MATH 8803: Representation Theory II

Frank Qiang
Instructor: Anton Zeitlin

Georgia Institute of Technology
Spring 2026

Contents

Lecture 1

Jan. 12 — Introduction and Review

1.1 Review and Overview

Remark. Recall that we are interested in representations of Lie groups G, which is closely related to representations of Lie algebras $\mathfrak{g}$.

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$, where $r = \dim \mathfrak{h}$ is called the *rank*. We have the Serre generators $\{h_i, e_i, f_i\}_{i=1}^r$ and relations

$$[h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = a_{i,j} f_j, \quad \mathrm{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \mathrm{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Here $\{\alpha_i\} \subseteq \mathfrak{h}^*$ and we identify $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_+$ is generated by $\{e_i\}$ and $\mathfrak{n}_-$ is generated by $\{f_i\}$. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R = R_+ \sqcup R_-$. We have $R_+ \subseteq Q_+$ and $R_- \subseteq Q_-$, where $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$. If the $a_{i,j}$ are degenerate, then we can define $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}c$ is called the *central extension* and $d = t\frac{d}{dt}$. We can think of these as maps $S^1 \to \mathfrak{g}$.

We can also consider the universal enveloping algebra $U(\mathfrak{g})$, and the related object. $U_q(\mathfrak{g})$ We have an R-matrix $R_{V,W}$ for the representations $V \otimes W$ and $W \otimes V$, and we have the relation

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}$$

in $V_1 \otimes V_2 \otimes V_3$. A main goal later in the course will be to relate the representations of $U_q(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.

In this case, we have the diagram:

$$\begin{tikzcd}
 & & \mathfrak{g} \arrow[dl] \arrow[dr] & & \\
 & U_q(\mathfrak{g}) \arrow[dl] \arrow[dr] & & \widehat{g} \arrow[ll] \arrow[dl] \arrow[dr] & \\
E_{q,\tau}(\mathfrak{g}) & & U_q(\widehat{g}) \arrow[ll] & & \widehat{\widehat{\mathfrak{g}}} \arrow[ll]
\end{tikzcd}$$

The object $U_q(\widehat{g})$ is related to quantum integrable models of spin chain type (XXX and XXZ), and $E_{q,\tau}(\mathfrak{g})$ is the *elliptic quantum group* (XYZ).

1.2 Representations of Semisimple Lie Algebras

Remark. Recall the *Weyl group* $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$. The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where ω_i are the fundamental weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We can consider the *highest weight representation*. The *Verma module* is $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ on which $\mathfrak{h}$ acts by $\lambda(h)$. Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each $\lambda \in \mathfrak{h}^*$, M_λ has a unique irreducible quotient L_λ. The *dominant integral weights* λ satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \le i \le r,$$

where $\lambda = \sum_{i=1}^r n_i \omega_i$ with $n_i \in \mathbb{Z}_+$.

Theorem 1.1. *The finite-dimensional irreps of $\mathfrak{g}$ are classified up to isomorphism by $\lambda \in P_+$. Moreover, $P(V)$ is Weyl invariant, and for any $\mu \in P(V)$, $w \in W$,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

Example 1.0.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the dominant integral weights are $n \in \mathbb{Z}_{\ge 0}$, $L_n = V_n$, and the Weyl group W acts by reflection.

Remark (Weyl character formula)**.** Let $\chi_V(g) = \mathrm{tr}_V(g)$. We can represent $g \sim e^h$, where $h \in \mathfrak{h}$. Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$. The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^\ell (w) e^{w(\lambda+\rho)}}{\Delta},$$

where $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w) w\rho}$ is the *Weyl denominator*. Here $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$. The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator* $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$, which acts by the scalar $(\lambda, \lambda + 2\rho)$.

1.3 Representations of SL_n and GL_n

Proposition 1.1. *For general simple $\mathfrak{g}$, let $\lambda = \sum_{i=1}^r m_i \omega_i$ be a dominant integral weight. Let $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$ and $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$. Let V be the subrepresentation of T_λ generated by v. Then $V \cong L_\lambda$.*

Remark. For $\mathfrak{sl}_n$, we have $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$. The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

We have $\alpha_i^\vee = e_i - e_{i-1}$ and $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$, where $\omega_i = (1, \dots, 1, 0, \dots, 0)$ with i ones. We can associate λ with the partition

$$\lambda = (m_1 + \cdots + m_{n-1}, m_2 + \cdots + m_{n-1}, \dots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0),$$

and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{n-1}$. Note that L_{ω_1} is the defining representation, where $v_{\omega_1} = (1, 0, \dots, 0)^T = v_1$, where $\{v_1, \dots, v_n\}$ is a basis of the defining representation. Then we have that $L_{\omega_m} = \wedge^m V$ with highest weight $v_1 \wedge \cdots \wedge v_m$. Here $e_i = E_{i,i+1}$. Then we see that $L_\lambda \subseteq \bigotimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$.

Remark. To move to GL_n, we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))/\mu_n,$$

where μ_n are the roots of unity embedded by $z \mapsto (z^{-1}, zI)$. We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of $\mathbb{C}^\times$. Its Lie algebra is spanned by h such that $e^{2\pi i h} = 1$. Within a representation, h acts by an operator H such that $e^{2\pi i H} = 1$. Thus all irreducible representations of $\mathbb{C}^\times$ are of the form $\chi_N(z) = z^N$. So for $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$, we have $L_{\lambda,N} = \chi_N \otimes L_\lambda$.

Exercise 1.1. Show that if $L_{\lambda,N} = \chi_N \otimes L_\lambda$, then $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r.

Remark. Letting $m_n = r \geq 0$ in the above exercise, the representation $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ for $\mathfrak{gl}_n$ corresponds to the partition $(m_1 + \cdots + m_n, \dots m_{n-1} + m_n, m_n)$.

Remark. For SL_n, the representation $\wedge^n V$ is trivial, but it is the determinant for GL_n. For GL_n, we also have χ^k and $(\chi^*)^k = \chi^{-k}$, these are called the *polynomial representations*.

Remark. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \geq \cdots \geq \lambda_n$ be a partition with at most n parts. Then $|\lambda| = \sum_i \lambda_i$ is an eigenvalue of $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$. We can realize λ as a Young diagram. Note that L_λ occurs in $V^{\otimes N}$, where V is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda : |\lambda| = N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$. There is a natural action of S_N on $V^{\otimes N}$.

Theorem 1.2 (Schur-Weyl duality)**.** *Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}(V^{\otimes N})$ and B be the image of $\mathbb{C}S_N$ in $\mathrm{End}(V^{\otimes N})$. Then*

1. *the centralizer of A is B and vice versa;*
2. *if λ has at most n parts, then the representation π_λ of B (and hence of S_N) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if $\dim V \geq N$, then the π_λ exhaust all irreducible representations of S_N.*

Lecture 2

Jan. 14 — Applications of Schur-Weyl Duality

2.1 The Schur Functor

Remark. Let V be the defining representation for GL_n. Then

$$V^{\otimes N} = \bigoplus_{\lambda:|\lambda|=N} L_\lambda \otimes \pi_\lambda.$$

Recall that if $\lambda = (\lambda_1, \ldots, \lambda_n)$, then we have

$$\lambda_1 = m_1 + \cdots + m_n, \quad \lambda_2 = m_2 + \cdots + m_n, \quad \ldots, \quad \lambda_n = m_n.$$

Definition 2.1. Suppose we are given the partition λ of N. The *Schur functor* S^λ is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space V. Note that this language, we have $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$.

Example 2.1.1. Consider the following:

1. $S^{(n)}V = S^n V$, where (n) is the partition of n with a single part.
2. $S^{(1^n)}V = \wedge^n V$, where (1^n) is the partition of n with n parts equal to 1.
3. $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$, where $\mathbb{C}_2$ acts trivially on $\mathbb{C}_+$ and by the sign on $\mathbb{C}_-$.
4. $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$, where S_3 acts trivially on $\mathbb{C}_+$ and by sign on $\mathbb{C}_-$ as before, and $\mathbb{C}^2 = \{(x,y,z) : x+y+z=0\}$.

   Note that $V \otimes V = S^2V \oplus \wedge^2 V$, so $S^2V \otimes V = S^3V \oplus S^{(2,1)}V$ and $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$.

Remark. Let $\dim V = N$ and λ have k parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+}(\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+}(\alpha, \rho)}.$$

We have $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$ and $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \ldots, 1, 0)$ (recall that ω_i is i ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \le i < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \le i < k < j \le N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1\le<k<j\le N}\frac{\lambda_i+j-i}{j-i}=\prod_{i=1}^{k}\frac{(N+1-i)\cdots(N+\lambda_i-i)}{(k+1-i)\cdots(k+\lambda_i-i)}.$$

Proposition 2.1. *We have* $\dim S^\lambda V = P_\lambda(N)$, *where* P_λ *is a polynomial of degree* $|\lambda|$ *with rational coefficients and integer roots. The roots of* P_λ *are all integers from the interval* $[1-\lambda_1, k-1]$ *(occurring with multiplicities).*

Example 2.1.2. Let $P_n(N)$ correspond to S^nV. Then $\lambda_1 = n$ and $k = 1$, and

$$P_n(N)=\dim S^nV=\binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N)=\dim\wedge^nV=\binom{N}{n}.$$

One can also consider $P_{(a,b)}(N)$ corresponding to partitions with two parts. The values $P_{(a,n)}(N)$ are called the Narayana numbers, which are of use in combinatorics.

2.2 Invariant Theory

Remark. Let V be a finite-dimensional vector space and $\{T_i\}\in(V^*)^{\otimes m_i}\otimes V^{\otimes n_i}$ for $i=1,\dots,k$. One would like to characterize *invariants* of such collections, i.e. polynomial functions $F(T_1,\dots,T_k)$ which are invariant under the action of $\mathrm{GL}(V)$.

One can think of such a tensor in $(V^*)^{\otimes m_i}\otimes V^{\otimes n_i}$ as a vertex with m_i incoming edges and n_i outgoing edges. Then constructing invariants $\{T_i\}$ reduces to studying graphs where T_i corresponds to a vertex v_i of the graph Γ. This allows us to assign to a given graph an invariant function F_Γ.

Theorem 2.1. *The functions* F_Γ *for various* Γ *span the space of invariant functions.*

Proof. We can view an invariant as an invariant element of the space $\bigotimes_{i=1}^k((V^*)^{\otimes m_i}\otimes V^{\otimes n_i})$, which we can view as $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M},V^{\otimes N})$, where $M=\sum d_im_i$ (the number of incoming edges) and $N=\sum d_in_i$ (the number of outgoing edges). Note that this space is empty when $M\neq N$, and the statement follows by Schur-Weyl duality when $M=N$. $\square$

Example 2.1.3. Let $m_i=n_i=1$. Then $T_1,\dots,T_k$ are matrices. Then the graph Γ must look like a cycle, hence the invariants are all of the form

$$F_{j_1,\dots,j_r}(T_1,\dots,T_k)=\mathrm{tr}(T_{j_1}\cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when V is large enough). In particular, if $P(T_1,\dots,T_k)=0$ in all dimensions, then $\mathrm{tr}(P(T_1,\dots,T_k)T_{k+1})=0$, which cannot be true as the trace decomposes in terms of the $F_{j_1,\dots,j_r}$. (However, note that $[X,Y]=0$ for 1×1 matrices and $[Z,[X,Y]^2]=0$ for 2×2 matrices.) This also implies the uniqueness of the μ_n in the BCH formula:

$$\log(\exp(x)\exp(y))=\sum_{n\ge1}\frac{\mu_n(x,y)}{n!}.$$

2.3 Weyl Character Formula for GL_n

Remark (Weyl character formula for GL_n)**.** Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})}, \tag{$*$}$$

where the denominator is $\Delta = \prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2}) = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$. Letting $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$,

$$\Delta = e^\rho \prod_{\alpha\in R_+}(1-e^{-\alpha}) = x_1^{n-1}x_2^{n-2}\cdots x_n^0\prod_{i<j}(1-x_j/x_i),$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$ and $x_i = e^{e_i}$ (e.g. $x_1 = e^{(1,0,\ldots,0)}$). After multiplying we get that

$$\Delta = \prod_{i<j}(x_i - x_j).$$

On the other hand, using $\Delta = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$, we have

$$\Delta = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1\le i,j\le n}) = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0 = \prod_{i<j}(x_i-x_j).$$

Now applying this to the numerator of $(*)$, we have

$$\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}.$$

Thus in total, the character χ_λ is given by

$$\chi_\lambda = \frac{\sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}}{\prod_{i<j}(x_i-x_j)} = \frac{\det(\{x_i^{\lambda_j+n-i}\})}{\prod_{i<j}(x_i-x_j)}.$$

These functions are known as the *Schur polynomials* $s_\lambda(x_1,\ldots,x_n)$.

Example 2.1.4 (Character of $S^{(n)}V$)**.** Using the above formula, we get the identity

$$s_{(m)}(x_1,\ldots,x_n) = \sum_{1\le j_1\le\cdots\le j_m\le n} x_{j_1}\cdots x_{j_m} = h_m(x_1,\ldots,x_m),$$

the mth complete symmetric function.

Example 2.1.5 (Character of $\lambda^n V$)**.** Similarly, one gets the identity

$$s_{(1^m)}(x_1,\ldots,x_n) = \sum_{1\le j_1<\cdots<j_m\le n} x_{j_1}\cdots x_{j_m} = e_m(x_1,\ldots,x_m),$$

the mth elementary symmetric function.

Example 2.1.6 (Trace in $V^{\otimes N}$)**.** Consider $x \otimes \sigma$, where $x = \mathrm{diag}(x_1, \dots, x_n)$ and σ has m_i cycles of length i. Then we have

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_\lambda \chi_\lambda(\sigma) s_\lambda(x) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_\lambda \chi_\lambda(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i<j}(x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Theorem 2.2 (Frobenius character formula)**.** *$\chi_\lambda(\sigma)$ is the coefficient of $x_1^{\lambda_1+N-1} \cdots x_N^{\lambda_N}$ in the polynomial*

$$\prod_{i<j}(x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

2.4 Howe Duality

Remark. Fix V, W and consider $S^n(V \otimes W)$, which is a representation of $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$.

Theorem 2.3 (Howe duality)**.** *We have a decomposition*

$$S^n(V \otimes W) = \bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes S^\lambda W.$$

Proof. We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left(\Big(\bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes \pi_\lambda \Big) \otimes \Big(\bigoplus_{\mu : |\mu| = n} S^\mu W \otimes \pi_\mu \Big) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu : |\lambda| = |\mu| = n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}. \end{aligned}$$

Since $\pi_\lambda = \pi_\lambda^*$, by Schur's lemma we have $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathbb{C}^{\delta_{\lambda,\mu}}$. □

Corollary 2.3.1 (Cauchy identity)**.** *Let $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$. Then*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_i}.$$

Lecture 3

Jan. 21 — Miniscule Weights

3.1 Miniscule Weights

Remark. Let $\mathfrak{g}$ be a simple complex Lie algebra.

Definition 3.1. A dominant integral weight ω for $\mathfrak{g}$ is called *miniscule* if $\langle \omega, \beta \rangle \leq 1$ for every positive coroot β (equivalently, if $|\langle \omega, \alpha \rangle| \leq 1$ for any coroot β).

Example 3.1.1. Clearly $\omega = 0$ is miniscule.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$ with fundamental weights $\{\omega_i\}_{i=1}^{n-1}$,[1] where

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0)$$

Let $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$. Note that $\langle \omega_i, e_j - e_k \rangle = 0$ when $j, k \leq i$ or $j, k > i$, and $\langle \omega_i, e_j - e_k \rangle = 1$ when $j \leq i < k$. So all of the ω_i are miniscule in this case.

Lemma 3.1. *Every nonzero miniscule weight is fundamental.*

Proof. Suppose ω is miniscule. Then there exists i with $\langle \omega, \alpha_i^\vee \rangle = 1$. Moreover, there can only be one such i, since if there were many, then $\langle \omega, \theta^\vee \rangle \geq 2$, where $\theta^\vee$ is the longest coroot (i.e. if $\theta = \sum_{m_i > 0} m_i \alpha_i$ is the longest root, then $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$). So ω is necessarily fundamental. □

Example 3.1.3. For G_2, F_4, and F_8, none of the fundamental weights are miniscule.

Lemma 3.2. *A fundamental weight ω_i is miniscule if and only if $m_i = 1$ where $\theta^\vee = \sum_j m_j \alpha_j^\vee$.*

Proof. By the miniscule condition, we know $m_i \leq 1$. If $m_i = 1$, then for any positive coroot $\beta = \sum n_j \alpha_j^\vee$ we have $n_j \leq m_j$, so $n_i \leq 1$. Thus $\langle \omega_i, \beta \rangle = n_i \leq 1$, so ω_i is miniscule. □

Lemma 3.3. *If $\omega \in Q$ with $|\langle \omega, \beta \rangle| \leq 1$ for all coroots β, then $\omega = 0$.*

Proof. Assume to the contrary that $\omega = \sum_i \alpha_i \neq 0$. We may assume that $\sum_i |m_i|$ is smallest possible. Then $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$, since the form is positive definite. Thus there exists j such that m_j and $\langle \omega, \alpha_j^\vee \rangle$ have the same sign. By replacing ω with $-\omega$ if necessary, we may assume both are positive. Then $\langle \omega, \alpha_j^\vee \rangle = 1$. Consider the reflection $s_j(\omega) = \omega - \alpha_j = \sum_i m_i' \alpha_i$. So $m_i' = m_j - 1$ and $m_i' = m_i$. But then $\sum_i |m_i'| = \sum_i |m_i| - 1 < \sum_i |m_i|$, contradicting the minimality of ω. □

[1] Recall a *fundamental weight* is a weight ω_i such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all simple coroots $\alpha_j^\vee$.

Proposition 3.1. *The following conditions are equivalent:*

1. *ω is miniscule;*
2. *all weights of L_ω belong to the Weyl orbit $W\omega$;*
3. *if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$, then $\lambda = \omega$.*

Proof. ($1 \Rightarrow 3$) If $\omega = 0$, then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$ where $\rho = \sum_{i=1}^r \omega_i$, so $\lambda = 0$. Now let $\omega = \omega_i$ be miniscule. Then $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for $k \neq i$, then the problem reduces to a lower rank Dynkin diagram. So we can assume $m_k > 0$ for every $k \neq i$. Let β be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \leq \langle \omega_i, \beta \rangle \leq 1.$$

If $\alpha_i^\vee$ does not occur in β, then the above is ≤ 0. In particular, we have $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \leq 0$ for $j \neq i$. If we also have $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \leq 0$, then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$. Otherwise, $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$. Then $m_j > 0$ for every j, so $\langle \omega_i - \lambda, \theta^\vee \rangle \geq 1$, since $\theta^\vee$ is a dominant coweight. Then $\langle \lambda, \theta^\vee \rangle \leq 0$, so we must have $\lambda = 0$ since $\theta^\vee$ contains all $\alpha_j^\vee$ with positive coefficients. But then $\omega_i \in Q$, which is impossible by Lemma 3.3.

($3 \Rightarrow 2$) If μ is any weight of L_ω, then there exists $w \in W$ such that $\lambda = w\mu$ is dominant (since every orbit of W intersects the dominant chamber at exactly 1 point). Then $\omega - \lambda \in Q_+$, so $\lambda = \omega$, hence $\mu = w^{-1}\omega \in W\omega$.

($2 \Rightarrow 1$) Suppose otherwise ω is not miniscule. Then $\langle \omega, \alpha^\vee \rangle > 1$ for some positive coroot $\alpha^\vee$. Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that $\omega - \alpha$ is a weight of L_ω (weight of $f_\alpha v_\omega$, where v_ω is a highest weight vector and $\{e_\alpha, f_\alpha, \alpha^\vee\}$ is an $\mathfrak{sl}_2$-triple). But $\omega - \alpha$ is not W-conjugate to ω, since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is W-invariant. Contradiction. □

Corollary 3.0.1. *If ω is miniscule, then $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$.*

Proposition 3.2. *$\omega \in P_+$ is miniscule if and only if the restriction of L_ω to any root $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. ($\Rightarrow$) Let ω be miniscule and $v \in L_\omega$ the highest weight vector (of weight $w\omega$) for $(\mathfrak{sl}_2)_\alpha$. Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then $h_\alpha v = 0$ or $h_\alpha v = v$, so the representation is 1-dimensional or 2-dimensional.

($\Leftarrow$) Suppose ω is not miniscule. Then there exists $\alpha \in Q_+$ with $\langle \omega, \alpha^\vee \rangle = m > 1$. Let v_ω be a highest weight vector, then $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$, which leads to a higher-dimensional $\mathfrak{sl}_2$-representation. □

Corollary 3.0.2. *If ω is miniscule, then for every dominant integral weight λ of $\mathfrak{g}$, we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda + \gamma}.$$

(It is assumed that if $\lambda + \gamma$ is not dominant, then $L_{\lambda+\gamma} = 0$.)

Proof. We know $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$. Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where Δ is the Weyl denominator. If $\lambda + \gamma \notin P_+$, then for some $\alpha_i^\vee$, we get $\langle \lambda + \gamma, \alpha_i^\vee \rangle < 0$. But we know $\langle \gamma, \alpha_i^\vee \rangle \geq -1$, so $\langle \lambda + \gamma, \alpha_i^\vee \rangle = -1$. Thus $\langle \lambda + \gamma + \rho, \alpha_i^\vee \rangle = 0$, so for any $w\gamma$, the term $ws_i\gamma$ comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result. □

Example 3.1.4. For $\mathfrak{sl}_2$, we have $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$, which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

Example 3.1.5. Let $V = V_{\omega_1}$ be the defining representation for GL_n. Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where λ is a partition and $\lambda + \square$ denotes the set of partitions obtained by adding a single box to λ. For example, for $\lambda = (3,3,2,1)$ we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

Similarly, for $\wedge^m V = L_{\omega_m}$ (where $\omega_m = (1,\ldots,1,0,\ldots,0)$ with m ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add m boxes to λ in $\lambda + m\square$. For example,

$$\wedge^2 V \otimes S^{(3,1)}V = S^{(4,2)}V \oplus S^{(4,1,1)}V \oplus S^{(3,2,1)}V \oplus S^{(3,1,1,1)}V.$$