# MATH 7337: Harmonic Analysis

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# Contents

# Lecture 1

# Aug. 19 — The Fourier Transform

## 1.1  The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over $\mathbb{R}$ unless otherwise specified.

**Definition 1.1.** The *Fourier transform* of $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}\, dx, \quad \xi \in \mathbb{R}.$$

**Remark.** Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x)e^{-2\pi i\xi x}|\, dx = \int |f(x)|\, dx = \|f\|_1 < \infty,$$

so $\widehat{f}(\xi)$ exists for all $\xi \in \mathbb{R}$ (in fact, $\widehat{f}$ is continuous).

**Remark.** The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ as $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$. This is linear in $f$. The *operator norm* of $\mathcal{F}$ is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \to L^\infty} = \sup_{\|f\|_1 = 1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1 = 1} \|f\|_1 = 1,$$

so $\mathcal{F}$ is a bounded linear operator. However, $\mathcal{F}$ is not isometric (norm-preserving) in general.

**Remark.** Observe that
$$\widehat{f}(0) = \int f(x)e^{-2\pi i \cdot 0 \cdot x}\, dx = \int f(x)\, dx.$$

So if $f \geq 0$ and we normalize $f$ so that $\widehat{f}(0) = 1$, then we have

$$|\widehat{f}(\xi)| \leq \int f(x)\, dx = \widehat{f}(0),$$

and so $\|\widehat{f}\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$. This is one particular case where $\mathcal{F}$ does preserve the norm.

**Definition 1.2.** For $r \neq 0$, *dilation* of $f$ by $r$ is $f_r(x) = rf(rx)$. Note that $\|f_r\|_1 = \|f\|_1$.

**Example 1.2.1.** The *Dirichlet function* is $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$.[1] Note that $d \notin L^1(\mathbb{R})$. We can also define the *sinc* function as $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$.

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[1] Recall that $C_0(\mathbb{R})$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\pm\infty} f(x) = 0$.

However, $d$ is the Fourier transform of a function in $L^1(\mathbb{R})$. Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that $\chi_{-[T,T]} \in L^1(\mathbb{R})$. Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^{T} e^{-2\pi i \xi x} \, dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\Big|_{-T}^{T} = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$.

**Remark.** We will see in general that $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, this is the Riemann-Lebesgue lemma. The image of $\mathcal{F}$ is a proper dense subspace of $C_0(\mathbb{R})$, which implies that $\mathcal{F}^{-1}$ must be unbounded as a linear operator by Banach space theory.

**Proposition 1.1.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$, i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \to 0} 0,$$

*where $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$.*

*Proof.* We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x}) \, dx \right| \leq \int |f(x)||e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x}| \, dx.$$

Note that $f \in L^1(\mathbb{R})$ and $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta) x}| = |1 - e^{2\pi i \eta x}| \to 0$ as $\eta \to 0$ independent of $\xi$, so the statement follows from the dominated convergence theorem (the integrand is dominated by $2f$). $\square$

## 1.2 Motivation for the Fourier Transform

**Remark.** We will define the *inverse Fourier transform* of $f \in L^1(\mathbb{R})$ as

$$\check{f}(x) = \int f(x) e^{2\pi i \xi x} \, d\xi.$$

Note that $\check{f}(\xi) = \widehat{f}(-\xi)$. With enough assumptions, this is an inverse to the Fourier transform.

**Proposition 1.2** (Fourier inversion formula)**.** *If $f, \widehat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi.$$

**Remark.** Note that $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$ and $e_\xi : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$. We have $e_\xi(x + y) = e_\xi(x) e_\xi(y)$, so $e_\xi$ is a homomorphism, and it is also continuous. Thus $e_\xi$ is a *character* on $\mathbb{R}$ (in fact, every character on $\mathbb{R}$ is of the form $e_\xi$ for some $\xi$). One can use this idea to define Fourier transforms in much more general settings.

**Remark.** The Fourier transform decomposes a function $f$ into the pure harmonics $e_\xi$, and the inversion formula says that we can recover $f$ as a "sum" of these pure harmonics.

# Lecture 2

# Aug. 21 — The Riemann-Lebesgue Lemma

## 2.1 Properties of the Fourier Transform

**Definition 2.1.** Define the following operators:

1. *Translation*: $T_a f(x) = f(x - a)$ for $a \in \mathbb{R}$;

2. *Modulation*: $M_b f(x) = e^{2\pi i b x} f(x)$ for $b \in \mathbb{R}$;

3. *Dilation*: $f_\lambda(x) = \lambda f(\lambda x)$ for $\lambda > 0$;

4. *Involution*: $\widetilde{f}(x) = \overline{f(-x)}$.

**Remark.** Translation and modulation are isometries on $L^p(\mathbb{R})$ for any $p$. Dilation as defined above is $L^1$-normalized, so it is only an isometry on $L^1(\mathbb{R})$.

**Exercise 2.1.** If $f \in L^1(\mathbb{R})$, then

1. $(T_a f)^\wedge(\xi) = (M_{-a}\widehat{f})(\xi) = e^{-2\pi i \xi a}\widehat{f}(\xi)$;

2. $(M_b f)^\wedge(\xi) = (T_b\widehat{f})(\xi) = \widehat{f}(\xi - b)$;

3. $(f_\lambda)^\wedge(\xi) = \lambda(f_{1/\lambda})^\wedge(\xi) = \widehat{f}(\xi/\lambda)$;[1]

4. $(\overline{f})^\wedge(\xi) = (\widehat{f})^\sim(\xi) = \overline{\widehat{f}(-\xi)}$;

5. $(\widetilde{f})^\wedge(\xi) = \overline{\widehat{f}(\xi)}$.

## 2.2 The Riemann-Lebesgue Lemma

**Definition 2.2.** Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support. For a continuous function, the *support* of $f$, denoted $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$. So for a continuous function $f$, $\mathrm{supp}(f)$ is compact if and only if $f = 0$ outside some finite interval.

**Theorem 2.1.** *$C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. In other words,*

1. *the closure of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is all of $L^p(\mathbb{R})$;*

2. *for any $f \in L^p(\mathbb{R})$ and $\epsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$;*

3. *if $f \in L^p(\mathbb{R})$, then there exists $g_n \in C_c(\mathbb{R})$ such that $g_n \to f$ in $L^p$-norm, i.e. $\|g_n - f\|_p \to 0$.*

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[1]Note that the result is an $L^\infty$-normalized dilation.

*For $p = \infty$, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ with respect to the $L^\infty$-norm (this is the same as the uniform norm for continuous functions).*

*Proof.* We sketch the proof. First approximate $f \in L^p(\mathbb{R})$ by a simple function (one that takes only finitely many distinct values) $\phi = \sum_{k=1}^N c_k \chi_{E_k}$, e.g. by rounding down to the nearest integer multiple of $2^{-n}$. Then use Urysohn's lemma to approximate $\chi_{E_k}$ by a continuous function. $\qquad\square$

**Exercise 2.2.** Fix $1 \le p < \infty$. Prove that if $f \in L^p(\mathbb{R})$, then $\lim_{a\to 0} \|f - T_a f\|_p = 0$. We say that translation is *strongly continuous* on $L^p(\mathbb{R})$. For $p = \infty$, use $C_0(\mathbb{R})$ and the uniform norm instead.

**Lemma 2.1** (Riemann-Lebesgue lemma). *If $f \in L^1(\mathbb{R})$, then $\widehat{f} \in C_0(\mathbb{R})$,*

*Proof.* We have already seen that $\widehat{f}$ is continuous. So it suffices to show decay at $\infty$. Write

$$\widehat{f}(\xi) = -\int f(x)e^{-2\pi i \xi x}e^{-2\pi i \xi(1/2\xi)}\,dx = -\int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi(x+1/2\xi)}\,dx.$$

Now make the change of variables $x \mapsto x - 1/2\xi$, so we get

$$\widehat{f}(\xi) = -\int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right)e^{-2\pi i \xi x}\,dx = -\int T_{1/2\xi}f(x)e^{-2\pi i \xi x}\,dx.$$

Taking an average with the usual expression for $\widehat{f}(\xi)$, we have

$$\widehat{f}(\xi) = \frac{1}{2}\int (f(x) - T_{1/2\xi}f(x))e^{-2\pi i \xi x}\,dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \le \frac{1}{2}\int |f(x) - T_{1/2\xi}f(x)|\,dx = \frac{1}{2}\|f - T_{1/2\xi}f\|_1 \xrightarrow[\xi\to\pm\infty]{} 0$$

by the strong continuity of translation on $L^1(\mathbb{R})$. $\qquad\square$

**Exercise 2.3.** The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$. By taking translations and dilations, we see that $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$. Consider *really simple functions* $\phi = \sum_{k=1}^N c_k \chi_{[a_k,b_k]}$, and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k,b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$. So if $f \in L^1(\mathbb{R})$, there exist really simple $\phi_n \to f$ in $L^1$-norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \le \|f - \phi_n\|_1 \longrightarrow 0.$$

Since $\phi_n \to \widehat{f}$ uniformly and $C_0(\mathbb{R})$ is a Banach space, we conclude $\widehat{f} \in C_0(\mathbb{R})$. Fill in the details.

## 2.3   Position and Momentum Operators

**Definition 2.3.** The *position operator* $P : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Pf(x) = xf(x)$. Note that $P$ is unbounded on $L^1(\mathbb{R})$ (in fact, $P$ is not defined on all of $L^1(\mathbb{R})$). Restrict $P$ to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in $L^1(\mathbb{R})$. Note that $D_P$ cannot be bounded as it does not admit an extension to $L^1(\mathbb{R})$.

**Exercise 2.4.** Show that $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$.

**Definition 2.4.** The *momentum operator* $M : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $Mf = f'/2\pi i$. Similarly, $M$ is unbounded and defined only on a dense subset of $L^1(\mathbb{R})$.

**Remark.** We have the relation $(Mf)^\wedge(\xi) = \xi P\widehat{f}(\xi)$, whenever the statement makes sense.

## 2.4   The HRT Conjecture

**Conjecture 2.1** (HRT conjecture)**.** *Assume $g$ is not zero a.e., $a_k, b_k$ are distinct, and consider finite linear combinations of translations and modulations of $g \in L^2(\mathbb{R})$ of the following form:*

$$\sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k). \tag{$*$}$$

*If $(*) = 0$, then must it be that $c_1 = \cdots = c_N = 0$? In other words, are these linearly independent?*

**Remark.** Consider the special case $b_k = 0$ for every $k$, so $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$ a.e. Then

$$\left(\sum c_k T_{a_k} g\right)^\wedge = \sum c_k M_{-a_k} \widehat{g} = \left(\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi}\right) \widehat{g}(\xi) = 0.$$

Since $\widehat{g}$ is not zero a.e., we must have $\sum_{k=1}^{N} c_k e^{-2\pi i a_k \xi} = 0$, which implies $c_k = 0$ for all $k$. In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

**Remark.** The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for $\widehat{g}$ instead of $g$.

# Lecture 3

# Aug. 3 — Convolution

## 3.1 Convolution

**Definition 3.1.** If $f, g$ are measurable on $\mathbb{R}$, their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\, dy.$$

**Remark.** When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\, dy = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy = (g * f)(x)$$

by the change of variables $y \mapsto x - y$. So $f * g = g * f$, if it exists. Similarly, $f * (g * h) = (f * g) * h$ if each of these convolutions exist.

**Remark.** If we take $g_T = \chi_{-T,T}/2T$ (note that $\|g_T\|_1 = 1$), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x - y)\, dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \text{Avg}_{[-T,T]} f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

**Remark.** We would like to show $f, g \in L^1(\mathbb{R})$ implies $f * g \in L^1(\mathbb{R})$. Note that $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$, since $C_0(\mathbb{R})$ is closed under multiplication, even though $L^1(\mathbb{R})$ is not.

**Remark.** The *Lebesgue differentiation theorem* says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then $(f * g_T)(x) \to f(x)$ a.e.

## 3.2 Properties of Convolution

**Remark.** Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if $1/p + 1/p' = 1$ with $1 \le p \le \infty$ and $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and we have

$$|\langle f, g \rangle| \le \int |f(x)||g(x)|\, dx \le \|f\|_p \|g\|_{p'}.$$

**Theorem 3.1.** *For $1 \le p \le \infty$, if $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$.*

*Proof.* By Hölder's inequality, we can write

$$\int |f(y)g(x-y)|\, dy \le \|f\|_p \|g(x)\|_{p'} < \infty,$$

so $(f * g)(x)$ exists for every $x \in \mathbb{R}$. □

**Exercise 3.1.** Show that $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{C} : h \text{ is continuous and bounded}\}$.

**Remark.** Denote $g^*(y) = \overline{g(-y)}$. Then we have

$$(f * g)(x) = \int f(y)g(x-y)\, dy = \int f(y)\overline{g^*(y-x)}\, dy = \langle f, T_x g^* \rangle.$$

**Theorem 3.2.** *Let $f, g \in L^1(\mathbb{R})$. Then*

1. *$f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}^2$;*

2. *for a.e. $x \in \mathbb{R}$, $f(y)g(x-y)$ is measurable and integrable on $\mathbb{R}$ as a function of $y$;*

3. *$f * g \in L^1(\mathbb{R})$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$, i.e. convolution is submultiplicative on $L^1(\mathbb{R})$;*

4. *$(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for every $\xi \in \mathbb{R}$.*

*Proof.* (1) Let $h(x, y) = f(x)$. Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(x) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in $\mathbb{R}^2$ since $\{f > a\}$ and $\mathbb{R}$ are measurable in $\mathbb{R}$. Similarly, $g(y)$ is measurable on $\mathbb{R}^2$, so $F(x, y) = f(x)g(y)$ is measurable on $\mathbb{R}^2$. Now make a linear change of variables $T(x, y) = (y, x - y)$, so $H = F \circ T = f(y)g(x - y)$ is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\iint |f(y)g(x-y)|\, dxdy = \int |f(y)| \left( \int |g(x-y)|\, dx \right) dy = \int |f(y)| \left( \int |g(z)|\, dz \right) dy$$

$$= \int |f(y)| \|g\|_1\, dy = \|f\|_1 \|g\|_1 < \infty,$$

hence $f(y)g(x - y)$ is integrable on $\mathbb{R}^2$.

(2) This follows by Fubini's theorem since $f(y)g(x - y)$ is integrable.

(3) By (2), $(f * g)(x)$ exists for a.e. $x$, and

$$\int |(f * g)(x)|\, dx = \int \left| \int f(y)g(x-y)\, dy \right| dx \le \iint |f(y)g(x-y)|\, dydx \le \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$(f * g)^\wedge(\xi) = \int (f * g)(x)e^{-2\pi i \xi x}\, dx = \int \left( \int f(y)g(x - y)\, dy \right) e^{-2\pi i \xi x}\, dx$$

$$= \iint f(y)e^{-2\pi i \xi y}g(x - y)e^{-2\pi i \xi(x-y)}\, dydx.$$

By Fubini's theorem, we can exchange orders and write

$$(f * g)^\wedge(\xi) = \int f(y)e^{-2\pi i \xi y}\left( \int g(x - y)e^{-2\pi i \xi(x-y)}\, dx \right) dy$$

$$= \int f(y)e^{-2\pi i \xi y}\left( \int g(z)e^{-2\pi i \xi z}\, dz \right) dy = \widehat{f}(\xi)\widehat{g}(\xi),$$

which is the desired equality. $\qquad \square$

**Corollary 3.2.1.** $L^1(\mathbb{R})$ *is closed under convolution.*

**Definition 3.2.** An *algebra* is a vector space $A$ with a product such that

  (a) $(fg)h = f(gh)$,

  (b) $f(g + h) = fg + fh$,

  (c) $\alpha(fg) = (\alpha f)g = f(\alpha g)$.

If $fg = gf$ always, then we say that $A$ is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

**Example 3.2.1.** With convolution as a product, $L^1(\mathbb{R})$ becomes a commutative Banach algebra without identity. Similarly, $C_0(\mathbb{R})$ is also a commutative Banach algebra without identity (under pointwise products). The space $\mathcal{B}(X)$ of bounded linear operators on a Banach space $X$ is also a Banach space under the operator norm, and we have $\|AB\| \leq \|A\|\|B\|$ with composition as a product. So $\mathcal{B}(X)$ is a noncommutative Banach algebra, with identity.

## 3.3   Young's Inequality

**Theorem 3.3** (Young's inequality, special case)**.** *Fix $1 \leq p \leq \infty$. If $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_p\|g\|_1$.*

*Proof.* The case $p = \infty$ is easy by Hölder's inequality and $p = 1$ is done, so assume $1 < p < \infty$. Then

$$|(f * g)(x)| \leq \int |f(y)||g(x - y)|\, dy = \int \left( |f(y)||g(x - y)|^{1/p} \right) \left( |g(x - y)|^{1/p'} \right) dy,$$

By Hölder's inequality, we can write

$$|(f * g)(x)| \leq \left( \int |f(y)|^p |g(x - y)|\, dy \right)^{1/p} \left( \int |g(x - y)|\, dy \right)^{1/p'}$$

$$\leq \|g\|_1^{1/p'} \left( \int |f(y)|^p |g(x - y)|\, dy \right)^{1/p}.$$

Now taking $L^p$-norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p \, dx \le \|g\|_1^{p/p'} \iint |f(y)|^p |g(x - y)| \, dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \le \|g\|_1^{p/p'} \int |f(y)|^p \left( \int |g(x - y)| \, dx \right) dy \le \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality $\|f * g\|_p \le \|f\|_p \|g\|_1$ after taking $p$th roots. $\qquad\square$

**Exercise 3.2** (Young's inequality, general case). Let $1 \le p, q, r \le \infty$ satisfy $1/r = 1/p + 1/q - 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then
$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

**Remark.** Recall *Minkowski's inequality* (the triangle inequality in $L^p(\mathbb{R})$):
$$\left\| \sum f_k \right\|_p \le \sum \|f_k\|_p.$$

*Minkowski's integral inequality* then says that for $1 \le p \le \infty$,

$$\left\| \int f_x \, dx \right\|_p = \left( \int \left| \int f(x, y) \, dx \right|^p dy \right)^{1/p} \le \int \left( \int |f(x, y)|^p \, dy \right)^{1/p} dx = \int \|f_x\|_p \, dx.$$

One can also use this to prove to Young's inequality.

**Remark.** The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left( \frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is $A_p A_q A_{r'}$, i.e. we have

$$\|f * g\|_r \le (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

## 3.4 The Dirac Delta

**Remark.** Is there an identity for convolution? Suppose there was a function $\delta \in L^1(\mathbb{R})$ (the *Dirac delta function*) such that $f * \delta = f$ for all $f \in L^1(\mathbb{R})$. Then we have $(f * \delta)^\wedge = \widehat{f}$, so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take $f(x) = e^{-x^2}$ with $\widehat{f}(\xi) = e^{-\xi^2}$ and note that $\widehat{f}(\xi)$ is everywhere nonzero. Then $\widehat{\delta}(\xi) = 1$ for all $\xi \in \mathbb{R}$, which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure $\delta$ to achieve a similar effect.

# Lecture 4

# Aug. 28 — Convolution, Part 2

# Lecture 5

# Sept. 2 — Smoothness and Decay

## 5.1 Smoothness and Decay

**Theorem 5.1** (Decay in time implies smoothness in frequency). *Assume $f \in L^1(\mathbb{R})$ and $x^m f(x) \in L^1(\mathbb{R})$, where $m > 0$. Then*

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \ldots, g^{(m)} \in C_0(\mathbb{R})\}.$$

*Furthermore, we have*

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k}\widehat{f} = \left((-2\pi i x)^k f(x)\right)^{\wedge}.$$

*Proof.* The proof is by induction on $m$. When $m = 1$, we can formally write

$$\frac{d}{d\xi}\widehat{f}(\xi) = \frac{d}{d\xi}\int f(x)e^{-2\pi i \xi x}\, dx$$

$$\stackrel{(*)}{=} \int f(x)\frac{d}{d\xi}e^{-2\pi i \xi x}\, dx = \int f(x)(-2\pi i x)e^{-2\pi i \xi x}\, dx = (-2\pi i x f(x))^{\wedge}(\xi).$$

It suffices to justify step $(*)$, which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f'}(\xi) = \lim_{\eta \to 0}\frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \to 0}\int f(x)\frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta}\, dx.$$

Note that we have the pointwise limit

$$f(x)\frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} \xrightarrow{\eta \to 0} f(x)\frac{d}{d\xi}e^{-2\pi i \xi x} = -2\pi i x f(x)e^{-2\pi i \xi x}.$$

Also note that we can bound

$$\left|f(x)\frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta}\right| = \left|f(x)\frac{e^{-2\pi i \eta x} - 1}{\eta}\right| \leq \left|f(x)\frac{-2\pi i \eta x}{\eta}\right| = |2\pi x f(x)|,$$

where we noted that $|e^{i\theta} - 1| \leq |\theta|$ for $\theta \in \mathbb{R}$. Thus $2\pi x f(x)$ dominates the integrand and is integrable since $x f(x) \in L^1(\mathbb{R})$ by assumption, we can conclude $(*)$ by the dominated convergence theorem. Then $\widehat{f'} \in C_0(\mathbb{R})$ by the Riemann-Lebesgue lemma, since $\widehat{f'} = (-2\pi i x f(x))^{\wedge}$ where $-2\pi i x f(x) \in L^1(\mathbb{R})$.

The inductive step is part of Homework 1. $\qquad\qquad\square$

**Remark.** Recall the position and momentum operators $Pf(x) = xf(x)$ and $Mf(x) = f'(x)/2\pi i$. If $f, Pf \in L^1(\mathbb{R})$, then the above theorem tells us that $(Pf)^{\wedge} = -M\widehat{f}$.

## 5.2   Absolute Continuity

**Definition 5.1.** A function $f : [a, b] \to \mathbb{C}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\{[a_j, b_j]\}_j$ are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}$.

**Theorem 5.2** (Fundamental theorem of calculus). *If $g : [a, b] \to \mathbb{C}$, then the following are equivalent:*

1. *$g \in \mathrm{AC}[a, b]$;*

2. *there exists $f \in L^1[a, b]$ such that for all $x \in [a, b]$,*

$$g(x) - g(a) = \int_a^x f(t)\, dt;$$

3. *$g$ is differentiable at a.e. point, $g' \in L^1[a, b]$, and*

$$g(x) - g(a) = \int_a^x g'(t)\, dt.$$

**Remark.** The Cantor-Lebesgue function $\varphi : [0, 1] \to [0, 1]$ is continuous with $\varphi' = 0$ a.e., but

$$\int_0^1 \varphi'(x)\, dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

**Lemma 5.1** (Growth lemma). *If $f : [a, b] \to \mathbb{R}$ is measurable and differentiable at every point in a measurable set $E \subseteq [a, b]$, then*

$$|f(E)|_e \leq \int_E |f'|,$$

*where $|f(E)|_e$ denotes the exterior Lebesgue measure of $f(E)$.*

**Theorem 5.3** (Banach-Zaretsky theorem). *If $f : [a, b] \to \mathbb{R}$, then the following are equivalent:*

1. *$f \in \mathrm{AC}[a, b]$;*

2. *$f$ is continuous, $f$ has bounded variation, and $|A| = 0$ implies $|f(A)| = 0$;*

3. *$f$ is continuous and differentiable a.e., $f' \in L^1[a, b]$, and $|A| = 0$ implies $|f(A)| = 0$.*

**Theorem 5.4.** *If $f : [a, b] \to \mathbb{C}$ is differentiable on $[a, b]$ and $f' \in L^1[a, b]$, then $f \in \mathrm{AC}[a, b]$.*

*Proof.* By the Banach-Zaretsky theorem, it suffices to show that $|A| = 0$ implies $|f(A)| = 0$. If $|A| = 0$, then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split $f$ into its real and imaginary parts.) $\qquad\square$

## 5.3   Smoothness and Decay, Continued

**Theorem 5.5** (Smoothness in time implies decay in frequency). *If $f \in L^1(\mathbb{R})$ is everywhere $m$-times differentiable and $f, f', \ldots, f^{(m)} \in L^1(\mathbb{R})$, then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad for\ k = 0, \ldots, m,$$

*hence* $|\widehat{f}(\xi)| \le |2\pi\xi|^{-k}|\widehat{f^{(k)}}(\xi)| \le |2\pi\xi|^{-k}\|\widehat{f^{(k)}}\|_\infty \le |2\pi\xi|^{-k}\|f^{(k)}\|_1$ *for* $k = 0, \ldots, m$.

*Proof.* We prove only the case $m = 1$, the rest follows by induction. Assume $f, f' \in L^1(\mathbb{R})$. By Theorem 5.4, we have $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$. Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t)\, dt.$$

Because $f'$ is integrable, we get that

$$\lim_{x \to \infty} f(x) = f(0) + \lim_{x \to \infty} \int_0^x f'(t)\, dt = f(0) + \int_0^\infty f'(t)\, dt.$$

Since $f$ is integrable and this limit exists, the limit must be 0. Hence $f \in C_0(\mathbb{R})$. We can compute

$$\widehat{f'}(\xi) = \int_{-\infty}^\infty f'(x) e^{-2\pi i \xi x}\, dx = \lim_{\substack{b \to \infty \\ a \to -\infty}} \int_a^b f'(x) e^{-2\pi i \xi x}\, dx.$$

Since $f$ is absolutely continuous, we can integrate by parts to get

$$\widehat{f'}(\xi) = \lim_{\substack{b \to \infty \\ a \to -\infty}} \left[ f(b) e^{-2\pi\xi b} - f(a) e^{-2\pi i \xi a} + (2\pi i\xi) \int_a^b f(x) e^{-2\pi i \xi x}\, dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result. $\qquad \square$

**Remark.** Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$.

## 5.4   Approximate Identities

**Remark.** Recall that if we take $g_T = \chi_{[-T,T]}/2T$, then we have $(f * g_T)(x) = \mathrm{Avg}_{[x-T,x+T]} f$. As $T \to 0$, this converges to $f$ if $f$ is continuous, and converges a.e. to $f$ if $f$ is integrable. In particular, this is almost like a identity for the convolution operation.

**Definition 5.2.** If $k_\lambda \in L^1(\mathbb{R})$ for $\lambda > 0$ (or sometimes $\lambda \in \mathbb{N}$) satisfy:

(a) Normalization: $\displaystyle\int_{-\infty}^\infty k_\lambda = 1$ for every $\lambda$,

(b) $L^1$-boundedness: $\displaystyle\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^\infty |k_\lambda| < \infty$,

(c) $L^1$-concentration: $\displaystyle\lim_{\lambda \to \infty} \int_{|x| \ge \delta} |k_\lambda| = 0$ for every $\delta > 0$,

then we say that $\{k_\lambda\}$ is an *approximate identity (for convolution)*.

**Exercise 5.1.** If $k \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} k = 1$, then $k_\lambda(x) = \lambda k(\lambda x)$ forms an approximate identity.

**Remark.** If we choose $k_\lambda$ to be nice, then $f * k_\lambda$ will also be nice and "close" to $f$.

# Lecture 6

# Sept. 4 — Approximate Identities

## 6.1 Properties of Approximate Identities

**Theorem 6.1.** *If $\{k_\lambda\}$ is an approximate identity, then for all $f \in L^1(\mathbb{R})$,*

$$\lim_{\lambda \to \infty} \|f * k_\lambda - f\|_1 = 0.$$

*That is, $f * k_\lambda \to f$ in $L^1$-norm.*

*Proof.* We have already seen that $f * k_\lambda \in L^1(\mathbb{R})$. Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)| \, dx = \int \left| f(x) \int k_\lambda(t) \, dt - \int f(x - t) k_\lambda(t) \, dt \right| dx,$$

where we used that $\int k_\lambda(t) \, dt = 1$. Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \leq \iint |f(x) - f(x - t)||k_\lambda(t)| \, dtdx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \leq \int |k_\lambda(t)| \left( \int |f(x) - T_t f(x)| \, dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \leq \int_{|t| < \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that $\lim_{t \to 0} \|f - T_t f\|_1 = 0$, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t| < \delta$ implies $\|f - T_t f\|_1 < \epsilon$. This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| \, dt + \int_{|t| \geq \delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use $\|f - T_t f\|_1 \leq \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$ to get

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t| < \delta} |k_\lambda(t)| \, dt + 2\|f\|_1 \int_{|t| \geq \delta} |k_\lambda(t)| \, dt \leq \epsilon K + 2\|f\|_1 \epsilon$$

where $K = \sup_\lambda \|k_\lambda\|_1 < \infty$ and $\lambda$ is large enough (as $\int_{|t| \geq \delta} |k_\lambda(t)| \, dt \to 0$). So $\|f - f * k_\lambda\|_1 \to 0$. $\square$

**Exercise 6.1.** Show that for $1 \leq p < \infty$, we still have $\|f - f * k_\lambda\|_p \to 0$ as $\lambda \to \infty$ for $f \in L^p(\mathbb{R})$. For $p = \infty$, show that if $f \in C_0(\mathbb{R})$, then $\|f - f * k_\lambda\|_\infty \to 0$ as $\lambda \to \infty$, that is $f * k_\lambda \to f$ uniformly.

**Exercise 6.2.** Show that if $f \in C_b(\mathbb{R})$, then for every compact set $K \subseteq \mathbb{R}$,

$$\lim_{\lambda \to \infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

**Definition 6.1.** A function $f$ is *Hölder continuous* with exponent $\alpha > 0$ if

$$|f(x) - f(y)| \leq K|x - y|^\alpha$$

for some constant $K$ and all $x, y$. If $\alpha = 1$, then we say that $f$ is *Lipschitz*.

**Remark.** If $f$ is Hölder continuous with exponent $\alpha > 1$, then the mean value theorem implies that $f$ is constant. Thus the interesting range for Hölder continuity is $0 < \alpha \leq 1$.

**Exercise 6.3.** Let $f$ be bounded and Hölder continuous with exponent $0 < \alpha \leq 1$, then show that

$$f * k_\lambda \to f \quad \text{uniformly on } \mathbb{R}.$$

**Remark.** If $f$ is differentiable and $f'$ is bounded, then $f$ is Lipschitz.

**Remark.** Recall the *Lebesgue differentiation theorem*, which says that if $f \in L^1_{\text{loc}}(\mathbb{R})$, then

$$(f * g_T)(x) = \frac{1}{2T} \int_{x-T}^{x+T} f(t) \, dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where $g_T = \chi_{[-T,T]}/(2T)$. The points where the limit holds are called the *Lebesgue points* of $f$.

**Theorem 6.2.** *Assume $k$ is bounded and compactly supported and $\int k = 1$. Set $k_\lambda(x) = \lambda k(\lambda x)$ for $\lambda > 0$. Then for any $f \in L^1(\mathbb{R})$,*

$$f * k_\lambda \to f \quad \text{pointwise a.e.}$$

*Moreover, the pointwise limit holds at every Lebesgue point of $f$.*

*Proof.* Assume $\text{supp}(k) \subseteq [-R, R]$. We can write

$$\lim_{\lambda \to \infty} |f(x) - (f * k_\lambda)(x)| = \lim_{\lambda \to \infty} \left| f(x) \int k_\lambda(x - t) \, dt - \int f(x) k_\lambda(x - t) \, dt \right|$$

$$\leq \lim_{\lambda \to \infty} \int |f(x) - f(t)| \lambda |k(\lambda x - \lambda t)| \, dt$$

$$\leq \lim_{\lambda \to \infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)| |k(\lambda x - \lambda t)| \, dt.$$

Making a change of variables $T = R/\lambda$, we have

$$\lim_{\lambda \to \infty} |f(x) - (f * k_\lambda)(x)| \leq \lim_{T \to 0} \frac{1}{2T} \int_{x-T}^{x+T} |f(x) - f(t)| \, dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point $x$ by the Lebesgue differentiation theorem.                                    $\square$

## 6.2 Density Results and Smooth Urysohn Lemma

**Theorem 6.3.** $C_c^m(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $m > 0$ *and* $1 \le p < \infty$.

*Proof.* Fix $\epsilon > 0$. Choose $k \in C_c^m(\mathbb{R})$ with $\int k = 1$, and set $k_\lambda(x) = \lambda k(\lambda x)$. Note that there exists a compactly supported $g \in L^p(\mathbb{R})$ with $\|f - g\|_p < \epsilon$ (e.g. take $g = f\chi_{[-R,R]}$ for large enough $R$, this works since $f\chi_{[-R,R]}$ converges pointwise to $f$ as $R \to \infty$ and is dominated by $f$, so the dominated convergence theorem implies that $f\chi_{[-R,R]} \to f$ in $L^p$-norm). Then note that $g * k_\lambda \in C_c^m(\mathbb{R})$ and $g * k_\lambda \to g$ in $L^p$-norm, so there exists $\lambda$ such that $\|g - g * k_\lambda\|_p < \epsilon$. Thus

$$\|f - g * k_\lambda\|_p \le \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result. $\qquad\square$

**Corollary 6.3.1.** $C_c^\infty(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $1 \le p < \infty$.

**Remark.** The above proof would work for $m = 0$ but becomes circular: The step $g * k_\lambda \in C_c^m(\mathbb{R})$ relies on the strong continuity of translation, which we proved by first showing it for $C_c(\mathbb{R})$ and then by an extension by density to $L^p(\mathbb{R})$. In particular, we needed to already know that $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

**Proposition 6.1** ($C^\infty$ Urysohn's lemma). *If* $K \subseteq \mathbb{R}$ *is compact and* $U \subseteq K$ *is open, then there exists* $f \in C_c^\infty(\mathbb{R})$ *such that* $0 \le f \le 1$, $f = 1$ *on* $K$, *and* $f = 0$ *on* $U^c$.

*Proof.* Since $K$ is compact and $U^c$ is closed, we have

$$d = \text{dist}(K, U^c) = \inf\{|x - y| : k \in K, y \notin U\} > 0.$$

Set $V = \{y \in \mathbb{R} : \text{dist}(y, K) < d/3\}$, and choose any $k \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(k) \subseteq [-d/3, d/3]$ and $\int k = 1$. Take $f = k * \chi_V \in C_c^\infty(\mathbb{R})$, which has $\text{supp}(f) \subseteq \text{supp}(k) + V \subseteq U$. If $x \in K$, then

$$f(x) = \int_V k(x - y)\, dy = \int k = 1.$$

One can check that $0 \le f \le 1$ and $f = 0$ on $U^c$ as an exercise, which would prove the result. $\qquad\square$

# Lecture 7

## Sept. 18 —

# Lecture 8

## Sept. 23 —

# Lecture 9

# Sept. 25 — Kernels and Schwartz Space

## 9.1 More Kernels

**Remark.** Recall that if $\{k_\lambda\}$ (where $k_\lambda(x) = \lambda k(\lambda x)$) is an approximate identity, then $f * k_\lambda \to f$ in $L^p$-norm. We also have
$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k_\lambda}.$$
Note that $\widehat{k_\lambda}(\xi) = \widehat{k}(\xi/\lambda)$, so $\widehat{k_\lambda}(\xi) \to 1$ pointwise. If the Dirichlet kernel $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$ were integrable, then we would have $\widehat{d}_{2\pi\lambda}(\xi) = \chi_{[-\lambda,\lambda]}(\xi)$, so we must have $d_{2\pi\lambda} \notin L^1(\mathbb{R})$.

Some alternatives are the following:

1. The Fejér kernel $w_\lambda(\xi) = d_{2\pi\lambda}^2(\xi) \in L^1(\mathbb{R})$. We saw this in the proof of the inversion formula.

2. The *de la Vallée Poussin kernel* $v_\lambda$, which has $\widehat{v}_\lambda$ as a trapezoid which is 1 on $[-\lambda, \lambda]$ and decays linearly to 0 on $[-2\lambda, -\lambda]$ and $[\lambda, 2\lambda]$. One has

$$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

   Explicitly, one can define $v(x) = 2w_2(x) - w(x)$ and $v_\lambda(x) = \lambda v(\lambda x)$.

3. The *Poisson kernel* $p(x) = 1/\pi(x^2 + 1)$.

4. The *Gauss kernel* $\phi(x) = e^{-\pi x^2}$.

**Exercise 9.1.** Show that if $f \in L^1(\mathbb{R})$, then $\text{supp}(\widehat{f})$ is compact if and only if $f = f * g$ for some $g \in L^1(\mathbb{R})$. Hint: Use the de la Vallée Poussin kernel.

**Exercise 9.2.** Let $\Phi = \widehat{\phi}$, where $\phi$ is the Gauss kernel. Show that $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$. Then solve this differential equation to get $\Phi(\xi) = \Phi(0)\phi(\xi)$. Finally show that $\Phi(0) = 1$, and conclude $\Phi(\xi) = \phi(\xi)$.

**Remark.** There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if $f, \widehat{f} \in L^1(\mathbb{R})$, then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for $f$ sufficiently nice, if we take $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge}$, then $\widehat{g} = g$.

**Theorem 9.1** (Weierstrass approximation theorem)**.** *If $f \in C[a, b]$ and $\epsilon > 0$, then there exists a polynomial $p$ such that $\|f - p\|_\infty = \sup_{x \in [a,b]} |f(x) - p(x)| < \epsilon$.*

*Proof.* Fix $f \in C[a,b]$, and choose $[a,b] \subseteq (-R,R)$. Extend $f$ to a function $g \in C_0(\mathbb{R})$ on $\mathbb{R}$ which equals $f$ on $[a,b]$ and is supported in $(-R,R)$. Let $\phi$ be the Gauss kernel, and choose $\lambda$ so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that $\phi_\lambda(x) = \lambda e^{-\pi \lambda^2 x^2}$ is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^\infty \lambda \frac{(-\pi \lambda^2 x^2)^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set $q(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$. If $N$ is large enough, then we will have

$$\sup_{|x| \leq 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \leq \int_{-R}^R |g(y)||\phi_\lambda(x-y) - q(x-y)|\,dy \leq \frac{\epsilon}{2\|g\|_1} \int_{-R}^R |g(y)|\,dy < \frac{\epsilon}{2}$$

for $|x| \leq R$. So $|f(x) - (g * q)(x)| \leq \epsilon$ for $x \in [a,b]$. Finally, observe that

$$(g * q)(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y)(x-y)^{2n}\,dy,$$

which is a polynomial by the binomial theorem. So we can take $p = g * q$. $\qquad\square$

## 9.2   Schwartz Space

**Definition 9.1.** Define the *Schwartz space* $\mathcal{S}(\mathbb{R})$ to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}.$$

**Remark.** Note that $|x^m f^{(n)}(x)| \leq C_{m,n}$ for some constant $C_{m,n}$, so

$$|f^{(n)}(x)| \leq \frac{C_{m,n}}{|x|^m}.$$

In particular, for every $n$ and polynomial $p$, there exists a constant $C_{n,p}$ such that

$$|f^{(n)}(x)| \leq \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply $f$ has exponential decay.

**Remark.** Note that $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$ defines a *seminorm* for each $m, n \geq 0$, but not a norm.[1]

---

[1]Recall that a *seminorm* $\rho$ is a function satisfying $0 \leq \rho(f) < \infty$, $\rho(cf) = |c|\rho(f)$, and $\rho(f+g) \leq \rho(f) + \rho(g)$. If we additionally have $\rho(f) = 0$ if and only if $\rho = 0$, then $\rho$ is a *norm*.

**Remark.** Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\|x^m f^{(n)}\|_1 = \int_{|x| \leq 1} |x^m f^{(n)}(x)| \, dx + \int_{|x| \geq 1} |x^m f^{(n)}(x)| \, dx$$

$$\leq 2\|x^m f^{(n)}\|_\infty + \int_{|x| \geq 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2} \, dx \leq 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty.$$

In particular, if the $L^\infty$-norms are controlled, then so are the $L^1$-norms.

**Exercise 9.3.** Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write $Df = f'$, and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \widehat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^{n} \binom{n}{j} D^j (-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on $f$ implies the Schwartz condition on $\widehat{f}$.

**Theorem 9.2.** If $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, we have $f = (\widehat{f})^\vee$ by the inversion formula, so the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $f \mapsto \widehat{f}$ is a bijection.

**Remark.** Note that $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$, and we will see later that $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$.

# Lecture 10

## Sept. 30 —

# Lecture 11

## Oct. 2 —

# Lecture 12

## Oct. 9 —

# Lecture 13

## Oct. 14 —

# Lecture 14

## Oct. 16 —

# Lecture 15

# Oct. 23 — The Paley-Wiener Theorem

## 15.1 Local Uncertainty Principle

**Remark.** Recall the classical uncertainty principle: For $f \in L^2(\mathbb{R})$, $\|f\|_2^2 \leq 4\pi \|xf(x)\|_2^2 \|\xi \widehat{f}(\xi)\|_2^2$. There is also a local version of this statement:

**Theorem 15.1** (Local uncertainty principle)**.** *If $f \in L^2(\mathbb{R})$ and $\epsilon > 0$, then for any $\xi_0 \in \mathbb{R}$, we have*

$$\int_{\xi_0 - \epsilon}^{\xi_0 + \epsilon} |\widehat{f}(\xi)|^2 \, d\xi \leq 8\pi\epsilon \|f\|_2 \|xf(x)\|_2.$$

## 15.2 The Paley-Wiener Theorem

**Remark.** Recall the smoothness and decay theorems: If $x, xf(x) \in L^1(\mathbb{R})$, then $(\widehat{f})'$ exists. "Extreme decay" corresponds to compact support, where $\operatorname{supp}(f) \subseteq [-T, T]$ if $f = 0$ a.e. outside $[-T, T]$. When this is the case, the *Paley-Wiener theorem* says that $\widehat{f}$ has extreme smoothness: $\widehat{f}$ extends to an analytic function on $\mathbb{C}$.

**Remark.** There is also motivation from Fourier series. The Fourier transform on $\mathbb{R}$ is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x} \, dx.$$

The functions $e_\xi(x) = e^{2\pi i \xi x} : \mathbb{R} \to S^1 \subseteq \mathbb{C}$ are continuous group homomorphism, and $dx$ is a Haar measure on $\mathbb{R}$ (invariant under the group action). For $\mathbb{Z}$, Haar measure is the counting measure, and for $c = (c_n)_{n \in \mathbb{Z}} \in \ell^1$, we can define its Fourier transform to be

$$\widehat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{R}.$$

In fact, we can take $\xi \in [0, 1)$. Also note that $\widehat{c}$ is 1-periodic.

Suppose now that $c$ is compactly supported, say only $c_1, \ldots, c_N$ are nonzero. Then

$$\widehat{c}(\xi) = \sum_{n=1}^{N} c_n e^{-2\pi i n \xi} = \sum_{n=1}^{N} c_n (e^{-2\pi i \xi})^n$$

If we set $z = e^{-2\pi i\xi} \in S^1$, then the above becomes

$$\widehat{c}(z) = \sum_{n=1}^{N} c_n z^n, \quad |z| = 1,$$

which is a polynomial. In particular, $\widehat{c}$ extends to an analytic function for all $z \in \mathbb{C}$.

**Remark.** Suppose $f \in L^2(\mathbb{R})$ and $\mathrm{supp}(f) \subseteq [-T, T]$ (we say that $f$ is "time-limited" to $[-T, T]$). Then

$$\int_{-\infty}^{\infty} |f(x)| \, dx = \int_{-T}^{T} |f(x)| \, dx \leq \left( \int_{-T}^{T} |f(x)|^2 \, dx \right)^{1/2} \left( \int_{-T}^{T} 1^2 \, dx \right)^{1/2} = (2T)^{1/2} \|f\|_2 < \infty,$$

so $f \in L^1(\mathbb{R})$. Thus we can take its Fourier transform as usual:

$$\widehat{f}(\xi) = \int_{-T}^{T} f(x) e^{-2\pi i\xi x} \, dx, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $e^{-2\pi i\xi x} = e^{-2\pi i\alpha x} e^{2\pi\beta x}$, where $e^{2\pi\beta x} > 0$. Then

$$\int_{-T}^{T} |f(x) e^{-2\pi i\xi x}| \, dx = \int_{-T}^{T} |f(x)| e^{2\pi\beta x} \, dx \leq e^{2\pi|\beta|T} \int_{-T}^{T} |f(x)| \, dx \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2.$$

So $\widehat{f}$ can be extended to all of $\mathbb{C}$. We will see that this extension is analytic.

We now have a function $\widehat{f} : \mathbb{C} \to \mathbb{C}$ satisfying

$$|\widehat{f}(\xi)| \leq (2T)^{1/2} e^{2\pi|\beta|T} \|f\|_2, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that $f \in L^1(\mathbb{R})$ implies $\widehat{f} \in C_0(\mathbb{R})$, so $\widehat{f}$ is bounded on $\mathbb{R}$. However, the extension to $\mathbb{C}$ cannot be bounded (for non-constant $f$) by Liouville's theorem.

**Definition 15.1.** A function $F : \mathbb{C} \to \mathbb{C}$ has *exponential type* $T$ if there exists $A > 0$ such that

$$|F(\xi)| \leq A e^{T|\xi|}, \quad \xi \in \mathbb{C}.$$

**Remark.** So from the above discussion, $\widehat{f}$ has exponential type $2\pi T$. We now show that $\widehat{f}$ is analytic on $\mathbb{C}$. Recall that $F$ is analytic if and only if $F'$ exists on $\mathbb{C}$. We can compute

$$(\widehat{f})'(\xi) = \lim_{\substack{\eta \to 0 \\ \xi \in \mathbb{C}}} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \to 0} \int_{-T}^{T} f(x) \frac{e^{-2\pi i(\xi+\eta)x} - e^{-2\pi i\xi x}}{\eta} \, dx.$$

We have seen that $f$ is integrable, and the exponential term is bounded on $[-T, T]$ by the mean value theorem. So by the dominated convergence theorem, we can exchange the limit and integral to get

$$(\widehat{f})'(\xi) = \int_{-T}^{T} f(x) \lim_{\eta \to 0} \frac{e^{-2\pi i(\xi+\eta)x} - e^{-2\pi i\xi x}}{\eta} \, dx$$

$$= \int_{-T}^{T} (-2\pi i x) f(x) e^{-2\pi i\xi x} \, dx = ((-2\pi i x) f(x))^{\wedge}(\xi).$$

Note that $x f(x) \in L^1(\mathbb{R})$ and also has support in $[-T, T]$, so the above exists for any $\xi \in \mathbb{C}$.

**Theorem 15.2** (Paley-Wiener theorem)**.** *We have the following:*

1. *If $f \in L^2(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq [-T, T]$, then $\widehat{f}$ extends to an analytic function on $\mathbb{C}$ with exponential type $2\pi T$.*

2. *If $F : \mathbb{C} \to \mathbb{C}$ is analytic with exponential type $2\pi T$, then there exists $f \in L^2(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq [-T, T]$ and $\widehat{f}(\xi) = F(\xi)$ for $\xi \in \mathbb{R}$, i.e. $\widehat{f} = F|_{\mathbb{R}}$.*

*Proof.* (1) This was the above discussion.

(2) The proof is more difficult and uses a lot of complex analysis. See Katznelson's book. □

**Remark.** Recall that analytic functions have Taylor expansions. Suppose $F : \mathbb{C} \to \mathbb{C}$ and $F = 0$ on a line segment. Then if $\eta$ lies on the line segment, we can write

$$F(\xi) = \sum_{k=0}^{\infty} F^{(k)}(\eta) \frac{(\xi - \eta)^k}{k!}.$$

Since $F^{(k)}(\eta) = 0$ for all $k$, we get $F \equiv 0$ on all of $\mathbb{C}$. In fact, if $F = 0$ on any set with an accumulation point, then $F \equiv 0$ on $\mathbb{C}$.

**Corollary 15.2.1.** *If $f \in L^2(\mathbb{R})$, $f \neq 0$, has compact support, then $\widehat{f}(\xi) = 0$ for only countably many $\xi$.*

*Proof.* If $\widehat{f}(\xi) = 0$ for uncountably many $\xi$, then there exists $\xi_n \to \xi$ such that $\widehat{f}(\xi_n) = 0$ for all $n$ (split $\mathbb{C}$ into countably many compact sets and apply a countability argument). Hence $\widehat{f}, f = 0$. □

**Corollary 15.2.2.** *If $f \in L^2(\mathbb{R})$, then $f$ and $\widehat{f}$ are both compactly supported if and only if $f = 0$.*

*Proof.* Suppose $f$ is compactly supported, then $\widehat{f}$ is analytic on $\mathbb{C}$. Suppose $\widehat{f}(\xi) = 0$ a.e. on $\mathbb{R}$ outside $[-\Omega, \Omega]$. Since $\widehat{f}$ is analytic and $\mathbb{R} \setminus [-\Omega, \Omega]$ has an accumulation point, we get $\widehat{f}, f = 0$. □

**Remark.** In particular, the above corollary implies that if $f \in C_c^{\infty}(\mathbb{R})$ is nonzero, then $\widehat{f}$ is not compactly supported. But $C_c^{\infty}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ and $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, so we do know $\mathcal{F}(C_c^{\infty}(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R})$.

## 15.3 Hilbert-Schmidt Operators

**Remark.** Let $H$ be a separable, infinite-dimensional Hilbert space. Recall the following definitions and results from functional analysis:

**Definition 15.2.** A linear operator $A : H \to H$ is *compact* if $A(D)$ is contained in a compact set, where $D = \{f \in H : \|f\| \leq 1\}$ is the closed unit disk in $H$.

**Theorem 15.3** (Spectral theorem for compact, self-adjoint operators)**.** *If $A : H \to H$ is a compact, self-adjoint operator, then there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $H$ such that*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \quad Af = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n, \quad Ae_n = \lambda_n e_n.$$

*Moreover, $\lambda_n \to 0$ if there are infinitely many $\lambda_n \neq 0$.*

**Remark.** Note that $\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n| \to 0$, so the identity operator $I$ is not compact.

**Example 15.2.1.** An example of a compact box in $\mathbb{R}^{\mathbb{N}}$ looks like

$$[0, 1] \times [0, 1/2] \times [0, 1/4] \times \cdots .$$

**Definition 15.3.** A linear operator $A : H \to H$ is *Hilbert-Schmidt* if there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$.

**Remark.** For an operator $A : H \to H$, if we write

$$Af(x) = \int_{-\infty}^{\infty} k(x, y) f(y) \, dy,$$

then $A$ is Hilbert-Schmidt if and only if $k \in L^2(\mathbb{R}^2)$.

# Lecture 16

## Oct. 28 —

# Lecture 17

## Oct. 30 —

# Lecture 18

# Nov. 4 — Energy Concentration, Part 2

## 18.1   Energy Concentration, Continued

**Remark.** Recall the following problem. Fix $T, \Omega > 0$. We want to determine

$$E_{T,\Omega} = \sup\left\{\int_{-T}^{T} |f|^2 : \|f\|_2 = 1, \operatorname{supp}(\widehat{f}) \subseteq [-\Omega, \Omega]\right\}.$$

We previously defined $A_T f = f\chi_{[-T,T]}$ and $B_\Omega f = (\widehat{f}\chi_{[-\Omega,\Omega]})^\vee = f * d_{2\pi\Omega}$, then

$$E_{T,\Omega} = \|A_T B_\Omega\|^2 = \|B_\Omega A_T B_\Omega\|^2 = \lambda_1,$$

where $\lambda_1$ is the largest eigenvalue of $B_\Omega A_T B_\Omega$ (recall that $B_\Omega A_T B_\Omega$ is compact and self-adjoint, so it has eigenvalues $\lambda_n \to 0$). The spectral theorem also gives orthonormal eigenvectors $\{\varphi_n\}_{n \in \mathbb{N}}$:

$$B_\Omega A_T B_\Omega f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n.$$

Note that $B_\Omega(A_T B_\Omega \varphi_n) = \lambda_n \varphi_n$ for $\lambda_n \neq 0$, so $\varphi_n \in \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ and $\operatorname{supp}(\widehat{\varphi}_n) \subseteq [-\Omega, \Omega]$.

What is $\|A_T \varphi_n\|_2^2$? We can compute

$$\|A_T \varphi_n\|_2^2 = \langle A_T \varphi_n, A_T \varphi_n \rangle = \langle A_T B_\Omega \varphi_n, A_T B_\Omega \varphi_n \rangle$$
$$= \langle B_\Omega \underbrace{A_T A_T}_{A_T} B_\Omega \varphi_n, \varphi_n \rangle = \lambda_n \langle \varphi_n, \varphi_n \rangle = \lambda_n \|\varphi_n\|_2^2 = \lambda_n.$$

Thus we see that $E_{T,\Omega} = \lambda_1 = \|A_T \varphi_1\|_2^2 < \|\varphi_1\|_2^2 = 1$, so $\varphi_1$ has the greatest energy in $[-T, T]$.

Note that $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for its closed span, which a priori lies in $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$.

**Proposition 18.1.** $\overline{\operatorname{span}}\{\varphi_n\} = \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$.

*Proof.* We show $\overline{\operatorname{span}}\{\varphi_n\}^\perp = \{0\}$. Suppose $f \in \mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ with $f \perp \varphi_n$ for all $n$. Then

$$f \in \overline{\operatorname{span}}\{\varphi_n\}^\perp = \overline{\operatorname{range}}(B_\Omega A_T B_\Omega)^\perp = \ker(B_\Omega A_T B_\Omega).$$

Thus we have

$$\|A_T f\|_2^2 = \langle A_T B_\Omega f, A_T B_\Omega f \rangle = \langle B_\Omega A_T B_\Omega f, f \rangle = 0,$$

so $f = 0$ a.e. on $[-T, T]$. By the Paley-Wiener theorem, this implies $f = 0$. $\qquad\square$

**Remark.** Proposition 18.1 implies that there are infinitely many nonzero eigenvalues for $B_\Omega A_T B_\Omega$.

Moreover, since $B_\Omega$ is the orthogonal projection onto $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$ and $\{\varphi_n\}$ is an orthonormal basis for $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$, we can write

$$B_\Omega f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n.$$

Note that $L^2_{[-T,T]}(\mathbb{R})$ is not orthogonal to $\mathcal{F}L^2_{[-\Omega,\Omega]}(\mathbb{R})$: For example,

$$\langle \chi_{[-T,T]}, d_{2\pi\Omega} \rangle = \int_{-T}^{T} \frac{\sin 2\pi\Omega x}{\pi x} \, dx \neq 0.$$

However, set $\psi_n = \lambda_n^{-1/2} A_T \varphi_n$. Then

$$\langle \lambda_m^{-1/2} A_T \varphi_m, \lambda_n^{-1/2} A_T \varphi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \langle B_\Omega A_T B_\Omega \varphi_m, \varphi_n \rangle = \lambda_m^{1/2} \lambda_n^{-1/2} \langle \varphi_m, \varphi_n \rangle = \delta_{mn}.$$

So $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{[-T,T]}(\mathbb{R})$.

**Exercise 18.1.** Show that $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2_{[-T,T]}(\mathbb{R})$.

**Remark.** Note that

$$\langle \varphi_m, \varphi_n \rangle = \int_{-\infty}^{\infty} \varphi_m(x)\overline{\varphi_n(x)} \, dx = 0, \quad m \neq n.$$

Furthermore,

$$\langle \psi_m, \psi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \int_{-T}^{T} \varphi_m(x)\overline{\varphi_n(x)} \, dx = 0, \quad m \neq n$$

even when the $\varphi_n$ are *not* supported in $[-T, T]$.

By symmetry (swapping the roles of $A_T$ and $B_\Omega$), one can also compute that

$$A_T B_\Omega A_T f = \sum_{n=1}^{\infty} \lambda_n \langle f, \psi_n \rangle \psi_n.$$

In particular, $A_T B_\Omega A_T$ and $B_\Omega A_T B_\Omega$ have the same eigenvalues.

**Proposition 18.2.** $B_\Omega A_T$ *commutes with*

$$Kf = (T^2 - x^2)f''(x) - 2xf'(x) - 4\pi^2\Omega^2 x^2 f(x).$$

**Remark.** Since $\varphi_n$ is already band-limited, we have

$$B_\Omega A_T \varphi_n = B_\Omega A_T B_\Omega \varphi_n = \lambda_n \varphi_n.$$

Moreover, Paley-Wiener implies $\varphi_n$ is infinitely differentiable, so $\varphi_n \in \text{domain}(K)$. Then

$$B_\Omega A_T K \varphi_n = K B_\Omega A_T \varphi_n = K(\lambda_n \varphi_n) = \lambda_n K \varphi_n.$$

Note that $K\varphi_n \neq 0$, so $K\varphi_n$ is an eigenvector for $B_\Omega A_T$. The multiplicities of the $\lambda_n$ is 1, so $K\varphi_n = \mu\varphi_n$. The eigenfunctions $\varphi_n$ of $K$ are known functions called the *prolate spheroidal wave functions*.

## 18.2    Approximating Band-Limited Functions

**Remark.** We know that $1 > \lambda_1 > \lambda_2 > \cdots \to 0$. Also,

$$\|A_T B_\Omega\|_{\mathrm{HS}}^2 = \|g_{T,\Omega}\|_2^2 = \int_{-T}^{T} \int_{-\infty}^{\infty} d_{2\pi\Omega}(x-y)^2 \, dxdy = 4T\Omega,$$

where $g_{T,\Omega}$ is the corresponding kernel. The singular values of $A_T B_\Omega$ are

$$s_n = \lambda_n((A_T B_\Omega)^*(A_T B_\Omega))^{1/2} = \lambda_n(B_\Omega A_T B_\Omega)^{1/2} = \lambda_n^{1/2}.$$

Thus $4T\Omega = \|A_T B_\Omega\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} s_n^2 = \sum_{n=1}^{\infty} \lambda_n$. On the other hand, $\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^2$, and

$$\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 = \|k_{T,\Omega}\|_2^2 = \int_{-T}^{T} \int_{-T}^{T} d_{2\pi\Omega}(x-y)^2 \, dxdy < 4T\Omega.$$

Due to the decay of $d_{2\pi\Omega}(x-y)^2$, for $T$ large, we should expect that $\|B_\Omega A_T B_\Omega\|_{\mathrm{HS}}^2 \to 4T\Omega$. Hence

$$\sum_{n=1}^{\infty} \lambda_n^2 \approx \sum_{n=1}^{\infty} \lambda_n.$$

Thus intuitively, $\lambda_n$ should be either close to 1 or close to 0. In particular, only finitely many terms can be close to 1, so

$$f \approx \sum_{n=1}^{N} \lambda \langle f, \varphi_n \rangle \varphi_n \in \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}.$$

Thus the space of band-limited functions are approximately finite-dimensional in some sense.

**Theorem 18.1.** $\#\{n : \lambda_n \geq 1 - \epsilon\} \leq 4T\Omega - C_\epsilon \log(T\Omega)$.

## 18.3    Space of Possible Time and Band Limits

**Remark.** Define the energies

$$\alpha = E_T(f) = \|A_T f\|_2^2 = \int_{-T}^{T} |f|^2$$

$$\beta = E_\Omega(f) = \|B_\Omega f\|_2^2 = \int_{-\Omega}^{\Omega} |\widehat{f}|^2.$$

Which pairs $(\alpha, \beta)$ can be achieved? We have seen that $(\lambda_1, 1)$ and $(\lambda_2, 1)$ are possible, while $(1, 1)$ is not by the Paley-Wiener theorem. Similarly, $(1, 0)$ and $(0, 1)$ are not possible by Paley-Wiener. Since the roles of $A_T$ and $B_\Omega$ are symmetric, we get that $(1, \lambda_1)$ and $(1, \lambda_2)$ are possible. Note that if $(\alpha, 1)$ is achieved, then so is $(\alpha', 1)$ for any $\alpha' < \alpha$, and a similar statement holds for $(1, \beta)$.

One can show the space of valid pairs is $(\alpha, \beta) \in [0, 1]^2$ such that

$$\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \lambda^{1/2},$$

which is in the shape of an ellipse.

# Lecture 19

# Nov. 6 — Fourier Series

## 19.1 The Abstract Fourier Transform

**Definition 19.1.** A *locally compact abelian (LCA) group* is an abelian group with a Hausdorff topology such that every point has a compact neighborhood, and $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous.

**Example 19.1.1.** Some examples of LCA groups are $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{T}^n$, $\mathbb{Z}_N^n$, where $\mathbb{T} = [0, 1)$ with addition modulo 1.

**Theorem 19.1.** *Every LCA group has a* Haar *measure, i.e. a nonzero Radon measure $\mu$ which is translation-invariant.*

**Definition 19.2.** A *character* on a LCA group is a continuous homomorphism

$$\xi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

i.e. a continuous map with $\xi(x + y) = \xi(x)\xi(y)$.

**Definition 19.3.** The *dual group* of $G$ is $\widehat{G} = \{\xi : \xi \text{ is a character of } G\}$.

**Example 19.3.1.** Consider $\mathbb{T} = [0, 1)$. Fix $n \in \mathbb{Z}$, then $e_n(x) = e^{2\pi i n x}$ is a character on $\mathbb{T}$. Note that we have $e_n(x) = e_1(x)^n$, and one can show these are all of the characters on $\mathbb{T}$. Thus

$$\widehat{\mathbb{T}} = \{e_n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

**Example 19.3.2.** The characters on $\mathbb{Z}$ are determined by their values at 1. So for any $\xi \in [0, 1)$, we get a character $e_\xi(n) = e^{2\pi i \xi n}$. In particular, $\widehat{\mathbb{Z}} = \mathbb{T}$.

**Example 19.3.3.** We have $\widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N$ and $\widehat{\mathbb{R}} \cong \mathbb{R}$. The characters on $\mathbb{R}$ are $e_\xi(x) = e^{2\pi i \xi x}$.

**Remark.** The dual group $\widehat{G}$ is a LCA group under the multiplication of characters, where the topology on $\widehat{G}$ is the topology of uniform convergence on compact sets (also called the compact-open topology).

**Theorem 19.2.** *If $G$ is discrete, then $\widehat{G}$ is compact. If $G$ is compact (so $G$ has finite measure), then $\widehat{G}$ is discrete, and the characters on $G$ are orthonormal in $L^2(G)$ (in fact, they form an orthonormal basis).*

**Example 19.3.4.** Consider the examples $\widehat{\mathbb{Z}} = \mathbb{T}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{Z}_N} = \mathbb{Z}_N$.

**Remark.** In some sense, $\mathbb{R}$ being non-compact and non-discrete makes things harder. On the other hand, having dilations in $\mathbb{R}$ can make harmonic analysis easier.

**Theorem 19.3** (Pontryagin duality)**.** $\widehat{\widehat{G}} \cong G$.

**Definition 19.4.** The *Fourier transform* of $f \in L^1(G)$ (complex-valued) is $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_G f(x)\overline{\xi(x)} \, d\mu(x), \quad \xi \in \widehat{G},$$

where $d\mu$ is the Haar measure on $G$.

**Example 19.4.1.** On the real line, the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x} \, dx, \quad \xi \in \mathbb{R}.$$

For $\mathbb{T}$, the Fourier coefficients are given by

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx} \, dx, \quad n \in \mathbb{Z}.$$

For $\mathbb{Z}$, given $c = (c_n)_{n \in \mathbb{Z}}$,

$$\widehat{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi inx}, \quad x \in \mathbb{T}$$

For $\mathbb{Z}_N$, the discrete Fourier transform is

$$\widehat{c}(k) = \sum_{n=0}^{N-1} c_n e^{-2\pi ink/N}, \quad k \in \mathbb{Z}_N.$$

## 19.2   Fourier Series

**Definition 19.5.** The *Fourier transform* of $f \in L^1(\mathbb{T})$ is the sequence $\widehat{f} = (\widehat{f}(n))_{n \in \mathbb{Z}}$, where

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx} \, dx$$

are the *Fourier coefficients* of $f$. The *inverse Fourier transform* of $f \in L^1(\mathbb{T})$ is $\widecheck{f} = (\widecheck{f}(n))_{n \in \mathbb{Z}}$, where

$$\widecheck{f}(n) = \int_0^1 f(x)e^{2\pi inx} \, dx.$$

Formally, $\sum_{n \in \mathbb{Z}} \widehat{f}(n)e^{2\pi inx}$ is the *Fourier series* of $f$.

**Remark.** If $f \in L^2(\mathbb{T})$ and we accept that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx} = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n,$$

where the convergence is in $L^2(\mathbb{T})$ (we do not get pointwise convergence in general).

**Remark.** If $f \in L^1(\mathbb{T})$, then we have

$$|\widehat{f}(n)| \leq \int_0^1 |f(x)e^{-2\pi inx}| \, dx = \|f\|_{L^1},$$

so we see that $\widehat{f} \in \ell^\infty(\mathbb{Z})$.

**Remark.** If we define $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, then we can view the Fourier series as

$$(\widehat{f})^{\vee}(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

**Example 19.5.1.** For $0 < \alpha \leq 1$, consider the function

$$g(x) = \sum_{m=0}^{\infty} 2^{-\alpha m} e^{2\pi i 2^m x}, \quad x \in \mathbb{T}$$

which converges absolutely in $C(\mathbb{T})$. One can show that $g$ is nowhere differentiable.

Weierstrass first showed that $g(x) = \sum_{m=0}^{\infty} a^{-m} \cos(b^m x)$ is nowhere differentiable for $a, b$ large enough.

## 19.3   Partial Sums of Fourier Series

**Definition 19.6.** For $f \in L^1(\mathbb{T})$, define the *partial sums*

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}.$$

**Remark.** Consider a discrete characteristic function

$$\chi_N(n) = \begin{cases} 1 & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that $\chi_N \in \ell^1(\mathbb{Z})$. Then we can compute that

$$\widehat{\chi}_N(x) = \check{\chi}_N(x) = \sum_{n=-\infty}^{\infty} \chi_N(n) e^{2\pi i n x} = \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} = d_N(x)$$

by the geometric series formula. Note that $d_N \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$.[1] We also have $\|d_N\|_1 \to \infty$.

**Remark.** Using the *Dirichlet kernel* $d_N$, we can write the partial sums as

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-N}^{N} \left( \int_0^1 f(t) e^{-2\pi i n t} \, dt \right) e^{2\pi i n x}$$

$$= \int_0^1 f(t) \sum_{n=-N}^{N} e^{2\pi i n (x-t)} \, dt = \int_0^1 f(t) d_N(x-t) \, dt = (f * d_N)(x).$$

Intuitively, $S_N f = (\widehat{f}\chi_N)^{\vee} = (\widehat{f})^{\vee} * \check{\chi}_N = f * d_N$, but the above is a direct proof.

---

[1] Note that $\mathbb{T}$ is compact, so a continuous function on $\mathbb{T}$ is automatically integrable.

## 19.4  Approximate Identities on $\mathbb{T}$

**Definition 19.7.** An *approximate identity* on $\mathbb{T}$ is a sequence $\{k_n\}_{n\in\mathbb{N}}$ such that

1.  $k_n \in L^1(\mathbb{T})$ and $\displaystyle\int_0^1 k_n(x)\,dx = 1$;

2.  $\sup_{n\in\mathbb{N}} \|k_n\|_1 < \infty$;

3.  for any $\delta > 0$, $\displaystyle\int_{|x|\geq\delta} |k_n(x)|\,dx \to 0$.

**Remark.** The Dirichlet kernel $d_N$ does not form an approximate identity.

**Definition 19.8.** For $f \in L^1(\mathbb{T})$, define the *Cesàro sums*

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + \cdots + S_N f(x)}{N+1}.$$

Note that this is the average of the partial sums.

**Remark.** We can rewrite the Cesàro sums as

$$\sigma_N f(x) = \sum_{n=-N}^{N} \underbrace{\left(1 - \frac{|n|}{N+1}\right)}_{W_N(n)} \widehat{f}(n) e^{2\pi i n x} = (f * \widehat{W}_N)(x).$$

We define the *Fejér kernel* to be

$$w_N(x) = \widehat{W}_N(x) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right)^2.$$

# Lecture 20

# Nov. 11 — Fourier Series, Part 2

## 20.1 Approximate Identities on $\mathbb{T}$, Continued

**Remark.** Using a similar proof as in the case for $\mathbb{R}$, for $f \in L^1(\mathbb{T})$, we have

$$\widehat{f} \in c_0(\mathbb{Z}) = \{c = (c_n) : c_n \to 0 \text{ as } |n| \to \infty\}.$$

However, in general $\widehat{f} \notin \ell^1(\mathbb{Z})$.

**Exercise 20.1.** Let $d_N$ be the Dirichlet kernel on $\mathbb{T}$. Show that

$$\frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \le \|d_N\|_1 \le 3 + \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}.$$

In particular, $\|d_N\|_1 \to \infty$ as $N \to \infty$, so $d_N$ does not form an approximate identity.

**Remark.** Define the function

$$W_N(x) = \begin{cases} 1 - |n|/(N+1) & |n| \le N, \\ 0 & |n| > N. \end{cases}$$

Note that $W_N = (\chi_N * \chi_N)/(2N+1)$ and $\widehat{W}_{2N} = (\widehat{\chi}_N)^2 = d_N^2/(2N+1)$. The Fejér kernel is

$$w_N = \widehat{W}_N = \frac{1}{N} \left( \frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2,$$

and one can check that $\int_0^1 w_N = 1$ and $\|w_N\|_1 = 1$.

**Theorem 20.1.** $(f * \widetilde{W}_N)(x) = (f * w_N)(x)$.

**Exercise 20.2.** $\{w_N\}_{N \in \mathbb{N}}$ forms an approximate identity on $\mathbb{T}$.

**Remark.** Unlike the real line, we cannot just take any $L^1$ function and dilate it to form an approximate identity on $\mathbb{T}$, as the dilations need not be 1-periodic.

**Exercise 20.3.** If $\{k_N\}$ is an approximate identity on $\mathbb{T}$, then:

1. For $1 \le p < \infty$, $f \in L^p(\mathbb{T})$ implies $f * k_N \to f$ in $L^p$-norm.

2. $f \in C(\mathbb{T})$ implies $f * k_N \to f$ uniformly.

## 20.2   The Inversion Formula on $\mathbb{T}$

**Theorem 20.2.** *If $f \in L^1(\mathbb{T})$ and $\widehat{f} \in \ell^1(\mathbb{Z})$, then*

$$f(x) = (\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx},$$

*where the above series converges uniformly in $C(\mathbb{T})$.*

*Proof.* Note that $(\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$ converges absolutely in the norm of $C(\mathbb{T})$ since

$$\sum_{n=-\infty}^{\infty} \|\widehat{f}(n)e^{2\pi inx}\|_\infty = \sum |\widehat{f}(n)| < \infty.$$

In particular, this shows that $(\widehat{f})^\vee \in C(\mathbb{T})$. Also $f * w_N \to f$ in $L^1$-form, where

$$(f * w_N)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n)e^{2\pi inx}.$$

Note that $(1 - |n|/(N+1))\widehat{f}(n) \to \widehat{f}(n)$ pointwise as $N \to \infty$, and

$$\left| \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) \right| \le |\widehat{f}(n)| \in \ell^1(\mathbb{Z}),$$

so by the dominated convergence theorem (for counting measure)

$$(f * w_N)(x) \xrightarrow[N\to\infty]{} \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx} = (\widehat{f})^\vee(x)$$

pointwise. Since $f * w_N \to f$ in $L^1$-norm, there exists a subsequence $f * w_{N_k} \to f$ pointwise a.e., hence we see that $f = (\widehat{f})^\vee$ a.e. Since $f$ and $(\widehat{f})^\vee$ are continuous, we get that $f = (\widehat{f})^\vee$ everywhere. $\qquad\square$

**Corollary 20.2.1.** *If $f \in L^1(\mathbb{T})$ and $\widehat{f}(n) = 0$ for every $n$, then $f = 0$ a.e. Moreover, if $f, g \in L^1(\mathbb{T})$ and $\widehat{f}(n) = \widehat{g}(n)$ for all $n$, then $f = g$ a.e.*

**Definition 20.1.** The *Fourier algebra* (or *Wiener algebra*) on $\mathbb{T}$ is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\}.$$

**Remark.** The Fourier algebra $A(\mathbb{T})$ is closed under convolution and forms a Banach algebra. Moreover, it is a dense subspace of $C(\mathbb{T})$.

## 20.3   $L^2$-Convergence of Fourier Series

**Remark.** Let $e_n(x) = e^{2\pi inx}$. The functions $e_n$ are orthonormal in $L^2(\mathbb{T})$: For $n \ne m$,

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi imx}\overline{e^{2\pi inx}} \, dx = \int_0^1 e^{2\pi i(m-n)x} \, dx = \left.\frac{e^{2\pi i(m-n)x}}{2\pi i(m-n)}\right|_{x=0}^{1} = 0.$$

**Theorem 20.3.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space $H$. Then the following are equivalent:*

1. *$\{f_n\}_{n\in\mathbb{N}}$ is complete (i.e. $\overline{\operatorname{span}}\{f_n\} = H$).*

2. *$f = \sum_{n=1}^{\infty} c_n f_n$ in $H$ for a unique choice of scalars $c_n$. That is, $\{f_n\}$ is a Schauder basis for $H$.*

3. *$f = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$, where the convergence is in $H$.*

4. *(Plancherel) $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$.*

5. *(Parseval) $\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle f_n, g \rangle$.*

**Example 20.1.1.** The above theorem does not hold when the sequence is not orthonormal. Recall the *Weierstrass approximation theorem*: $\{x^n\}_{n=0}^{\infty}$ is complete in $C(\mathbb{T})$. But not every $f \in C(\mathbb{T})$ can be written as $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (such functions are infinitely differentiable in some disk).

**Theorem 20.4.** *We have the following:*

1. *For $1 \le p < \infty$, $\{e_n\}_{n\in\mathbb{N}}$ is complete in $L^p(\mathbb{T})$.*

2. *$\{e_n\}_{n\in\mathbb{N}}$ is complete in $C(\mathbb{T})$.*

*Proof.* (1) If $f \in L^p(\mathbb{T})$, then we have

$$(f * w_N) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x} \xrightarrow[N\to\infty]{} f$$

in $L^p$-norm. But $f * w_n \in \operatorname{span}\{e_n\}_{n\in\mathbb{Z}}$, so $\overline{\operatorname{span}}\{e_n\}_{n\in\mathbb{Z}} = L^p(\mathbb{T})$. The same proof works for (2). $\square$

**Corollary 20.4.1.** *$\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. Furthermore,*

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

*where the above convergence is in $L^2$-norm. Moreover, one has*

1. *(Plancherel) $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$,*

2. *(Parseval) $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)}$,*

*so the Fourier transform operator $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ given by $f \mapsto \widehat{f}$ is unitary.*

**Theorem 20.5.** *If $1 < p < \infty$, then $S_N f = (f * d_N)(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}$ converges in $L^p$-norm, so*

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

*in $L^p$-norm with respect to the ordering $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$. However, the convergence is conditional: Only this ordering (and finite permutations of it) need converge when $p \ne 2$.*

# Lecture 21

# Nov. 13 — Fourier Series, Part 3

## 21.1 Shannon Sampling Theorem

**Definition 21.1.** Define the *Paley-Wiener space* to be

$$\mathrm{PW}(\mathbb{R}) = \mathcal{F}L^2_{[-1/2,1/2]}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

**Remark.** Set $e_n(x) = e^{-2\pi inx}\chi_{[-1/2,1/2]}(x) = M_n\chi_{[-1/2,1/2]}(x)$, then

$$\check{e}_n(x) = T_n\check{\chi}_{[-1/2,1/2]}(x) = \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

Since $\{e_n\}$ is an orthonormal basis for $L^2_{[-1/2,1/2]}(\mathbb{R})$, the $\{\check{e}_n\}$ form an orthonormal basis for $\mathrm{PW}(\mathbb{R})$. Note that if $f \in \mathrm{PW}(\mathbb{R})$, then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, \check{e}_n \rangle \check{e}_n(x).$$

By the unitarity of the Fourier transform, we can compute that

$$\langle f, \check{e}_n \rangle = \langle \widehat{f}, e_n \rangle = \int_{-1/2}^{1/2} \widehat{f}(\xi)e^{2\pi in\xi}\, d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi in\xi}\, d\xi = (\widehat{f})^{\vee}(n) = f(n).$$

Note that the third equality follows from $\mathrm{supp}(\widehat{f}) \subseteq [-1/2, 1/2]$. Thus

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

This is called the *Shannon sampling theorem* (or *classical sampling theorem*), i.e. that any function in $\mathrm{PW}(\mathbb{R})$ is completely determined by its values on the integers.

**Remark.** Fix $b > 0$, and set $e_{nb}(x) = e^{2\pi inbx}$. This is an orthogonal basis for $L^2[0, 1/b]$.

**Example 21.1.1.** If we set $b = 1/2$, then we can write

$$\{e_{n/2}\}_{n\in\mathbb{Z}} = \{e_n\}_{n\in\mathbb{Z}} \cup \{e_{(n+1)/2}\}_{n\in\mathbb{Z}}.$$

This is a union of two orthonormal bases for $L^2[0, 1]$. For $f \in L^2[0, 1]$, we can write

$$2f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n + \sum_{n=-\infty}^{\infty} \langle f, e_{(n+1)/2} \rangle e_{(n+1)/2} = \sum_{n\in\mathbb{Z}} \langle f, e_{n/2} \rangle e_{n/2}.$$

**Remark.** If we take $0 < b < 1$ and consider $e^{2\pi inbx} = e_{bn}(x)$ in $L^2[0,1]$, then for $f \in L^2[0,1] \subseteq L^2[0,1/b]$,

$$f = \frac{1}{b} \sum_{n=-\infty}^{\infty} \langle f, e_{bn} \rangle e_{bn}$$

in $L^2[0,1/b]$. Since $f = 0$ on $[1,1/b]$, we have the same expansion in $L^2[0,1]$. Note, however, that this expansion is not orthogonal and not unique. These sets $\{e_{bn}\}_{n\in\mathbb{Z}}$ are called *frames*.

## 21.2 Weyl's Equidistribution Theorem

**Theorem 21.1.** *If $\alpha \in \mathbb{T}$ is irrational, then $\{k\alpha\}_{k\in\mathbb{N}}$ is equidistributed in $[0,1)$, i.e.*

$$\lim_{N\to\infty} \frac{\#\{1 \le k \le N : k\alpha \bmod 1 \in (a,b)\}}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha) = b - a$$

*for every $0 \le a < b \le 1$.*

**Remark.** Consider $f \in C(\mathbb{T})$. Then we can view $\frac{1}{N} \sum_{k=1}^{N} f(k\alpha)$ as some type of Riemann sum, so we might expect that

$$\frac{1}{N} \sum_{n=1}^{N} f(k\alpha) \xrightarrow[N\to\infty]{} \int_0^1 f(x)\,dx.$$

The *Birkhoff ergodic theorem* says that the above average in fact converges to the integral (given that $\{k\alpha\}$ is equidistributed). In fact, the result holds for any $f \in L^1(\mathbb{T})$.

**Exercise 21.1.** Prove the above statement for $e_n(x) = e^{2\pi inx}$ (and thus for trigonometric polynomials $p(x) = \sum_{n=-N}^{N} c_n e^{2\pi inx}$). Then apply density to prove this for $f \in C(\mathbb{T})$.

*Proof of Theorem 21.1.* Approximate $\chi_{(a,b)}$ by a trapezoidal continuous function $f$ which equals $\chi_{(a,b)}$ on $(a+\epsilon, b-\epsilon)$ and decays linearly to 0 on $[a, a+\epsilon]$ and $[b-\epsilon, b]$. Then

$$b - a - \epsilon = \int_0^1 f(x)\,dx = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(k\alpha) \le \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha).$$

Taking a similar approximation with $g$ which equals $\chi_{(a,b)}$ on $[a,b]$ and decays linearly to 0 on $[a-\epsilon, a]$ and $[b, b+\epsilon]$, we also get an upper bound:

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{(a,b)}(k\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} g(k\alpha) \le \int_0^1 g(x)\,dx = b - a + \epsilon.$$

This holds for arbitrary $\epsilon$, hence the limit is equal to $b - a$. □

**Corollary 21.1.1** (Kronecker's theorem). *$\{k\alpha \bmod 1\}_{k\in\mathbb{N}}$ is dense in $[0,1)$.*

## 21.3 $L^p$-Convergence of Fourier Series

**Remark.** Recall that $\{e_n\}_{n\in\mathbb{Z}}$ is complete in $L^p(\mathbb{T})$ and $C(\mathbb{T})$, i.e. $\text{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense.

For $1 \leq p < \infty$, if $f \in L^p(\mathbb{T})$, do the partial sums

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} = (f * d_N)(x)$$

converge to $f$ in $L^p$-norm?

The motivation is as follows. Suppose that $S_N f \to f$ in $L^p$ for every $f \in L^p(\mathbb{T})$. Then we have a linear operator $S_N : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ on a Banach space. Moreover, the convergence is also pointwise. Thus for any individual $f \in L^p(\mathbb{T})$,

$$\sup_{N \in \mathbb{N}} \|S_N f\|_p < \infty.$$

Then by the Banach-Steinhaus theorem (also called the uniform boundedness principle), we get

$$\sup_{N \in \mathbb{N}} \|S_N\| = \sup_{N \in \mathbb{N}} \sup_{\|f\|_p = 1} \|S_N f\|_p < \infty$$

In fact, this is equivalent to the $L^p$-convergence of Fourier series, and idea behind the proof is to instead argue about the uniform boundedness of $S_N$.

Note that $\|S_N f\|_p = \|f * d_N\|_p \leq \|f\|_p \|d_N\|_1$, so $\|S_N\| \leq \|d_N\|_1 \to \infty$.

For $p = 1$, we can argue as follows. Fix an approximate identity $\{k_\lambda\}$ with $k_\lambda \geq 0$, so that

$$\|k_\lambda\|_1 = \int_0^1 |k_\lambda| = \int_0^1 k_\lambda = 1.$$

Then we can see that

$$\|S_N\| \geq \|S_N k_\lambda\|_1 = \|k_\lambda * d_N\|_1 \xrightarrow[\lambda \to \infty]{} \|d_N\|_1$$

since $k_\lambda * d_N \to d_N$ in $L^1$. Thus $\|S_N\| \geq \|d_N\|_1 \to \infty$, so the partial sums cannot converge in $L^1$. In particular, $\{e_n\}_{n \in \mathbb{N}}$ is *not* a Schauder basis for $L^1(\mathbb{T})$. A similar argument works to show that $\{e_n\}_{n \in \mathbb{N}}$ is not a Schauder basis for $C(\mathbb{T})$. However, one has the following:

**Theorem 21.2.** $\{e_n\}_{n \in \mathbb{Z}}$ *is a Schauder basis for* $L^p(\mathbb{T})$ *for* $1 < p < \infty$, *with the ordering*

$$\mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}.$$

**Example 21.1.2** (Haar system). Another example of a Schauder basis is the following. Let

$$\varphi = \chi_{[0,1)} \quad \text{and} \quad \psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

Note that $\langle \varphi, \psi \rangle = 0$. Then we can define

$$\psi_1 = 2^{1/2} \psi(2x) \quad \text{and} \quad \psi_2 = 2^{1/2} \psi(2x - 1),$$

which are still orthogonal with $\varphi, \psi$ and with each other. Continuing this process, we get an orthonormal basis for $L^2(\mathbb{T})$. Moreover, this is a Schauder basis for $L^p(\mathbb{T})$ for $1 < p < \infty$ and it is unconditional, i.e. the convergence does not depending on the ordering of the functions.

**Remark.** There also exists a continuous analogue of the Haar system, the functions are called $D_4$ and $W_4$. The functions $\varphi, \psi$ from the Haar system are known as $D_2, W_2$. These are called *wavelets*.

# Lecture 22

# Nov. 18 — Fourier Series, Part 4

## 22.1  The Poisson Summation Formula

**Remark.** Suppose $f \in L^1(\mathbb{R})$ Set $\varphi(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Then by Fubini-Tonelli,

$$\|\varphi\|_1 = \int_0^1 \left| \sum_{n=-\infty}^{\infty} f(x+n) \right| \leq \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)| = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_1 < \infty,$$

so $\varphi \in L^1(\mathbb{T})$. Moreover, the integrals are the same if we do not take absolute values.

**Exercise 22.1.** Let $f, \varphi$ as above. Prove that

$$\widehat{\varphi}(n) = \int_0^1 \sum_{n=-\infty}^{\infty} f(x) e^{-2\pi i n x} \, dx = \widehat{f}(n).$$

**Theorem 22.1** (Poisson summation). *Let $f : \mathbb{R} \to \mathbb{C}$. If there exist $C, \epsilon > 0$ such that*

$$|f(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}} \quad and \quad |\widehat{f}(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}},$$

*then for every $x \in \mathbb{R}$,*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

*In particular, setting $x = 0$ one obtains*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

*Proof.* Both $f, \widehat{f} \in L^1(\mathbb{R})$, hence $f, \widehat{f} \in C_0(\mathbb{R})$ by Fourier inversion. Also $\varphi \in L^1(\mathbb{T})$ and

$$\sum_{n=-\infty}^{\infty} |\widehat{\varphi}(n)| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty,$$

so $\widehat{\varphi} \in \ell^1(\mathbb{Z})$. Therefore the inversion formula gives

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) e^{2\pi i n x}$$

pointwise. (In fact one can get uniform convergence on compact sets.) $\qquad\square$

**Remark.** Formally consider $\mu = \sum_{n=-\infty}^{\infty} \delta_n$, where $\delta_n$ is the Dirac delta at $n$. Then

$$\langle f, \mu \rangle = \int_{-\infty}^{\infty} f(x)\, d\mu(x) = \sum_{n=-\infty}^{\infty} f(n).$$

A similar computation shows that

$$\langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

These series are equal by Poisson summation, so if the Parseval identity holds, then one gets

$$\langle f, \widecheck{\mu} \rangle = \langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n) = \langle f, \mu \rangle.$$

Thus Poisson summation says that $\mu = \widecheck{\mu}$.

## 22.2   Wiener's Lemma

**Remark.** Let $C(\mathbb{T}) = \{f \text{ continuous on } \mathbb{T}\}$, which is closed under products and

$$\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty.$$

In particular, $C(\mathbb{T})$ is a commutative Banach algebra. Also note that if $f \in C(\mathbb{T})$ and $f(x) \ne 0$ for every $x$, then $1/f \in C(\mathbb{T})$. We say that $C(\mathbb{T})$ is *inverse-closed*.

Recall that the Wiener (Fourier) algebra is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\} = \{\widecheck{c} : c = (c_n) \in \ell^1(\mathbb{Z})\}.$$

The equality holds by the formulas $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$ and $\widecheck{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$. Note that $A(\mathbb{T}) \subseteq C(\mathbb{T})$ is dense in the uniform norm.

On the other hand, if we define

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|,$$

then for any $f, g \in A(\mathbb{T})$, we have

$$fg = (\widehat{fg})^\vee = (\widehat{f} * \widehat{g})^\vee.$$

Since $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$, we have $\widehat{f} * \widehat{g} \in \ell^1(\mathbb{Z})$, so $fg \in A(\mathbb{T})$. Moreover,

$$\|fg\|_{A(\mathbb{T})} = \|\widehat{f} * \widehat{g}\|_1 \le \|\widehat{f}\|_1 \|\widehat{g}\|_1 = \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}.$$

Thus we see that $A(\mathbb{T})$ is also a commutative Banach algebra with respect to pointwise products.

If $A(\mathbb{T})$ inverse-closed? In other words, if $f \in A(\mathbb{T})$ and $f(x) \ne 0$ for all $x$, must $1/f \in A(\mathbb{T})$?

**Lemma 22.1.** *If $P(\xi) = \sum_{k=-N}^{N} a_k e^{2\pi i k \xi}$, then $\|P\|_{A(\mathbb{T})} \le (2N+1)^{1/2} \|P\|_\infty$.*

*Proof.* By Cauchy-Schwarz, we have

$$\|P\|_{A(\mathbb{T})} = \sum_{k=-N}^{N} |a_k| \cdot 1 \le (2N+1)^{1/2} \left( \sum_{k=-N}^{N} |a_k|^2 \right)^{1/2} = (2N+1)^{1/2} \|P\|_2$$

where the second equality is by Plancherel's theorem. The result follows since $\|P\|_2 \le \|P\|_\infty$ on $\mathbb{T}$.   $\square$

**Lemma 22.2** (Wiener's lemma). *If $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any $\xi$, then there is $h \in \ell^1(\mathbb{Z})$ such that*

$$\widehat{h}(\xi) = \frac{1}{\widehat{g}(\xi)}, \quad \xi \in \mathbb{T}.$$

*Equivalently, if $G \in A(\mathbb{T})$ and $G(\xi) \neq 0$ for any $\xi$, then $1/G \in A(\mathbb{T})$.*

*Proof.* Assume $g \in \ell^1(\mathbb{Z})$ and $\widehat{g}(\xi) \neq 0$ for any $\xi$. Let $G = \widehat{g}$. We prove the result in two steps:

1. Assume $0 \leq G \leq 1$. Then since $G$ is nonzero,

$$d = \inf_{\xi \in \mathbb{T}} G(\xi) > 0.$$

Let $H = 1 - G = \widehat{\delta} - \widehat{g}$, where $\delta$ is the delta sequence. Note that

$$\widehat{\delta}(\xi) = \sum_{n=-\infty}^{\infty} \delta_n e^{2\pi i n \xi} = \delta_0 e^{2\pi i 0 \xi} = 1.$$

Then $H = \widehat{h} - \widehat{g} = (\delta - g)^\wedge \in A(\mathbb{T})$. So if we let $h = \delta - g \in \ell^1(\mathbb{Z})$, then $\widehat{h} = H$, and

$$\|H\|_\infty = \|1 - G\|_\infty = 1 - d < 1.$$

Hence $\sum_{n=0}^{\infty} H(\xi)^n$ converges, and $\sum_{n=0}^{\infty} H^n = 1/(1 - H(\xi)) = 1/G(\xi) \in C(\mathbb{T})$.

However, we need convergence in $A(\mathbb{T})$. Fix $0 < \epsilon < d/2$. Let $p = h\chi_{[-N,N]}$ for $N$ large enough so that $\|p - h\|_1 < \epsilon$. Let $P = \widehat{p} = \sum_{n=-N}^{N} h(n) e^{2\pi i n \xi}$. Let $r = p - h$ and $R = P - H$. Then

$$\|P\|_\infty = \|H + R\|_\infty \leq \|H\|_\infty + \|R\|_\infty \leq (1 - d) + \|P - H\|_\infty$$
$$\leq (1 - d) + \|\widehat{p} - \widehat{h}\|_\infty \leq (1 - d) + \|p - h\|_1 = 1 - d + \epsilon.$$

Now we can compute that

$$\|H^n\|_{A(\mathbb{T})} = \|(P - R)^n\|_{A(\mathbb{T})} = \left\| \sum_{j=0}^{n} \binom{n}{j} P^j (-R)^{n-j} \right\|_{A(\mathbb{T})}$$

where we can use the binomial theorem since $A(\mathbb{T})$ is a commutative Banach algebra. Then

$$\|H^n\|_{A(\mathbb{T})} \leq \sum_{j=0}^{n} \binom{n}{j} \|P\|_{A(\mathbb{T})}^j \|R\|_{A(\mathbb{T})}^{n-j} \leq \sum_{j=0}^{n} \binom{n}{j} (2Nj + 1)^{1/2} \|P^j\|_\infty \epsilon^{n-j},$$

where the second equality is by Lemma 22.1 and $\|R\|_{A(\mathbb{T})} = \|p - h\|_1 < \epsilon$. Thus

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2} \sum_{j=1}^{n} \binom{n}{j} \|P\|_\infty^j \epsilon^{n-j} = (2Nn + 1)^{1/2} (\|P\|_\infty + \epsilon)^n,$$

where $\|P^j\|_\infty \leq \|P\|_\infty^j$ by submultiplicativity. Since $\|P\|_\infty \leq 1 - d + \epsilon$,

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn + 1)^{1/2} (1 - d + 2\epsilon)^n,$$

where $1 - d + 2\epsilon < 1$. So $\sum_{n=0}^{\infty} \|H^n\|_{A(\mathbb{T})} < \infty$, so $\sum_{n=0}^{\infty} H^n$ converges in $A(\mathbb{T})$ to $1/(1 - H) = 1/G$.

One can reduce the general case to this first case, see the course notes.                            $\square$

## 22.3 Distributions

**Remark.** Let $1 \leq p < \infty$. Note that $L^p(\mathbb{R})^* \cong L^{p'}(\mathbb{R})$, where $L^p(\mathbb{R})^*$ is the dual space of $L^p(\mathbb{R})$, i.e. the space of bounded linear functions on $L^p(\mathbb{R})$. If we fix $g \in L^{p'}(\mathbb{R})$, then we can define

$$\mu_g(f) = \langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

which is linear in $f$ and antilinear in $g$ (this is known as a *sesquilinear form*).

Recall that $X^* = \{\text{bounded linear functionals } \mu \text{ on } X\}$, which is equivalent to continuous linear functionals when $X$ is a Banach space. We consider the following space of distributions:

**Definition 22.1.** Define the following:

- $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, the space of *distributions*.

- $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$, the space of *tempered distributions*.

- $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$, the space of *compactly supported distributions*.

**Example 22.1.1.** If $f \in C_c^\infty(\mathbb{R})$, then define $\langle f, \delta \rangle = f(0)$. This is the Dirac delta as a distribution.

**Remark.** Recall that the Schwartz space is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R})\}.$$

This is *not* a Banach space (it does not have a norm). But we can define seminorms

$$\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty, \quad m, n \geq 0.$$

There is no way to combine these seminorms into a single norm.[1] However, one can define a metric:

$$d(f,g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\rho_{m,n}(f-g)}{1 + \rho_{m,n}(f-g)}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Note that convergence with respect to $d$ is the same as convergence in $\rho_{m,n}$ for every $m, n$:

$$\lim_{k \to \infty} \rho_{m,n}(f - f_k) = 0.$$

**Definition 22.2.** We say that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ if for every $m, n \geq 0$,

$$\rho_{m,n}(f - f_k) = \|x^m f^{(n)}(x) - x^m (f_k)^{(n)}(x)\|_\infty \longrightarrow 0$$

Therefore, a linear functional $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is *continuous* if $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ implies $\langle f_k, \mu \rangle \to \langle f, \mu \rangle$.

**Remark.** Since convergence in $\mathcal{S}(\mathbb{R})$ is a very strong condition, we expect that it is easy for $\mu$ to satisfy the above condition. This intuitively explains why $\mathcal{S}'(\mathbb{R})$ is so large.

**Example 22.2.1.** Assume that $f_k \to f$ in $\mathcal{S}(\mathbb{R})$. Then $\langle f_k, \delta \rangle = f_k(0)$, so

$$|\langle f - f_k, \delta \rangle| = |f(0) - f_k(0)| \leq \rho_{0,0}(f - f_k) \longrightarrow 0.$$

Thus we see that $\delta \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

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[1]Compare this with $C_b^1(\mathbb{R}) = \{f : \|f\|_\infty, \|f'\|_\infty < \infty\}$, where one can define a norm $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$.

# Lecture 23

# Nov. 20 — Distributions, Part 2

## 23.1 Convergence with Families of Seminorms

**Example 23.0.1.** Like $\mathcal{S}(\mathbb{R})$, many other spaces are defined by a family of seminorms. Recall that

$$L^1_{\text{loc}}(\mathbb{R}) = \{\text{measurable } f : f \text{ is integrable on compact } K \subseteq \mathbb{R}\}.$$

If we define $\rho_k(f) = \|f\chi_K\|_1$, then $f \in L^1_{\text{loc}}(\mathbb{R})$ if and only if $\rho_K(f) < \infty$. In fact, it is enough to consider $\rho_N(f) = \|f\chi_{[-N,N]}\|_1$. Given such a countable family of seminorms, one can always define a metric

$$d(f, g) = \sum_{N=1}^{\infty} 2^{-N} \frac{\rho_N(f - g)}{1 + \rho_N(f - g)}$$

which satisfies $d(f, f_k) \to 0$ if and only if $\rho_N(f - f_k) \to 0$ for every $N$.

**Remark.** Suppose $\| \cdot \|$ is a norm on $L^1_{\text{loc}}(\mathbb{R})$, and consider $\chi_{[k,k+1]}$. Then $\rho_N(\chi_{[k,k+1]}) \to 0$, so

$$c_k \chi_{[k,k+1]} \longrightarrow 0$$

in $L^1_{\text{loc}}(\mathbb{R})$ for every choice of $c_k$. But then $\|c_k \chi_{[k,k+1]}\| \to 0$, a contradiction for $c_k = 1/\|\chi_{[k,k+1]}\|$.

**Example 23.0.2.** For $C^\infty(\mathbb{R})$, we can take $\rho_{N,n}(f) = \|f^{(n)}\chi_{[-N,N]}\|_\infty$. Then

$$C^\infty(\mathbb{R})^* = \{\text{continuous linear functionals } \mu : C^\infty(\mathbb{R}) \to \mathbb{C}\} = \mathcal{E}'(\mathbb{R})$$

is the space of compactly supported distributions, where $\mu$ is continuous if whenever $f_k \to f$ in $C^\infty(\mathbb{R})$, we have $\mu(f_k) = \langle \mu, f_k \rangle \to \langle \mu, f \rangle$.

**Remark.** For $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$, it is better to think of $C_c^\infty(\mathbb{R}) = \bigcup_{N=1}^{\infty} C_c^\infty([-N, N])$.

**Definition 23.1.** Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space $X$.

  (a) $f_k \to f$ in $X$ means that $\rho_\alpha(f - f_k) \to 0$ for every $\alpha \in J$.

  (b) $X$ is *Hausdorff* if $\rho_\alpha(f) = 0$ for every $f \in X$ if and only if $f = 0$.

**Definition 23.2.** Let $\{\rho_n\}_{n \in \mathbb{N}}$ be a countable, Hausdorff family of seminorms on $X$. If the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f - g)}{1 + \rho_n(f - g)}$$

is complete, then we call $X$ a *Fréchet space.*

**Exercise 23.1.** $\mathcal{S}(\mathbb{R})$ and $C^\infty(\mathbb{R})$ are Fréchet spaces (but $C_c^\infty(\mathbb{R})$ is not).

**Remark.** Recall that if $\mu : X \to \mathbb{C}$ is a linear functional on a normed space $X$, then $\mu$ is continuous if and only if $\mu$ is bounded (i.e. there exists $C > 0$ such that $|\langle f, \mu \rangle| \le C\|f\|$ for every $f \in X$).

**Theorem 23.1.** *Let $X$ be a Fréchet space with seminorms $\{\rho_n\}_{n\in\mathbb{N}}$ and $\mu : X \to \mathbb{C}$ a linear functional. Then the following are equivalent:*

1. *$\mu$ is continuous, i.e. $f_k \to f$ implies $\langle f_k, \mu \rangle \to \langle f, \mu \rangle$ as $k \to \infty$.*

2. *$f_k \to 0$ in $X$ implies $\langle f_k, \mu \rangle \to 0$.*

3. *There exists $C > 0$ and $N \in \mathbb{N}$ (depending on $\mu$) such that $|\langle f, \mu \rangle| \le C \sum_{n=1}^N \rho_n(f)$.*

*Proof.* $(1 \Leftrightarrow 2)$ This follows from linearity.

$(3 \Rightarrow 2)$ Suppose $f_k \to 0$, so $\rho_n(f_k) \to 0$ for every $n$. So

$$|\langle f_k, \mu \rangle| \le C \sum_{n=1}^N \rho_n(f_k) \xrightarrow[k\to\infty]{} 0,$$

since each of the finitely many terms converge to 0.

$(2 \Rightarrow 3)$ We prove the contrapositive. Suppose (3) is false. So for every $C = N = k \in \mathbb{N}$, there exists $f_k \in X$ for which $|\langle f_k, \mu \rangle| > k \sum_{n=1}^k \rho_n(f_k)$. Set $\varphi_k = f_k/|\langle f_k, \mu \rangle|$. Then

$$\sum_{n=1}^k \rho_n(\varphi_k) = \frac{1}{|\langle f_k, \mu \rangle|} \sum_{n=1}^k \rho_n(f_k) < \frac{1}{|\langle f_k, \mu \rangle|} \cdot \frac{|\langle f_k, \mu \rangle|}{k} = \frac{1}{k} \xrightarrow[k\to\infty]{} 0.$$

Thus $\rho_n(\varphi_k) \to 0$ as $k \to \infty$ for every $n \in \mathbb{N}$. So $\varphi_k \to 0$ in $X$. But $\langle \varphi_k, \mu \rangle = 1 \not\to 0$. $\qquad\square$

**Example 23.2.1.** What does the topology look like given a family of seminorms, i.e. what are the open sets? Consider seminorms $\rho_1(x_1, x_2) = |x_1|$ and $\rho_2(x_1, x_2) = |x_2|$. Consider strips

$$B_r^i(x_1, x_2) = \{(y_1, y_2) : |y_i - x_i| < r\}$$

for each seminorm $\rho_i$. Then the open balls are of the form

$$B_r^1(x_1, x_2) \cap B_r^2(x_1, x_2).$$

A basis in the general case involves intersections of finitely many strips (as in the product topology).

## 23.2 Distributional Derivatives

**Example 23.2.2.** If $f, g$ are differentiable and decay at $\infty$ (think $f, g \in \mathcal{S}(\mathbb{R})$), then

$$\langle f, g' \rangle = \int_{-\infty}^\infty f(x)\overline{g'(x)}\,dx = f(x)g(x)\big|_{x=-\infty}^\infty - \int_{-\infty}^\infty f'(x)\overline{g(x)}\,dx = -\langle f', g \rangle,$$

by integration by parts, where $f(x)g(x)\big|_{x=-\infty}^\infty = 0$ by the decay of $f, g$. Now define $\delta' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \delta' \rangle := -\langle f', \delta \rangle = -f'(0).$$

for $f \in \mathcal{S}(\mathbb{R})$. Then $\delta'$ is a linear functional, and

$$|\langle f, \delta' \rangle| = |-f'(0)| \leq \|f'\|_\infty = \|x^0 f^{(1)}(x)\|_\infty = \rho_{0,1}(f) \leq 1 \sum_{m=0}^{0} \sum_{n=0}^{1} \rho_{m,n}(f).$$

Hence $\delta'$ is bounded on $\mathcal{S}(\mathbb{R})$, so $\delta' \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$.

**Proposition 23.1.** *Fix $\mu \in \mathcal{S}'(\mathbb{R})$, and define $\mu' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by*

$$\langle f, \mu' \rangle = -\langle f', \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

*Then $\mu' \in \mathcal{S}'(\mathbb{R})$. In particular, every tempered distribution has a (distributional) derivative.*

*Proof.* Since $\mu$ is bounded, there are $C, N > 0$ such that $|\langle f, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \rho_{m,n}(f)$. Then

$$|\langle f, \mu' \rangle| = |\langle f', \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \rho_{m,n}(f') = C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m (f')^{(n)}\|_\infty = C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m f^{n+1}\|_\infty,$$

so $\mu'$ is bounded. Thus $\mu' \in \mathcal{S}'(\mathbb{R})$ also. $\qquad\square$

**Example 23.2.3.** Consider the Heaviside function $H = \chi_{[0,\infty)}$ We can define

$$\langle f, H \rangle = \int_{-\infty}^{\infty} f(x) \overline{H(x)} \, dx = \int_{0}^{\infty} f(x) \, dx.$$

Then we can check that

$$|\langle f, H \rangle| \leq \int_{0}^{\infty} |f(x)| \, dx \leq \int_{0}^{1} |f(x)| \, dx + \int_{1}^{\infty} \frac{|f(x)|}{x^2} x^2 \, dx$$

$$\leq \|f\|_\infty + \|x^2 f(x)\|_\infty \int_{1}^{\infty} \frac{1}{x^2} \, dx \leq C \sum_{m=0}^{2} \sum_{n=0}^{0} \|x^m f^{(n)}\|_\infty.$$

Thus $H$ is bounded, so $H \in \mathcal{S}'(\mathbb{R})$. So $H$ has a distributional derivative $H' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$, where

$$\langle f, H' \rangle := -\langle f', H \rangle = -\int_{0}^{\infty} f'(x) \, dx = -(f(\infty) - f(0)) = f(0) = \langle f, \delta \rangle.$$

In particular, we see that $H' = \delta$ as tempered distributions.

**Remark.** Note that $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ and $L^1_{\text{loc}}(\mathbb{R}) + \text{polynomial growth} \subseteq \mathcal{S}'(\mathbb{R})$, so each of the functions in the latter has distributional derivatives.

**Exercise 23.2.** If $g$ is smooth, then $g'$ (as a tempered distribution) is the usual derivative.

**Remark.** To avoid confusion, we sometimes write $D\mu$ for the distributional derivative and $g'$ for the pointwise a.e. derivative.

# Lecture 24

# Nov. 25 — Distributions, Part 3

## 24.1 Space of Distributions

**Remark.** Let $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$. The topology on this space is *not* given by a family of seminorms. Instead, we write $C_c^\infty(\mathbb{R}) = \bigcup_{K \text{ compact}} C^\infty(K)$. Note that $C^\infty(K)$ is a Fréchet space with seminorms

$$\rho_{K,n}(f) = \|f^{(n)}(x)\chi_K(x)\|_\infty$$

for $f \in C^\infty(K) = \{f \in C^\infty(\mathbb{R}) : \mathrm{supp}(f) \subseteq K\}$.

**Theorem 24.1.** $f_k \to f$ in $C_c^\infty(\mathbb{R})$ if and only if there exists a compact set $K \subseteq \mathbb{R}$ such that $f_k \in C^\infty(K)$ and $f_k \to f$ in $C^\infty(K)$.

**Theorem 24.2.** *Given a linear function* $\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$, *the following are equivalent:*

1. *$\mu$ is continuous;*

2. *$\mu|_{C^\infty(K)}$ is continuous for each compact $K$;*

3. *for all compact $K \subseteq \mathbb{R}$, there exists $C_K \geq 0$, $N_K \geq 0$ such that*

$$|\langle f, \mu \rangle| \leq C_K \sum_{n=0}^{N_K} \|f^{(n)}(x)\chi_K(x)\|_\infty, \quad \text{for } f \in C^\infty(K).$$

**Definition 24.1.** If there exists a single $N$ that can be used for each compact set $K$ in Theorem 24.2(3), then we say that $\mu$ has *finite order*. The *order* of $\mu$ is the smallest such $N$.

**Example 24.1.1.** We have the following:

1. The order of $\delta^{(j)}$ is $j$.

2. Consider the $\delta$-train $\mu = \sum_{n=-\infty}^\infty \delta_n$, so $\langle f, \mu \rangle = \sum_{n=-\infty}^\infty f(n)$. Note that this sum is finite since $f \in C^\infty(K)$ for some compact $K$. In fact, $|\langle f, \mu \rangle| \leq C_K \|f\|_\infty$, so the order is 0.

3. Let $\nu = \sum_{n=1}^\infty \delta_n^{(n)}$, so $\langle f, \nu \rangle = \sum_{n=1}^\infty (-1)^n f^{(n)}(n)$. One can check that $\nu$ has infinite order.

**Remark.** As sets, we have $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$. What about their duals?

**Theorem 24.3.** *We have the following:*

1. *$f_k \to f$ in $C_c^\infty(\mathbb{R})$ implies $f_k \to f$ in $\mathcal{S}(\mathbb{R})$.*

2. *$f_k \to f$ in $\mathcal{S}(\mathbb{R})$ implies $f_k \to f$ in $C^\infty(\mathbb{R})$.*

   *3. $\mu \in \mathcal{S}'(\mathbb{R})$ implies $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$.*

   *4. $\mu \in \mathcal{E}'(\mathbb{R})$ implies $\mu|_{\mathcal{S}(\mathbb{R})} \in \mathcal{S}'(\mathbb{R})$.*

*In particular, we have the containments $\mathcal{D}'(\mathbb{R}) \supseteq \mathcal{S}'(\mathbb{R}) \supseteq \mathcal{E}'(\mathbb{R})$.*

## 24.2   Functions as Distributions

**Theorem 24.4.** *$L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$, where the embedding is given by*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx, \quad f \in C_c^\infty(\mathbb{R}).$$

*Proof.* If $g \in L^1_{\text{loc}}(\mathbb{R})$ and $f \in C^\infty(K)$, then

$$\langle f, g \rangle \leq \int_K |f(x)g(x)|\, dx \leq \|f\|_\infty \|g\chi_K\|_1.$$

Thus we see that $g$ defines a continuous linear functional on $C_c^\infty(\mathbb{R})$. $\square$

**Example 24.1.2.** Note that $\delta \in \mathcal{D}'(\mathbb{R})$, but it cannot be identified with a function in $L^1_{\text{loc}}(\mathbb{R})$.

**Example 24.1.3.** Consider the space

$$M_b(\mathbb{R}) = \{\text{bounded Radon measures on } \mathbb{R}\}$$
$$= \{\text{bounded locally finite Borel measures on } \mathbb{R}\}.$$

If $\mu \in M_b(\mathbb{R})$ and $f \in C^\infty(K)$, then we can define a linear functional by

$$\langle f, \mu \rangle = \int f(x)\, d\mu(x).$$

Then we have that

$$|\langle f, \mu \rangle| \leq \int_K |f(x)|\, d|\mu|(x) \leq \|f\|_\infty |\mu|(K),$$

which is finite as $\mu$ is a bounded Radon measure.

**Example 24.1.4.** We have $1/x \notin L^1_{\text{loc}}(\mathbb{R})$, but we can define $\text{pv}(1/x) : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, \text{pv}(1/x) \rangle = \lim_{T \to \infty} \int_{1/T \leq |x| \leq T} \frac{f(x)}{x}\, dx, \quad f \in C^\infty(K).$$

One can show that the above limit exists, so $\text{pv}(1/x) \in \mathcal{D}'(\mathbb{R})$.

## 24.3   Operations on Distributions

**Remark.** Suppose $g \in L^1_{\text{loc}}(\mathbb{R})$, and set $T_a g(x) = g(x - a)$. If $f \in C_c^\infty(\mathbb{R})$, then

$$\langle f, T_a g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x - a)}\, dx = \int_{-\infty}^{\infty} f(x + a)\overline{g(x)}\, dx = \langle T_{-a}f, g \rangle.$$

**Definition 24.2.** If $\mu \in \mathcal{D}'(\mathbb{R})$, then define $T_a\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by $\langle f, T_a\mu \rangle := \langle T_{-a}f, \mu \rangle$ for $f \in C_c^\infty(\mathbb{R})$.

**Remark.** Note that for $f \in C^\infty(K)$, we have

$$|\langle f, T_a\mu \rangle| = |\langle T_{-a}f, \mu \rangle| \le C_{K+a} \sum_{n=0}^{N_{K+a}} \|T_{-a}f\|_\infty \le C_{K+a} \sum_{n=0}^{N_{K+a}} \|f\|_\infty$$

since $T_{-a}f \in C^\infty(K+a)$. Thus $T_a\mu$ is a continuous linear functional on $C_c^\infty(\mathbb{R})$,

**Remark.** One can similarly define other operations on distributions, e.g. dilation, modulation, etc. For example, the *involution* $\widetilde{g}(x) = \overline{g(-x)}$ satisfies

$$\langle f, \widetilde{g} \rangle = \int f(x)\overline{\overline{g(-x)}}\,dx = \overline{\int \overline{f(x)}g(-x)\,dx} = \overline{\int \overline{f(-x)}g(x)\,dx} = \overline{\int \widetilde{f}(x)\overline{g(x)}\,dx} = \overline{\langle \widetilde{f}, g \rangle}.$$

Thus if $\mu \in \mathcal{D}'(\mathbb{R})$, then we can define $\langle f, \widetilde{\mu} \rangle := \overline{\langle \widetilde{f}, \mu \rangle}$ for $f \in C_c^\infty(\mathbb{R})$.

**Example 24.2.1.** Consider a translation of $\delta$:

$$\langle f, T_a\delta \rangle := \langle T_{-a}f, \delta \rangle = T_{-a}f(0) = f(a).$$

But sometimes by abuse of notation we may write

$$\langle f, T_a\delta \rangle = \int f(x)\delta(x-a)\,dx = \int f(x+a)\delta(x)\,dx = f(a).$$

This should be understood in the distributional sense, the above integral is *not* a Lebesgue integral.

## 24.4 Products and Convolution of Distributions

**Remark.** Suppose $g \in L^1_{\text{loc}}(\mathbb{R})$, $\theta \in C^\infty(\mathbb{R})$. Then

$$\langle f, \theta g \rangle = \int f(x)\overline{\theta(x)g(x)}\,dx = \int (f(x)\overline{\theta(x)})\overline{g(x)}\,dx = \langle f\overline{\theta}, g \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Note that $f\overline{\theta} \in C_c^\infty(\mathbb{R})$, so the above definition makes sense.

**Definition 24.3.** If $\mu \in \mathcal{D}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$, then define $\theta\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ by $\langle f, \theta\mu \rangle := \langle f\overline{\theta}, \mu \rangle$.

**Remark.** We can check that for $f \in C^\infty(K)$ (so $f\overline{\theta} \in C^\infty(K)$),

$$|\langle f, \theta\mu \rangle| = |\langle f\overline{\theta}, \mu \rangle| \le C_K \sum_{n=0}^{N_K} \|(f\overline{\theta})^{(n)}\|_\infty \le C_K \sum_{n=0}^{N_K} \sum_{j=0}^{n} \binom{n}{j} \|f^{(j)}\overline{\theta^{(n-j)}}\|_\infty$$

$$= C_K \sum_{n=0}^{N_K} \sum_{j=0}^{n} \binom{n}{j} \|\theta^{(n-j)}\|_\infty \|f^{(j)}\|_\infty.$$

Thus we see that $\theta\mu \in \mathcal{D}'(\mathbb{R})$.

**Remark.** Now consider $g \in L^1_{\text{loc}}(\mathbb{R})$, $f \in C_c^\infty(\mathbb{R})$. Then we have

$$(f * g)(x) = \int f(x-y)g(y)\,dy = \int \overline{\widetilde{f}(y-x)}g(y)\,dy = \langle T_x\widetilde{f}, g \rangle.$$

**Definition 24.4.** The *convolution* of $\mu \in \mathcal{D}'(\mathbb{R})$ with $f \in C_c^\infty(\mathbb{R})$ is

$$(f * \mu)(x) := \overline{\langle T_x \widetilde{f}, \mu \rangle}, \quad x \in \mathbb{R}.$$

**Theorem 24.5.** *If* $\mu \in \mathcal{D}'(\mathbb{R})$ *and* $f \in C_c^\infty(\mathbb{R})$, *then:*

1. *Convolution commutes with translation, i.e.* $T_a(f * \mu) = (T_a f) * \mu = f * (T_a \mu)$.

2. $f * \mu \in C^\infty(\mathbb{R})$ *and* $(f * \mu)' = f' * \mu$ *(and also* $(f * \mu)' = f * D\mu$*).*

**Remark.** Suppose $f, g, h$ are "nice" functions. Then

$$\langle f, g * h \rangle = \int f(x) \overline{(g * h)(x)} \, dx = \iint f(x) \overline{g(x - y) h(y)} \, dy dx$$

$$= \iint \left( f(x) \widetilde{g}(y - x) \, dx \right) \overline{h(y)} \, dy = \int (f * \widetilde{g})(y) \overline{h(y)} \, dy = \langle f * \widetilde{g}, h \rangle.$$

**Theorem 24.6.** *If* $\mu \in \mathcal{D}'(\mathbb{R})$ *and* $f, g \in C_c^\infty(\mathbb{R})$, *then* $\langle f, g * \mu \rangle = \langle f * \widetilde{g}, \mu \rangle$.