MATH 7337: Harmonic Analysis

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Contents

1 Aug. 19 — The Fourier Transform **2**
1.1 The Fourier Transform on $L^1(\mathbb{R})$ 2
1.2 Motivation for the Fourier Transform 3

2 Aug. 21 — The Riemann-Lebesgue Lemma **4**
2.1 Properties of the Fourier Transform 4
2.2 The Riemann-Lebesgue Lemma 4
2.3 Position and Momentum Operators 6
2.4 The HRT Conjecture 6

3 Aug. 3 — Convolution **7**
3.1 Convolution 7
3.2 Properties of Convolution 7
3.3 Young's Inequality 9
3.4 The Dirac Delta 10

4 Aug. 28 — Convolution, Part 2 **11**

5 Sept. 2 — Smoothness and Decay **12**
5.1 Smoothness and Decay 12
5.2 Absolute Continuity 13
5.3 Smoothness and Decay, Continued 14
5.4 Approximate Identities 14

6 Sept. 4 — Approximate Identities **16**
6.1 Properties of Approximate Identities 16
6.2 Density Results and Smooth Urysohn Lemma 18

7 Sept. 18 — **19**

8 Sept. 23 — **20**

9 Sept. 25 — Kernels and Schwartz Space **21**
9.1 More Kernels 21
9.2 Schwartz Space 22

Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}\,dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \le \int |f(x)e^{-2\pi i\xi x}|\,dx = \int |f(x)|\,dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} |\widehat{f}(\xi)| \le \|f\|_1$. This is linear in f. The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1\to L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \le \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x)e^{-2\pi i\cdot 0\cdot x}\,dx = \int f(x)\,dx.$$

So if $f \ge 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \le \int f(x)\,dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi\in\mathbb{R}} |\widehat{f}(\xi)| \le 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

Definition 1.2. For $r \ne 0$, *dilation* of f by r is $f_r(x) = rf(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.[1] Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

[1] Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \le T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^{T} e^{-2\pi i \xi x}\, dx = \left. \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right|_{-T}^{T} = \frac{\sin(2\pi T\xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \to 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i(\xi - \eta)x})\, dx \right| \le \int |f(x)||e^{-2\pi i \xi x} - e^{-2\pi i (\xi-\eta)x}|\, dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i(\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \to 0$ as $\eta \to 0$ independent of ξ, so the statement follows from the dominated convergence theorem (the integrand is dominated by $2f$). $\square$

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(x) e^{2\pi i \xi x}\, d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula)**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x}\, d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x) e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ, and the inversion formula says that we can recover f as a "sum" of these pure harmonics.

Lecture 2

Aug. 21 — The Riemann-Lebesgue Lemma

2.1 Properties of the Fourier Transform

Definition 2.1. Define the following operators:

1. *Translation*: $T_a f(x) = f(x-a)$ for $a \in \mathbb{R}$;
2. *Modulation*: $M_b f(x) = e^{2\pi i b x} f(x)$ for $b \in \mathbb{R}$;
3. *Dilation*: $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;
4. *Involution*: $\widetilde{f}(x) = \overline{f(-x)}$.

Remark. Translation and modulation are isometries on $L^p(\mathbb{R})$ for any p. Dilation as defined above is L^1-normalized, so it is only an isometry on $L^1(\mathbb{R})$.

Exercise 2.1. If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a}\widehat{f})(\xi) = e^{-2\pi i \xi a}\widehat{f}(\xi)$;
2. $(M_b f)^\wedge(\xi) = (T_b \widehat{f})(\xi) = \widehat{f}(\xi - b)$;
3. $(f_\lambda)^\wedge(\xi) = \lambda(f_{1/\lambda})^\wedge(\xi) = \widehat{f}(\xi/\lambda)$;[1]
4. $(\overline{f})^\wedge(\xi) = (\widehat{f})^\sim(\xi) = \overline{\widehat{f}(-\xi)}$;
5. $(\widetilde{f})^\wedge(\xi) = \overline{\widehat{f}(\xi)}$.

2.2 The Riemann-Lebesgue Lemma

Definition 2.2. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of f, denoted $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function f, $\operatorname{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

Theorem 2.1. *$C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. In other words,*

1. *the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;*
2. *for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;*
3. *if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \to f$ in L^p-norm, i.e. $\|g_n - f\|_p \to 0$.*

[1]Note that the result is an L^∞-normalized dilation.

For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the L^∞-norm (this is the same as the uniform norm for continuous functions).

Proof. We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of 2^{-n}. Then use Urysohn's lemma to approximate χ_{E_k} by a continuous function. $\square$

Exercise 2.2. Fix $1 \le p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a\to 0} \|f - T_a f\|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

Lemma 2.1 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

Proof. We have already seen that $\widehat{f}$ is continuous. So it suffices to show decay at ∞. Write

$$\widehat{f}(\xi) = -\int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)}\,dx = -\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x + 1/2\xi)}\,dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = -\int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x}\,dx = -\int T_{1/2\xi} f(x) e^{-2\pi i \xi x}\,dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2}\int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x}\,dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \le \frac{1}{2}\int |f(x) - T_{1/2\xi} f(x)|\,dx = \frac{1}{2}\|f - T_{1/2\xi} f\|_1 \xrightarrow[\xi \to \pm\infty]{} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. $\square$

Exercise 2.3. The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k,b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \to f$ in L^1-norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \le \|f - \phi_n\|_1 \longrightarrow 0.$$

Since $\phi_n \to \widehat{f}$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

2.3 Position and Momentum Operators

Definition 2.3. The *position operator* $P : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that P is unbounded on $L^1(\mathbb{R})$ (in fact, P is not defined on all of $L^1(\mathbb{R})$). Restrict P to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that D_P cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

Exercise 2.4. Show that $\sup_{\|f\|_1=1, f\in D_P} \|Pf\|_1 = \infty$.

Definition 2.4. The *momentum operator* $M : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, M is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

Remark. We have the relation $(Mf)^\wedge(\xi) = \xi P\widehat{f}(\xi)$, whenever the statement makes sense.

2.4 The HRT Conjecture

Conjecture 2.1 (HRT conjecture)**.** *Assume g is not zero a.e., a_k, b_k are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:*

$$\sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k). \tag{$*$}$$

If $(*) = 0$, then must it be that $c_1 = \cdots = c_N = 0$? In other words, are these linearly independent?

Remark. Consider the special case $b_k = 0$ for every k, so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g\right)^\wedge = \sum c_k M_{-a_k} \widehat{g} = \left(\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi}\right) \widehat{g}(\xi) = 0.$$

Since $\widehat{g}$ is not zero a.e., we must have $\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all k. In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

Remark. The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for $\widehat{g}$ instead of g.

Lecture 3

Aug. 3 — Convolution

3.1 Convolution

Definition 3.1. If f, g are measurable on $\mathbb{R}$, their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

Remark. When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = (g * f)(x)$$

by the change of variables $y \mapsto x - y$. So $f * g = g * f$, if it exists. Similarly, $f * (g * h) = (f * g) * h$ if each of these convolutions exist.

Remark. If we take $g_T = \chi_{-T,T}/2T$ (note that $\|g_T\|_1 = 1$), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x-y)\,dy = \frac{1}{2T}\int_{x-T}^{x+T} f(y)dy = \mathrm{Avg}_{[-T,T]}f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

Remark. We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.

Remark. The *Lebesgue differentiation theorem* says that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $(f * g_T)(x) \to f(x)$ a.e.

3.2 Properties of Convolution

Remark. Use the notation

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if $1/p + 1/p' = 1$ with $1 \le p \le \infty$ and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and we have

$$|\langle f, g\rangle| \le \int |f(x)||g(x)|\,dx \le \|f\|_p\|g\|_{p'}.$$

Theorem 3.1. *For $1 \le p \le \infty$, if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$.*

Proof. By Hölder's inequality, we can write

$$\int |f(y)g(x-y)|\,dy \le \|f\|_p \|g(x)\|_{p'} < \infty,$$

so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. □

Exercise 3.1. Show that $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{C} : h \text{ is continuous and bounded}\}$.

Remark. Denote $g^*(y) = \overline{g(-y)}$. Then we have

$$(f * g)(x) = \int f(y)g(x-y)\,dy = \int f(y)\overline{g^*(y-x)}\,dy = \langle f, T_x g^* \rangle.$$

Theorem 3.2. *Let $f, g \in L^1(\mathbb{R})$. Then*

1. *$f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}^2$;*
2. *for a.e. $x \in \mathbb{R}$, $f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}$ as a function of y;*
3. *$f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$, i.e. convolution is* submultiplicative *on $L^1(\mathbb{R})$;*
4. *$(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for every $\xi \in \mathbb{R}$.*

Proof. (1) Let $h(x, y) = f(x)$. Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(x) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in $\mathbb{R}^2$ since $\{f > a\}$ and $\mathbb{R}$ are measurable in $\mathbb{R}$. Similarly, $g(y)$ is measurable on $\mathbb{R}^2$, so $F(x, y) = f(x)g(y)$ is measurable on $\mathbb{R}^2$. Now make a linear change of variables $T(x, y) = (y, x - y)$, so $H = F \circ T = f(y)g(x - y)$ is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\begin{aligned} \iint |f(y)g(x-y)|\,dxdy &= \int |f(y)| \left(\int |g(x-y)|\,dx \right) dy = \int |f(y)| \left(\int |g(z)|\,dz \right) dy \\ &= \int |f(y)| \|g\|_1\,dy = \|f\|_1 \|g\|_1 < \infty, \end{aligned}$$

hence $f(y)g(x-y)$ is integrable on $\mathbb{R}^2$.

(2) This follows by Fubini's theorem since $f(y)g(x-y)$ is integrable.

(3) By (2), $(f * g)(x)$ exists for a.e. x, and

$$\int |(f * g)(x)|\,dx = \int \left| \int f(y)g(x-y)\,dy \right| dx \le \iint |f(y)g(x-y)|\,dydx \le \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$\begin{aligned}(f*g)^\wedge(\xi) &= \int (f*g)(x)e^{-2\pi i\xi x}\,dx = \int \left(\int f(y)g(x-y)\,dy\right)e^{-2\pi i\xi x}\,dx \\ &= \iint f(y)e^{-2\pi i\xi y}g(x-y)e^{-2\pi i\xi(x-y)}\,dydx.\end{aligned}$$

By Fubini's theorem, we can exchange orders and write

$$\begin{aligned}(f*g)^\wedge(\xi) &= \int f(y)e^{-2\pi i\xi y}\left(\int g(x-y)e^{-2\pi i\xi(x-y)}\,dx\right)dy \\ &= \int f(y)e^{-2\pi i\xi y}\left(\int g(z)e^{-2\pi i\xi z}\,dz\right)dy = \widehat{f}(\xi)\widehat{g}(\xi),\end{aligned}$$

which is the desired equality. □

Corollary 3.2.1. $L^1(\mathbb{R})$ *is closed under convolution.*

Definition 3.2. An *algebra* is a vector space A with a product such that

(a) $(fg)h = f(gh)$,

(b) $f(g+h) = fg + fh$,

(c) $\alpha(fg) = (\alpha f)g = f(\alpha g)$.

If $fg = gf$ always, then we say that A is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

Example 3.2.1. With convolution as a product, $L^1(\mathbb{R})$ becomes a commutative Banach algebra without identity. Similarly, $C_0(\mathbb{R})$ is also a commutative Banach algebra without identity (under pointwise products). The space $\mathcal{B}(X)$ of bounded linear operators on a Banach space X is also a Banach space under the operator norm, and we have $\|AB\| \le \|A\|\|B\|$ with composition as a product. So $\mathcal{B}(X)$ is a noncommutative Banach algebra, with identity.

3.3 Young's Inequality

Theorem 3.3 (Young's inequality, special case)**.** *Fix* $1 \le p \le \infty$. *If* $f \in L^p(\mathbb{R})$ *and* $g \in L^1(\mathbb{R})$, *then* $f*g \in L^p(\mathbb{R})$ *and* $\|f*g\|_p \le \|f\|_p\|g\|_1$.

Proof. The case $p=\infty$ is easy by Hölder's inequality and $p=1$ is done, so assume $1<p<\infty$. Then

$$|(f*g)(x)| \le \int |f(y)||g(x-y)|\,dy = \int \left(|f(y)||g(x-y)|^{1/p}\right)\left(|g(x-y)|^{1/p'}\right)dy,$$

By Hölder's inequality, we can write

$$\begin{aligned}|(f*g)(x)| &\le \left(\int |f(y)|^p|g(x-y)|\,dy\right)^{1/p}\left(\int |g(x-y)|\,dy\right)^{1/p'} \\ &\le \|g\|_1^{1/p'}\left(\int |f(y)|^p|g(x-y)|\,dy\right)^{1/p}.\end{aligned}$$

Now taking L^p-norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p \, dx \le \|g\|_1^{p/p'} \iint |f(y)|^p |g(x-y)| \, dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \le \|g\|_1^{p/p'} \int |f(y)|^p \left(\int |g(x-y)| \, dx \right) dy \le \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality $\|f * g\|_p \le \|f\|_p \|g\|_1$ after taking pth roots. □

Exercise 3.2 (Young's inequality, general case)**.** Let $1 \le p, q, r \le \infty$ satisfy $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Remark. Recall *Minkowski's inequality* (the triangle inequality in $L^p(\mathbb{R})$):

$$\left\| \sum f_k \right\|_p \le \sum \|f_k\|_p.$$

Minkowski's integral inequality then says that for $1 \le p \le \infty$,

$$\left\| \int f_x \, dx \right\|_p = \left(\int \left| \int f(x,y) \, dx \right|^p dy \right)^{1/p} \le \int \left(\int |f(x,y)|^p \, dy \right)^{1/p} dx = \int \|f_x\|_p \, dx.$$

One can also use this to prove to Young's inequality.

Remark. The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is $A_p A_q A_{r'}$, i.e. we have

$$\|f * g\|_r \le (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

3.4 The Dirac Delta

Remark. Is there an identity for convolution? Suppose there was a function $\delta \in L^1(\mathbb{R})$ (the *Dirac delta function*) such that $f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Then we have $(f * \delta)^\wedge = \widehat{f}$, so

$$\widehat{f}(\xi)\widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take $f(x) = e^{-x^2}$ with $\widehat{f}(\xi) = e^{-\xi^2}$ and note that $\widehat{f}(\xi)$ is everywhere nonzero. Then $\widehat{\delta}(\xi) = 1$ for all $\xi \in \mathbb{R}$, which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure δ to achieve a similar effect.

Lecture 4

Aug. 28 — Convolution, Part 2

Lecture 5

Sept. 2 — Smoothness and Decay

5.1 Smoothness and Decay

Theorem 5.1 (Decay in time implies smoothness in frequency). *Assume $f \in L^1(\mathbb{R})$ and $x^m f(x) \in L^1(\mathbb{R})$, where $m > 0$. Then*

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \ldots, g^{(m)} \in C_0(\mathbb{R})\}.$$

Furthermore, we have

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k}\widehat{f} = \left((-2\pi i x)^k f(x)\right)^\wedge.$$

Proof. The proof is by induction on m. When $m = 1$, we can formally write

$$\begin{aligned}\frac{d}{d\xi}\widehat{f}(\xi) &= \frac{d}{d\xi}\int f(x)e^{-2\pi i\xi x}\,dx \\ &\overset{(*)}{=} \int f(x)\frac{d}{d\xi}e^{-2\pi i\xi x}\,dx = \int f(x)(-2\pi i x)e^{-2\pi i\xi x}\,dx = (-2\pi i x f(x))^\wedge(\xi).\end{aligned}$$

It suffices to justify step $(*)$, which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f}'(\xi) = \lim_{\eta\to 0}\frac{\widehat{f}(\xi+\eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta\to 0}\int f(x)\frac{e^{-2\pi i(\xi+\eta)x} - e^{-2\pi i\xi x}}{\eta}\,dx.$$

Note that we have the pointwise limit

$$f(x)\frac{e^{-2\pi i(\xi+\eta)x} - e^{-2\pi i\xi x}}{\eta} \xrightarrow{\eta\to 0} f(x)\frac{d}{d\xi}e^{-2\pi i\xi x} = -2\pi i x f(x)e^{-2\pi i\xi x}.$$

Also note that we can bound

$$\left|f(x)\frac{e^{-2\pi i(\xi+\eta)x} - e^{-2\pi i\xi x}}{\eta}\right| = \left|f(x)\frac{e^{-2\pi i\eta x} - 1}{\eta}\right| \le \left|f(x)\frac{-2\pi i\eta x}{\eta}\right| = |2\pi x f(x)|,$$

where we noted that $|e^{i\theta} - 1| \le |\theta|$ for $\theta \in \mathbb{R}$. Thus $2\pi x f(x)$ dominates the integrand and is integrable since $xf(x) \in L^1(\mathbb{R})$ by assumption, we can conclude $(*)$ by the dominated convergence theorem. Then $\widehat{f}' \in C_0(\mathbb{R})$ by the Riemann-Lebesgue lemma, since $\widehat{f}' = (-2\pi i x f(x))^\wedge$ where $-2\pi i x f(x) \in L^1(\mathbb{R})$.

The inductive step is part of Homework 1. $\square$

Remark. Recall the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = f'(x)/2\pi i$. If $f, Pf \in L^1(\mathbb{R})$, then the above theorem tells us that $(Pf)^\wedge = -M\widehat{f}$.

5.2 Absolute Continuity

Definition 5.1. A function $f : [a, b] \to \mathbb{C}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}$.

Theorem 5.2 (Fundamental theorem of calculus)**.** *If $g : [a, b] \to \mathbb{C}$, then the following are equivalent:*

1. $g \in \mathrm{AC}[a, b]$*;*
2. *there exists $f \in L^1[a, b]$ such that for all $x \in [a, b]$,*

$$g(x) - g(a) = \int_a^x f(t)\, dt;$$

3. *g is differentiable at a.e. point, $g' \in L^1[a, b]$, and*

$$g(x) - g(a) = \int_a^x g'(t)\, dt.$$

Remark. The Cantor-Lebesgue function $\varphi : [0, 1] \to [0, 1]$ is continuous with $\varphi' = 0$ a.e., but

$$\int_0^1 \varphi'(x)\, dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

Lemma 5.1 (Growth lemma)**.** *If $f : [a, b] \to \mathbb{R}$ is measurable and differentiable at every point in a measurable set $E \subseteq [a, b]$, then*

$$|f(E)|_e \leq \int_E |f'|,$$

where $|f(E)|_e$ denotes the exterior Lebesgue measure of $f(E)$.

Theorem 5.3 (Banach-Zaretsky theorem)**.** *If $f : [a, b] \to \mathbb{R}$, then the following are equivalent:*

1. $f \in \mathrm{AC}[a, b]$*;*
2. *f is continuous, f has bounded variation, and $|A| = 0$ implies $|f(A)| = 0$;*
3. *f is continuous and differentiable a.e., $f' \in L^1[a, b]$, and $|A| = 0$ implies $|f(A)| = 0$.*

Theorem 5.4. *If $f : [a, b] \to \mathbb{C}$ is differentiable on $[a, b]$ and $f' \in L^1[a, b]$, then $f \in \mathrm{AC}[a, b]$.*

Proof. By the Banach-Zaretsky theorem, it suffices to show that $|A| = 0$ implies $|f(A)| = 0$. If $|A| = 0$, then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split f into its real and imaginary parts.) ∎

5.3 Smoothness and Decay, Continued

Theorem 5.5 (Smoothness in time implies decay in frequency). *If $f \in L^1(\mathbb{R})$ is everywhere m-times differentiable and $f, f', \ldots, f^{(m)} \in L^1(\mathbb{R})$, then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad \textit{for } k = 0, \ldots, m,$$

hence $|\widehat{f}(\xi)| \leq |2\pi\xi|^{-k} |\widehat{f^{(k)}}(\xi)| \leq |2\pi\xi|^{-k} \|\widehat{f^{(k)}}\|_\infty \leq |2\pi\xi|^{-k} \|f^{(k)}\|_1$ *for* $k = 0, \ldots, m$.

Proof. We prove only the case $m = 1$, the rest follows by induction. Assume $f, f' \in L^1(\mathbb{R})$. By Theorem 5.4, we have $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$. Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t)\, dt.$$

Because f' is integrable, we get that

$$\lim_{x\to\infty} f(x) = f(0) + \lim_{x\to\infty} \int_0^x f'(t)\, dt = f(0) + \int_0^\infty f'(t)\, dt.$$

Since f is integrable and this limit exists, the limit must be 0. Hence $f \in C_0(\mathbb{R})$. We can compute

$$\widehat{f'}(\xi) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi i \xi x}\, dx = \lim_{\substack{b\to\infty \\ a\to-\infty}} \int_a^b f'(x) e^{-2\pi i \xi x}\, dx.$$

Since f is absolutely continuous, we can integrate by parts to get

$$\widehat{f'}(\xi) = \lim_{\substack{b\to\infty \\ a\to-\infty}} \left[f(b) e^{-2\pi \xi b} - f(a) e^{-2\pi i \xi a} + (2\pi i \xi) \int_a^b f(x) e^{-2\pi i \xi x}\, dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result. □

Remark. Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$.

5.4 Approximate Identities

Remark. Recall that if we take $g_T = \chi_{[-T,T]}/2T$, then we have $(f * g_T)(x) = \mathrm{Avg}_{[x-T,x+T]} f$. As $T \to 0$, this converges to f if f is continuous, and converges a.e. to f if f is integrable. In particular, this is almost like a identity for the convolution operation.

Definition 5.2. If $k_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ (or sometimes $\lambda \in \mathbb{N}$) satisfy:

(a) Normalization: $\displaystyle\int_{-\infty}^{\infty} k_\lambda = 1$ for every λ,

(b) L^1-boundedness: $\displaystyle\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^{\infty} |k_\lambda| < \infty$,

(c) L^1-concentration: $\displaystyle\lim_{\lambda\to\infty} \int_{|x|\geq\delta} |k_\lambda| = 0$ for every $\delta > 0$, then we say that $\{k_\lambda\}$ is an *approximate identity (for convolution)*.

Exercise 5.1. If $k \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} k = 1$, then $k_\lambda(x) = \lambda k(\lambda x)$ forms an approximate identity.

Remark. If we choose k_λ to be nice, then $f * k_\lambda$ will also be nice and "close" to f.

Lecture 6

Sept. 4 — Approximate Identities

6.1 Properties of Approximate Identities

Theorem 6.1. *If $\{k_\lambda\}$ is an approximate identity, then for all $f \in L^1(\mathbb{R})$,*

$$\lim_{\lambda\to\infty} \|f * k_\lambda - f\|_1 = 0.$$

That is, $f * k_\lambda \to f$ in L^1-norm.

Proof. We have already seen that $f * k_\lambda \in L^1(\mathbb{R})$. Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)|\, dx = \int \left| f(x) \int k_\lambda(t)\, dt - \int f(x-t) k_\lambda(t)\, dt \right| dx,$$

where we used that $\int k_\lambda(t)\, dt = 1$. Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \le \iint |f(x) - f(x-t)||k_\lambda(t)|\, dt dx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \le \int |k_\lambda(t)| \left(\int |f(x) - T_t f(x)|\, dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \le \int_{|t|<\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t|\ge\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that $\lim_{t\to 0} \|f - T_t f\|_1 = 0$, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t| < \delta$ implies $\|f - T_t f\|_1 < \epsilon$. This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \le \epsilon \int_{|t|<\delta} |k_\lambda(t)|\, dt + \int_{|t|\ge\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use $\|f - T_t f\|_1 \le \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$ to get

$$\|f - f * k_\lambda\|_1 \le \epsilon \int_{|t|<\delta} |k_\lambda(t)|\, dt + 2\|f\|_1 \int_{|t|\ge\delta} |k_\lambda(t)|\, dt \le \epsilon K + 2\|f\|_1 \epsilon$$

where $K = \sup_\lambda \|k_\lambda\|_1 < \infty$ and λ is large enough (as $\int_{|t|\ge\delta} |k_\lambda(t)|\, dt \to 0$). So $\|f - f * k_\lambda\|_1 \to 0$. $\square$

Exercise 6.1. Show that for $1 \le p < \infty$, we still have $\|f - f * k_\lambda\|_p \to 0$ as $\lambda \to \infty$ for $f \in L^p(\mathbb{R})$. For $p = \infty$, show that if $f \in C_0(\mathbb{R})$, then $\|f - f * k_\lambda\|_\infty \to 0$ as $\lambda \to \infty$, that is $f * k_\lambda \to f$ uniformly.

Exercise 6.2. Show that if $f \in C_b(\mathbb{R})$, then for every compact set $K \subseteq \mathbb{R}$,

$$\lim_{\lambda\to\infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

Definition 6.1. A function f is *Hölder continuous* with exponent $\alpha > 0$ if

$$|f(x) - f(y)| \le K|x-y|^\alpha$$

for some constant K and all x, y. If $\alpha = 1$, then we say that f is *Lipschitz*.

Remark. If f is Hölder continuous with exponent $\alpha > 1$, then the mean value theorem implies that f is constant. Thus the interesting range for Hölder continuity is $0 < \alpha \le 1$.

Exercise 6.3. Let f be bounded and Hölder continuous with exponent $0 < \alpha \le 1$, then show that

$$f * k_\lambda \to f \quad \text{uniformly on } \mathbb{R}.$$

Remark. If f is differentiable and f' is bounded, then f is Lipschitz.

Remark. Recall the *Lebesgue differentiation theorem*, which says that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then

$$(f * g_T)(x) = \frac{1}{2T}\int_{x-T}^{x+T} f(t)\,dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where $g_T = \chi_{[-T,T]}/(2T)$. The points where the limit holds are called the *Lebesgue points* of f.

Theorem 6.2. *Assume k is bounded and compactly supported and $\int k = 1$. Set $k_\lambda(x) = \lambda k(\lambda x)$ for $\lambda > 0$. Then for any $f \in L^1(\mathbb{R})$,*

$$f * k_\lambda \to f \quad \text{pointwise a.e.}$$

Moreover, the pointwise limit holds at every Lebesgue point of f.

Proof. Assume $\operatorname{supp}(k) \subseteq [-R, R]$. We can write

$$\begin{aligned}
\lim_{\lambda\to\infty} |f(x) - (f * k_\lambda)(x)| &= \lim_{\lambda\to\infty} \left| f(x) \int k_\lambda(x-t)\,dt - \int f(x) k_\lambda(x-t)\,dt \right| \\
&\le \lim_{\lambda\to\infty} \int |f(x) - f(t)|\lambda|k(\lambda x - \lambda t)|\,dt \\
&\le \lim_{\lambda\to\infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)||k(\lambda x - \lambda t)|\,dt.
\end{aligned}$$

Making a change of variables $T = R/\lambda$, we have

$$\lim_{\lambda\to\infty} |f(x) - (f * k_\lambda)(x)| \le \lim_{T\to 0} \frac{1}{2T}\int_{x-T}^{x+T} |f(x) - f(t)|\,dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point x by the Lebesgue differentiation theorem. □

6.2 Density Results and Smooth Urysohn Lemma

Theorem 6.3. *$C_c^m(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $m > 0$ and $1 \le p < \infty$.*

Proof. Fix $\epsilon > 0$. Choose $k \in C_c^m(\mathbb{R})$ with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Note that there exists a compactly supported $g \in L^p(\mathbb{R})$ with $\|f - g\|_p < \epsilon$ (e.g. take $g = f\chi_{[-R,R]}$ for large enough R, this works since $f\chi_{[-R,R]}$ converges pointwise to f as $R \to \infty$ and is dominated by f, so the dominated convergence theorem implies that $f\chi_{[-R,R]} \to f$ in L^p-norm). Then note that $g * k_\lambda \in C_c^m(\mathbb{R})$ and $g * k_\lambda \to g$ in L^p-norm, so there exists λ such that $\|g - g * k_\lambda\|_p < \epsilon$. Thus

$$\|f - g * k_\lambda\|_p \le \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result. □

Corollary 6.3.1. *$C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$.*

Remark. The above proof would work for $m = 0$ but becomes circular: The step $g * k_\lambda \in C_c^m(\mathbb{R})$ relies on the strong continuity of translation, which we proved by first showing it for $C_c(\mathbb{R})$ and then by an extension by density to $L^p(\mathbb{R})$. In particular, we needed to already know that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proposition 6.1 (C^∞ Urysohn's lemma)**.** *If $K \subseteq \mathbb{R}$ is compact and $U \subseteq K$ is open, then there exists $f \in C_c^\infty(\mathbb{R})$ such that $0 \le f \le 1$, $f = 1$ on K, and $f = 0$ on U^c.*

Proof. Since K is compact and U^c is closed, we have

$$d = \operatorname{dist}(K, U^c) = \inf\{|x - y| : k \in K, y \notin U\} > 0.$$

Set $V = \{y \in \mathbb{R} : \operatorname{dist}(y, K) < d/3\}$, and choose any $k \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(k) \subseteq [-d/3, d/3]$ and $\int k = 1$. Take $f = k * \chi_V \in C_c^\infty(\mathbb{R})$, which has $\operatorname{supp}(f) \subseteq \operatorname{supp}(k) + V \subseteq U$. If $x \in K$, then

$$f(x) = \int_V k(x - y)\, dy = \int k = 1.$$

One can check that $0 \le f \le 1$ and $f = 0$ on U^c as an exercise, which would prove the result. □

Lecture 7

Sept. 18 —

Lecture 8

Sept. 23 —

Lecture 9

Sept. 25 — Kernels and Schwartz Space

9.1 More Kernels

Remark. Recall that if $\{k_\lambda\}$ (where $k_\lambda(x) = \lambda k(\lambda x)$) is an approximate identity, then $f * k_\lambda \to f$ in L^p-norm. We also have

$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k}_\lambda.$$

Note that $\widehat{k}_\lambda(\xi) = \widehat{k}(\xi/\lambda)$, so $\widehat{k}_\lambda(\xi) \to 1$ pointwise. If the Dirichlet kernel $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$ were integrable, then we would have $\widehat{d}_{2\pi\lambda}(\xi) = \chi_{[-\lambda,\lambda]}(\xi)$, so we must have $d_{2\pi\lambda} \notin L^1(\mathbb{R})$.

Some alternatives are the following:

1. The Fejér kernel $w_\lambda(\xi) = d^2_{2\pi\lambda}(\xi) \in L^1(\mathbb{R})$. We saw this in the proof of the inversion formula.
2. The *de la Vallée Poussin kernel* v_λ, which has $\widehat{v}_\lambda$ as a trapezoid which is 1 on $[-\lambda, \lambda]$ and decays linearly to 0 on $[-2\lambda, -\lambda]$ and $[\lambda, 2\lambda]$. One has

   $$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

   Explicitly, one can define $v(x) = 2w_2(x) - w(x)$ and $v_\lambda(x) = \lambda v(\lambda x)$.
3. The *Poisson kernel* $p(x) = 1/\pi(x^2 + 1)$.
4. The *Gauss kernel* $\phi(x) = e^{-\pi x^2}$.

Exercise 9.1. Show that if $f \in L^1(\mathbb{R})$, then $\operatorname{supp}(\widehat{f})$ is compact if and only if $f = f * g$ for some $g \in L^1(\mathbb{R})$. Hint: Use the de la Vallée Poussin kernel.

Exercise 9.2. Let $\Phi = \widehat{\phi}$, where ϕ is the Gauss kernel. Show that $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$. Then solve this differential equation to get $\Phi(\xi) = \Phi(0)\phi(\xi)$. Finally show that $\Phi(0) = 1$, and conclude $\Phi(\xi) = \phi(\xi)$.

Remark. There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if $f, \widehat{f} \in L^1(\mathbb{R})$, then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for f sufficiently nice, if we take $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge}$, then $\widehat{g} = g$.

Theorem 9.1 (Weierstrass approximation theorem). *If $f \in C[a, b]$ and $\epsilon > 0$, then there exists a polynomial p such that $\|f - p\|_\infty = \sup_{x\in[a,b]} |f(x) - p(x)| < \epsilon$.*

Proof. Fix $f \in C[a,b]$, and choose $[a,b] \subseteq (-R,R)$. Extend f to a function $g \in C_0(\mathbb{R})$ on $\mathbb{R}$ which equals f on $[a,b]$ and is supported in $(-R,R)$. Let ϕ be the Gauss kernel, and choose λ so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that $\phi_\lambda(x) = \lambda e^{-\pi\lambda^2 x^2}$ is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \lambda \frac{(-\pi\lambda^2 x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set $q(x) = \sum_{n=0}^{N} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$. If N is large enough, then we will have

$$\sup_{|x| \le 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \le \int_{-R}^{R} |g(y)||\phi_\lambda(x-y) - q(x-y)|\, dy \le \frac{\epsilon}{2\|g\|_1} \int_{-R}^{R} |g(y)|\, dy < \frac{\epsilon}{2}$$

for $|x| \le R$. So $|f(x) - (g * q)(x)| \le \epsilon$ for $x \in [a,b]$. Finally, observe that

$$(g * q)(x) = \sum_{n=0}^{N} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y)(x-y)^{2n}\, dy,$$

which is a polynomial by the binomial theorem. So we can take $p = g * q$. □

9.2 Schwartz Space

Definition 9.1. Define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \ge 0\}.$$

Remark. Note that $|x^m f^{(n)}(x)| \le C_{m,n}$ for some constant $C_{m,n}$, so

$$|f^{(n)}(x)| \le \frac{C_{m,n}}{|x|^m}.$$

In particular, for every n and polynomial p, there exists a constant $C_{n,p}$ such that

$$|f^{(n)}(x)| \le \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply f has exponential decay.

Remark. Note that $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$ defines a *seminorm* for each $m, n \ge 0$, but not a norm.[1]

[1] Recall that a *seminorm* ρ is a function satisfying $0 \le \rho(f) < \infty$, $\rho(cf) = |c|\rho(f)$, and $\rho(f+g) \le \rho(f) + \rho(g)$. If we additionally have $\rho(f) = 0$ if and only if $\rho = 0$, then ρ is a *norm*.

Remark. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}\|x^m f^{(n)}\|_1 &= \int_{|x|\le 1} |x^m f^{(n)}(x)|\,dx + \int_{|x|\ge 1} |x^m f^{(n)}(x)|\,dx \\ &\le 2\|x^m f^{(n)}\|_\infty + \int_{|x|\ge 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2}\,dx \le 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty.\end{aligned}$$

In particular, if the L^∞-norms are controlled, then so are the L^1-norms.

Exercise 9.3. Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write $Df = f'$, and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \widehat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^{n} \binom{n}{j} D^j(-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on f implies the Schwartz condition on $\widehat{f}$.

Theorem 9.2. *If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, we have $f = (\widehat{f})^\vee$ by the inversion formula, so the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $f \mapsto \widehat{f}$ is a bijection.*

Remark. Note that $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, and we will see later that $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$.