MATH 8803: Representation Theory I

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Contents

Lecture 1

Aug. 18 — Historical Perspective

1.1 Origin of Representation Theory

One motivation for representation theory is symmetries in physics. From a mathematical perspective, we consider *groups* and *algebras* (a vector space with a bilinear operation). In this course, we will study two types of groups:

1. *finite groups*, e.g. the symmetric group;
2. *Lie groups*, e.g. the rotation group.

Definition 1.1. A *representation* of a group G is a homomorphism $G \to \mathrm{End}(V)$, where V is some finite-dimensional vector space.

The history of representation theory is as follows:

1. In the late 19th century, people were interested in *crystallography*, in particular crystallographic groups and their classification. There are related objects called *Bieberbach groups* (e.g. $O(n)$ with translations, i.e. $\mathbb{R}^n \rtimes O(n)$).

   Sophus Lie discovered *Lie groups* in his main manuscript "Transformation groups." From Lie groups, one then derives *Lie algebras*.

2. In the early 20th century (1905), *special relativity* was discovered, which involves the *Lorentz group* $\mathrm{SO}(1,3)$ (the transformations preserving the form $-t^2 + x^2 + y^2 + z^2$). This is a Lie group.

   Around the same time, E. Cartan developed the modern theory of *semisimple Lie groups* and *Lie algebras*, and H. Weyl studied their representations.

3. In the period 1920–1930, quantum ("matrix") mechanics was discovered. Here one has a Hilbert space $\mathcal{H}$ and a self-adjoint Hamiltonian (energy) operator H on $\mathcal{H}$. The symmetry operator A satisfies the commutator relation $[H, A] = 0$, and if we set $U = e^{iA}$, we have $UHU^\dagger = H$.

4. After the discovery of *spin* by W. Pauli, E. Wigner realized that spin was directly related to the representation theory of the universal cover $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

   In the 1960s, there was a "zoo" of elementary particles. M. Gell-Mann and Y. Neeman realized that all of these can be described by representations of $\mathrm{SU}(3)$. The led to the discovery of *quarks* and the later notion of grand unified theories and string theory in the 1970s.

   There are also connections to condensed matter theory and quantum information.

This course will cover the following topics:

1. basics about associative algebras and their representations, finite groups and their representations in general, the symmetric group and its representations, Young tableaux;
2. Lie groups and Lie algebras;
3. the structure of semisimple Lie algebras;
4. representations of $\mathrm{SL}(n)$.

1.2 Introduction to Lie Groups and Lie Algebras

In general, groups are complicated, whereas algebras are less complicated. We begin with finite groups.

Definition 1.2. Let G be a finite group and $\mathbb{F}$ a field. The *group algebra* $\mathbb{F}G$ is

$$\mathbb{F}G = \left\{ \sum_g a_g g : a_g \in \mathbb{F} \right\}.$$

This forms an algebra over $\mathbb{F}$ with the obvious multiplication operation.

Example 1.2.1. Consider the rotation group, generated by the matrices

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Letting δ be an infinitesimal value and using a Taylor expansion, we can write

$$R_z(\delta\theta) = 1 + \delta\theta \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1 + \delta\theta M_z,$$

$$R_x(\delta\phi) = 1 + \delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = 1 + \delta\phi M_x,$$

$$R_y(\delta\psi) = 1 + \delta\psi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 1 + \delta\psi M_y.$$

We can measure the commutativity of these matrices via

$$\begin{aligned} R_x(\delta\phi)R_y(\delta\psi)R_x^{-1}(\delta\phi)R_y^{-1}(\delta\psi) &= (1 + M_x\delta\phi)(1 + M_y\delta\psi)(1 - M_x\delta\phi)(1 - M_y\delta\psi) \\ &= 1 + \delta\phi\delta\psi(M_xM_y - M_yM_x). \end{aligned}$$

Exercise 1.1. Show that $[M_x, M_y] = -M_z$.

Remark. Thus we have a vector space spanned by M_x, M_y, M_z with an operation $[\cdot,\cdot]$ satisfying the identity $[M_x, M_y] = -M_z$. Note that this property is satisfied by the cross product on $\mathbb{R}^3$. The cross product also satisfies the following *Jacobi identity*:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

The above properties define a *Lie algebra*.

Definition 1.3. Let $\{e_k\}$ be a basis of a Lie algebra and $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The *universal enveloping algebra* of the Lie algebra is the free associative algebra on $\{e_k\}$, modulo the relations $[e_i, e_j] = \sum_k c_{ij}^k e_k$.

Remark. One way to return to the Lie group from the Lie algebra is exponentiation, e.g. $R_z(\theta) = e^{\theta M_z}$.

1.3 Algebras and Modules

Let k be a commutative ring (most of the time $k = \mathbb{C}$). All rings will be associative and unital.

Definition 1.4. A *(associative and unital) k-algebra* is a unital ring A with a homomorphism $i : k \to A$ such that $i(r) \cdot a = a \cdot i(r)$, i.e. the image of i commutes with A.

Example 1.4.1. Any ring is a $\mathbb{Z}$-algebra.

Definition 1.5. A *homomorphism* of k-algebras is a k-linear homomorphism of unital rings.

Definition 1.6. Let A, B be unital rings, and M an abelian group. Then

1. a *left A-module structure* on M is a $\mathbb{Z}$-bilinear map $A \times M \to M$, associative in the sense that

$$a_1(a_2 m) = (a_1 a_2)m, \quad \text{for all } a_1, a_2 \in A,\ m \in M,$$

   and such that $1_A m = m$ for all $m \in M$;

2. a *right A-module structure* on M is a $\mathbb{Z}$-bilinear map $M \times B \to M$, associative in the sense that

$$(mb_1)b_2 = m(b_1 b_2), \quad \text{for all } b_1, b_2 \in B,\ m \in M,$$

   and such that $m1_B = m$ for all $m \in M$;

3. an *A-B-bimodule structure* on M is a left A-module and right B-module structure on M, along with the condition that $(am)b = a(mb)$ for all $a \in A$, $b \in B$, and $m \in M$.

Remark. In general, an A-module will mean a left A-module by default.

Definition 1.7. Let M, N be left A-modules. An *A-module homomorphism* is a map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$.

Example 1.7.1. A ring A is both a left/right A-module and an A-A-bimodule (the *regular bimodule*).

Definition 1.8. The *direct sum* $\bigoplus_{i \in I} M_i$ of left A-modules M_i is the collection of $(m_i)_{i \in I}$ with finitely many nonzero entries, with component-wise addition and scalar multiplication.

Example 1.8.1. Let I be an index set. Then $A^{\oplus I}$ is the *coordinate A-module.*

Definition 1.9. A *submodule* of M is a nontrivial subgroup closed under addition and invariant under the action of A.

Example 1.9.1. Submodules of the regular left/right A-module are the left/right ideals of A.

Definition 1.10. Let M be a left A-module and M_0 a submodule of M. The *quotient module* M/M_0 is the set of equivalence classes $m + M_0$, where the action of A is given by $a(m + M_0) = am + M_0$.

Lemma 1.1. *Let M, N be A-modules and $M_0 \subseteq M$ a submodule. Let $\varphi : M \to N$ be A-linear such that $\varphi(M_0) = \{0\}$. Then there exists a unique A-linear map $\underline{\varphi} : M/M_0 \to N$ such that $\varphi = \underline{\varphi} \circ \pi$, where $\pi : M \to M/M_0$ is the canonical projection.*

Lecture 2

Aug. 20 — Algebras and Modules

2.1 More on Algebras and Modules

Definition 2.1. A *free* module is a module which has a basis.

Example 2.1.1. Consider the coordinate module $A^{\oplus I}$. Then a basis is given by $e_i = \{\delta_{ij}\}_{j\in I}$ for $i \in I$.

Proposition 2.1. *Let M be a left A-module. Let I be an index set and let $m_i \in M$ for $i \in I$. Then*

1. *There exists a unique A-linear map $A^{\oplus I} \to M$ which sends $e_i \mapsto m_i$.*
2. *This map is surjective if and only if the elements m_i span M. In particular, every M is isomorphic to a quotient of a free module.*
3. *This map is an isomorphism if and only if $\{m_i\}$ form a basis of M. In particular, every coordinate module is a free module.*

Proof. Left as an exercise. □

Example 2.1.2. Suppose M is spanned by a single element m. Then $M \cong A/I$, where I is the left ideal

$$I = \{a \in A : am = 0\}.$$

Example 2.1.3. We can now construct the following examples of algebras:

1. Let $\mathrm{Mat}_n(A)$ be the set of $n \times n$ matrices with entries in A. If A is a k-algebra, then $\mathrm{Mat}_n(A)$ is also a k-algebra.
2. If G is a group, then the group algebra kG (for a ring k) given by

   $$kG = \left\{ \sum_{g\in G} a_g g : a_g \in k \right\}$$

   is a free module with basis identified with the elements of G.

   The importance of of this object is as follows: Let G be a group and B an algebra. Consider the set of maps satisfying $1_G \mapsto 1_B$ and respecting the group multiplication. This set is in bijection with maps $kG \to B$ (they extend by linearity). If V is a vector space and $B = \mathrm{End}(V)$, then this statement says that there is a bijection between the representations of the group G and the representations of the group algebra kG.

3. If I is a two-sided ideal, then A/I has a natural algebra structure.
4. If A_1, A_2 are k-algebras, then the direct sum $A_1 \oplus A_2$ is again a k-algebra (with component-wise multiplication). One can extend this by induction to a finite direct sum, but note that we lose the multiplicative identity in an infinite direct sum (so we do not get an algebra in the infinite case).

2.2 Module of Homomorphisms

Definition 2.2. Let k be a commutative ring and A a k-algebra. Let M, N be left A-modules. Denote by $\mathrm{Hom}_A(M, N)$ the set of all A-module homomorphisms $M \to N$. Give $\mathrm{Hom}_A(M, N)$ a k-module structure via

$$[\varphi_1 + \varphi_2](m) = \varphi_1(m) + \varphi_2(m), \quad [r\varphi](m) = r\varphi(m)$$

for $\varphi_1, \varphi_2 \in \mathrm{Hom}_A(M, N)$, $r \in k$, and $m \in M$.

Remark. Let L, M, N be left A-modules. Then we can define a k-bilinear map

$$\begin{aligned} \mathrm{Hom}_A(M, N) \times \mathrm{Hom}_A(L, M) &\longrightarrow \mathrm{Hom}_A(L, N) \\ (\varphi, \psi) &\longmapsto \varphi \circ \psi. \end{aligned}$$

Exercise 2.1. Let N_2 be an A-module, $N_1 \subseteq N_2$ an A-submodule, and $N_3 = N_2/N_1$. Let $i : N_1 \hookrightarrow N_2$ be the inclusion and $\pi : N_2 \to N_3$ the projection. Define the maps

$$\begin{aligned} \widetilde{\imath} : \mathrm{Hom}(M, N_1) &\to \mathrm{Hom}(M, N_2) \\ \varphi_1 &\longmapsto i \circ \varphi_1 \\ \widetilde{\pi} : \mathrm{Hom}(M, N_2) &\to \mathrm{Hom}(M, N_3) \\ \varphi_2 &\longmapsto \pi \circ \varphi_2. \end{aligned}$$

Then show that $\widetilde{\imath}$ is injective and $\mathrm{Im}\,\widetilde{\imath} = \ker \widetilde{\pi}$.

Remark. Let B be a k-algebra and M and A-B-bimodule. Then for all A-modules N, we have that $\mathrm{Hom}_A(M, N)$ is a left B-module via

$$[b\varphi](m) = \varphi(mb).$$

Similarly, if N is an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a right C-module via

$$[\varphi c](m) = \varphi(m)c.$$

So if M is an A-B-bimodule and N an A-C-bimodule, then $\mathrm{Hom}_A(M, N)$ is a B-C-bimodule.

Remark. Let M be a left A-module. We write $\mathrm{End}_A(M)$ in place of $\mathrm{Hom}_A(M, M)$, and composition gives $\mathrm{End}_A(M)$ the structure of a k-algebra. If $M = A^{\oplus n}$, then we can identify

$$\mathrm{End}_A(M) = \mathrm{Mat}_n(A^{\mathrm{opp}}),$$

where the opposite algebra exchanges the order of multiplication in the original algebra (this is because $\mathrm{End}_A(M)$ must respect the action by A). Then M becomes an A-$(\mathrm{Mat}_n(A))^{\mathrm{opp}}$-bimodule.

Remark. If M, N are two left A-modules, then $\mathrm{Hom}_A(M, N)$ is an $\mathrm{End}_A(N)$-$\mathrm{End}_A(M)$-bimodule (by taking into account compositions).

2.3 Tensor Product of Modules

Remark. Let A be a k-algebra, M a right A-module, and N a left A-module. We want to produce a k-module $M \otimes_A N$, which will be the *tensor product* of M and N over A.

Definition 2.3. Let L be a k-module. We say that a map $\varphi : M \times N \to L$ is *A-bilinear* if it is k-linear in both arguments and satisfies

$$\varphi(ma, n) = \varphi(m, an)$$

for any $a \in A$, $m \in M$, and $n \in N$.

Definition 2.4 (Universal property of the tensor product)**.** There is an A-bilinear map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_A N \\ (m, n) &\longmapsto m \otimes n \end{aligned}$$

such that for any A-bilinear map $\varphi : M \times N \to L$, there exists a unique k-linear map $\psi : M \otimes_A N \to L$ such that $\varphi(m, n) = \psi(m \otimes n)$. As a diagram, this says that

$$\begin{tikzcd} M \times N \arrow[rr, "(m,n)\mapsto m\otimes n"] \arrow[dr, "\varphi"'] & & M \otimes_A N \arrow[dl, "\psi"] \\ & L & \end{tikzcd}$$

Exercise 2.2. If we choose $M \otimes'_A N$ with bilinear map $(m, n) \mapsto m \otimes' n$, then there exists a unique isomorphism $i : M \otimes_A N \to M \otimes'_A N$ given by $i(m \otimes n) = m \otimes' n$.

Corollary 2.0.1. *Assume $M \otimes_A N$ satisfies the universal property. Then $\{m \otimes n\}$ span $M \otimes_A N$.*

Theorem 2.1. *The tensor product $M \otimes_A N$ exists for all right A-modules M and left A-modules N.*

Proof. We sketch the proof. First take M to be free. Then we can define $M \otimes_A N$ as $N^{\oplus I}$, where we have $(e_i a_i) \otimes n = (a_i n)_{i \in I}$. The universal property is easy to check for this case, and the general case can be done by writing M as a quotient of a free module. □

Example 2.4.1. If M, N are both free and $\{e_i\}_{i \in I}$, $\{f_j\}_{j \in J}$ are bases of M, N, respectively, then $M \otimes_A N$ is a free k-module with basis vectors $\{e_i \otimes f_j\}_{i \in I, j \in J}$.

Exercise 2.3. Let $M = A/I$, where I is a right ideal. Show that $M \otimes_A N = N/IN$. Find out what happens when $N = A/J$, where J is a left ideal, what can you say about $M \otimes_A N$ in terms of A, I, J?

Proposition 2.2. *Assume B is a k-algebra and M a B-A-module. Then $M \otimes_A N$ is a left B-module.*

Proof. Define $\varphi_b : M \times N \to M \otimes_A N$ by $(m, n) \mapsto bm \otimes n$. This is bilinear, so by the universal property, there exists $\psi_b : M \otimes_A N \to M \otimes_A N$ such that $\psi_b(m \otimes n) = bm \otimes n$, which gives the B-action. □

Definition 2.5. Let L be a B-module. A map $\varphi : M \times N \to L$ is called *B-A-linear* if it is k-linear in both arguments and

$$\varphi(ma, n) = \varphi(m, an), \quad \varphi(bm, n) = b\varphi(m, n)$$

for all $m \in M$, $n \in N$, $b \in B$, and $a \in A$.

Proposition 2.3. *The left B-module $M \otimes_A N$ has the following universal property:*

Let L be any left B-module and $\varphi : M \times N \to L$ a B-A-linear map. Then there exists a unique B-linear map $\psi : M \otimes_A N \to L$ such that $\psi(m \otimes n) = \varphi(m, n)$.

Example 2.5.1. Let A_1, A_2 be k-algebras. Then

1. $A_1 \otimes_k A_2$ has the structure of a k-algebra via

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2),$$

where $1 \otimes 1$ is a unit element.

2. Let M_i be a left A_i-module for $i = 1, 2$. Then $M_1 \otimes_k M_2$ is a module for $A_1 \otimes_k A_2$.

2.4 Tensor-Hom Adjunction

Proposition 2.4 (Tensor-Hom adjunction)**.** *Let A, B be associative algebras, N a B-module, M an A-module, and L an A-B-bimodule. Then*

1. *$L \otimes_B N$ is an A-module;*

2. *$\mathrm{Hom}_A(L, M)$ is a B-module.*

Moreover, there is a natural k-linear isomorphism

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)).$$

Proof. By the universal property, there is a natural map

$$\mathrm{Hom}_A(L \otimes_B N, M) \xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M).$$

So it suffices to find

$$\begin{aligned} \mathrm{Hom}_B(N, \mathrm{Hom}_A(L, M)) &\xrightarrow{\cong} \mathrm{Bilin}_{A,B}(L \times N, M) \\ f &\longmapsto \varphi_f. \end{aligned}$$

Construct this map by $\psi_f(e, n) = [f(n)](e)$, with inverse $h \mapsto \psi(\cdot, h)$ for $\psi \in \mathrm{Bilin}_{A,B}(L \times N, M)$. □

Example 2.5.2. If we have an algebra homomorphism $B \to A$, where A is a an A-B-bimodule. One can show as an exercise that $\mathrm{Hom}_A(A, M)$ is naturally identified with M as an A-module and B-module. Thus by the Tensor-Hom adjunction, we have a natural isomorphism

$$\mathrm{Hom}_A(A \otimes_B N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, M).$$

Definition 2.6. The A-module $A \otimes_B N$ is said to be *induced* from N.

Remark. Assume there is Hom from $A \to B$. Then B is an A-B-bimodule. Take it as L in the Tensor-Hom adjunction. Note that $B \otimes_B N \cong N$ as A-modules, and we have a natural isomorphism

$$\mathrm{Hom}_A(N, M) \xrightarrow{\cong} \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)).$$

Definition 2.7. The B-module $\mathrm{Hom}_A(B, M)$ is said to be *coinduced* from M.

Lecture 3

Aug. 25 — Complete Reducibility

3.1 Reducibility of Modules

Remark. Consider an associative algebra A over a field $\mathbb{F}$. We proceed to study completely reducible representations of A. Let U be an A-module.

Definition 3.1. An A-module U is *irreducible* if it only has two distinct submodules ($\{0\}$ and U).

Remark. With this definition, $\{0\}$ is not irreducible.

Definition 3.2. An A-module U is *completely reducible* if for any submodule $U' \subseteq U$, there exists an A-submodule U'' such that $U = U' \oplus U''$.

Exercise 3.1. Show that any submodule and any quotient module of a completely reducible A-module is also completely reducible.

Example 3.2.1. Consider $A = \operatorname{End}_{\mathbb{F}}(U)$. Then U is an A-module and is irreducible (there is a linear operator $\alpha : U \to U$ taking $u \mapsto v$ for any $u, v \in U$, so there are no nontrivial invariant subspaces).

Proposition 3.1. *Let U_1, U_2 be completely reducible A-modules. Then $U_1 \oplus U_2$ is completely reducible.*

Proof. Left as an exercise. □

Corollary 3.0.1. *Let U be a finite-dimensional A-module. Then the following are equivalent:*

1. *U is completely reducible;*
2. *U is isomorphic to a direct sum of irreducible submodules.*

Exercise 3.2. Show that every irreducible A-module is isomorphic to a quotient module for a regular module (i.e. one isomorphic to A). In particular, every irreducible module over a finite-dimensional associative $\mathbb{F}$-algebra is finite-dimensional.

3.2 Schur's Lemma

Theorem 3.1 (Schur's lemma)**.** *Let A be an associative $\mathbb{F}$-algebra and U, V irreducible A-modules. Then*

1. *if U, V are not isomorphic, then $\operatorname{Hom}_A(U, V) = 0$;*
2. *$\operatorname{End}_A(U)$ is a skew field (i.e. a division ring). Furthermore, if U is finite-dimensional and $\mathbb{F}$ is algebraically closed, then $\dim \operatorname{End}_A(U) = 1$.*

Proof. (1) Assume we have a nonzero homomorphism $\varphi : U \to V$. Then $\ker \varphi \subsetneq U$, and $\operatorname{Im} \varphi \subseteq V$ is nontrivial, so by irreducibility φ must be an isomorphism.

(2) Let $\varphi \in \operatorname{End}_A(U)$. From (1), we know that φ is an isomorphism, so φ has an inverse, i.e. $\operatorname{End}_A(U)$ is a skew field. For the second part, since $\mathbb{F}$ is algebraically closed, we can find an eigenvalue z for φ. Then $\varphi - z \operatorname{Id}_U$ is not invertible, so we have $\varphi - z \operatorname{Id}_U = 0$ by (1). $\square$

Exercise 3.3. Consider $1, i, j, k$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The *quaternion algebra* over $\mathbb{R}$ is given by

$$\mathbb{H}_{\mathbb{R}} = \{q = w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

Note that $\overline{q} = w - xi - yj - zk$ satisfies $q\overline{q} = w^2 + x^2 + y^2 + z^2$, so $q^{-1} = \overline{q}/(w^2 + x^2 + y^2 + z^2)$, i.e. $\mathbb{H}_{\mathbb{R}}$ is a skew field. Show that $\operatorname{End}_{\mathbb{H}_{\mathbb{R}}}(\mathbb{H}_{\mathbb{R}}) \cong \mathbb{H}_{\mathbb{R}}^{\mathrm{opp}}$.

Remark. We have an embedding $\mathbb{H}_{\mathbb{R}} \hookrightarrow \operatorname{Mat}_2(\mathbb{C})$ given by

$$q \longmapsto \begin{pmatrix} w + xi & y + zi \\ -y + zi & w - xi \end{pmatrix}.$$

If we replace $\mathbb{R}$ with $\mathbb{C}$, then $\mathbb{H}_{\mathbb{C}} \cong \operatorname{Mat}_2(\mathbb{C})$, which is reducible (consider the sum of column spaces).

Definition 3.3. Let U be an A-module. We say that U is *endotrivial* if $\operatorname{End}_A(U)$ consists only of scalar maps, i.e. maps of the form $z \operatorname{Id}$.

Remark. Suppose $\mathbb{F}$ is algebraically closed and uncountable (e.g. $\mathbb{C}$), A has countable dimension over $\mathbb{F}$, and U an irreducible A-module. Then U is endotrivial.

Definition 3.4. Define the *center* of A to be

$$\mathcal{Z}(A) = \{z \in A : za = az \text{ for all } a \in A\}.$$

Note that this is a commutative algebra.

Exercise 3.4. Schur's lemma gives a description of the center of A. Let U be an endotrivial A-module (e.g. a finite-dimensional irreducible module over A if $\mathbb{F}$ is algebraically closed). Show that $z \in \mathcal{Z}(A)$ acts as a scalar on U. We call the algebra homomorphism $\mathcal{Z}(A) \to \mathbb{F}$ the *central character* of U.

3.3 Completely Reducible Modules

Remark. Consider finite direct sums of endotrivial irreducible modules:

$$\bigoplus_{i=1}^{k} U_i \otimes M_i,$$

where the U_i are endotrivial modules and the M_i are vector spaces known as *multiplicity spaces*. Note that $U_1^{\oplus i} = U_1 \otimes \mathbb{F}^i$. The A-action on the direct sum for $a \in A$ is given by

$$a(u_1 \otimes m_1, \ldots, u_k \otimes m_k) = (au_1 \otimes m_1, \ldots, au_k \otimes m_k).$$

We will use Schur's lemma to understand homomorphisms between such modules.

Write $U^j = \bigoplus_{i=1}^k U_i \otimes M_i^j$ for $j = 1, 2$. We can produce a linear map

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \longrightarrow \mathrm{Hom}_A(U^1, U^2)$$

in the following manner: For $\underline{\varphi} = (\varphi_1, \ldots, \varphi_k) \in \bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2)$, we can define

$$\psi_{\underline{\varphi}}\left(\sum_{i=1}^k u_i \otimes m_i^1\right) = \sum_{i=1}^k u_i \otimes \varphi_i(m_i^1).$$

Theorem 3.2. *We have the following:*

1. *The map $\underline{\varphi} \mapsto \psi_{\underline{\varphi}}$ defines a vector space isomorphism*

$$\bigoplus_{i=1}^k \mathrm{Hom}_{\mathbb{F}}(M_i^1, M_i^2) \xrightarrow{\cong} \mathrm{Hom}_A(U^1, U^2).$$

2. *Every A-module homomorphism $U_1 \to U_2$ sends $U_i \otimes M_i^1$ to $U_i \otimes M_i^2$ for any i.*

Proof. Left as an exercise (use Schur's lemma). □

Corollary 3.2.1. *We have the following:*

1. *there is an isomorphism* $\mathrm{Hom}_A(U_i, U) \xrightarrow{\cong} M_i$;
2. *there is an isomorphism* $\bigoplus_{i=1}^k U_i \otimes \mathrm{Hom}_A(U_i, U) \cong U$ *given by*

$$\sum_{i=1}^k u_i \otimes \varphi_i \longmapsto \sum_{i=1}^k \varphi_i(u_i).$$

Proposition 3.2. *For any A-submodule $U' \subseteq U$, there exists a unique collection of determined subspaces $M_i' \subseteq M_i$ such that $U' = \bigoplus_{i=1}^k U_i \otimes M_i'$ as submodules of U.*

Proof. Note that $\mathrm{Hom}_A(U_i, U') \subseteq \mathrm{Hom}_A(U_i, U)$, set $M_i' = \mathrm{Hom}_A(U_i, U')$, and use Corollary 3.2.1. □

Theorem 3.3. *Let U_i be irreducible modules for A and consider maps $\beta_i : A \to \mathrm{End}_{\mathbb{F}}(U_i)$. Set*

$$\beta = \beta_1 \oplus \cdots \oplus \beta_k : A \longrightarrow \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i),$$

where the U_i are pairwise non-isomorphic. Then the homomorphism β is surjective.

Proof. Replace A by $A/\ker\beta$, so that β is injective. Then β equips $\bigoplus_{i=1}^k \mathrm{End}(U_i)$ with an A-bimodule structure, and there is a natural isomorphism $\mathrm{End}_{\mathbb{F}}(U_i) \cong U_i \otimes U_i^*$. View U_i as the multiplicity space for the right A-module and U_i^* as the multiplicity space for the left A-module. By Proposition 3.2,

$$A = \bigoplus_{i=1}^k U_i \otimes V_i$$

as a left A-module for some $V_i \subseteq U_i^*$. Similarly for the right A-module, we have

$$A = \bigoplus_{i=1}^{k} W_i \otimes U_i^*$$

for some $W_i \subseteq U_i$. Then we must have $U_i \oplus V_i = W_i \oplus U_i^*$, so $U_i \cong W_i$ and $V_i \cong U_i^*$ (the identity $1 \in A$ guarantees that no component is zero). Thus β is surjective. $\square$

Corollary 3.3.1. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional $\mathbb{F}$-algebra. Then the set of isomorphism classes of irreducible A-modules is finite and non-empty.*

Proof. First this set is nonempty since A is nonzero, so it has an irreducible subrepresentation. To see that it is finite, note that for all collections $U_1, \ldots, U_k$, the map $A \to \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$ is surjective, so

$$\dim A \geq \sum_{i=1}^{k} (\dim U_i)^2.$$

This proves the desired result, since A is finite-dimensional. $\square$

3.4 Simple Algebras

Definition 3.5. An algebra A is *simple* if the only two-sided ideals are $\{0\}$ and A (i.e. A is irreducible as a bimodule over itself).

Theorem 3.4. *Let $\mathbb{F}$ be an algebraically closed field and A a finite-dimensional $\mathbb{F}$-algebra. Then the following are equivalent:*

1. *A is simple;*
2. *$A \cong \operatorname{End}_{\mathbb{F}}(U)$ for some finite-dimensional vector space U.*

Proof. ($1 \Rightarrow 2$): The algebra A has an irreducible representation U, i.e. we have a map $A \to \operatorname{End}_{\mathbb{F}}(U)$. Since A is simple, this map must have trivial kernel, i.e. it is injective. We also already know that it is surjective, so this map is an isomorphism.

($2 \Rightarrow 1$): Assume I is a two-sided ideal in $\operatorname{End}_{\mathbb{F}}(U) \cong U \otimes U^*$ and view $I \subseteq U \otimes U^*$. Show as an exercise that we must have $I = \{0\}$. $\square$

Theorem 3.5. *Every finite-dimensional module V for $A = \operatorname{End}_{\mathbb{F}}(U)$ is isomorphic to a direct sum of several copies of U.*

Proof. Recall that every finitely generated module V is a quotient of $A^{\oplus \ell}$ for some $\ell \in \mathbb{N}$. We can write $A = U \otimes U^*$. Let $A^{\oplus \ell} = U \otimes M$ and consider the quotient map $\pi : U \otimes M \to V$. Then $\ker \pi \subseteq U \otimes M$ must be of the form $U \oplus M_0$, so we have $V \cong (U \otimes M)/(U \otimes M_0) = U \otimes (M/M_0)$. $\square$

Lecture 4

Aug. 27 — Semisimple Algebras

4.1 Semisimple Algebras

Definition 4.1. A finite-dimensional $\mathbb{F}$-algebra A is called *semisimple* if it is isomorphic to a direct sum of simple algebras.

Remark. If $\mathbb{F}$ is algebraically closed, then A is a direct sum of matrix algebras, i.e. $\bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$.

Theorem 4.1. *Let $U_1, \ldots, U_k$ be finite-dimensional vector spaces over $\mathbb{F}$. Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$, so that U_i is an irreducible A-module. Then every finite-dimensional A-module V is isomorphic to a direct sum of several copies of $U_1, \ldots, U_k$.*

Proof. Left as an exercise. □

Corollary 4.1.1. *Let $\mathbb{F}$ be algebraically closed, and A be semisimple and finite-dimensional. Then*

1. *The number of isomorphism classes of irreducible A-modules is equal to* $\dim \mathcal{Z}(A)$.
2. *Different irreducible modules have different central characters.*

Proof. (1) Let $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. By Theorem 4.1, the number of irreducible representations is k. We can also write

$$\mathcal{Z}\left(\bigoplus_{i=1}^k A_k\right) = \bigoplus_{i=1}^k \mathcal{Z}(A_i),$$

where $A_i = \operatorname{End}_{\mathbb{F}}(U_i)$. Since $\dim \mathcal{Z}(A_i) = 1$, we have $\dim \mathcal{Z}(\bigoplus_{i=1}^k A_k) = k$ as well.

(2) Use the projections $\mathcal{Z} \to \mathcal{Z}(A_i) \to \mathbb{F}$, which correspond to the central characteres. □

4.2 Characterizations of Semisimple Algebras

Definition 4.2. Let A be a finite-dimensional algebra. We say that a two-sided ideal $I \subseteq A$ is *nilpotent* if $I^n = \{0\}$ for some n.

Exercise 4.1. If I, J are nilpotent, then show that $I + J$ is also nilpotent.

Definition 4.3. The maximal nilpotent ideal of A, denoted $\operatorname{rad}(A)$, is called the *radical* of A.

Theorem 4.2. *Let $\mathbb{F}$ be algebraically closed and A a finite-dimensional algebra. Then the following are equivalent:*

1. *A is semisimple;*
2. *all finite-dimensional representations of A are completely reducible;*
3. $\operatorname{rad}(A) = \{0\}$.

Proof. ($1 \Rightarrow 2$) We have already shown this.

($2 \Rightarrow 3$) Let $I = \operatorname{rad}(A)$, so $I^n = \{0\}$ for some $n \in \mathbb{N}$. Let N be a finite-dimensional A-module. Then $I^\ell N$ is an A-submodule for $\ell = 0, \dots, n$. Since N is completely reducible and $I^{\ell+1}N \subseteq I^\ell N$, we have

$$I^\ell N = N_\ell \oplus I^{\ell+1}.$$

Acting on both sides by I, we get $IN_\ell \subseteq I^{\ell+1}$, so $IN_\ell = \{0\}$. Continuing, we get $IN = 0$, so $A = N$.

($3 \Rightarrow 1$) Take $N_1, \dots, N_k$ to be pairwise non-isomorphic irreducible A-modules. We have an epimorphism $A \to \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(N_i)$. Let I be the kernel, so I acts trivially on every irreducible A-module. We claim that I is nilpotent. Take A to be the regular module. Take a filtration

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = \{0\},$$

where A_i/A_{i+1} is irreducible. Now I acts trivially on A_i/A_{i+1}, so $IA_i \subseteq A_{i+1}$ for all i, thus $I^n = \{0\}$. □

Remark. Assume $\operatorname{char}(\mathbb{F}) = 0$. Consider the following bilinear form on A:

$$(a, b)_U = \operatorname{tr}_U(ab),$$

where U is any A-module. Note that U could be A.

Theorem 4.3. *Let $\operatorname{char}(\mathbb{F}) = 0$, and let A be a finite-dimensional $\mathbb{F}$-algebra. Then A is semisimple if and only if $(a, b)_A$ is nondegenerate.*

Proof. ($\Rightarrow$) Assume A is semisimple, so $A = \bigoplus_{i=1}^k \operatorname{End}(U_i)$. Note that the restriction of $(\cdot,\cdot)_A$ to the direct summand $\operatorname{End}_{\mathbb{F}}(U_i)$ coincides with $(\cdot,\cdot)_{\operatorname{End}_{\mathbb{F}}(U_i)}$. Let $E_{j\ell}$ denote the matrix with all 0s except a single 1 in the (j, ℓ) entry. Then we can compute that

$$(E_{j\ell}, E_{j'\ell'})_{\operatorname{End}_{\mathbb{F}}(U_i)} = \delta_{ej'} \operatorname{tr}_{\operatorname{End}_{\mathbb{F}}(U_i)}(E_{j\ell'}) = \delta_{e\ell}\delta_{j\ell'} \dim U_i.$$

So if $\{E_{j\ell}\}$ is a basis, then $\{(\dim U_i)^{-1} E_{j\ell}\}$ is the dual basis. This is nondenegerate if $\operatorname{char}(\mathbb{F}) = 0$.

($\Leftarrow$) Suppose $(\cdot,\cdot)_A$ is nondegenerate. If I is a nilpotent ideal, then for any $a \in I$ such that $a^n = 0$. Then $\operatorname{tr}_A(a) = 0$ for any $a \in I$, so $I \in \ker(\cdot,\cdot) = 0$. Since $(\cdot,\cdot)$ is nondegenerate, we have $I = \{0\}$. □

4.3 Double Centralizer Theorem

Theorem 4.4 (Double centralizer theorem)**.** *Let V be a finite-dimensional vector space over $\mathbb{F}$. Let $A \subseteq \operatorname{End}_{\mathbb{F}}(V)$ be a semisimple algebra, and set $B = \operatorname{End}_A(V)$. Then $A = \operatorname{End}_B(V)$.*

Proof. Let $A = \bigoplus_{i=1}^k \operatorname{End}(U_i)$ and V be a faithful representation of A, so V is completely reducible:

$$V \cong \bigoplus_{i=1}^k U_i \oplus M_i,$$

where the M_i are multiplicity spaces. Let $a = (\varphi_1, \dots, \varphi_k) \in A$ (for $\varphi_i \in \operatorname{End}(U_i)$) act on $\operatorname{End}_{\mathbb{F}}(V)$ by

$$(\varphi_1, \dots, \varphi_k) \longmapsto \sum_{i=1}^{k} \varphi_i \otimes \operatorname{Id}_{M_i}.$$

Note that the M_i are nonzero since V is faithful. Then $B = \bigoplus_{i=1}^{n} \operatorname{End}(M_i)$ embeds into $\operatorname{End}_{\mathbb{F}}(V)$ via

$$(\psi_1, \dots, \psi_k) \longmapsto \sum_{i=1}^{k} \operatorname{Id}_{U_i} \otimes \psi_i,$$

which completes the proof. □

4.4 Representations of Finite Groups

Remark. Recall that to any group G we can associate the group algebra $\mathbb{F}G$. For any representation of G, there is a representation of $\mathbb{F}G$ and vice versa.

Remark. Consider the following operations with representations. Let U, V be representations of G.

1. the *tensor product* $U \otimes_{\mathbb{F}} V$, where $g(u \otimes v) = (gu) \otimes (gv)$;
2. the *dual* U^* defined by $\langle g\alpha, u \rangle = \langle \alpha, g^{-1}u \rangle$ for $u \in U$, $\alpha \in U^*$, $g \in G$;
3. $\operatorname{Hom}_{\mathbb{F}}(U, V)$, with action given by $[g\varphi](h) = g[\varphi(g^{-1}u)]$ for $\varphi \in \operatorname{Hom}_{\mathbb{F}}(U, V)$.

Exercise 4.2. Show the following:

1. The tensor product of representations satisfies associativity, distributivity, and commutativity.
2. There is an isomorphism of representations $U^* \otimes V \to \operatorname{Hom}(U, V)$.
3. $\operatorname{Hom}_G(U, V) \subseteq \operatorname{Hom}(U, V)$ coincides with the space of G-invariant elements.

Remark. For the rest of this section, assume $\mathbb{F}$ is algebraically closed and $\operatorname{char} \mathbb{F} = 0$.

Theorem 4.5. *The group algebra $\mathbb{F}G$ is semisimple.*

Proof. It suffices to show that $(\cdot, \cdot)_{\mathbb{F}G}$ is nondegenerate. Take $g, g' \in G$, and note that $gg' : h \mapsto gg'h$, so

$$(g, g')_{\mathbb{F}G} = \operatorname{tr}_{\mathbb{F}G}(gg') = \delta_{1,gg'}|G|,$$

which is nondegenerate. Moreover, the basis $\{g\}$ in $\mathbb{F}G$ corresponds to the dual basis $\{|G|^{-1}g^{-1}\}$. □

Corollary 4.5.1. *(Let $\mathbb{F}$ be algebraically closed and* $\operatorname{char} \mathbb{F} = 0$.*)*

1. *Every finite-dimensional representation of G is completely reducible.*
2. *The number of isomorphism classes of irreducible representations is equal to the number of conjugacy classes of G.*
3. *If $U_1, \dots, U_k$ are all of the pairwise non-isomorphic irreducible representations of G, then*

$$|G| = \sum_{i=1}^{k} (\dim U_i)^2.$$

Proof. (1) This follows from the semisimplicity of $\mathbb{F}G$.

(2) It suffices to show that $\dim \mathcal{Z}(\mathbb{F}G)$ equals the number of conjugacy classes of G. We have

$$\mathcal{Z}(\mathbb{F}G) = \left\{ \sum_{g \in G} a_g g : a_g \text{ is constant on conjugacy classes} \right\},$$

i.e. we must have $a_{hgh^{-1}} = a_g$ for any $h \in G$. So the dimension is the number of conjugacy classes.

(3) This automatically follows from looking at the dimension of $\mathbb{F}G$. □

Lecture 5

Sept. 3 — Representations of Finite Groups

5.1 Representations of S_4

Remark. We will write *irrep* for "irreducible representation."

Example 5.0.1. Consider the symmetric group S_4, with $|S_4| = 24$. The conjugacy classes of S_4 are parametrized by partitions of 4: If we have a partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1, \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = 4,$$

then the corresponding conjugacy class has cycle type λ. For example, the conjugacy classes are given by

1. $\lambda_1 = 4$: $[4]$;
2. $\lambda_1 = 3$, $\lambda_2 = 1$: $[3, 1]$;
3. $\lambda_1 = 2$, $\lambda_2 = 2$: $[2, 2]$;
4. $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$: $[2, 1, 1]$;
5. $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 = 1$: $[1, 1, 1, 1]$.

In particular, this means that S_4 has 5 irreps. We can enumerate them as follows:

1. We have the 1-dimensional representations: the trivial representation and the sign sign_4.
2. Let S_4 act on $\mathbb{C}^4$ by permuting the basis vectors. The span of (x, x, x, x) gives a 1-dimensional subrepresentation, but it has a unique 3-dimensional complement refl_4.
3. We can take a tensor product $\mathrm{refl}_4 \otimes \mathrm{sign}_4$, which is also 3-dimensional. One can check that this is different from refl_4 by looking at the determinant.
4. We have found two 1-dimensional and two 3-dimensional irreps, which account for $1+1+9+9 = 20$ of the 24 dimensions. Thus there is a missing 2-dimensional representation.

   Note that there is a projection $\pi : S_4 \to S_3$ which is a homomorphism with kernel $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Figure this out and find the last irrep as an exercise.

Exercise 5.1. Let G be a finite abelian group. Prove that all irreps of G are 1-dimensional.

5.2 Characters

Definition 5.1. Let G be a group, and let U a finite-dimensional representation of G. The *character* $\chi_U : G \to \mathbb{F}$ is defined by $\chi_U(g) = \mathrm{tr}_U(g)$.

Exercise 5.2. Prove the following:

1. χ_U is constant on conjugacy classes of G.
2. $\chi_{U\oplus V} = \chi_U \oplus \chi_V$.
3. $\chi_{U\otimes V} = \chi_U \chi_V$.

Remark. For the rest of this section, assume G is finite and $\mathbb{F} = \mathbb{C}$. So we know every representation of G is completely reducible. Denote by $\mathbb{C}[G]$ the algebra of complex-valued functions on G, and $\mathbb{C}[G]^G$ the subalgebra of functions constant on conjugacy classes (i.e. the G-invariant functions). Clearly the character χ_U lies in $\mathbb{C}[G]^G$ for any finite-dimensional representation U of G.

Definition 5.2. Define a Hermitian scalar product on $\mathbb{C}[G]^G$ (a priori only on the characters) by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g\in G} \chi_1(g)\overline{\chi_2(g)}.$$

Proposition 5.1. *Let U, V be finite-dimensional representations of G. Then*

$$(\chi_U, \chi_V) = \dim \operatorname{Hom}_G(U, V).$$

Proof. We first note that $\chi_{U^*} = \overline{\chi}_U$. To see this, observe that since G is finite, we have $g^n = 1$ for some n. In particular, the eigenvalues $\lambda_i(g)$ of g have $|\lambda_i(g)| = 1$. Thus $\lambda_i(g^{-1}) = \overline{\lambda_i(g)}$, so we see the result after taking traces. Another way to see this is the following: For a representation $\rho : G \to U$, we can make each $\rho(g)$ into a unitary operator as follows. Begin with a pairing $\langle \cdot, \cdot \rangle_0$ on U and define

$$\langle v, w\rangle = \frac{1}{|G|} \sum_{g\in G} \langle \rho(g)v, \rho(g)w\rangle_0, \quad v, w \in U.$$

Then $\rho(g)$ is unitary with respect to $\langle \cdot, \cdot \rangle$, and we get the result.

Continuing, we have $V \otimes U^* = \operatorname{Hom}_{\mathbb{C}}(U, V)$, so $\chi_{\operatorname{Hom}(U,V)} = \chi_V \overline{\chi}_U$. Consider the averaging element

$$\epsilon = |G|^{-1} \sum_{g\in G} g \in \mathbb{C}[G].$$

This is a projector on G-invariants (W^G) in any representation W. Thus $\operatorname{tr}_W(\epsilon) = \dim W^G$. Applying this to $W = \operatorname{Hom}(U, V)$ and noting that $\operatorname{Hom}_G(U, V) = \operatorname{Hom}(U, V)^G$, we get

$$\dim \operatorname{Hom}_G(U, V) = \operatorname{tr}_{\operatorname{Hom}(U,V)}(\epsilon) = |G|^{-1} \sum_{g\in G} \chi_{\operatorname{Hom}(U,V)}(g) = |G|^{-1} \sum_{g\in G} \chi_V(g)\overline{\chi_U(g)} = (\chi_V, \chi_U),$$

which proves the desired claim. □

Corollary 5.0.1. *The characters of irreps form an orthonormal basis in $\mathbb{C}[G]^G$.*

Proof. Schur's lemma implies orthonormality. Since the number of irreps equals the number of conjugacy classes, the characters must form a basis. □

5.3 Induced Representations

Remark. In this section, we only assume k is a commutative ring.

Let $H \subseteq G$, where H, G are finite groups, let kH, kG be the corresponding group algebras, and let U be a representation of H. Treating kG as a kG-kH-bimodule, we can construct the tensor product

$$kG \otimes_{kH} U.$$

Similarly, treating kG as a kH-kG-bimodule, we can construct the representation

$$\mathrm{Hom}_{kH}(kG, U).$$

In fact, these two representations are isomorphic, we call it the *induced representation*, denoted $\mathrm{Ind}_H^G U$.

Proposition 5.2. *There is a natural isomorphism $kG \otimes_{kH} U \cong \mathrm{Hom}_{kH}(kG, U)$.*

Proof. First treat kG as a kH-kG-bimodule, so we can consider $\mathrm{Hom}_{kH}(kG, kH)$ since kG, kH are both left kH-modules. So for any element $\varphi : kG \to kH$, we have

$$\varphi(hg) = h\varphi(g), \quad h \in H, g \in G.$$

with a left G-action and right H-action given by

$$[g\varphi](g') = \varphi(g'g) \quad \text{and} \quad [\varphi h](g') = \varphi(hg').$$

Note that kG is a free left kH-module with basis given by the orbits of H. Show as an exercise that

$$\begin{aligned} \mathrm{Hom}_{kH}(kG, kH) \otimes_{kH} U &\xrightarrow{\cong} \mathrm{Hom}_{kH}(kG, U) \\ \alpha \otimes u &\longmapsto (x \mapsto \alpha(x)u) \end{aligned}$$

is an isomorphism. From here it suffices to show that

$$kG \xrightarrow{\cong} \mathrm{Hom}_{kH}(kG, kH)$$

as kG-kH-bimodules. Define this map via $g \mapsto \varphi_g \in \mathrm{Hom}_{kH}(kG, kH)$, where

$$\varphi_g(g') = \begin{cases} g'g & \text{if } g'g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that φ is H-equivariant, G-equivariant, and an isomorphism of k-modules.

To see H-equivariance, note that $\varphi_{gh}(g')$ and $\varphi_g(g')h$ are nonzero and equal if and only if $gg' \in H$. For the G-equivariance, note that $\varphi_{g_1 g}(g')$ and $[g_1\varphi_g](g')$ are given by

$$\begin{aligned} \varphi_{g_1 g}(g') &= g'g_1 g \quad \text{if } g'g_1 g \in H, \\ [g_1\varphi_g](g') &= g'g_1 g \quad \text{if } g'g_1 g \in H \end{aligned}$$

and zero otherwise, so they coincide. To prove that φ is an isomorphism of k-modules, we need to check that the φ_g form a basis in $\mathrm{Hom}_{kH}(kG, kH)$. Let $g_1, \ldots, g_\ell$ be representatives of the left H-orbits in G. Then we claim that the following map is an isomorphism of k-modules:

$$\begin{aligned} \mathrm{Hom}_{kH}(kG, kH) &\xrightarrow{\cong} (kH)^{\oplus \ell} \\ \varphi &\mapsto \{\varphi(g_i)\}_{i=1}^{\ell}. \end{aligned}$$

This follows since for any $g \in G$ and $i \in \{1, \ldots, \ell\}$, there is a unique element $h \in H$ such that $hg_i = g^{-1}$, so φ_g is sent to the corresponding summand. $\square$

Corollary 5.0.2 (Frobenius reciprocity)**.** *Let U, V be representations of H, G, respectively. Then*

1. $\mathrm{Hom}_G(\mathrm{Ind}_H^G(U), V) \cong \mathrm{Hom}_H(U, V)$;
2. $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(U)) \cong \mathrm{Hom}_H(V, U)$.

Proof. This follows from the Tensor-Hom adjunction, check it as an exercise. □

Remark. What really is $\mathrm{Ind}_H^G U$? Consider the set of maps (of sets) $G \to U$, denote it by $\mathrm{Fun}(G, U)$. The action of G on itself gives $\mathrm{Fun}(G, U)$ the structure of a kG-module. Then we can define

$$\mathrm{Fun}_H(G, U) = \{f \in \mathrm{Fun}(G, U) : f(hg) = hf(g)\} \subseteq \mathrm{Fun}(G, U),$$

which we can identify with the induced representation $\mathrm{Hom}_{kH}(kG, U)$.

Lecture 6

Sept. 8 — Representations of S_n

6.1 Motivation for Studying S_n and Summary

Remark. The finite *simple* groups (those with no nontrivial normal subgroups) are classified as follows:

1. abelian groups: cyclic groups of finite order;
2. alternating groups: $U_n \subseteq S_n$ (the subgroup of even permutations) for $n \geq 5$;
3. 26 exceptional finite simple groups;
4. finite simple groups of *Lie type* (analogues of Lie groups for finite fields).

The final parts of the classification were done by Gorenstein (1960–1980s) and Aschbacher-Smith (2004).

Remark. We study S_n because it is easier to work with than directly studying U_n, and we can recover representations of U_n from those of S_n via Frobenius reciprocity.

Remark. We have previously seen the following using our abstract theory:

1. Representations of S_n are the same as representations of $\mathbb{C}S_n$.
2. The algebra $\mathbb{C}S_n$ is semisimple: $\mathbb{C}S_n \cong \bigoplus_V \operatorname{End}_{\mathbb{C}}(V)$, where V runs over the isomorphism classes of irreps of S_n.
3. The number of irreps of S_n (up to isomorphism) coincides with the number of conjugacy classes.

Remark. In the case of S_n, the conjugacy classes are enumerated by partitions of n:

$$(n_1, n_2, \ldots, n_k), \quad n_1 \geq n_2 \geq \cdots \geq n_k.$$

We can write repeated parts via $(m_1^{d_1}, \ldots, m_e^{d_e})$, where $m_1 > m_2 > \cdots > m_e$. So for S_6, we have

$$(2, 2, 1, 1) \longleftrightarrow (2^2, 1^2).$$

6.2 The Inductive Approach: Background

Remark. We will follow the *inductive approach*, due to Okounkov-Vershik. Consider the inclusions

$$\{1\} = S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1} \subseteq S_n.$$

Note that if $H \subseteq G$ are finite groups, then an irrep of $\mathbb{C}G$ decomposes into irreps of $\mathbb{C}H$.

In general, if $B \subseteq A$ are finite-dimensional associative algebras and $\tau : B \to A$ is a homomorphism, then any A-module is also a B-module by the homomorphism τ. We have isomorphisms

$$A \xrightarrow{\cong} \bigoplus_{V \in \mathrm{Irr}(A)} \mathrm{End}_{\mathbb{C}}(V),$$
$$B \xrightarrow{\cong} \bigoplus_{U \in \mathrm{Irr}(B)} \mathrm{End}_{\mathbb{C}}(U).$$

Let $M_{V,U} = \mathrm{Hom}_B(U, V)$ be multiplicity spaces. Then there is a B-linear isomorphism

$$\bigoplus_i U_i \otimes M_{V,U_i} \xrightarrow{\cong} V$$
$$\sum_i u_i \otimes \varphi_i \longmapsto \sum_i \varphi_i(u_i).$$

We can compute $M_{V,U}$ from an algebraic perspective.

Definition 6.1. Define the *centralizer* of B in A to be

$$\mathcal{Z}_B(A) = \{a \in A : a\tau(b) = \tau(b)a \text{ for all } b \in B\}.$$

Exercise 6.1. Prove the following:

1. $\mathcal{Z}_A(A) = \mathcal{Z}(A)$.
2. $\mathcal{Z}_B(A)$ is a subalgebra of A.

Lemma 6.1. *There is an isomorphism $\mathcal{Z}_B(A) \cong \bigoplus_{U,V} \mathrm{End}(M_{V,U})$, with U, V such that $M_{V,U} \neq 0$.*

Proof. We have the isomorphism

$$A \xrightarrow{\cong} \bigoplus_V \mathrm{End}(V),$$

and we can view $\tau : B \to A$ as $(\tau_V)_{V \in \mathrm{Irr}(A)}$, where $\tau_V : B \to \mathrm{End}(V)$. Similarly, we can view an element $a \in A$ as $(a_V) \in \bigoplus_V \mathrm{End}(V)$. Then $a \in \mathcal{Z}_B(A)$ if and only if $a_V \in \mathcal{Z}_B(\mathrm{End}(V))$ for all V, so

$$\mathcal{Z}_B(A) = \bigoplus_V \mathcal{Z}_B(\mathrm{End}(V)).$$

Then $\mathcal{Z}_B(\mathrm{End}(V)) \cong \mathrm{End}_B(V) \cong \bigoplus_U \mathrm{End}(M_{V,U})$, which completes the proof. □

Remark. Show that the following actions of $\mathcal{Z}_B(A)$ on $\mathrm{End}(M_{V,U}) = \mathrm{Hom}_B(U, V)$ are the same:

1. $\mathrm{End}_B(V)$ acts on $\mathrm{Hom}_B(U, V)$ via

$$\mathrm{End}_B(V) \times \mathrm{Hom}_B(U, V) \longrightarrow \mathrm{Hom}_B(U, V)$$
$$(\alpha, \varphi) \longmapsto \alpha \circ \varphi;$$

2. for $z \in \mathcal{Z}_B(A)$, $\varphi \in \mathrm{Hom}_B(U, V)$, we can define $z\varphi \in \mathrm{Hom}_B(U, V)$ by

$$[z\varphi](u) = z\varphi(u),$$

where the right-hand side is the A-action on V.

Corollary 6.0.1. *The following conditions are equivalent:*

1. *for all $U \in \mathrm{Irr}(B)$ and $V \in \mathrm{Irr}(A)$, we have $\dim \mathrm{Hom}_B(U, V) \le 1$;*
2. *$\mathcal{Z}_B(A)$ is commutative.*

Proof. $\mathcal{Z}_B(A) = \bigoplus_{U,V} \mathrm{End}(M_{V,U})$ is commutative if and only if $\mathrm{End}(M_{V,U})$ has dimension 1 or 0. $\square$

Example 6.1.1. Let $A = \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ and $B = \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 2}$. Define $\tau : B \to A$ by

$$\tau(x_1, x_2, x_3) = (\mathrm{diag}(x_1, x_2, x_2), \mathrm{diag}(x_1, x_3)), \quad x_1 \in \mathrm{Mat}_2(\mathbb{C}), x_2, x_3 \in \mathbb{C}.$$

We have B-modules U_1, U_2, U_3 of dimensions $2, 1, 1$ and A-modules V_1, V_2 of dimensions $4, 3$. Note that M_{V_1,U_2} is 2-dimensional, and M_{V_1,U_1}, M_{V_2,U_1}, M_{V_2,U_3} are 1-dimensional. So far, we have

$$\mathcal{Z}_B(A) \cong \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C}^{\oplus 3}.$$

To verify this directly, we know that $\mathcal{Z}_B(A)$ consists of pairs $(y_1, y_2) \in \mathrm{Mat}_4(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C})$ such that y_1 commutes with $\mathrm{diag}(x_1, x_2, x_2)$ and y_2 commutes with $\mathrm{diag}(x_1, x_3)$. So

$$y_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & d & e \end{pmatrix}, \quad y_2 = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix}.$$

So $\mathcal{Z}_B(A)$ is parametrized by the 2×2 matrix and the 3 scalars a, f, g.

6.3 The Inductive Approach: Properties of $\mathbb{C}S_n$

Remark. Let $S_m \subseteq S_n$ for $m < n$, and let $\mathcal{Z}_m(n)$ be the corresponding centralizer for group algebras.

Lemma 6.2. *Let $H \subseteq G$ be finite groups. Then $\mathbb{Z}_{\mathbb{C}H}(\mathbb{C}G) \subseteq \mathbb{C}G$ consists of elements of the form $\sum_{g \in G} a_g g$ such that $a_{hgh^{-1}} = a_g$ for all $h \in H$. In particular, $\mathcal{Z}_{\mathbb{C}H}(\mathbb{C}G)$ has a basis indexed by the H-conjugacy classes in G, given by (for a conjugacy class C)*

$$C \longmapsto b_C = \sum_{g \in C} g \in \mathcal{Z}_{\mathbb{C}H}(\mathbb{C}G).$$

Example 6.1.2. Note that for $\mathbb{C}S_m \subseteq \mathbb{C}S_n$, conjugation permutes the first m elements. For example, for $S_3 \subseteq S_6$, we can write a conjugacy class as $(*\,*\,4)(5\,*)(6)$, which contains elements like $(1\ 2\ 4)(5\ 3)$ and $(2\ 3\ 4)(5\ 1)$. For $m = n - 1$, consider the conjugacy class $(*\ n)$, which consists of

$$(1\ n), \quad (2\ n), \quad \ldots, \quad (n-1\ n).$$

Then the basis element $b_{(*\,n)}$ (called the nth *Jucys-Murphy element*) is given by

$$b_{(*\,n)} = \sum_{i=1}^{n-1} (i\ n).$$

Lecture 7

Sept. 10 — Representations of S_n, Part 2

7.1 Properties of $\mathbb{C}S_n$, Continued

Remark. We will now determine algebra generators of $\mathcal{Z}_m(n)$. It contains

1. $\mathcal{Z}_m(m)$: the center of $\mathbb{C}S_m$;
2. $S_{[m+1,n]}$: the subgroup of S_n containing permutations fixing $1, \ldots, m$;
3. $J_k = \sum_{i=1}^{k-1} (i\ k)$ for $k = m+1, \ldots, n$.

Note that $J_{m+1}, \ldots, J_n$ pairwise commute (check this as an exercise).

Theorem 7.1. *The algebra $\mathcal{Z}_m(n)$ is generated by the subalgebras $\mathcal{Z}_m(m)$, $\mathbb{C}S_{[m+1,n]}$, and the elements $J_{m+1}, \ldots, J_n$.*

Proof. Let C be an S_m-conjugacy class in S_n. Define $\deg(C)$ to be the number of elements in $\{1, \ldots, n\}$ which are moved by the corresponding permutations (for instance, $(*\ n) = (*\ *)$ has degree 2). Note that we either have $\deg(C) = 0$ or $\deg(C) \geq 2$.

Let A be the subalgebra of $\mathcal{Z}_m(n)$ generated by $\mathcal{Z}_m(m)$, $\mathbb{C}S_{n-1}$, and $J_{m+1}, \ldots, J_n$. We need to show that $b_C \in A$ for every C. Assume it is not true, and pick C of minimal degree such that $b_C \notin A$. First we show that $\deg(C) > 2$. If $\deg(C) = 2$, then we have two possibilities:

1. $C = (*\ k)$ for $k > m$. Then
$$b_{(*\,k)} = \sum_{i=1}^{m} (i\ k) = J_k - \sum_{i=m+1}^{k-1} (i\ k).$$
Then $J_k \in A$ and $\sum_{i=m+1}^{k-1} (i\ k) \in \mathbb{C}S_{[m+1,n]} \subseteq A$, so we are good.
2. $C = (k\ \ell)$ for $m < k < \ell \leq n$. Then $b_{(k\,\ell)} \in \mathbb{C}S_{[m+1,n]} \subseteq A$.
3. $C = (*\ *)$. Then $b_{(**)} \in \mathcal{Z}_m(m) \subseteq A$.

So $\deg(C) > 2$. Now assume that C has more than 1 cycle of degree ≥ 2. Write $C = C'C''$, then
$$b_{C'} b_{C''} = \alpha b_C + \sum_{C_0, \deg C_0 < \deg C} \alpha_{C_0} b_{C_0}.$$

Since $b_{C'}, b_{C''}, b_{C_0} \in A$ by minimality of C, we also get $\alpha b_C \in A$, so $b_C \in A$ since $\alpha \neq 0$ (note that we may have characteristic issues here if we are not working over $\mathbb{C}$).

So we may assume C is a single cycle. Pick a cycle $(i_1\ i_2\ \cdots\ i_k) \in S_n$. Then if $j \notin \{i_1, \ldots, i_k\}$,

$$(i_1\ i_2\ \cdots\ i_k)(i_s\ j) = (i_1\ i_2\ \cdots\ i_{s-1}\ j\ i_{s+1}\ \cdots\ i_k).$$

If $j \in \{i_1, \ldots, i_k\}$, then $(i_1\ i_2\ \cdots\ i_k)(i_s\ j)$ either splits into two cycles or reduces the degree by 1.

So suppose a cycle in C has elements from $\{1, \ldots, m\}$ and $k \in \{m+1, \ldots, n\}$. We can assume that k is next to $*$. Denote by C' the cycle obtained after eliminating $*$. Then

$$b_{C'} b_{(*\,k)} = \alpha b_C + \sum_{C_0} \alpha_{C_0} b_{C_0},$$

where C_0 either contains disjoint cycles or cycles of smaller degree. Thus we get $b_{C'}, b_{(*\,k)}, b_{C_0} \in A$ by the minimality of C, so $b_C \in A$ as well.

Thus we may assume the elements in our 1-cycle C sit in either $\{1, \ldots, m\}$ or $\{m+1, \ldots, n\}$. In the first case, $b_C \in \mathcal{Z}_m(m) \subseteq A$, and in the second case, $b_C \in \mathbb{C}S_{[m+1,n]} \subseteq A$. □

Corollary 7.1.1. *We have the following:*

1. $\mathcal{Z}_{n-1}(n)$ *is commutative;*
2. *for all* $U \in \operatorname{Irr}(\mathbb{C}S_{n-1})$ *and* $V \in \operatorname{Irr}(\mathbb{C}S_n)$*, the multiplicity of* U *in* V *is either* 0 *or* 1*;*
3. *the element* J_n *acts on each irreducible* $\mathbb{C}S_{n-1}$*-submodule of* $V \in \operatorname{Irr}(\mathbb{C}S_n)$ *by a scalar.*

Proof. (1) $\mathcal{Z}_{n-1}(n)$ is generated by $\mathcal{Z}(n-1)$ and J_n, which commute.

(2) This follows from the statement about abelian centralizers for algebras.

(3) This follows from Schur's lemma. □

Example 7.0.1. We will determine how J_n acts on various modules and how they decompose:

1. $V = \mathrm{refl}_n$, which is a $\mathbb{C}S_n$-module and is given by

   $$\mathrm{refl}_n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

   As a $\mathbb{C}S_{n-1}$-module, refl_n decomposes as follows
   - $U_1 = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{C}^n : x_1 + \cdots + x_{n-1} = 0\}$. This is refl_{n-1}.
   - $U_0 = \{(-x, \ldots, -x, (n-1)x) \in \mathbb{C}^n\}$. This is the trivial representation.

   Note that $J_n = \sum_{i=1}^{n-1} (i\ n)$ acts on $(x_1, \ldots, x_n)$ by

   $$(x_1, \ldots, x_n) \longmapsto ((n-2)x_1 + x_n, \ldots, (n-2)x_{n-2} + x_n, x_1 + \cdots + x_n).$$

   On refl_{n-1}, the eigenvalue is $n-2$, and on the trivial subrepresentation, the eigenvalue is -1.

2. When $n = 4$, there was a representation V of dimension 2, given by the pull-back of refl_3 under the projection $S_4 \to S_3$. The kernel of the projection is the normal subgroup

   $$\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\},$$

where S_3 permutes (1 2)(3 4), (1 3)(2 4), and (1 4)(2 3). Now

$$J_4 = (1\ 4) + (2\ 4) + (3\ 4),$$

and we are looking for an action of J_4 on V. We can take

$$J_4|_V = (2\ 3) + (1\ 3) + (1\ 2),$$

which is an element of $\mathbb{C}S_3$. Note that refl_3 is given by

$$\text{refl}_3 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + x_2 + x_3 = 0\}.$$

When $J_4|_V$ acts on refl_3, we get $x_1 + x_2 + x_3 = 0$ in every coordinate, for any $(x_1, x_2, x_3) \in \text{refl}_3$, so the eigenvalue in this case is 0.

7.2 Branching Graphs

Remark. Let V^n be an irrep for $\mathbb{C}S_n$. We know that V^n decomposes into a direct sum of non-isomorphic $\mathbb{C}S_{n-1}$-modules. These then decompose into $\mathbb{C}S_{n-2}$-modules, and so on.

Definition 7.1. The *branching graph* is a directed graph, where the vertices are labeled by isomorphism classes of $\mathbb{C}S_n$-modules (for all n), and the edge $U \to V$ exists if V is an irreducible module for $\mathbb{C}S_n$ and U is an irreducible module for $\mathbb{C}S_{n-1}$ which occurs in the decomposition of V.

Example 7.1.1. The following is the branching graph up to S_4:

$$\begin{tikzcd}
\text{triv}_4 & & \text{refl}_4 & \mathbb{C}^2 & \text{refl}_4 \otimes \text{sign}_4 & & \text{sign}_4 \\
& \text{triv}_3 \arrow[ul] \arrow[ur] & & \text{refl}_3 \arrow[ul] \arrow[u] \arrow[ur] & & \text{sign}_3 \arrow[ul] \arrow[ur] & \\
& & \text{triv}_2 \arrow[ul] \arrow[ur] & & \text{sign}_2 \arrow[ul] \arrow[ur] & & \\
& & & \text{triv}_1 \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Note that there is a left-right symmetry in the graph, which comes from tensoring with sign_n.

Definition 7.2. Let $V^m \in \text{Irr}(\mathbb{C}S_m)$ and $V^n \in \text{Irr}(\mathbb{C}S_n)$ for $m < n$. Define $\text{Path}(V^m, V^n)$ to be the set of all paths from V^m to V^n in the branching graph. If $m = 1$, we write $\text{Path}(V^n) = \text{Path}(V^1, V^n)$, and we denote $\text{Path}_n = \bigsqcup_{V^n \in \text{Irr}(\mathbb{C}S_n)} \text{Path}(V^n)$.

Remark. For $\overline{P} = (V^m \to V^{m+1} \to \cdots \to V^n) \in \text{Path}(V^m, V^n)$, denote by $V^m(\overline{P})$ a copy of V^m in V^n according to the path $\overline{P}$. Then we can write the decomposition of V^n by

$$V^n = \bigoplus_{V^m \in \text{Irr}(\mathbb{C}S_m)} \bigoplus_{\overline{P} \in \text{Path}(V^m, V^n)} V^m(\overline{P}).$$

Definition 7.3. Denote by $\varphi_{\overline{P}} : V^m \to V^n$ the homomorphism sending V^m to its copy in V^n according to the path $\overline{P}$, which is defined uniquely up to rescaling, and define

$$w_{\overline{P}} = (w_{m+1}, \ldots, w_n) \in \mathbb{C}^{n-m}$$

where w_k is the scalar by which J_k acts on $V^{k-1} \subseteq V^k$. Call $w_{\overline{P}}$ the *weight* of $\overline{P}$.

Remark. Recall that $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$ is an irreducible $\mathcal{Z}_m(n)$-module from properties of centralizers.

Lemma 7.1. *We have the following:*

1. *The elements $\varphi_{\overline{P}}$ form a basis in $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$.*
2. *Each $\varphi_{\overline{P}}$ is an eigenvector for J_k with eigenvalue w_k, for each $k = m+1, \ldots, n$, where*

$$(w_{m+1}, \ldots, w_n) = w_{\overline{P}}.$$

Proof. (1) We can write

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n) &= \bigoplus_{V'^m \in \mathrm{Irr}(S_m)} \bigoplus_{\overline{P} \in \mathrm{Path}(V'^m, V^n)} \mathrm{Hom}(V^m, V'^m(\overline{P})) \\ &= \bigoplus_{\overline{P} \in \mathrm{Path}(V^m, V^n)} \mathrm{Hom}(V^m, V^m(\overline{P})),\end{aligned}$$

where the second equality is by Schur's lemma. By Schur's lemma again, $\mathrm{Hom}(V^m, V^m(\overline{P})) \cong \mathbb{C}$. Since the $\varphi_{\overline{P}}$ correspond to these summands, this proves (1).

(2) For any $u \in V^m$, we have $[J_k \varphi_{\overline{P}}](u) = J_k[\varphi_{\overline{P}}(u)]$. By construction, $V^m(\overline{P})$ lies in some copy of V^{k-1} in V^k for $k = m+1, \ldots, n$, so $J_k \varphi_{\overline{P}} = w_k \varphi_{\overline{P}}$ implies (2). $\square$

Lecture 8

Sept. 15 — Representations of S_n, Part 3

8.1 More on Branching Graphs

Remark. Consider $\mathrm{Hom}_{\mathbb{C}S_m}(V^m, V^n)$. When $m = 1$, we may identify $\mathrm{Hom}_{\mathbb{C}S_1}(V^1, V^n) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V^n)$ with V^n itself. For $P \in \mathrm{Path}(V^n)$, we will write v_P for φ_P.

Corollary 8.0.1. *We have the following:*

1. *the vectors v_P for $P \in \mathrm{Path}(V^n)$ form a basis in V^n;*
2. *each v_P is an eigenvector for J_k with eigenvalue w_k for $k = 1, \ldots, n$. Note $w_1 = 0$ since $J_1 = 0$.*

Example 8.0.1. Consider the following:

1. $V^n = \mathrm{refl}_n$. We have $\mathrm{refl}_n \cong \mathrm{refl}_{n-1} \oplus \mathrm{triv}_{n-1}$. When $n = 2$, we have $\mathrm{refl}_2 = \mathrm{triv}_1$. Then any path $P \in \mathrm{Path}(V^n)$ must be of the form

$$P = \mathrm{triv}_1 \to \cdots \to \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \cdots \to \mathrm{refl}_n .$$

The corresponding weights are $w_P = (0, 1, \ldots, i-1, -1, i, \ldots, n-2)$: Recall from before that J_k acts on $\mathrm{refl}_{k-1} \subseteq \mathrm{refl}_k$ by $k-2$ and $\mathrm{triv}_{k-1} \subseteq \mathrm{refl}_k$ by -1.

Exercise 8.1. Check that $v_P = (1, \ldots, 1, -i, 0, \ldots, 0)$ in Example 8.0.1 (there are i ones).

Exercise 8.2. Let $V = \mathbb{C}^2$ be a representation of S_4. Write down two elements in $\mathrm{Path}(\mathbb{C}^2)$ and find the corresponding weights.

Corollary 8.0.2. *Let $m < n$ and $V^m \in \mathrm{Irr}(\mathbb{C}S_m)$, $V^n \in \mathrm{Irr}(\mathbb{C}S_n)$, $\underline{P} \in \mathrm{Path}(V^m)$, $\overline{P} \in \mathrm{Path}(V^m, V^n)$. Let P be the path obtained by concatenating $\underline{P}$ and $\overline{P}$. Then v_P is proportional to $\varphi_{\overline{P}}(v_{\underline{P}})$.*

Proof. Both are clearly nonzero and lie in $V^1(P)$, which is one-dimensional. $\square$

8.2 Properties of Weights

Theorem 8.1. *Let $P, P' \in \mathrm{Path}_n$. If $w_P = w_{P'}$, then $P = P'$.*

Proof. The proof is by induction. The $n = 1$ case is trivial. Now suppose the statement is true for $n - 1$. Let $\underline{P}, \underline{P}' \in \mathrm{Path}_{n-1}$ be truncations of $P, P' \in \mathrm{Path}_n$. Assume that

$$\begin{cases} w_P = (w_1, \ldots, w_n), \\ w_{P'} = (w'_1, \ldots, w'_n), \end{cases}$$

so $w_{\underline{P}} = (w_1, \ldots, w_{n-1})$ and $w_{\underline{P'}} = (w'_1, \ldots, w'_{n-1})$. If $w_P = w_{P'}$, then we have $w_{\underline{P}} = w_{\underline{P'}}$ and thus $\underline{P} = \underline{P'}$ by the inductive hypothesis.

Now assume V, V' are the endpoints of P, P', respectively, $V, V' \in \operatorname{Irr}(\mathbb{C}S_n)$. We need to show that $V \cong V'$. Let $U \in \operatorname{Irr}(\mathbb{C}S_{n-1})$ be the endpoint of $\underline{P} = \underline{P'}$. Note that each $z \in \mathcal{Z}_{n-1}(n)$ acts on $U \subseteq V$ and $U \subseteq V'$ as a scalar. Denote these scalars by $\chi(z)$ and $\chi'(z)$, and note that $\chi(z) = \chi'(z)$: We know that $\mathcal{Z}_{n-1}(n)$ is generated by $\mathcal{Z}_{n-1}$ and J_n, any $z \in \mathcal{Z}_{n-1}$ acts on U as a scalar with $\chi(z) = \chi'(z)$, and J_n acts on both U's embedded in V, V' by w_n, so $\chi(J_n) = \chi'(J_n) = w_n$.

Let $\mathcal{Z}_n(n)$ be the center of $\mathbb{C}S_n$, which is contained in $\mathcal{Z}_{n-1}(n)$. Every $z \in \mathcal{Z}_n(n)$ acts on V and V' as scalars $\chi_V(z)$ and $\chi_{V'}(z)$, which must be the same scalars by which z acts on U. Then χ_V and $\chi_{V'}$ are the same central characters, so we find that $V \cong V'$. $\square$

Definition 8.1. Define $\mathrm{Wt}_n = \{w_P : p \in \mathrm{Path}_n\}$. We say that two elements in Wt_n are *r-equivalent* (the r is for "representation") if the weights of the two paths are in the same irreducible module.

Remark. Theorem 8.1 states that there is a one-to-one correspondence $\mathrm{Path}_n \longleftrightarrow \mathrm{Wt}_n$. Moreover, r-equivalence is an equivalence relation and gives a one-to-one correspondence between equivalence classes and isomorphism classes of irreducible representations.

Theorem 8.1 also implies that the basis vectors v_P for $P \in \mathrm{Path}(V^n)$ are in bijection with weights in the corresponding equivalence class. Thus it suffices to study weights going forward.

Remark. We now see what happens when we vary paths. Consider a path

$$P = (V^1 \to \cdots \to V^n) \in \mathrm{Path}_n .$$

Pick $i \in \{1, \ldots, n-1\}$, and consider the space of all paths of the form

$$P' = (V'^1 \to \cdots \to V'^n), \quad \text{where } V'^j = V^j \text{ for } j \neq i.$$

Denote this set by $\mathrm{Path}(P, i)$. We will prove the following theorem later:

Theorem 8.2. *Let $w_P = (w_1, \ldots, w_n)$. Then the following are true:*

1. *$w_i \neq w_{i+1}$;*
2. *if $w_{i+1} = w_i \pm 1$, then* $\mathrm{Path}(P, i) = \{P\}$*;*
3. *if $w_{i+1} \neq w_i \pm 1$, then* $\mathrm{Path}(P, i)$ *consists of two elements P, P' and $w_{P'}$ is obtained from w_P by permuting w_i, w_{i+1};*
4. *if $i < n-1$, then $w_i = w_{i+1} \pm 1$ implies $w_{i+2} \neq w_i$.*

Remark. To simplify notation, denote $V = V^n$, $\mathcal{Z}_{i-1}(i+1) \subseteq \mathbb{C}S_n$, and

$$V_{P,i} = \mathrm{Span}\{v_{P'} : P' \in \mathrm{Path}(P, i)\}.$$

Note that the $v_{P'}$ actually form a basis of $V_{P,i}$.

Proposition 8.1. *The subspace $V_{P,i} \subseteq V$ is an irreducible $\mathcal{Z}_{i-1}(i+1)$-module.*

Proof. Let $P = P_0 P_1 P_2$, where $P_0 \in \mathrm{Path}(V^{i-1})$, $P_1 \in \mathrm{Path}(V^{i-1}, V^{i+1})$, and $P_2 \in \mathrm{Path}(V^{i+1}, V^n)$. Then $\mathrm{Path}(P, i)$ consists of paths of the form $P_0 P'_1 P_2$, where $P'_1 \in \mathrm{Path}(V^{i-1}, V^{i+1})$. We have

$$V_{P_0 P'_1 P_2} = \varphi_{P_2}(\varphi_{P'_1}(v_{P_0})).$$

Now consider the linear map

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1}) &\longrightarrow V\\ \psi &\longmapsto \varphi_{P_2}(\psi(v_{P_0})).\end{aligned}$$

Note that we have $\varphi_{P_1'} \mapsto v_{P_0P_1'P_2}$ in $V_{P,i}$, where the $v_{P_0P_1'P_2}$ form a basis of $V_{P,i}$ and the $\varphi_{P_1'}$ form a basis in $\mathrm{Hom}_{\mathbb{C}S_{i-1}}(V^{i-1}, V^{i+1})$. In particular, this map is injective with image $V_{P,i}$.

It only remains to show that this map is $\mathcal{Z}_{i-1}(i+1)$-linear, which is left as an exercise. □

8.3 The Degenerate Affine Hecke Algebra

Remark. We want to study $\mathcal{Z}_{i-1}(i+1) \subseteq \mathbb{C}S_n$ better. We know $\mathcal{Z}_{i-1}(i+1)$ is generated by $\mathcal{Z}_{i-1}(i-1)$, J_i, J_{i+1}, and $(i, i+1)$, and we know that $V_{P,i}$ is an irreducible representation for $\mathcal{Z}_{i-1}(i+1)$. Note that the elements in $\mathcal{Z}_{i-1}(i-1)$ act as scalars, so we only need to worry about J_i, J_{i+1}, and $(i, i+1)$.

Lemma 8.1. *We have the following relations:*

1. $J_iJ_{i+1} = J_{i+1}J_i$;
2. $(i, i+1)^2 = 1$;
3. $(i, i+1)J_i = J_{i+1}(i, i+1) - 1$.

Proof. We already know (1) and (2). For

$$(i, i+1)J_i(i, i+1) = \sum_{j=1}^{i-1}(j, i+1) = J_{i+1} - (i, i+1),$$

which becomes (3) after right-multiplying by $(i, i+1)$. □

Definition 8.2. Define the *degenerate affine Hecke algebra* $\mathcal{H}(2)$ to be the algebra with generators X_1, X_2, T and relations $X_1X_2 = X_2X_1$, $T^2 = 1$, and $TX_1 = X_2T - 1$ (equivalently, $X_1T = TX_2 - 1$).

Remark. There is a unique homomorphism $\mathcal{H}(2) \to \mathcal{Z}_{i-1}(i+1)$ given by

$$X_1 \mapsto J_i, \quad X_2 \mapsto J_{i+1}, \quad T \mapsto (i, i+1).$$

Corollary 8.2.1. *Let M be an irreducible module for $\mathcal{Z}_{i-1}(i+1)$. Then M stays irreducible as an $\mathcal{H}(2)$-module.*

Proof. Note that $\mathcal{Z}_{i-1}(i-1)$ is the central subalgebra of $\mathcal{Z}_{i-1}(i+1)$. Any element of the center acts as a scalar on an irreducible $\mathcal{Z}_{i-1}(i+1)$-module, so a subspace invariant under $\mathcal{Z}_{i-1}(i+1)$ is also invariant under $\mathcal{H}(2)$. This proves the claim. □

Remark. A basis of $\mathcal{H}(2)$ is given by $\{X_1^{d_1}X_2^{d_2}\sigma : \sigma \in \{1, T\}\}$.

Remark. One can generalize this construction to $\mathcal{Z}_i(d)$ to get $\mathcal{H}(d)$, with generators $X_1, \ldots, X_d$ and $T_1, \ldots, T_{d-1}$, with similar relations.

Example 8.2.1. We consider finite-dimensional irreps of $\mathcal{H}(2)$. Note that X_1, X_2 commute, so they have a common eigenvector $m \in M$. So $X_1 m = am$ and $X_2 m = bm$ for $a, b \in \mathbb{C}$. We have two cases:

1. $Tm \sim m$. Since $T^2 = 1$, we have two options:
   (a) $Tm = m$. Then $TX_1 m = am$, and applying $TX_1 = X_2 T - 1$ to m, we get
   $$(X_2 T - 1)m = (b-1)m.$$
   Thus we must have $b = a + 1$.
   (b) $Tm = -m$. Then one can check that $b = a - 1$ as an exercise.
2. m, Tm are linearly independent. Then
   $$\begin{aligned} X_1(Tm) &= (TX_2 - 1)m = b(Tm) - m, \\ X_2(Tm) &= (TX_1 + 1)m = a(Tm) + m. \end{aligned}$$
   In particular, $\mathrm{Span}\{m, Tm\}$ is stable under $\mathcal{H}(2)$. Since M is irreducible, $\{m, Tm\}$ is a basis of M. In this case, one can check that
   $$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, \quad X_2 \mapsto \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}$$
   defines an $\mathcal{H}(2)$-module on $\mathbb{C}^2$, denoted as $M(a,b)$.

Lemma 8.2. *$M(a,b)$ is irreducible if and only if $a \neq b \pm 1$. If $a \neq b \pm 1$, then $M(a,b) \cong M(a',b')$ if and only if $(a,b) = (a',b')$ or $(b,a) = (a',b')$.*

Lecture 9

Sept. 17 — Combinatorial Weights

9.1 More on the Degenerate Affine Hecke Algebra

Proof of Lemma 8.2. Assume $a \neq b$. Then X_1, X_2 have two distinct eigenvalues, hence they are diagonalizable. Since $a \neq b$, every subspace in $M(a,b)$ stable under X_1 (or X_2) must be the sum of these eigenspaces. If one has a 1-dimensional submodule for $\mathcal{H}(2)$, then T must preserve it. If $a = b \pm 1$, then $m \pm Tm$ is an eigenvector for X_1, X_2, T, so $M(a,b)$ is not irreducible.

For the last part, we can simply switch the two eigenvalues. $\square$

Proposition 9.1. *The finite-dimensional irreps of $\mathcal{H}(2)$ are classified by pairs of complex numbers (a,b), $(a,b) \mapsto L(a,b)$, where $L(a,b) \cong L(b,a)$ if $b \neq a, a \pm 1$. Moreover, we have*

1. *If $b = a+1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto 1$, $X_1 = a$, $X_2 = b$.*
2. *If $b = a-1$, then $L(a,b) = \mathbb{C}$ with $T \mapsto -1$, $X_1 = a$, $X_2 = b$.*
3. *If $b \neq a \pm 1$, then $L(a,b) \cong M(a,b)$.*
4. *The action of X_1, X_2 on $L(a,b)$ is diagonalizable if and only if $a \neq b$.*

Proof. This is Example 8.2.1 and Lemma 8.2. $\square$

Proof of Theorem 8.2. Let $w_P = (w_1, \dots, w_n)$, $P' \in \mathrm{Path}(V,i)$, and $w_{P'} = (w'_1, \dots, w'_n)$, where the w'_j depend only on V_j, V_{j-1}. Note that $V'_j = V_j$ for all $j \neq i$ implies $w'_j = w_j$ for all $j \neq i$. We have shown that $V_{P,i}$ is an irreducible $\mathcal{Z}_{i-1}(i+1)$-module and also an irreducible $\mathcal{H}(2)$-module, and that X_1, X_2 (J_i, J_{i+1}) are diagonalizable with eigenvalues (w_i, w_{i+1}) and (w'_i, w'_{i+1}). This proves (1)-(3).

(4) If $w_{i+1} = w_i \pm 1$, then $w_{i+2} \neq w_i$ (check this as an exercise). By (2), $w_{i+1} = w_i \neq 1$ implies that $V_{P,i+1}$ is also 1-dimensional, and $\mathbb{C}v_P$ is invariant under $(i,i+1)$, $(i+1,i+2)$. Now observe that

$$(i,i+1)(i+1,i+2)(i,i+1) = (i,i+2) = (i+1,i+2)(i,i+1)(i+1,i+2),$$

which is the same element. But $(i,i+1)$ and $(i+1,i+2)$ act on v_P by ± 1 and ∓ 1, respectively, so the above implies that $\mp 1 = \pm 1$, which is a contradiction. $\square$

9.2 Combinatorial Weights

Definition 9.1. We say two elements of $\mathbb{C}^n$ are *c-equivalent* (the c is for "combinatorial") if one can be obtained from the other through a sequence of *admissible* transpositions (those where the difference between two adjacent entries in the transposition is not ± 1).

Definition 9.2. A *combinatorial weight* is an element of $\mathbb{C}^n$ such that every element $(w_1, \dots, w_n) \in \mathbb{C}^n$ combinatorially equivalent to it satisfies:

1. $w_1 = 0$;
2. for all $i = 1, \dots, n-1$, $w_i \neq w_{i+1}$;
3. for all $i = 1, \dots, n-2$, we have $w_{i+1} = w_i \pm 1$ implies $w_{i+2} \neq w_i$.

Denote the set of combinatorial weights by $c\mathrm{Wt}_n$.

Corollary 9.0.1. *We have the following:*

1. $\mathrm{Wt}_n \subseteq c\mathrm{Wt}_n$*, so* Wt_n *is a collection of c-equivalence classes.*
2. *c-equivalence implies r-equivalence. Moreover,* $|\mathrm{Wt}_n/{\sim_r}| \leq |\mathrm{Wt}_n/{\sim_c}| \leq |c\mathrm{Wt}_n/{\sim_c}|$.
3. *There is a one-to-one correspondence* $\mathrm{Wt}_n/{\sim_r} \longleftrightarrow \mathrm{Irr}(\mathbb{C}S_n)$.

Lemma 9.1. *Every c-equivalence class contains elements of the form*

$$(0, 1, \dots, n_1 - 1, -1, 0, 1, \dots, n_2 - 2, -2, \dots, (1-k), \dots, n_k - k),$$

where $n_1 \geq n_2 \geq \dots \geq n_k$ *and* $n_1 + \dots + n_k = n$.

Proof. First we show that all components of combinatorial weights are integers. Suppose not, and let i be the minimal number such that $w_i \notin \mathbb{Z}$. Then we can make admissible transformations from right to left until it reaches the first slot, which is a contradiction since $w_1 = 0 \in \mathbb{Z}$.

Consider the lexicographic order on $c\mathrm{Wt}_n$, i.e. $(w_1, \dots, w_n) > (w'_1, \dots, w'_n)$ if there exists i such that $w_j = w'_j$ for each $1 \leq j < i$ and $w_i > w'_i$. Let $(w_1, \dots, w_n)$ be a maximal element in this equivalence class. We need to show that this maximal element is of the desired form.

To do this, first take n_1 such that $n_1 - 1 = \max\{w_i\}$. Let k be the smallest index such that $w_k = n_1 - 1$. We claim that $k = n_1$ and $w_i = i - 1$ for all $i < n_1$. Assume not. Then pick the largest index $j < k$ with $w_j \neq n_1 - 1 - (k - j)$. By the choice of k, we have $w_j < n_1$. We also have $w_j \geq j - 1$ (otherwise one can permute j and $j+1$, which increases the order). Note that if $w_j \geq j$, then we can make admissible transformations to the left until we arrive to (w_j, w_j), $(w_j, w_{j\pm 1}, w_j)$, or w_j in the first position, which are all impossible. Thus $w_j = n_1 - 1 - (k - j)$ for all $j < k$. But $w_1 = 0$, so $k = n_1$.

Thus we have shown that we can take an element starting with $0, 1, \dots, n_1 - 1$. Now if $n_1 = n$, then we are done. Otherwise, we need to prove that $w_{n_1+1} = -1$. Note that $w_{n_1+1} \leq n_1 - 1$ by our choice of n_1, and $w_{n_1} \neq n_1 - 1$ since $w_{n+1} \neq w_n$. If we move w_{n+1} to the left, then we encounter

$$(w_{n_1+1}, w_{n_1+1} + 1, w_{n_1+1})$$

for any $w_{n+1} \geq 0$. If $w_{n+1} < -1$, then we can move it to the first position, which is impossible since we always have $w_1 = 0$. So the only possibility is $w_{n+1} = -1$.

Now we can repeat the above argument to get the rest of the form. $\square$

Remark. Lemma 9.1 implies the following:

1. $c\mathrm{Wt}_n = \mathrm{Wt}_n$;

2. $\sim_c = \sim_r$;
3. $n_1, \dots, n_k$ uniquely characterize the equivalence class.

Example 9.2.1. Consider the following:

1. triv_4 for S_4, i.e. (x, x, x, x). Here $(w_1, w_2, w_3, w_4) = (0, 1, 2, 3)$, so $k = 1$, $n_1 = 4$.
2. refl_4 with path $P = \mathrm{triv}_1 \to \cdots \to \mathrm{triv}_i \to \mathrm{refl}_{i+1} \to \cdots \to \mathrm{refl}_n$. In this case, we have seen that
$$(w_1, \dots, w_n) = (0, 1, \dots, i-1, i, \dots, n-2).$$
For refl_4, we can get $(0, -1, 1, 2)$, $(0, 1, -1, 2)$, $(0, 1, 2, -1)$. The last one has $k = 2$, $n_1 = 3$, $n_2 = 1$.

Exercise 9.1. Compute the combinatorial weights for $\mathbb{C}^2$ (for S_4).

Lecture 10

Sept. 22 — Lie Groups

10.1 Young Tableaux

Remark. Recall there is a one-to-one correspondence between partitions $(n_1, \ldots, n_k)$ of n (satisfying $n_1 \leq \cdots \leq n_k$ and $n_1 + \cdots + n_k$) and $\operatorname{Irr}(\mathbb{C}S_n)$. Also recall the *Young tableau* for a partition, which is consists of k rows of n_k boxes stacked on top of each other, where the 1st row is at the top.

Definition 10.1. A *standard Young tableau* is a Young tableau filled with numbers $\{1, \ldots, n\}$ so that they strictly increase from bottom to top and from left to right. Denote by $\mathrm{SYT}(n)$ the set of standard Young tableaux (corresponding to a partition of n).

Definition 10.2. To a Young tableau T, assign its *content* as follows. Let (x_i, y_i) be the coordinates of the box numbered i. Then the content of the box is $x_i - y_i$. The content of the tableau is

$$c(T) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n).$$

Exercise 10.1. Show that the map $T \mapsto cT$ is injective.

Proposition 10.1. *The map $T \mapsto c(T)$ is a bijection $\mathrm{SYT}(n) \to c\mathrm{Wt}_n$, and the shape of T coincides with the partition assigned to n.*

Proof. This is an exercise in combinatorics. $\square$

Definition 10.3. The Young tableau with numbers $1, \ldots, n_1$ in the bottom row, $n_1 + 1, \ldots, n_1 + n_2$ in the second-to-bottom row, and so on is called the *normal Young tableau.*

Corollary 10.0.1. *Let λ be a Young tableau with n boxes and V_λ the corresponding $\mathbb{C}S_n$-module. Then there is a basis $\{v_T\}$ in V_λ which is labeled by $\mathrm{SYT}(n)$ associated to λ. Moreover, each v_T is an eigenvector of the Jucys-Murphy's elements such that the eigenvalue of J_i is the content $x_i - y_i$ of the ith box.*

Remark. Take $(w_1, \ldots, w_n) \in c\mathrm{Wt}_n$, and consider $(w_1, \ldots, w_{n-1})$. What does this mean in terms of $\mathrm{SYT}(n)$? The new tableau T' is obtained from the original tableau T by removing the box labeled n.

Corollary 10.0.2. *Let λ be a partition of n and V_λ the corresponding irrep of $\mathbb{C}S_n$. As $\mathbb{C}S_{n-1}$-modules,*

$$V_\lambda \cong \bigoplus_\mu V_\mu,$$

where μ runs through all (unlabeled) Young tableaux obtained from λ by removing one box.

Definition 10.4. The *Young graph* is the directed graph whose vertices are Young tableaux and we have an edge $\mu \to \lambda$ if μ is obtained from λ by removing one box.

Corollary 10.0.3. *There is a graph isomorphism between the Young graph and the branching graph.*

Exercise 10.2. Prove that tensoring any V_λ with sign_n gives a transposed Young tableau.

10.2 Lie Groups

Remark. We will denote a C^∞ manifold by M, and its tangent space at $m \in M$ by $T_m M$. Denote by

$$TM = \bigsqcup_{m \in M} T_m M$$

the tangent bundle of M, and $\mathrm{Vect}(M)$ the sections of TM. If $f : X \to Y$ is a C^∞ map, then we denote its differential at $x \in X$ by $T_x f : T_x X \to T_{f(x)} Y$.

Recall that a map $f : X \to Y$ is an *immersion* if $\operatorname{rank} T_x f = \dim X$ for all $x \in X$. In this case, by the inverse function theorem, we can choose local coordinates around x and $f(x)$ such that

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

An *immersed submanifold* $N \subseteq M$ is a subset with the structure of a manifold such that $i : N \hookrightarrow M$ is an immersion (the topology of N need not be inherited from M). An *embedded submanifold* $N \subseteq M$ is an immersed submanifold such that $i : N \hookrightarrow M$ is also a homeomorphism onto its image.

Example 10.4.1. The figure eight curve $\mathbb{R} \to \mathbb{R}^2$ is an immersed submanifold but not embedded.

Definition 10.5. A *(real) Lie group* G is a group with a manifold structure such that the multiplication $G \times G \to G$ and inversion $G \to G$ are C^∞ maps. A *morphism* of Lie groups is a C^∞ map f such that

$$f(gh) = f(g)f(h) \quad \text{and} \quad f(1) = 1.$$

Definition 10.6. A *complex Lie group* is the same as a real Lie group, except with a complex manifold structure, i.e. there are charts to $\mathbb{C}^n$ such that the transition maps are analytic.

Example 10.6.1. The following are examples of Lie groups:

1. $\mathbb{R}^n$ with addition.
2. $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ with multiplication, which has two components $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$.
3. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with multiplication.
4. $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ with matrix multiplication.
5. $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) : A\overline{A}^T = 1, \det A = 1\}$, or

   $$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \cong S^3 \subseteq \mathbb{R}^4.$$

   Note that $\mathrm{SU}(2)$ is a real Lie group.
6. The *classical groups* $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{R})$, $\mathrm{O}(n, \mathbb{C})$, $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{Sp}(n, \mathbb{C})$, etc.

Theorem 10.1. *Let G be a (real or complex) Lie group. Denote by G^0 the connected component of the identity. Then G^0 is a normal subgroup of G and is a Lie group. The quotient is a discrete group.*

Proof. Note that the image of a connected topological space under a continuous map is connected, so the inverse map sends $G^0 \to G^0$. The same argument works for multiplication, so G^0 is a Lie group.

To show that G^0 is normal, let $h \in G^0$. Note that for any g, the map $h \mapsto ghg^{-1}$ is continuous, so ghg^{-1} must lie in the same connected component G^0. Thus G^0 is a normal subgroup of G.

Finally, the quotient is discrete since G^0 is open and its cosets partition G. □

Theorem 10.2. *If G is a connected (real or complex) Lie group, then its universal cover $\widetilde{G}$ has a canonical structure of a Lie group such that the covering map $p : \widetilde{G} \to G$ is a morphism of Lie groups, and $\ker p \cong \pi_1(G)$ is discrete and central.*

Definition 10.7. A *closed Lie subgroup* H of a (real or complex) Lie group G is a subgroup which is a submanifold (complex submanifold in the complex case).

Theorem 10.3 (Cartan's theorem)**.** *Let G be a (real or complex) Lie group.*

1. *Any closed Lie subgroup is closed in G.*
2. *Any closed subgroup of a Lie group is a closed real Lie subgroup.*

Corollary 10.3.1. *We have the following:*

1. *If G is a connected (real or complex) Lie group and U is a neighborhood of 1, then U generates G.*
2. *Let $f : G_1 \to G_2$ be a morphism of (real or complex) Lie groups, where G_2 is connected and the differential $T_1 f : T_1 G_1 \to T_1 G_2$ at the identity is surjective. Then f is surjective.*

Proof. (1) Assume H is a subgroup generated by U. Then H is open in G, since for any $h \in H$, hU is a neighborhood of h in G. Since H is an open subset of a manifold, H is a submanifold. Then H is a closed Lie subgroup of G, so it is also closed. Thus $H = G$ since G is connected.

(2) Check this as an exercise. □

Lecture 11

Sept. 24 — Lie Groups, Part 2

11.1 More on Lie Groups

Theorem 11.1. *Let G be a (real or complex) Lie group with $\dim G = n$, and $H \subseteq G$ a closed Lie subgroup with $\dim H = k$. Then the coset space G/H has the structure of a manifold with dimension $n-k$, such that $p : G \to G/H$ is a fiber bundle with fibers diffeomorphic to H. The tangent space at $\overline{1} = p(1)$ is given by T_1G/T_1H.*

Proof. Consider $p : G \to G/H$, which sends $g \mapsto \overline{g} = p(g)$. Note that $gH \subseteq G$ is a submanifold (since multiplication by g is a diffeomorphism). Choose a submanifold M which is transversal to gH (i.e. such that $T_gG = T_g(gH) \oplus T_gM$). Let U be a sufficiently small neighborhood of g so that $UH = \{uh : u \in U, h \in H\}$ is open in G, which exists by the inverse function theorem applied to the multiplication map $U \times H \to G$. Then let $\overline{U} = p(U)$. Since $p^{-1}(\overline{U}) = U$ is open, $\overline{U}$ is an open neighborhood of $\overline{g}$ in G/H. This gives $p : G \to G/H$ the natural structure of a fiber bundle.

For the tangent space, consider the map $T_1p : T_1G \to T_{\overline{1}}G/H$, and note that $\ker(T_1p) = T_1H$. □

Corollary 11.1.1. *Let H be a closed Lie subgroup of G.*

1. *If H is connected, then the set of connected components satisfies $\pi_0(G) = \pi_0(G/H)$.*
2. *If G, H are connected, then there is an exact sequence*

$$\pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow \{1\}.$$

Remark. Often, $\pi_2(G/H)$ and $\pi_1(G/H)$ are known, which allows us to compute $\pi_1(G)$.

Example 11.0.1. Let $G_1 = \mathbb{R}$ and $G_2 = \mathbb{R}^2/\mathbb{Z}^2$ (the torus). Define $f : G_1 \to G_2$ by

$$f(t) = (t \bmod \mathbb{Z}, \alpha t \bmod \mathbb{Z}),$$

for some fixed irrational α. Then the image of f is everywhere dense in G_2.

Definition 11.1. A *Lie subgroup* H in a (real or complex) Lie group G is an immersed submanifold which is also a subgroup.

Theorem 11.2. *Let $f : G_1 \to G_2$ be a morphism (in the real or complex sense). Then $H = \ker f$ is a normal closed Lie subgroup of G_1, and f gives rise to an injective map $G_1/H \to G_2$ which is an immersion. In particular, $\operatorname{Im} f$ is a Lie subgroup of G_2. If $\operatorname{Im} f$ is an embedded submanifold, then it is a closed Lie subgroup. Moreover, f gives an isomorphism (of Lie groups) $G_1/H \cong \operatorname{Im} f$.*

Proof. We will prove this later using Lie algebras. □

11.2 Actions of Lie Groups on Manifolds

Definition 11.2. An *action* of a real Lie group G on a real manifold M is an assignment

$$g \mapsto \rho(g) \in \mathrm{Diff}(M)$$

with $\rho(1) = \mathrm{Id}$ and $\rho(g)\rho(h) = \rho(gh)$, such that the map $G \times M \to M$ by $(g, m) \mapsto g.m$ is smooth. When G is a complex Lie group and M is a complex manifold, we require $G \times M \to M$ to be analytic.

Example 11.2.1. The following are examples of actions on Lie groups:

1. $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$.
2. $\mathrm{O}(n, \mathbb{R})$ acts on $S^{n-1} \subseteq \mathbb{R}^n$.
3. $\mathrm{U}(n)$ acts on $S^{2n-1} \subseteq \mathbb{C}^n$.

Definition 11.3. A *representation* of a (real or complex) Lie group G is a vector space V (complex if G is complex and real or complex if G is real) together with a homomorphism $\rho : G \to \mathrm{GL}(V)$. If V is finite-dimensional, we require ρ to be smooth (or analytic if G is complex).

A *morphism* between two representations ρ_V and ρ_W is a map $f : V \to W$ such that it commutes with the G-action, i.e. $f\rho_V(g) = \rho_W(g)f$ for all $g \in G$.

Remark. Any action of G on a manifold M gives the following infinite-dimensional representations:

1. Space of functions (the space of analytic functions $\mathcal{O}(M)$ in the complex case or $C^\infty(M)$ in the real case), given by $\rho(g)f(m) = f(g^{-1}.m)$.
2. Vector fields on M (denoted $\mathrm{Vect}(M)$), given by the *pushforward*
$$(\rho(g)v)(m) = g_* v = T_{g^{-1}.m}(g)(v(g^{-1}.m)).$$
3. Assume m is a fixed point of G, i.e. $g.m = m$ for all $g \in G$. Then G acts on $T_m M$ by differentials
$$T_m g : T_m M \to T_m M.$$
This representation is finite-dimensional if $\dim M < \infty$.

11.3 Orbits and Homogeneous Spaces

Definition 11.4. Define the *orbit* of a point $m \in M$ to be

$$\mathcal{O}_m = G.m = \{g.m : g \in G\}.$$

and the *stabilizer* of m to be $G_m = \{g \in G : g.m = m\}$.

Theorem 11.3. *Let M be a manifold with action of Lie group G (or complex manifold with action of complex G). Then for all $m \in M$, the stabilizer G_m is a closed Lie subgroup of G, and $g \mapsto g.m$ forms an injective immersion $G/G_m \hookrightarrow M$ whose image coincides with $\mathcal{O}_m$.*

Corollary 11.3.1. *The orbit $\mathcal{O}_m$ is an immersed submanifold in M with tangent space*

$$T_m \mathcal{O}_m = T_1 G / T_1 G_m.$$

If $\mathcal{O}_m$ is a submanifold, then $g \mapsto g.m$ gives a diffeomorphism $G/G_m \to \mathcal{O}_m$.

Definition 11.5. If the action of G on M is transitive (i.e. there is just one orbit), then we call M a *homogeneous space* for G.

Corollary 11.3.2. *Let M be a G-homogeneous space. Then the map $G \to M$ by $g \mapsto g.m$ is a fiber bundle over M with fiber G_m.*

Example 11.5.1. Consider the following:

1. $\mathrm{SO}(n, \mathbb{R})$ acting on $S^{n-1} \subseteq \mathbb{R}^n$. Then S^{n-1} is a homogeneous space, and the stabilizer of any point in S^{n-1} (which can be moved to $(1, 0, \ldots, 0)$) is $\mathrm{SO}(n-1, \mathbb{R})$. So we have the diagram

$$\begin{tikzcd} \mathrm{SO}(n-1,\mathbb{R}) \arrow[r] & \mathrm{SO}(n,\mathbb{R}) \arrow[d, "p"] \\ & S^{n-1} \end{tikzcd}$$

2. $\mathrm{SU}(n)$ acting on $S^{2n-1} \subseteq \mathbb{C}^n$. Here we have

$$\begin{tikzcd} \mathrm{SU}(n-1) \arrow[r] & \mathrm{SU}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

Remark. The action of G can be used to define a smooth structure on M. If M is a set with a transitive action by G, then M is in bijection with G/H, where $H = \mathrm{Stab}_G(m)$. Then M has a natural structure of a manifold of dimension $\dim G - \dim H$.

Example 11.5.2. A *(full) flag* in $\mathbb{R}^n$ is a collection of subspaces

$$\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{R}^n,$$

where $\dim V_i = i$. Denote by $\mathcal{F}_n(\mathbb{R})$ the space of all flags in $\mathbb{R}^n$. There is an action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathcal{F}_n(\mathbb{R})$. We can move any flag to the *standard flag*

$$V^{\mathrm{st}} = \{0\} \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, \ldots, e_n \rangle,$$

which has stabilizer $\mathrm{Stab}\, V^{\mathrm{s,t}} = \mathrm{B}(n, R) \subseteq \mathrm{GL}(n, \mathbb{R})$, the subgroup upper-triangular matrices, so

$$\mathcal{F}_n(\mathbb{R}) \cong \frac{\mathrm{GL}(n, \mathbb{R})}{\mathrm{B}(n, \mathbb{R})}.$$

Now $\dim \mathrm{B}(n, \mathbb{R}) = n(n+1)/2$, so we can see that

$$\dim \mathcal{F}_n(\mathbb{R}) = \dim \mathrm{GL}(n, \mathbb{R}) - \dim \mathrm{B}(n, \mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

11.4 Actions of a Lie Group on Itself

Remark. We can define actions $L_g : G \to G$ and $R_g : G \to G$ by

$$L_g(h) = gh \quad \text{and} \quad R_g(h) = hg^{-1}.$$

There is also an *adjoint action* $\mathrm{Ad}_g : G \to G$ by $\mathrm{Ad}_g(h) = L_g R_g(h) = ghg^{-1}$.

For $v \in T_m G$, we will write $g.v$ for $T_m L_g$ and $v.g$ for $T_m R_{g^{-1}}$.

Exercise 11.1. Check that the above agrees with matrix multiplication for $G = \mathrm{GL}(n, \mathbb{R})$.

Remark. Note that Ad_g sends $1 \mapsto 1$, so there is a representation $\mathrm{Ad}_g : T_1G \to T_1G$, called the *adjoint representation* of a Lie group G.

Definition 11.6. A vector field $v \in \mathrm{Vect}(G)$ is called *left-invariant* if $g.v = v$ for all $g \in G$, and v is called *right-invariant* if $v.g = v$ for all $g \in G$.

Theorem 11.4. *The map $v \mapsto v(1)$ (where 1 is the identity of G) defines an isomorphism of the vector space of left-invariant vector fields on G with T_1G. Similarly, one has the same isomorphism for the vector space of right-invariant vector fields.*

Theorem 11.5. *The map $v \mapsto v(1)$ defines an isomorphism of the vector space of bi-invariant vector fields on G with the vector space of invariants under the adjoint action, i.e.*

$$(T_1G)^{\mathrm{Ad}\, G} = \{x \in T_1G : \mathrm{Ad}_g(x) = x \textit{ for all } g \in G\}.$$

Lecture 12

Sept. 29 — The Exponential Map

12.1 Classical Lie Groups

Example 12.0.1. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. The *classical Lie groups* are

1. the *general linear group* $\mathrm{GL}(n, \mathbb{K})$,
2. the *special linear group* $\mathrm{SL}(n, \mathbb{K})$,
3. the *orthogonal group* $\mathrm{O}(n, \mathbb{K})$,
4. the *special orthogonal groups* $\mathrm{SO}(n, \mathbb{K})$ and $\mathrm{SO}(p, q)$,
5. the *symplectic group* $\mathrm{Sp}(n, \mathbb{K})$,
6. the *unitary groups* $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ which are real Lie groups,
7. the *compact symplectic group* $\mathrm{USp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$ which is a real Lie group.

Remark. How do we compute the dimensions of these classical Lie groups?

12.2 The Exponential Map for Matrix Groups

Remark. For $\mathrm{GL}(n, \mathbb{K})$, write its Lie algebra as $\mathfrak{gl}(n, \mathbb{K})$.

Definition 12.1. For $x \in \mathrm{Mat}_n(\mathbb{K}) = \mathfrak{gl}(n, \mathbb{K})$, define the *exponential map*

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This defines an analytic map $\mathfrak{gl}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ with inverse map in a neighborhood of I given by

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}.$$

Theorem 12.1. *We have the following:*

1. $\log(\exp(x)) = x$ *and* $\exp(\log(x)) = x$*.*
2. $\exp(x) = 1 + x + \dots$, $\exp(0) = 1$*, and* $d\exp(0) = \mathrm{Id}$*.*

3. *If* $xy = yx$, *then* $\exp(x+y) = \exp(x)\exp(y)$*; if* X *and* Y *commute (for* X, Y *in some neighborhood of* I*), then* $\log(XY) = \log(X) + \log(Y)$*; also,* $\exp(-x)\exp(x) = \mathrm{Id}$*, so* $\exp(x) \in \mathrm{GL}(n, \mathbb{K})$*.*
4. *For any* $x \in \mathfrak{gl}(n, \mathbb{K})$*, the map* $\mathbb{K} \to \mathrm{GL}(n, \mathbb{K})$ *by* $t \mapsto \exp(tx)$ *is a morphism of Lie groups. So in particular one has* $\exp((t+s)x) = \exp(tx)\exp(sx)$*.*
5. $\exp(AxA^{-1}) = A\exp(x)A^{-1}$ *and* $\exp(x^T) = (\exp(x))^T$*.*

Theorem 12.2. *For any classical subgroup* $G \subseteq \mathrm{GL}(n, \mathbb{K})$*, there exists a vector space* $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{K})$ *such that for some neighborhood* U *of* 1 *in* $\mathrm{GL}(n, \mathbb{K})$ *and some neighborhood* V *of* 0 *in* $\mathfrak{gl}(n, \mathbb{K})$*, the following maps are inverses of each other:*

$$(U \cap G) \underset{\exp}{\overset{\log}{\rightleftarrows}} (V \cap \mathfrak{g})$$

Proof. We have already proved this for $\mathrm{GL}(n, \mathbb{K})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$.

Now consider $\mathrm{SL}(n, \mathbb{K})$. Let $g \in \mathrm{SL}(n, \mathbb{K})$ be close enough to the identity, so that $g = \exp(x)$ for some $x \in \mathfrak{gl}(n, \mathbb{K})$. Then $1 = \det(g) = \det(\exp(x))$. Now recall that

$$\det(\exp(x)) = \exp(\operatorname{tr} x),$$

which can be proved using the Jordan normal form. So this $\deg(g) = 1$ if and only if $\operatorname{tr} x = 0$. Thus we can take $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K}) = \{x \in \mathfrak{gl}(n, \mathbb{K}) : \operatorname{tr} x = 0\}$.

Next consider $\mathrm{O}(n, \mathbb{K})$ and $\mathrm{SO}(n, \mathbb{K})$. For $g \in \mathrm{O}(n, \mathbb{K})$, we have $g^T g = I$. Writing $g = \exp(x)$ and $g^T = \exp(x^T)$, we have $\exp(x^T)\exp(x) = I$ since x and x^T commute. This translates to $x + x^T = 0$, so

$$\mathfrak{o}(n, \mathbb{K}) = \{x \in \mathfrak{gl}(n, \mathbb{K}) : x + x^T = 0\}.$$

Note that $\mathfrak{o}(n, \mathbb{K}) = \mathfrak{so}(n, \mathbb{K})$ ($x + x^T = 0$ implies $\operatorname{tr} x = 0$) since $\mathrm{SO}(n, \mathbb{K})$ is a neighborhood of I.

For $\mathrm{U}(n)$, one can check we have the condition $x + x^\dagger = 0$ (where $x^\dagger$ denotes the conjugate transpose of x) on the Lie algebra. This time, we do not automatically get $\operatorname{tr} x = 0$, so the Lie algebra of $\mathrm{SU}(n)$ has the two conditions $x + x^\dagger = 0$ and $\operatorname{tr} x = 0$.

One can check the remaining classical groups similarly. □

Corollary 12.2.1. *Each classical group is a Lie group with tangent space at the identity* $T_1 G = \mathfrak{g}$ *and* $\dim G = \dim \mathfrak{g}$*. Also,* $\mathrm{U}(n)$*,* $\mathrm{SU}(n)$*, and* $\mathrm{USp}(n)$ *are real Lie groups, while* $\mathrm{GL}(n, \mathbb{K})$*,* $\mathrm{SL}(n, \mathbb{K})$*,* $\mathrm{SO}(n, \mathbb{K})$*,* $\mathrm{O}(n, \mathbb{K})$*, and* $\mathrm{Sp}(n, \mathbb{K})$ *are real or complex depending on* $\mathbb{K}$*.*

12.3 The Exponential Map in General

Remark. For $\mathfrak{g} = T_1 G$, we want to define $\exp : \mathfrak{g} \to G$ for a general Lie group G.

Proposition 12.1. *Let* G *be a (real or complex) Lie group,* $\mathfrak{g} = T_1 G$*, and* $x \in \mathfrak{g}$*. Then there exists a unique morphism of Lie groups* $\gamma_x : \mathbb{K} \to G$ *such that* $\gamma_x(0) = x$*. Here* $\gamma_x(t)$ *is known as a* 1-parameter subgroup*.*

Proof. Motivated by the matrix case, where $\gamma_x(t) = \exp(tx)$ satisfies $\dot\gamma(t) = \gamma(t)\dot\gamma(0) = \gamma(t)x$, we define the differential equation

$$\dot\gamma(t) = T_1 L_{\gamma(t)} \dot\gamma(0),$$

for which it suffices to construct γ satisfying $\gamma(t+s) = \gamma(t)\gamma(s)$ by the uniqueness of solutions to the differential equation. So it suffices to show that such a γ exists. Let $\gamma(t) = \Phi^t(1)$ and $\gamma(t+s) = \Phi^{t+s}(1)$, where Φ is the flow of a left-invariant vector field. By left-invariance, we have

$$\Phi^t(g_1 g_2) = g_1 \Phi^t(g_2) \quad \text{and} \quad \Phi^{t+s}(1) = \Phi^s(\Phi^t(1)) = \Phi^s(\gamma(t)\cdot 1) = \gamma(t)\Phi^s(1) = \gamma(t)\gamma(s).$$

Thus $\gamma(t+s) = \gamma(t)\gamma(s)$, and we get the desired map $\gamma_x : \mathbb{K} \to G$. □

Remark. The uniqueness of the 1-parameter subgroups implies that $\gamma_x(\lambda t) = \gamma_{\lambda x}(t)$ since

$$\left.\frac{d\gamma_x(\lambda t)}{dt}\right|_{t=0} = \lambda x.$$

Example 12.1.1. Let $G = (\mathbb{R}, +)$ with $\mathfrak{g} = \mathbb{R}$. Then for $a \in \mathfrak{g}$, we have $\gamma_a(t) = ta$ and $\exp(a) = a$.

Example 12.1.2. Let $G = S^1 = \mathbb{R}/\mathbb{Z} = \{z \in \mathbb{C} : |z| = 1\}$, where the identification $\mathbb{R}/\mathbb{Z} \to \{z \in \mathbb{C} : |z| = 1\}$ is given by $\theta \mapsto e^{2\pi i\theta}$ for $\theta \in \mathbb{R}/\mathbb{Z}$. Then $\mathfrak{g} = \mathbb{R}$, and for $a \in \mathfrak{g}$,

$$\exp(a) = a \bmod \mathbb{Z} \quad \text{or} \quad \exp(a) = e^{2\pi i a},$$

depending on if we view S^1 as $\mathbb{R}/\mathbb{Z}$ or as $\{z \in \mathbb{C} : |z| = 1\}$.

Proposition 12.2. *Let G be a (real or complex) Lie group.*

1. *Let v be a left-invariant vector field on G. Then time flow of the vector field v is given by $g \mapsto g\exp(tx)$, where $x = v(1)$.*
2. *Let v be a right-invariant vector field on G. Then the time flow of the vector field v is given by $g \mapsto \exp(tx)g$, where $x = v(1)$.*

Theorem 12.3 (Summary)**.** *Let G be a (real or complex) Lie group and $\mathfrak{g} = T_1 G$. Then*

1. $\exp(x) = 1 + x + \dots$, $\exp(0) = 1$, *and* $T_0 \exp : T_1 G \xrightarrow{\mathrm{Id}} T_1 G$
2. *The exponential map is a diffeomorphism (analytic map for complex G) between some neighborhood of 0 in $\mathfrak{g}$ and some neighborhood of 1 in G.*
3. $\exp((t+s)x) = \exp(tx)\exp(sx)$ *for all* $t, s \in \mathbb{K}$.
4. *For any morphism of Lie groups $\varphi : G_1 \to G_2$ and any $x \in T_1 G_1$, we have*
$$\exp(T_1\varphi(x)) = \varphi(\exp(x)).$$
5. *For any $g \in G$ and $x \in \mathfrak{g}$, we have* $g\exp(x)g^{-1} = \exp(\mathrm{Ad}_g x)$.

Proof. (4) Note that $\varphi(\exp(tx))$ is a one-parameter subgroup with tangent vector at identity

$$\left.\frac{d}{dt}\right|_{t=0} \varphi(\exp(tx)) = T_1\varphi \cdot x.$$

By the uniqueness of one-parameter subgroups, this must be equal to $\exp(T_1\varphi \cdot x)$.

(5) This follows from (4) by taking φ to be conjugation by g. □

Proposition 12.3. *Let G_1, G_2 be (real or complex) Lie groups. If G_1 is connected, then any Lie group morphism $\varphi : G_1 \to G_2$ is uniquely determined by the linear map $T_1\varphi : T_1G_1 \to T_1G_2$.*

Example 12.1.3. Consider $\mathrm{SO}(3, \mathbb{R})$, and let

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which form a basis for $\mathfrak{g} = \mathfrak{so}(3, \mathbb{R})$. Then one can check that

$$\exp(tJ_z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with similar formulas for $\exp(tJ_x)$ and $\exp(tJ_y)$.

Lecture 13

Oct. 1 — Lie Algebras

13.1 Commutator Structure

Remark. Recall that for small enough $x, y \in \mathfrak{g} = T_1 G$, we have

$$\exp(x)\exp(y) = \exp(\mu(x,y)), \quad \mu(x,y) \in \mathfrak{g}.$$

Lemma 13.1. *The Taylor series for $\mu(x,y)$ is given by*

$$\mu(x,y) = x + y + \lambda(x,y) + \dots,$$

where $\dots$ stands for higher order terms in x, y, and $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear and skew-symmetric.

Proof. We can write $\mu(x,y) = \alpha_1(x) + \alpha_2(y) + Q_1(x) + Q_2(y) + \lambda(x,y) + \dots$, where α_1, α_2 are linear maps, Q_1, Q_2 are quadratic forms, and λ is bilinear. Setting $y = 0$ gives $\mu(x,0) = x$, so $\alpha_1 = x$ and $Q_1 = 0$. Similarly, setting $x = 0$ gives $\alpha_2 = y$ and $Q_2 = 0$. So it suffices to show that λ is skew-symmetric. To see this, note that $\exp(x)\exp(x) = \exp(2x)$, so $\lambda(x,x) = 0$, which implies λ is skew-symmetric. $\square$

Definition 13.1. A *commutator* of two elements $x, y \in \mathfrak{g}$ is $[x,y] = 2\lambda(x,y)$.

Proposition 13.1. *We have the following:*

1. *Let $\varphi : G_1 \to G_2$ be a morphism of (real or complex) Lie groups and $T_1\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then*

$$T_1\varphi[x,y] = [T_1\varphi x, T_1\varphi y] \quad \textit{for all } x, y \in \mathfrak{g}_1.$$

2. *The adjoint action of G on $\mathfrak{g}$ satisfies $\mathrm{Ad}_g[x,y] = [\mathrm{Ad}_g\, x, \mathrm{Ad}_g\, y]$.*
3. $\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp([x,y] + \dots)$, *where $\dots$ denotes higher order terms in x, y.*

Proof. (1) This follows from the fact that morphisms "commute" with the exponential map.

(2) Apply (1) to the conjugation morphism $\varphi(h) = ghg^{-1}$. $\square$

Corollary 13.0.1. *If G is a commutative Lie group, then $[x,y] = 0$ for all $x, y \in \mathfrak{g}$.*

Example 13.1.1. Consider a Lie subgroup $G \subseteq \mathrm{GL}(n, \mathbb{K})$. Then $[x,y] = xy - yx$ (expand $\log(e^x e^y)$).

Remark. Consider $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and associate to a morphism φ of Lie groups to a morphism $T_1\varphi$ of $\mathfrak{g}$. Note that there is a representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$.

Lemma 13.2. $\operatorname{ad} = T_1 \operatorname{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a map of tangent spaces satisfying*

1. $\operatorname{ad}_x y = [x, y]$,
2. $\operatorname{Ad}_{\exp(x)} = \exp(\operatorname{ad}_x)$.

Proof. By definition, we have

$$\operatorname{Ad}_g y = \frac{d}{dt}\bigg|_{t=0} g \exp(ty) g^{-1}.$$

Then we can write

$$\begin{aligned}\operatorname{ad}_x y &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \exp(sx)\exp(ty)\exp(-sx) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{d}{dt}\bigg|_{t=0} \exp(ty + ts[x,y] + \dots) = [x,y],\end{aligned}$$

which proves (1). Then (2) follows since Ad is a morphism of Lie groups. □

13.2 Lie Algebras

Example 13.1.2. For matrices, we have $e^x A e^{-x} = e^{\operatorname{ad}_x} A$, where $\operatorname{ad}_x = [x, \cdot]$.

Theorem 13.1. *Let G be a (real or complex) Lie group, $\mathfrak{g} = T_1 G$, and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the commutator. Then $[\cdot,\cdot]$ satisfies the following (equivalent) versions of the* Jacobi identity*:*

1. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$,
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,
3. $\operatorname{ad}_x [y, z] = [\operatorname{ad}_x y, z] + [y, \operatorname{ad}_x z]$,
4. $\operatorname{ad}_{[x,y]} = \operatorname{ad}_x \operatorname{ad}_y - \operatorname{ad}_y \operatorname{ad}_x$.

Proof. These are clearly all equivalent, so it suffices to prove (4). Let $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and note that $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ preserves the commutator. In $\mathfrak{gl}(\mathfrak{g})$, we have $[A, B] = AB - BA$, so

$$\operatorname{ad}_{[x,y]} = \operatorname{ad}_x \operatorname{ad}_y - \operatorname{ad}_y \operatorname{ad}_x .$$

This proves the identity (4). □

Definition 13.2. A *Lie algebra* over a field $\mathbb{K}$ is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity.

Example 13.2.1. Any vector space has a structure of a Lie algebra on it by $[v, v] = 0$. This is called the *abelian Lie algebra*.

Example 13.2.2. Any associative algebra over $\mathbb{K}$ can be made into Lie algebra by $[x, y] = xy - yx$.

Theorem 13.2. *Let G be a (real or complex) Lie group. Then $\mathfrak{g} = T_1 G$ has a canonical structure of a Lie algebra with commutator defined as $2\lambda(x, y)$. We sometimes write $\mathfrak{g} = \operatorname{Lie}(G)$. Moreover, every morphism of Lie groups $\varphi : G_1 \to G_2$ induces a morphism of Lie algebras $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$. If G is connected, then the map $\operatorname{Hom}(G_1, G_2) \to \operatorname{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ by $\varphi \mapsto \varphi_*$ is injective.*

Definition 13.3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a *Lie subalgebra* if it is closed under the commutator, and $\mathfrak{h} \subseteq \mathfrak{g}$ is called an *ideal* if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$.

Corollary 13.2.1. *If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{h}$ has a canonical structure of a Lie algebra.*

Theorem 13.3. *Let G be a (real or complex) Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$.*

1. *Let H be a subgroup in G (not necessarily a closed Lie subgroup). Then $\mathfrak{h} = T_1H$ is a Lie subalgebra in $\mathfrak{g}$.*
2. *Let H be a normal closed Lie subgroup in G. Then $\mathfrak{h} = T_1H$ is an ideal in $\mathfrak{g}$ and $\mathrm{Lie}(G/H) = \mathfrak{g}/\mathfrak{h}$. Conversely, if H is a closed Lie subgroup such that H, G are connected and $\mathfrak{h} = T_1H$ is an ideal, then H is normal.*

Proof. (1) If $x \in T_1H$, then we have $\exp(tx) \in H$ for all $t \in \mathbb{K}$. Then using $\lambda(x, y)$ as the commutator implies that $[x, y] \in \mathfrak{h}$ for $x, y \in \mathfrak{h}$.

(2) If H is a normal closed Lie subgroup, then we have

$$\exp(x)\exp(y)\exp(-x) \in H$$

for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. So $[x, y] \in \mathfrak{h}$, i.e. $\mathfrak{h}$ is an ideal. If $\mathfrak{h}$ is an ideal, then

$$\mathrm{Ad}_{\exp(x)} \mathfrak{h} \subseteq \mathfrak{h} \quad \text{for all } x \in \mathfrak{g}$$

since $\mathrm{Ad}_{\exp(x)} = \exp(\mathrm{ad}_x)$. Since $g\exp(y)g^{-1} = \exp(\mathrm{Ad}_g y)$ for all $y \in \mathfrak{h}$ and $g \in G$, we have

$$g\exp(y)g^{-1} \in H,$$

i.e. we have $ghg^{-1} \in H$ for all $h \in H$. Thus H is normal. □

13.3 The Lie Algebra of Diffeomorphisms

Definition 13.4. Let M be a manifold. Then $\mathrm{Diff}(M)$ is the *group of diffeomorphisms* of M.

Remark. Note that $\mathrm{Diff}(M)$ is *not* a Lie group, since it is infinite-dimensional. However, we can still think of a "Lie algebra" in this setting. Let $\varphi^t : M \to M$ be a 1-parameter family of diffeomorphisms. Then $\phi^t(m)$ for $m \in M$ defines a curve in M. Taking its derivative, we have

$$\left.\frac{d}{dt}\right|_{t=0} \varphi^t(m) \in T_mM.$$

If we look at all $m \in M$, we get a vector field $\left.\frac{d}{dt}\right|_{t=0} \varphi^t$ on M.

Definition 13.5. Define the *Lie algebra of diffeomorphisms* to be $\mathrm{Lie}(\mathrm{Diff}(M)) = \mathrm{Vect}(M)$.

Remark. For $\xi \in \mathrm{Vect}(M)$, $\exp(t\xi)$ generates a 1-parameter family of diffeomorphisms with derivative ξ at $t = 0$. So we get a differential equation

$$\left.\frac{d}{dt}\right|_{t=0} \varphi^t(m) = \xi(m),$$

which defines a time flow Φ^t for ξ. Then we can define $\exp(t\xi) = \Phi^t_\xi$.

Proposition 13.2. *We have the following:*

1. *Let* $\xi, \eta \in \mathrm{Vect}(M)$*. There exists a unique vector field* $[\xi, \eta]$ *such that*

$$\Phi_\xi^t \Phi_\eta^s \Phi_{-\xi}^t \Phi_{-\eta}^s = \Phi_{[\xi,\eta]}^{ts} + \dots.$$

2. *The commutator defines a structure of a Lie algebra on* $\mathrm{Vect}(M)$*.*
3. $[\xi, \eta] = \frac{d}{dt}\big|_{t=0} (\Phi_\xi^t)_* \eta$*, and* $\partial_{[\xi,\eta]} f = (\partial_\eta \partial_\xi - \partial_\xi \partial_\eta) f$ *satisfies*

$$\left[\sum_i f_i \partial_i, \sum_i g_i \partial_i\right] = \sum_{i,j} (g_i \partial_i f_j - f_i \partial_i g_j) \partial_j.$$

Remark. The minus sign in (3) is since $\Phi : M \to M$ acts on f by $(\Phi f)(m) = f(\Phi^{-1}(m))$.

Example 13.5.1. For $x \mapsto x + t$, we have $\Phi^t f(x) = f(x - t)$ and

$$\partial_x f = -\left.\frac{d}{dt}\right|_{t=0} \Phi^t f.$$

Theorem 13.4. *Let* G *be a finite-dimensional Lie group action on* M *and let* $\rho : G \to \mathrm{Diff}(M)$*.*

1. *This action defines a linear map* $\rho_* : \mathfrak{g} \to \mathrm{Vect}(M)$*.*
2. $\rho_*[x, y] = [\rho_* x, \rho_* y]$*, where the right-hand side is the commutator of vector fields.*

Example 13.5.2. Let $\mathrm{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n$. Let $a \in \mathfrak{gl}(n, \mathbb{R})$ and $\Phi_a^t = e^{ta}$. Then for $\vec{x} \in \mathbb{R}^n$,

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_a^t f(\vec{x}) = \left.\frac{d}{dt}\right|_{t=0} f(e^{-ta}\vec{x}) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{x} - ta\vec{x} + \dots) = -\sum_{i,j} a_{i,j} x_j \partial_{x_i} f(\vec{x}).$$

Check as an exercise that ρ_* maps $a \mapsto -a_{i,j} x_j \partial_{x_i}$ for matrices, which matches the above.

Lecture 14

Oct. 8 — Stabilizers and Center

14.1 Stabilizers

Proposition 14.1. *Consider the left action of G on itself: $L_g h = gh$. Then for all $x \in \mathfrak{g} = \mathrm{Lie}(G)$, there is a corresponding vector field $\xi = L_* x \in \mathrm{Vect}(G)$ which is right-invariant and satisfies $\xi(1) = x$.*

Proof. The curve $\exp(tx)$ lies in G and

$$L_* x(h) = \frac{d}{dt}\bigg|_{t=0} (\exp(tx)h) = xh,$$

which is a right-invariant vector field. $\square$

Corollary 14.0.1. $\mathfrak{g} = \mathrm{Lie}(G)$ *is isomorphic to the Lie algebra of right-invariant vector fields.*

Theorem 14.1. *Let G be a (real or complex) Lie group acting on a manifold M by ρ. Then*

1. *$G_m = \{g \in G : gm = m\}$ is a closed Lie subgroup with Lie algebra $\mathfrak{h} = \{x \in \mathfrak{g} : \rho_*(x)(m) = 0\}$, where $\rho_*(x)$ is the vector field (generator) corresponding to x.*
2. *The map $G/G_m \to M$ is an immersion. In particular, $\mathcal{O}_m$ is an immersed submanifold.*

Proof. (1) We have to show that there is some neighborhood U around 1 in G such that $U \cap G_m$ is a submanifold with tangent space $T_1 G_m = \mathfrak{h}$. Note the following:

(i) $\mathfrak{h}$ is closed under the commutator (the vector field commutator)

(ii) $\xi = \rho_*(x) \in \mathfrak{h}$ vanishes at m, so $\rho(\exp(tx))(m) = \Phi_\xi^t(m) = m$, so $\exp(tx) \in G_m$.

Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$ as vector spaces. Then $\rho_* : \mathfrak{g} \to T_m M$ has $\ker \rho_* = \mathfrak{h}$, so $\rho_*|_{\mathfrak{u}}$ is injective. Thus the map

$$\begin{aligned} \mathfrak{u} &\longrightarrow M \\ y &\longmapsto \rho(\exp(y))(m) \end{aligned}$$

is injective in some neighborhood of 0 in $\mathfrak{u}$ by the implicit function theorem, so $\exp(y) \in G_m$ if and only if $y = 0$. So in a small neighborhood of 1, we can write any $g \in G$ as

$$g = \exp(y)\exp(x), \quad y \in \mathfrak{u}, x \in \mathfrak{h}$$

by the inverse function theorem. Then we have

$$gm = \exp(y)\exp(x)m = \exp(y)m,$$

so $g \in G_m$ if and only if $g \in \exp(\mathfrak{h})$ (elements of this kind generate the submanifold).

(2) We have seen that $T_1(G/G_m) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{u}$, so the injectivity of $\rho_* : \mathfrak{u} \to T_m M$ shows that the map $\rho : G/G_m \to M$ is an immersion, as desired. $\square$

Corollary 14.1.1. *Let $f : G_1 \to G_2$ be a morphism of (real or complex) Lie groups and f_* the induced map of Lie algebras. Then $\ker f$ is a closed Lie subgroup with Lie algebra $\ker f_*$, and the map $G_1/\ker f \to G_2$ is an immersion. If $\operatorname{Im} f$ is a submanifold, then we have an isomorphism $\operatorname{Im} f \cong G_1/\ker f$.*

Proof. Let G_1 act on G_2 by $\rho(g)h = f(g)h$ for $g \in G_1$, $h \in G_2$. The stabilizer of 1 is exactly $\ker f$, so it is a closed Lie subgroup by Theorem 14.1. The rest also follows from Theorem 14.1. $\square$

Corollary 14.1.2. *Let V be a representation of G and $v \in V$. Then the stabilizer G_v is a closed Lie subgroup in G with Lie algebra $\{x \in \mathfrak{g} : xv = 0\}$.*

Example 14.0.1. Let V be a vector space over $\mathbb{K}$ and B a bilnear form on V. Then

$$\mathrm{O}(V, B) = \{g \in \mathrm{GL}(V) : B(gv, gw) = B(v, w) \text{ for all } v, w \in V\}$$

has Lie algebra

$$\mathfrak{o}(V, B) = \{x \in \mathfrak{gl}(V) : B(xv, w) + B(v, xw) = 0\}.$$

We claim that $\mathrm{O}(V, B)$ is always a Lie group with such Lie algebra: Let G act on the space of bilinear forms by $gF(v, w) = F(g^{-1}v, g^{-1}w)$, then $\mathrm{O}(V, B)$ is the stabilizer of B.

Example 14.0.2. Let A be a finite-dimensional associative algebra with multiplication $\mu : A \times A \to A$, and define

$$\mathrm{Aut}(A) = \{g \in \mathrm{GL}(A) : \mu(ga, gb) = g\mu(a, b) \text{ for all } a, b \in A\}.$$

We claim that $\mathrm{Aut}(A)$ is a Lie group with Lie algebra

$$\mathrm{Der}(A) = \{x \in \mathfrak{gl}(A) : \mu(xa, b) + \mu(a, xb) = x\mu(a, b) \text{ for all } a, b \in A\}.$$

Let W be the space of all linear maps $A \otimes A \to A$, and let $\mathrm{GL}(A)$ acts on W by

$$(gf)(a \otimes b) = gf(g^{-1}a \otimes g^{-1}b).$$

Then $\mathrm{Aut}(A)$ is exactly the stabilizer G_μ. The same argument shows that

$$\mathrm{Aut}(\mathfrak{g}) = \{g \in \mathrm{GL}(\mathfrak{g}) : [ga, gb] = g[a, b] \text{ for all } a, b \in \mathfrak{g}\}$$

is a Lie group with Lie algebra

$$\mathrm{Der}(\mathfrak{g}) = \{x \in \mathfrak{gl}(\mathfrak{g}) : [xa, b] + [a, xb] = x[a, b] \text{ for all } a, b \in \mathfrak{g}\}.$$

Note that x could be ad_c for some $c \in \mathfrak{g}$, these are called the *inner derivations*.

14.2 Center

Definition 14.1. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is

$$\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

Remark. The center $\mathfrak{z}(\mathfrak{g})$ is an ideal in $\mathfrak{g}$.

Theorem 14.2. *Let G be a connected Lie group. Then its center $\mathcal{Z}(G)$ is a closed Lie subgroup with Lie algebra $\mathfrak{z}(\mathfrak{g})$. If G is not connected, then $\mathcal{Z}(G)$ is still a closed Lie subgroup, however its Lie algebra is smaller than $\mathfrak{z}(\mathfrak{g})$.*

Proof. Let $g \in G$ and $x \in \mathfrak{g}$. Note that

$$\exp(\operatorname{Ad}_g tx) = g \exp(tx) g^{-1},$$

so g commutes with $\exp(tx)$ if and only if $\operatorname{Ad}_g x = x$. For connected Lie groups, the elements $\exp(tx)$ for all $x \in \mathfrak{g}$ generate the entire group, so $g \in \mathcal{Z}(G)$ if and only if $\operatorname{Ad}_g x = x$ for all $x \in \mathfrak{g}$. Thus we have $\mathcal{Z}(G) = \ker \operatorname{Ad}$ for the adjoint action $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. □

Example 14.1.1. O(2) and SO(2) have the same Lie algebra, but O(2) has center $\{\pm I\}$.

Remark. Call $G/\mathcal{Z}(G)$ the *adjoint group* associated to G. Denote

$$\operatorname{Ad} G = G/\mathcal{Z}(G) = \operatorname{Im}(\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})) \quad \text{and} \quad \operatorname{ad} \mathfrak{g} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \operatorname{Im}(\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})).$$

Example 14.1.2. Consider $\mathrm{SL}(2,\mathbb{R})$. Then $\operatorname{Ad}(\mathrm{SL}(2,\mathbb{R})) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\} = \mathrm{PSL}(2,\mathbb{R})$.

14.3 The Baker-Campbell-Hausdorff Formula

Theorem 14.3. *Let $x, y \in \mathfrak{g}$ such that $[x,y] = 0$. Then*

$$\exp(x)\exp(y) = \exp(x+y) = \exp(y)\exp(x).$$

Proof. Let ξ, η be the right-invariant vector fields corresponding to x, y, and Φ^t_ξ, Φ^t_η the corresponding time flows. Then the formula

$$\Phi^t_\xi \Phi^s_\eta \Phi^{-t}_\xi \Phi^{-s}_\eta = ts[\xi,\eta] + \cdots$$

implies that $[\xi,\eta] = 0$ since this is an isomorphism of Lie algebras. Now observe that

$$(\Phi^s_\xi)_* \left.\frac{d}{dt}\right|_{t=0} (\Phi^t_\eta)_*\eta = 0,$$

so $\frac{d}{dt}(\Phi^t_\eta)_*\eta = 0$, which implies that $(\Phi^t_\eta)_*\eta = \eta$ since the flow of ξ preserves η. Then

$$\Phi^t_\xi \Phi^s_\eta \Phi^{-t}_\xi = \Phi^s_\eta,$$

and applying this to 1 gives $\exp(tx)\exp(sy)\exp(-tx) = \exp(sy)$. □

Theorem 14.4 (Baker-Campbell-Hausdorff formula)**.** *For small enough $x, y \in \mathfrak{g}$, we have*

$$\exp(x)\exp(y)\exp(\mu(x,y)),$$

where μ is given by

$$\mu(x,y) = x + y + \sum_{n \geq 2} \mu(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \cdots.$$

In the above, the μ_n are degree-n commutators of x and y.

Lecture 15

Oct. 13 — Fundamental Theorems of Lie Theory

15.1 Fundamental Theorems of Lie Theory

Remark. We know the following about Lie theory so far:

1. Every (real or complex) Lie group G defines a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and every morphism $\varphi : G_1 \to G_2$ gives a morphism $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$. The map $\mathrm{Hom}(G_1, G_2) \to \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$ is injective.
2. Every Lie subgroup $H \subseteq G$ defines a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.
3. The group law can be recovered from $[\cdot, \cdot]$ on $\mathfrak{g}$.

Now we want to understand the following:

1. Given a morphism $\mathrm{Lie}(G_1) = \mathfrak{g}_1 \to \mathfrak{g}_2 = \mathrm{Lie}(G_2)$, can we lift it to a morphism $G_1 \to G_2$?

   We will see that the answer is no in general.
2. Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, does there always exist a subgroup H such that $\mathfrak{h} = \mathrm{Lie}(H)$?

   We will see that the answer is yes.
3. Can every finite-dimensional Lie algebra be obtained as the Lie algebra of a Lie group?

Example 15.0.1. Let $G = S^1 = \mathbb{R}/\mathbb{Z}$ and $G_2 = \mathbb{R}$. Then $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathbb{R}$. Consider the identity map

$$\mathrm{Id} : \mathfrak{g}_1 \to \mathfrak{g}_2.$$

If this lifted to a morphism $G_1 \to G_2$, then we must have $\theta \mapsto \theta$. But $f(\mathbb{Z}) = 0$, so this is impossible. Thus morphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$ cannot always be lifted to morphisms $G_1 \to G_2$ in general.

Theorem 15.1. *For any (real or complex) Lie group G, there is a bijection between connected Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ given by $H \mapsto \mathfrak{h} = \mathrm{Lie}(H) = T_1 H$.*

Theorem 15.2. *If G_1, G_2 are Lie groups and G_1 is simply connected, then*

$$\mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2) = \mathrm{Hom}(G_1, G_2),$$

where $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ for $i = 1, 2$.

Theorem 15.3. *Any finite-dimensional (real or complex) Lie algebra is isomorphic to the Lie algebra of a (real or complex) Lie group.*

Corollary 15.3.1. *For any (real or complex) finite-dimensional Lie algebra $\mathfrak{g}$, there exists a unique (up to isomorphism) connected and simply connected (real or complex) Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$. *Any other connected Lie group G' with Lie algebra $\mathfrak{g}$ must be of the form G/Z' for some discrete central subgroup $Z' \subseteq Z \subseteq G$.*

Proof. We sketch the proof. Theorem 15.3 says that there is a Lie group $\widetilde{G}$ with $\mathrm{Lie}(\widetilde{G}) = \mathfrak{g}$. Take a neighborhood of the identity, and construct the universal cover to get G. Then for any other G', there is a covering map $G \to G'$. One can check that the deck transformations are a normal subgroup and that they form a subgroup of the center Z. Then we may take $Z' = \pi_1(G')$. □

Corollary 15.3.2. *The categories of simply connected finite-dimensional Lie groups and Lie algebras are equivalent.*

15.2 Complex and Real Forms

Definition 15.1. Let $\mathfrak{g}$ be a real Lie algebra. Its *complexification* is

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g},$$

where the commutator extends naturally from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$.

Example 15.1.1. Consider the following:

1. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$.
2. For $\mathfrak{g} = \mathfrak{u}(n)$ (the skew-Hermitian matrices), $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$ as $i\mathfrak{g}$ are the Hermitian matrices.

Definition 15.2. Let G be a connected complex Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. If $K \subseteq G$ be a closed real Lie subgroup such that $\mathfrak{k} = \mathrm{Lie}(K)$ is a *real form* of $\mathfrak{g}$ (meaning that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}$), then K is called a *real form* of G.

Remark. It can be shown that if $\mathfrak{g} = \mathrm{Lie}(G)$ where G is simply connected and complex, then for any form $\mathfrak{k} \subseteq \mathfrak{g}$, one can obtain a real form $K \subseteq G$ with $\mathrm{Lie}(K) = \mathfrak{k}$.

It is not true that one can represent any real Lie group as a subgroup of some complex Lie group, an example is the universal cover of $\mathrm{SL}(2, \mathbb{R})$.

Example 15.2.1. We will study $\mathfrak{so}(3, \mathbb{R})$, $\mathfrak{su}(2)$, and $\mathfrak{sl}(2, \mathbb{C})$. Recall that $\mathfrak{so}(3, \mathbb{R})$ has a basis

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with commutator relations $[J_{\{x}, J_y] = J_{z\}}$ (the brackets mean that x, y, z can be replaced with any cyclic permutations). A basis of $\mathfrak{su}(2)$ is given by the *Pauli matrices* times i:

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with commutator relations $[i\sigma_{\{1}, i\sigma_2] = -2i\sigma_{3\}}$. Note that there is an isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3, \mathbb{R})$ by $i\sigma_1 \mapsto -2J_x$, $i\sigma_2 \mapsto -2J_y$, $i\sigma_3 \mapsto -2J_z$. A basis for $\mathfrak{sl}(2, \mathbb{C})$ is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with commutation relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. One can check as an exercise that

$$(\mathfrak{su}(2))_{\mathbb{C}} \cong (\mathfrak{so}(3, \mathbb{R}))_{\mathbb{C}} \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}).$$

The h element will play a similar role to the Jucys-Murphy's elements in representation theory.

15.3 Representations of Lie Groups and Lie Algebras

Definition 15.3. Recall the definition of a representation:

1. A *representation* of a Lie group G is a vector space V with a morphism of Lie groups

$$\rho : G \to \mathrm{GL}(V).$$

2. A *representation* of of a Lie algebra is a vector space V with a morphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

A *morphism* between two representations V, W of G is a map $f : V \to W$ such that

$$f\rho(g) = \rho(g)f.$$

Similarly one defines *morphisms* between representations of Lie algebras. Denote the set of morphisms of representations by $\mathrm{Hom}_G(V, W)$ and $\mathrm{Hom}_{\mathfrak{g}}(V, W)$, these are also known as *intertwining operators*.

Theorem 15.4. *Let G be a (real or complex) Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. Then*

1. *Every representation $\rho : G \to \mathrm{GL}(V)$ defines a representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and every morphism of representations of G is a morphism of representations of $\mathfrak{g}$.*

2. *If G is simply-connected and connected, then $\rho \mapsto \rho_*$ gives an equivalence of the categories of representations of G and representations of $\mathfrak{g}$. In particular, every representation of $\mathfrak{g}$ can be lifted to a representation of G.*

Remark. Note the following:

1. The representations of $\mathfrak{su}(2)$ is the same set as the representations of $\mathrm{SU}(2)$ (since $\mathrm{SU}(2) \cong S^3$ is simply-connected).

2. For G simply-connected, Theorem 15.4 can be used to describe representations of $\widetilde{G} = G/\widetilde{Z}$ for $\widetilde{Z} \subseteq Z$ (where Z is the center of G). These are the representations of G such that $\rho(\widetilde{Z}) = \mathrm{Id}$.

Lemma 15.1. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then any complex representation of $\mathfrak{g}$ has a unique structure of a representation of $\mathfrak{g}_{\mathbb{C}}$. Moreover,*

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V, W).$$

In particular, the categories of complex representations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are equivalent.

Proof. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. For $x, y \in \mathfrak{g}$, define ρ on $x + iy \in \mathfrak{g}_{\mathbb{C}}$ by

$$\rho(x + iy) = \rho(x) + i\rho(y).$$

One can check that this gives the desired representation of $\mathfrak{g}_{\mathbb{C}}$. $\square$

Example 15.3.1. Using Lemma 15.1, we see that the categories of complex representations of $\mathrm{SL}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$, $\mathrm{SU}(2)$, $\mathfrak{su}(2)$ are all equivalent.

Remark. We can define the following operations on representations:

1. Let $W \subseteq G$ be a subrepresentation of G or $\mathfrak{g}$ (meaning that W is invariant). Then we can construct the *quotient representation* V/W.
2. The *dual*, the *direct sum* and the *tensor product*.

Lemma 15.2. *Let* V, W *be representations of* G *(resp. of* $\mathrm{Lie}(G) = \mathfrak{g}$*). There is a canonical structure of a representation on* V^*, $V \oplus W$, *and* $V \otimes W$.

Proof. The direct sum is easy to define. For the tensor product, one defines

$$\rho(g)(v \otimes w) = (\rho(g)v) \otimes (\rho(g)w)$$

for $g \in G$. If $x \in \mathfrak{g}$, then we can choose a curve γ such that $\frac{d}{dt}\big|_{t=0}\gamma(t) = x$ and $\gamma(0) = 1$, and define

$$\begin{aligned}
\rho(x)(v \otimes w) &= \frac{d}{dt}\bigg|_{t=0} (\rho(\gamma(t))v \otimes \rho(\gamma(t))w) \\
&= \rho(\dot\gamma(0))v \otimes \rho(\gamma(0))w + \rho(\gamma(0))v \otimes \rho(\dot\gamma(0))w \\
&= \rho(x)v \otimes w + v \otimes \rho(x)w \\
&= (\rho(x) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho(x))(v \otimes w).
\end{aligned}$$

For the dual, let $\langle \cdot, \cdot \rangle : V \otimes V^* \to \mathbb{C}$ be the natural pairing. Then for $g \in G$, we need

$$\langle \rho(g)v, \rho(g)v^* \rangle = \langle v, v^* \rangle,$$

so we can define $\rho_{V^*}(g) = \rho(g^{-1})^T$. On the level of Lie algebras, we need

$$\langle \rho(x)v, v^* \rangle + \langle v, \rho(x)v^* \rangle = 0,$$

so we can define $\rho_{V^*}(x) = -\rho(x)^T$. One can check as an exercise that these definitions work. □

Lecture 16

Oct. 15 — Representations of Lie Groups

16.1 More Examples of Representations

Example 16.0.1. Let V, W be representations of G or $\mathrm{Lie}(G)$. Then

1. $\mathrm{End}(V) \cong V \otimes V^*$ is a representation, with G-action
$$g : A \mapsto \rho_V(g) A \rho_V(g)^{-1}$$
for $g \in G$ and $\mathfrak{g}$-action $x : A \mapsto \rho_V(x)A - A\rho_V(x)$ for $x \in \mathfrak{g}$.
2. $\mathrm{Hom}(V, W)$ is a representation. Write down the G- and $\mathfrak{g}$-actions as an exercise.
3. The space of bilinear forms is a representation with G-action $gB(v, w) = B(g^{-1}v, g^{-1}w)$.

16.2 Invariants

Definition 16.1. Let V be a representation of a Lie group G. Then $v \in V$ is called an *invariant* of the Lie group action if $\rho(g)v = v$ for all $g \in G$. The space of invariants is denoted by V^G. For a Lie algebra $\mathfrak{g}$, we say v is an *invariant* if $\rho(x)v = 0$ for all $x \in \mathfrak{g}$. Similarly, $V^{\mathfrak{g}}$ denotes the space of invariants.

Proposition 16.1. *If G is connected, then $V^G = V^{\mathfrak{g}}$ where $\mathfrak{g} = \mathrm{Lie}(G)$.*

Example 16.1.1. Let V, W be representations. Then G acts on $\mathrm{Hom}(V, W)$ by
$$g : A \mapsto \rho_W(g) A \rho_V(g)^{-1},$$
and we have $(\mathrm{Hom}(V, W))^G = \mathrm{Hom}_G(V, W)$. In particular, $V^G = (\mathrm{Hom}(\mathbb{C}, V))^G = \mathrm{Hom}_G(\mathbb{C}, V)$, where $\mathbb{C}$ is the trivial representation of G.

Example 16.1.2. Let B be a bilinear form. The space of bilinear forms on $V \otimes W$ form a representation of G if V, W are representations of G. The invariants are those B such that $B(gv, gw) = B(v, w)$.

Proposition 16.2. *A bilinear form B is an invariant if and only if $v \mapsto B(v, \cdot)$ is a morphism of representations.*

16.3 Irreducible Representations

Definition 16.2. A representation V of G or $\mathfrak{g} = \mathrm{Lie}(G)$ is called *irreducible* if it has no subrepresentations except for V and $\{0\}$. Otherwise, we say that V is *reducible*.

Example 16.2.1. $\mathrm{SL}(n, \mathbb{C})$ acting on $\mathbb{C}^n$ is irreducible.

Remark. Note that if W is a subrepresentation of V, then we have an exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

Do we have $V = W \oplus V/W$? If this is true for every W, then V is *completely reducible* (see below).

Definition 16.3. A representation V is called *completely reducible* if $V \cong \bigoplus_i V_i$ for V_i irreducible.

Remark. In general, we can write a completely reducible representation V as

$$V \cong \bigoplus_i n_i V_i,$$

where the n_i are multiplicities, i.e. contributions of isomorphic summands.

Example 16.3.1. Let $G = \mathbb{R}$ with $\mathfrak{g} = \mathbb{R}$. Then a representation of V is a linear map $\mathbb{R} \to \mathrm{End}(V)$, so it is of the form $t \mapsto tA$ for any element $A \in \mathrm{End}(V)$. On the group level this is $t \mapsto \exp(tA)$. Note that A has some Jordan normal form, so it will not split into irreducibles unless A is diagonalizable.

Remark. The following are the classic goals of representation theory:

1. For a given group G, classify all irreducible representations.
2. For a given representation V of G, if it is completely reducible, find the decomposition of V into irreducibles.
3. For which Lie groups G are all representations completely reducible?

Lemma 16.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a Lie group G (with corresponding Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$). Let $A : V \to V$ be a diagonalizable intertwining operator, and let $V_\lambda \subseteq V$ be the eigenspace of A with eigenvalue λ. Then the following holds for both G and $\mathfrak{g}$:*

1. *Each V_λ is a subrepresentation.*
2. *As a consequence of (1), $V = \bigoplus_\lambda V_\lambda$.*

Lemma 16.2. *Let V be a representation of G and $z \in \mathcal{Z}(G)$ be a central element such that $\rho(z)$ is diagonalizable. Then $V = \bigoplus_\lambda V_\lambda$ where V_λ is the eigenspace of $\rho(z)$ with eigenvalue λ.*

Example 16.3.2. Let $\mathrm{GL}(n, \mathbb{C})$ act on $\mathbb{C}^n \otimes \mathbb{C}^n$. Then $P : v \otimes w \mapsto w \otimes v$ is an intertwining operator (i.e. it commutes with the action of G). Then $S^2\mathbb{C}^n$ and $\wedge^2\mathbb{C}^n$ are eigenspaces of P, so

$$\mathbb{C}^n \otimes \mathbb{C}^n = S^2\mathbb{C}^n \oplus \wedge^2\mathbb{C}^n,$$

but we do not know at this point if these are irreducible.

Lemma 16.3 (Schur's lemma)**.** *We have the following:*

1. *If V is a complex irreducible representation of G, then the space of intertwining operators if* $\mathrm{Hom}_G(V, V) = \mathbb{C}\,\mathrm{Id}$. *In other words, any endomorphism of an irreducible representation of G is multiplication by a scalar.*
2. *If V, W are non-isomorphic complex irreducible representations, then* $\mathrm{Hom}_G(V, W) = 0$.

Example 16.3.3. Let $G = \mathrm{GL}(n, \mathbb{C})$. Then $\mathbb{C}^n$ is an irreducible representation of G, so every operator commuting with $\mathrm{GL}(n, \mathbb{C})$ must be scalar. Thus $\mathcal{Z}(\mathrm{GL}(n, \mathbb{C})) = \{\lambda\,\mathrm{Id} : \lambda \in \mathbb{C}^\times\}$. Similarly, we can use this to see that $\mathfrak{z}(\mathfrak{gl}(n, \mathbb{C})) = \{\lambda\,\mathrm{Id} : \lambda \in \mathbb{C}\}$ for the Lie algebra.

Exercise 16.1. Extend the above to all other classical Lie groups.

Corollary 16.0.1. *Assume V is a completely reducible representation of a Lie group G (or Lie algebra $\mathfrak{g}$). Then we have the following:*

1. *If $V = \bigoplus V_i$ with V_i irreducible and pairwise non-isomorphic, then any intertwining operator $\Phi : V \to V$ is of the form $\Phi = \bigoplus_i \lambda_i \operatorname{Id}_{V_i}$.*
2. *If $V = \bigoplus_i n_i V_i = \bigoplus_i (\mathbb{C}^{n_i} \otimes V_i)$, then any intertwining operator $\Phi : V \to V$ is given by*

$$\Phi = \bigoplus_i (A_i \otimes \operatorname{Id}_{V_i}), \quad A_i \in \operatorname{End}(\mathbb{C}^{n_i}).$$

Proposition 16.3. *If G is a commutative group (resp. if $\mathfrak{g}$ is a commutative Lie algebra), then every irreducible complex representations of G (resp. $\mathfrak{g}$) is 1-dimensional.*

Proof. Note that $\rho(g)$ commutes with the action of G for every g, so $\rho(g) \sim \operatorname{Id}$. □

Example 16.3.4. Let $G = \mathbb{R}$ and $\mathfrak{g} = \mathbb{R}$. Then a 1-dimensional representation of $\mathfrak{g}$ is given by $a \mapsto \lambda a$ for $a \in \mathfrak{g}$. Then for $b \in G$, we have $\rho(b) = \exp(\lambda b)$. So λ labels representations. For $U(1) = S^1 = \mathbb{R}/\mathbb{Z}$ and $\mathfrak{g} = \mathbb{R}$, we have $\rho(a) = 1$ for all $a \in \mathbb{Z}$, so $\rho(a) = \exp(2\pi i k a)$. Here k labels the representations.

16.4 Unitary Representations

Remark. Recall that for finite groups, every representation is completely reducible.

Definition 16.4. A complex representation of a real Lie group G is called *unitary* if there exists an invariant inner product (i.e. a Hermitian positive-definite bilinear form) on V, i.e.

$$(\rho(g)v, \rho(g)w) = (v, w).$$

One can make a similar definition for Lie algebras by differentiating.

Theorem 16.1. *Every unitary representation is completely reducible.*

Proof. We induct on dimension. Assume that V is not irreducible. Then we can write $V = W \oplus W^\perp$ for some nontrivial subrepresentation W. We claim $W^\perp$ is also a subrepresentation: For $w \in W^\perp$, $v \in W$,

$$(gw, v) = (w, g^{-1}v) = 0$$

since $g^{-1}v \in W$, so $gw \in W^\perp$. Thus $W^\perp$ is a subrepresentation, and $\dim W, \dim W^\perp < \dim V$. □

Remark. Let G be finite. Recall that if $B(V, W)$ is any inner product on V, then we can define

$$(v, w) = \frac{1}{|G|} \sum_{g \in G} B(gv, gw),$$

which one can check is positive-definite and G-invariant. So we get that every complex representation of a finite group is completely reducible. We can apply a similar idea to Lie groups.

16.5 Compact Lie Groups and Haar Measure

Definition 16.5. A *right Haar measure* on a real Lie group G is a Borel measure dg which is invariant under the right action of G on itself, i.e. we have

$$\int f(gh)\,dg = \int f(g)\,dg$$

for any integrable function f and $h \in G$. One defines a *left Haar measure* similarly.

Lemma 16.4. *Let V be a 1-dimensional real representation of a compact Lie group G. Then for any $g \in G$, we have $|\rho(g)| = 1$.*

Proof. If $|\rho(g)| < 1$, then we have $\rho(g)^n \to 0$ as $n \to \infty$. But $\rho(G)$ is a compact subset of $\mathbb{R}^\times$, so it cannot contain a sequence converging to 0. A similar argument works for $|\rho(g)| > 1$. □

Theorem 16.2. *Let G be a real Lie group. Then*

1. *G is orientable. Moreover, the orientation can be chosen such that the right action of G on itself preserves orientation.*
2. *If G is compact, then for a fixed choice of right-invariant orientation on G, there exists a unique right-invariant top form ω such that*
$$\int_G \omega = 1.$$
3. *ω is also left-invariant if G is connected (otherwise it is left-invariant up to sign).*

Proof. Choose an element in $\bigwedge^n \mathfrak{g}^*$ where $n = \dim G$. It can be uniquely translated by right-invariance to give a non-vanishing form $\widetilde{\omega}$. Then let

$$I = \int_G \widetilde{\omega},$$

then $\omega = \widetilde{\omega}/I$ gives the desired ω. Now note that $\bigwedge^n \mathfrak{g}^*$ is a 1-dimensional representation of G via the adjoint action. Left-invariance of ω follows from Lemma 16.4, and the sign is obtained as follows. Let $i : g \mapsto g^{-1}$ be the inversion map. If ω is left-invariant, then $i^*(\omega)$ is right-invariant, and ω and $i^*(\omega)$ are equal up to a sign: $i_* : \mathfrak{g} \to \mathfrak{g}$ is given by $x \mapsto -x$, so $i^* : \bigwedge^n \mathfrak{g}^* \to \bigwedge^n \mathfrak{g}^*$ is given by $\omega \mapsto (-1)^n \omega$. □

Theorem 16.3. *Let G be a compact real Lie group. Then it has a canonical Borel measure dg, called the* Haar measure, *which is both left and right invariant, invariant under $g \mapsto g^{-1}$, and satisfies*

$$\int_G dg = 1.$$

Theorem 16.4. *Any finite-dimensional representation of a compact Lie group is unitary and is thus completely reducible.*

Proof. Let $B(v, w)$ be a positive-definite Hermitian inner product. Then one can check

$$\overline{B}(v, w) = \int_G B(gv, gw)\,dg$$

is unitary, and thus the representation is completely reducible by Theorem 16.1. □

Lecture 17

Oct. 20 — Representations of $\mathfrak{sl}(2, \mathbb{C})$

17.1 Irreducible Representations of $\mathfrak{sl}(2, \mathbb{C})$

Remark. Consider $\mathfrak{sl}(2, \mathbb{C}), \mathrm{SL}(2, \mathbb{C}), \mathfrak{su}(2), \mathrm{SU}(2)$. Recall from last time that the categories of complex representations of these Lie groups and Lie algebras are equivalent and completely reducible.

We will now study (finite-dimensional) irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. Recall that $\mathfrak{sl}(2, \mathbb{C})$ has a basis given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with relations $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

Definition 17.1. Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$. A vector $v \in V$ is called a *vector of weight* λ (for $\lambda \in \mathbb{C}$) if it is an eigenvector for h with eigenvalue λ, i.e. $hv = \lambda v$. Denote by $V[\lambda]$ the subspace of vectors with eigenvalue λ.

Lemma 17.1. *$eV[\lambda] \subseteq V[\lambda + 2]$ and $fV[\lambda] \subseteq V[\lambda - 2]$.*

Proof. Let $v \in V[\lambda]$. Then $hv = \lambda$, so we have

$$hev = ehv + [h, e]v = ehv + 2ev(\lambda + 2)ev,$$

so $ev \in V[\lambda + 2]$. This proves the first statement, and one can prove the second similarly. $\square$

Theorem 17.1. *Every finite-dimensional representation V of $\mathfrak{sl}(2, \mathbb{C})$ can be written in the form*

$$V = \bigoplus_{\lambda} V[\lambda].$$

Such a decomposition is called the weight decomposition *of V.*

Proof. Since every representation for $\mathfrak{sl}(2, \mathbb{C})$ is completely reducible, it suffices to prove the statement for irreps. So let V be irreducible. Let $V' = \sum_\lambda V[\lambda] = \bigoplus_\lambda V[\lambda]$. By Lemma 17.1, V' is stable under the actions of e, f, so it is a subrepresentation of V. Since V is irreducible, $V' = V$. $\square$

Remark. Let λ be a weight of V (i.e. $V[\lambda] \neq 0$) which is maximal in the following sense:

$$\operatorname{Re} \lambda \geq \operatorname{Re} \lambda' \quad \text{for all weights } \lambda' \text{ of } V.$$

Then the corresponding eigenvector $v \in V[\lambda]$ is called the *highest weight vector*, and it always exists for finite-dimensional V.

Lemma 17.2. *Assume $v \in V[\lambda]$ is the highest weight vector. Then*

1. $ev = 0$.
2. $v^k := f^k v/k!$ *(for $k \geq 0$) satisfies the following formulas:*

$$hv^k = (\lambda - 2k)v^k, \quad ev^k = (\lambda - k + 1)v^{k-1}, \quad fv^k = (k+1)v^{k+1}.$$

Proof. We prove the second formula in (2), the rest is left as an exercise. We induct on k. For $k = 1$,

$$ev^1 = efv = [e,f]v + fev = hv = \lambda v.$$

Now assume it is true for k, and we prove the formula for $k+1$. We have

$$\begin{aligned} ev^{k+1} &= \frac{1}{k+1} efv^k = \frac{1}{k+1}(hv^k + fev^k) = \frac{1}{k+1}((\lambda - 2k)v^k + (\lambda - k + 1)fv^{k-1}) \\ &= \frac{1}{k+1}(\lambda - 2k + (\lambda - k + 1)k)v^k = (\lambda - k)v^k, \end{aligned}$$

which is the corresponding formula for $k+1$. □

Lemma 17.3. *Let $\lambda \in \mathbb{C}$. Define M_λ (the* Verma module*) to be the infinite-dimensional vector space with basis $v^0, v^1, v^2, \ldots$. Then*

1. *The formulas from Lemma 17.2(2) and $ev^0 = 0$ define the structure of an infinite-dimensional representation on M_λ.*
2. *If V is a finite-dimensional irreducible module of $\mathfrak{sl}(2,\mathbb{C})$ which contains a nonzero highest weight vector of highest weight λ, then $V = M_\lambda/W$, where W is some subrepresentation.*

Proof. This follows from Lemma 17.2. □

Theorem 17.2. *We have the following:*

1. *For any $n \in \mathbb{Z}_{\geq 0}$, let V_n be the finite-dimensional vector space with basis $v^0, v^1, \ldots, v^n$. Define an action of $\mathfrak{sl}(2,\mathbb{C})$ on V by the formulas from Lemma 17.2(2), with $fv^n = 0$ and $ev^0 = 0$. Then V_n is an irreducible representation.*
2. *For $n \neq m$, the representations V_n, V_m are non-isomorphic.*
3. *Every finite-dimensional irrep of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to one of the V_n.*

Proof. (1) Start with the Verma module M_λ for $\lambda = n$. Consider the subspace

$$M' = \mathrm{Span}\{v^{n+1}, v^{n+2}, \ldots\}.$$

Note that $ev^{n+1} = (n+1-(n+1))v^n = 0$. Then M_n/M' is a finite-dimensional representation with basis $v^0, v^1, \ldots, v^n$, so it is isomorphic to V_n. Now we show that it is irreducible: If there is a subrepresentation $V' = \mathrm{Span}\{v_i\}$ where $\{v_i\} \subseteq \{v^0, \ldots, v^n\}$, then $V' = V_n$ since otherwise V' is not invariant under the action of e, f, h. Thus V_n is irreducible.

(2) This follows since V_n and V_m have different dimensions. □

Remark. The actions of f and e have the following actions on the V_n (f on top, e on bottom):

$$\begin{tikzcd}
V_n \arrow[r, bend right, "1"'] & V_{n-1} \arrow[l, bend right, "n"'] \arrow[r, bend right, "2"'] & V_{n-2} \arrow[l, bend right, "n-1"'] \arrow[r, bend right, "3"'] & \cdots \arrow[l, bend right, "n-2"'] \arrow[r, bend right, "n-1"'] & V_1 \arrow[l, bend right, "2"'] \arrow[r, bend right, "n"'] & V_0 \arrow[l, bend right, "1"']
\end{tikzcd}$$

Note that the eigenvalues of h are $n, n-2, \dots, -n$.

For SU(2), we have $J_z = \frac{1}{2}h$, with eigenvalues $\frac{n}{2}, \dots, -\frac{n}{2}$. This is called the *spin* in quantum mechanics. We have seen that any representation of $\mathfrak{su}(2)$ lifts to a representation of SU(2). A representation of $\mathfrak{so}(3,\mathbb{R})$ will lift to a representation of $\mathrm{SO}(3,\mathbb{R})$ if and only if the spin is an integer.

Theorem 17.3. *Let V be any finite-dimensional complex representation of $\mathfrak{sl}(2,\mathbb{C})$. Then*

1. *V admits a weight decomposition with respect to integer weights: $V = \bigoplus_{n\in\mathbb{Z}} V[n]$.*
2. *$\dim V[n] = \dim V[-n]$, and $e^n : V[n] \to V[-n]$, $f^n : V[-n] \to V[n]$ are isomorphisms.*

17.2 The Universal Enveloping Algebra

Definition 17.2. Let $\mathfrak{g}$ be a Lie algebra over some field $\mathbb{K}$. The *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{K}$ is the associative algebra with unit over $\mathbb{K}$ with generators $i(x)$ for $x \in \mathfrak{g}$, subject to the relations

1. $i(x) + i(y) = i(x+y)$,
2. $i(cx) = ci(x)$ for $c \in \mathbb{K}$,
3. $i(x)i(y) - i(y)i(x) = i([x,y])$.

Remark. We will see that the map $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective, so we will just write x instead of $i(x)$.

Remark. Without the condition (3), this is just the tensor algebra $T\mathfrak{g} = \bigoplus_{n\geq 0} g^{\otimes n}$. So we can say

$$U(\mathfrak{g}) = T\mathfrak{g}/\{xy - yx - [x,y]\}.$$

Example 17.2.1. Let $\mathfrak{g}$ be a commutative Lie algebra. Then $U(\mathfrak{g})$ is generated by $x \in \mathfrak{g}$ subject to conditions $\{xy = yx\}$. Thus $U(\mathfrak{g}) = S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. One can view $S(\mathfrak{g})$ as polynomials on the dual space $\mathfrak{g}^*$. From this perspective, $S(\mathfrak{g}) = \mathbb{K}[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ form a basis of $\mathfrak{g}$. In particular, this shows that $U(\mathfrak{g})$ is infinite-dimensional.

Example 17.2.2. Recall that $\mathfrak{sl}(2,\mathbb{C})$ is the associative algebra generated by $1, e, f, h$ subject to

$$\begin{cases} ef - fe = h, \\ he - eh = 2e, \\ hf - fh = -2f. \end{cases}$$

Note that the product $h^m e^k f^\ell$ is *not* a matrix multiplication (we have $e^2 = f^2 = 0$ as matrices).

Theorem 17.4. *Let A be an associative algebra with unit over $\mathbb{K}$, and let $\rho : \mathfrak{g} \to A$ be a linear map satisfying $\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x,y])$. Then ρ can be extended in a unique way to a morphism of associative algebras, i.e. to a map $\rho : U(\mathfrak{g}) \to A$.*

Corollary 17.4.1. *Any representation of $\mathfrak{g}$ (not necessarily finite-dimensional) has a canonical structure of a $U(\mathfrak{g})$-module. Conversely, every $U(\mathfrak{g})$-module has a canonical structure of a $\mathfrak{g}$-representation.*

Lecture 18

Oct. 22 — The Universal Enveloping Algebra

18.1 The Universal Enveloping Algebra, Continued

Example 18.0.1. Let $\mathfrak{sl}(2, \mathbb{C})$ be generated by $\{e, f, h\}$, and

$$C = \frac{1}{2}h^2 + ef + fe.$$

This is known as the *Casimir operator*. Note that C commutes with e, f, h and hence lies in the center of $\mathfrak{sl}(2, \mathbb{C})$. We can explicitly check this for e as follows:

$$\begin{aligned} eC &= e^2 f + efe + \frac{1}{2}eh^2 = e(fe + h) + (fe + h)e + \frac{1}{2}(he - 2e)h \\ &= efe + fe^2 + \frac{1}{2}heh + eh + he - eh = efe + fe^2 + \frac{1}{2}h(he - 2e) + he \\ &= efe + fe^2 + \frac{1}{2}h^2 e = \big(ef + fe + \frac{1}{2}h^2\big)e = Ce. \end{aligned}$$

Thus $\rho(C)$ for any representation V of $\mathfrak{sl}(2, \mathbb{C})$ acts as a constant.

Proposition 18.1. *We have the following:*

1. *The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ can be extended to an action on $U(\mathfrak{g})$ by*

$$\operatorname{ad}_x(ab) = \operatorname{ad}_x(a)b + a\operatorname{ad}_x(b) \quad \textit{and} \quad \operatorname{ad}_x a = xa - ax.$$

2. *Let $\mathcal{Z}(U(\mathfrak{g}))$ be the center of $U(\mathfrak{g})$. Then $\mathcal{Z}(U(\mathfrak{g})) = (U(\mathfrak{g}))^{\operatorname{ad}\mathfrak{g}}$.*

18.2 The Poincaré-Birkhoff-Witt Theorem

Remark. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{K}$, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Note that we cannot put on a grading on $U(\mathfrak{g})$ since if we assign $\deg(x_i) = 1$ for $x_1, \dots, x_n$ a basis of $\mathfrak{g}$, then $\deg(x_1 \dots x_\ell) = \ell$, but we have $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

Instead we can define a *filtration*. Let $U_k(\mathfrak{g})$ be the subspace in $U(\mathfrak{g})$ spanned by the products $x_{i_1} \cdots x_{i_p}$ for $p \le k$. Then

$$\mathbb{K} = U_0(\mathfrak{g}) \subseteq U_1(\mathfrak{g}) \subseteq \cdots \subseteq U(\mathfrak{g}) = \bigcup U_p(\mathfrak{g}).$$

Proposition 18.2. *We have the following:*

1. *$U(\mathfrak{g})$ is a* filtered algebra, *i.e. if $x \in U_p(\mathfrak{g})$, $y \in U_q(\mathfrak{g})$, then $xy \in U_{p+q}(\mathfrak{g})$.*
2. *If $x \in U_p(\mathfrak{g})$, $y \in U_q(\mathfrak{g})$, then $xy - yx \in U_{p+q-1}(\mathfrak{g})$.*
3. *Let $x_1, \ldots, x_n$ be an ordered basis in $\mathfrak{g}$. Then the monomials $x_1^{k_1} \ldots x_n^{k_n}$ for $\sum k_i \leq p$ span $U_p(\mathfrak{g})$.*

Proof. (1) This is obvious.

(2) We use induction on p. First let $p = 1$. Then we have

$$x(y_1 \cdots y_q) - (y_1 \cdots y_q)x = \sum_i y_1 \cdots [x, y_i] \cdots y_q$$

where $[x, y_i] = xy_i - y_i x$. So the above expression lies in $U_q(\mathfrak{g})$. Now assume it is true for p. Then

$$x_1 \cdots x_{p+1} y \equiv x_1 \cdots x_p y x_{p+1} \equiv y x_1 \cdots x_p x_{p+1} \pmod{U_{p+q}(\mathfrak{g})}.$$

So the result is true for $p + 1$.

(3) We again induct on p. The case $p = 1$ is obvious. Note that $U_{p+1}(\mathfrak{g})$ is generated by xy for $x \in \mathfrak{g}$ and $y \in U_p(\mathfrak{g})$. By the induction hypothesis, y can be represented as such a sum of monomials

$$x_i(x_1^{k_1} \cdots x_i^{k_i} \cdots x_n^{k_n}) \equiv x_1^{k_1} \cdots x_i^{k_i+1} \cdots x_n^{k_n} \pmod{U_p(\mathfrak{g})}.$$

This proves the claim. □

Corollary 18.0.1. *Each of the $U_p(\mathfrak{g})$ are finite-dimensional.*

Corollary 18.0.2. *Define $\operatorname{Gr} U(\mathfrak{g}) = \bigoplus_p U_p(\mathfrak{g})/U_{p-1}(\mathfrak{g})$. This is a graded commutative algebra.*

Theorem 18.1 (Poincaré-Birkhoff-Witt)**.** *Let $x_1, \ldots, x_n$ be an ordered basis in $\mathfrak{g}$. Then the monomials as in Proposition 18.2(3) form a basis in $U(\mathfrak{g})$.*

Proof. We need to show linear independence. The idea is to construct a representation V so that the operators corresponding to these monomials are linearly independent. We can take $V = U(\mathfrak{g}) \cdot 1$ with

$$\rho(x_i) x_{j_1} \cdots x_{j_n} = x_i x_{j_1} \cdots x_{j_n}$$

for $i \leq j_1, \ldots, j_n$. For $i > j_1$, note that we have

$$\rho(x_2)x_1 = \rho(x_2)(\rho(x_1)) \cdot 1 = \rho(x_1)\rho(x_2) \cdot 1 + \rho([x_2, x_1]) \cdot 1 = x_1 x_2 + \sum a_i x_i.$$

We can define the general case in a similar way. □

Theorem 18.2 (Poicaré-Birkhoff-Witt, alternative)**.** $\operatorname{Gr} U(\mathfrak{g})$ *is isomorphic to the symmetric algebra $S(\mathfrak{g})$. There is a well-defined map*

$$\begin{aligned} S^p(\mathfrak{g}) &\longrightarrow \operatorname{Gr}^p(\mathfrak{g}) \\ a_1 \cdots a_p &\longmapsto a_1 \cdots a_p \bmod U_{p-1}(\mathfrak{g}). \end{aligned}$$

with well-defined inverse given by

$$\begin{aligned} \operatorname{Gr}^p(\mathfrak{g}) &\longrightarrow S^p(\mathfrak{g}) \\ a_1 \ldots a_p &\longmapsto a_1 \cdots a_p \\ a_1 \ldots a_e &\longmapsto 0, \quad e < p. \end{aligned}$$

Corollary 18.2.1. *The map $\mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Corollary 18.2.2. *Let $\mathfrak{g}_1, \mathfrak{g}_2 \subseteq \mathfrak{g}$ be subalgebras. Then $\mathfrak{g} = \mathfrak{g}_1 \otimes \mathfrak{g}_2$ as vector space, and*

$$U(\mathfrak{g}) \otimes U(\mathfrak{g}_1) \longrightarrow U(\mathfrak{g}_2)$$

is an isomorphism of vector spaces.

Corollary 18.2.3. *$U(\mathfrak{g})$ has no zero divisors.*

Theorem 18.3. *The map $S(\mathfrak{g}) \to U(\mathfrak{g})$ which sends*

$$\mathrm{sym}(x_1 \cdots x_p) = \frac{1}{p} \sum_{s \in S_p} x_{s(1)} \cdots x_{s(p)}$$

is an isomorphism of $\mathfrak{g}$-modules (with respect to the adjoint action).

18.3 Ideals and Subalgebras

Remark. Recall that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if $[x, y] \in \mathfrak{h}$ for every $x \in \mathfrak{g}$, $y \in \mathfrak{h}$.

Lemma 18.1. *If $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism, then $\ker f$ is an ideal of $\mathfrak{g}_1$, $\operatorname{Im} f$ is a subalgebra of $\mathfrak{g}_2$, and $\mathfrak{g}_1/\ker f \cong \operatorname{Im} f$.*

Lemma 18.2. *Let I_1, I_2 be ideals in $\mathfrak{g}$. Define*

1. *$I_1 + I_2 = \{x_1 + x_2 : x_1 \in I_1, x_2 \in I_2\}$.*
2. *$[I_1, I_2]$ is the subspace spanned by $[x, y]$ for $x \in I_1$, $y \in I_2$.*
3. *$I_1 \cap I_2$.*

Then (1)-(3) are all idals in $\mathfrak{g}$.

Definition 18.1. The *commutant* of a Lie algebra $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Lemma 18.3. *$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an abelian Lie algebra. Moreover, $[\mathfrak{g}, \mathfrak{g}]$ is the smallest ideal with the property that $\mathfrak{g}/I$ is an abelian Lie algebra.*

Example 18.1.1. $[\mathfrak{gl}(n, \mathbb{K}), \mathfrak{gl}(n, \mathbb{K})] = [\mathfrak{sl}(n, \mathbb{K}), \mathfrak{sl}(n, \mathbb{K})] = \mathfrak{sl}(n, \mathbb{K})$. To see this, let $z = [x, y]$, so we have $\operatorname{tr} z = 0$. Note that $E_{i,i} - E_{j,j} = [E_{i,j}, E_{j,i}]$, where $E_{i,j}$ contains a single 1 in the (i, j) position and 0 otherwise, so $2E_{i,j} = [E_{i,i} - E_{j,j}, E_{i,j}]$.

18.4 Solvable and Nilpotent Lie Algebras

Definition 18.2. Let $\mathfrak{g}$ be a Lie algebra. The *derived series* $D^i\mathfrak{g}$ of $\mathfrak{g}$ is defined by

$$D^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}].$$

Remark. Each $D^i\mathfrak{g}$ is an ideal in $\mathfrak{g}$ and $D^i\mathfrak{g}/D^{i+1}\mathfrak{g}$ is abelian.

Proposition 18.3. *The following conditions are equivalent:*

1. *$D^n\mathfrak{g} = 0$ for large enough n.*

2. *There exists a sequence of subalgebras* $\mathfrak{g} = \mathfrak{a}^0 \supseteq \mathfrak{a}^1 \supseteq \cdots \supseteq \mathfrak{a}^n = \{0\}$ *such that* $\mathfrak{a}^{i+1}$ *is an ideal in* $\mathfrak{a}^i$ *and the quotient* $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ *is abelian.*
3. *For large enough* n*, all commutators of the form*

$$[\ldots[[x_1, x_2], [x_3, x_4]]\ldots]$$

(2^n *terms) is zero.*

Proof. ($1 \Leftrightarrow 2$) This is clear.

($1 \Rightarrow 2$) Take $\mathfrak{a}^i = D^i\mathfrak{g}$.

($2 \Rightarrow 1$) If $\mathfrak{a}^i$ satisfies (2), then $\mathfrak{a}^{i+1} \supseteq [\mathfrak{a}^i, \mathfrak{a}^i]$, so by induction we have (1). □

Definition 18.3. A Lie algebra $\mathfrak{g}$ is called *solvable* if it satisfies one of the above conditions.

Definition 18.4. Define the *lower central series* $D_i\mathfrak{g} \subseteq \mathfrak{g}$ by

$$D_0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad D_{i+1}\mathfrak{g} = [\mathfrak{g}, D_i\mathfrak{g}].$$

Proposition 18.4. *The following conditions are equivalent:*

1. $D_n\mathfrak{g} = 0$ *for large enough* n*.*
2. *There exists a sequence of ideals such that* $\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n = \{0\}$ *such that* $[y, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$*.*
3. *For large enough* n*, all commutators of the form*

$$[\ldots[[x_1, x_2], x_3], x_4]\ldots x_n]$$

(n *terms) are zero.*

Definition 18.5. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if it satisfies one of the above conditions.

Example 18.5.1. The Lie algebra of strictly lower triangular matrices is nilpotent. The non-strictly lower triangular matrices form a solvable Lie algebra.

Theorem 18.4. *We have the following:*

1. *A real Lie algebra is solvable (resp. nilpotent) if and only if its complexification is solvable (resp. nilpotent).*
2. *If* $\mathfrak{g}$ *is solvable, then any subalgebra and any quotient are solvable. The same applies to nilpotence.*
3. *If* $\mathfrak{g}$ *is nilpotent, then* $\mathfrak{g}$ *is solvable.*
4. *If* $I \subseteq \mathfrak{g}$ *is an ideal and both* I*,* $\mathfrak{g}/I$ *are solvable, then* $\mathfrak{g}$ *is solvable.*

Proof. (4) Consider the map $\varphi : \mathfrak{g} \to \mathfrak{g}/I$, then $\varphi(D^n\mathfrak{g}) = D^n\mathfrak{g}/I = 0$ for n large enough. So $D^n\mathfrak{g} \subseteq I$, and $D^{n+k}\mathfrak{g} \subseteq D^kI$, which vanishes for k large enough. So $\mathfrak{g}$ is solvable. □

Lecture 19

Oct. 27 — Semisimple Lie Algebras

19.1 Lie's and Engel's Theorem

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Theorem 19.1 (Lie's theorem)**.** *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of some real or complex solvable Lie algebra. Then there exists a basis where all $\rho(x)$ are given by upper-triangular matrices.*

Proposition 19.1. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of a solvable Lie algebra $\mathfrak{g}$. Then there exists a vector $v \in V$ which is a common eigenvector for all $\rho(x)$ for $x \in \mathfrak{g}$.*

Proof. The proof is by induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ is solvable, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Consider the subspace $\mathfrak{g}'$ of codimension 1 which contains $[\mathfrak{g}, \mathfrak{g}]$, and write $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}x$. So $\mathfrak{g}'$ is an ideal.

We assume for induction that there exists a common eigenvector $v \in V$, i.e. for all $\rho(h)$ for $h \in \mathfrak{g}'$, we have $\rho(h)v = \lambda(h)v$. Then we can write

$$W = \mathrm{Span}\{v, \rho(x)v, \rho(x)^2 v, \dots\} = \{v^0, v^1, v^2, \dots\}.$$

We claim that W is stable under any action of $\mathfrak{h} \in \mathfrak{g}'$. To see this, write

$$hv^k = \lambda(h)v^k + \sum_{\ell < k} a_{k,\ell} v^\ell.$$

We prove this by induction:

$$hv^k = hxv^{k-1} = xhv^{k-1} + [h, x]v^{k-1} = \lambda(h)xv^{k-1} + \lambda([h, x])v^{k-1} + \dots.$$

this formula holds by induction on k, so W is stable. Now assume n is the smallest integer such that $v^{n+1} \in \mathrm{Span}\{v^0, v^1, \dots, v^n\}$. Therefore,$v^0, v^1, \dots, v^n$ form a basis of W. Therefore, $\rho(h)$ is upper-triangular in this basis, and moreover, $\operatorname{tr} \rho(h) = (n + 1)\lambda(h)$. Then

$$\operatorname{tr}_w([\rho(x), \rho(h)]) = 0,$$

so $\lambda([h, x]) = 0$ for all $\mathfrak{h} \in \mathfrak{g}'$. Then $hv^k = \lambda(h)v^k$, so we can choose the vector as an eigenvector of x. ☐

Proof of 19.1. The proof is by induction on $\dim V$. By Proposition 19.1, there exists a common eigenvector $v \in V/\mathbb{C}v$. By the induction hypothesis, there exists a basis $v_3, v_2, \dots$ of $\mathbb{C}$ such that the action of $\mathfrak{g}$ is upper-triangular. Now choose a preimage $\widehat{v}_i$ for any v_i. The action of any $x \in \mathfrak{g}$ in the basis $v, \widehat{v}_1, \widehat{v}_2, \dots$ will be upper-triangular. ☐

Corollary 19.1.1. *We have the following:*

1. *Any irreducible complex representation of a solvable Lie algebra is* 1*-dimensional.*
2. *If a complex Lie algebra* $\mathfrak{g}$ *is solvable, then there exists a sequence*

$$0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = \mathfrak{g},$$

where each I_k *is an ideal in* $\mathfrak{g}$ *and* I_{k+1}/I_k *is* 1*-dimensional.*

3. $\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Proof. If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable. Now assume g is solvable and assume $\mathfrak{g}$ is complex. Applying Lie's theorem to the adjoint representation, we have

$$\mathrm{ad}_g \subseteq \mathfrak{b} = \text{upper-triangular matrices...},$$

so $[\mathrm{ad}_\mathfrak{g}, \mathrm{ad}_\mathfrak{g}] = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} \subseteq \eta$, the set of strictly upper-triangular matrices. Thus

$$[x_1, [\dots [x_{n-1}, x_n]]] = 0$$

for sufficiently large n, so $[y, [x_1, \dots [x_{n-1}, x_n]]] = 0$ for sufficiently large n. So $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

Theorem 19.2. *Let* V *be a finite-dimensional vector space (either real or complex) and* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ *be a Lie subalgebra which consists of nilpotent operators. Then there exists a basis in* V *such that all operators* $x \in \mathfrak{g}$ *are strictly upper-triangular.*

Theorem 19.3 (Engel's theorem)**.** *A Lie algebra is nilpotent if and only if for every* $x \in \mathfrak{g}$ *the operator* $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ *is nilpotent.*

Proof. If $\mathfrak{g}$ is nilpotent, then $[x, [x, \dots, [x, y]]] = 0$, so $\mathrm{ad}_x^n y = 0$. If ad_x is nilpotent for all x, then by the previous theorem there exists a sequence of subspaces

$$0 \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$$

such that $\mathrm{ad}_x g_i =\subseteq \mathfrak{g}_{i-1}$, so $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$. □

19.2 Semisimple Lie Algebras

Definition 19.1. A Lie algebra is called *semisimple* if it contains no nonzero solvable ideals.

Remark. A semisimple Lie algebra has $\mathfrak{z}(\mathfrak{g}) = 0$.

Definition 19.2. A Lie algebra is called *simple* if it is not abelian and contains no ideals except itself and $\{0\}$.

Lemma 19.1. *Any simple Lie algebra is semisimple.*

Proof. If $\mathfrak{g}$ is simple, then it has no ideals except for $\mathfrak{g}$ and $\{0\}$. So if $\mathfrak{g}$ contains a nonzero solvable ideal, then it must coincide with $\mathfrak{g}$. But by solvability, the ideal $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, a contradiction. □

Example 19.2.1. We claim that $\mathfrak{sl}(2,\mathbb{C})$ is simple. Notice that ad_h is diagonal in the basis $\{e, f, h\}$:

$$\mathrm{ad}_h = \mathrm{diag}(2, -2, 0).$$

So any ideal must also be stable under ad_h. We use the following lemma from linear algebra:

> **Lemma 19.2.** *If A is diagonalizable with different eigenvalues $Av_i = \lambda_i v_i$ with $\lambda_i \neq \lambda_j$ for $i \neq j$, then the only subspaces invariant under A are spanned by eigenvectors.*

So any ideal in $\mathfrak{sl}(2,\mathbb{C})$ must be spanned by e, f, h. Assume first the ideal contains h. Since $[h, e] = 2e$ and $[h, f] = 2f$, this ideal is $\mathfrak{sl}(2,\mathbb{C})$. If it contains e, then $[e, f] = h$, so this ideal is also $\mathfrak{sl}(2,\mathbb{C})$. One can check the same for f, so we see that $\mathfrak{sl}(2,\mathbb{C})$ is simple.

Proposition 19.2. *In any Lie algebra $\mathfrak{g}$, there exists a unique solvable ideal that contains any other solvable ideal. This solvable ideal is called the* radical *of $\mathfrak{g}$ and is denoted* $\mathrm{rad}(\mathfrak{g})$.

Proof. First we show existence. If I_1, I_2 are solvable, then so is $I_1 + I_2$ since we can write

$$(I_1 + I_2)/I_1 = I_1/(I_1 \cap I_2),$$

so $I_1 + I_2$ contains I_1 and $I_1/(I_1 \cap I_2)$, which are both solvable. By induction, any sum of solvable ideals is solvable. Thus we can take $\mathrm{rad}(\mathfrak{g}) = \sum I$. Uniqueness is clear. $\square$

Remark. We can also say $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = \{0\}$.

Theorem 19.4. *For any Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. Conversely, if $\mathfrak{b}$ is a solvable ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{b}$ is semisimple, then $\mathfrak{b} = \mathrm{rad}(\mathfrak{g})$. In particular, we have an exact sequence*

$$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_{\mathrm{ss}} \longrightarrow 0$$

where $\mathfrak{b}$ is solvable and $\mathfrak{g}_{\mathrm{ss}}$ is semisimple.

Theorem 19.5 (Levi)**.** *Any Lie algebra can be written as a direct sum $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$, where $\mathfrak{g}_{\mathrm{ss}}$ is semisimple subalgebra (but not necessarily an ideal in $\mathfrak{g}$).*

Example 19.2.2. Consider the semidirect product $G = \mathrm{SO}(3,\mathbb{R}) \ltimes \mathbb{R}^3$ given by

$$\vec{x} \mapsto A\vec{x} + \vec{b},$$

for $A \in \mathrm{SO}(3,\mathbb{R})$ and $\vec{b} \in \mathbb{R}^3$. Then we have $\mathfrak{g} = \mathfrak{so}(3,\mathbb{R}) \oplus \mathbb{R}^3$ with the relations

$$[(A_1, b_1), (A_2, b_2)] = ([A_1, A_2], A_1 b_2 - A_2 b_1),$$

for $A_i \in \mathfrak{so}(3,\mathbb{R})$ and $b_1, b_2 \in \mathbb{R}^3$.

Theorem 19.6. *Let V be an irreducible complex representation of $\mathfrak{g}$. Then any $h \in \mathrm{rad}(\mathfrak{g})$ acts in V by scalar operators $\rho(h) = \lambda(h)\,\mathrm{Id}$. Also, any $h \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ acts by zero.*

Proof. We have seen that there is a common eigenvector $v \in V$ for all $h \in \mathrm{rad}(\mathfrak{g})$, so $\rho(h)v = \lambda(h)v$. This gives a map $\lambda : \mathrm{rad}(\mathfrak{g}) \to \mathbb{C}$. Define $V_\lambda = \{w \in V : \rho(h)w = \lambda(h)w \text{ for all } h \in \mathrm{rad}(\mathfrak{g})\}$. Using a similar argument to before, we get that $\rho(x)V_\lambda \subseteq V_\lambda$ for all $x \in \mathfrak{g}$, so V_λ is a subrepresentation. Since V_λ is nonzero and V is irreducible, we have $V = V_\lambda$. The second statement is trivial. $\square$

Definition 19.3. A Lie algebra is called *reductive* if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$. In other words, $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple, or $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$ for some semisimple subalgebra $\mathfrak{g}_{\mathrm{ss}} \subseteq \mathfrak{g}$.