

MATH 7337: Harmonic Analysis

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Lecture 1

Aug. 19 — The Fourier Transform

1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over \mathbb{R} unless otherwise specified.

Definition 1.1. The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

Remark. Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x) e^{-2\pi i \xi x}| dx = \int |f(x)| dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, \widehat{f} is continuous).

Remark. The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$. This is linear in f . The *operator norm* of \mathcal{F} is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \rightarrow L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so \mathcal{F} is a bounded linear operator. However, \mathcal{F} is not isometric (norm-preserving) in general.

Remark. Observe that

$$\widehat{f}(0) = \int f(x) e^{-2\pi i \cdot 0 \cdot x} dx = \int f(x) dx.$$

So if $f \geq 0$ and we normalize f so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \leq \int f(x) dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$. This is one particular case where \mathcal{F} does preserve the norm.

Definition 1.2. For $r \neq 0$, *dilation* of f by r is $f_r(x) = r f(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

Example 1.2.1. The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.¹ Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\text{sinc } \xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

¹Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $\lim_{x \rightarrow \pm\infty} f(x) = 0$.

However, d is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^T e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_{-T}^T = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

Remark. We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of \mathcal{F} is a proper dense subspace of $C_0(\mathbb{R})$, which implies that \mathcal{F}^{-1} must be unbounded as a linear operator by Banach space theory.

Proposition 1.1. *If $f \in L^1(\mathbb{R})$, then \widehat{f} is uniformly continuous on \mathbb{R} , i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \rightarrow 0} 0,$$

where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.

Proof. We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}) dx \right| \leq \int |f(x)| |e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \rightarrow 0$ as $\eta \rightarrow 0$ independent of ξ , so the statement follows from the dominated convergence theorem (the integrand is dominated by $2|f|$). \square

1.2 Motivation for the Fourier Transform

Remark. We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(\xi) e^{2\pi i \xi x} d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

Proposition 1.2 (Fourier inversion formula). *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Remark. Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x)e_\xi(y)$, so e_ξ is a homomorphism, and it is also continuous. Thus e_ξ is a *character* on \mathbb{R} (in fact, every character on \mathbb{R} is of the form e_ξ for some ξ). One can use this idea to define Fourier transforms in much more general settings.

Remark. The Fourier transform decomposes a function f into the pure harmonics e_ξ , and the inversion formula says that we can recover f as a “sum” of these pure harmonics.

Lecture 2

Aug. 21 — The Riemann-Lebesgue Lemma

2.1 Properties of the Fourier Transform

Definition 2.1. Define the following operators:

1. *Translation:* $T_a f(x) = f(x - a)$ for $a \in \mathbb{R}$;
2. *Modulation:* $M_b f(x) = e^{2\pi i b x} f(x)$ for $b \in \mathbb{R}$;
3. *Dilation:* $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;
4. *Involution:* $\tilde{f}(x) = \overline{f(-x)}$.

Remark. Translation and modulation are isometries on $L^p(\mathbb{R})$ for any p . Dilation as defined above is L^1 -normalized, so it is only an isometry on $L^1(\mathbb{R})$.

Exercise 2.1. If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a} \hat{f})(\xi) = e^{-2\pi i \xi a} \hat{f}(\xi)$;
2. $(M_b f)^\wedge(\xi) = (T_b \hat{f})(\xi) = \hat{f}(\xi - b)$;
3. $(f_\lambda)^\wedge(\xi) = \lambda (f_{1/\lambda})^\wedge(\xi) = \hat{f}(\xi/\lambda)$;¹
4. $(\bar{f})^\wedge(\xi) = (\hat{f})^\sim(\xi) = \overline{\hat{f}(-\xi)}$;
5. $(\tilde{f})^\wedge(\xi) = \overline{\hat{f}(\xi)}$.

2.2 The Riemann-Lebesgue Lemma

Definition 2.2. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of f , denoted $\text{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function f , $\text{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

Theorem 2.1. $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. In other words,

1. the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;
2. for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;
3. if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \rightarrow f$ in L^p -norm, i.e. $\|g_n - f\|_p \rightarrow 0$.

¹Note that the result is an L^∞ -normalized dilation.

For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the L^∞ -norm (this is the same as the uniform norm for continuous functions).

Proof. We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of 2^{-n} . Then use Urysohn's lemma to approximate χ_{E_k} by a continuous function. \square

Exercise 2.2. Fix $1 \leq p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a \rightarrow 0} \|f - T_a f\|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

Lemma 2.1 (Riemann-Lebesgue lemma). *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

Proof. We have already seen that \widehat{f} is continuous. So it suffices to show decay at ∞ . Write

$$\widehat{f}(\xi) = - \int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi (1/2\xi)} dx = - \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi (x+1/2\xi)} dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = - \int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x} dx = - \int T_{1/2\xi} f(x) e^{-2\pi i \xi x} dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x} dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \leq \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)| dx = \frac{1}{2} \|f - T_{1/2\xi} f\|_1 \xrightarrow{\xi \rightarrow \pm\infty} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. \square

Exercise 2.3. The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \rightarrow f$ in L^1 -norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \leq \|f - \phi_n\|_1 \rightarrow 0.$$

Since $\phi_n \rightarrow f$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

2.3 Position and Momentum Operators

Definition 2.3. The *position operator* $P : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that P is unbounded on $L^1(\mathbb{R})$ (in fact, P is not defined on all of $L^1(\mathbb{R})$). Restrict P to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that D_P cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

Exercise 2.4. Show that $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$.

Definition 2.4. The *momentum operator* $M : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, M is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

Remark. We have the relation $(Mf)^\wedge(\xi) = \xi P\hat{f}(\xi)$, whenever the statement makes sense.

2.4 The HRT Conjecture

Conjecture 2.1 (HRT conjecture). Assume g is not zero a.e., a_k, b_k are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:

$$\sum_{k=1}^N c_k e^{2\pi i b_k x} g(x - a_k). \quad (*)$$

If $(*) = 0$, then must it be that $c_1 = \dots = c_N = 0$? In other words, are these linearly independent?

Remark. Consider the special case $b_k = 0$ for every k , so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g \right)^\wedge = \sum c_k M_{-a_k} \hat{g} = \left(\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} \right) \hat{g}(\xi) = 0.$$

Since \hat{g} is not zero a.e., we must have $\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all k . In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

Remark. The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for \hat{g} instead of g .

Lecture 3

Aug. 3 — Convolution

3.1 Convolution

Definition 3.1. If f, g are measurable on \mathbb{R} , their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y) dy.$$

Remark. When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y) dy = \int_{-\infty}^{\infty} f(x-y)g(y) dy = (g * f)(x)$$

by the change of variables $y \mapsto x - y$. So $f * g = g * f$, if it exists. Similarly, $f * (g * h) = (f * g) * h$ if each of these convolutions exist.

Remark. If we take $g_T = \chi_{-T,T}/2T$ (note that $\|g_T\|_1 = 1$), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x-y) dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \text{Avg}_{[-T,T]} f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

Remark. We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f\widehat{g}} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.

Remark. The *Lebesgue differentiation theorem* says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then $(f * g_T)(x) \rightarrow f(x)$ a.e.

3.2 Properties of Convolution

Remark. Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if $1/p + 1/p' = 1$ with $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and we have

$$|\langle f, g \rangle| \leq \int |f(x)||g(x)| dx \leq \|f\|_p \|g\|_{p'}.$$

Theorem 3.1. For $1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$.

Proof. By Hölder's inequality, we can write

$$\int |f(y)g(x-y)| dy \leq \|f\|_p \|g(x)\|_{p'} < \infty,$$

so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. □

Exercise 3.1. Show that $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{C} : h \text{ is continuous and bounded}\}$.

Remark. Denote $g^*(y) = \overline{g(-y)}$. Then we have

$$(f * g)(x) = \int f(y)g(x-y) dy = \int f(y)\overline{g^*(y-x)} dy = \langle f, T_x g^* \rangle.$$

Theorem 3.2. Let $f, g \in L^1(\mathbb{R})$. Then

1. $f(y)g(x-y)$ is measurable and integrable on \mathbb{R}^2 ;
2. for a.e. $x \in \mathbb{R}$, $f(y)g(x-y)$ is measurable and integrable on \mathbb{R} as a function of y ;
3. $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, i.e. convolution is submultiplicative on $L^1(\mathbb{R})$;
4. $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for every $\xi \in \mathbb{R}$.

Proof. (1) Let $h(x, y) = f(y)g(x-y)$. Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(y)g(x-y) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in \mathbb{R}^2 since $\{f > a\}$ and \mathbb{R} are measurable in \mathbb{R} . Similarly, $g(y)$ is measurable on \mathbb{R}^2 , so $F(x, y) = f(y)g(x-y)$ is measurable on \mathbb{R}^2 . Now make a linear change of variables $T(x, y) = (y, x-y)$, so $H = F \circ T = f(y)g(x-y)$ is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\begin{aligned} \iint |f(y)g(x-y)| dx dy &= \int |f(y)| \left(\int |g(x-y)| dx \right) dy = \int |f(y)| \left(\int |g(z)| dz \right) dy \\ &= \int |f(y)| \|g\|_1 dy = \|f\|_1 \|g\|_1 < \infty, \end{aligned}$$

hence $f(y)g(x-y)$ is integrable on \mathbb{R}^2 .

(2) This follows by Fubini's theorem since $f(y)g(x-y)$ is integrable.

(3) By (2), $(f * g)(x)$ exists for a.e. x , and

$$\int |(f * g)(x)| dx = \int \left| \int f(y)g(x-y) dy \right| dx \leq \iint |f(y)g(x-y)| dy dx \leq \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int (f * g)(x) e^{-2\pi i \xi x} dx = \int \left(\int f(y) g(x - y) dy \right) e^{-2\pi i \xi x} dx \\ &= \iint f(y) e^{-2\pi i \xi y} g(x - y) e^{-2\pi i \xi (x - y)} dy dx.\end{aligned}$$

By Fubini's theorem, we can exchange orders and write

$$\begin{aligned}(f * g)^\wedge(\xi) &= \int f(y) e^{-2\pi i \xi y} \left(\int g(x - y) e^{-2\pi i \xi (x - y)} dx \right) dy \\ &= \int f(y) e^{-2\pi i \xi y} \left(\int g(z) e^{-2\pi i \xi z} dz \right) dy = \widehat{f}(\xi) \widehat{g}(\xi),\end{aligned}$$

which is the desired equality. □

Corollary 3.2.1. $L^1(\mathbb{R})$ is closed under convolution.

Definition 3.2. An *algebra* is a vector space A with a product such that

- (a) $(fg)h = f(gh)$,
- (b) $f(g + h) = fg + fh$,
- (c) $\alpha(fg) = (\alpha f)g = f(\alpha g)$.

If $fg = gf$ always, then we say that A is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

Example 3.2.1. With convolution as a product, $L^1(\mathbb{R})$ becomes a commutative Banach algebra without identity. Similarly, $C_0(\mathbb{R})$ is also a commutative Banach algebra without identity (under pointwise products). The space $\mathcal{B}(X)$ of bounded linear operators on a Banach space X is also a Banach space under the operator norm, and we have $\|AB\| \leq \|A\|\|B\|$ with composition as a product. So $\mathcal{B}(X)$ is a noncommutative Banach algebra, with identity.

3.3 Young's Inequality

Theorem 3.3 (Young's inequality, special case). Fix $1 \leq p \leq \infty$. If $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$.

Proof. The case $p = \infty$ is easy by Hölder's inequality and $p = 1$ is done, so assume $1 < p < \infty$. Then

$$|(f * g)(x)| \leq \int |f(y)| |g(x - y)| dy = \int (|f(y)| |g(x - y)|^{1/p}) (|g(x - y)|^{1/p'}) dy,$$

By Hölder's inequality, we can write

$$\begin{aligned}|(f * g)(x)| &\leq \left(\int |f(y)|^p |g(x - y)| dy \right)^{1/p} \left(\int |g(x - y)| dy \right)^{1/p'} \\ &\leq \|g\|_1^{1/p'} \left(\int |f(y)|^p |g(x - y)| dy \right)^{1/p}.\end{aligned}$$

Now taking L^p -norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p dx \leq \|g\|_1^{p/p'} \iint |f(y)|^p |g(x - y)| dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \leq \|g\|_1^{p/p'} \int |f(y)|^p \left(\int |g(x - y)| dx \right) dy \leq \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality $\|f * g\|_p \leq \|f\|_p \|g\|_1$ after taking p th roots. \square

Exercise 3.2 (Young's inequality, general case). Let $1 \leq p, q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Remark. Recall *Minkowski's inequality* (the triangle inequality in $L^p(\mathbb{R})$):

$$\left\| \sum f_k \right\|_p \leq \sum \|f_k\|_p.$$

Minkowski's integral inequality then says that for $1 \leq p \leq \infty$,

$$\left\| \int f_x dx \right\|_p = \left(\int \left| \int f(x, y) dx \right|^p dy \right)^{1/p} \leq \int \left(\int |f(x, y)|^p dy \right)^{1/p} dx = \int \|f_x\|_p dx.$$

One can also use this to prove Young's inequality.

Remark. The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is $A_p A_q A_{r'}$, i.e. we have

$$\|f * g\|_r \leq (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

3.4 The Dirac Delta

Remark. Is there an identity for convolution? Suppose there was a function $\delta \in L^1(\mathbb{R})$ (the *Dirac delta function*) such that $f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Then we have $(f * \delta)^\wedge = \widehat{f}$, so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take $f(x) = e^{-x^2}$ with $\widehat{f}(\xi) = e^{-\xi^2}$ and note that $\widehat{f}(\xi)$ is everywhere nonzero. Then $\widehat{\delta}(\xi) = 1$ for all $\xi \in \mathbb{R}$, which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure δ to achieve a similar effect.

Lecture 4

Aug. 28 — Convolution, Part 2

Lecture 5

Sept. 2 — Smoothness and Decay

5.1 Smoothness and Decay

Theorem 5.1 (Decay in time implies smoothness in frequency). Assume $f \in L^1(\mathbb{R})$ and $x^m f(x) \in L^1(\mathbb{R})$, where $m > 0$. Then

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \dots, g^{(m)} \in C_0(\mathbb{R})\}.$$

Furthermore, we have

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k} \widehat{f} = ((-2\pi i x)^k f(x))^\wedge.$$

Proof. The proof is by induction on m . When $m = 1$, we can formally write

$$\begin{aligned} \frac{d}{d\xi} \widehat{f}(\xi) &= \frac{d}{d\xi} \int f(x) e^{-2\pi i \xi x} dx \\ &\stackrel{(*)}{=} \int f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} dx = \int f(x) (-2\pi i x) e^{-2\pi i \xi x} dx = (-2\pi i x f(x))^\wedge(\xi). \end{aligned}$$

It suffices to justify step (*), which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f}'(\xi) = \lim_{\eta \rightarrow 0} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \rightarrow 0} \int f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} dx.$$

Note that we have the pointwise limit

$$f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} \xrightarrow{\eta \rightarrow 0} f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} = -2\pi i x f(x) e^{-2\pi i \xi x}.$$

Also note that we can bound

$$\left| f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} \right| = \left| f(x) \frac{e^{-2\pi i \eta x} - 1}{\eta} \right| \leq \left| f(x) \frac{-2\pi i \eta x}{\eta} \right| = |2\pi x f(x)|,$$

where we noted that $|e^{i\theta} - 1| \leq |\theta|$ for $\theta \in \mathbb{R}$. Thus $2\pi x f(x)$ dominates the integrand and is integrable since $x f(x) \in L^1(\mathbb{R})$ by assumption, we can conclude (*) by the dominated convergence theorem. Then $\widehat{f}' \in C_0(\mathbb{R})$ by the Riemann-Lebesgue lemma, since $\widehat{f}' = (-2\pi i x f(x))^\wedge$ where $-2\pi i x f(x) \in L^1(\mathbb{R})$.

The inductive step is part of Homework 1. □

Remark. Recall the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = f'(x)/2\pi i$. If $f, Pf \in L^1(\mathbb{R})$, then the above theorem tells us that $(Pf)^\wedge = -M\widehat{f}$.

5.2 Absolute Continuity

Definition 5.1. A function $f : [a, b] \rightarrow \mathbb{C}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define $\text{AC}_{\text{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}$.

Theorem 5.2 (Fundamental theorem of calculus). *If $g : [a, b] \rightarrow \mathbb{C}$, then the following are equivalent:*

1. $g \in \text{AC}[a, b]$;
2. there exists $f \in L^1[a, b]$ such that for all $x \in [a, b]$,

$$g(x) - g(a) = \int_a^x f(t) dt;$$

3. g is differentiable at a.e. point, $g' \in L^1[a, b]$, and

$$g(x) - g(a) = \int_a^x g'(t) dt.$$

Remark. The Cantor-Lebesgue function $\varphi : [0, 1] \rightarrow [0, 1]$ is continuous with $\varphi' = 0$ a.e., but

$$\int_0^1 \varphi'(x) dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

Lemma 5.1 (Growth lemma). *If $f : [a, b] \rightarrow \mathbb{R}$ is measurable and differentiable at every point in a measurable set $E \subseteq [a, b]$, then*

$$|f(E)|_e \leq \int_E |f'|,$$

where $|f(E)|_e$ denotes the exterior Lebesgue measure of $f(E)$.

Theorem 5.3 (Banach-Zaretsky theorem). *If $f : [a, b] \rightarrow \mathbb{R}$, then the following are equivalent:*

1. $f \in \text{AC}[a, b]$;
2. f is continuous, f has bounded variation, and $|A| = 0$ implies $|f(A)| = 0$;
3. f is continuous and differentiable a.e., $f' \in L^1[a, b]$, and $|A| = 0$ implies $|f(A)| = 0$.

Theorem 5.4. *If $f : [a, b] \rightarrow \mathbb{C}$ is differentiable on $[a, b]$ and $f' \in L^1[a, b]$, then $f \in \text{AC}[a, b]$.*

Proof. By the Banach-Zaretsky theorem, it suffices to show that $|A| = 0$ implies $|f(A)| = 0$. If $|A| = 0$, then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split f into its real and imaginary parts.) □

5.3 Smoothness and Decay, Continued

Theorem 5.5 (Smoothness in time implies decay in frequency). *If $f \in L^1(\mathbb{R})$ is everywhere m -times differentiable and $f, f', \dots, f^{(m)} \in L^1(\mathbb{R})$, then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad \text{for } k = 0, \dots, m,$$

hence $|\widehat{f}(\xi)| \leq |2\pi \xi|^{-k} |\widehat{f^{(k)}}(\xi)| \leq |2\pi \xi|^{-k} \|\widehat{f^{(k)}}\|_\infty \leq |2\pi \xi|^{-k} \|f^{(k)}\|_1$ for $k = 0, \dots, m$.

Proof. We prove only the case $m = 1$, the rest follows by induction. Assume $f, f' \in L^1(\mathbb{R})$. By Theorem 5.4, we have $f \in \text{AC}_{\text{loc}}(\mathbb{R})$. Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t) dt.$$

Because f' is integrable, we get that

$$\lim_{x \rightarrow \infty} f(x) = f(0) + \lim_{x \rightarrow \infty} \int_0^x f'(t) dt = f(0) + \int_0^\infty f'(t) dt.$$

Since f is integrable and this limit exists, the limit must be 0. Hence $f \in C_0(\mathbb{R})$. We can compute

$$\widehat{f'}(\xi) = \int_{-\infty}^\infty f'(x) e^{-2\pi i \xi x} dx = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_a^b f'(x) e^{-2\pi i \xi x} dx.$$

Since f is absolutely continuous, we can integrate by parts to get

$$\widehat{f'}(\xi) = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \left[f(b) e^{-2\pi i \xi b} - f(a) e^{-2\pi i \xi a} + (2\pi i \xi) \int_a^b f(x) e^{-2\pi i \xi x} dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result. □

Remark. Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that $f \in \text{AC}_{\text{loc}}(\mathbb{R})$.

5.4 Approximate Identities

Remark. Recall that if we take $g_T = \chi_{[-T, T]}/2T$, then we have $(f * g_T)(x) = \text{Avg}_{[x-T, x+T]} f$. As $T \rightarrow 0$, this converges to f if f is continuous, and converges a.e. to f if f is integrable. In particular, this is almost like a identity for the convolution operation.

Definition 5.2. If $k_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ (or sometimes $\lambda \in \mathbb{N}$) satisfy:

- (a) Normalization: $\int_{-\infty}^\infty k_\lambda = 1$ for every λ ,
- (b) L^1 -boundedness: $\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^\infty |k_\lambda| < \infty$,
- (c) L^1 -concentration: $\lim_{\lambda \rightarrow \infty} \int_{|x| \geq \delta} |k_\lambda| = 0$ for every $\delta > 0$,

then we say that $\{k_\lambda\}$ is an *approximate identity (for convolution)*.

Exercise 5.1. If $k \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} k = 1$, then $k_\lambda(x) = \lambda k(\lambda x)$ forms an approximate identity.

Remark. If we choose k_λ to be nice, then $f * k_\lambda$ will also be nice and “close” to f .

Lecture 6

Sept. 4 — Approximate Identities

6.1 Properties of Approximate Identities

Theorem 6.1. *If $\{k_\lambda\}$ is an approximate identity, then for all $f \in L^1(\mathbb{R})$,*

$$\lim_{\lambda \rightarrow \infty} \|f * k_\lambda - f\|_1 = 0.$$

*That is, $f * k_\lambda \rightarrow f$ in L^1 -norm.*

Proof. We have already seen that $f * k_\lambda \in L^1(\mathbb{R})$. Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)| dx = \int \left| f(x) \int k_\lambda(t) dt - \int f(x-t) k_\lambda(t) dt \right| dx,$$

where we used that $\int k_\lambda(t) dt = 1$. Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \leq \iint |f(x) - f(x-t)| |k_\lambda(t)| dt dx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \leq \int |k_\lambda(t)| \left(\int |f(x) - T_t f(x)| dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \leq \int_{|t| < \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that $\lim_{t \rightarrow 0} \|f - T_t f\|_1 = 0$, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t| < \delta$ implies $\|f - T_t f\|_1 < \epsilon$. This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use $\|f - T_t f\|_1 \leq \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$ to get

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| dt + 2\|f\|_1 \int_{|t| \geq \delta} |k_\lambda(t)| dt \leq \epsilon K + 2\|f\|_1 \epsilon$$

where $K = \sup_\lambda \|k_\lambda\|_1 < \infty$ and λ is large enough (as $\int_{|t| \geq \delta} |k_\lambda(t)| dt \rightarrow 0$). So $\|f - f * k_\lambda\|_1 \rightarrow 0$. \square

Exercise 6.1. Show that for $1 \leq p < \infty$, we still have $\|f - f * k_\lambda\|_p \rightarrow 0$ as $\lambda \rightarrow \infty$ for $f \in L^p(\mathbb{R})$. For $p = \infty$, show that if $f \in C_0(\mathbb{R})$, then $\|f - f * k_\lambda\|_\infty \rightarrow 0$ as $\lambda \rightarrow \infty$, that is $f * k_\lambda \rightarrow f$ uniformly.

Exercise 6.2. Show that if $f \in C_b(\mathbb{R})$, then for every compact set $K \subseteq \mathbb{R}$,

$$\lim_{\lambda \rightarrow \infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

Definition 6.1. A function f is *Hölder continuous* with exponent $\alpha > 0$ if

$$|f(x) - f(y)| \leq K|x - y|^\alpha$$

for some constant K and all x, y . If $\alpha = 1$, then we say that f is *Lipschitz*.

Remark. If f is Hölder continuous with exponent $\alpha > 1$, then the mean value theorem implies that f is constant. Thus the interesting range for Hölder continuity is $0 < \alpha \leq 1$.

Exercise 6.3. Let f be bounded and Hölder continuous with exponent $0 < \alpha \leq 1$, then show that

$$f * k_\lambda \rightarrow f \quad \text{uniformly on } \mathbb{R}.$$

Remark. If f is differentiable and f' is bounded, then f is Lipschitz.

Remark. Recall the *Lebesgue differentiation theorem*, which says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then

$$(f * g_T)(x) = \frac{1}{2T} \int_{x-T}^{x+T} f(t) dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where $g_T = \chi_{[-T, T]}/(2T)$. The points where the limit holds are called the *Lebesgue points* of f .

Theorem 6.2. Assume k is bounded and compactly supported and $\int k = 1$. Set $k_\lambda(x) = \lambda k(\lambda x)$ for $\lambda > 0$. Then for any $f \in L^1(\mathbb{R})$,

$$f * k_\lambda \rightarrow f \quad \text{pointwise a.e.}$$

Moreover, the pointwise limit holds at every Lebesgue point of f .

Proof. Assume $\text{supp}(k) \subseteq [-R, R]$. We can write

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| &= \lim_{\lambda \rightarrow \infty} \left| f(x) \int k_\lambda(x - t) dt - \int f(x) k_\lambda(x - t) dt \right| \\ &\leq \lim_{\lambda \rightarrow \infty} \int |f(x) - f(t)| \lambda |k(\lambda x - \lambda t)| dt \\ &\leq \lim_{\lambda \rightarrow \infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)| |k(\lambda x - \lambda t)| dt. \end{aligned}$$

Making a change of variables $T = R/\lambda$, we have

$$\lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| \leq \lim_{T \rightarrow 0} \frac{1}{2T} \int_{x-T}^{x+T} |f(x) - f(t)| dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point x by the Lebesgue differentiation theorem. □

6.2 Density Results and Smooth Urysohn Lemma

Theorem 6.3. $C_c^m(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $m > 0$ and $1 \leq p < \infty$.

Proof. Fix $\epsilon > 0$. Choose $k \in C_c^m(\mathbb{R})$ with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Note that there exists a compactly supported $g \in L^p(\mathbb{R})$ with $\|f - g\|_p < \epsilon$ (e.g. take $g = f\chi_{[-R,R]}$ for large enough R , this works since $f\chi_{[-R,R]}$ converges pointwise to f as $R \rightarrow \infty$ and is dominated by f , so the dominated convergence theorem implies that $f\chi_{[-R,R]} \rightarrow f$ in L^p -norm). Then note that $g * k_\lambda \in C_c^m(\mathbb{R})$ and $g * k_\lambda \rightarrow g$ in L^p -norm, so there exists λ such that $\|g - g * k_\lambda\|_p < \epsilon$. Thus

$$\|f - g * k_\lambda\|_p \leq \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result. \square

Corollary 6.3.1. $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Remark. The above proof would work for $m = 0$ but becomes circular: The step $g * k_\lambda \in C_c^m(\mathbb{R})$ relies on the strong continuity of translation, which we proved by first showing it for $C_c(\mathbb{R})$ and then by an extension by density to $L^p(\mathbb{R})$. In particular, we needed to already know that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proposition 6.1 (C^∞ Urysohn's lemma). *If $K \subseteq \mathbb{R}$ is compact and $U \subseteq K$ is open, then there exists $f \in C_c^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$, $f = 1$ on K , and $f = 0$ on U^c .*

Proof. Since K is compact and U^c is closed, we have

$$d = \text{dist}(K, U^c) = \inf\{|x - y| : k \in K, y \notin U\} > 0.$$

Set $V = \{y \in \mathbb{R} : \text{dist}(y, K) < d/3\}$, and choose any $k \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(k) \subseteq [-d/3, d/3]$ and $\int k = 1$. Take $f = k * \chi_V \in C_c^\infty(\mathbb{R})$, which has $\text{supp}(f) \subseteq \text{supp}(k) + V \subseteq U$. If $x \in K$, then

$$f(x) = \int_V k(x - y) dy = \int k = 1.$$

One can check that $0 \leq f \leq 1$ and $f = 0$ on U^c as an exercise, which would prove the result. \square

Lecture 7

Sept. 18 —

Lecture 8

Sept. 23 —

Lecture 9

Sept. 25 — Kernels and Schwartz Space

9.1 More Kernels

Remark. Recall that if $\{k_\lambda\}$ (where $k_\lambda(x) = \lambda k(\lambda x)$) is an approximate identity, then $f * k_\lambda \rightarrow f$ in L^p -norm. We also have

$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k}_\lambda.$$

Note that $\widehat{k}_\lambda(\xi) = \widehat{k}(\xi/\lambda)$, so $\widehat{k}_\lambda(\xi) \rightarrow 1$ pointwise. If the Dirichlet kernel $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$ were integrable, then we would have $\widehat{d_{2\pi\lambda}}(\xi) = \chi_{[-\lambda, \lambda]}(\xi)$, so we must have $d_{2\pi\lambda} \notin L^1(\mathbb{R})$.

Some alternatives are the following:

1. The Fejér kernel $w_\lambda(\xi) = d_{2\pi\lambda}^2(\xi) \in L^1(\mathbb{R})$. We saw this in the proof of the inversion formula.
2. The *de la Vallée Poussin kernel* v_λ , which has \widehat{v}_λ as a trapezoid which is 1 on $[-\lambda, \lambda]$ and decays linearly to 0 on $[-2\lambda, -\lambda]$ and $[\lambda, 2\lambda]$. One has

$$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

Explicitly, one can define $v(x) = 2w_2(x) - w(x)$ and $v_\lambda(x) = \lambda v(\lambda x)$.

3. The *Poisson kernel* $p(x) = 1/\pi(x^2 + 1)$.
4. The *Gauss kernel* $\phi(x) = e^{-\pi x^2}$.

Exercise 9.1. Show that if $f \in L^1(\mathbb{R})$, then $\text{supp}(\widehat{f})$ is compact if and only if $f = f * g$ for some $g \in L^1(\mathbb{R})$. Hint: Use the de la Vallée Poussin kernel.

Exercise 9.2. Let $\Phi = \widehat{\phi}$, where ϕ is the Gauss kernel. Show that $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$. Then solve this differential equation to get $\Phi(\xi) = \Phi(0)\phi(\xi)$. Finally show that $\Phi(0) = 1$, and conclude $\Phi(\xi) = \phi(\xi)$.

Remark. There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if $f, \widehat{f} \in L^1(\mathbb{R})$, then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for f sufficiently nice, if we take $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge}$, then $\widehat{g} = g$.

Theorem 9.1 (Weierstrass approximation theorem). *If $f \in C[a, b]$ and $\epsilon > 0$, then there exists a polynomial p such that $\|f - p\|_\infty = \sup_{x \in [a, b]} |f(x) - p(x)| < \epsilon$.*

Proof. Fix $f \in C[a, b]$, and choose $[a, b] \subseteq (-R, R)$. Extend f to a function $g \in C_0(\mathbb{R})$ on \mathbb{R} which equals f on $[a, b]$ and is supported in $(-R, R)$. Let ϕ be the Gauss kernel, and choose λ so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that $\phi_\lambda(x) = \lambda e^{-\pi\lambda^2 x^2}$ is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \lambda \frac{(-\pi\lambda^2 x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set $q(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$. If N is large enough, then we will have

$$\sup_{|x| \leq 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \leq \int_{-R}^R |g(y)| |\phi_\lambda(x - y) - q(x - y)| dy \leq \frac{\epsilon}{2\|g\|_1} \int_{-R}^R |g(y)| dy < \frac{\epsilon}{2}$$

for $|x| \leq R$. So $|f(x) - (g * q)(x)| \leq \epsilon$ for $x \in [a, b]$. Finally, observe that

$$(g * q)(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y) (x - y)^{2n} dy,$$

which is a polynomial by the binomial theorem. So we can take $p = g * q$. □

9.2 Schwartz Space

Definition 9.1. Define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}.$$

Remark. Note that $|x^m f^{(n)}(x)| \leq C_{m,n}$ for some constant $C_{m,n}$, so

$$|f^{(n)}(x)| \leq \frac{C_{m,n}}{|x|^m}.$$

In particular, for every n and polynomial p , there exists a constant $C_{n,p}$ such that

$$|f^{(n)}(x)| \leq \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply f has exponential decay.

Remark. Note that $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$ defines a *seminorm* for each $m, n \geq 0$, but not a norm.¹

¹Recall that a *seminorm* ρ is a function satisfying $0 \leq \rho(f) < \infty$, $\rho(cf) = |c|\rho(f)$, and $\rho(f + g) \leq \rho(f) + \rho(g)$. If we additionally have $\rho(f) = 0$ if and only if $f = 0$, then ρ is a *norm*.

Remark. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned} \|x^m f^{(n)}\|_1 &= \int_{|x| \leq 1} |x^m f^{(n)}(x)| dx + \int_{|x| \geq 1} |x^m f^{(n)}(x)| dx \\ &\leq 2\|x^m f^{(n)}\|_\infty + \int_{|x| \geq 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2} dx \leq 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty. \end{aligned}$$

In particular, if the L^∞ -norms are controlled, then so are the L^1 -norms.

Exercise 9.3. Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write $Df = f'$, and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \hat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^n \binom{n}{j} D^j (-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on f implies the Schwartz condition on \hat{f} .

Theorem 9.2. If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, we have $f = (\hat{f})^\vee$ by the inversion formula, so the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ by $f \mapsto \hat{f}$ is a bijection.

Remark. Note that $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, and we will see later that $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$.

Lecture 10

Sept. 30 —

Lecture 11

Oct. 2 —

Lecture 12

Oct. 9 —

Lecture 13

Oct. 14 —

Lecture 14

Oct. 16 —

Lecture 15

Oct. 23 — The Paley-Wiener Theorem

15.1 Local Uncertainty Principle

Remark. Recall the classical uncertainty principle: For $f \in L^2(\mathbb{R})$, $\|f\|_2^2 \leq 4\pi \|xf(x)\|_2^2 \|\widehat{f}(\xi)\|_2^2$. There is also a local version of this statement:

Theorem 15.1 (Local uncertainty principle). *If $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, then for any $\xi_0 \in \mathbb{R}$, we have*

$$\int_{\xi_0 - \epsilon}^{\xi_0 + \epsilon} |\widehat{f}(\xi)|^2 d\xi \leq 8\pi\epsilon \|f\|_2 \|xf(x)\|_2.$$

15.2 The Paley-Wiener Theorem

Remark. Recall the smoothness and decay theorems: If $x, xf(x) \in L^1(\mathbb{R})$, then $(\widehat{f})'$ exists. “Extreme decay” corresponds to compact support, where $\text{supp}(f) \subseteq [-T, T]$ if $f = 0$ a.e. outside $[-T, T]$. When this is the case, the *Paley-Wiener theorem* says that \widehat{f} has extreme smoothness: \widehat{f} extends to an analytic function on \mathbb{C} .

Remark. There is also motivation from Fourier series. The Fourier transform on \mathbb{R} is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx.$$

The functions $e_\xi(x) = e^{2\pi i \xi x} : \mathbb{R} \rightarrow S^1 \subseteq \mathbb{C}$ are continuous group homomorphism, and dx is a Haar measure on \mathbb{R} (invariant under the group action). For \mathbb{Z} , Haar measure is the counting measure, and for $c = (c_n)_{n \in \mathbb{Z}} \in \ell^1$, we can define its Fourier transform to be

$$\widehat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{R}.$$

In fact, we can take $\xi \in [0, 1)$. Also note that \widehat{c} is 1-periodic.

Suppose now that c is compactly supported, say only c_1, \dots, c_N are nonzero. Then

$$\widehat{c}(\xi) = \sum_{n=1}^N c_n e^{-2\pi i n \xi} = \sum_{n=1}^N c_n (e^{-2\pi i \xi})^n$$

If we set $z = e^{-2\pi i\xi} \in S^1$, then the above becomes

$$\widehat{c}(z) = \sum_{n=1}^N c_n z^n, \quad |z| = 1,$$

which is a polynomial. In particular, \widehat{c} extends to an analytic function for all $z \in \mathbb{C}$.

Remark. Suppose $f \in L^2(\mathbb{R})$ and $\text{supp}(f) \subseteq [-T, T]$ (we say that f is “time-limited” to $[-T, T]$). Then

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_{-T}^T |f(x)| dx \leq \left(\int_{-T}^T |f(x)|^2 dx \right)^{1/2} \left(\int_{-T}^T 1^2 dx \right)^{1/2} = (2T)^{1/2} \|f\|_2 < \infty,$$

so $f \in L^1(\mathbb{R})$. Thus we can take its Fourier transform as usual:

$$\widehat{f}(\xi) = \int_{-T}^T f(x) e^{-2\pi i\xi x} dx, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $e^{-2\pi i\xi x} = e^{-2\pi i\alpha x} e^{2\pi\beta x}$, where $e^{2\pi\beta x} > 0$. Then

$$\int_{-T}^T |f(x) e^{-2\pi i\xi x}| dx = \int_{-T}^T |f(x)| e^{2\pi\beta x} dx \leq e^{2\pi|\beta|T} \int_{-T}^T |f(x)| dx \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2.$$

So \widehat{f} can be extended to all of \mathbb{C} . We will see that this extension is analytic.

We now have a function $\widehat{f} : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$|\widehat{f}(\xi)| \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $f \in L^1(\mathbb{R})$ implies $\widehat{f} \in C_0(\mathbb{R})$, so \widehat{f} is bounded on \mathbb{R} . However, the extension to \mathbb{C} cannot be bounded (for non-constant f) by Liouville’s theorem.

Definition 15.1. A function $F : \mathbb{C} \rightarrow \mathbb{C}$ has *exponential type* T if there exists $A > 0$ such that

$$|F(\xi)| \leq A e^{T|\xi|}, \quad \xi \in \mathbb{C}.$$

Remark. So from the above discussion, \widehat{f} has exponential type $2\pi T$. We now show that \widehat{f} is analytic on \mathbb{C} . Recall that F is analytic if and only if F' exists on \mathbb{C} . We can compute

$$(\widehat{f})'(\xi) = \lim_{\substack{\eta \rightarrow 0 \\ \xi \in \mathbb{C}}} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \rightarrow 0} \int_{-T}^T f(x) \frac{e^{-2\pi i(\xi + \eta)x} - e^{-2\pi i\xi x}}{\eta} dx.$$

We have seen that f is integrable, and the exponential term is bounded on $[-T, T]$ by the mean value theorem. So by the dominated convergence theorem, we can exchange the limit and integral to get

$$\begin{aligned} (\widehat{f})'(\xi) &= \int_{-T}^T f(x) \lim_{\eta \rightarrow 0} \frac{e^{-2\pi i(\xi + \eta)x} - e^{-2\pi i\xi x}}{\eta} dx \\ &= \int_{-T}^T (-2\pi i x) f(x) e^{-2\pi i\xi x} dx = ((-2\pi i x) f(x))^\wedge(\xi). \end{aligned}$$

Note that $x f(x) \in L^1(\mathbb{R})$ and also has support in $[-T, T]$, so the above exists for any $\xi \in \mathbb{C}$.

Theorem 15.2 (Paley-Wiener theorem). *We have the following:*

1. *If $f \in L^2(\mathbb{R})$ and $\text{supp}(f) \subseteq [-T, T]$, then \widehat{f} extends to an analytic function on \mathbb{C} with exponential type $2\pi T$.*
2. *If $F : \mathbb{C} \rightarrow \mathbb{C}$ is analytic with exponential type $2\pi T$, then there exists $f \in L^2(\mathbb{R})$ with $\text{supp}(f) \subseteq [-T, T]$ and $\widehat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}$, i.e. $\widehat{f} = F|_{\mathbb{R}}$.*

Proof. (1) This was the above discussion.

(2) The proof is more difficult and uses a lot of complex analysis. See Katznelson's book. \square

Remark. Recall that analytic functions have Taylor expansions. Suppose $F : \mathbb{C} \rightarrow \mathbb{C}$ and $F = 0$ on a line segment. Then if η lies on the line segment, we can write

$$F(\xi) = \sum_{k=0}^{\infty} F^{(k)}(\eta) \frac{(\xi - \eta)^k}{k!}.$$

Since $F^{(k)}(\eta) = 0$ for all k , we get $F \equiv 0$ on all of \mathbb{C} . In fact, if $F = 0$ on any set with an accumulation point, then $F \equiv 0$ on \mathbb{C} .

Corollary 15.2.1. *If $f \in L^2(\mathbb{R})$, $f \neq 0$, has compact support, then $\widehat{f}(\xi) = 0$ for only countably many ξ .*

Proof. If $\widehat{f}(\xi) = 0$ for uncountably many ξ , then there exists $\xi_n \rightarrow \xi$ such that $\widehat{f}(\xi_n) = 0$ for all n (split \mathbb{C} into countably many compact sets and apply a countability argument). Hence $\widehat{f}, f = 0$. \square

Corollary 15.2.2. *If $f \in L^2(\mathbb{R})$, then f and \widehat{f} are both compactly supported if and only if $f = 0$.*

Proof. Suppose f is compactly supported, then \widehat{f} is analytic on \mathbb{C} . Suppose $\widehat{f}(\xi) = 0$ a.e. on \mathbb{R} outside $[-\Omega, \Omega]$. Since \widehat{f} is analytic and $\mathbb{R} \setminus [-\Omega, \Omega]$ has an accumulation point, we get $\widehat{f}, f = 0$. \square

Remark. In particular, the above corollary implies that if $f \in C_c^\infty(\mathbb{R})$ is nonzero, then \widehat{f} is not compactly supported. But $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and \mathcal{F} maps $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, so we do know $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R})$.

15.3 Hilbert-Schmidt Operators

Remark. Let H be a separable, infinite-dimensional Hilbert space. Recall the following definitions and results from functional analysis:

Definition 15.2. A linear operator $A : H \rightarrow H$ is *compact* if $A(D)$ is contained in a compact set, where $D = \{f \in H : \|f\| \leq 1\}$ is the closed unit disk in H .

Theorem 15.3 (Spectral theorem for compact, self-adjoint operators). *If $A : H \rightarrow H$ is a compact, self-adjoint operator, then there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H such that*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \quad Af = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n, \quad Ae_n = \lambda_n e_n.$$

Moreover, $\lambda_n \rightarrow 0$ if there are infinitely many $\lambda_n \neq 0$.

Remark. Note that $\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n| \rightarrow 0$, so the identity operator I is not compact.

Example 15.2.1. An example of a compact box in $\mathbb{R}^{\mathbb{N}}$ looks like

$$[0, 1] \times [0, 1/2] \times [0, 1/4] \times \cdots .$$

Definition 15.3. A linear operator $A : H \rightarrow H$ is *Hilbert-Schmidt* if there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$.

Remark. For an operator $A : H \rightarrow H$, if we write

$$Af(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy,$$

then A is Hilbert-Schmidt if and only if $k \in L^2(\mathbb{R}^2)$.