MATH 7337: Harmonic Analysis

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Contents

Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \le \int |f(x)e^{-2\pi i \xi x}|\, dx = \int |f(x)|\, dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \le \|f\|_1$. This is linear in f. The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \to L^\infty} = \sup_{\|f\|_1 = 1} \|\widehat{f}\|_\infty \le \sup_{\|f\|_1 = 1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x)e^{-2\pi i \cdot 0 \cdot x}\, dx = \int f(x)\, dx.$$

So if $f \ge 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \le \int f(x)\, dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \le 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

Definition 1.2. For $r \ne 0$, *dilation* of f by r is $f_r(x) = rf(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.[1] Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

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[1] Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x \to \pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^{T} e^{-2\pi i \xi x}\, dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\bigg|_{-T}^{T} = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \to 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x})\, dx \right| \leq \int |f(x)||e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}|\, dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \to 0$ as $\eta \to 0$ independent of ξ, so the statement follows from the dominated convergence theorem (the integrand is dominated by $2f$). $\square$

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(x)e^{2\pi i \xi x}\, d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula)**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi)e^{2\pi i \xi x}\, d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x)e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ, and the inversion formula says that we can recover f as a "sum" of these pure harmonics.

Lecture 2

Aug. 21 — The Riemann-Lebesgue Lemma

2.1   Properties of the Fourier Transform

Definition 2.1. Define the following operators:

1. *Translation*: $T_a f(x) = f(x-a)$ for $a \in \mathbb{R}$;

2. *Modulation*: $M_b f(x) = e^{2\pi ibx} f(x)$ for $b \in \mathbb{R}$;

3. *Dilation*: $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;

4. *Involution*: $\widetilde{f}(x) = \overline{f(-x)}$.

Remark. Translation and modulation are isometries on $L^p(\mathbb{R})$ for any p. Dilation as defined above is L^1-normalized, so it is only an isometry on $L^1(\mathbb{R})$.

Exercise 2.1. If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a}\widehat{f})(\xi) = e^{-2\pi i\xi a}\widehat{f}(\xi)$;

2. $(M_b f)^\wedge(\xi) = (T_b\widehat{f})(\xi) = \widehat{f}(\xi - b)$;

3. $(f_\lambda)^\wedge(\xi) = \lambda(f_{1/\lambda})^\wedge(\xi) = \widehat{f}(\xi/\lambda)$;[1]

4. $(\overline{f})^\wedge(\xi) = (\widehat{f})^\sim(\xi) = \overline{\widehat{f}(-\xi)}$;

5. $(\widetilde{f})^\wedge(\xi) = \overline{\widehat{f}(\xi)}$.

2.2   The Riemann-Lebesgue Lemma

Definition 2.2. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of f, denoted $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function f, $\mathrm{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

Theorem 2.1. *$C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. In other words,*

1. *the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;*

2. *for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;*

3. *if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \to f$ in L^p-norm, i.e. $\|g_n - f\|_p \to 0$.*

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[1]Note that the result is an L^∞-normalized dilation.

For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the L^∞-norm (this is the same as the uniform norm for continuous functions).

Proof. We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of 2^{-n}. Then use Urysohn's lemma to approximate χ_{E_k} by a continuous function. $\square$

Exercise 2.2. Fix $1 \le p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a \to 0} \| f - T_a f \|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

Lemma 2.1 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

Proof. We have already seen that $\widehat{f}$ is continuous. So it suffices to show decay at ∞. Write

$$\widehat{f}(\xi) = -\int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)} \, dx = -\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x + 1/2\xi)} \, dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = -\int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi} \right) e^{-2\pi i \xi x} \, dx = -\int T_{1/2\xi} f(x) e^{-2\pi i \xi x} \, dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x} \, dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \le \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)| \, dx = \frac{1}{2} \| f - T_{1/2\xi} f \|_1 \xrightarrow[\xi \to \pm\infty]{} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. $\square$

Exercise 2.3. The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^{N} c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \to f$ in L^1-norm. On the Fourier side, we have

$$\| \widehat{f} - \widehat{\phi}_n \|_\infty \le \| f - \phi_n \|_1 \longrightarrow 0.$$

Since $\phi_n \to \widehat{f}$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

2.3 Position and Momentum Operators

Definition 2.3. The *position operator* $P : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that P is unbounded on $L^1(\mathbb{R})$ (in fact, P is not defined on all of $L^1(\mathbb{R})$). Restrict P to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that D_P cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

Exercise 2.4. Show that $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$.

Definition 2.4. The *momentum operator* $M : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, M is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

Remark. We have the relation $(Mf)^\wedge(\xi) = \xi P \widehat{f}(\xi)$, whenever the statement makes sense.

2.4 The HRT Conjecture

Conjecture 2.1 (HRT conjecture). *Assume g is not zero a.e., a_k, b_k are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:*

$$\sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k). \tag{$*$}$$

If $(*) = 0$, then must it be that $c_1 = \cdots = c_N = 0$? In other words, are these linearly independent?

Remark. Consider the special case $b_k = 0$ for every k, so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g\right)^\wedge = \sum c_k M_{-a_k} \widehat{g} = \left(\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi}\right) \widehat{g}(\xi) = 0.$$

Since $\widehat{g}$ is not zero a.e., we must have $\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all k. In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

Remark. The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for $\widehat{g}$ instead of g.

Lecture 3

Aug. 3 — Convolution

3.1 Convolution

Definition 3.1. If f, g are measurable on $\mathbb{R}$, their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\, dy.$$

Remark. When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\, dy = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy = (g * f)(x)$$

by the change of variables $y \mapsto x - y$. So $f * g = g * f$, if it exists. Similarly, $f * (g * h) = (f * g) * h$ if each of these convolutions exist.

Remark. If we take $g_T = \chi_{-T,T}/2T$ (note that $\|g_T\|_1 = 1$), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x - y)\, dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y)dy = \mathrm{Avg}_{[-T,T]}f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

Remark. We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.

Remark. The *Lebesgue differentiation theorem* says that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $(f * g_T)(x) \to f(x)$ a.e.

3.2 Properties of Convolution

Remark. Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if $1/p + 1/p' = 1$ with $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and we have

$$|\langle f, g \rangle| \leq \int |f(x)||g(x)|\, dx \leq \|f\|_p \|g\|_{p'}.$$

Theorem 3.1. *For $1 \le p \le \infty$, if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$.*

Proof. By Hölder's inequality, we can write

$$\int |f(y)g(x-y)|\,dy \le \|f\|_p \|g(x)\|_{p'} < \infty,$$

so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. $\qquad\square$

Exercise 3.1. Show that $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{C} : h$ is continuous and bounded$\}$.

Remark. Denote $g^*(y) = \overline{g(-y)}$. Then we have

$$(f * g)(x) = \int f(y)g(x-y)\,dy = \int f(y)\overline{g^*(y-x)}\,dy = \langle f, T_x g^* \rangle.$$

Theorem 3.2. *Let $f, g \in L^1(\mathbb{R})$. Then*

1. *$f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}^2$;*
2. *for a.e. $x \in \mathbb{R}$, $f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}$ as a function of y;*
3. *$f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$, i.e. convolution is submultiplicative on $L^1(\mathbb{R})$;*
4. *$(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for every $\xi \in \mathbb{R}$.*

Proof. (1) Let $h(x, y) = f(x)$. Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(x) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in $\mathbb{R}^2$ since $\{f > a\}$ and $\mathbb{R}$ are measurable in $\mathbb{R}$. Similarly, $g(y)$ is measurable on $\mathbb{R}^2$, so $F(x, y) = f(x)g(y)$ is measurable on $\mathbb{R}^2$. Now make a linear change of variables $T(x, y) = (y, x - y)$, so $H = F \circ T = f(y)g(x - y)$ is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\iint |f(y)g(x-y)|\,dxdy = \int |f(y)| \left(\int |g(x-y)|\,dx \right) dy = \int |f(y)| \left(\int |g(z)|\,dz \right) dy$$
$$= \int |f(y)|\|g\|_1\,dy = \|f\|_1\|g\|_1 < \infty,$$

hence $f(y)g(x - y)$ is integrable on $\mathbb{R}^2$.

(2) This follows by Fubini's theorem since $f(y)g(x - y)$ is integrable.

(3) By (2), $(f * g)(x)$ exists for a.e. x, and

$$\int |(f * g)(x)|\,dx = \int \left| \int f(y)g(x-y)\,dy \right| dx \le \iint |f(y)g(x-y)|\,dydx \le \|f\|_1\|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$(f * g)^\wedge(\xi) = \int (f * g)(x)e^{-2\pi i\xi x}\, dx = \int \left(\int f(y)g(x - y)\, dy \right) e^{-2\pi i\xi x}\, dx$$

$$= \iint f(y)e^{-2\pi i\xi y}g(x - y)e^{-2\pi i\xi(x-y)}\, dydx.$$

By Fubini's theorem, we can exchange orders and write

$$(f * g)^\wedge(\xi) = \int f(y)e^{-2\pi i\xi y}\left(\int g(x - y)e^{-2\pi i\xi(x-y)}\, dx \right) dy$$

$$= \int f(y)e^{-2\pi i\xi y}\left(\int g(z)e^{-2\pi i\xi z}\, dz \right) dy = \widehat{f}(\xi)\widehat{g}(\xi),$$

which is the desired equality.   □

Corollary 3.2.1. $L^1(\mathbb{R})$ *is closed under convolution.*

Definition 3.2. An *algebra* is a vector space A with a product such that

(a) $(fg)h = f(gh)$,

(b) $f(g + h) = fg + fh$,

(c) $\alpha(fg) = (\alpha f)g = f(\alpha g)$.

If $fg = gf$ always, then we say that A is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

Example 3.2.1. With convolution as a product, $L^1(\mathbb{R})$ becomes a commutative Banach algebra without identity. Similarly, $C_0(\mathbb{R})$ is also a commutative Banach algebra without identity (under pointwise products). The space $\mathcal{B}(X)$ of bounded linear operators on a Banach space X is also a Banach space under the operator norm, and we have $\|AB\| \le \|A\|\|B\|$ with composition as a product. So $\mathcal{B}(X)$ is a noncommutative Banach algebra, with identity.

3.3   Young's Inequality

Theorem 3.3 (Young's inequality, special case). *Fix $1 \le p \le \infty$. If $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \le \|f\|_p\|g\|_1$.*

Proof. The case $p = \infty$ is easy by Hölder's inequality and $p = 1$ is done, so assume $1 < p < \infty$. Then

$$|(f * g)(x)| \le \int |f(y)||g(x - y)|\, dy = \int \left(|f(y)||g(x - y)|^{1/p} \right)\left(|g(x - y)|^{1/p'} \right) dy,$$

By Hölder's inequality, we can write

$$|(f * g)(x)| \le \left(\int |f(y)|^p|g(x - y)|\, dy \right)^{1/p}\left(\int |g(x - y)|\, dy \right)^{1/p'}$$

$$\le \|g\|_1^{1/p'}\left(\int |f(y)|^p|g(x - y)|\, dy \right)^{1/p}.$$

Now taking L^p-norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p \, dx \leq \|g\|_1^{p/p'} \iint |f(y)|^p |g(x - y)| \, dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \leq \|g\|_1^{p/p'} \int |f(y)|^p \left(\int |g(x - y)| \, dx \right) dy \leq \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality $\|f * g\|_p \leq \|f\|_p \|g\|_1$ after taking pth roots. $\qquad\square$

Exercise 3.2 (Young's inequality, general case). Let $1 \leq p, q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then
$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Remark. Recall *Minkowski's inequality* (the triangle inequality in $L^p(\mathbb{R})$):

$$\left\| \sum f_k \right\|_p \leq \sum \|f_k\|_p.$$

Minkowski's integral inequality then says that for $1 \leq p \leq \infty$,

$$\left\| \int f_x \, dx \right\|_p = \left(\int \left| \int f(x, y) \, dx \right|^p \, dy \right)^{1/p} \leq \int \left(\int |f(x, y)|^p \, dy \right)^{1/p} dx = \int \|f_x\|_p \, dx.$$

One can also use this to prove to Young's inequality.

Remark. The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is $A_p A_q A_{r'}$, i.e. we have

$$\|f * g\|_r \leq (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

3.4   The Dirac Delta

Remark. Is there an identity for convolution? Suppose there was a function $\delta \in L^1(\mathbb{R})$ (the *Dirac delta function*) such that $f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Then we have $(f * \delta)^\wedge = \widehat{f}$, so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take $f(x) = e^{-x^2}$ with $\widehat{f}(\xi) = e^{-\xi^2}$ and note that $\widehat{f}(\xi)$ is everywhere nonzero. Then $\widehat{\delta}(\xi) = 1$ for all $\xi \in \mathbb{R}$, which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure δ to achieve a similar effect.

Lecture 4

Aug. 28 — Convolution, Part 2

Lecture 5

Sept. 2 — Smoothness and Decay

5.1   Smoothness and Decay

Theorem 5.1 (Decay in time implies smoothness in frequency)**.** *Assume $f \in L^1(\mathbb{R})$ and $x^m f(x) \in L^1(\mathbb{R})$, where $m > 0$. Then*

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \ldots, g^{(m)} \in C_0(\mathbb{R})\}.$$

Furthermore, we have

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k} \widehat{f} = \left((-2\pi i x)^k f(x)\right)^{\wedge}.$$

Proof. The proof is by induction on m. When $m = 1$, we can formally write

$$\frac{d}{d\xi} \widehat{f}(\xi) = \frac{d}{d\xi} \int f(x) e^{-2\pi i \xi x} \, dx$$

$$\overset{(*)}{=} \int f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} \, dx = \int f(x)(-2\pi i x) e^{-2\pi i \xi x} \, dx = (-2\pi i x f(x))^{\wedge}(\xi).$$

It suffices to justify step $(*)$, which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f}'(\xi) = \lim_{\eta \to 0} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \to 0} \int f(x) \frac{e^{-2\pi i (\xi + \eta) x} - e^{-2\pi i \xi x}}{\eta} \, dx.$$

Note that we have the pointwise limit

$$f(x) \frac{e^{-2\pi i (\xi + \eta) x} - e^{-2\pi i \xi x}}{\eta} \overset{\eta \to 0}{\longrightarrow} f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} = -2\pi i x f(x) e^{-2\pi i \xi x}.$$

Also note that we can bound

$$\left| f(x) \frac{e^{-2\pi i (\xi + \eta) x} - e^{-2\pi i \xi x}}{\eta} \right| = \left| f(x) \frac{e^{-2\pi i \eta x} - 1}{\eta} \right| \le \left| f(x) \frac{-2\pi i \eta x}{\eta} \right| = |2\pi x f(x)|,$$

where we noted that $|e^{i\theta} - 1| \le |\theta|$ for $\theta \in \mathbb{R}$. Thus $2\pi x f(x)$ dominates the integrand and is integrable since $x f(x) \in L^1(\mathbb{R})$ by assumption, we can conclude $(*)$ by the dominated convergence theorem. Then $\widehat{f}' \in C_0(\mathbb{R})$ by the Riemann-Lebesgue lemma, since $\widehat{f}' = (-2\pi i x f(x))^{\wedge}$ where $-2\pi i x f(x) \in L^1(\mathbb{R})$.

The inductive step is part of Homework 1. $\qquad\square$

Remark. Recall the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = f'(x)/2\pi i$. If $f, Pf \in L^1(\mathbb{R})$, then the above theorem tells us that $(Pf)^{\wedge} = -M\widehat{f}$.

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5.2   Absolute Continuity

Definition 5.1. A function $f : [a, b] \to \mathbb{C}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}$.

Theorem 5.2 (Fundamental theorem of calculus). *If $g : [a, b] \to \mathbb{C}$, then the following are equivalent:*

1. *$g \in \mathrm{AC}[a, b]$;*

2. *there exists $f \in L^1[a, b]$ such that for all $x \in [a, b]$,*

$$g(x) - g(a) = \int_a^x f(t)\, dt;$$

3. *g is differentiable at a.e. point, $g' \in L^1[a, b]$, and*

$$g(x) - g(a) = \int_a^x g'(t)\, dt.$$

Remark. The Cantor-Lebesgue function $\varphi : [0, 1] \to [0, 1]$ is continuous with $\varphi' = 0$ a.e., but

$$\int_0^1 \varphi'(x)\, dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

Lemma 5.1 (Growth lemma). *If $f : [a, b] \to \mathbb{R}$ is measurable and differentiable at every point in a measurable set $E \subseteq [a, b]$, then*

$$|f(E)|_e \leq \int_E |f'|,$$

where $|f(E)|_e$ denotes the exterior Lebesgue measure of $f(E)$.

Theorem 5.3 (Banach-Zaretsky theorem). *If $f : [a, b] \to \mathbb{R}$, then the following are equivalent:*

1. *$f \in \mathrm{AC}[a, b]$;*

2. *f is continuous, f has bounded variation, and $|A| = 0$ implies $|f(A)| = 0$;*

3. *f is continuous and differentiable a.e., $f' \in L^1[a, b]$, and $|A| = 0$ implies $|f(A)| = 0$.*

Theorem 5.4. *If $f : [a, b] \to \mathbb{C}$ is differentiable on $[a, b]$ and $f' \in L^1[a, b]$, then $f \in \mathrm{AC}[a, b]$.*

Proof. By the Banach-Zaretsky theorem, it suffices to show that $|A| = 0$ implies $|f(A)| = 0$. If $|A| = 0$, then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split f into its real and imaginary parts.) $\qquad \square$

5.3   Smoothness and Decay, Continued

Theorem 5.5 (Smoothness in time implies decay in frequency). *If $f \in L^1(\mathbb{R})$ is everywhere m-times differentiable and $f, f', \ldots, f^{(m)} \in L^1(\mathbb{R})$, then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad for \ k = 0, \ldots, m,$$

hence $|\widehat{f}(\xi)| \leq |2\pi\xi|^{-k} |\widehat{f^{(k)}}(\xi)| \leq |2\pi\xi|^{-k} \|\widehat{f^{(k)}}\|_\infty \leq |2\pi\xi|^{-k} \|f^{(k)}\|_1$ *for* $k = 0, \ldots, m$.

Proof. We prove only the case $m = 1$, the rest follows by induction. Assume $f, f' \in L^1(\mathbb{R})$. By Theorem 5.4, we have $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$. Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t) \, dt.$$

Because f' is integrable, we get that

$$\lim_{x \to \infty} f(x) = f(0) + \lim_{x \to \infty} \int_0^x f'(t) \, dt = f(0) + \int_0^\infty f'(t) \, dt.$$

Since f is integrable and this limit exists, the limit must be 0. Hence $f \in C_0(\mathbb{R})$. We can compute

$$\widehat{f'}(\xi) = \int_{-\infty}^\infty f'(x) e^{-2\pi i \xi x} \, dx = \lim_{\substack{b \to \infty \\ a \to -\infty}} \int_a^b f'(x) e^{-2\pi i \xi x} \, dx.$$

Since f is absolutely continuous, we can integrate by parts to get

$$\widehat{f'}(\xi) = \lim_{\substack{b \to \infty \\ a \to -\infty}} \left[f(b) e^{-2\pi \xi b} - f(a) e^{-2\pi i \xi a} + (2\pi i \xi) \int_a^b f(x) e^{-2\pi i \xi x} \, dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result. $\qquad\square$

Remark. Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$.

5.4   Approximate Identities

Remark. Recall that if we take $g_T = \chi_{[-T,T]}/2T$, then we have $(f * g_T)(x) = \mathrm{Avg}_{[x-T,x+T]} f$. As $T \to 0$, this converges to f if f is continuous, and converges a.e. to f if f is integrable. In particular, this is almost like a identity for the convolution operation.

Definition 5.2. If $k_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ (or sometimes $\lambda \in \mathbb{N}$) satisfy:

(a) Normalization: $\displaystyle\int_{-\infty}^\infty k_\lambda = 1$ for every λ,

(b) L^1-boundedness: $\displaystyle\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^\infty |k_\lambda| < \infty$,

(c) L^1-concentration: $\displaystyle\lim_{\lambda \to \infty} \int_{|x| \geq \delta} |k_\lambda| = 0$ for every $\delta > 0$,

then we say that $\{k_\lambda\}$ is an *approximate identity (for convolution)*.

Exercise 5.1. If $k \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} k = 1$, then $k_\lambda(x) = \lambda k(\lambda x)$ forms an approximate identity.

Remark. If we choose k_λ to be nice, then $f * k_\lambda$ will also be nice and "close" to f.

Lecture 6

Sept. 4 — Approximate Identities

6.1 Properties of Approximate Identities

Theorem 6.1. *If $\{k_\lambda\}$ is an approximate identity, then for all $f \in L^1(\mathbb{R})$,*

$$\lim_{\lambda \to \infty} \|f * k_\lambda - f\|_1 = 0.$$

That is, $f * k_\lambda \to f$ in L^1-norm.

Proof. We have already seen that $f * k_\lambda \in L^1(\mathbb{R})$. Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)| \, dx = \int \left| f(x) \int k_\lambda(t) \, dt - \int f(x-t) k_\lambda(t) \, dt \right| dx,$$

where we used that $\int k_\lambda(t) \, dt = 1$. Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \le \iint |f(x) - f(x-t)| |k_\lambda(t)| \, dt dx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \le \int |k_\lambda(t)| \left(\int |f(x) - T_t f(x)| \, dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \le \int_{|t| < \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t| \ge \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that $\lim_{t \to 0} \|f - T_t f\|_1 = 0$, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t| < \delta$ implies $\|f - T_t f\|_1 < \epsilon$. This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \le \epsilon \int_{|t| < \delta} |k_\lambda(t)| \, dt + \int_{|t| \ge \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use $\|f - T_t f\|_1 \le \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$ to get

$$\|f - f * k_\lambda\|_1 \le \epsilon \int_{|t| < \delta} |k_\lambda(t)| \, dt + 2\|f\|_1 \int_{|t| \ge \delta} |k_\lambda(t)| \, dt \le \epsilon K + 2\|f\|_1 \epsilon$$

where $K = \sup_\lambda \|k_\lambda\|_1 < \infty$ and λ is large enough (as $\int_{|t| \ge \delta} |k_\lambda(t)| \, dt \to 0$). So $\|f - f * k_\lambda\|_1 \to 0$. $\square$

Exercise 6.1. Show that for $1 \leq p < \infty$, we still have $\|f - f * k_\lambda\|_p \to 0$ as $\lambda \to \infty$ for $f \in L^p(\mathbb{R})$. For $p = \infty$, show that if $f \in C_0(\mathbb{R})$, then $\|f - f * k_\lambda\|_\infty \to 0$ as $\lambda \to \infty$, that is $f * k_\lambda \to f$ uniformly.

Exercise 6.2. Show that if $f \in C_b(\mathbb{R})$, then for every compact set $K \subseteq \mathbb{R}$,

$$\lim_{\lambda \to \infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

Definition 6.1. A function f is *Hölder continuous* with exponent $\alpha > 0$ if

$$|f(x) - f(y)| \leq K|x - y|^\alpha$$

for some constant K and all x, y. If $\alpha = 1$, then we say that f is *Lipschitz*.

Remark. If f is Hölder continuous with exponent $\alpha > 1$, then the mean value theorem implies that f is constant. Thus the interesting range for Hölder continuity is $0 < \alpha \leq 1$.

Exercise 6.3. Let f be bounded and Hölder continuous with exponent $0 < \alpha \leq 1$, then show that

$$f * k_\lambda \to f \quad \text{uniformly on } \mathbb{R}.$$

Remark. If f is differentiable and f' is bounded, then f is Lipschitz.

Remark. Recall the *Lebesgue differentiation theorem*, which says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then

$$(f * g_T)(x) = \frac{1}{2T} \int_{x-T}^{x+T} f(t)\, dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where $g_T = \chi_{[-T,T]}/(2T)$. The points where the limit holds are called the *Lebesgue points* of f.

Theorem 6.2. *Assume k is bounded and compactly supported and $\int k = 1$. Set $k_\lambda(x) = \lambda k(\lambda x)$ for $\lambda > 0$. Then for any $f \in L^1(\mathbb{R})$,*

$$f * k_\lambda \to f \quad \text{pointwise a.e.}$$

Moreover, the pointwise limit holds at every Lebesgue point of f.

Proof. Assume $\text{supp}(k) \subseteq [-R, R]$. We can write

$$\lim_{\lambda \to \infty} |f(x) - (f * k_\lambda)(x)| = \lim_{\lambda \to \infty} \left| f(x) \int k_\lambda(x - t)\, dt - \int f(x) k_\lambda(x - t)\, dt \right|$$

$$\leq \lim_{\lambda \to \infty} \int |f(x) - f(t)| \lambda |k(\lambda x - \lambda t)|\, dt$$

$$\leq \lim_{\lambda \to \infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)| |k(\lambda x - \lambda t)|\, dt.$$

Making a change of variables $T = R/\lambda$, we have

$$\lim_{\lambda \to \infty} |f(x) - (f * k_\lambda)(x)| \leq \lim_{T \to 0} \frac{1}{2T} \int_{x-T}^{x+T} |f(x) - f(t)|\, dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point x by the Lebesgue differentiation theorem. $\qquad \square$

6.2 Density Results and Smooth Urysohn Lemma

Theorem 6.3. $C_c^m(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $m > 0$ *and* $1 \leq p < \infty$.

Proof. Fix $\epsilon > 0$. Choose $k \in C_c^m(\mathbb{R})$ with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Note that there exists a compactly supported $g \in L^p(\mathbb{R})$ with $\|f - g\|_p < \epsilon$ (e.g. take $g = f\chi_{[-R,R]}$ for large enough R, this works since $f\chi_{[-R,R]}$ converges pointwise to f as $R \to \infty$ and is dominated by f, so the dominated convergence theorem implies that $f\chi_{[-R,R]} \to f$ in L^p-norm). Then note that $g * k_\lambda \in C_c^m(\mathbb{R})$ and $g * k_\lambda \to g$ in L^p-norm, so there exists λ such that $\|g - g * k_\lambda\|_p < \epsilon$. Thus

$$\|f - g * k_\lambda\|_p \leq \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result. $\qquad\square$

Corollary 6.3.1. $C_c^\infty(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $1 \leq p < \infty$.

Remark. The above proof would work for $m = 0$ but becomes circular: The step $g * k_\lambda \in C_c^m(\mathbb{R})$ relies on the strong continuity of translation, which we proved by first showing it for $C_c(\mathbb{R})$ and then by an extension by density to $L^p(\mathbb{R})$. In particular, we needed to already know that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proposition 6.1 (C^∞ Urysohn's lemma). *If* $K \subseteq \mathbb{R}$ *is compact and* $U \subseteq K$ *is open, then there exists* $f \in C_c^\infty(\mathbb{R})$ *such that* $0 \leq f \leq 1$, $f = 1$ *on* K, *and* $f = 0$ *on* U^c.

Proof. Since K is compact and U^c is closed, we have

$$d = \mathrm{dist}(K, U^c) = \inf\{|x - y| : k \in K, y \notin U\} > 0.$$

Set $V = \{y \in \mathbb{R} : \mathrm{dist}(y, K) < d/3\}$, and choose any $k \in C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(k) \subseteq [-d/3, d/3]$ and $\int k = 1$. Take $f = k * \chi_V \in C_c^\infty(\mathbb{R})$, which has $\mathrm{supp}(f) \subseteq \mathrm{supp}(k) + V \subseteq U$. If $x \in K$, then

$$f(x) = \int_V k(x - y)\, dy = \int k = 1.$$

One can check that $0 \leq f \leq 1$ and $f = 0$ on U^c as an exercise, which would prove the result. $\qquad\square$

Lecture 7

Sept. 18 —

Lecture 8

Sept. 23 —

Lecture 9

Sept. 25 — Kernels and Schwartz Space

9.1 More Kernels

Remark. Recall that if $\{k_\lambda\}$ (where $k_\lambda(x) = \lambda k(\lambda x)$) is an approximate identity, then $f * k_\lambda \to f$ in L^p-norm. We also have
$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k_\lambda}.$$
Note that $\widehat{k_\lambda}(\xi) = \widehat{k}(\xi/\lambda)$, so $\widehat{k_\lambda}(\xi) \to 1$ pointwise. If the Dirichlet kernel $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$ were integrable, then we would have $\widehat{d}_{2\pi\lambda}(\xi) = \chi_{[-\lambda,\lambda]}(\xi)$, so we must have $d_{2\pi\lambda} \notin L^1(\mathbb{R})$.

Some alternatives are the following:

1. The Fejér kernel $w_\lambda(\xi) = d_{2\pi\lambda}^2(\xi) \in L^1(\mathbb{R})$. We saw this in the proof of the inversion formula.

2. The *de la Vallée Poussin kernel* v_λ, which has $\widehat{v}_\lambda$ as a trapezoid which is 1 on $[-\lambda, \lambda]$ and decays linearly to 0 on $[-2\lambda, -\lambda]$ and $[\lambda, 2\lambda]$. One has

$$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

   Explicitly, one can define $v(x) = 2w_2(x) - w(x)$ and $v_\lambda(x) = \lambda v(\lambda x)$.

3. The *Poisson kernel* $p(x) = 1/\pi(x^2 + 1)$.

4. The *Gauss kernel* $\phi(x) = e^{-\pi x^2}$.

Exercise 9.1. Show that if $f \in L^1(\mathbb{R})$, then $\text{supp}(\widehat{f})$ is compact if and only if $f = f * g$ for some $g \in L^1(\mathbb{R})$. Hint: Use the de la Vallée Poussin kernel.

Exercise 9.2. Let $\Phi = \widehat{\phi}$, where ϕ is the Gauss kernel. Show that $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$. Then solve this differential equation to get $\Phi(\xi) = \Phi(0)\phi(\xi)$. Finally show that $\Phi(0) = 1$, and conclude $\Phi(\xi) = \phi(\xi)$.

Remark. There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if $f, \widehat{f} \in L^1(\mathbb{R})$, then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for f sufficiently nice, if we take $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge}$, then $\widehat{g} = g$.

Theorem 9.1 (Weierstrass approximation theorem)**.** *If $f \in C[a, b]$ and $\epsilon > 0$, then there exists a polynomial p such that $\|f - p\|_\infty = \sup_{x \in [a,b]} |f(x) - p(x)| < \epsilon$.*

Proof. Fix $f \in C[a, b]$, and choose $[a, b] \subseteq (-R, R)$. Extend f to a function $g \in C_0(\mathbb{R})$ on $\mathbb{R}$ which equals f on $[a, b]$ and is supported in $(-R, R)$. Let ϕ be the Gauss kernel, and choose λ so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that $\phi_\lambda(x) = \lambda e^{-\pi \lambda^2 x^2}$ is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^\infty \lambda \frac{(-\pi \lambda^2 x^2)^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set $q(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$. If N is large enough, then we will have

$$\sup_{|x| \le 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \le \int_{-R}^R |g(y)| |\phi_\lambda(x - y) - q(x - y)| \, dy \le \frac{\epsilon}{2\|g\|_1} \int_{-R}^R |g(y)| \, dy < \frac{\epsilon}{2}$$

for $|x| \le R$. So $|f(x) - (g * q)(x)| \le \epsilon$ for $x \in [a, b]$. Finally, observe that

$$(g * q)(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y)(x - y)^{2n} \, dy,$$

which is a polynomial by the binomial theorem. So we can take $p = g * q$. $\quad\square$

9.2 Schwartz Space

Definition 9.1. Define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \ge 0\}.$$

Remark. Note that $|x^m f^{(n)}(x)| \le C_{m,n}$ for some constant $C_{m,n}$, so

$$|f^{(n)}(x)| \le \frac{C_{m,n}}{|x|^m}.$$

In particular, for every n and polynomial p, there exists a constant $C_{n,p}$ such that

$$|f^{(n)}(x)| \le \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply f has exponential decay.

Remark. Note that $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$ defines a *seminorm* for each $m, n \ge 0$, but not a norm.[1]

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[1]Recall that a *seminorm* ρ is a function satisfying $0 \le \rho(f) < \infty$, $\rho(cf) = |c|\rho(f)$, and $\rho(f + g) \le \rho(f) + \rho(g)$. If we additionally have $\rho(f) = 0$ if and only if $\rho = 0$, then ρ is a *norm*.

Remark. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\|x^m f^{(n)}\|_1 = \int_{|x| \leq 1} |x^m f^{(n)}(x)| \, dx + \int_{|x| \geq 1} |x^m f^{(n)}(x)| \, dx$$

$$\leq 2\|x^m f^{(n)}\|_\infty + \int_{|x| \geq 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2} \, dx \leq 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty.$$

In particular, if the L^∞-norms are controlled, then so are the L^1-norms.

Exercise 9.3. Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write $Df = f'$, and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \widehat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^{n} \binom{n}{j} D^j (-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on f implies the Schwartz condition on $\widehat{f}$.

Theorem 9.2. *If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, we have $f = (\widehat{f})^\vee$ by the inversion formula, so the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $f \mapsto \widehat{f}$ is a bijection.*

Remark. Note that $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, and we will see later that $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$.

Lecture 10

Sept. 30 —

Lecture 11

Oct. 2 —

Lecture 12

Oct. 9 —

Lecture 13

Oct. 14 —

Lecture 14

Oct. 16 —

Lecture 15

Oct. 23 — The Paley-Wiener Theorem

15.1   Local Uncertainty Principle

Remark. Recall the classical uncertainty principle: For $f \in L^2(\mathbb{R})$, $\|f\|_2^2 \leq 4\pi \|xf(x)\|_2^2 \|\xi \widehat{f}(\xi)\|_2^2$. There is also a local version of this statement:

Theorem 15.1 (Local uncertainty principle). *If $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, then for any $\xi_0 \in \mathbb{R}$, we have*

$$\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{f}(\xi)|^2 \, d\xi \leq 8\pi\epsilon \|f\|_2 \|xf(x)\|_2.$$

15.2   The Paley-Wiener Theorem

Remark. Recall the smoothness and decay theorems: If $x, xf(x) \in L^1(\mathbb{R})$, then $(\widehat{f})'$ exists. "Extreme decay" corresponds to compact support, where $\operatorname{supp}(f) \subseteq [-T, T]$ if $f = 0$ a.e. outside $[-T, T]$. When this is the case, the *Paley-Wiener theorem* says that $\widehat{f}$ has extreme smoothness: $\widehat{f}$ extends to an analytic function on $\mathbb{C}$.

Remark. There is also motivation from Fourier series. The Fourier transform on $\mathbb{R}$ is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx.$$

The functions $e_\xi(x) = e^{2\pi i \xi x} : \mathbb{R} \to S^1 \subseteq \mathbb{C}$ are continuous group homomorphism, and dx is a Haar measure on $\mathbb{R}$ (invariant under the group action). For $\mathbb{Z}$, Haar measure is the counting measure, and for $c = (c_n)_{n \in \mathbb{Z}} \in \ell^1$, we can define its Fourier transform to be

$$\widehat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{R}.$$

In fact, we can take $\xi \in [0, 1)$. Also note that $\widehat{c}$ is 1-periodic.

Suppose now that c is compactly supported, say only $c_1, \ldots, c_N$ are nonzero. Then

$$\widehat{c}(\xi) = \sum_{n=1}^{N} c_n e^{-2\pi i n \xi} = \sum_{n=1}^{N} c_n (e^{-2\pi i \xi})^n$$

If we set $z = e^{-2\pi i \xi} \in S^1$, then the above becomes

$$\widehat{c}(z) = \sum_{n=1}^{N} c_n z^n, \quad |z| = 1,$$

which is a polynomial. In particular, $\widehat{c}$ extends to an analytic function for all $z \in \mathbb{C}$.

Remark. Suppose $f \in L^2(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [-T, T]$ (we say that f is "time-limited" to $[-T, T]$). Then

$$\int_{-\infty}^{\infty} |f(x)| \, dx = \int_{-T}^{T} |f(x)| \, dx \le \left(\int_{-T}^{T} |f(x)|^2 \, dx \right)^{1/2} \left(\int_{-T}^{T} 1^2 \, dx \right)^{1/2} = (2T)^{1/2} \|f\|_2 < \infty,$$

so $f \in L^1(\mathbb{R})$. Thus we can take its Fourier transform as usual:

$$\widehat{f}(\xi) = \int_{-T}^{T} f(x) e^{-2\pi i \xi x} \, dx, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $e^{-2\pi i \xi x} = e^{-2\pi i \alpha x} e^{2\pi \beta x}$, where $e^{2\pi \beta x} > 0$. Then

$$\int_{-T}^{T} |f(x) e^{-2\pi i \xi x}| \, dx = \int_{-T}^{T} |f(x)| e^{2\pi \beta x} \, dx \le e^{2\pi |\beta| T} \int_{-T}^{T} |f(x)| \, dx \le (2T)^{1/2} e^{2\pi |\beta| T} \|f\|_2.$$

So $\widehat{f}$ can be extended to all of $\mathbb{C}$. We will see that this extension is analytic.

We now have a function $\widehat{f} : \mathbb{C} \to \mathbb{C}$ satisfying

$$|\widehat{f}(\xi)| \le (2T)^{1/2} e^{2\pi |\beta| T} \|f\|_2, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $f \in L^1(\mathbb{R})$ implies $\widehat{f} \in C_0(\mathbb{R})$, so $\widehat{f}$ is bounded on $\mathbb{R}$. However, the extension to $\mathbb{C}$ cannot be bounded (for non-constant f) by Liouville's theorem.

Definition 15.1. A function $F : \mathbb{C} \to \mathbb{C}$ has *exponential type T* if there exists $A > 0$ such that

$$|F(\xi)| \le A e^{T |\xi|}, \quad \xi \in \mathbb{C}.$$

Remark. So from the above discussion, $\widehat{f}$ has exponential type $2\pi T$. We now show that $\widehat{f}$ is analytic on $\mathbb{C}$. Recall that F is analytic if and only if F' exists on $\mathbb{C}$. We can compute

$$(\widehat{f})'(\xi) = \lim_{\substack{\eta \to 0 \\ \xi \in \mathbb{C}}} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \to 0} \int_{-T}^{T} f(x) \frac{e^{-2\pi i (\xi + \eta) x} - e^{-2\pi i \xi x}}{\eta} \, dx.$$

We have seen that f is integrable, and the exponential term is bounded on $[-T, T]$ by the mean value theorem. So by the dominated convergence theorem, we can exchange the limit and integral to get

$$(\widehat{f})'(\xi) = \int_{-T}^{T} f(x) \lim_{\eta \to 0} \frac{e^{-2\pi i (\xi + \eta) x} - e^{-2\pi i \xi x}}{\eta} \, dx$$

$$= \int_{-T}^{T} (-2\pi i x) f(x) e^{-2\pi i \xi x} \, dx = ((-2\pi i x) f(x))^{\wedge}(\xi).$$

Note that $x f(x) \in L^1(\mathbb{R})$ and also has support in $[-T, T]$, so the above exists for any $\xi \in \mathbb{C}$.

Theorem 15.2 (Paley-Wiener theorem). *We have the following:*

1. *If $f \in L^2(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [-T, T]$, then $\widehat{f}$ extends to an analytic function on $\mathbb{C}$ with exponential type $2\pi T$.*

2. *If $F : \mathbb{C} \to \mathbb{C}$ is analytic with exponential type $2\pi T$, then there exists $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(f) \subseteq [-T, T]$ and $\widehat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}$, i.e. $\widehat{f} = F|_{\mathbb{R}}$.*

Proof. (1) This was the above discussion.

(2) The proof is more difficult and uses a lot of complex analysis. See Katznelson's book. $\qquad\square$

Remark. Recall that analytic functions have Taylor expansions. Suppose $F : \mathbb{C} \to \mathbb{C}$ and $F = 0$ on a line segment. Then if η lies on the line segment, we can write

$$F(\xi) = \sum_{k=0}^{\infty} F^{(k)}(\eta) \frac{(\xi - \eta)^k}{k!}.$$

Since $F^{(k)}(\eta) = 0$ for all k, we get $F \equiv 0$ on all of $\mathbb{C}$. In fact, if $F = 0$ on any set with an accumulation point, then $F \equiv 0$ on $\mathbb{C}$.

Corollary 15.2.1. *If $f \in L^2(\mathbb{R})$, $f \neq 0$, has compact support, then $\widehat{f}(\xi) = 0$ for only countably many ξ.*

Proof. If $\widehat{f}(\xi) = 0$ for uncountably many ξ, then there exists $\xi_n \to \xi$ such that $\widehat{f}(\xi_n) = 0$ for all n (split $\mathbb{C}$ into countably many compact sets and apply a countability argument). Hence $\widehat{f}, f = 0$. $\qquad\square$

Corollary 15.2.2. *If $f \in L^2(\mathbb{R})$, then f and $\widehat{f}$ are both compactly supported if and only if $f = 0$.*

Proof. Suppose f is compactly supported, then $\widehat{f}$ is analytic on $\mathbb{C}$. Suppose $\widehat{f}(\xi) = 0$ a.e. on $\mathbb{R}$ outside $[-\Omega, \Omega]$. Since $\widehat{f}$ is analytic and $\mathbb{R} \setminus [-\Omega, \Omega]$ has an accumulation point, we get $\widehat{f}, f = 0$. $\qquad\square$

Remark. In particular, the above corollary implies that if $f \in C_c^\infty(\mathbb{R})$ is nonzero, then $\widehat{f}$ is not compactly supported. But $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, so we do know $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R})$.

15.3   Hilbert-Schmidt Operators

Remark. Let H be a separable, infinite-dimensional Hilbert space. Recall the following definitions and results from functional analysis:

Definition 15.2. A linear operator $A : H \to H$ is *compact* if $A(D)$ is contained in a compact set, where $D = \{f \in H : \|f\| \leq 1\}$ is the closed unit disk in H.

Theorem 15.3 (Spectral theorem for compact, self-adjoint operators). *If $A : H \to H$ is a compact, self-adjoint operator, then there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H such that*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \quad Af = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n, \quad Ae_n = \lambda_n e_n.$$

Moreover, $\lambda_n \to 0$ if there are infinitely many $\lambda_n \neq 0$.

Remark. Note that $\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n| \to 0$, so the identity operator I is not compact.

Example 15.2.1. An example of a compact box in $\mathbb{R}^{\mathbb{N}}$ looks like

$$[0, 1] \times [0, 1/2] \times [0, 1/4] \times \cdots .$$

Definition 15.3. A linear operator $A : H \to H$ is *Hilbert-Schmidt* if there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$.

Remark. For an operator $A : H \to H$, if we write

$$Af(x) = \int_{-\infty}^{\infty} k(x, y) f(y) \, dy,$$

then A is Hilbert-Schmidt if and only if $k \in L^2(\mathbb{R}^2)$.

Lecture 16

Oct. 28 —

Lecture 17

Oct. 30 —

Lecture 18

Nov. 4 — Energy Concentration, Part 2

18.1 Energy Concentration, Continued

Remark. Recall the following problem. Fix $T, \Omega > 0$. We want to determine

$$E_{T,\Omega} = \sup \left\{ \int_{-T}^{T} |f|^2 : \|f\|_2 = 1, \operatorname{supp}(\widehat{f}) \subseteq [-\Omega, \Omega] \right\}.$$

We previously defined $A_T f = f \chi_{[-T,T]}$ and $B_\Omega f = (\widehat{f} \chi_{[-\Omega,\Omega]})^\vee = f * d_{2\pi\Omega}$, then

$$E_{T,\Omega} = \|A_T B_\Omega\|^2 = \|B_\Omega A_T B_\Omega\|^2 = \lambda_1,$$

where λ_1 is the largest eigenvalue of $B_\Omega A_T B_\Omega$ (recall that $B_\Omega A_T B_\Omega$ is compact and self-adjoint, so it has eigenvalues $\lambda_n \to 0$). The spectral theorem also gives orthonormal eigenvectors $\{\varphi_n\}_{n \in \mathbb{N}}$:

$$B_\Omega A_T B_\Omega f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n.$$

Note that $B_\Omega(A_T B_\Omega \varphi_n) = \lambda_n \varphi_n$ for $\lambda_n \neq 0$, so $\varphi_n \in \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ and $\operatorname{supp}(\widehat{\varphi}_n) \subseteq [-\Omega, \Omega]$.

What is $\|A_T \varphi_n\|_2^2$? We can compute

$$\|A_T \varphi_n\|_2^2 = \langle A_T \varphi_n, A_T \varphi_n \rangle = \langle A_T B_\Omega \varphi_n, A_T B_\Omega \varphi_n \rangle$$
$$= \langle B_\Omega \underbrace{A_T A_T}_{A_T} B_\Omega \varphi_n, \varphi_n \rangle = \lambda_n \langle \varphi_n, \varphi_n \rangle = \lambda_n \|\varphi_n\|_2^2 = \lambda_n.$$

Thus we see that $E_{T,\Omega} = \lambda_1 = \|A_T \varphi_1\|_2^2 < \|\varphi_1\|_2^2 = 1$, so φ_1 has the greatest energy in $[-T, T]$.

Note that $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for its closed span, which a priori lies in $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$.

Proposition 18.1. $\overline{\operatorname{span}}\{\varphi_n\} = \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$.

Proof. We show $\overline{\operatorname{span}}\{\varphi_n\}^\perp = \{0\}$. Suppose $f \in \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ with $f \perp \varphi_n$ for all n. Then

$$f \in \overline{\operatorname{span}}\{\varphi_n\}^\perp = \overline{\operatorname{range}}(B_\Omega A_T B_\Omega)^\perp = \ker(B_\Omega A_T B_\Omega).$$

Thus we have

$$\|A_T f\|_2^2 = \langle A_T B_\Omega f, A_T B_\Omega f \rangle = \langle B_\Omega A_T B_\Omega f, f \rangle = 0,$$

so $f = 0$ a.e. on $[-T, T]$. By the Paley-Wiener theorem, this implies $f = 0$. $\qquad \square$

Remark. Proposition 18.1 implies that there are infinitely many nonzero eigenvalues for $B_\Omega A_T B_\Omega$.

Moreover, since B_Ω is the orthogonal projection onto $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ and $\{\varphi_n\}$ is an orthonormal basis for $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$, we can write

$$B_\Omega f = \sum_{n=1}^\infty \langle f, \varphi_n\rangle \varphi_n.$$

Note that $L^2_{[-T,T]}(\mathbb{R})$ is not orthogonal to $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$: For example,

$$\langle \chi_{[-T,T]}, d_{2\pi\Omega}\rangle = \int_{-T}^T \frac{\sin 2\pi\Omega x}{\pi x}\,dx \neq 0.$$

However, set $\psi_n = \lambda_n^{-1/2} A_T \varphi_n$. Then

$$\langle \lambda_m^{-1/2} A_T \varphi_m, \lambda_n^{-1/2} A_T \varphi_n\rangle = \lambda_m^{-1/2}\lambda_n^{-1/2}\langle B_\Omega A_T B_\Omega \varphi_m, \varphi_n\rangle = \lambda_m^{1/2}\lambda_n^{-1/2}\langle \varphi_m, \varphi_n\rangle = \delta_{mn}.$$

So $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{[-T,T]}(\mathbb{R})$.

Exercise 18.1. Show that $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2_{[-T,T]}(\mathbb{R})$.

Remark. Note that

$$\langle \varphi_m, \varphi_n\rangle = \int_{-\infty}^\infty \varphi_m(x)\overline{\varphi_n(x)}\,dx = 0, \quad m \neq n.$$

Furthermore,

$$\langle \psi_m, \psi_n\rangle = \lambda_m^{-1/2}\lambda_n^{-1/2}\int_{-T}^T \varphi_m(x)\overline{\varphi_n(x)}\,dx = 0, \quad m \neq n$$

even when the φ_n are *not* supported in $[-T,T]$.

By symmetry (swapping the roles of A_T and B_Ω), one can also compute that

$$A_T B_\Omega A_T f = \sum_{n=1}^\infty \lambda_n\langle f, \psi_n\rangle \psi_n.$$

In particular, $A_T B_\Omega A_T$ and $B_\Omega A_T B_\Omega$ have the same eigenvalues.

Proposition 18.2. $B_\Omega A_T$ commutes with

$$Kf = (T^2 - x^2)f''(x) - 2xf'(x) - 4\pi^2\Omega^2 x^2 f(x).$$

Remark. Since φ_n is already band-limited, we have

$$B_\Omega A_T \varphi_n = B_\Omega A_T B_\Omega \varphi_n = \lambda_n\varphi_n.$$

Moreover, Paley-Wiener implies φ_n is infinitely differentiable, so $\varphi_n \in \text{domain}(K)$. Then

$$B_\Omega A_T K\varphi_n = KB_\Omega A_T \varphi_n = K(\lambda_n\varphi_n) = \lambda_n K\varphi_n.$$

Note that $K\varphi_n \neq 0$, so $K\varphi_n$ is an eigenvector for $B_\Omega A_T$. The multiplicities of the λ_n is 1, so $K\varphi_n = \mu\varphi_n$. The eigenfunctions φ_n of K are known functions called the *prolate spheroidal wave functions*.

18.2   Approximating Band-Limited Functions

Remark. We know that $1 > \lambda_1 > \lambda_2 > \cdots \to 0$. Also,

$$\|A_T B_\Omega\|_{\mathrm{HS}}^2 = \|g_{T,\Omega}\|_2^2 = \int_{-T}^{T} \int_{-\infty}^{\infty} d_{2\pi\Omega}(x-y)^2 \, dxdy = 4T\Omega,$$

where $g_{T,\Omega}$ is the corresponding kernel. The singular values of $A_T B_\Omega$ are

$$s_n = \lambda_n((A_T B_\Omega)^*(A_T B_\Omega))^{1/2} = \lambda_n(B_\Omega A_T B_\Omega)^{1/2} = \lambda_n^{1/2}.$$

Thus $4T\Omega = \|A_T B_\Omega\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} s_n^2 = \sum_{n=1}^{\infty} \lambda_n$. On the other hand, $\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^2$, and

$$\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 = \|k_{T,\Omega}\|_2^2 = \int_{-T}^{T} \int_{-T}^{T} d_{2\pi\Omega}(x-y)^2 \, dxdy < 4T\Omega.$$

Due to the decay of $d_{2\pi\Omega}(x-y)^2$, for T large, we should expect that $\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 \to 4T\Omega$. Hence

$$\sum_{n=1}^{\infty} \lambda_n^2 \approx \sum_{n=1}^{\infty} \lambda_n.$$

Thus intuitively, λ_n should be either close to 1 or close to 0. In particular, only finitely many terms can be close to 1, so

$$f \approx \sum_{n=1}^{N} \lambda \langle f, \varphi_n \rangle \varphi_n \in \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}.$$

Thus the space of band-limited functions are approximately finite-dimensional in some sense.

Theorem 18.1. $\#\{n : \lambda_n \geq 1 - \epsilon\} \leq 4T\Omega - C_\epsilon \log(T\Omega)$.

18.3   Space of Possible Time and Band Limits

Remark. Define the energies

$$\alpha = E_T(f) = \|A_T f\|_2^2 = \int_{-T}^{T} |f|^2$$

$$\beta = E_\Omega(f) = \|B_\Omega f\|_2^2 = \int_{-\Omega}^{\Omega} |\widehat{f}|^2.$$

Which pairs (α, β) can be achieved? We have seen that $(\lambda_1, 1)$ and $(\lambda_2, 1)$ are possible, while $(1, 1)$ is not by the Paley-Wiener theorem. Similarly, $(1, 0)$ and $(0, 1)$ are not possible by Paley-Wiener. Since the roles of A_T and B_Ω are symmetric, we get that $(1, \lambda_1)$ and $(1, \lambda_2)$ are possible. Note that if $(\alpha, 1)$ is achieved, then so is $(\alpha', 1)$ for any $\alpha' < \alpha$, and a similar statement holds for $(1, \beta)$.

One can show the space of valid pairs is $(\alpha, \beta) \in [0, 1]^2$ such that

$$\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \lambda^{1/2},$$

which is in the shape of an ellipse.

Lecture 19

Nov. 6 — Fourier Series

19.1 The Abstract Fourier Transform

Definition 19.1. A *locally compact abelian (LCA) group* is an abelian group with a Hausdorff topology such that every point has a compact neighborhood, and $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous.

Example 19.1.1. Some examples of LCA groups are $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{T}^n$, $\mathbb{Z}_N^n$, where $\mathbb{T} = [0, 1)$ with addition modulo 1.

Theorem 19.1. *Every LCA group has a* Haar measure, *i.e. a nonzero Radon measure μ which is translation-invariant.*

Definition 19.2. A *character* on a LCA group is a continuous homomorphism

$$\xi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

i.e. a continuous map with $\xi(x + y) = \xi(x)\xi(y)$.

Definition 19.3. The *dual group* of G is $\widehat{G} = \{\xi : \xi \text{ is a character of } G\}$.

Example 19.3.1. Consider $\mathbb{T} = [0, 1)$. Fix $n \in \mathbb{Z}$, then $e_n(x) = e^{2\pi i n x}$ is a character on $\mathbb{T}$. Note that we have $e_n(x) = e_1(x)^n$, and one can show these are all of the characters on $\mathbb{T}$. Thus

$$\widehat{\mathbb{T}} = \{e_n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Example 19.3.2. The characters on $\mathbb{Z}$ are determined by their values at 1. So for any $\xi \in [0, 1)$, we get a character $e_\xi(n) = e^{2\pi i \xi n}$. In particular, $\widehat{\mathbb{Z}} = \mathbb{T}$.

Example 19.3.3. We have $\widehat{\mathbb{Z}}_N \cong \mathbb{Z}_N$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$. The characters on $\mathbb{R}$ are $e_\xi(x) = e^{2\pi i \xi x}$.

Remark. The dual group $\widehat{G}$ is a LCA group under the multiplication of characters, where the topology on $\widehat{G}$ is the topology of uniform convergence on compact sets (also called the compact-open topology).

Theorem 19.2. *If G is discrete, then $\widehat{G}$ is compact. If G is compact (so G has finite measure), then $\widehat{G}$ is discrete, and the characters on G are orthonormal in $L^2(G)$ (in fact, they form an orthonormal basis).*

Example 19.3.4. Consider the examples $\widehat{\mathbb{Z}} = \mathbb{T}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$.

Remark. In some sense, $\mathbb{R}$ being non-compact and non-discrete makes things harder. On the other hand, having dilations in $\mathbb{R}$ can make harmonic analysis easier.

Theorem 19.3 (Pontryagin duality). $\widehat{\widehat{G}} \cong G$.

Definition 19.4. The *Fourier transform* of $f \in L^1(G)$ (complex-valued) is $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\, d\mu(x), \quad \xi \in \widehat{G},$$

where $d\mu$ is the Haar measure on G.

Example 19.4.1. On the real line, the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R}.$$

For $\mathbb{T}$, the Fourier coefficients are given by

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\, dx, \quad n \in \mathbb{Z}.$$

For $\mathbb{Z}$, given $c = (c_n)_{n \in \mathbb{Z}}$,

$$\widehat{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n x}, \quad x \in \mathbb{T}$$

For $\mathbb{Z}_N$, the discrete Fourier transform is

$$\widehat{c}(k) = \sum_{n=0}^{N-1} c_n e^{-2\pi i n k / N}, \quad k \in \mathbb{Z}_N.$$

19.2   Fourier Series

Definition 19.5. The *Fourier transform* of $f \in L^1(\mathbb{T})$ is the sequence $\widehat{f} = (\widehat{f}(n))_{n \in \mathbb{Z}}$, where

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\, dx$$

are the *Fourier coefficients* of f. The *inverse Fourier transform* of $f \in L^1(\mathbb{T})$ is $\check{f} = (\check{f}(n))_{n \in \mathbb{Z}}$, where

$$\check{f}(n) = \int_0^1 f(x)e^{2\pi i n x}\, dx.$$

Formally, $\sum_{n \in \mathbb{Z}} \widehat{f}(n)e^{2\pi i n x}$ is the *Fourier series* of f.

Remark. If $f \in L^2(\mathbb{T})$ and we accept that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n,$$

where the convergence is in $L^2(\mathbb{T})$ (we do not get pointwise convergence in general).

Remark. If $f \in L^1(\mathbb{T})$, then we have

$$|\widehat{f}(n)| \leq \int_0^1 |f(x)e^{-2\pi i n x}|\, dx = \|f\|_{L^1},$$

so we see that $\widehat{f} \in \ell^\infty(\mathbb{Z})$.

Remark. If we define $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, then we can view the Fourier series as

$$(\widehat{f})^{\vee}(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

Example 19.5.1. For $0 < \alpha \leq 1$, consider the function

$$g(x) = \sum_{m=0}^{\infty} 2^{-\alpha m} e^{2\pi i 2^m x}, \quad x \in \mathbb{T}$$

which converges absolutely in $C(\mathbb{T})$. One can show that g is nowhere differentiable.

Weierstrass first showed that $g(x) = \sum_{m=0}^{\infty} a^{-m} \cos(b^m x)$ is nowhere differentiable for a, b large enough.

19.3 Partial Sums of Fourier Series

Definition 19.6. For $f \in L^1(\mathbb{T})$, define the *partial sums*

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}.$$

Remark. Consider a discrete characteristic function

$$\chi_N(n) = \begin{cases} 1 & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that $\chi_N \in \ell^1(\mathbb{Z})$. Then we can compute that

$$\widehat{\chi}_N(x) = \check{\chi}_N(x) = \sum_{n=-\infty}^{\infty} \chi_N(n) e^{2\pi i n x} = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} = d_N(x)$$

by the geometric series formula. Note that $d_N \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$.[1] We also have $\|d_N\|_1 \to \infty$.

Remark. Using the *Dirichlet kernel* d_N, we can write the partial sums as

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-N}^{N} \left(\int_0^1 f(t) e^{-2\pi i n t} \, dt \right) e^{2\pi i n x}$$

$$= \int_0^1 f(t) \sum_{n=-N}^{N} e^{2\pi i n (x-t)} \, dt = \int_0^1 f(t) d_N(x-t) \, dt = (f * d_N)(x).$$

Intuitively, $S_N f = (\widehat{f} \chi_N)^{\vee} = (\widehat{f})^{\vee} * \check{\chi}_N = f * d_N$, but the above is a direct proof.

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[1] Note that $\mathbb{T}$ is compact, so a continuous function on $\mathbb{T}$ is automatically integrable.

19.4   Approximate Identities on $\mathbb{T}$

Definition 19.7. An *approximate identity* on $\mathbb{T}$ is a sequence $\{k_n\}_{n\in\mathbb{N}}$ such that

1. $k_n \in L^1(\mathbb{T})$ and $\displaystyle\int_0^1 k_n(x)\,dx = 1$;

2. $\sup_{n\in\mathbb{N}}\|k_n\|_1 < \infty$;

3. for any $\delta > 0$, $\displaystyle\int_{|x|\geq\delta}|k_n(x)|\,dx \to 0$.

Remark. The Dirichlet kernel d_N does not form an approximate identity.

Definition 19.8. For $f \in L^1(\mathbb{T})$, define the *Cesàro sums*

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + \cdots + S_N f(x)}{N+1}.$$

Note that this is the average of the partial sums.

Remark. We can rewrite the Cesàro sums as

$$\sigma_N f(x) = \sum_{n=-N}^{N} \underbrace{\left(1 - \frac{|n|}{N+1}\right)}_{W_N(n)} \widehat{f}(n)e^{2\pi inx} = (f * \widehat{W}_N)(x).$$

We define the *Fejér kernel* to be

$$w_N(x) = \widehat{W}_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right)^2.$$

Lecture 20

Nov. 11 — Fourier Series, Part 2

20.1 Approximate Identities on $\mathbb{T}$, Continued

Remark. Using a similar proof as in the case for $\mathbb{R}$, for $f \in L^1(\mathbb{T})$, we have

$$\widehat{f} \in c_0(\mathbb{Z}) = \{c = (c_n) : c_n \to 0 \text{ as } |n| \to \infty\}.$$

However, in general $\widehat{f} \notin \ell^1(\mathbb{Z})$.

Exercise 20.1. Let d_N be the Dirichlet kernel on $\mathbb{T}$. Show that

$$\frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \leq \|d_N\|_1 \leq 3 + \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}.$$

In particular, $\|d_N\|_1 \to \infty$ as $N \to \infty$, so d_N does not form an approximate identity.

Remark. Define the function

$$W_N(x) = \begin{cases} 1 - |n|/(N+1) & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that $W_N = (\chi_N * \chi_N)/(2N+1)$ and $\widehat{W}_{2N} = (\widehat{\chi}_N)^2 = d_N^2/(2N+1)$. The Fejér kernel is

$$w_N = \widehat{W}_N = \frac{1}{N} \left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2,$$

and one can check that $\int_0^1 w_N = 1$ and $\|w_N\|_1 = 1$.

Theorem 20.1. $(f * \widetilde{W}_N)(x) = (f * w_N)(x)$.

Exercise 20.2. $\{w_N\}_{N \in \mathbb{N}}$ forms an approximate identity on $\mathbb{T}$.

Remark. Unlike the real line, we cannot just take any L^1 function and dilate it to form an approximate identity on $\mathbb{T}$, as the dilations need not be 1-periodic.

Exercise 20.3. If $\{k_N\}$ is an approximate identity on $\mathbb{T}$, then:

1. For $1 \leq p < \infty$, $f \in L^p(\mathbb{T})$ implies $f * k_N \to f$ in L^p-norm.

2. $f \in C(\mathbb{T})$ implies $f * k_N \to f$ uniformly.

20.2   The Inversion Formula on $\mathbb{T}$

Theorem 20.2. *If $f \in L^1(\mathbb{T})$ and $\widehat{f} \in \ell^1(\mathbb{Z})$, then*

$$f(x) = (\widehat{f})^{\vee}(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x},$$

where the above series converges uniformly in $C(\mathbb{T})$.

Proof. Note that $(\widehat{f})^{\vee}(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x}$ converges absolutely in the norm of $C(\mathbb{T})$ since

$$\sum_{n=-\infty}^{\infty} \|\widehat{f}(n)e^{2\pi i n x}\|_{\infty} = \sum |\widehat{f}(n)| < \infty.$$

In particular, this shows that $(\widehat{f})^{\vee} \in C(\mathbb{T})$. Also $f * w_N \to f$ in L^1-form, where

$$(f * w_N)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)e^{2\pi i n x}.$$

Note that $(1 - |n|/(N+1))\widehat{f}(n) \to \widehat{f}(n)$ pointwise as $N \to \infty$, and

$$\left|\left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)\right| \leq |\widehat{f}(n)| \in \ell^1(\mathbb{Z}),$$

so by the dominated convergence theorem (for counting measure)

$$(f * w_N)(x) \xrightarrow[N\to\infty]{} \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x} = (\widehat{f})^{\vee}(x)$$

pointwise. Since $f * w_N \to f$ in L^1-norm, there exists a subsequence $f * w_{N_k} \to f$ pointwise a.e., hence we see that $f = (\widehat{f})^{\vee}$ a.e. Since f and $(\widehat{f})^{\vee}$ are continuous, we get that $f = (\widehat{f})^{\vee}$ everywhere. $\square$

Corollary 20.2.1. *If $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for every n, then $f = 0$ a.e. Moreover, if $f, g \in L^1(\mathbb{T})$ and $\widehat{f}(n) = \widehat{g}(n)$ for all n, then $f = g$ a.e.*

Definition 20.1. The *Fourier algebra* (or *Wiener algebra*) on $\mathbb{T}$ is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\}.$$

Remark. The Fourier algebra $A(\mathbb{T})$ is closed under convolution and forms a Banach algebra. Moreover, it is a dense subspace of $C(\mathbb{T})$.

20.3 L^2-Convergence of Fourier Series

Remark. Let $e_n(x) = e^{2\pi i n x}$. The functions e_n are orthonormal in $L^2(\mathbb{T})$: For $n \neq m$,

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i m x}\overline{e^{2\pi i n x}}\, dx = \int_0^1 e^{2\pi i (m-n)x}\, dx = \frac{e^{2\pi i (m-n)x}}{2\pi i (m-n)}\bigg|_{x=0}^{1} = 0.$$

Theorem 20.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. Then the following are equivalent:*

1. *$\{f_n\}_{n\in\mathbb{N}}$ is complete (i.e. $\overline{\mathrm{span}}\{f_n\} = H$).*

2. *$f = \sum_{n=1}^{\infty} c_n f_n$ in H for a unique choice of scalars c_n. That is, $\{f_n\}$ is a Schauder basis for H.*

3. *$f = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n$, where the convergence is in H.*

4. *(Plancherel) $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2$.*

5. *(Parseval) $\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, f_n\rangle \langle f_n, g\rangle$.*

Example 20.1.1. The above theorem does not hold when the sequence is not orthonormal. Recall the *Weierstrass approximation theorem*: $\{x^n\}_{n=0}^{\infty}$ is complete in $C(\mathbb{T})$. But not every $f \in C(\mathbb{T})$ can be written as $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (such functions are infinitely differentiable in some disk).

Theorem 20.4. *We have the following:*

1. *For $1 \le p < \infty$, $\{e_n\}_{n\in\mathbb{N}}$ is complete in $L^p(\mathbb{T})$.*

2. *$\{e_n\}_{n\in\mathbb{N}}$ is complete in $C(\mathbb{T})$.*

Proof. (1) If $f \in L^p(\mathbb{T})$, then we have

$$(f * w_N) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x} \xrightarrow[N\to\infty]{} f$$

in L^p-norm. But $f * w_n \in \mathrm{span}\{e_n\}_{n\in\mathbb{Z}}$, so $\overline{\mathrm{span}}\{e_n\}_{n\in\mathbb{Z}} = L^p(\mathbb{T})$. The same proof works for (2). $\quad\square$

Corollary 20.4.1. *$\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. Furthermore,*

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where the above convergence is in L^2-norm. Moreover, one has

1. *(Plancherel) $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$,*

2. *(Parseval) $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$,*

so the Fourier transform operator $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ given by $f \mapsto \widehat{f}$ is unitary.

Theorem 20.5. *If $1 < p < \infty$, then $S_N f = (f * d_N)(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}$ converges in L^p-norm, so*

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

in L^p-norm with respect to the ordering $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$. However, the convergence is conditional: Only this ordering (and finite permutations of it) need converge when $p \ne 2$.

Lecture 21

Nov. 13 — Fourier Series, Part 3

21.1 Shannon Sampling Theorem

Definition 21.1. Define the *Paley-Wiener space* to be

$$\text{PW}(\mathbb{R}) = \mathcal{F}L^2_{[-1/2,1/2]}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

Remark. Set $e_n(x) = e^{-2\pi i n x}\chi_{[-1/2,1/2]}(x) = M_n\chi_{[-1/2,1/2]}(x)$, then

$$\check{e}_n(x) = T_n\check{\chi}_{[-1/2,1/2]}(x) = \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

Since $\{e_n\}$ is an orthonormal basis for $L^2_{[-1/2,1/2]}(\mathbb{R})$, the $\{\check{e}_n\}$ form an orthonormal basis for $\text{PW}(\mathbb{R})$. Note that if $f \in \text{PW}(\mathbb{R})$, then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, \check{e}_n\rangle \check{e}_n(x).$$

By the unitarity of the Fourier transform, we can compute that

$$\langle f, \check{e}_n\rangle = \langle \widehat{f}, e_n\rangle = \int_{-1/2}^{1/2} \widehat{f}(\xi)e^{2\pi i n \xi}\,d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i n \xi}\,d\xi = (\widehat{f})^\vee(n) = f(n).$$

Note that the third equality follows from $\text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]$. Thus

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

This is called the *Shannon sampling theorem* (or *classical sampling theorem*), i.e. that any function in $\text{PW}(\mathbb{R})$ is completely determined by its values on the integers.

Remark. Fix $b > 0$, and set $e_{nb}(x) = e^{2\pi i n b x}$. This is an orthogonal basis for $L^2[0, 1/b]$.

Example 21.1.1. If we set $b = 1/2$, then we can write

$$\{e_{n/2}\}_{n\in\mathbb{Z}} = \{e_n\}_{n\in\mathbb{Z}} \cup \{e_{(n+1)/2}\}_{n\in\mathbb{Z}}.$$

This is a union of two orthonormal bases for $L^2[0, 1]$. For $f \in L^2[0, 1]$, we can write

$$2f = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n + \sum_{n=-\infty}^{\infty} \langle f, e_{(n+1)/2}\rangle e_{(n+1)/2} = \sum_{n\in\mathbb{Z}} \langle f, e_{n/2}\rangle e_{n/2}.$$

Remark. If we take $0 < b < 1$ and consider $e^{2\pi i n b x} = e_{bn}(x)$ in $L^2[0, 1]$, then for $f \in L^2[0, 1] \subseteq L^2[0, 1/b]$,

$$f = \frac{1}{b} \sum_{n=-\infty}^{\infty} \langle f, e_{bn} \rangle e_{bn}$$

in $L^2[0, 1/b]$. Since $f = 0$ on $[1, 1/b]$, we have the same expansion in $L^2[0, 1]$. Note, however, that this expansion is not orthogonal and not unique. These sets $\{e_{bn}\}_{n \in \mathbb{Z}}$ are called *frames*.

21.2   Weyl's Equidistribution Theorem

Theorem 21.1. *If $\alpha \in \mathbb{T}$ is irrational, then $\{k\alpha\}_{k \in \mathbb{N}}$ is* equidistributed *in $[0, 1)$, i.e.*

$$\lim_{N \to \infty} \frac{\#\{1 \le k \le N : k\alpha \bmod 1 \in (a, b)\}}{N} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha) = b - a$$

for every $0 \le a < b \le 1$.

Remark. Consider $f \in C(\mathbb{T})$. Then we can view $\frac{1}{N} \sum_{k=1}^{N} f(k\alpha)$ as some type of Riemann sum, so we might expect that

$$\frac{1}{N} \sum_{n=1}^{N} f(k\alpha) \xrightarrow[N \to \infty]{} \int_0^1 f(x) \, dx.$$

The *Birkhoff ergodic theorem* says that the above average in fact converges to the integral (given that $\{k\alpha\}$ is equidistributed). In fact, the result holds for any $f \in L^1(\mathbb{T})$.

Exercise 21.1. Prove the above statement for $e_n(x) = e^{2\pi i n x}$ (and thus for trigonometric polynomials $p(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$). Then apply density to prove this for $f \in C(\mathbb{T})$.

Proof of Theorem 21.1. Approximate $\chi_{(a,b)}$ by a trapezoidal continuous function f which equals $\chi_{(a,b)}$ on $(a + \epsilon, b - \epsilon)$ and decays linearly to 0 on $[a, a + \epsilon]$ and $[b - \epsilon, b]$. Then

$$b - a - \epsilon = \int_0^1 f(x) \, dx = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k\alpha) \le \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha).$$

Taking a similar approximation with g which equals $\chi_{(a,b)}$ on $[a, b]$ and decays linearly to 0 on $[a - \epsilon, a]$ and $[b, b + \epsilon]$, we also get an upper bound:

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} g(k\alpha) \le \int_0^1 g(x) \, dx = b - a + \epsilon.$$

This holds for arbitrary ϵ, hence the limit is equal to $b - a$. $\qquad \square$

Corollary 21.1.1 (Kronecker's theorem)**.** $\{k\alpha \bmod 1\}_{k \in \mathbb{N}}$ *is dense in $[0, 1)$.*

21.3 L^p-Convergence of Fourier Series

Remark. Recall that $\{e_n\}_{n \in \mathbb{Z}}$ is complete in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, i.e. $\mathrm{span}\{e_n\}_{n \in \mathbb{Z}}$ is dense.

For $1 \leq p < \infty$, if $f \in L^p(\mathbb{T})$, do the partial sums

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} = (f * d_N)(x)$$

converge to f in L^p-norm?

The motivation is as follows. Suppose that $S_N f \to f$ in L^p for every $f \in L^p(\mathbb{T})$. Then we have a linear operator $S_N : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ on a Banach space. Moreover, the convergence is also pointwise. Thus for any individual $f \in L^p(\mathbb{T})$,

$$\sup_{N \in \mathbb{N}} \|S_N f\|_p < \infty.$$

Then by the Banach-Steinhaus theorem (also called the uniform boundedness principle), we get

$$\sup_{N \in \mathbb{N}} \|S_N\| = \sup_{N \in \mathbb{N}} \sup_{\|f\|_p = 1} \|S_N f\|_p < \infty$$

In fact, this is equivalent to the L^p-convergence of Fourier series, and idea behind the proof is to instead argue about the uniform boundedness of S_N.

Note that $\|S_N f\|_p = \|f * d_N\|_p \leq \|f\|_p \|d_N\|_1$, so $\|S_N\| \leq \|d_N\|_1 \to \infty$.

For $p = 1$, we can argue as follows. Fix an approximate identity $\{k_\lambda\}$ with $k_\lambda \geq 0$, so that

$$\|k_\lambda\|_1 = \int_0^1 |k_\lambda| = \int_0^1 k_\lambda = 1.$$

Then we can see that

$$\|S_N\| \geq \|S_N k_\lambda\|_1 = \|k_\lambda * d_N\|_1 \xrightarrow[\lambda \to \infty]{} \|d_N\|_1$$

since $k_\lambda * d_N \to d_N$ in L^1. Thus $\|S_N\| \geq \|d_N\|_1 \to \infty$, so the partial sums cannot converge in L^1. In particular, $\{e_n\}_{n \in \mathbb{N}}$ is *not* a Schauder basis for $L^1(\mathbb{T})$. A similar argument works to show that $\{e_n\}_{n \in \mathbb{N}}$ is not a Schauder basis for $C(\mathbb{T})$. However, one has the following:

Theorem 21.2. $\{e_n\}_{n \in \mathbb{Z}}$ *is a Schauder basis for* $L^p(\mathbb{T})$ *for* $1 < p < \infty$, *with the ordering*

$$\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}.$$

Example 21.1.2 (Haar system). Another example of a Schauder basis is the following. Let

$$\varphi = \chi_{[0,1)} \quad \text{and} \quad \psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

Note that $\langle \varphi, \psi \rangle = 0$. Then we can define

$$\psi_1 = 2^{1/2} \psi(2x) \quad \text{and} \quad \psi_2 = 2^{1/2} \psi(2x - 1),$$

which are still orthogonal with φ, ψ and with each other. Continuing this process, we get an orthonormal basis for $L^2(\mathbb{T})$. Moreover, this is a Schauder basis for $L^p(\mathbb{T})$ for $1 < p < \infty$ and it is unconditional, i.e. the convergence does not depending on the ordering of the functions.

Remark. There also exists a continuous analogue of the Haar system, the functions are called D_4 and W_4. The functions φ, ψ from the Haar system are known as D_2, W_2. These are called *wavelets*.

Lecture 22

Nov. 18 — Fourier Series, Part 4

22.1 The Poisson Summation Formula

Remark. Suppose $f \in L^1(\mathbb{R})$ Set $\varphi(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Then by Fubini-Tonelli,

$$\|\varphi\|_1 = \int_0^1 \left| \sum_{n=-\infty}^{\infty} f(x+n) \right| \le \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)| = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_1 < \infty,$$

so $\varphi \in L^1(\mathbb{T})$. Moreover, the integrals are the same if we do not take absolute values.

Exercise 22.1. Let f, φ as above. Prove that

$$\widehat{\varphi}(n) = \int_0^1 \sum_{n=-\infty}^{\infty} f(x) e^{-2\pi i n x} \, dx = \widehat{f}(n).$$

Theorem 22.1 (Poisson summation). *Let $f : \mathbb{R} \to \mathbb{C}$. If there exist $C, \epsilon > 0$ such that*

$$|f(x)| \le \frac{C}{(1+|x|)^{1+\epsilon}} \quad and \quad |\widehat{f}(x)| \le \frac{C}{(1+|x|)^{1+\epsilon}},$$

then for every $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

In particular, setting $x = 0$ one obtains

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

Proof. Both $f, \widehat{f} \in L^1(\mathbb{R})$, hence $f, \widehat{f} \in C_0(\mathbb{R})$ by Fourier inversion. Also $\varphi \in L^1(\mathbb{T})$ and

$$\sum_{n=-\infty}^{\infty} |\widehat{\varphi}(n)| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty,$$

so $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. Therefore the inversion formula gives

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) e^{2\pi i n x}$$

pointwise. (In fact one can get uniform convergence on compact sets.) $\qquad \square$

Remark. Formally consider $\mu = \sum_{n=-\infty}^{\infty} \delta_n$, where δ_n is the Dirac delta at n. Then

$$\langle f, \mu \rangle = \int_{-\infty}^{\infty} f(x) \, d\mu(x) = \sum_{n=-\infty}^{\infty} f(n).$$

A similar computation shows that

$$\langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

These series are equal by Poisson summation, so if the Parseval identity holds, then one gets

$$\langle f, \widecheck{\mu} \rangle = \langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n) = \langle f, \mu \rangle.$$

Thus Poisson summation says that $\mu = \widecheck{\mu}$.

22.2 Wiener's Lemma

Remark. Let $C(\mathbb{T}) = \{f \text{ continuous on } \mathbb{T}\}$, which is closed under products and

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty.$$

In particular, $C(\mathbb{T})$ is a commutative Banach algebra. Also note that if $f \in C(\mathbb{T})$ and $f(x) \neq 0$ for every x, then $1/f \in C(\mathbb{T})$. We say that $C(\mathbb{T})$ is *inverse-closed.*

Recall that the Wiener (Fourier) algebra is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\} = \{\widecheck{c} : c = (c_n) \in \ell^1(\mathbb{Z})\}.$$

The equality holds by the formulas $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$ and $\widecheck{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$. Note that $A(\mathbb{T}) \subseteq C(\mathbb{T})$ is dense in the uniform norm.

On the other hand, if we define

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|,$$

then for any $f, g \in A(\mathbb{T})$, we have

$$fg = (\widehat{fg})^\vee = (\widehat{f} * \widehat{g})^\vee.$$

Since $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$, we have $\widehat{f} * \widehat{g} \in \ell^1(\mathbb{Z})$, so $fg \in A(\mathbb{T})$. Moreover,

$$\|fg\|_{A(\mathbb{T})} = \|\widehat{f} * \widehat{g}\|_1 \leq \|\widehat{f}\|_1 \|\widehat{g}\|_1 = \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}.$$

Thus we see that $A(\mathbb{T})$ is also a commutative Banach algebra with respect to pointwise products.

If $A(\mathbb{T})$ inverse-closed? In other words, if $f \in A(\mathbb{T})$ and $f(x) \neq 0$ for all x, must $1/f \in A(\mathbb{T})$?

Lemma 22.1. *If $P(\xi) = \sum_{k=-N}^{N} a_k e^{2\pi i k \xi}$, then $\|P\|_{A(\mathbb{T})} \leq (2N+1)^{1/2} \|P\|_\infty$.*

Proof. By Cauchy-Schwarz, we have

$$\|P\|_{A(\mathbb{T})} = \sum_{k=-N}^{N} |a_k| \cdot 1 \leq (2N+1)^{1/2} \left(\sum_{k=-N}^{N} |a_k|^2 \right)^{1/2} = (2N+1)^{1/2} \|P\|_2$$

where the second equality is by Plancherel's theorem. The result follows since $\|P\|_2 \leq \|P\|_\infty$ on $\mathbb{T}$. □

Lemma 22.2 (Wiener's lemma)**.** *If $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any ξ, then there is $h \in \ell^1(\mathbb{Z})$ such that*

$$\widehat{h}(\xi) = \frac{1}{\widehat{g}(\xi)}, \quad \xi \in \mathbb{T}.$$

Equivalently, if $G \in A(\mathbb{T})$ and $G(\xi) \neq 0$ for any ξ, then $1/G \in A(\mathbb{T})$.

Proof. Assume $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any ξ. Let $G = \widehat{g}$. We prove the result in two steps:

1. Assume $0 \leq G \leq 1$. Then since G is nonzero,

$$d = \inf_{\xi \in \mathbb{T}} G(\xi) > 0.$$

Let $H = 1 - G = \widehat{\delta} - \widehat{g}$, where δ is the delta sequence. Note that

$$\widehat{\delta}(\xi) = \sum_{n=-\infty}^{\infty} \delta_n e^{2\pi i n \xi} = \delta_0 e^{2\pi i 0 \xi} = 1.$$

Then $H = \widehat{h} - \widehat{g} = (\delta - g)^\wedge \in A(\mathbb{T})$. So if we let $h = \delta - g \in \ell^1(\mathbb{Z})$, then $\widehat{h} = H$, and

$$\|H\|_\infty = \|1 - G\|_\infty = 1 - d < 1.$$

Hence $\sum_{n=0}^{\infty} H(\xi)^n$ converges, and $\sum_{n=0}^{\infty} H^n = 1/(1 - H(\xi)) = 1/G(\xi) \in C(\mathbb{T})$.

However, we need convergence in $A(\mathbb{T})$. Fix $0 < \epsilon < d/2$. Let $p = h\chi_{[-N,N]}$ for N large enough so that $\|p - h\|_1 < \epsilon$. Let $P = \widehat{p} = \sum_{n=-N}^{N} h(n)e^{2\pi i n \xi}$. Let $r = p - h$ and $R = P - H$. Then

$$\|P\|_\infty = \|H + R\|_\infty \leq \|H\|_\infty + \|R\|_\infty \leq (1 - d) + \|P - H\|_\infty$$
$$\leq (1 - d) + \|\widehat{p} - \widehat{h}\|_\infty \leq (1 - d) + \|p - h\|_1 = 1 - d + \epsilon.$$

Now we can compute that

$$\|H^n\|_{A(\mathbb{T})} = \|(P - R)^n\|_{A(\mathbb{T})} = \left\| \sum_{j=0}^{n} \binom{n}{j} P^j (-R)^{n-j} \right\|_{A(\mathbb{T})}$$

where we can use the binomial theorem since $A(\mathbb{T})$ is a commutative Banach algebra. Then

$$\|H^n\|_{A(\mathbb{T})} \leq \sum_{j=0}^{n} \binom{n}{j} \|P\|_{A(\mathbb{T})}^j \|R\|_{A(\mathbb{T})}^{n-j} \leq \sum_{j=0}^{n} \binom{n}{j} (2Nj + 1)^{1/2} \|P^j\|_\infty \epsilon^{n-j},$$

where the second equality is by Lemma 22.1 and $\|R\|_{A(\mathbb{T})} = \|p - h\|_1 < \epsilon$. Thus

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2} \sum_{j=1}^{n} \binom{n}{j} \|P\|_\infty^j \epsilon^{n-j} = (2Nn + 1)^{1/2}(\|P\|_\infty + \epsilon)^n,$$

where $\|P^j\|_\infty \leq \|P\|_\infty^j$ by submultiplicativity. Since $\|P\|_\infty \leq 1 - d + \epsilon$,

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2}(1 - d + 2\epsilon)^n,$$

where $1 - d + 2\epsilon < 1$. So $\sum_{n=0}^{\infty} \|H^n\|_{A(\mathbb{T})} < \infty$, so $\sum_{n=0}^{\infty} H^n$ converges in $A(\mathbb{T})$ to $1/(1 - H) = 1/G$.

One can reduce the general case to this first case, see the course notes. $\square$

22.3   Distributions

Remark. Let $1 \le p < \infty$. Note that $L^p(\mathbb{R})^* \cong L^{p'}(\mathbb{R})$, where $L^p(\mathbb{R})^*$ is the dual space of $L^p(\mathbb{R})$, i.e. the space of bounded linear functions on $L^p(\mathbb{R})$. If we fix $g \in L^{p'}(\mathbb{R})$, then we can define

$$\mu_g(f) = \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

which is linear in f and antilinear in g (this is known as a *sesquilinear form*).

Recall that $X^* = \{\text{bounded linear functionals } \mu \text{ on } X\}$, which is equivalent to continuous linear functionals when X is a Banach space. We consider the following space of distributions:

Definition 22.1. Define the following:

- $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, the space of *distributions*.

- $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$, the space of *tempered distributions*.

- $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$, the space of *compactly supported distributions*.

Example 22.1.1. If $f \in C_c^\infty(\mathbb{R})$, then define $\langle f, \delta \rangle = f(0)$. This is the Dirac delta as a distribution.

Remark. Recall that the Schwartz space is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R})\}.$$

This is *not* a Banach space (it does not have a norm). But we can define seminorms

$$\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty, \quad m, n \ge 0.$$

There is no way to combine these seminorms into a single norm.[1] However, one can define a metric:

$$d(f, g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\rho_{m,n}(f-g)}{1 + \rho_{m,n}(f-g)}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Note that convergence with respect to d is the same as convergence in $\rho_{m,n}$ for every m, n:

$$\lim_{k \to \infty} \rho_{m,n}(f - f_k) = 0.$$

Definition 22.2. We say that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ if for every $m, n \ge 0$,

$$\rho_{m,n}(f - f_k) = \|x^m f^{(n)}(x) - x^m (f_k)^{(n)}(x)\|_\infty \longrightarrow 0$$

Therefore, a linear functional $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is *continuous* if $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ implies $\langle f_k, \mu \rangle \to \langle f, \mu \rangle$.

Remark. Since convergence in $\mathcal{S}(\mathbb{R})$ is a very strong condition, we expect that it is easy for μ to satisfy the above condition. This intuitively explains why $\mathcal{S}'(\mathbb{R})$ is so large.

Example 22.2.1. Assume that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$. Then $\langle f_k, \delta \rangle = f_k(0)$, so

$$|\langle f - f_k, \delta \rangle| = |f(0) - f_k(0)| \le \rho_{0,0}(f - f_k) \longrightarrow 0.$$

Thus we see that $\delta \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

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[1]Compare this with $C_b^1(\mathbb{R}) = \{f : \|f\|_\infty, \|f'\|_\infty < \infty\}$, where one can define a norm $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$.

Lecture 23

Nov. 20 — Distributions, Part 2

23.1 Convergence with Families of Seminorms

Example 23.0.1. Like $\mathcal{S}(\mathbb{R})$, many other spaces are defined by a family of seminorms. Recall that

$$L^1_{\text{loc}}(\mathbb{R}) = \{\text{measurable } f : f \text{ is integrable on compact } K \subseteq \mathbb{R}\}.$$

If we define $\rho_k(f) = \|f\chi_K\|_1$, then $f \in L^1_{\text{loc}}(\mathbb{R})$ if and only if $\rho_K(f) < \infty$. In fact, it is enough to consider $\rho_N(f) = \|f\chi_{[-N,N]}\|_1$. Given such a countable family of seminorms, one can always define a metric

$$d(f, g) = \sum_{N=1}^{\infty} 2^{-N} \frac{\rho_N(f - g)}{1 + \rho_N(f - g)}$$

which satisfies $d(f, f_k) \to 0$ if and only if $\rho_N(f - f_k) \to 0$ for every N.

Remark. Suppose $\| \cdot \|$ is a norm on $L^1_{\text{loc}}(\mathbb{R})$, and consider $\chi_{[k,k+1]}$. Then $\rho_N(\chi_{[k,k+1]}) \to 0$, so

$$c_k \chi_{[k,k+1]} \longrightarrow 0$$

in $L^1_{\text{loc}}(\mathbb{R})$ for every choice of c_k. But then $\|c_k \chi_{[k,k+1]}\| \to 0$, a contradiction for $c_k = 1/\|\chi_{[k,k+1]}\|$.

Example 23.0.2. For $C^\infty(\mathbb{R})$, we can take $\rho_{N,n}(f) = \|f^{(n)}\chi_{[-N,N]}\|_\infty$. Then

$$C^\infty(\mathbb{R})^* = \{\text{continuous linear functionals } \mu : C^\infty(\mathbb{R}) \to \mathbb{C}\} = \mathcal{E}'(\mathbb{R})$$

is the space of compactly supported distributions, where μ is continuous if whenever $f_k \to f$ in $C^\infty(\mathbb{R})$, we have $\mu(f_k) = \langle \mu, f_k \rangle \to \langle \mu, f \rangle$.

Remark. For $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, it is better to think of $C_c^\infty(\mathbb{R}) = \bigcup_{N=1}^{\infty} C_c^\infty([-N, N])$.

Definition 23.1. Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X.

   (a) $f_k \to f$ in X means that $\rho_\alpha(f - f_k) \to 0$ for every $\alpha \in J$.

   (b) X is *Hausdorff* if $\rho_\alpha(f) = 0$ for every $f \in X$ if and only if $f = 0$.

Definition 23.2. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be a countable, Hausdorff family of seminorms on X. If the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f - g)}{1 + \rho_n(f - g)}$$

is complete, then we call X a *Fréchet space.*

Exercise 23.1. $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$ are Fréchet spaces (but $C_c^\infty(\mathbb{R})$ is not).

Remark. Recall that if $\mu : X \to \mathbb{C}$ is a linear functional on a normed space X, then μ is continuous if and only if μ is bounded (i.e. there exists $C > 0$ such that $|\langle f, \mu \rangle| \leq C\|f\|$ for every $f \in X$).

Theorem 23.1. *Let X be a Fréchet space with seminorms $\{\rho_n\}_{n \in \mathbb{N}}$ and $\mu : X \to \mathbb{C}$ a linear functional. Then the following are equivalent:*

1. *μ is continuous, i.e. $f_k \to f$ implies $\langle f_k, \mu \rangle \to \langle f, \mu \rangle$ as $k \to \infty$.*

2. *$f_k \to 0$ in X implies $\langle f_k, \mu \rangle \to 0$.*

3. *There exists $C > 0$ and $N \in \mathbb{N}$ (depending on μ) such that $|\langle f, \mu \rangle| \leq C \sum_{n=1}^{N} \rho_n(f)$.*

Proof. $(1 \Leftrightarrow 2)$ This follows from linearity.

$(3 \Rightarrow 2)$ Suppose $f_k \to 0$, so $\rho_n(f_k) \to 0$ for every n. So

$$|\langle f_k, \mu \rangle| \leq C \sum_{n=1}^{N} \rho_n(f_k) \xrightarrow[k \to \infty]{} 0,$$

since each of the finitely many terms converge to 0.

$(2 \Rightarrow 3)$ We prove the contrapositive. Suppose (3) is false. So for every $C = N = k \in \mathbb{N}$, there exists $f_k \in X$ for which $|\langle f_k, \mu \rangle| > k \sum_{n=1}^{k} \rho_n(f_k)$. Set $\varphi_k = f_k / |\langle f_k, \mu \rangle|$. Then

$$\sum_{n=1}^{k} \rho_n(\varphi_k) = \frac{1}{|\langle f_k, \mu \rangle|} \sum_{n=1}^{k} \rho_n(f_k) < \frac{1}{|\langle f_k, \mu \rangle|} \cdot \frac{|\langle f_k, \mu \rangle|}{k} = \frac{1}{k} \xrightarrow[k \to \infty]{} 0.$$

Thus $\rho_n(\varphi_k) \to 0$ as $k \to \infty$ for every $n \in \mathbb{N}$. So $\varphi_k \to 0$ in X. But $\langle \varphi_k, \mu \rangle = 1 \not\to 0$. $\qquad\square$

Example 23.2.1. What does the topology look like given a family of seminorms, i.e. what are the open sets? Consider seminorms $\rho_1(x_1, x_2) = |x_1|$ and $\rho_2(x_1, x_2) = |x_2|$. Consider strips

$$B_r^i(x_1, x_2) = \{(y_1, y_2) : |y_i - x_i| < r\}$$

for each seminorm ρ_i. Then the open balls are of the form

$$B_r^1(x_1, x_2) \cap B_r^2(x_1, x_2).$$

A basis in the general case involves intersections of finitely many strips (as in the product topology).

23.2   Distributional Derivatives

Example 23.2.2. If f, g are differentiable and decay at ∞ (think $f, g \in \mathcal{S}(\mathbb{R})$), then

$$\langle f, g' \rangle = \int_{-\infty}^{\infty} f(x)\overline{g'(x)}\,dx = f(x)g(x)|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\overline{g(x)}\,dx = -\langle f', g \rangle,$$

by integration by parts, where $f(x)g(x)|_{x=-\infty}^{\infty} = 0$ by the decay of f, g. Now define $\delta' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \delta' \rangle := -\langle f', \delta \rangle = -f'(0).$$

for $f \in \mathcal{S}(\mathbb{R})$. Then δ' is a linear functional, and

$$|\langle f, \delta' \rangle| = |-f'(0)| \leq \|f'\|_\infty = \|x^0 f^{(1)}(x)\|_\infty = \rho_{0,1}(f) \leq 1 \sum_{m=0}^{0} \sum_{n=0}^{1} \rho_{m,n}(f).$$

Hence δ' is bounded on $\mathcal{S}(\mathbb{R})$, so $\delta' \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

Proposition 23.1. *Fix $\mu \in \mathcal{S}'(\mathbb{R})$, and define $\mu' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by*

$$\langle f, \mu' \rangle = -\langle f', \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

Then $\mu' \in \mathcal{S}'(\mathbb{R})$. In particular, every tempered distribution has a (distributional) derivative.

Proof. Since μ is bounded, there are $C, N > 0$ such that $|\langle f, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \rho_{m,n}(f)$. Then

$$|\langle f, \mu' \rangle| = |\langle f', \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \rho_{m,n}(f') = C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m (f')^{(n)}\|_\infty = C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m f^{n+1}\|_\infty,$$

so μ' is bounded. Thus $\mu' \in \mathcal{S}'(\mathbb{R})$ also. $\qquad\square$

Example 23.2.3. Consider the Heaviside function $H = \chi_{[0,\infty)}$ We can define

$$\langle f, H \rangle = \int_{-\infty}^{\infty} f(x) \overline{H(x)} \, dx = \int_{0}^{\infty} f(x) \, dx.$$

Then we can check that

$$|\langle f, H \rangle| \leq \int_{0}^{\infty} |f(x)| \, dx \leq \int_{0}^{1} |f(x)| \, dx + \int_{1}^{\infty} \frac{|f(x)|}{x^2} x^2 \, dx$$

$$\leq \|f\|_\infty + \|x^2 f(x)\|_\infty \int_{1}^{\infty} \frac{1}{x^2} \, dx \leq C \sum_{m=0}^{2} \sum_{n=0}^{0} \|x^m f^{(n)}\|_\infty.$$

Thus H is bounded, so $H \in \mathcal{S}'(\mathbb{R})$. So H has a distributional derivative $H' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$, where

$$\langle f, H' \rangle := -\langle f', H \rangle = -\int_{0}^{\infty} f'(x) \, dx = -(f(\infty) - f(0)) = f(0) = \langle f, \delta \rangle.$$

In particular, we see that $H' = \delta$ as tempered distributions.

Remark. Note that $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ and $L^1_{\text{loc}}(\mathbb{R}) + \text{polynomial growth} \subseteq \mathcal{S}'(\mathbb{R})$, so each of the functions in the latter has distributional derivatives.

Exercise 23.2. If g is smooth, then g' (as a tempered distribution) is the usual derivative.

Remark. To avoid confusion, we sometimes write $D\mu$ for the distributional derivative and g' for the pointwise a.e. derivative.