E}(U) := \{\text{sections of } p^{-1}(U) \rightarrow U\}$$

for $U \subseteq X$ open. For $V \subseteq U \subseteq X$ open, the restriction map is given by

$$\begin{aligned} \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto s|_V. \end{aligned}$$

We call \mathcal{E} the *sheaf of sections* of \mathbb{E} . Also note that $\mathcal{E}(U)$ has the structure of an $\mathcal{O}_X(U)$ -module. We

will see that this gives rise to the structure of an \mathcal{O}_X -module.

3.3 Review of Sheaves

Definition 3.6. A *presheaf* of abelian groups \mathcal{F} on a topological space X is the data of:

- for $U \subseteq X$ open, an abelian group $\mathcal{F}(U)$ (with $\mathcal{F}(\emptyset) = 0$),
- for $V \subseteq U \subseteq X$ open, a group homomorphism $p_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$.

Remark. Note the following:

1. We may replace abelian groups in the above definition by rings, sets, R -modules, etc.
2. We denote $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, whose elements are called *sections*.
3. $s|_V := p_{V,U}(s)$ is called the *restriction* for $s \in \mathcal{F}(U)$ and $V \subseteq U \subseteq X$ open.
4. We may view \mathcal{F} as a functor $\text{Open}_X \rightarrow \text{Ab-gps}$ given by $U \mapsto \mathcal{F}(U)$.

Definition 3.7. For \mathcal{F} a presheaf on X and $x \in X$, the *stalk* of \mathcal{F} at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x \text{ open}} \mathcal{F}(U) = \{(s, U) : s \in \mathcal{F}(U)\} / \sim.$$

Example 3.7.1. The following are examples of presheaves:

1. Let M be a smooth manifold. Then
 - \mathcal{O}_M = sheaf of smooth \mathbb{R} -valued functions on M ,
 - \mathcal{E} = sheaf of sections of a vector bundle $\mathbb{E} \rightarrow M$.
2. Let X be an algebraic variety, $\mathbb{E} \rightarrow X$ a vector bundle, and $Z \subseteq X$ closed. Then
 - \mathcal{O}_X and \mathcal{E} are sheaves,
 - ℓ_Z = ideal sheaf of Z , given by $\ell_Z(U) = \{\varphi \in \mathcal{O}_X(U) : \varphi|_Z = 0\}$.
3. Let X be a topological space and A an abelian group.
 - $\underline{A}^{\text{pre}}$ given by $U \mapsto \{\text{constant functions } U \rightarrow A\}$, i.e. $\underline{A}^{\text{pre}}(U) \cong A$ for $U \neq \emptyset$,
 - \underline{A} given by $U \mapsto \{\text{locally constant functions } U \rightarrow A\}$,
 - $i_p A$ = skyscraper sheaf, given by $U \mapsto \begin{cases} A & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$

Definition 3.8. A presheaf \mathcal{F} is a *sheaf* if for any

- open set $U \subseteq X$,
- open cover $U = \bigcup_{i \in I} U_i$,
- and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{i,j}} = s_j|_{U_{i,j}}$ for all $i, j \in I$,

then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Remark. The presheaf $\underline{A}^{\text{pre}}$ is not a sheaf in general. All other examples above are sheaves.

Definition 3.9. A *morphism* of (pre)sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ on a topological space X is the data of group homomorphisms $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each $U \subseteq X$ open such that for all $V \subseteq U \subseteq X$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

Example 3.9.1. Let X be a variety.

1. If $a : \mathbb{E} \rightarrow \mathbb{F}$ is a morphism of vector bundles on X , then we get a morphism of sheaves $\mathcal{E} \rightarrow \mathcal{F}$ by $s \mapsto a \circ s \in \mathcal{F}(U)$ for $s \in \mathcal{E}(U)$.
2. A closed subvariety $Z \subseteq X$ induces a morphism $\ell_Z \rightarrow \mathcal{O}_X$ given by inclusion.

Proposition 3.1. If \mathcal{F} is a presheaf on X , then there exists a morphism to a sheaf $i : \mathcal{F} \rightarrow \mathcal{F}^+$ such that for any morphism $g : \mathcal{F} \rightarrow \mathcal{G}$, there exists a unique morphism $g^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $g = g^+ \circ i$.