

# MATH 7337: Harmonic Analysis

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# Lecture 1

## Aug. 19 — The Fourier Transform

### 1.1 The Fourier Transform on $L^1(\mathbb{R})$

All integrals will be taken over  $\mathbb{R}$  unless otherwise specified.

**Definition 1.1.** The *Fourier transform* of  $f \in L^1(\mathbb{R})$  is

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

**Remark.** Note that by the triangle inequality,

$$|\widehat{f}(\xi)| \leq \int |f(x)e^{-2\pi i \xi x}| dx = \int |f(x)| dx = \|f\|_1 < \infty,$$

so  $\widehat{f}(\xi)$  exists for all  $\xi \in \mathbb{R}$  (in fact,  $\widehat{f}$  is continuous).

**Remark.** The Fourier transform is an operator  $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$  as  $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \|f\|_1$ . This is linear in  $f$ . The *operator norm* of  $\mathcal{F}$  is

$$\|\mathcal{F}\| = \|\mathcal{F}\|_{L^1 \rightarrow L^\infty} = \sup_{\|f\|_1=1} \|\widehat{f}\|_\infty \leq \sup_{\|f\|_1=1} \|f\|_1 = 1,$$

so  $\mathcal{F}$  is a bounded linear operator. However,  $\mathcal{F}$  is not isometric (norm-preserving) in general.

**Remark.** Observe that

$$\widehat{f}(0) = \int f(x)e^{-2\pi i \cdot 0 \cdot x} dx = \int f(x) dx.$$

So if  $f \geq 0$  and we normalize  $f$  so that  $\widehat{f}(0) = 1$ , then we have

$$|\widehat{f}(\xi)| \leq \int f(x) dx = \widehat{f}(0),$$

and so  $\|\widehat{f}\|_\infty = \text{ess sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq 1$ . This is one particular case where  $\mathcal{F}$  does preserve the norm.

**Definition 1.2.** For  $r \neq 0$ , *dilation* of  $f$  by  $r$  is  $f_r(x) = rf(rx)$ . Note that  $\|f_r\|_1 = \|f\|_1$ .

**Example 1.2.1.** The *Dirichlet function* is  $d(\xi) = \sin(\xi)/\pi\xi \in C_0(\mathbb{R})$ .<sup>1</sup> Note that  $d \notin L^1(\mathbb{R})$ . We can also define the *sinc* function as  $\text{sinc } \xi = \sin(\pi\xi)/(\pi\xi) = d\pi(x)$ .

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<sup>1</sup>Recall that  $C_0(\mathbb{R})$  is the space of continuous functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ .

However,  $d$  is the Fourier transform of a function in  $L^1(\mathbb{R})$ . Consider the

$$\chi_{-[T,T]}(x) = \begin{cases} 1 & \text{if } |x| \leq T, \\ 0 & |x| > T. \end{cases}$$

Note that  $\chi_{-[T,T]} \in L^1(\mathbb{R})$ . Then we have

$$\widehat{\chi}_{-[T,T]}(\xi) = \int_{-T}^T e^{-2\pi i \xi x} dx = \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_{-T}^T = \frac{\sin(2\pi T \xi)}{\pi \xi} = d_{2\pi T}(\xi),$$

so we see that  $\widehat{\chi}_{-[T,T]} \in C_0(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})$ .

**Remark.** We will see in general that  $\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ , this is the Riemann-Lebesgue lemma. The image of  $\mathcal{F}$  is a proper dense subspace of  $C_0(\mathbb{R})$ , which implies that  $\mathcal{F}^{-1}$  must be unbounded as a linear operator by Banach space theory.

**Proposition 1.1.** *If  $f \in L^1(\mathbb{R})$ , then  $\widehat{f}$  is uniformly continuous on  $\mathbb{R}$ , i.e.*

$$\|\widehat{f} - T_\eta \widehat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| \xrightarrow{\eta \rightarrow 0} 0,$$

where  $T_\eta \widehat{f}(\xi) = \widehat{f}(\xi - \eta)$ .

*Proof.* We can write the difference as

$$|\widehat{f}(\xi) - \widehat{f}(\xi - \eta)| = \left| \int f(x)(e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}) dx \right| \leq \int |f(x)| |e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| dx.$$

Note that  $f \in L^1(\mathbb{R})$  and  $|e^{-2\pi i \xi x} - e^{-2\pi i (\xi - \eta)x}| = |1 - e^{2\pi i \eta x}| \rightarrow 0$  as  $\eta \rightarrow 0$  independent of  $\xi$ , so the statement follows from the dominated convergence theorem (the integrand is dominated by  $2f$ ).  $\square$

## 1.2 Motivation for the Fourier Transform

**Remark.** We will define the *inverse Fourier transform* of  $f \in L^1(\mathbb{R})$  as

$$\check{f}(x) = \int f(x) e^{2\pi i \xi x} d\xi.$$

Note that  $\check{f}(\xi) = \widehat{f}(-\xi)$ . With enough assumptions, this is an inverse to the Fourier transform.

**Proposition 1.2** (Fourier inversion formula). *If  $f, \widehat{f} \in L^1(\mathbb{R})$ , then*

$$f(x) = (\widehat{f})^\vee(x) = \int \widehat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

**Remark.** Note that  $e_\xi(x) = e^{2\pi i \xi x} = \cos 2\pi \xi x + i \sin 2\pi \xi x$  and  $e_\xi : \mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . We have  $e_\xi(x+y) = e_\xi(x)e_\xi(y)$ , so  $e_\xi$  is a homomorphism, and it is also continuous. Thus  $e_\xi$  is a *character* on  $\mathbb{R}$  (in fact, every character on  $\mathbb{R}$  is of the form  $e_\xi$  for some  $\xi$ ). One can use this idea to define Fourier transforms in much more general settings.

**Remark.** The Fourier transform decomposes a function  $f$  into the pure harmonics  $e_\xi$ , and the inversion formula says that we can recover  $f$  as a “sum” of these pure harmonics.

# Lecture 2

## Aug. 21 — The Riemann-Lebesgue Lemma

### 2.1 Properties of the Fourier Transform

**Definition 2.1.** Define the following operators:

1. *Translation*:  $T_a f(x) = f(x - a)$  for  $a \in \mathbb{R}$ ;
2. *Modulation*:  $M_b f(x) = e^{2\pi i b x} f(x)$  for  $b \in \mathbb{R}$ ;
3. *Dilation*:  $f_\lambda(x) = \lambda f(\lambda x)$  for  $\lambda > 0$ ;
4. *Involution*:  $\tilde{f}(x) = \overline{f(-x)}$ .

**Remark.** Translation and modulation are isometries on  $L^p(\mathbb{R})$  for any  $p$ . Dilation as defined above is  $L^1$ -normalized, so it is only an isometry on  $L^1(\mathbb{R})$ .

**Exercise 2.1.** If  $f \in L^1(\mathbb{R})$ , then

1.  $(T_a f)^\wedge(\xi) = (M_{-a} \widehat{f})(\xi) = e^{-2\pi i \xi a} \widehat{f}(\xi)$ ;
2.  $(M_b f)^\wedge(\xi) = (T_b \widehat{f})(\xi) = \widehat{f}(\xi - b)$ ;
3.  $(f_\lambda)^\wedge(\xi) = \lambda (f_{1/\lambda})^\wedge(\xi) = \widehat{f}(\xi/\lambda)$ ;<sup>1</sup>
4.  $(\overline{f})^\wedge(\xi) = (\widehat{f})^\sim(\xi) = \overline{\widehat{f}(-\xi)}$ ;
5.  $(\tilde{f})^\wedge(\xi) = \overline{\widehat{f}(\xi)}$ .

### 2.2 The Riemann-Lebesgue Lemma

**Definition 2.2.** Let  $C_c(\mathbb{R})$  be the space of continuous functions with compact support. For a continuous function, the *support* of  $f$ , denoted  $\text{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$ . So for a continuous function  $f$ ,  $\text{supp}(f)$  is compact if and only if  $f = 0$  outside some finite interval.

**Theorem 2.1.**  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . In other words,

1. the closure of  $C_c(\mathbb{R})$  in  $L^p(\mathbb{R})$  is all of  $L^p(\mathbb{R})$ ;
2. for any  $f \in L^p(\mathbb{R})$  and  $\epsilon > 0$ , there exists  $g \in C_c(\mathbb{R})$  such that  $\|f - g\|_p < \epsilon$ ;
3. if  $f \in L^p(\mathbb{R})$ , then there exists  $g_n \in C_c(\mathbb{R})$  such that  $g_n \rightarrow f$  in  $L^p$ -norm, i.e.  $\|g_n - f\|_p \rightarrow 0$ .

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<sup>1</sup>Note that the result is an  $L^\infty$ -normalized dilation.

For  $p = \infty$ ,  $C_c(\mathbb{R})$  is dense in  $C_0(\mathbb{R})$  with respect to the  $L^\infty$ -norm (this is the same as the uniform norm for continuous functions).

*Proof.* We sketch the proof. First approximate  $f \in L^p(\mathbb{R})$  by a simple function (one that takes only finitely many distinct values)  $\phi = \sum_{k=1}^N c_k \chi_{E_k}$ , e.g. by rounding down to the nearest integer multiple of  $2^{-n}$ . Then use Urysohn's lemma to approximate  $\chi_{E_k}$  by a continuous function.  $\square$

**Exercise 2.2.** Fix  $1 \leq p < \infty$ . Prove that if  $f \in L^p(\mathbb{R})$ , then  $\lim_{a \rightarrow 0} \|f - T_a f\|_p = 0$ . We say that translation is *strongly continuous* on  $L^p(\mathbb{R})$ . For  $p = \infty$ , use  $C_0(\mathbb{R})$  and the uniform norm instead.

**Lemma 2.1** (Riemann-Lebesgue lemma). *If  $f \in L^1(\mathbb{R})$ , then  $\widehat{f} \in C_0(\mathbb{R})$ ,*

*Proof.* We have already seen that  $\widehat{f}$  is continuous. So it suffices to show decay at  $\infty$ . Write

$$\widehat{f}(\xi) = - \int f(x) e^{-2\pi i \xi x} e^{-2\pi i \xi(1/2\xi)} dx = - \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi(x+1/2\xi)} dx.$$

Now make the change of variables  $x \mapsto x - 1/2\xi$ , so we get

$$\widehat{f}(\xi) = - \int_{-\infty}^{\infty} f\left(x - \frac{1}{2\xi}\right) e^{-2\pi i \xi x} dx = - \int T_{1/2\xi} f(x) e^{-2\pi i \xi x} dx.$$

Taking an average with the usual expression for  $\widehat{f}(\xi)$ , we have

$$\widehat{f}(\xi) = \frac{1}{2} \int (f(x) - T_{1/2\xi} f(x)) e^{-2\pi i \xi x} dx.$$

Taking absolute values, we obtain

$$|\widehat{f}(\xi)| \leq \frac{1}{2} \int |f(x) - T_{1/2\xi} f(x)| dx = \frac{1}{2} \|f - T_{1/2\xi} f\|_1 \xrightarrow[\xi \rightarrow \pm\infty]{} 0$$

by the strong continuity of translation on  $L^1(\mathbb{R})$ .  $\square$

**Exercise 2.3.** The following is an alternative proof of the Riemann-Lebesgue lemma. Recall that we have  $\widehat{\chi}_{-T,T} = d_{2\pi T} \in C_0(\mathbb{R})$ . By taking translations and dilations, we see that  $\widehat{\chi}_{[a,b]} \in C_0(\mathbb{R})$ . Consider *really simple functions*  $\phi = \sum_{k=1}^N c_k \chi_{[a_k, b_k]}$ , and by linearity we can write

$$\widehat{\phi} = \sum_{k=1}^N c_k \widehat{\chi}_{[a_k, b_k]} \in C_0(\mathbb{R}).$$

Note that really simple functions are also dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ . So if  $f \in L^1(\mathbb{R})$ , there exist really simple  $\phi_n \rightarrow f$  in  $L^1$ -norm. On the Fourier side, we have

$$\|\widehat{f} - \widehat{\phi}_n\|_\infty \leq \|f - \phi_n\|_1 \rightarrow 0.$$

Since  $\phi_n \rightarrow \widehat{f}$  uniformly and  $C_0(\mathbb{R})$  is a Banach space, we conclude  $\widehat{f} \in C_0(\mathbb{R})$ . Fill in the details.

## 2.3 Position and Momentum Operators

**Definition 2.3.** The *position operator*  $P : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is given by  $Pf(x) = xf(x)$ . Note that  $P$  is unbounded on  $L^1(\mathbb{R})$  (in fact,  $P$  is not defined on all of  $L^1(\mathbb{R})$ ). Restrict  $P$  to the domain

$$D_P = \{f \in L^1(\mathbb{R}) : xf(x) \in L^1(\mathbb{R})\},$$

which is dense in  $L^1(\mathbb{R})$ . Note that  $D_P$  cannot be bounded as it does not admit an extension to  $L^1(\mathbb{R})$ .

**Exercise 2.4.** Show that  $\sup_{\|f\|_1=1, f \in D_P} \|Pf\|_1 = \infty$ .

**Definition 2.4.** The *momentum operator*  $M : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is given by  $Mf = f'/2\pi i$ . Similarly,  $M$  is unbounded and defined only on a dense subset of  $L^1(\mathbb{R})$ .

**Remark.** We have the relation  $(Mf)^\wedge(\xi) = \xi P\hat{f}(\xi)$ , whenever the statement makes sense.

## 2.4 The HRT Conjecture

**Conjecture 2.1** (HRT conjecture). *Assume  $g$  is not zero a.e.,  $a_k, b_k$  are distinct, and consider finite linear combinations of translations and modulations of  $g \in L^2(\mathbb{R})$  of the following form:*

$$\sum_{k=1}^N c_k e^{2\pi i b_k x} g(x - a_k). \quad (*)$$

If  $(*) = 0$ , then must it be that  $c_1 = \dots = c_N = 0$ ? In other words, are these linearly independent?

**Remark.** Consider the special case  $b_k = 0$  for every  $k$ , so  $\sum c_k T_{a_k} g = \sum c_k g(x - a_k) = 0$  a.e. Then

$$\left( \sum c_k T_{a_k} g \right)^\wedge = \sum c_k M_{-a_k} \hat{g} = \left( \sum_{k=1}^N c_k e^{-2\pi i a_k \xi} \right) \hat{g}(\xi) = 0.$$

Since  $\hat{g}$  is not zero a.e., we must have  $\sum_{k=1}^N c_k e^{-2\pi i a_k \xi} = 0$ , which implies  $c_k = 0$  for all  $k$ . In particular, this means that translations alone are linearly independent (the same is true for modulations alone).

**Remark.** The general case of the HRT conjecture is still open. Note that after taking a Fourier transform, we end up with the same problem, just for  $\hat{g}$  instead of  $g$ .

# Lecture 3

## Aug. 3 — Convolution

### 3.1 Convolution

**Definition 3.1.** If  $f, g$  are measurable on  $\mathbb{R}$ , their *convolution* is (formally)

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy.$$

**Remark.** When it exists, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y) dy = \int_{-\infty}^{\infty} f(x - y)g(y) dy = (g * f)(x)$$

by the change of variables  $y \mapsto x - y$ . So  $f * g = g * f$ , if it exists. Similarly,  $f * (g * h) = (f * g) * h$  if each of these convolutions exist.

**Remark.** If we take  $g_T = \chi_{-T,T}/2T$  (note that  $\|g_T\|_1 = 1$ ), then

$$(f * g_T)(x) = \int_{-\infty}^{\infty} f(y)g_T(x - y) dy = \frac{1}{2T} \int_{x-T}^{x+T} f(y) dy = \text{Avg}_{[-T,T]} f(x),$$

so we can see convolution as a averaging or smoothing operation (also known as *mollification*).

**Remark.** We would like to show  $f, g \in L^1(\mathbb{R})$  implies  $f * g \in L^1(\mathbb{R})$ . Note that  $(f * g)^\wedge = \widehat{f}\widehat{g} \in C_0(\mathbb{R})$ , since  $C_0(\mathbb{R})$  is closed under multiplication, even though  $L^1(\mathbb{R})$  is not.

**Remark.** The *Lebesgue differentiation theorem* says that if  $f \in L^1_{\text{loc}}(\mathbb{R})$ , then  $(f * g_T)(x) \rightarrow f(x)$  a.e.

### 3.2 Properties of Convolution

**Remark.** Use the notation

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx,$$

whenever this integral exists. Then *Hölder's inequality* says that if  $1/p + 1/p' = 1$  with  $1 \leq p \leq \infty$  and  $f \in L^p(\mathbb{R})$ ,  $g \in L^{p'}(\mathbb{R})$ , then  $fg \in L^1(\mathbb{R})$  and we have

$$|\langle f, g \rangle| \leq \int |f(x)||g(x)| dx \leq \|f\|_p \|g\|_{p'}.$$

**Theorem 3.1.** For  $1 \leq p \leq \infty$ , if  $f \in L^p(\mathbb{R})$  and  $g \in L^1(\mathbb{R})$ , then  $f * g \in L^\infty(\mathbb{R})$ .

*Proof.* By Hölder's inequality, we can write

$$\int |f(y)g(x-y)| dy \leq \|f\|_p \|g(x)\|_{p'} < \infty,$$

so  $(f * g)(x)$  exists for every  $x \in \mathbb{R}$ . □

**Exercise 3.1.** Show that  $f * g \in C_b(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{C} : h \text{ is continuous and bounded}\}$ .

**Remark.** Denote  $g^*(y) = \overline{g(-y)}$ . Then we have

$$(f * g)(x) = \int f(y)g(x-y) dy = \int f(y)\overline{g^*(y-x)} dy = \langle f, T_x g^* \rangle.$$

**Theorem 3.2.** Let  $f, g \in L^1(\mathbb{R})$ . Then

1.  $f(y)g(x-y)$  is measurable and integrable on  $\mathbb{R}^2$ ;
2. for a.e.  $x \in \mathbb{R}$ ,  $f(y)g(x-y)$  is measurable and integrable on  $\mathbb{R}$  as a function of  $y$ ;
3.  $f * g \in L^1(\mathbb{R})$  and  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ , i.e. convolution is submultiplicative on  $L^1(\mathbb{R})$ ;
4.  $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$  for every  $\xi \in \mathbb{R}$ .

*Proof.* (1) Let  $h(x, y) = f(x)$ . Then we have

$$\{h > a\} = h^{-1}((a, \infty)) = \{(x, y) : f(x) > a\} = \{f > a\} \times \mathbb{R},$$

which is measurable in  $\mathbb{R}^2$  since  $\{f > a\}$  and  $\mathbb{R}$  are measurable in  $\mathbb{R}$ . Similarly,  $g(y)$  is measurable on  $\mathbb{R}^2$ , so  $F(x, y) = f(x)g(y)$  is measurable on  $\mathbb{R}^2$ . Now make a linear change of variables  $T(x, y) = (y, x-y)$ , so  $H = F \circ T = f(y)g(x-y)$  is measurable (note that linear maps preserve measurability).

Now we can integrate by Tonelli's theorem and see that

$$\begin{aligned} \iint |f(y)g(x-y)| dx dy &= \int |f(y)| \left( \int |g(x-y)| dx \right) dy = \int |f(y)| \left( \int |g(z)| dz \right) dy \\ &= \int |f(y)| \|g\|_1 dy = \|f\|_1 \|g\|_1 < \infty, \end{aligned}$$

hence  $f(y)g(x-y)$  is integrable on  $\mathbb{R}^2$ .

(2) This follows by Fubini's theorem since  $f(y)g(x-y)$  is integrable.

(3) By (2),  $(f * g)(x)$  exists for a.e.  $x$ , and

$$\int |(f * g)(x)| dx = \int \left| \int f(y)g(x-y) dy \right| dx \leq \iint |f(y)g(x-y)| dy dx \leq \|f\|_1 \|g\|_1,$$

which is the desired inequality.

(4) Justify the following calculation as an exercise via Fubini/Tonelli's theorem:

$$\begin{aligned}(f * g)^{\wedge}(\xi) &= \int (f * g)(x) e^{-2\pi i \xi x} dx = \int \left( \int f(y) g(x-y) dy \right) e^{-2\pi i \xi x} dx \\ &= \iint f(y) e^{-2\pi i \xi y} g(x-y) e^{-2\pi i \xi(x-y)} dy dx.\end{aligned}$$

By Fubini's theorem, we can exchange orders and write

$$\begin{aligned}(f * g)^{\wedge}(\xi) &= \int f(y) e^{-2\pi i \xi y} \left( \int g(x-y) e^{-2\pi i \xi(x-y)} dx \right) dy \\ &= \int f(y) e^{-2\pi i \xi y} \left( \int g(z) e^{-2\pi i \xi z} dz \right) dy = \widehat{f}(\xi) \widehat{g}(\xi),\end{aligned}$$

which is the desired equality.  $\square$

**Corollary 3.2.1.**  $L^1(\mathbb{R})$  is closed under convolution.

**Definition 3.2.** An *algebra* is a vector space  $A$  with a product such that

- (a)  $(fg)h = f(gh)$ ,
- (b)  $f(g+h) = fg + fh$ ,
- (c)  $\alpha(fg) = (\alpha f)g = f(\alpha g)$ .

If  $fg = gf$  always, then we say that  $A$  is *commutative*. A Banach space which is also an algebra with a submultiplicative product is a *Banach algebra*.

**Example 3.2.1.** With convolution as a product,  $L^1(\mathbb{R})$  becomes a commutative Banach algebra without identity. Similarly,  $C_0(\mathbb{R})$  is also a commutative Banach algebra without identity (under pointwise products). The space  $\mathcal{B}(X)$  of bounded linear operators on a Banach space  $X$  is also a Banach space under the operator norm, and we have  $\|AB\| \leq \|A\|\|B\|$  with composition as a product. So  $\mathcal{B}(X)$  is a noncommutative Banach algebra, with identity.

### 3.3 Young's Inequality

**Theorem 3.3** (Young's inequality, special case). Fix  $1 \leq p \leq \infty$ . If  $f \in L^p(\mathbb{R})$  and  $g \in L^1(\mathbb{R})$ , then  $f * g \in L^p(\mathbb{R})$  and  $\|f * g\|_p \leq \|f\|_p \|g\|_1$ .

*Proof.* The case  $p = \infty$  is easy by Hölder's inequality and  $p = 1$  is done, so assume  $1 < p < \infty$ . Then

$$|(f * g)(x)| \leq \int |f(y)| |g(x-y)| dy = \int (|f(y)| |g(x-y)|^{1/p}) \left( |g(x-y)|^{1/p'} \right) dy,$$

By Hölder's inequality, we can write

$$\begin{aligned}|(f * g)(x)| &\leq \left( \int |f(y)|^p |g(x-y)| dy \right)^{1/p} \left( \int |g(x-y)| dy \right)^{1/p'} \\ &\leq \|g\|_1^{1/p'} \left( \int |f(y)|^p |g(x-y)| dy \right)^{1/p}.\end{aligned}$$

Now taking  $L^p$ -norms, we get

$$\|f * g\|_p^p = \int |(f * g)(x)|^p dx \leq \|g\|_1^{p/p'} \iint |f(y)|^p |g(x-y)| dy dx.$$

By Tonelli's theorem, we can exchange orders and write

$$\|f * g\|_p^p \leq \|g\|_1^{p/p'} \int |f(y)|^p \left( \int |g(x-y)| dx \right) dy \leq \|g\|_1^{1+p/p'} \|f\|_p^p = \|g\|_1^p \|f\|_p^p,$$

so we get the desired inequality  $\|f * g\|_p \leq \|f\|_p \|g\|_1$  after taking  $p$ th roots.  $\square$

**Exercise 3.2** (Young's inequality, general case). Let  $1 \leq p, q, r \leq \infty$  satisfy  $1/r = 1/p + 1/q - 1$ . If  $f \in L^p(\mathbb{R})$  and  $g \in L^q(\mathbb{R})$ , then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

**Remark.** Recall *Minkowski's inequality* (the triangle inequality in  $L^p(\mathbb{R})$ ):

$$\left\| \sum f_k \right\|_p \leq \sum \|f_k\|_p.$$

*Minkowski's integral inequality* then says that for  $1 \leq p \leq \infty$ ,

$$\left\| \int f_x dx \right\|_p = \left( \int \left| \int f(x, y) dy \right|^p dx \right)^{1/p} \leq \int \left( \int |f(x, y)|^p dy \right)^{1/p} dx = \int \|f_x\|_p dx.$$

One can also use this to prove to Young's inequality.

**Remark.** The *Babenko-Beckner constant* is the optimal constant in front of Hölder's inequality:

$$A_p = \left( \frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

The optimal constant in Young's inequality is  $A_p A_q A_{r'}$ , i.e. we have

$$\|f * g\|_r \leq (A_p A_q A_{r'}) \|f\|_p \|g\|_q.$$

## 3.4 The Dirac Delta

**Remark.** Is there an identity for convolution? Suppose there was a function  $\delta \in L^1(\mathbb{R})$  (the *Dirac delta function*) such that  $f * \delta = f$  for all  $f \in L^1(\mathbb{R})$ . Then we have  $(f * \delta)^\wedge = \widehat{f}$ , so

$$\widehat{f}(\xi) \widehat{\delta}(\xi) = \widehat{f}(\xi) \quad \text{for all } f \in L^1(\mathbb{R}).$$

Take  $f(x) = e^{-x^2}$  with  $\widehat{f}(\xi) = e^{-\xi^2}$  and note that  $\widehat{f}(\xi)$  is everywhere nonzero. Then  $\widehat{\delta}(\xi) = 1$  for all  $\xi \in \mathbb{R}$ , which contradicts the Riemann-Lebesgue lemma.

The correct way to work with the Dirac delta is to use the measure

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E. \end{cases}$$

One can then integrate against the measure  $\delta$  to achieve a similar effect.

# Lecture 4

Aug. 28 — Convolution, Part 2

# Lecture 5

## Sept. 2 — Smoothness and Decay

### 5.1 Smoothness and Decay

**Theorem 5.1** (Decay in time implies smoothness in frequency). *Assume  $f \in L^1(\mathbb{R})$  and  $x^m f(x) \in L^1(\mathbb{R})$ , where  $m > 0$ . Then*

$$\widehat{f} \in C_0^m(\mathbb{R}) = \{g : g, g', \dots, g^{(m)} \in C_0(\mathbb{R})\}.$$

Furthermore, we have

$$\widehat{f}^{(k)} = \frac{d^k}{d\xi^k} \widehat{f} = ((-2\pi i x)^k f(x))^\wedge.$$

*Proof.* The proof is by induction on  $m$ . When  $m = 1$ , we can formally write

$$\begin{aligned} \frac{d}{d\xi} \widehat{f}(\xi) &= \frac{d}{d\xi} \int f(x) e^{-2\pi i \xi x} dx \\ &\stackrel{(*)}{=} \int f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} dx = \int f(x) (-2\pi i x) e^{-2\pi i \xi x} dx = (-2\pi i x f(x))^\wedge(\xi). \end{aligned}$$

It suffices to justify step (\*), which we will do by appealing to the dominated convergence theorem. We can write

$$\widehat{f}'(\xi) = \lim_{\eta \rightarrow 0} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\eta \rightarrow 0} \int f(x) \frac{e^{-2\pi i (\xi+\eta)x} - e^{-2\pi i \xi x}}{\eta} dx.$$

Note that we have the pointwise limit

$$f(x) \frac{e^{-2\pi i (\xi+\eta)x} - e^{-2\pi i \xi x}}{\eta} \xrightarrow{\eta \rightarrow 0} f(x) \frac{d}{d\xi} e^{-2\pi i \xi x} = -2\pi i x f(x) e^{-2\pi i \xi x}.$$

Also note that we can bound

$$\left| f(x) \frac{e^{-2\pi i (\xi+\eta)x} - e^{-2\pi i \xi x}}{\eta} \right| = \left| f(x) \frac{e^{-2\pi i \eta x} - 1}{\eta} \right| \leq \left| f(x) \frac{-2\pi i \eta x}{\eta} \right| = |2\pi x f(x)|,$$

where we noted that  $|e^{i\theta} - 1| \leq |\theta|$  for  $\theta \in \mathbb{R}$ . Thus  $2\pi x f(x)$  dominates the integrand and is integrable since  $x f(x) \in L^1(\mathbb{R})$  by assumption, we can conclude (\*) by the dominated convergence theorem. Then  $\widehat{f}' \in C_0(\mathbb{R})$  by the Riemann-Lebesgue lemma, since  $\widehat{f}' = (-2\pi i x f(x))^\wedge$  where  $-2\pi i x f(x) \in L^1(\mathbb{R})$ .

The inductive step is part of Homework 1. □

**Remark.** Recall the position and momentum operators  $Pf(x) = xf(x)$  and  $Mf(x) = f'(x)/2\pi i$ . If  $f, Pf \in L^1(\mathbb{R})$ , then the above theorem tells us that  $(Pf)^\wedge = -M\widehat{f}$ .

## 5.2 Absolute Continuity

**Definition 5.1.** A function  $f : [a, b] \rightarrow \mathbb{C}$  is *absolutely continuous* if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\{[a_j, b_j]\}_j$  are countably many non-overlapping intervals, then

$$\sum_j (b_j - a_j) < \delta \quad \text{implies} \quad \sum_j |f(b_j) - f(a_j)| < \epsilon.$$

Define  $\text{AC}_{\text{loc}}(\mathbb{R}) = \{f \in C(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b]\}.$

**Theorem 5.2** (Fundamental theorem of calculus). *If  $g : [a, b] \rightarrow \mathbb{C}$ , then the following are equivalent:*

1.  $g \in \text{AC}[a, b];$
2. *there exists  $f \in L^1[a, b]$  such that for all  $x \in [a, b]$ ,*

$$g(x) - g(a) = \int_a^x f(t) dt;$$

3.  *$g$  is differentiable at a.e. point,  $g' \in L^1[a, b]$ , and*

$$g(x) - g(a) = \int_a^x g'(t) dt.$$

**Remark.** The Cantor-Lebesgue function  $\varphi : [0, 1] \rightarrow [0, 1]$  is continuous with  $\varphi' = 0$  a.e., but

$$\int_0^1 \varphi'(x) dx = 0 \neq 1 = \varphi(1) - \varphi(0).$$

**Lemma 5.1** (Growth lemma). *If  $f : [a, b] \rightarrow \mathbb{R}$  is measurable and differentiable at every point in a measurable set  $E \subseteq [a, b]$ , then*

$$|f(E)|_e \leq \int_E |f'|,$$

where  $|f(E)|_e$  denotes the exterior Lebesgue measure of  $f(E)$ .

**Theorem 5.3** (Banach-Zaretsky theorem). *If  $f : [a, b] \rightarrow \mathbb{R}$ , then the following are equivalent:*

1.  $f \in \text{AC}[a, b];$
2.  $f$  is continuous,  $f$  has bounded variation, and  $|A| = 0$  implies  $|f(A)| = 0$ ;
3.  $f$  is continuous and differentiable a.e.,  $f' \in L^1[a, b]$ , and  $|A| = 0$  implies  $|f(A)| = 0$ .

**Theorem 5.4.** *If  $f : [a, b] \rightarrow \mathbb{C}$  is differentiable on  $[a, b]$  and  $f' \in L^1[a, b]$ , then  $f \in \text{AC}[a, b]$ .*

*Proof.* By the Banach-Zaretsky theorem, it suffices to show that  $|A| = 0$  implies  $|f(A)| = 0$ . If  $|A| = 0$ , then by the growth lemma,

$$|f(A)| \leq \int_A |f'| = 0,$$

which completes the proof. (Technically we should split  $f$  into its real and imaginary parts.)  $\square$

### 5.3 Smoothness and Decay, Continued

**Theorem 5.5** (Smoothness in time implies decay in frequency). *If  $f \in L^1(\mathbb{R})$  is everywhere  $m$ -times differentiable and  $f, f', \dots, f^{(m)} \in L^1(\mathbb{R})$ , then*

$$\widehat{f^{(k)}}(\xi) = (2\pi i \xi)^k \widehat{f}(\xi), \quad \text{for } k = 0, \dots, m,$$

hence  $|\widehat{f}(\xi)| \leq |2\pi \xi|^{-k} |\widehat{f^{(k)}}(\xi)| \leq |2\pi \xi|^{-k} \|\widehat{f^{(k)}}\|_\infty \leq |2\pi \xi|^{-k} \|f^{(k)}\|_1$  for  $k = 0, \dots, m$ .

*Proof.* We prove only the case  $m = 1$ , the rest follows by induction. Assume  $f, f' \in L^1(\mathbb{R})$ . By Theorem 5.4, we have  $f \in \text{AC}_{\text{loc}}(\mathbb{R})$ . Hence by the fundamental theorem of calculus,

$$f(x) - f(0) = \int_0^x f'(t) dt.$$

Because  $f'$  is integrable, we get that

$$\lim_{x \rightarrow \infty} f(x) = f(0) + \lim_{x \rightarrow \infty} \int_0^x f'(t) dt = f(0) + \int_0^\infty f'(t) dt.$$

Since  $f$  is integrable and this limit exists, the limit must be 0. Hence  $f \in C_0(\mathbb{R})$ . We can compute

$$\widehat{f}'(\xi) = \int_{-\infty}^\infty f'(x) e^{-2\pi i \xi x} dx = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_a^b f'(x) e^{-2\pi i \xi x} dx.$$

Since  $f$  is absolutely continuous, we can integrate by parts to get

$$\widehat{f}'(\xi) = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \left[ f(b) e^{-2\pi \xi b} - f(a) e^{-2\pi \xi a} + (2\pi i \xi) \int_a^b f(x) e^{-2\pi i \xi x} dx \right] = (2\pi i \xi) \widehat{f}(\xi),$$

which proves the desired result.  $\square$

**Remark.** Note that for the absolute continuity arguments, we need to first restrict to a finite interval and then take limits, since we only know that  $f \in \text{AC}_{\text{loc}}(\mathbb{R})$ .

### 5.4 Approximate Identities

**Remark.** Recall that if we take  $g_T = \chi_{[-T,T]} / 2T$ , then we have  $(f * g_T)(x) = \text{Avg}_{[x-T,x+T]} f$ . As  $T \rightarrow 0$ , this converges to  $f$  if  $f$  is continuous, and converges a.e. to  $f$  if  $f$  is integrable. In particular, this is almost like a identity for the convolution operation.

**Definition 5.2.** If  $k_\lambda \in L^1(\mathbb{R})$  for  $\lambda > 0$  (or sometimes  $\lambda \in \mathbb{N}$ ) satisfy:

- (a) Normalization:  $\int_{-\infty}^\infty k_\lambda = 1$  for every  $\lambda$ ,
- (b)  $L^1$ -boundedness:  $\sup_\lambda \|k_\lambda\|_1 = \sup_\lambda \int_{-\infty}^\infty |k_\lambda| < \infty$ ,
- (c)  $L^1$ -concentration:  $\lim_{\lambda \rightarrow \infty} \int_{|x| \geq \delta} |k_\lambda| = 0$  for every  $\delta > 0$ ,

then we say that  $\{k_\lambda\}$  is an *approximate identity* (for convolution).

**Exercise 5.1.** If  $k \in L^1(\mathbb{R})$  and  $\int_{-\infty}^{\infty} k = 1$ , then  $k_\lambda(x) = \lambda k(\lambda x)$  forms an approximate identity.

**Remark.** If we choose  $k_\lambda$  to be nice, then  $f * k_\lambda$  will also be nice and “close” to  $f$ .

# Lecture 6

## Sept. 4 — Approximate Identities

### 6.1 Properties of Approximate Identities

**Theorem 6.1.** *If  $\{k_\lambda\}$  is an approximate identity, then for all  $f \in L^1(\mathbb{R})$ ,*

$$\lim_{\lambda \rightarrow \infty} \|f * k_\lambda - f\|_1 = 0.$$

That is,  $f * k_\lambda \rightarrow f$  in  $L^1$ -norm.

*Proof.* We have already seen that  $f * k_\lambda \in L^1(\mathbb{R})$ . Then

$$\|f - f * k_\lambda\|_1 = \int |f(x) - (f * k_\lambda)(x)| dx = \int \left| f(x) \int k_\lambda(t) dt - \int f(x-t) k_\lambda(t) dt \right| dx,$$

where we used that  $\int k_\lambda(t) dt = 1$ . Collecting terms and taking absolute values inside,

$$\|f - f * k_\lambda\|_1 \leq \iint |f(x) - f(x-t)| |k_\lambda(t)| dt dx.$$

By Tonelli's theorem, we can exchange orders to get

$$\|f - f * k_\lambda\|_1 \leq \int |k_\lambda(t)| \left( \int |f(x) - T_t f(x)| dx \right) dt = \int |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

We split this integral into two parts:

$$\|f - f * k_\lambda\|_1 \leq \int_{|t|<\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt + \int_{|t|\geq\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

By the strong continuity of translation, we know that  $\lim_{t \rightarrow 0} \|f - T_t f\|_1 = 0$ , so for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|t| < \delta$  implies  $\|f - T_t f\|_1 < \epsilon$ . This lets us estimate the first integral:

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t|<\delta} |k_\lambda(t)| dt + \int_{|t|\geq\delta} |k_\lambda(t)| \|f - T_t f\|_1 dt.$$

For the second integral, we can use  $\|f - T_t f\|_1 \leq \|f\|_1 + \|T_t f\|_1 = 2\|f\|_1$  to get

$$\|f - f * k_\lambda\|_1 \leq \epsilon \int_{|t|<\delta} |k_\lambda(t)| dt + 2\|f\|_1 \int_{|t|\geq\delta} |k_\lambda(t)| dt \leq \epsilon K + 2\|f\|_1 \epsilon$$

where  $K = \sup_\lambda \|k_\lambda\|_1 < \infty$  and  $\lambda$  is large enough (as  $\int_{|t|\geq\delta} |k_\lambda(t)| dt \rightarrow 0$ ). So  $\|f - f * k_\lambda\|_1 \rightarrow 0$ .  $\square$

**Exercise 6.1.** Show that for  $1 \leq p < \infty$ , we still have  $\|f - f * k_\lambda\|_p \rightarrow 0$  as  $\lambda \rightarrow \infty$  for  $f \in L^p(\mathbb{R})$ . For  $p = \infty$ , show that if  $f \in C_0(\mathbb{R})$ , then  $\|f - f * k_\lambda\|_\infty \rightarrow 0$  as  $\lambda \rightarrow \infty$ , that is  $f * k_\lambda \rightarrow f$  uniformly.

**Exercise 6.2.** Show that if  $f \in C_b(\mathbb{R})$ , then for every compact set  $K \subseteq \mathbb{R}$ ,

$$\lim_{\lambda \rightarrow \infty} \|(f - f * k_\lambda)\chi_K\|_\infty = 0.$$

**Definition 6.1.** A function  $f$  is *Hölder continuous* with exponent  $\alpha > 0$  if

$$|f(x) - f(y)| \leq K|x - y|^\alpha$$

for some constant  $K$  and all  $x, y$ . If  $\alpha = 1$ , then we say that  $f$  is *Lipschitz*.

**Remark.** If  $f$  is Hölder continuous with exponent  $\alpha > 1$ , then the mean value theorem implies that  $f$  is constant. Thus the interesting range for Hölder continuity is  $0 < \alpha \leq 1$ .

**Exercise 6.3.** Let  $f$  be bounded and Hölder continuous with exponent  $0 < \alpha \leq 1$ , then show that

$$f * k_\lambda \rightarrow f \quad \text{uniformly on } \mathbb{R}.$$

**Remark.** If  $f$  is differentiable and  $f'$  is bounded, then  $f$  is Lipschitz.

**Remark.** Recall the *Lebesgue differentiation theorem*, which says that if  $f \in L^1_{\text{loc}}(\mathbb{R})$ , then

$$(f * g_T)(x) = \frac{1}{2T} \int_{x-T}^{x+T} f(t) dt \longrightarrow f(x) \quad \text{for a.e. } x.$$

where  $g_T = \chi_{[-T,T]} / (2T)$ . The points where the limit holds are called the *Lebesgue points* of  $f$ .

**Theorem 6.2.** Assume  $k$  is bounded and compactly supported and  $\int k = 1$ . Set  $k_\lambda(x) = \lambda k(\lambda x)$  for  $\lambda > 0$ . Then for any  $f \in L^1(\mathbb{R})$ ,

$$f * k_\lambda \rightarrow f \quad \text{pointwise a.e.}$$

Moreover, the pointwise limit holds at every Lebesgue point of  $f$ .

*Proof.* Assume  $\text{supp}(k) \subseteq [-R, R]$ . We can write

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| &= \lim_{\lambda \rightarrow \infty} \left| f(x) \int k_\lambda(x-t) dt - \int f(x) k_\lambda(x-t) dt \right| \\ &\leq \lim_{\lambda \rightarrow \infty} \int |f(x) - f(t)| \lambda |k(\lambda x - \lambda t)| dt \\ &\leq \lim_{\lambda \rightarrow \infty} \lambda \int_{x-R/\lambda}^{x+R/\lambda} |f(x) - f(t)| |k(\lambda x - \lambda t)| dt. \end{aligned}$$

Making a change of variables  $T = R/\lambda$ , we have

$$\lim_{\lambda \rightarrow \infty} |f(x) - (f * k_\lambda)(x)| \leq \lim_{T \rightarrow 0} \frac{1}{2T} \int_{x-T}^{x+T} |f(x) - f(t)| dt \cdot \|k\|_\infty = 0$$

for every Lebesgue point  $x$  by the Lebesgue differentiation theorem.  $\square$

## 6.2 Density Results and Smooth Urysohn Lemma

**Theorem 6.3.**  $C_c^m(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $m > 0$  and  $1 \leq p < \infty$ .

*Proof.* Fix  $\epsilon > 0$ . Choose  $k \in C_c^m(\mathbb{R})$  with  $\int k = 1$ , and set  $k_\lambda(x) = \lambda k(\lambda x)$ . Note that there exists a compactly supported  $g \in L^p(\mathbb{R})$  with  $\|f - g\|_p < \epsilon$  (e.g. take  $g = f\chi_{[-R,R]}$  for large enough  $R$ , this works since  $f\chi_{[-R,R]}$  converges pointwise to  $f$  as  $R \rightarrow \infty$  and is dominated by  $f$ , so the dominated convergence theorem implies that  $f\chi_{[-R,R]} \rightarrow f$  in  $L^p$ -norm). Then note that  $g * k_\lambda \in C_c^m(\mathbb{R})$  and  $g * k_\lambda \rightarrow g$  in  $L^p$ -norm, so there exists  $\lambda$  such that  $\|g - g * k_\lambda\|_p < \epsilon$ . Thus

$$\|f - g * k_\lambda\|_p \leq \|f - g\|_p + \|g - g * k_\lambda\|_p < 2\epsilon$$

which implies the desired result.  $\square$

**Corollary 6.3.1.**  $C_c^\infty(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ .

**Remark.** The above proof would work for  $m = 0$  but becomes circular: The step  $g * k_\lambda \in C_c^m(\mathbb{R})$  relies on the strong continuity of translation, which we proved by first showing it for  $C_c(\mathbb{R})$  and then by an extension by density to  $L^p(\mathbb{R})$ . In particular, we needed to already know that  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ .

**Proposition 6.1** ( $C^\infty$  Urysohn's lemma). *If  $K \subseteq \mathbb{R}$  is compact and  $U \subseteq K$  is open, then there exists  $f \in C_c^\infty(\mathbb{R})$  such that  $0 \leq f \leq 1$ ,  $f = 1$  on  $K$ , and  $f = 0$  on  $U^c$ .*

*Proof.* Since  $K$  is compact and  $U^c$  is closed, we have

$$d = \text{dist}(K, U^c) = \inf\{|x - y| : k \in K, y \notin U\} > 0.$$

Set  $V = \{y \in \mathbb{R} : \text{dist}(y, K) < d/3\}$ , and choose any  $k \in C_c^\infty(\mathbb{R})$  such that  $\text{supp}(k) \subseteq [-d/3, d/3]$  and  $\int k = 1$ . Take  $f = k * \chi_V \in C_c^\infty(\mathbb{R})$ , which has  $\text{supp}(f) \subseteq \text{supp}(k) + V \subseteq U$ . If  $x \in K$ , then

$$f(x) = \int_V k(x - y) dy = \int k = 1.$$

One can check that  $0 \leq f \leq 1$  and  $f = 0$  on  $U^c$  as an exercise, which would prove the result.  $\square$

# Lecture 7

Sept. 18 —

# **Lecture 8**

**Sept. 23 —**

# Lecture 9

## Sept. 25 — Kernels and Schwartz Space

### 9.1 More Kernels

**Remark.** Recall that if  $\{k_\lambda\}$  (where  $k_\lambda(x) = \lambda k(\lambda x)$ ) is an approximate identity, then  $f * k_\lambda \rightarrow f$  in  $L^p$ -norm. We also have

$$(f * k_\lambda)^\wedge = \widehat{f} \cdot \widehat{k}_\lambda.$$

Note that  $\widehat{k}_\lambda(\xi) = \widehat{k}(\xi/\lambda)$ , so  $\widehat{k}_\lambda(\xi) \rightarrow 1$  pointwise. If the Dirichlet kernel  $d_{2\pi\lambda}(\xi) = \sin(2\pi\lambda\xi)/(\pi\xi)$  were integrable, then we would have  $\widehat{d}_{2\pi\lambda}(\xi) = \chi_{[-\lambda, \lambda]}(\xi)$ , so we must have  $d_{2\pi\lambda} \notin L^1(\mathbb{R})$ .

Some alternatives are the following:

1. The Fejér kernel  $w_\lambda(\xi) = d_{2\pi\lambda}^2(\xi) \in L^1(\mathbb{R})$ . We saw this in the proof of the inversion formula.
2. The *de la Vallée Poussin kernel*  $v_\lambda$ , which has  $\widehat{v}_\lambda$  as a trapezoid which is 1 on  $[-\lambda, \lambda]$  and decays linearly to 0 on  $[-2\lambda, -\lambda]$  and  $[\lambda, 2\lambda]$ . One has

$$(f * v_\lambda)^\wedge = \widehat{f} \cdot \widehat{v}_\lambda = \widehat{f} \quad \text{on } [-\lambda, \lambda].$$

Explicitly, one can define  $v(x) = 2w_2(x) - w(x)$  and  $v_\lambda(x) = \lambda v(\lambda x)$ .

3. The *Poisson kernel*  $p(x) = 1/\pi(x^2 + 1)$ .
4. The *Gauss kernel*  $\phi(x) = e^{-\pi x^2}$ .

**Exercise 9.1.** Show that if  $f \in L^1(\mathbb{R})$ , then  $\text{supp}(\widehat{f})$  is compact if and only if  $f = f * g$  for some  $g \in L^1(\mathbb{R})$ . Hint: Use the de la Vallée Poussin kernel.

**Exercise 9.2.** Let  $\Phi = \widehat{\phi}$ , where  $\phi$  is the Gauss kernel. Show that  $\Phi'(\xi) = -2\pi\xi\Phi(\xi)$ . Then solve this differential equation to get  $\Phi(\xi) = \Phi(0)\phi(\xi)$ . Finally show that  $\Phi(0) = 1$ , and conclude  $\Phi(\xi) = \phi(\xi)$ .

**Remark.** There are other ways to find functions which are their own Fourier transforms. The inversion formula says that if  $f, \widehat{f} \in L^1(\mathbb{R})$ , then we have

$$f(x) = (\widehat{f})^\vee(x) = f^{\wedge\wedge}(-x) = f^{\wedge\wedge\wedge\wedge}(x).$$

In particular, for  $f$  sufficiently nice, if we take  $g = f + f^\wedge + f^{\wedge\wedge} + f^{\wedge\wedge\wedge\wedge}$ , then  $\widehat{g} = g$ .

**Theorem 9.1** (Weierstrass approximation theorem). *If  $f \in C[a, b]$  and  $\epsilon > 0$ , then there exists a polynomial  $p$  such that  $\|f - p\|_\infty = \sup_{x \in [a, b]} |f(x) - p(x)| < \epsilon$ .*

*Proof.* Fix  $f \in C[a, b]$ , and choose  $[a, b] \subseteq (-R, R)$ . Extend  $f$  to a function  $g \in C_0(\mathbb{R})$  on  $\mathbb{R}$  which equals  $f$  on  $[a, b]$  and is supported in  $(-R, R)$ . Let  $\phi$  be the Gauss kernel, and choose  $\lambda$  so that

$$\|g - g * \phi_\lambda\|_\infty < \frac{\epsilon}{2}.$$

Note that  $\phi_\lambda(x) = \lambda e^{-\pi\lambda^2 x^2}$  is analytic and has a Taylor expansion

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \lambda \frac{(-\pi\lambda^2 x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}.$$

Set  $q(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} x^{2n}$ . If  $N$  is large enough, then we will have

$$\sup_{|x| \leq 2R} |\phi_\lambda(x) - q(x)| < \frac{\epsilon}{2\|g\|_1}$$

since the Taylor series converges uniformly on compact sets. Then

$$|(g * \phi_\lambda)(x) - (g * q)(x)| \leq \int_{-R}^R |g(y)| |\phi_\lambda(x-y) - q(x-y)| dy \leq \frac{\epsilon}{2\|g\|_1} \int_{-R}^R |g(y)| dy < \frac{\epsilon}{2}$$

for  $|x| \leq R$ . So  $|f(x) - (g * q)(x)| \leq \epsilon$  for  $x \in [a, b]$ . Finally, observe that

$$(g * q)(x) = \sum_{n=0}^N \frac{(-1)^n \pi^n \lambda^{2n+1}}{n!} \int g(y) (x-y)^{2n} dy,$$

which is a polynomial by the binomial theorem. So we can take  $p = g * q$ . □

## 9.2 Schwartz Space

**Definition 9.1.** Define the *Schwartz space*  $\mathcal{S}(\mathbb{R})$  to be

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}.$$

**Remark.** Note that  $|x^m f^{(n)}(x)| \leq C_{m,n}$  for some constant  $C_{m,n}$ , so

$$|f^{(n)}(x)| \leq \frac{C_{m,n}}{|x|^m}.$$

In particular, for every  $n$  and polynomial  $p$ , there exists a constant  $C_{n,p}$  such that

$$|f^{(n)}(x)| \leq \frac{C_{n,p}}{|p(x)|}.$$

Note, however, that this does not imply  $f$  has exponential decay.

**Remark.** Note that  $\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty$  defines a *seminorm* for each  $m, n \geq 0$ , but not a norm.<sup>1</sup>

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<sup>1</sup>Recall that a *seminorm*  $\rho$  is a function satisfying  $0 \leq \rho(f) < \infty$ ,  $\rho(cf) = |c|\rho(f)$ , and  $\rho(f+g) \leq \rho(f) + \rho(g)$ . If we additionally have  $\rho(f) = 0$  if and only if  $f = 0$ , then  $\rho$  is a *norm*.

**Remark.** Let  $f \in \mathcal{S}(\mathbb{R})$ . Then

$$\begin{aligned}\|x^m f^{(n)}\|_1 &= \int_{|x| \leq 1} |x^m f^{(n)}(x)| dx + \int_{|x| \geq 1} |x^m f^{(n)}(x)| dx \\ &\leq 2\|x^m f^{(n)}\|_\infty + \int_{|x| \geq 1} \frac{|x^{m+2} f^{(n)}(x)|}{|x|^2} dx \leq 2\|x^m f^{(n)}\|_\infty + C\|x^{m+2} f^{(n)}\|_\infty.\end{aligned}$$

In particular, if the  $L^\infty$ -norms are controlled, then so are the  $L^1$ -norms.

**Exercise 9.3.** Recall the smoothness and decay theorems: The Fourier transform interchanges smoothness and decay. Write  $Df = f'$ , and show that

$$(D^n((-2\pi i x)^m f(x)))^\wedge(\xi) = (2\pi i \xi)^n D^m \hat{f}(\xi).$$

Note that we have

$$D^n((-2\pi i x)^m f(x)) = \sum_{j=0}^n \binom{n}{j} D^j (-2\pi i x)^m f^{(n-j)}(x),$$

so in particular, the Schwartz condition on  $f$  implies the Schwartz condition on  $\hat{f}$ .

**Theorem 9.2.** If  $f \in \mathcal{S}(\mathbb{R})$ , then  $\hat{f} \in \mathcal{S}(\mathbb{R})$ . Moreover, we have  $f = (\hat{f})^\vee$  by the inversion formula, so the Fourier transform  $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$  by  $f \mapsto \hat{f}$  is a bijection.

**Remark.** Note that  $C_c^\infty(\mathbb{R}) \subsetneq \mathcal{S}(\mathbb{R})$ , and we will see later that  $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R}) \setminus C_c^\infty(\mathbb{R})$ .

# Lecture 10

Sept. 30 —

# Lecture 11

Oct. 2 —

# Lecture 12

Oct. 9 —

# Lecture 13

Oct. 14 —

# Lecture 14

Oct. 16 —

# Lecture 15

## Oct. 23 — The Paley-Wiener Theorem

### 15.1 Local Uncertainty Principle

**Remark.** Recall the classical uncertainty principle: For  $f \in L^2(\mathbb{R})$ ,  $\|f\|_2^2 \leq 4\pi\|xf(x)\|_2^2\|\xi\widehat{f}(\xi)\|_2^2$ . There is also a local version of this statement:

**Theorem 15.1** (Local uncertainty principle). *If  $f \in L^2(\mathbb{R})$  and  $\epsilon > 0$ , then for any  $\xi_0 \in \mathbb{R}$ , we have*

$$\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{f}(\xi)|^2 d\xi \leq 8\pi\epsilon\|f\|_2\|xf(x)\|_2.$$

### 15.2 The Paley-Wiener Theorem

**Remark.** Recall the smoothness and decay theorems: If  $x, xf(x) \in L^1(\mathbb{R})$ , then  $(\widehat{f})'$  exists. “Extreme decay” corresponds to compact support, where  $\text{supp}(f) \subseteq [-T, T]$  if  $f = 0$  a.e. outside  $[-T, T]$ . When this is the case, the *Paley-Wiener theorem* says that  $\widehat{f}$  has extreme smoothness:  $\widehat{f}$  extends to an analytic function on  $\mathbb{C}$ .

**Remark.** There is also motivation from Fourier series. The Fourier transform on  $\mathbb{R}$  is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x} dx.$$

The functions  $e_\xi(x) = e^{2\pi i \xi x} : \mathbb{R} \rightarrow S^1 \subseteq \mathbb{C}$  are continuous group homomorphism, and  $dx$  is a Haar measure on  $\mathbb{R}$  (invariant under the group action). For  $\mathbb{Z}$ , Haar measure is the counting measure, and for  $c = (c_n)_{n \in \mathbb{Z}} \in \ell^1$ , we can define its Fourier transform to be

$$\widehat{c}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{R}.$$

In fact, we can take  $\xi \in [0, 1)$ . Also note that  $\widehat{c}$  is 1-periodic.

Suppose now that  $c$  is compactly supported, say only  $c_1, \dots, c_N$  are nonzero. Then

$$\widehat{c}(\xi) = \sum_{n=1}^N c_n e^{-2\pi i n \xi} = \sum_{n=1}^N c_n (e^{-2\pi i \xi})^n$$

If we set  $z = e^{-2\pi i \xi} \in S^1$ , then the above becomes

$$\widehat{c}(z) = \sum_{n=1}^N c_n z^n, \quad |z| = 1,$$

which is a polynomial. In particular,  $\widehat{c}$  extends to an analytic function for all  $z \in \mathbb{C}$ .

**Remark.** Suppose  $f \in L^2(\mathbb{R})$  and  $\text{supp}(f) \subseteq [-T, T]$  (we say that  $f$  is “time-limited” to  $[-T, T]$ ). Then

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_{-T}^T |f(x)| dx \leq \left( \int_{-T}^T |f(x)|^2 dx \right)^{1/2} \left( \int_{-T}^T 1^2 dx \right)^{1/2} = (2T)^{1/2} \|f\|_2 < \infty,$$

so  $f \in L^1(\mathbb{R})$ . Thus we can take its Fourier transform as usual:

$$\widehat{f}(\xi) = \int_{-T}^T f(x) e^{-2\pi i \xi x} dx, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that  $e^{-2\pi i \xi x} = e^{-2\pi i \alpha x} e^{2\pi \beta x}$ , where  $e^{2\pi \beta x} > 0$ . Then

$$\int_{-T}^T |f(x) e^{-2\pi i \xi x}| dx = \int_{-T}^T |f(x)| e^{2\pi \beta x} dx \leq e^{2\pi |\beta| T} \int_{-T}^T |f(x)| dx \leq (2T)^{1/2} e^{2\pi |\beta| T} \|f\|_2.$$

So  $\widehat{f}$  can be extended to all of  $\mathbb{C}$ . We will see that this extension is analytic.

We now have a function  $\widehat{f} : \mathbb{C} \rightarrow \mathbb{C}$  satisfying

$$|\widehat{f}(\xi)| \leq (2T)^{1/2} e^{2\pi |\beta| T} \|f\|_2, \quad \xi = \alpha + i\beta \in \mathbb{C}.$$

Note that  $f \in L^1(\mathbb{R})$  implies  $\widehat{f} \in C_0(\mathbb{R})$ , so  $\widehat{f}$  is bounded on  $\mathbb{R}$ . However, the extension to  $\mathbb{C}$  cannot be bounded (for non-constant  $f$ ) by Liouville’s theorem.

**Definition 15.1.** A function  $F : \mathbb{C} \rightarrow \mathbb{C}$  has *exponential type*  $T$  if there exists  $A > 0$  such that

$$|F(\xi)| \leq A e^{T|\xi|}, \quad \xi \in \mathbb{C}.$$

**Remark.** So from the above discussion,  $\widehat{f}$  has exponential type  $2\pi T$ . We now show that  $\widehat{f}$  is analytic on  $\mathbb{C}$ . Recall that  $F$  is analytic if and only if  $F'$  exists on  $\mathbb{C}$ . We can compute

$$(\widehat{f})'(\xi) = \lim_{\substack{\eta \rightarrow 0 \\ \xi \in \mathbb{C}}} \frac{\widehat{f}(\xi + \eta) - \widehat{f}(\xi)}{\eta} = \lim_{\substack{\eta \rightarrow 0 \\ \xi \in \mathbb{C}}} \int_{-T}^T f(x) \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} dx.$$

We have seen that  $f$  is integrable, and the exponential term is bounded on  $[-T, T]$  by the mean value theorem. So by the dominated convergence theorem, we can exchange the limit and integral to get

$$\begin{aligned} (\widehat{f})'(\xi) &= \int_{-T}^T f(x) \lim_{\eta \rightarrow 0} \frac{e^{-2\pi i (\xi + \eta)x} - e^{-2\pi i \xi x}}{\eta} dx \\ &= \int_{-T}^T (-2\pi i x) f(x) e^{-2\pi i \xi x} dx = ((-2\pi i x) f(x))^\wedge(\xi). \end{aligned}$$

Note that  $xf(x) \in L^1(\mathbb{R})$  and also has support in  $[-T, T]$ , so the above exists for any  $\xi \in \mathbb{C}$ .

**Theorem 15.2** (Paley-Wiener theorem). *We have the following:*

1. *If  $f \in L^2(\mathbb{R})$  and  $\text{supp}(f) \subseteq [-T, T]$ , then  $\widehat{f}$  extends to an analytic function on  $\mathbb{C}$  with exponential type  $2\pi T$ .*
2. *If  $F : \mathbb{C} \rightarrow \mathbb{C}$  is analytic with exponential type  $2\pi T$ , then there exists  $f \in L^2(\mathbb{R})$  with  $\text{supp}(f) \subseteq [-T, T]$  and  $\widehat{f}(\xi) = F(\xi)$  for  $\xi \in \mathbb{R}$ , i.e.  $\widehat{f} = F|_{\mathbb{R}}$ .*

*Proof.* (1) This was the above discussion.

(2) The proof is more difficult and uses a lot of complex analysis. See Katznelson's book.  $\square$

**Remark.** Recall that analytic functions have Taylor expansions. Suppose  $F : \mathbb{C} \rightarrow \mathbb{C}$  and  $F = 0$  on a line segment. Then if  $\eta$  lies on the line segment, we can write

$$F(\xi) = \sum_{k=0}^{\infty} F^{(k)}(\eta) \frac{(\xi - \eta)^k}{k!}.$$

Since  $F^{(k)}(\eta) = 0$  for all  $k$ , we get  $F \equiv 0$  on all of  $\mathbb{C}$ . In fact, if  $F = 0$  on any set with an accumulation point, then  $F \equiv 0$  on  $\mathbb{C}$ .

**Corollary 15.2.1.** *If  $f \in L^2(\mathbb{R})$ ,  $f \neq 0$ , has compact support, then  $\widehat{f}(\xi) = 0$  for only countably many  $\xi$ .*

*Proof.* If  $\widehat{f}(\xi) = 0$  for uncountably many  $\xi$ , then there exists  $\xi_n \rightarrow \xi$  such that  $\widehat{f}(\xi_n) = 0$  for all  $n$  (split  $\mathbb{C}$  into countably many compact sets and apply a countability argument). Hence  $\widehat{f}, f = 0$ .  $\square$

**Corollary 15.2.2.** *If  $f \in L^2(\mathbb{R})$ , then  $f$  and  $\widehat{f}$  are both compactly supported if and only if  $f = 0$ .*

*Proof.* Suppose  $f$  is compactly supported, then  $\widehat{f}$  is analytic on  $\mathbb{C}$ . Suppose  $\widehat{f}(\xi) = 0$  a.e. on  $\mathbb{R}$  outside  $[-\Omega, \Omega]$ . Since  $\widehat{f}$  is analytic and  $\mathbb{R} \setminus [-\Omega, \Omega]$  has an accumulation point, we get  $\widehat{f}, f = 0$ .  $\square$

**Remark.** In particular, the above corollary implies that if  $f \in C_c^\infty(\mathbb{R})$  is nonzero, then  $\widehat{f}$  is not compactly supported. But  $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$  and  $\mathcal{F}$  maps  $\mathcal{S}(\mathbb{R})$  onto  $\mathcal{S}(\mathbb{R})$ , so we do know  $\mathcal{F}(C_c^\infty(\mathbb{R})) \subseteq \mathcal{S}(\mathbb{R})$ .

### 15.3 Hilbert-Schmidt Operators

**Remark.** Let  $H$  be a separable, infinite-dimensional Hilbert space. Recall the following definitions and results from functional analysis:

**Definition 15.2.** A linear operator  $A : H \rightarrow H$  is *compact* if  $A(D)$  is contained in a compact set, where  $D = \{f \in H : \|f\| \leq 1\}$  is the closed unit disk in  $H$ .

**Theorem 15.3** (Spectral theorem for compact, self-adjoint operators). *If  $A : H \rightarrow H$  is a compact, self-adjoint operator, then there exists an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  of  $H$  such that*

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \quad Af = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n, \quad Ae_n = \lambda_n e_n.$$

Moreover,  $\lambda_n \rightarrow 0$  if there are infinitely many  $\lambda_n \neq 0$ .

**Remark.** Note that  $\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n| \rightarrow 0$ , so the identity operator  $I$  is not compact.

**Example 15.2.1.** An example of a compact box in  $\mathbb{R}^{\mathbb{N}}$  looks like

$$[0, 1] \times [0, 1/2] \times [0, 1/4] \times \cdots .$$

**Definition 15.3.** A linear operator  $A : H \rightarrow H$  is *Hilbert-Schmidt* if there exists an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  such that  $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$ .

**Remark.** For an operator  $A : H \rightarrow H$ , if we write

$$Af(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy,$$

then  $A$  is Hilbert-Schmidt if and only if  $k \in L^2(\mathbb{R}^2)$ .

# Lecture 16

Oct. 28 —

# **Lecture 17**

**Oct. 30 —**

# Lecture 18

## Nov. 4 — Energy Concentration, Part 2

### 18.1 Energy Concentration, Continued

**Remark.** Recall the following problem. Fix  $T, \Omega > 0$ . We want to determine

$$E_{T,\Omega} = \sup \left\{ \int_{-T}^T |f|^2 : \|f\|_2 = 1, \text{supp}(\widehat{f}) \subseteq [-\Omega, \Omega] \right\}.$$

We previously defined  $A_T f = f \chi_{[-T,T]}$  and  $B_\Omega f = (\widehat{f} \chi_{[-\Omega,\Omega]})^\vee = f * d_{2\pi\Omega}$ , then

$$E_{T,\Omega} = \|A_T B_\Omega\|^2 = \|B_\Omega A_T B_\Omega\|^2 = \lambda_1,$$

where  $\lambda_1$  is the largest eigenvalue of  $B_\Omega A_T B_\Omega$  (recall that  $B_\Omega A_T B_\Omega$  is compact and self-adjoint, so it has eigenvalues  $\lambda_n \rightarrow 0$ ). The spectral theorem also gives orthonormal eigenvectors  $\{\varphi_n\}_{n \in \mathbb{N}}$ :

$$B_\Omega A_T B_\Omega f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n.$$

Note that  $B_\Omega(A_T B_\Omega \varphi_n) = \lambda_n \varphi_n$  for  $\lambda_n \neq 0$ , so  $\varphi_n \in \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$  and  $\text{supp}(\widehat{\varphi}_n) \subseteq [-\Omega, \Omega]$ .

What is  $\|A_T \varphi_n\|_2^2$ ? We can compute

$$\begin{aligned} \|A_T \varphi_n\|_2^2 &= \langle A_T \varphi_n, A_T \varphi_n \rangle = \langle A_T B_\Omega \varphi_n, A_T B_\Omega \varphi_n \rangle \\ &= \langle B_\Omega \underbrace{A_T A_T}_{} B_\Omega \varphi_n, \varphi_n \rangle = \lambda_n \langle \varphi_n, \varphi_n \rangle = \lambda_n \|\varphi_n\|_2^2 = \lambda_n. \end{aligned}$$

Thus we see that  $E_{T,\Omega} = \lambda_1 = \|A_T \varphi_1\|_2^2 < \|\varphi_1\|_2^2 = 1$ , so  $\varphi_1$  has the greatest energy in  $[-T, T]$ .

Note that  $\{\varphi_n\}_{n \in \mathbb{N}}$  is an orthonormal basis for its closed span, which a priori lies in  $\mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$ .

**Proposition 18.1.**  $\overline{\text{span}}\{\varphi_n\} = \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$ .

*Proof.* We show  $\overline{\text{span}}\{\varphi_n\}^\perp = \{0\}$ . Suppose  $f \in \mathcal{FL}_{[-\Omega,\Omega]}^2(\mathbb{R})$  with  $f \perp \varphi_n$  for all  $n$ . Then

$$f \in \overline{\text{span}}\{\varphi_n\}^\perp = \overline{\text{range}}(B_\Omega A_T B_\Omega)^\perp = \ker(B_\Omega A_T B_\Omega).$$

Thus we have

$$\|A_T f\|_2^2 = \langle A_T B_\Omega f, A_T B_\Omega f \rangle = \langle B_\Omega A_T B_\Omega f, f \rangle = 0,$$

so  $f = 0$  a.e. on  $[-T, T]$ . By the Paley-Wiener theorem, this implies  $f = 0$ .  $\square$

**Remark.** Proposition 18.1 implies that there are infinitely many nonzero eigenvalues for  $B_\Omega A_T B_\Omega$ . Moreover, since  $B_\Omega$  is the orthogonal projection onto  $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$  and  $\{\varphi_n\}$  is an orthonormal basis for  $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$ , we can write

$$B_\Omega f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n.$$

Note that  $L_{[-T, T]}^2(\mathbb{R})$  is not orthogonal to  $\mathcal{FL}_{[-\Omega, \Omega]}^2(\mathbb{R})$ : For example,

$$\langle \chi_{[-T, T]}, d_{2\pi\Omega} \rangle = \int_{-T}^T \frac{\sin 2\pi\Omega x}{\pi x} dx \neq 0.$$

However, set  $\psi_n = \lambda_n^{-1/2} A_T \varphi_n$ . Then

$$\langle \lambda_m^{-1/2} A_T \varphi_m, \lambda_n^{-1/2} A_T \varphi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \langle B_\Omega A_T B_\Omega \varphi_m, \varphi_n \rangle = \lambda_m^{1/2} \lambda_n^{-1/2} \langle \varphi_m, \varphi_n \rangle = \delta_{mn}.$$

So  $\{\psi_n\}_{n \in \mathbb{N}}$  is an orthonormal sequence in  $L_{[-T, T]}^2(\mathbb{R})$ .

**Exercise 18.1.** Show that  $\{\psi_n\}_{n \in \mathbb{N}}$  is an orthonormal basis for  $L_{[-T, T]}^2(\mathbb{R})$ .

**Remark.** Note that

$$\langle \varphi_m, \varphi_n \rangle = \int_{-\infty}^{\infty} \varphi_m(x) \overline{\varphi_n(x)} dx = 0, \quad m \neq n.$$

Furthermore,

$$\langle \psi_m, \psi_n \rangle = \lambda_m^{-1/2} \lambda_n^{-1/2} \int_{-T}^T \varphi_m(x) \overline{\varphi_n(x)} dx = 0, \quad m \neq n$$

even when the  $\varphi_n$  are *not* supported in  $[-T, T]$ .

By symmetry (swapping the roles of  $A_T$  and  $B_\Omega$ ), one can also compute that

$$A_T B_\Omega A_T f = \sum_{n=1}^{\infty} \lambda_n \langle f, \psi_n \rangle \psi_n.$$

In particular,  $A_T B_\Omega A_T$  and  $B_\Omega A_T B_\Omega$  have the same eigenvalues.

**Proposition 18.2.**  $B_\Omega A_T$  commutes with

$$Kf = (T^2 - x^2)f''(x) - 2xf'(x) - 4\pi^2\Omega^2x^2f(x).$$

**Remark.** Since  $\varphi_n$  is already band-limited, we have

$$B_\Omega A_T \varphi_n = B_\Omega A_T B_\Omega \varphi_n = \lambda_n \varphi_n.$$

Moreover, Paley-Wiener implies  $\varphi_n$  is infinitely differentiable, so  $\varphi_n \in \text{domain}(K)$ . Then

$$B_\Omega A_T K \varphi_n = K B_\Omega A_T \varphi_n = K(\lambda_n \varphi_n) = \lambda_n K \varphi_n.$$

Note that  $K \varphi_n \neq 0$ , so  $K \varphi_n$  is an eigenvector for  $B_\Omega A_T$ . The multiplicities of the  $\lambda_n$  is 1, so  $K \varphi_n = \mu \varphi_n$ . The eigenfunctions  $\varphi_n$  of  $K$  are known functions called the *prolate spheroidal wave functions*.

## 18.2 Approximating Band-Limited Functions

**Remark.** We know that  $1 > \lambda_1 > \lambda_2 > \dots \rightarrow 0$ . Also,

$$\|A_T B_\Omega\|_{\text{HS}}^2 = \|g_{T,\Omega}\|_2^2 = \int_{-T}^T \int_{-\infty}^{\infty} d_{2\pi\Omega}(x-y)^2 dx dy = 4T\Omega,$$

where  $g_{T,\Omega}$  is the corresponding kernel. The singular values of  $A_T B_\Omega$  are

$$s_n = \lambda_n((A_T B_\Omega)^*(A_T B_\Omega))^{1/2} = \lambda_n(B_\Omega A_T B_\Omega)^{1/2} = \lambda_n^{1/2}.$$

Thus  $4T\Omega = \|A_T B_\Omega\|_{\text{HS}}^2 = \sum_{n=1}^{\infty} s_n^2 = \sum_{n=1}^{\infty} \lambda_n$ . On the other hand,  $\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^2$ , and

$$\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 = \|k_{T,\Omega}\|_2^2 = \int_{-T}^T \int_{-T}^T d_{2\pi\Omega}(x-y)^2 dx dy < 4T\Omega.$$

Due to the decay of  $d_{2\pi\Omega}(x-y)^2$ , for  $T$  large, we should expect that  $\|B_\Omega A_T B_\Omega\|_{\text{HS}}^2 \rightarrow 4T\Omega$ . Hence

$$\sum_{n=1}^{\infty} \lambda_n^2 \approx \sum_{n=1}^{\infty} \lambda_n.$$

Thus intuitively,  $\lambda_n$  should be either close to 1 or close to 0. In particular, only finitely many terms can be close to 1, so

$$f \approx \sum_{n=1}^N \lambda \langle f, \varphi_n \rangle \varphi_n \in \text{span}\{\varphi_1, \dots, \varphi_N\}.$$

Thus the space of band-limited functions are approximately finite-dimensional in some sense.

**Theorem 18.1.**  $\#\{n : \lambda_n \geq 1 - \epsilon\} \leq 4T\Omega - C_\epsilon \log(T\Omega)$ .

## 18.3 Space of Possible Time and Band Limits

**Remark.** Define the energies

$$\begin{aligned} \alpha &= E_T(f) = \|A_T f\|_2^2 = \int_{-T}^T |f|^2 \\ \beta &= E_\Omega(f) = \|B_\Omega f\|_2^2 = \int_{-\Omega}^{\Omega} |\widehat{f}|^2. \end{aligned}$$

Which pairs  $(\alpha, \beta)$  can be achieved? We have seen that  $(\lambda_1, 1)$  and  $(\lambda_2, 1)$  are possible, while  $(1, 1)$  is not by the Paley-Wiener theorem. Similarly,  $(1, 0)$  and  $(0, 1)$  are not possible by Paley-Wiener. Since the roles of  $A_T$  and  $B_\Omega$  are symmetric, we get that  $(1, \lambda_1)$  and  $(1, \lambda_2)$  are possible. Note that if  $(\alpha, 1)$  is achieved, then so is  $(\alpha', 1)$  for any  $\alpha' < \alpha$ , and a similar statement holds for  $(1, \beta)$ .

One can show the space of valid pairs is  $(\alpha, \beta) \in [0, 1]^2$  such that

$$\cos^{-1} \alpha + \cos^{-1} \beta \geq \cos^{-1} \lambda^{1/2},$$

which is in the shape of an ellipse.

# Lecture 19

## Nov. 6 — Fourier Series

### 19.1 The Abstract Fourier Transform

**Definition 19.1.** A *locally compact abelian (LCA) group* is an abelian group with a Hausdorff topology such that every point has a compact neighborhood, and  $(x, y) \mapsto x + y$  and  $x \mapsto -x$  are continuous.

**Example 19.1.1.** Some examples of LCA groups are  $\mathbb{R}^n$ ,  $\mathbb{Z}^n$ ,  $\mathbb{T}^n$ ,  $\mathbb{Z}_N^n$ , where  $\mathbb{T} = [0, 1)$  with addition modulo 1.

**Theorem 19.1.** Every LCA group has a Haar measure, i.e. a nonzero Radon measure  $\mu$  which is translation-invariant.

**Definition 19.2.** A *character* on a LCA group is a continuous homomorphism

$$\xi : G \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

i.e. a continuous map with  $\xi(x + y) = \xi(x)\xi(y)$ .

**Definition 19.3.** The *dual group* of  $G$  is  $\widehat{G} = \{\xi : \xi \text{ is a character of } G\}$ .

**Example 19.3.1.** Consider  $\mathbb{T} = [0, 1)$ . Fix  $n \in \mathbb{Z}$ , then  $e_n(x) = e^{2\pi i n x}$  is a character on  $\mathbb{T}$ . Note that we have  $e_n(x) = e_1(x)^n$ , and one can show these are all of the characters on  $\mathbb{T}$ . Thus

$$\widehat{\mathbb{T}} = \{e_n : n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

**Example 19.3.2.** The characters on  $\mathbb{Z}$  are determined by their values at 1. So for any  $\xi \in [0, 1)$ , we get a character  $e_\xi(n) = e^{2\pi i \xi n}$ . In particular,  $\widehat{\mathbb{Z}} = \mathbb{T}$ .

**Example 19.3.3.** We have  $\widehat{\mathbb{Z}}_N \cong \mathbb{Z}_N$  and  $\widehat{\mathbb{R}} \cong \mathbb{R}$ . The characters on  $\mathbb{R}$  are  $e_\xi(x) = e^{2\pi i \xi x}$ .

**Remark.** The dual group  $\widehat{G}$  is a LCA group under the multiplication of characters, where the topology on  $\widehat{G}$  is the topology of uniform convergence on compact sets (also called the compact-open topology).

**Theorem 19.2.** If  $G$  is discrete, then  $\widehat{G}$  is compact. If  $G$  is compact (so  $G$  has finite measure), then  $\widehat{G}$  is discrete, and the characters on  $G$  are orthonormal in  $L^2(G)$  (in fact, they form an orthonormal basis).

**Example 19.3.4.** Consider the examples  $\widehat{\mathbb{Z}} = \mathbb{T}$ ,  $\widehat{\mathbb{T}} = \mathbb{Z}$ ,  $\widehat{\mathbb{R}} = \mathbb{R}$ ,  $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$ .

**Remark.** In some sense,  $\mathbb{R}$  being non-compact and non-discrete makes things harder. On the other hand, having dilations in  $\mathbb{R}$  can make harmonic analysis easier.

**Theorem 19.3** (Pontryagin duality).  $\widehat{\widehat{G}} \cong G$ .

**Definition 19.4.** The *Fourier transform* of  $f \in L^1(G)$  (complex-valued) is  $\widehat{f} : \widehat{G} \rightarrow \mathbb{C}$  defined by

$$\widehat{f}(\xi) = \int_G f(x) \overline{\xi(x)} d\mu(x), \quad \xi \in \widehat{G},$$

where  $d\mu$  is the Haar measure on  $G$ .

**Example 19.4.1.** On the real line, the Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

For  $\mathbb{T}$ , the Fourier coefficients are given by

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \quad n \in \mathbb{Z}.$$

For  $\mathbb{Z}$ , given  $c = (c_n)_{n \in \mathbb{Z}}$ ,

$$\widehat{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i n x}, \quad x \in \mathbb{T}$$

For  $\mathbb{Z}_N$ , the discrete Fourier transform is

$$\widehat{c}(k) = \sum_{n=0}^{N-1} c_n e^{-2\pi i n k / N}, \quad k \in \mathbb{Z}_N.$$

## 19.2 Fourier Series

**Definition 19.5.** The *Fourier transform* of  $f \in L^1(\mathbb{T})$  is the sequence  $\widehat{f} = (\widehat{f}(n))_{n \in \mathbb{Z}}$ , where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

are the *Fourier coefficients* of  $f$ . The *inverse Fourier transform* of  $f \in L^1(\mathbb{T})$  is  $\check{f} = (\check{f}(n))_{n \in \mathbb{Z}}$ , where

$$\check{f}(n) = \int_0^1 f(x) e^{2\pi i n x} dx.$$

Formally,  $\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$  is the *Fourier series* of  $f$ .

**Remark.** If  $f \in L^2(\mathbb{T})$  and we accept that  $\{e_n\}_{n \in \mathbb{Z}}$  is an orthonormal basis for  $L^2(\mathbb{T})$ , then

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n,$$

where the convergence is in  $L^2(\mathbb{T})$  (we do not get pointwise convergence in general).

**Remark.** If  $f \in L^1(\mathbb{T})$ , then we have

$$|\widehat{f}(n)| \leq \int_0^1 |f(x) e^{-2\pi i n x}| dx = \|f\|_{L^1},$$

so we see that  $\widehat{f} \in \ell^\infty(\mathbb{Z})$ .

**Remark.** If we define  $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ , then we can view the Fourier series as

$$(\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

**Example 19.5.1.** For  $0 < \alpha \leq 1$ , consider the function

$$g(x) = \sum_{m=0}^{\infty} 2^{-\alpha m} e^{2\pi i 2^m x}, \quad x \in \mathbb{T}$$

which converges absolutely in  $C(\mathbb{T})$ . One can show that  $g$  is nowhere differentiable.

Weierstrass first showed that  $g(x) = \sum_{m=0}^{\infty} a^{-m} \cos(b^m x)$  is nowhere differentiable for  $a, b$  large enough.

### 19.3 Partial Sums of Fourier Series

**Definition 19.6.** For  $f \in L^1(\mathbb{T})$ , define the *partial sums*

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}.$$

**Remark.** Consider a discrete characteristic function

$$\chi_N(n) = \begin{cases} 1 & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that  $\chi_N \in \ell^1(\mathbb{Z})$ . Then we can compute that

$$\widehat{\chi}_N(x) = \check{\chi}_N(x) = \sum_{n=-\infty}^{\infty} \chi_N(n) e^{2\pi i n x} = \sum_{n=-N}^N e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} = d_N(x)$$

by the geometric series formula. Note that  $d_N \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$ .<sup>1</sup> We also have  $\|d_N\|_1 \rightarrow \infty$ .

**Remark.** Using the *Dirichlet kernel*  $d_N$ , we can write the partial sums as

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x} = \sum_{n=-N}^N \left( \int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x} \\ &= \int_0^1 f(t) \sum_{n=-N}^N e^{2\pi i n (x-t)} dt = \int_0^1 f(t) d_N(x-t) dt = (f * d_N)(x). \end{aligned}$$

Intuitively,  $S_N f = (\widehat{f} \chi_N)^\vee = (\widehat{f})^\vee * \check{\chi}_N = f * d_N$ , but the above is a direct proof.

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<sup>1</sup>Note that  $\mathbb{T}$  is compact, so a continuous function on  $\mathbb{T}$  is automatically integrable.

## 19.4 Approximate Identities on $\mathbb{T}$

**Definition 19.7.** An *approximate identity* on  $\mathbb{T}$  is a sequence  $\{k_n\}_{n \in \mathbb{N}}$  such that

1.  $k_n \in L^1(\mathbb{T})$  and  $\int_0^1 k_n(x) dx = 1$ ;
2.  $\sup_{n \in \mathbb{N}} \|k_n\|_1 < \infty$ ;
3. for any  $\delta > 0$ ,  $\int_{|x| \geq \delta} |k_n(x)| dx \rightarrow 0$ .

**Remark.** The Dirichlet kernel  $d_N$  does not form an approximate identity.

**Definition 19.8.** For  $f \in L^1(\mathbb{T})$ , define the *Cesàro sums*

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + \cdots + S_N f(x)}{N+1}.$$

Note that this is the average of the partial sums.

**Remark.** We can rewrite the Cesàro sums as

$$\sigma_N f(x) = \sum_{n=-N}^N \underbrace{\left(1 - \frac{|n|}{N+1}\right)}_{W_N(n)} \widehat{f}(n) e^{2\pi i n x} = (f * \widehat{W}_N)(x).$$

We define the *Fejér kernel* to be

$$w_N(x) = \widehat{W}_N(x) = \frac{1}{N+1} \left( \frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2.$$

# Lecture 20

## Nov. 11 — Fourier Series, Part 2

### 20.1 Approximate Identities on $\mathbb{T}$ , Continued

**Remark.** Using a similar proof as in the case for  $\mathbb{R}$ , for  $f \in L^1(\mathbb{T})$ , we have

$$\widehat{f} \in c_0(\mathbb{Z}) = \{c = (c_n) : c_n \rightarrow 0 \text{ as } |n| \rightarrow \infty\}.$$

However, in general  $\widehat{f} \notin \ell^1(\mathbb{Z})$ .

**Exercise 20.1.** Let  $d_N$  be the Dirichlet kernel on  $\mathbb{T}$ . Show that

$$\frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{k} \leq \|d_N\|_1 \leq 3 + \frac{4}{\pi^2} \sum_{k=1}^N \frac{1}{k}.$$

In particular,  $\|d_N\|_1 \rightarrow \infty$  as  $N \rightarrow \infty$ , so  $d_N$  does not form an approximate identity.

**Remark.** Define the function

$$W_N(x) = \begin{cases} 1 - |n|/(N+1) & |n| \leq N, \\ 0 & |n| > N. \end{cases}$$

Note that  $W_N = (\chi_N * \chi_N)/(2N+1)$  and  $\widehat{W}_{2N} = (\widehat{\chi}_N)^2 = d_N^2/(2N+1)$ . The Fejér kernel is

$$w_N = \widehat{W}_N = \frac{1}{N} \left( \frac{\sin((N+1)\pi x)}{\sin(\pi x)} \right)^2,$$

and one can check that  $\int_0^1 w_N = 1$  and  $\|w_N\|_1 = 1$ .

**Theorem 20.1.**  $(f * \widetilde{W}_N)(x) = (f * w_N)(x)$ .

**Exercise 20.2.**  $\{w_N\}_{N \in \mathbb{N}}$  forms an approximate identity on  $\mathbb{T}$ .

**Remark.** Unlike the real line, we cannot just take any  $L^1$  function and dilate it to form an approximate identity on  $\mathbb{T}$ , as the dilations need not be 1-periodic.

**Exercise 20.3.** If  $\{k_N\}$  is an approximate identity on  $\mathbb{T}$ , then:

1. For  $1 \leq p < \infty$ ,  $f \in L^p(\mathbb{T})$  implies  $f * k_N \rightarrow f$  in  $L^p$ -norm.
2.  $f \in C(\mathbb{T})$  implies  $f * k_N \rightarrow f$  uniformly.

## 20.2 The Inversion Formula on $\mathbb{T}$

**Theorem 20.2.** If  $f \in L^1(\mathbb{T})$  and  $\widehat{f} \in \ell^1(\mathbb{Z})$ , then

$$f(x) = (\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where the above series converges uniformly in  $C(\mathbb{T})$ .

*Proof.* Note that  $(\widehat{f})^\vee(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$  converges absolutely in the norm of  $C(\mathbb{T})$  since

$$\sum_{n=-\infty}^{\infty} \|\widehat{f}(n) e^{2\pi i n x}\|_\infty = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty.$$

In particular, this shows that  $(\widehat{f})^\vee \in C(\mathbb{T})$ . Also  $f * w_N \rightarrow f$  in  $L^1$ -norm, where

$$(f * w_N)(x) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x}.$$

Note that  $(1 - |n|/(N+1)) \widehat{f}(n) \rightarrow \widehat{f}(n)$  pointwise as  $N \rightarrow \infty$ , and

$$\left| \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) \right| \leq |\widehat{f}(n)| \in \ell^1(\mathbb{Z}),$$

so by the dominated convergence theorem (for counting measure)

$$(f * w_N)(x) \xrightarrow[N \rightarrow \infty]{} \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x} = (\widehat{f})^\vee(x)$$

pointwise. Since  $f * w_N \rightarrow f$  in  $L^1$ -norm, there exists a subsequence  $f * w_{N_k} \rightarrow f$  pointwise a.e., hence we see that  $f = (\widehat{f})^\vee$  a.e. Since  $f$  and  $(\widehat{f})^\vee$  are continuous, we get that  $f = (\widehat{f})^\vee$  everywhere.  $\square$

**Corollary 20.2.1.** If  $f \in L^1(\mathbb{T})$  and  $\widehat{f}(n) = 0$  for every  $n$ , then  $f = 0$  a.e. Moreover, if  $f, g \in L^1(\mathbb{T})$  and  $\widehat{f}(n) = \widehat{g}(n)$  for all  $n$ , then  $f = g$  a.e.

**Definition 20.1.** The *Fourier algebra* (or *Wiener algebra*) on  $\mathbb{T}$  is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\}.$$

**Remark.** The Fourier algebra  $A(\mathbb{T})$  is closed under convolution and forms a Banach algebra. Moreover, it is a dense subspace of  $C(\mathbb{T})$ .

## 20.3 $L^2$ -Convergence of Fourier Series

**Remark.** Let  $e_n(x) = e^{2\pi i n x}$ . The functions  $e_n$  are orthonormal in  $L^2(\mathbb{T})$ : For  $n \neq m$ ,

$$\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i m x} \overline{e^{2\pi i n x}} dx = \int_0^1 e^{2\pi i(m-n)x} dx = \frac{e^{2\pi i(m-n)x}}{2\pi i(m-n)} \Big|_{x=0}^1 = 0.$$

**Theorem 20.3.** Let  $\{f_n\}_{n \in \mathbb{N}}$  be an orthonormal sequence in a Hilbert space  $H$ . Then the following are equivalent:

1.  $\{f_n\}_{n \in \mathbb{N}}$  is complete (i.e.  $\overline{\text{span}}\{f_n\} = H$ ).
2.  $f = \sum_{n=1}^{\infty} c_n f_n$  in  $H$  for a unique choice of scalars  $c_n$ . That is,  $\{f_n\}$  is a Schauder basis for  $H$ .
3.  $f = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$ , where the convergence is in  $H$ .
4. (Plancherel)  $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$ .
5. (Parseval)  $\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle f_n, g \rangle$ .

**Example 20.1.1.** The above theorem does not hold when the sequence is not orthonormal. Recall the Weierstrass approximation theorem:  $\{x^n\}_{n=0}^{\infty}$  is complete in  $C(\mathbb{T})$ . But not every  $f \in C(\mathbb{T})$  can be written as  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  (such functions are infinitely differentiable in some disk).

**Theorem 20.4.** We have the following:

1. For  $1 \leq p < \infty$ ,  $\{e_n\}_{n \in \mathbb{N}}$  is complete in  $L^p(\mathbb{T})$ .
2.  $\{e_n\}_{n \in \mathbb{N}}$  is complete in  $C(\mathbb{T})$ .

*Proof.* (1) If  $f \in L^p(\mathbb{T})$ , then we have

$$(f * w_N) = \sum_{n=-N}^N \left(1 - \frac{|n|}{N+1}\right) \widehat{f}(n) e^{2\pi i n x} \xrightarrow[N \rightarrow \infty]{} f$$

in  $L^p$ -norm. But  $f * w_n \in \text{span}\{e_n\}_{n \in \mathbb{Z}}$ , so  $\overline{\text{span}}\{e_n\}_{n \in \mathbb{Z}} = L^p(\mathbb{T})$ . The same proof works for (2).  $\square$

**Corollary 20.4.1.**  $\{e_n\}_{n \in \mathbb{Z}}$  is an orthonormal basis for  $L^2(\mathbb{T})$ . Furthermore,

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x},$$

where the above convergence is in  $L^2$ -norm. Moreover, one has

1. (Plancherel)  $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$ ,
2. (Parseval)  $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)}$ ,

so the Fourier transform operator  $\mathcal{F} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z})$  given by  $f \mapsto \widehat{f}$  is unitary.

**Theorem 20.5.** If  $1 < p < \infty$ , then  $S_N f = (f * d_N)(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x}$  converges in  $L^p$ -norm, so

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}$$

in  $L^p$ -norm with respect to the ordering  $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$ . However, the convergence is conditional: Only this ordering (and finite permutations of it) need converge when  $p \neq 2$ .

# Lecture 21

## Nov. 13 — Fourier Series, Part 3

### 21.1 Shannon Sampling Theorem

**Definition 21.1.** Define the *Paley-Wiener space* to be

$$\text{PW}(\mathbb{R}) = \mathcal{F}L_{[-1/2, 1/2]}^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

**Remark.** Set  $e_n(x) = e^{-2\pi i n x} \chi_{[-1/2, 1/2]}(x) = M_n \chi_{[-1/2, 1/2]}(x)$ , then

$$\check{e}_n(x) = T_n \check{\chi}_{[-1/2, 1/2]}(x) = \frac{\sin(\pi(x - n))}{\pi(x - n)}.$$

Since  $\{e_n\}$  is an orthonormal basis for  $L_{[-1/2, 1/2]}^2(\mathbb{R})$ , the  $\{\check{e}_n\}$  form an orthonormal basis for  $\text{PW}(\mathbb{R})$ . Note that if  $f \in \text{PW}(\mathbb{R})$ , then we can write

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f, \check{e}_n \rangle \check{e}_n(x).$$

By the unitarity of the Fourier transform, we can compute that

$$\langle f, \check{e}_n \rangle = \langle \widehat{f}, e_n \rangle = \int_{-1/2}^{1/2} \widehat{f}(\xi) e^{2\pi i n \xi} d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i n \xi} d\xi = (\widehat{f})^\vee(n) = f(n).$$

Note that the third equality follows from  $\text{supp}(\widehat{f}) \subseteq [-1/2, 1/2]$ . Thus

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(x - n))}{\pi(x - n)}.$$

This is called the *Shannon sampling theorem* (or *classical sampling theorem*), i.e. that any function in  $\text{PW}(\mathbb{R})$  is completely determined by its values on the integers.

**Remark.** Fix  $b > 0$ , and set  $e_{nb}(x) = e^{2\pi i n b x}$ . This is an orthogonal basis for  $L^2[0, 1/b]$ .

**Example 21.1.1.** If we set  $b = 1/2$ , then we can write

$$\{e_{n/2}\}_{n \in \mathbb{Z}} = \{e_n\}_{n \in \mathbb{Z}} \cup \{e_{(n+1)/2}\}_{n \in \mathbb{Z}}.$$

This is a union of two orthonormal bases for  $L^2[0, 1]$ . For  $f \in L^2[0, 1]$ , we can write

$$2f = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n + \sum_{n=-\infty}^{\infty} \langle f, e_{(n+1)/2} \rangle e_{(n+1)/2} = \sum_{n \in \mathbb{Z}} \langle f, e_{n/2} \rangle e_{n/2}.$$

**Remark.** If we take  $0 < b < 1$  and consider  $e^{2\pi i n b x} = e_{bn}(x)$  in  $L^2[0, 1]$ , then for  $f \in L^2[0, 1] \subseteq L^2[0, 1/b]$ ,

$$f = \frac{1}{b} \sum_{n=-\infty}^{\infty} \langle f, e_{bn} \rangle e_{bn}$$

in  $L^2[0, 1/b]$ . Since  $f = 0$  on  $[1, 1/b]$ , we have the same expansion in  $L^2[0, 1]$ . Note, however, that this expansion is not orthogonal and not unique. These sets  $\{e_{bn}\}_{n \in \mathbb{Z}}$  are called *frames*.

## 21.2 Weyl's Equidistribution Theorem

**Theorem 21.1.** *If  $\alpha \in \mathbb{T}$  is irrational, then  $\{k\alpha\}_{k \in \mathbb{N}}$  is equidistributed in  $[0, 1)$ , i.e.*

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq k \leq N : k\alpha \bmod 1 \in (a, b)\}}{N} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha) = b - a$$

for every  $0 \leq a < b \leq 1$ .

**Remark.** Consider  $f \in C(\mathbb{T})$ . Then we can view  $\frac{1}{N} \sum_{k=1}^N f(k\alpha)$  as some type of Riemann sum, so we might expect that

$$\frac{1}{N} \sum_{n=1}^N f(k\alpha) \xrightarrow[N \rightarrow \infty]{} \int_0^1 f(x) dx.$$

The *Birkhoff ergodic theorem* says that the above average in fact converges to the integral (given that  $\{k\alpha\}$  is equidistributed). In fact, the result holds for any  $f \in L^1(\mathbb{T})$ .

**Exercise 21.1.** Prove the above statement for  $e_n(x) = e^{2\pi i n x}$  (and thus for trigonometric polynomials  $p(x) = \sum_{n=-N}^N c_n e^{2\pi i n x}$ ). Then apply density to prove this for  $f \in C(\mathbb{T})$ .

*Proof of Theorem 21.1.* Approximate  $\chi_{(a,b)}$  by a trapezoidal continuous function  $f$  which equals  $\chi_{(a,b)}$  on  $(a + \epsilon, b - \epsilon)$  and decays linearly to 0 on  $[a, a + \epsilon]$  and  $[b - \epsilon, b]$ . Then

$$b - a - \epsilon = \int_0^1 f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(k\alpha) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha).$$

Taking a similar approximation with  $g$  which equals  $\chi_{(a,b)}$  on  $[a, b]$  and decays linearly to 0 on  $[a - \epsilon, a]$  and  $[b, b + \epsilon]$ , we also get an upper bound:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \chi_{(a,b)}(k\alpha) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N g(k\alpha) \leq \int_0^1 g(x) dx = b - a + \epsilon.$$

This holds for arbitrary  $\epsilon$ , hence the limit is equal to  $b - a$ . □

**Corollary 21.1.1** (Kronecker's theorem).  *$\{k\alpha \bmod 1\}_{k \in \mathbb{N}}$  is dense in  $[0, 1)$ .*

## 21.3 $L^p$ -Convergence of Fourier Series

**Remark.** Recall that  $\{e_n\}_{n \in \mathbb{Z}}$  is complete in  $L^p(\mathbb{T})$  and  $C(\mathbb{T})$ , i.e.  $\text{span}\{e_n\}_{n \in \mathbb{Z}}$  is dense.

For  $1 \leq p < \infty$ , if  $f \in L^p(\mathbb{T})$ , do the partial sums

$$S_N f(x) = \sum_{n=-N}^N \widehat{f}(n) e^{2\pi i n x} = (f * d_N)(x)$$

converge to  $f$  in  $L^p$ -norm?

The motivation is as follows. Suppose that  $S_N f \rightarrow f$  in  $L^p$  for every  $f \in L^p(\mathbb{T})$ . Then we have a linear operator  $S_N : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$  on a Banach space. Moreover, the convergence is also pointwise. Thus for any individual  $f \in L^p(\mathbb{T})$ ,

$$\sup_{N \in \mathbb{N}} \|S_N f\|_p < \infty.$$

Then by the Banach-Steinhaus theorem (also called the uniform boundedness principle), we get

$$\sup_{N \in \mathbb{N}} \|S_N\| = \sup_{N \in \mathbb{N}} \sup_{\|f\|_p=1} \|S_N f\|_p < \infty$$

In fact, this is equivalent to the  $L^p$ -convergence of Fourier series, and idea behind the proof is to instead argue about the uniform boundedness of  $S_N$ .

Note that  $\|S_N f\|_p = \|f * d_N\|_p \leq \|f\|_p \|d_N\|_1$ , so  $\|S_N\| \leq \|d_N\|_1 \rightarrow \infty$ .

For  $p = 1$ , we can argue as follows. Fix an approximate identity  $\{k_\lambda\}$  with  $k_\lambda \geq 0$ , so that

$$\|k_\lambda\|_1 = \int_0^1 |k_\lambda| = \int_0^1 k_\lambda = 1.$$

Then we can see that

$$\|S_N\| \geq \|S_N k_\lambda\|_1 = \|k_\lambda * d_N\|_1 \xrightarrow{\lambda \rightarrow \infty} \|d_N\|_1$$

since  $k_\lambda * d_N \rightarrow d_N$  in  $L^1$ . Thus  $\|S_N\| \geq \|d_N\|_1 \rightarrow \infty$ , so the partial sums cannot converge in  $L^1$ . In particular,  $\{e_n\}_{n \in \mathbb{N}}$  is *not* a Schauder basis for  $L^1(\mathbb{T})$ . A similar argument works to show that  $\{e_n\}_{n \in \mathbb{N}}$  is not a Schauder basis for  $C(\mathbb{T})$ . However, one has the following:

**Theorem 21.2.**  $\{e_n\}_{n \in \mathbb{Z}}$  is a Schauder basis for  $L^p(\mathbb{T})$  for  $1 < p < \infty$ , with the ordering

$$\mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}.$$

**Example 21.1.2** (Haar system). Another example of a Schauder basis is the following. Let

$$\varphi = \chi_{[0,1)} \quad \text{and} \quad \psi = \chi_{[0,1/2)} - \chi_{[1/2,1]}.$$

Note that  $\langle \varphi, \psi \rangle = 0$ . Then we can define

$$\psi_1 = 2^{1/2} \psi(2x) \quad \text{and} \quad \psi_2 = 2^{1/2} \psi(2x - 1),$$

which are still orthogonal with  $\varphi, \psi$  and with each other. Continuing this process, we get an orthonormal basis for  $L^2(\mathbb{T})$ . Moreover, this is a Schauder basis for  $L^p(\mathbb{T})$  for  $1 < p < \infty$  and it is unconditional, i.e. the convergence does not depend on the ordering of the functions.

**Remark.** There also exists a continuous analogue of the Haar system, the functions are called  $D_4$  and  $W_4$ . The functions  $\varphi, \psi$  from the Haar system are known as  $D_2, W_2$ . These are called *wavelets*.

# Lecture 22

## Nov. 18 — Fourier Series, Part 4

### 22.1 The Poisson Summation Formula

**Remark.** Suppose  $f \in L^1(\mathbb{R})$ . Set  $\varphi(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ . Then by Fubini-Tonelli,

$$\|\varphi\|_1 = \int_0^1 \left| \sum_{n=-\infty}^{\infty} f(x+n) \right| \leq \sum_{n=-\infty}^{\infty} \int_0^1 |f(x+n)| = \sum_{n=-\infty}^{\infty} \int_n^{n+1} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_1 < \infty,$$

so  $\varphi \in L^1(\mathbb{T})$ . Moreover, the integrals are the same if we do not take absolute values.

**Exercise 22.1.** Let  $f, \varphi$  as above. Prove that

$$\widehat{\varphi}(n) = \int_0^1 \sum_{n=-\infty}^{\infty} f(x) e^{-2\pi i n x} dx = \widehat{f}(n).$$

**Theorem 22.1** (Poisson summation). *Let  $f : \mathbb{R} \rightarrow \mathbb{C}$ . If there exist  $C, \epsilon > 0$  such that*

$$|f(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}} \quad \text{and} \quad |\widehat{f}(x)| \leq \frac{C}{(1+|x|)^{1+\epsilon}},$$

*then for every  $x \in \mathbb{R}$ ,*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}.$$

*In particular, setting  $x = 0$  one obtains*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

*Proof.* Both  $f, \widehat{f} \in L^1(\mathbb{R})$ , hence  $f, \widehat{f} \in C_0(\mathbb{R})$  by Fourier inversion. Also  $\varphi \in L^1(\mathbb{T})$  and

$$\sum_{n=-\infty}^{\infty} |\widehat{\varphi}(n)| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty,$$

so  $\widehat{\varphi} \in \ell^1(\mathbb{Z})$ . Therefore the inversion formula gives

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n) e^{2\pi i n x}$$

pointwise. (In fact one can get uniform convergence on compact sets.) □

**Remark.** Formally consider  $\mu = \sum_{n=-\infty}^{\infty} \delta_n$ , where  $\delta_n$  is the Dirac delta at  $n$ . Then

$$\langle f, \mu \rangle = \int_{-\infty}^{\infty} f(x) d\mu(x) = \sum_{n=-\infty}^{\infty} f(n).$$

A similar computation shows that

$$\langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n).$$

These series are equal by Poisson summation, so if the Parseval identity holds, then one gets

$$\langle f, \check{\mu} \rangle = \langle \widehat{f}, \mu \rangle = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} f(n) = \langle f, \mu \rangle.$$

Thus Poisson summation says that  $\mu = \check{\mu}$ .

## 22.2 Wiener's Lemma

**Remark.** Let  $C(\mathbb{T}) = \{f \text{ continuous on } \mathbb{T}\}$ , which is closed under products and

$$\|fg\|_{\infty} \leq \|f\|_{\infty} \|g\|_{\infty}.$$

In particular,  $C(\mathbb{T})$  is a commutative Banach algebra. Also note that if  $f \in C(\mathbb{T})$  and  $f(x) \neq 0$  for every  $x$ , then  $1/f \in C(\mathbb{T})$ . We say that  $C(\mathbb{T})$  is *inverse-closed*.

Recall that the Wiener (Fourier) algebra is

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f} \in \ell^1(\mathbb{Z})\} = \{\widehat{c} : c = (c_n) \in \ell^1(\mathbb{Z})\}.$$

The equality holds by the formulas  $f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x}$  and  $\check{c}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ . Note that  $A(\mathbb{T}) \subseteq C(\mathbb{T})$  is dense in the uniform norm.

On the other hand, if we define

$$\|f\|_{A(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|,$$

then for any  $f, g \in A(\mathbb{T})$ , we have

$$fg = (\widehat{f}g)^{\vee} = (\widehat{f} * \widehat{g})^{\vee}.$$

Since  $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z})$ , we have  $\widehat{f} * \widehat{g} \in \ell^1(\mathbb{Z})$ , so  $fg \in A(\mathbb{T})$ . Moreover,

$$\|fg\|_{A(\mathbb{T})} = \|\widehat{f} * \widehat{g}\|_1 \leq \|\widehat{f}\|_1 \|\widehat{g}\|_1 = \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}.$$

Thus we see that  $A(\mathbb{T})$  is also a commutative Banach algebra with respect to pointwise products.

If  $A(\mathbb{T})$  inverse-closed? In other words, if  $f \in A(\mathbb{T})$  and  $f(x) \neq 0$  for all  $x$ , must  $1/f \in A(\mathbb{T})$ ?

**Lemma 22.1.** If  $P(\xi) = \sum_{k=-N}^N a_k e^{2\pi i k \xi}$ , then  $\|P\|_{A(\mathbb{T})} \leq (2N+1)^{1/2} \|P\|_{\infty}$ .

*Proof.* By Cauchy-Schwarz, we have

$$\|P\|_{A(\mathbb{T})} = \sum_{k=-N}^N |a_k| \cdot 1 \leq (2N+1)^{1/2} \left( \sum_{k=-N}^N |a_k|^2 \right)^{1/2} = (2N+1)^{1/2} \|P\|_2$$

where the second equality is by Plancherel's theorem. The result follows since  $\|P\|_2 \leq \|P\|_{\infty}$  on  $\mathbb{T}$ .  $\square$

**Lemma 22.2** (Wiener's lemma). *If  $g \in \ell^1(\mathbb{Z})$  and  $\widehat{g}(\xi) \neq 0$  for any  $\xi$ , then there is  $h \in \ell^1(\mathbb{Z})$  such that*

$$\widehat{h}(\xi) = \frac{1}{\widehat{g}(\xi)}, \quad \xi \in \mathbb{T}.$$

Equivalently, if  $G \in A(\mathbb{T})$  and  $G(\xi) \neq 0$  for any  $\xi$ , then  $1/G \in A(\mathbb{T})$ .

*Proof.* Assume  $g \in \ell^1(\mathbb{Z})$  and  $\widehat{g}(\xi) \neq 0$  for any  $\xi$ . Let  $G = \widehat{g}$ . We prove the result in two steps:

1. Assume  $0 \leq G \leq 1$ . Then since  $G$  is nonzero,

$$d = \inf_{\xi \in \mathbb{T}} G(\xi) > 0.$$

Let  $H = 1 - G = \widehat{\delta} - \widehat{g}$ , where  $\delta$  is the delta sequence. Note that

$$\widehat{\delta}(\xi) = \sum_{n=-\infty}^{\infty} \delta_n e^{2\pi i n \xi} = \delta_0 e^{2\pi i 0 \xi} = 1.$$

Then  $H = \widehat{h} - \widehat{g} = (\delta - g)^{\wedge} \in A(\mathbb{T})$ . So if we let  $h = \delta - g \in \ell^1(\mathbb{Z})$ , then  $\widehat{h} = H$ , and

$$\|H\|_{\infty} = \|1 - G\|_{\infty} = 1 - d < 1.$$

Hence  $\sum_{n=0}^{\infty} H(\xi)^n$  converges, and  $\sum_{n=0}^{\infty} H^n = 1/(1 - H(\xi)) = 1/G(\xi) \in C(\mathbb{T})$ .

However, we need convergence in  $A(\mathbb{T})$ . Fix  $0 < \epsilon < d/2$ . Let  $p = h\chi_{[-N,N]}$  for  $N$  large enough so that  $\|p - h\|_1 < \epsilon$ . Let  $P = \widehat{p} = \sum_{n=-N}^N h(n)e^{2\pi i n \xi}$ . Let  $r = p - h$  and  $R = P - H$ . Then

$$\begin{aligned} \|P\|_{\infty} &= \|H + R\|_{\infty} \leq \|H\|_{\infty} + \|R\|_{\infty} \leq (1 - d) + \|P - H\|_{\infty} \\ &\leq (1 - d) + \|\widehat{p} - \widehat{h}\|_{\infty} \leq (1 - d) + \|p - h\|_1 = 1 - d + \epsilon. \end{aligned}$$

Now we can compute that

$$\|H^n\|_{A(\mathbb{T})} = \|(P - R)^n\|_{A(\mathbb{T})} = \left\| \sum_{j=0}^n \binom{n}{j} P^j (-R)^{n-j} \right\|_{A(\mathbb{T})}$$

where we can use the binomial theorem since  $A(\mathbb{T})$  is a commutative Banach algebra. Then

$$\|H^n\|_{A(\mathbb{T})} \leq \sum_{j=0}^n \binom{n}{j} \|P\|_{A(\mathbb{T})}^j \|R\|_{A(\mathbb{T})}^{n-j} \leq \sum_{j=0}^n \binom{n}{j} (2Nj+1)^{1/2} \|P^j\|_{\infty} \epsilon^{n-j},$$

where the second equality is by Lemma 22.1 and  $\|R\|_{A(\mathbb{T})} = \|p - h\|_1 < \epsilon$ . Thus

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn+1)^{1/2} \sum_{j=1}^n \binom{n}{j} \|P\|_{\infty}^j \epsilon^{n-j} = (2Nn+1)^{1/2} (\|P\|_{\infty} + \epsilon)^n,$$

where  $\|P^j\|_{\infty} \leq \|P\|_{\infty}^j$  by submultiplicativity. Since  $\|P\|_{\infty} \leq 1 - d + \epsilon$ ,

$$\|H^n\|_{A(\mathbb{T})} \leq (2Nn+1)^{1/2} (1 - d + 2\epsilon)^n,$$

where  $1 - d + 2\epsilon < 1$ . So  $\sum_{n=0}^{\infty} \|H^n\|_{A(\mathbb{T})} < \infty$ , so  $\sum_{n=0}^{\infty} H^n$  converges in  $A(\mathbb{T})$  to  $1/(1 - H) = 1/G$ .

One can reduce the general case to this first case, see the course notes.  $\square$

## 22.3 Distributions

**Remark.** Let  $1 \leq p < \infty$ . Note that  $L^p(\mathbb{R})^* \cong L^{p'}(\mathbb{R})$ , where  $L^p(\mathbb{R})^*$  is the dual space of  $L^p(\mathbb{R})$ , i.e. the space of bounded linear functions on  $L^p(\mathbb{R})$ . If we fix  $g \in L^{p'}(\mathbb{R})$ , then we can define

$$\mu_g(f) = \langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx,$$

which is linear in  $f$  and antilinear in  $g$  (this is known as a *sesquilinear form*).

Recall that  $X^* = \{\text{bounded linear functionals } \mu \text{ on } X\}$ , which is equivalent to continuous linear functionals when  $X$  is a Banach space. We consider the following space of distributions:

**Definition 22.1.** Define the following:

- $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ , the space of *distributions*.
- $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$ , the space of *tempered distributions*.
- $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$ , the space of *compactly supported distributions*.

**Example 22.1.1.** If  $f \in C_c^\infty(\mathbb{R})$ , then define  $\langle f, \delta \rangle = f(0)$ . This is the Dirac delta as a distribution.

**Remark.** Recall that the Schwartz space is

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R})\}.$$

This is *not* a Banach space (it does not have a norm). But we can define seminorms

$$\rho_{m,n}(f) = \|x^m f^{(n)}\|_\infty, \quad m, n \geq 0.$$

There is no way to combine these seminorms into a single norm.<sup>1</sup> However, one can define a metric:

$$d(f, g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\rho_{m,n}(f - g)}{1 + \rho_{m,n}(f - g)}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Note that convergence with respect to  $d$  is the same as convergence in  $\rho_{m,n}$  for every  $m, n$ :

$$\lim_{k \rightarrow \infty} \rho_{m,n}(f - f_k) = 0.$$

**Definition 22.2.** We say that  $f_k \rightarrow f$  in  $\mathcal{S}(\mathbb{R})$  if for every  $m, n \geq 0$ ,

$$\rho_{m,n}(f - f_k) = \|x^m f^{(n)}(x) - x^m (f_k)^{(n)}(x)\|_\infty \rightarrow 0$$

Therefore, a linear functional  $\mu : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$  is *continuous* if  $f_k \rightarrow f$  in  $\mathcal{S}(\mathbb{R})$  implies  $\langle f_k, \mu \rangle \rightarrow \langle f, \mu \rangle$ .

**Remark.** Since convergence in  $\mathcal{S}(\mathbb{R})$  is a very strong condition, we expect that it is easy for  $\mu$  to satisfy the above condition. This intuitively explains why  $\mathcal{S}'(\mathbb{R})$  is so large.

**Example 22.2.1.** Assume that  $f_k \rightarrow f$  in  $\mathcal{S}(\mathbb{R})$ . Then  $\langle f_k, \delta \rangle = f_k(0)$ , so

$$|\langle f - f_k, \delta \rangle| = |f(0) - f_k(0)| \leq \rho_{0,0}(f - f_k) \rightarrow 0.$$

Thus we see that  $\delta \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$ .

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<sup>1</sup>Compare this with  $C_b^1(\mathbb{R}) = \{f : \|f\|_\infty, \|f'\|_\infty < \infty\}$ , where one can define a norm  $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$ .

# Lecture 23

## Nov. 20 — Distributions, Part 2

### 23.1 Convergence with Families of Seminorms

**Example 23.0.1.** Like  $\mathcal{S}(\mathbb{R})$ , many other spaces are defined by a family of seminorms. Recall that

$$L^1_{\text{loc}}(\mathbb{R}) = \{\text{measurable } f : f \text{ is integrable on compact } K \subseteq \mathbb{R}\}.$$

If we define  $\rho_k(f) = \|f\chi_K\|_1$ , then  $f \in L^1_{\text{loc}}(\mathbb{R})$  if and only if  $\rho_K(f) < \infty$ . In fact, it is enough to consider  $\rho_N(f) = \|f\chi_{[-N,N]}\|_1$ . Given such a countable family of seminorms, one can always define a metric

$$d(f, g) = \sum_{N=1}^{\infty} 2^{-N} \frac{\rho_N(f - g)}{1 + \rho_N(f - g)}$$

which satisfies  $d(f, f_k) \rightarrow 0$  if and only if  $\rho_N(f - f_k) \rightarrow 0$  for every  $N$ .

**Remark.** Suppose  $\|\cdot\|$  is a norm on  $L^1_{\text{loc}}(\mathbb{R})$ , and consider  $\chi_{[k,k+1]}$ . Then  $\rho_N(\chi_{[k,k+1]}) \rightarrow 0$ , so

$$c_k \chi_{[k,k+1]} \longrightarrow 0$$

in  $L^1_{\text{loc}}(\mathbb{R})$  for every choice of  $c_k$ . But then  $\|c_k \chi_{[k,k+1]}\| \rightarrow 0$ , a contradiction for  $c_k = 1/\|\chi_{[k,k+1]}\|$ .

**Example 23.0.2.** For  $C^\infty(\mathbb{R})$ , we can take  $\rho_{N,n}(f) = \|f^{(n)}\chi_{[-N,N]}\|_\infty$ . Then

$$C^\infty(\mathbb{R})^* = \{\text{continuous linear functionals } \mu : C^\infty(\mathbb{R}) \rightarrow \mathbb{C}\} = \mathcal{E}'(\mathbb{R})$$

is the space of compactly supported distributions, where  $\mu$  is continuous if whenever  $f_k \rightarrow f$  in  $C^\infty(\mathbb{R})$ , we have  $\mu(f_k) = \langle \mu, f_k \rangle \rightarrow \langle \mu, f \rangle$ .

**Remark.** For  $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ , it is better to think of  $C_c^\infty(\mathbb{R}) = \bigcup_{N=1}^{\infty} C_c^\infty([-N, N])$ .

**Definition 23.1.** Let  $\{\rho_\alpha\}_{\alpha \in J}$  be a family of seminorms on a vector space  $X$ .

- (a)  $f_k \rightarrow f$  in  $X$  means that  $\rho_\alpha(f - f_k) \rightarrow 0$  for every  $\alpha \in J$ .
- (b)  $X$  is *Hausdorff* if  $\rho_\alpha(f) = 0$  for every  $f \in X$  if and only if  $f = 0$ .

**Definition 23.2.** Let  $\{\rho_n\}_{n \in \mathbb{N}}$  be a countable, Hausdorff family of seminorms on  $X$ . If the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f - g)}{1 + \rho_n(f - g)}$$

is complete, then we call  $X$  a *Fréchet space*.

**Exercise 23.1.**  $\mathcal{S}(\mathbb{R})$  and  $C^\infty(\mathbb{R})$  are Fréchet spaces (but  $C_c^\infty(\mathbb{R})$  is not).

**Remark.** Recall that if  $\mu : X \rightarrow \mathbb{C}$  is a linear functional on a normed space  $X$ , then  $\mu$  is continuous if and only if  $\mu$  is bounded (i.e. there exists  $C > 0$  such that  $|\langle f, \mu \rangle| \leq C\|f\|$  for every  $f \in X$ ).

**Theorem 23.1.** Let  $X$  be a Fréchet space with seminorms  $\{\rho_n\}_{n \in \mathbb{N}}$  and  $\mu : X \rightarrow \mathbb{C}$  a linear functional. Then the following are equivalent:

1.  $\mu$  is continuous, i.e.  $f_k \rightarrow f$  implies  $\langle f_k, \mu \rangle \rightarrow \langle f, \mu \rangle$  as  $k \rightarrow \infty$ .
2.  $f_k \rightarrow 0$  in  $X$  implies  $\langle f_k, \mu \rangle \rightarrow 0$ .
3. There exists  $C > 0$  and  $N \in \mathbb{N}$  (depending on  $\mu$ ) such that  $|\langle f, \mu \rangle| \leq C \sum_{n=1}^N \rho_n(f)$ .

*Proof.* (1  $\Leftrightarrow$  2) This follows from linearity.

(3  $\Rightarrow$  2) Suppose  $f_k \rightarrow 0$ , so  $\rho_n(f_k) \rightarrow 0$  for every  $n$ . So

$$|\langle f_k, \mu \rangle| \leq C \sum_{n=1}^N \rho_n(f_k) \xrightarrow{k \rightarrow \infty} 0,$$

since each of the finitely many terms converge to 0.

(2  $\Rightarrow$  3) We prove the contrapositive. Suppose (3) is false. So for every  $C = N = k \in \mathbb{N}$ , there exists  $f_k \in X$  for which  $|\langle f_k, \mu \rangle| > k \sum_{n=1}^k \rho_n(f_k)$ . Set  $\varphi_k = f_k / |\langle f_k, \mu \rangle|$ . Then

$$\sum_{n=1}^k \rho_n(\varphi_k) = \frac{1}{|\langle f_k, \mu \rangle|} \sum_{n=1}^k \rho_n(f_k) < \frac{1}{|\langle f_k, \mu \rangle|} \cdot \frac{|\langle f_k, \mu \rangle|}{k} = \frac{1}{k} \xrightarrow{k \rightarrow \infty} 0.$$

Thus  $\rho_n(\varphi_k) \rightarrow 0$  as  $k \rightarrow \infty$  for every  $n \in \mathbb{N}$ . So  $\varphi_k \rightarrow 0$  in  $X$ . But  $\langle \varphi_k, \mu \rangle = 1 \not\rightarrow 0$ .  $\square$

**Example 23.2.1.** What does the topology look like given a family of seminorms, i.e. what are the open sets? Consider seminorms  $\rho_1(x_1, x_2) = |x_1|$  and  $\rho_2(x_1, x_2) = |x_2|$ . Consider strips

$$B_r^i(x_1, x_2) = \{(y_1, y_2) : |y_i - x_i| < r\}$$

for each seminorm  $\rho_i$ . Then the open balls are of the form

$$B_r^1(x_1, x_2) \cap B_r^2(x_1, x_2).$$

A basis in the general case involves intersections of finitely many strips (as in the product topology).

## 23.2 Distributional Derivatives

**Example 23.2.2.** If  $f, g$  are differentiable and decay at  $\infty$  (think  $f, g \in \mathcal{S}(\mathbb{R})$ ), then

$$\langle f, g' \rangle = \int_{-\infty}^{\infty} f(x) \overline{g'(x)} dx = f(x)g(x)|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \overline{g(x)} dx = -\langle f', g \rangle,$$

by integration by parts, where  $f(x)g(x)|_{x=-\infty}^{\infty} = 0$  by the decay of  $f, g$ . Now define  $\delta' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$  by

$$\langle f, \delta' \rangle := -\langle f', \delta \rangle = -f'(0).$$

for  $f \in \mathcal{S}(\mathbb{R})$ . Then  $\delta'$  is a linear functional, and

$$|\langle f, \delta' \rangle| = |-f'(0)| \leq \|f'\|_\infty = \|x^0 f^{(1)}(x)\|_\infty = \rho_{0,1}(f) \leq 1 \sum_{m=0}^0 \sum_{n=0}^1 \rho_{m,n}(f).$$

Hence  $\delta'$  is bounded on  $\mathcal{S}(\mathbb{R})$ , so  $\delta' \in \mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$ .

**Proposition 23.1.** *Fix  $\mu \in \mathcal{S}'(\mathbb{R})$ , and define  $\mu' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$  by*

$$\langle f, \mu' \rangle = -\langle f', \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

*Then  $\mu' \in \mathcal{S}'(\mathbb{R})$ . In particular, every tempered distribution has a (distributional) derivative.*

*Proof.* Since  $\mu$  is bounded, there are  $C, N > 0$  such that  $|\langle f, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \rho_{m,n}(f)$ . Then

$$|\langle f, \mu' \rangle| = |\langle f', \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \rho_{m,n}(f') = C \sum_{m=0}^M \sum_{n=0}^N \|x^m (f')^{(n)}\|_\infty = C \sum_{m=0}^M \sum_{n=0}^N \|x^m f^{n+1}\|_\infty,$$

so  $\mu'$  is bounded. Thus  $\mu' \in \mathcal{S}'(\mathbb{R})$  also.  $\square$

**Example 23.2.3.** Consider the Heaviside function  $H = \chi_{[0,\infty)}$ . We can define

$$\langle f, H \rangle = \int_{-\infty}^{\infty} f(x) \overline{H(x)} dx = \int_0^{\infty} f(x) dx.$$

Then we can check that

$$\begin{aligned} |\langle f, H \rangle| &\leq \int_0^{\infty} |f(x)| dx \leq \int_0^1 |f(x)| dx + \int_1^{\infty} \frac{|f(x)|}{x^2} x^2 dx \\ &\leq \|f\|_\infty + \|x^2 f(x)\|_\infty \int_1^{\infty} \frac{1}{x^2} dx \leq C \sum_{m=0}^2 \sum_{n=0}^0 \|x^m f^{(n)}\|_\infty. \end{aligned}$$

Thus  $H$  is bounded, so  $H \in \mathcal{S}'(\mathbb{R})$ . So  $H$  has a distributional derivative  $H' : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ , where

$$\langle f, H' \rangle := -\langle f', H \rangle = - \int_0^{\infty} f'(x) dx = -(f(\infty) - f(0)) = f(0) = \langle f, \delta \rangle.$$

In particular, we see that  $H' = \delta$  as tempered distributions.

**Remark.** Note that  $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$  and  $L^1_{\text{loc}}(\mathbb{R}) + \text{polynomial growth} \subseteq \mathcal{S}'(\mathbb{R})$ , so each of the functions in the latter has distributional derivatives.

**Exercise 23.2.** If  $g$  is smooth, then  $g'$  (as a tempered distribution) is the usual derivative.

**Remark.** To avoid confusion, we sometimes write  $D\mu$  for the distributional derivative and  $g'$  for the pointwise a.e. derivative.

# Lecture 24

## Nov. 25 — Distributions, Part 3

### 24.1 Space of Distributions

**Remark.** Let  $\mathcal{D}'(\mathbb{R}) = C_c^\infty(\mathbb{R})^*$ . The topology on this space is *not* given by a family of seminorms. Instead, we write  $C_c^\infty(\mathbb{R}) = \bigcup_{K \text{ compact}} C^\infty(K)$ . Note that  $C^\infty(K)$  is a Fréchet space with seminorms

$$\rho_{K,n}(f) = \|f^{(n)}(x)\chi_K(x)\|_\infty$$

for  $f \in C^\infty(K) = \{f \in C^\infty(\mathbb{R}) : \text{supp}(f) \subseteq K\}$ .

**Theorem 24.1.**  $f_k \rightarrow f$  in  $C_c^\infty(\mathbb{R})$  if and only if there exists a compact set  $K \subseteq \mathbb{R}$  such that  $f_k \in C^\infty(K)$  and  $f_k \rightarrow f$  in  $C^\infty(K)$ .

**Theorem 24.2.** Given a linear function  $\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$ , the following are equivalent:

1.  $\mu$  is continuous;
2.  $\mu|_{C^\infty(K)}$  is continuous for each compact  $K$ ;
3. for all compact  $K \subseteq \mathbb{R}$ , there exists  $C_K \geq 0$ ,  $N_K \geq 0$  such that

$$|\langle f, \mu \rangle| \leq C_K \sum_{n=0}^{N_K} \|f^{(n)}(x)\chi_K(x)\|_\infty, \quad \text{for } f \in C^\infty(K).$$

**Definition 24.1.** If there exists a single  $N$  that can be used for each compact set  $K$  in Theorem 24.2(3), then we say that  $\mu$  has *finite order*. The *order* of  $\mu$  is the smallest such  $N$ .

**Example 24.1.1.** We have the following:

1. The order of  $\delta^{(j)}$  is  $j$ .
2. Consider the  $\delta$ -train  $\mu = \sum_{n=-\infty}^{\infty} \delta_n$ , so  $\langle f, \mu \rangle = \sum_{n=-\infty}^{\infty} f(n)$ . Note that this sum is finite since  $f \in C^\infty(K)$  for some compact  $K$ . In fact,  $|\langle f, \mu \rangle| \leq C_K \|f\|_\infty$ , so the order is 0.
3. Let  $\nu = \sum_{n=1}^{\infty} \delta_n^{(n)}$ , so  $\langle f, \nu \rangle = \sum_{n=1}^{\infty} (-1)^n f^{(n)}(n)$ . One can check that  $\nu$  has infinite order.

**Remark.** As sets, we have  $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$ . What about their duals?

**Theorem 24.3.** We have the following:

1.  $f_k \rightarrow f$  in  $C_c^\infty(\mathbb{R})$  implies  $f_k \rightarrow f$  in  $\mathcal{S}(\mathbb{R})$ .
2.  $f_k \rightarrow f$  in  $\mathcal{S}(\mathbb{R})$  implies  $f_k \rightarrow f$  in  $C^\infty(\mathbb{R})$ .

3.  $\mu \in \mathcal{S}'(\mathbb{R})$  implies  $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$ .

4.  $\mu \in \mathcal{E}'(\mathbb{R})$  implies  $\mu|_{\mathcal{S}(\mathbb{R})} \in \mathcal{S}'(\mathbb{R})$ .

In particular, we have the containments  $\mathcal{D}'(\mathbb{R}) \supseteq \mathcal{S}'(\mathbb{R}) \supseteq \mathcal{E}'(\mathbb{R})$ .

## 24.2 Functions as Distributions

**Theorem 24.4.**  $L^1_{\text{loc}}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ , where the embedding is given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx, \quad f \in C_c^\infty(\mathbb{R}).$$

*Proof.* If  $g \in L^1_{\text{loc}}(\mathbb{R})$  and  $f \in C^\infty(K)$ , then

$$\langle f, g \rangle \leq \int_K |f(x)g(x)| dx \leq \|f\|_\infty \|g\chi_K\|_1.$$

Thus we see that  $g$  defines a continuous linear functional on  $C_c^\infty(\mathbb{R})$ .  $\square$

**Example 24.1.2.** Note that  $\delta \in \mathcal{D}'(\mathbb{R})$ , but it cannot be identified with a function in  $L^1_{\text{loc}}(\mathbb{R})$ .

**Example 24.1.3.** Consider the space

$$\begin{aligned} M_b(\mathbb{R}) &= \{\text{bounded Radon measures on } \mathbb{R}\} \\ &= \{\text{bounded locally finite Borel measures on } \mathbb{R}\}. \end{aligned}$$

If  $\mu \in M_b(\mathbb{R})$  and  $f \in C^\infty(K)$ , then we can define a linear functional by

$$\langle f, \mu \rangle = \int f(x) d\mu(x).$$

Then we have that

$$|\langle f, \mu \rangle| \leq \int_K |f(x)| d|\mu|(x) \leq \|f\|_\infty |\mu|(K),$$

which is finite as  $\mu$  is a bounded Radon measure.

**Example 24.1.4.** We have  $1/x \notin L^1_{\text{loc}}(\mathbb{R})$ , but we can define  $\text{pv}(1/x) : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$  by

$$\langle f, \text{pv}(1/x) \rangle = \lim_{T \rightarrow \infty} \int_{1/T \leq |x| \leq T} \frac{f(x)}{x} dx, \quad f \in C^\infty(K).$$

One can show that the above limit exists, so  $\text{pv}(1/x) \in \mathcal{D}'(\mathbb{R})$ .

## 24.3 Operations on Distributions

**Remark.** Suppose  $g \in L^1_{\text{loc}}(\mathbb{R})$ , and set  $T_a g(x) = g(x - a)$ . If  $f \in C_c^\infty(\mathbb{R})$ , then

$$\langle f, T_a g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x - a)} dx = \int_{-\infty}^{\infty} f(x + a) \overline{g(x)} dx = \langle T_{-a} f, g \rangle.$$

**Definition 24.2.** If  $\mu \in \mathcal{D}'(\mathbb{R})$ , then define  $T_a\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$  by  $\langle f, T_a\mu \rangle := \langle T_{-a}f, \mu \rangle$  for  $f \in C_c^\infty(\mathbb{R})$ .

**Remark.** Note that for  $f \in C^\infty(K)$ , we have

$$|\langle f, T_a\mu \rangle| = |\langle T_{-a}f, \mu \rangle| \leq C_{K+a} \sum_{n=0}^{N_{K+a}} \|T_{-a}f\|_\infty \leq C_{K+a} \sum_{n=0}^{N_{K+a}} \|f\|_\infty$$

since  $T_{-a}f \in C^\infty(K+a)$ . Thus  $T_a\mu$  is a continuous linear functional on  $C_c^\infty(\mathbb{R})$ ,

**Remark.** One can similarly define other operations on distributions, e.g. dilation, modulation, etc. For example, the *involution*  $\tilde{g}(x) = \overline{g(-x)}$  satisfies

$$\langle f, \tilde{g} \rangle = \int f(x) \overline{\tilde{g}(-x)} dx = \overline{\int f(x) g(-x) dx} = \overline{\int f(-x) g(x) dx} = \int \tilde{f}(x) \overline{g(x)} dx = \overline{\langle \tilde{f}, g \rangle}.$$

Thus if  $\mu \in \mathcal{D}'(\mathbb{R})$ , then we can define  $\langle f, \tilde{\mu} \rangle := \overline{\langle \tilde{f}, \mu \rangle}$  for  $f \in C_c^\infty(\mathbb{R})$ .

**Example 24.2.1.** Consider a translation of  $\delta$ :

$$\langle f, T_a\delta \rangle := \langle T_{-a}f, \delta \rangle = T_{-a}f(0) = f(a).$$

But sometimes by abuse of notation we may write

$$\langle f, T_a\delta \rangle = \int f(x) \delta(x-a) dx = \int f(x+a) \delta(x) dx = f(a).$$

This should be understood in the distributional sense, the above integral is *not* a Lebesgue integral.

## 24.4 Products and Convolution of Distributions

**Remark.** Suppose  $g \in L^1_{\text{loc}}(\mathbb{R})$ ,  $\theta \in C^\infty(\mathbb{R})$ . Then

$$\langle f, \theta g \rangle = \int f(x) \overline{\theta(x)g(x)} dx = \int (f(x) \overline{\theta(x)}) \overline{g(x)} dx = \langle f\bar{\theta}, g \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Note that  $f\bar{\theta} \in C_c^\infty(\mathbb{R})$ , so the above definition makes sense.

**Definition 24.3.** If  $\mu \in \mathcal{D}'(\mathbb{R})$  and  $\theta \in C^\infty(\mathbb{R})$ , then define  $\theta\mu : C_c^\infty(\mathbb{R}) \rightarrow \mathbb{C}$  by  $\langle f, \theta\mu \rangle := \langle f\bar{\theta}, \mu \rangle$ .

**Remark.** We can check that for  $f \in C^\infty(K)$  (so  $f\bar{\theta} \in C^\infty(K)$ ),

$$\begin{aligned} |\langle f, \theta\mu \rangle| &= |\langle f\bar{\theta}, \mu \rangle| \leq C_K \sum_{n=0}^{N_K} \|(f\bar{\theta})^{(n)}\|_\infty \leq C_K \sum_{n=0}^{N_K} \sum_{j=0}^n \binom{n}{j} \|f^{(j)}\bar{\theta}^{(n-j)}\|_\infty \\ &= C_K \sum_{n=0}^{N_K} \sum_{j=0}^n \binom{n}{j} \|\theta^{(n-j)}\|_\infty \|f^{(j)}\|_\infty. \end{aligned}$$

Thus we see that  $\theta\mu \in \mathcal{D}'(\mathbb{R})$ .

**Remark.** Now consider  $g \in L^1_{\text{loc}}(\mathbb{R})$ ,  $f \in C_c^\infty(\mathbb{R})$ . Then we have

$$(f * g)(x) = \int f(x-y)g(y) dy = \int \overline{\tilde{f}(y-x)} g(y) dy = \overline{\langle T_x\tilde{f}, g \rangle}.$$

**Definition 24.4.** The *convolution* of  $\mu \in \mathcal{D}'(\mathbb{R})$  with  $f \in C_c^\infty(\mathbb{R})$  is

$$(f * \mu)(x) := \overline{\langle T_x \tilde{f}, \mu \rangle}, \quad x \in \mathbb{R}.$$

**Theorem 24.5.** If  $\mu \in \mathcal{D}'(\mathbb{R})$  and  $f \in C_c^\infty(\mathbb{R})$ , then:

1. Convolution commutes with translation, i.e.  $T_a(f * \mu) = (T_a f) * \mu = f * (T_a \mu)$ .
2.  $f * \mu \in C^\infty(\mathbb{R})$  and  $(f * \mu)' = f' * \mu$  (and also  $(f * \mu)' = f * D\mu$ ).

**Remark.** Suppose  $f, g, h$  are “nice” functions. Then

$$\begin{aligned} \langle f, g * h \rangle &= \int f(x) \overline{(g * h)(x)} dx = \iint f(x) \overline{g(x-y)h(y)} dy dx \\ &= \iint (f(x) \tilde{g}(y-x) dx) \overline{h(y)} dy = \int (f * \tilde{g})(y) \overline{h(y)} dy = \langle f * \tilde{g}, h \rangle. \end{aligned}$$

**Theorem 24.6.** If  $\mu \in \mathcal{D}'(\mathbb{R})$  and  $f, g \in C_c^\infty(\mathbb{R})$ , then  $\langle f, g * \mu \rangle = \langle f * \tilde{g}, \mu \rangle$ .

# Lecture 25

## Dec. 2 — Distributions, Part 4

### 25.1 More on Convolutions of Distributions

**Remark.** Recall that if  $\nu \in \mathcal{D}'(\mathbb{R})$  and  $f, g \in C_c^\infty(\mathbb{R})$ , then we define

$$\langle f, g * \nu \rangle := \langle f * \tilde{g}, \nu \rangle.$$

We want to make a similar definition for  $\mu, \nu \in \mathcal{D}'(\mathbb{R})$ :

$$\langle f, \mu * \nu \rangle := \langle f * \tilde{\mu}, \nu \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Note that we only know  $f * \mu \in C^\infty(\mathbb{R})$ , not necessarily  $C_c^\infty(\mathbb{R})$ . We know

$$\text{supp}(\mu) = \bigcap \{\text{closed sets } F \subseteq \mathbb{R} : \mu = 0 \text{ off } F\}.$$

**Theorem 25.1.**  $\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^* = \{\mu \in \mathcal{D}'(\mathbb{R}) : \mu \text{ is compactly supported}\}$ .

**Definition 25.1.** If  $\mu, \nu \in \mathcal{D}'(\mathbb{R})$  and at least one is compactly supported, then we can define

$$\langle f, \mu * \nu \rangle := \langle f * \tilde{\mu}, \nu \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

**Exercise 25.1.** Show that if at least one of  $\nu$  or  $\mu$  is compactly supported, then

$$f * (\mu * \nu) = (f * \mu) * \nu, \quad f \in C_c^\infty(\mathbb{R}).$$

**Exercise 25.2.** Show that  $(H * \delta') * 1 \neq H * (\delta' * 1)$ , where  $H$  is the Heaviside function,  $\delta$  is the Dirac delta, and  $1$  is the constant function.

### 25.2 More on Distributional Derivatives

**Lemma 25.1.** Let  $g \in C(\mathbb{R})$  such that  $Dg \in C(\mathbb{R})$ . Then  $g \in C^1(\mathbb{R})$  and  $Dg = g'$ .

*Proof.* We give the idea of the proof. Let  $h \in \mathbb{R}$  and suppose  $f \in C_c^\infty(\mathbb{R})$ . Then

$$\int_0^h (\tilde{f} * g)'(t) dt = (\tilde{f} * g)(h) - (\tilde{f} * g)(0)$$

by the fundamental theorem of calculus. Check as an exercise that  $(\tilde{f} * g)'(t) = (\tilde{f} * Dg)(t)$ , then

$$\int \overline{f(x)} g(x) dx = \overline{\langle f, g \rangle} = (\tilde{f} * g)(h) - \int_0^h (\tilde{f} * Dg)(t) dt.$$

One can continue to show that  $Dg$  and  $g'$  agree. □

**Exercise 25.3.** Show that if  $\mu \in \mathcal{D}'(\mathbb{R})$  and  $f \in C_c^\infty(\mathbb{R})$ , then

$$\overline{\langle f, \mu \rangle} = (\tilde{f} * \mu)(t) - \int_0^h (\tilde{f} * D\mu)(t) dt.$$

**Theorem 25.2.** If  $\mu \in \mathcal{D}'(\mathbb{R})$ , then  $D\mu \in C(\mathbb{R})$  (i.e. there exists  $g \in C(\mathbb{R})$  such that  $\langle f, g \rangle = \langle f, D\mu \rangle = -\langle f', \mu \rangle$ ) if and only if  $\mu \in C^1(\mathbb{R})$ .

**Theorem 25.3.** Fix  $\mu \in \mathcal{D}'(\mathbb{R})$ . Then for each compact  $K \subseteq \mathbb{R}$ , there exists  $g \in C(\mathbb{R})$  and  $k \geq 0$  such that  $\langle f, \mu \rangle = \langle f, D^k g \rangle$  for all  $f \in C^\infty(K)$ .

**Remark.** Theorem 25.3 says that in some sense, distributions are exactly the derivatives of functions.

## 25.3 Distributional Fourier Transform

**Remark.** Recall that the Fourier transform is unitary on  $L^2(\mathbb{R})$ , i.e.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi = \langle \hat{f}, \hat{g} \rangle.$$

Also recall that if  $f \in \mathcal{S}(\mathbb{R})$ , then  $\hat{f} \in \mathcal{S}(\mathbb{R})$  also (this is not the case with  $C_c^\infty(\mathbb{R})$ ).

**Definition 25.2.** The *Fourier transform* of a tempered distribution  $\mu \in \mathcal{S}'(\mathbb{R})$  is implicitly defined by

$$\langle \hat{f}, \hat{\mu} \rangle := \langle f, \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

This defines  $\hat{\mu}$  on all of  $\mathcal{S}(\mathbb{R})$  since  $\hat{f}$  can be any element of  $\mathcal{S}(\mathbb{R})$ . More explicitly,

$$\langle f, \hat{\mu} \rangle = \langle \check{f}, \mu \rangle, \quad f \in \mathcal{S}(\mathbb{R}).$$

**Lemma 25.2.** If  $\mu \in \mathcal{S}'(\mathbb{R})$ , then  $\hat{\mu} \in \mathcal{S}'(\mathbb{R})$ .

*Proof.* Since  $\mu \in \mathcal{S}'(\mathbb{R})$ , there exist  $C, M, N \geq 0$  such that

$$|\langle f, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \|x^m f^{(n)}\|_\infty.$$

Then we can compute

$$|\langle f, \hat{\mu} \rangle| = |\langle \check{f}, \mu \rangle| \leq C \sum_{m=0}^M \sum_{n=0}^N \|x^m \check{f}^{(n)}\|_\infty.$$

For  $m = n = 0$ , we have  $\|\check{f}\|_\infty \leq \|f\|_1 \leq C'(\|f\|_\infty + \|x^2 f(x)\|_\infty)$  (split into  $|x| \leq 1$  and  $|x| > 1$ ). We can apply a similar argument to the other  $m, n$  to bound the remaining terms.  $\square$

**Example 25.2.1.** We compute  $\widehat{\delta}$ . Let  $f \in \mathcal{S}(\mathbb{R})$ . Then

$$\langle \hat{f}, \widehat{\delta} \rangle = \langle f, \delta \rangle = f(0) = (\hat{f})^\vee(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi = \langle \hat{f}, 1 \rangle.$$

This is true for every  $f \in \mathcal{S}(\mathbb{R})$ , hence  $\widehat{\delta} = 1$ .

**Example 25.2.2.** We compute  $\widehat{1}$ . For  $f \in \mathcal{S}(\mathbb{R})$ , we have

$$\langle \widehat{f}, \widehat{1} \rangle = \langle f, 1 \rangle = \int f(x) dx = \widehat{f}(0) = \langle \widehat{f}, \delta \rangle.$$

Thus we see that  $\widehat{1} = \delta$ .

**Example 25.2.3.** Formally, we can write

$$\widehat{\delta}(\xi) = \int_{-\infty}^{\infty} \delta(x) e^{-2\pi i \xi x} dx = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} d\delta(x) = e^{-2\pi i \xi \cdot 0} = 1.$$

This somewhat makes sense, with some hand-waving. On the other hand

$$\widehat{1}(\xi) = \int_{-\infty}^{\infty} 1 \cdot e^{-2\pi i \xi x} dx = \delta(\xi)$$

does not really make sense at all.

**Exercise 25.4.** Show that  $\mu \mapsto \widehat{\mu}$  is a continuous bijection  $\mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ .

**Exercise 25.5.** Show that if  $1 \leq p \leq 2$  and  $f \in L^p(\mathbb{R})$ , then the usual Fourier transform agrees with the distributional Fourier transform (if we interpret the usual  $\widehat{f} \in L^{p'}(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ ).

**Remark.** If  $2 < p \leq \infty$ , then we still have  $L^p(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ , so every  $f \in L^p(\mathbb{R})$  has a distributional Fourier transform  $\widehat{f} \in \mathcal{S}'(\mathbb{R})$ , but in general it cannot be identified with a function.

**Remark.**  $\mathcal{F}L^\infty(\mathbb{R})$  is called the space of *pseudo-measures*.

**Exercise 25.6.** Show that if  $\mu \in \mathcal{S}'(\mathbb{R})$  and  $\theta \in \mathcal{S}(\mathbb{R})$ , then

$$(\theta\mu)^\wedge = \widehat{\theta} * \widehat{\mu} \quad \text{and} \quad (\theta * \mu)^\wedge = \widehat{\theta}\widehat{\mu}.$$

Moreover,  $(D\mu)^\wedge = 2\pi i \xi \widehat{\mu}$  and  $D\widehat{\mu} = (-2\pi i x \mu)^\wedge$ .

**Theorem 25.4.**  $(\sum_{n=-\infty}^{\infty} \delta_n)^\wedge = \sum_{n=-\infty}^{\infty} \delta_n$ .

*Proof.* We can compute that

$$\left\langle \widehat{f}, \left( \sum_{n=-\infty}^{\infty} \delta_n \right)^\wedge \right\rangle = \left\langle f, \sum_{n=-\infty}^{\infty} \delta_n \right\rangle = \sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) = \left\langle \widehat{f}, \sum_{n=-\infty}^{\infty} \delta_n \right\rangle,$$

where we used the Poisson summation formula in the third equality.  $\square$