

MATH 8803: Representation Theory II

Frank Qiang
Instructor: Anton Zeitlin

Georgia Institute of Technology
Spring 2026

Contents

1	Jan. 12 — Introduction and Review	2
1.1	Review and Overview	2
1.2	Representations of Semisimple Lie Algebras	3
1.3	Representations of SL_n and GL_n	3
2	Jan. 14 — Applications of Schur-Weyl Duality	5
2.1	The Schur Functor	5
2.2	Invariant Theory	6
2.3	Weyl Character Formula for GL_n	7
2.4	Howe Duality	8
3	Jan. 21 — Minuscule Weights	9
3.1	Minuscule Weights	9
3.2	Applications of Minuscule Weights	10
4	Jan. 26 — Other Classical Lie Algebras	12
4.1	Applications of Minuscule Weights, Continued	12
4.2	Other Classical Lie Algebras	13
4.3	Representations of Symplectic Lie Algebras	13
4.4	Representations of Orthogonal Lie Algebras	14
5	Jan. 28 — Other Classical Lie Algebras, Part 2	15
5.1	More on Orthogonal Lie Algebras	15
5.2	Clifford Algebras	16
6	Feb. 2 — Duals, Maximal Weights, Exponents	18
6.1	Dual Representations	18
6.2	Maximal Weights	18
6.3	Principal \mathfrak{sl}_2 -Subalgebra	19
6.4	Complex, Real, and Quaternionic Types	19
6.5	Review of Compact Lie Groups	20

Lecture 1

Jan. 12 — Introduction and Review

1.1 Review and Overview

Remark. Recall that we are interested in representations of Lie groups G , which is closely related to representations of Lie algebras \mathfrak{g} .

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$, where $r = \dim \mathfrak{h}$ is called the *rank*. We have the Serre generators $\{h_i, e_i, f_i\}_{i=1}^r$ and relations

$$[h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = a_{i,j} f_j, \quad \text{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \text{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $\alpha_i^\vee = 2\alpha_i / (\alpha_i, \alpha_i)$. Here $\{\alpha_i\} \subseteq \mathfrak{h}^*$ and we identify $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where \mathfrak{n}_+ is generated by $\{e_i\}$ and \mathfrak{n}_- is generated by $\{f_i\}$. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

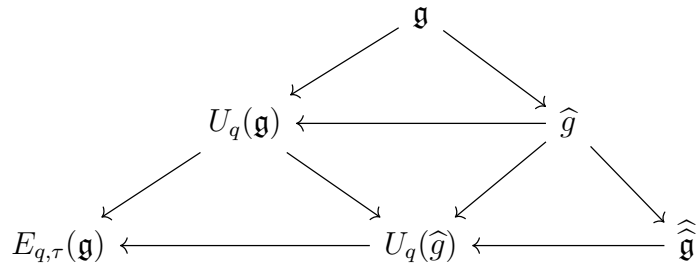
where $R = R_+ \sqcup R_-$. We have $R_+ \subseteq Q_+$ and $R_- \subseteq Q_-$, where $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$. If the $a_{i,j}$ are degenerate, then we can define $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}c$ is called the *central extension* and $d = t \frac{d}{dt}$. We can think of these as maps $S^1 \rightarrow \mathfrak{g}$.

We can also consider the universal enveloping algebra $U(\mathfrak{g})$, and the related object. $U_q(\mathfrak{g})$ We have an R -matrix $R_{V,W}$ for the representations $V \otimes W$ and $W \otimes V$, and we have the relation

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}$$

in $V_1 \otimes V_2 \otimes V_3$. A main goal later in the course will be to relate the representations of $U_q(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.

In this case, we have the diagram:



The object $U_q(\widehat{\mathfrak{g}})$ is related to quantum integrable models of spin chain type (XXX and XXZ), and $E_{q,\tau}(\mathfrak{g})$ is the *elliptic quantum group* (XYZ).

1.2 Representations of Semisimple Lie Algebras

Remark. Recall the *Weyl group* $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$. The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where ω_i are the fundamental weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We can consider the *highest weight representation*. The *Verma module* is $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$, where \mathbb{C}_λ is the 1-dimensional representation of $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ on which \mathfrak{h} acts by $\lambda(h)$. Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each $\lambda \in \mathfrak{h}^*$, M_λ has a unique irreducible quotient L_λ . The *dominant integral weights* λ satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \leq i \leq r,$$

where $\lambda = \sum_{i=1}^r n_i \omega_i$ with $n_i \in \mathbb{Z}_+$.

Theorem 1.1. *The finite-dimensional irreps of \mathfrak{g} are classified up to isomorphism by $\lambda \in P_+$. Moreover, $P(V)$ is Weyl invariant, and for any $\mu \in P(V)$, $w \in W$,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

Example 1.0.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the dominant integral weights are $n \in \mathbb{Z}_{\geq 0}$, $L_n = V_n$, and the Weyl group W acts by reflection.

Remark (Weyl character formula). Let $\chi_V(g) = \text{tr}_V(g)$. We can represent $g \sim e^h$, where $h \in \mathfrak{h}$. Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$. The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\Delta},$$

where $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w)} e^{w\rho}$ is the *Weyl denominator*. Here $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$. The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator* $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$, which acts by the scalar $(\lambda, \lambda + 2\rho)$.

1.3 Representations of SL_n and GL_n

Proposition 1.1. *For general simple \mathfrak{g} , let $\lambda = \sum_{i=1}^r m_i \omega_i$ be a dominant integral weight. Let $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$ and $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$. Let V be the subrepresentation of T_λ generated by v . Then $V \cong L_\lambda$.*

Remark. For \mathfrak{sl}_n , we have $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$. The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 + \dots + x_n = 0\}.$$

We have $\alpha_i^\vee = e_i - e_{i-1}$ and $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$, where $\omega_i = (1, \dots, 1, 0, \dots, 0)$ with i ones. We can associate λ with the partition

$$\lambda = (m_1 + \dots + m_{n-1}, m_2 + \dots + m_{n-1}, \dots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0),$$

and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1}$. Note that L_{ω_1} is the defining representation, where $v_{\omega_1} = (1, 0, \dots, 0)^T = v_1$, where $\{v_1, \dots, v_n\}$ is a basis of the defining representation. Then we have that $L_{\omega_m} = \wedge^m V$ with highest weight $v_1 \wedge \dots \wedge v_m$. Here $e_i = E_{i,i+1}$. Then we see that $L_\lambda \subseteq \bigotimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$.

Remark. To move to GL_n , we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})) / \mu_n,$$

where μ_n are the roots of unity embedded by $z \mapsto (z^{-1}, zI)$. We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of \mathbb{C}^\times . Its Lie algebra is spanned by h such that $e^{2\pi i h} = 1$. Within a representation, h acts by an operator H such that $e^{2\pi i H} = 1$. Thus all irreducible representations of \mathbb{C}^\times are of the form $\chi_N(z) = z^N$. So for $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$, we have $L_{\lambda,N} = \chi_N \otimes L_\lambda$.

Exercise 1.1. Show that if $L_{\lambda,N} = \chi_N \otimes L_\lambda$, then $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r .

Remark. Letting $m_n = r \geq 0$ in the above exercise, the representation $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ for \mathfrak{gl}_n corresponds to the partition $(m_1 + \dots + m_n, \dots, m_{n-1} + m_n, m_n)$.

Remark. For SL_n , the representation $\wedge^n V$ is trivial, but it is the determinant for GL_n . For GL_n , we also have χ^k and $(\chi^*)^k = \chi^{-k}$, these are called the *polynomial representations*.

Remark. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \geq \dots \geq \lambda_n$ be a partition with at most n parts. Then $|\lambda| = \sum_i \lambda_i$ is an eigenvalue of $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$. We can realize λ as a Young diagram. Note that L_λ occurs in $V^{\otimes N}$, where V is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$. There is a natural action of S_N on $V^{\otimes N}$.

Theorem 1.2 (Schur-Weyl duality). *Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}(V^{\otimes N})$ and B be the image of $\mathbb{C}S_N$ in $\mathrm{End}(V^{\otimes N})$. Then*

1. *the centralizer of A is B and vice versa;*
2. *if λ has at most n parts, then the representation π_λ of B (and hence of S_N) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if $\dim V \geq N$, then the π_λ exhaust all irreducible representations of S_N .*

Lecture 2

Jan. 14 — Applications of Schur-Weyl Duality

2.1 The Schur Functor

Remark. Let V be the defining representation for GL_n . Then

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda.$$

Recall that if $\lambda = (\lambda_1, \dots, \lambda_n)$, then we have

$$\lambda_1 = m_1 + \dots + m_n, \quad \lambda_2 = m_2 + \dots + m_n, \quad \dots, \quad \lambda_n = m_n.$$

Definition 2.1. Suppose we are given the partition λ of N . The *Schur functor* S^λ is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space V . Note that this language, we have $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$.

Example 2.1.1. Consider the following:

1. $S^{(n)}V = S^n V$, where (n) is the partition of n with a single part.
2. $S^{(1^n)}V = \wedge^n V$, where (1^n) is the partition of n with n parts equal to 1.
3. $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$, where \mathbb{C}_2 acts trivially on \mathbb{C}_+ and by the sign on \mathbb{C}_- .
4. $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$, where S_3 acts trivially on \mathbb{C}_+ and by sign on \mathbb{C}_- as before, and $\mathbb{C}^2 = \{(x, y, z) : x + y + z = 0\}$.

Note that $V \otimes V = S^2 V \oplus \wedge^2 V$, so $S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V$ and $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$.

Remark. Let $\dim V = N$ and λ have k parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

We have $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$ and $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \dots, 1, 0)$ (recall that ω_i is i ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i < k < j \leq N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1 \leq k < j \leq N} \frac{\lambda_i + j - i}{j - i} = \prod_{i=1}^k \frac{(N+1-i) \cdots (N+\lambda_i-i)}{(k+1-i) \cdots (k+\lambda_i-i)}.$$

Proposition 2.1. *We have $\dim S^\lambda V = P_\lambda(N)$, where P_λ is a polynomial of degree $|\lambda|$ with rational coefficients and integer roots. The roots of P_λ are all integers from the interval $[1 - \lambda_1, k - 1]$ (occurring with multiplicities).*

Example 2.1.2. Let $P_n(N)$ correspond to $S^n V$. Then $\lambda_1 = n$ and $k = 1$, and

$$P_n(N) = \dim S^n V = \binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N) = \dim \wedge^n V = \binom{N}{n}.$$

One can also consider $P_{(a,b)}(N)$ corresponding to partitions with two parts. The values $P_{(a,n)}(N)$ are called the Narayana numbers, which are of use in combinatorics.

2.2 Invariant Theory

Remark. Let V be a finite-dimensional vector space and $\{T_i\} \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ for $i = 1, \dots, k$. One would like to characterize *invariants* of such collections, i.e. polynomial functions $F(T_1, \dots, T_k)$ which are invariant under the action of $\mathrm{GL}(V)$.

One can think of such a tensor in $(V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ as a vertex with m_i incoming edges and n_i outgoing edges. Then constructing invariants $\{T_i\}$ reduces to studying graphs where T_i corresponds to a vertex v_i of the graph Γ . This allows us to assign to a given graph an invariant function F_Γ .

Theorem 2.1. *The functions F_Γ for various Γ span the space of invariant functions.*

Proof. We can view an invariant as an invariant element of the space $\bigotimes_{i=1}^k ((V^*)^{\otimes m_i} \otimes V^{\otimes n_i})$, which we can view as $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M}, V^{\otimes N})$, where $M = \sum d_i m_i$ (the number of incoming edges) and $N = \sum d_i n_i$ (the number of outgoing edges). Note that this space is empty when $M \neq N$, and the statement follows by Schur-Weyl duality when $M = N$. \square

Example 2.1.3. Let $m_i = n_i = 1$. Then T_1, \dots, T_k are matrices. Then the graph Γ must look like a cycle, hence the invariants are all of the form

$$F_{j_1, \dots, j_r}(T_1, \dots, T_k) = \mathrm{tr}(T_{j_1} \cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when V is large enough). In particular, if $P(T_1, \dots, T_k) = 0$ in all dimensions, then $\mathrm{tr}(P(T_1, \dots, T_k)T_{k+1}) = 0$, which cannot be true as the trace decomposes in terms of the F_{j_1, \dots, j_r} . (However, note that $[X, Y] = 0$ for 1×1 matrices and $[Z, [X, Y]^2] = 0$ for 2×2 matrices.) This also implies the uniqueness of the μ_n in the BCH formula:

$$\log(\exp(x) \exp(y)) = \sum_{n \geq 1} \frac{\mu_n(x, y)}{n!}.$$

2.3 Weyl Character Formula for GL_n

Remark (Weyl character formula for GL_n). Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}, \quad (*)$$

where the denominator is $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$. Letting $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$,

$$\Delta = e^\rho \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \prod_{i < j} (1 - x_j/x_i),$$

where $\rho = (n-1, n-2, \dots, 1, 0)$ and $x_i = e^{e_i}$ (e.g. $x_1 = e^{(1,0,\dots,0)}$). After multiplying we get that

$$\Delta = \prod_{i < j} (x_i - x_j).$$

On the other hand, using $\Delta = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$, we have

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1 \leq i, j \leq n}) = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0 = \prod_{i < j} (x_i - x_j).$$

Now applying this to the numerator of $(*)$, we have

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}.$$

Thus in total, the character χ_λ is given by

$$\chi_\lambda = \frac{\sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}}{\prod_{i < j} (x_i - x_j)} = \frac{\det(\{x_i^{\lambda_j + n - i}\})}{\prod_{i < j} (x_i - x_j)}.$$

These functions are known as the *Schur polynomials* $s_\lambda(x_1, \dots, x_n)$.

Example 2.1.4 (Character of $S^{(n)}V$). Using the above formula, we get the identity

$$s_{(m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} x_{j_1} \cdots x_{j_m} = h_m(x_1, \dots, x_n),$$

the m th complete symmetric function.

Example 2.1.5 (Character of $\lambda^n V$). Similarly, one gets the identity

$$s_{(1^m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \dots < j_m \leq n} x_{j_1} \cdots x_{j_m} = e_m(x_1, \dots, x_n),$$

the m th elementary symmetric function.

Example 2.1.6 (Trace in $V^{\otimes N}$). Consider $x \otimes \sigma$, where $x = \text{diag}(x_1, \dots, x_n)$ and σ has m_i cycles of length i . Then we have

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \dots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_{\lambda} \chi_{\lambda}(\sigma) s_{\lambda}(x) = \prod_i (x_1^i + \dots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_{\lambda} \chi_{\lambda}(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \dots + x_n^i)^{m_i}.$$

Theorem 2.2 (Frobenius character formula). $\chi_{\lambda}(\sigma)$ is the coefficient of $x_1^{\lambda_1 + N - 1} \dots x_N^{\lambda_N}$ in the polynomial

$$\prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \dots + x_n^i)^{m_i}.$$

2.4 Howe Duality

Remark. Fix V, W and consider $S^n(V \otimes W)$, which is a representation of $\text{GL}(V) \otimes \text{GL}(W)$.

Theorem 2.3 (Howe duality). *We have a decomposition*

$$S^n(V \otimes W) = \bigoplus_{\lambda: |\lambda|=n} S^{\lambda}V \otimes S^{\lambda}W.$$

Proof. We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left(\left(\bigoplus_{\lambda: |\lambda|=n} S^{\lambda}V \otimes \pi_{\lambda} \right) \otimes \left(\bigoplus_{\mu: |\mu|=n} S^{\mu}W \otimes \pi_{\mu} \right) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu: |\lambda|=|\mu|=n} S^{\lambda}V \otimes S^{\mu}W \otimes (\pi_{\lambda} \otimes \pi_{\mu})^{S_n}. \end{aligned}$$

Since $\pi_{\lambda} = \pi_{\lambda}^*$, by Schur's lemma we have $(\pi_{\lambda} \otimes \pi_{\mu})^{S_n} = \mathbb{C}^{\delta_{\lambda, \mu}}$. □

Corollary 2.3.1 (Cauchy identity). *Let $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$. Then*

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_j}.$$

Lecture 3

Jan. 21 — Minuscale Weights

3.1 Minuscale Weights

Remark. Let \mathfrak{g} be a simple complex Lie algebra.

Definition 3.1. A dominant integral weight ω for \mathfrak{g} is called *minuscale* if $\langle \omega, \beta \rangle \leq 1$ for every positive coroot β (equivalently, if $|\langle \omega, \alpha \rangle| \leq 1$ for any coroot β).

Example 3.1.1. Clearly $\omega = 0$ is minuscale.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$ with fundamental weights $\{\omega_i\}_{i=1}^{n-1}$,¹ where

$$\omega_i = (\underbrace{1, \dots, 1}_i, 0, \dots, 0)$$

Let $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$. Note that $\langle \omega_i, e_j - e_k \rangle = 0$ when $j, k \leq i$ or $j, k > i$, and $\langle \omega_i, e_j - e_k \rangle = 1$ when $j \leq i < k$. So all of the ω_i are minuscale in this case.

Lemma 3.1. *Every nonzero minuscale weight is fundamental.*

Proof. Suppose ω is minuscale. Then there exists i with $\langle \omega, \alpha_i^\vee \rangle = 1$. Moreover, there can only be one such i , since if there were many, then $\langle \omega, \theta^\vee \rangle \geq 2$, where θ^\vee is the longest coroot (i.e. if $\theta = \sum_{m_i > 0} m_i \alpha_i$ is the longest root, then $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$). So ω is necessarily fundamental. \square

Example 3.1.3. For G_2 , F_4 , and F_8 , none of the fundamental weights are minuscale.

Lemma 3.2. *A fundamental weight ω_i is minuscale if and only if $m_i = 1$ where $\theta^\vee = \sum_j m_j \alpha_j^\vee$.*

Proof. By the minuscale condition, we know $m_i \leq 1$. If $m_i = 1$, then for any positive coroot $\beta = \sum n_j \alpha_j^\vee$ we have $n_j \leq m_j$, so $n_i \leq 1$. Thus $\langle \omega_i, \beta \rangle = n_i \leq 1$, so ω_i is minuscale. \square

Lemma 3.3. *If $\omega \in Q$ with $|\langle \omega, \beta \rangle| \leq 1$ for all coroots β , then $\omega = 0$.*

Proof. Assume to the contrary that $\omega = \sum_i \alpha_i \neq 0$. We may assume that $\sum_i |m_i|$ is smallest possible. Then $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$, since the form is positive definite. Thus there exists j such that m_j and $\langle \omega, \alpha_j^\vee \rangle$ have the same sign. By replacing ω with $-\omega$ if necessary, we may assume both are positive. Then $\langle \omega, \alpha_j^\vee \rangle = 1$. Consider the reflection $s_j(\omega) = \omega - \alpha_j = \sum_i m'_i \alpha_i$. So $m'_i = m_j - 1$ and $m'_i = m_i$. But then $\sum_i |m'_i| = \sum_i |m_i| - 1 < \sum_i |m_i|$, contradicting the minimality of ω . \square

¹Recall a *fundamental weight* is a weight ω_i such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all simple coroots α_j^\vee .

Proposition 3.1. *The following conditions are equivalent:*

1. ω is minuscule;
2. all weights of L_ω belong to the Weyl orbit $W\omega$;
3. if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$, then $\lambda = \omega$.

Proof. (1 \Rightarrow 3) If $\omega = 0$, then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$ where $\rho = \sum_{i=1}^r \omega_i$, so $\lambda = 0$. Now let $\omega = \omega_i$ be minuscule. Then $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for $k \neq i$, then the problem reduces to a lower rank Dynkin diagram. So we can assume $m_k > 0$ for every $k \neq i$. Let β be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \leq \langle \omega_i, \beta \rangle \leq 1.$$

If α_j^\vee does not occur in β , then the above is ≤ 0 . In particular, we have $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \leq 0$ for $j \neq i$. If we also have $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \leq 0$, then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$. Otherwise, $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$. Then $m_j > 0$ for every j , so $\langle \omega_i - \lambda, \theta^\vee \rangle \geq 1$, since θ^\vee is a dominant coweight. Then $\langle \lambda, \theta^\vee \rangle \leq 0$, so we must have $\lambda = 0$ since θ^\vee contains all α_j^\vee with positive coefficients. But then $\omega_i \in Q$, which is impossible by Lemma 3.3.

(3 \Rightarrow 2) If μ is any weight of L_ω , then there exists $w \in W$ such that $\lambda = w\mu$ is dominant (since every orbit of W intersects the dominant chamber at exactly 1 point). Then $\omega - \lambda \in Q_+$, so $\lambda = \omega$, hence $\mu = w^{-1}\omega \in W\omega$.

(2 \Rightarrow 1) Suppose otherwise ω is not minuscule. Then $\langle \omega, \alpha^\vee \rangle > 1$ for some positive coroot α^\vee . Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that $\omega - \alpha$ is a weight of L_ω (weight of $f_\alpha v_\omega$, where v_ω is a highest weight vector and $\{e_\alpha, f_\alpha, \alpha^\vee\}$ is an \mathfrak{sl}_2 -triple). But $\omega - \alpha$ is not W -conjugate to ω , since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is W -invariant. Contradiction. \square

Corollary 3.0.1. *If ω is minuscule, then $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$.*

3.2 Applications of Minuscule Weights

Proposition 3.2. *$\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root \mathfrak{sl}_2 -subalgebra of \mathfrak{g} is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. (\Rightarrow) Let ω be minuscule and $v \in L_\omega$ the highest weight vector (of weight $w\omega$) for $(\mathfrak{sl}_2)_\alpha$. Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then $h_\alpha v = 0$ or $h_\alpha v = v$, so the representation is 1-dimensional or 2-dimensional.

(\Leftarrow) Suppose ω is not minuscule. Then there exists $\alpha \in Q_+$ with $\langle \omega, \alpha^\vee \rangle = m > 1$. Let v_ω be a highest weight vector, then $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$, which leads to a higher-dimensional \mathfrak{sl}_2 -representation. \square

Corollary 3.0.2. *If ω is minuscule, then for every dominant integral weight λ of \mathfrak{g} , we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma}.$$

(It is assumed that if $\lambda + \gamma$ is not dominant, then $L_{\lambda+\gamma} = 0$.)

Proof. We know $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$. Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where Δ is the Weyl denominator. If $\lambda + \gamma \notin P_+$, then for some α_i^\vee , we get $\langle \lambda + \gamma, \alpha_i^\vee \rangle < 0$. But we know $\langle \gamma, \alpha_i^\vee \rangle \geq -1$, so $\langle \lambda + \gamma, \alpha_i^\vee \rangle = -1$. Thus $\langle \lambda + \gamma + \rho, \alpha_i^\vee \rangle = 0$, so for any $w\gamma$, the term $ws_i\gamma$ comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result. \square

Example 3.1.4. For \mathfrak{sl}_2 , we have $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$, which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

Example 3.1.5. Let $V = V_{\omega_1}$ be the defining representation for GL_n . Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where λ is a partition and $\lambda + \square$ denotes the set of partitions obtained by adding a single box to λ . For example, for $\lambda = (3, 3, 2, 1)$ we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

Similarly, for $\wedge^m V = L_{\omega_m}$ (where $\omega_m = (1, \dots, 1, 0, \dots, 0)$ with m ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add m boxes to λ in $\lambda + m\square$. For example,

$$\wedge^2 V \otimes S^{(3,1)}V = S^{(4,2)}V \oplus S^{(4,1,1)}V \oplus S^{(3,2,1)}V \oplus S^{(3,1,1,1)}V.$$

Lecture 4

Jan. 26 — Other Classical Lie Algebras

4.1 Applications of Minuscule Weights, Continued

Proposition 4.1. *We have the following:*

1. Let λ be a partition of N . Then $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.
2. Let μ be a partition of $N + 1$. Then $\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda$.

Proof. (1) Let V be a vector space of sufficiently large dimension. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda, V^{\otimes(N+1)}) \cong \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N} \otimes V) = V \otimes S^\lambda V.$$

Now by Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}\left(\bigoplus_{\mu \in \lambda + \square} \pi_\mu, V^{\otimes(N+1)}\right) = \bigoplus_{\mu \in \lambda + \square} S^\mu V.$$

Since $V \otimes S^\lambda V = \bigoplus_{\mu \in \lambda + \square} S^\mu V$, we conclude that $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.

(2) This is left as an exercise. Use a different version of Frobenius reciprocity. \square

Definition 4.1. Let λ be a partition, and λ^\dagger be the *conjugate partition* (the one corresponding to the transposed diagram). For example, $(3, 3, 2, 1)^\dagger = (4, 3, 2)$.

Corollary 4.0.1. Let \mathbb{C}_- be the sign representation of S_N . Then $\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}$.

Proof. This is left as an exercise. The proof is by induction on $N = |\lambda|$. Let $C = \sum_{i < j} (i \ j)$, and note that its eigenvalues are the same as the Casimir operator of SL_N . \square

Proposition 4.2 (Skew Howe duality). *We have a decomposition $\wedge^n(V \otimes W) = \bigoplus_\lambda S^\lambda V \otimes S^{\lambda^\dagger} W$ (as $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$ -modules).*

Proposition 4.3. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights, so the number of minuscule weights is equal to $\det A$, where A is the Cartan matrix.*

Proof. Let $C = a + Q \in P/Q$ be a coset. Let $\omega \in C \cap P_+$ be the element which minimizes $\langle \omega, \rho^\vee \rangle$. If λ is the dominant weight for L_ω , then $\lambda \in C \cap P_+$ implies that

$$(\lambda, \rho^\vee) \geq (\omega, \rho^\vee).$$

Thus $(\omega - \lambda, \rho^\vee) \leq 0$, so $\omega - \lambda \in Q_+$. Thus $\lambda = \omega$, so ω is minuscule. Now suppose $\omega_1, \omega_2 \in C$ are minuscule and $\omega_1 \neq \omega_2$ with $\omega_1 - \omega_2 \in Q$. By Lemma 3.3, we must have $\langle \omega_1 - \omega_2, \beta \rangle \geq 2$ for all coroots β . But then $\langle \omega_1, \beta \rangle = 1$ (which implies $\beta > 0$) and $\langle \omega_2, \beta \rangle = -1$ (which implies $\beta < 0$), a contradiction. \square

Remark. Let A be the Cartan matrix. For every root, we can write

$$\alpha_i = \sum_{j=1}^r A_{i,j} \omega_j.$$

We have a covering map $\mathbb{R}^r / \Lambda_2 \rightarrow \mathbb{R}^r / \Lambda_1$, where $\Lambda_2 = P$ and $\Lambda_1 = Q$. Then $\det A$ is precisely the degree of this covering, which counts the number of cosets.

4.2 Other Classical Lie Algebras

Example 4.1.1. Recall that $\mathfrak{g} = \mathfrak{sp}_{2n}$ corresponds to the Dynkin diagram C_n ($\bullet \cdots \bullet \rightleftarrows \bullet$), where the arrow points from longer roots to shorter roots. We have $R_+ = e_i \pm e_j, 2e_j$. The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

We have $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = e_n$, and $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $1 \leq i \leq n$.

Example 4.1.2. The Dynkin diagram B_n ($\bullet \cdots \bullet \rightleftarrows \bullet$) corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Most things are the same as above, but we will have $\alpha_n = e_n$ and $\alpha_n^\vee = 2e_n$. We have the same ω_i for $i < n$, but we get $\omega_n = (1/2, \dots, 1/2)$. We have $R_+ = e_i \pm e_j, e_i$.

Example 4.1.3. The Dynkin diagram D_n ($\bullet \cdots \bullet \begin{smallmatrix} \nearrow \bullet \\ \searrow \bullet \end{smallmatrix}$) corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. In this case we have $R_+ = e_i \pm e_j$, and simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-2} = e_{n-1}, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n.$$

We have $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $i = 1, \dots, n-2$, but we get $\omega_{n-1} = (1/2, \dots, 1/2, 1/2)$ and $\omega_n = (1/2, \dots, 1/2, -1/2)$.

Remark. We have the following:

- For G_2, F_4, F_8 , we have $\det A = 1$ (here A is the Cartan matrix), so the only minuscule weight is 0.
- For B_n , we have $\det A = 2$ (the nontrivial minuscule weight is $(1/2, \dots, 1/2)$, and the representation has weights $(\pm 1/2, \dots, \pm 1/2)$ with all possible combinations of \pm and dimension 2^n).
- For D_n , we have $\det A = 4$. The minuscule weights are $\omega_1, \omega_{n-1}, \omega_n$. Here ω_1 is the $2n$ -dimensional defining representation. The other two are spin representations of dimension 2^{n-1} , with weights $(\pm 1/2, \dots, \pm 1/2)$, taking even or odd numbers of $-$ signs.

4.3 Representations of Symplectic Lie Algebras

Remark. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have the Dynkin diagram C_n and

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0).$$

The elements of the Cartan subalgebra are given by $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. So $L_{\omega_1} = V$ (the defining representation) with highest weight e_1 . Note that $\wedge^2 V$ is not irreducible:

$$\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C},$$

where \mathbb{C} is the trivial representation spanned by $B^{-1} = \sum_i e_{i+n} \wedge e_i$ (note that B^{-1} is invariant under \mathfrak{sp}_{2n}). However, one can check that $\wedge_0^2 V$ is irreducible.

Now let us consider L_{ω_j} for $j \geq 2$. Let $B = \sum_i e_i^* \wedge e_{i+n}^*$. We have an operator

$$i_B : \wedge^{i+1} V \longrightarrow \wedge^{i-1} V,$$

and we can denote $\wedge_0^i V = \ker(i_B|_{\wedge^i V})$ (note that $i_B|_{\wedge^i V}$ is injective when $i \geq n$). The $\wedge_0^i V$ are irreducible for $i \leq n$, and one can check that these form all of the irreducible representations of \mathfrak{sp}_{2n} (compute their dimensions and compare them to the highest weight representations).

We can also define an operator

$$\begin{aligned} m_B : \wedge^{i-1} V &\longrightarrow \wedge^{i+1} V \\ u &\mapsto B^{-1} \wedge u. \end{aligned}$$

One can check that m_B and i_B together with h (acting as $i - n$ on $\wedge^i V$) form an \mathfrak{sl}_2 -triple. Then

$$\wedge V = \bigoplus_{i=0}^n L_{\omega_i} \otimes L_{n-j}$$

(where $\omega_0 = 0$ and L_{n-j} is the representation of \mathfrak{sl}_2 of weight $n - j$) as representations of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$.

4.4 Representations of Orthogonal Lie Algebras

Remark. First consider B_n , which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$. In this case, the Cartan subalgebra is given by elements of the form $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n, 0)$. Let V be the $(2n+1)$ -dimensional defining representation. Then for $1 \leq i \leq n-1$, the representation $\wedge^i V$ is irreducible (one can check this using the dimension formula) with highest weight

$$\omega_i = (\underbrace{1, \dots, 1}_i, 0, \dots, 0).$$

On the other hand, $\wedge^n V$ is irreducible but not fundamental, with highest weight $(1, \dots, 1) = 2\omega_n$.

Now we consider the spin representation S (whose elements are called *spinors*). It has weights

$$(\pm 1/2, \pm 1/2, \dots, \pm 1/2)$$

(all possible combinations of \pm). The character of S is given by

$$\chi_S(x_1, \dots, x_n) = (x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}).$$

Remark. We will want to look at the Lie group $\text{Spin}_{2n+1}(\mathbb{C})$, the universal cover of $\text{SO}_{2n+1}(\mathbb{C})$. For $n = 1$, we have $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$. We will see that S is 2-dimensional, and $\pi_1(\text{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.

Lecture 5

Jan. 28 — Other Classical Lie Algebras, Part 2

5.1 More on Orthogonal Lie Algebras

Proposition 5.1. *For $n \geq 3$, we have $\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. There is a deformation retract from the surface X_n defined by $z_1^2 + \cdots + z_n^2 = 1$ in \mathbb{C}^n to the sphere $X_n^{\mathbb{R}} = X_n \cap \mathbb{R}^n$ defined by $x_1^2 + \cdots + x_n^2 = 1$ in \mathbb{R}^n : Let $\vec{z} = \vec{x} + i\vec{y} \in X_n$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, and note that $|\vec{z}|^2 = 1$ if and only if $|\vec{x}|^2 - |\vec{y}|^2 = 1$ and $\vec{x} \cdot \vec{y} = 0$. We also have

$$(\vec{x} + ti\vec{y})^2 = |\vec{x}|^2 - t^2|\vec{y}|^2 = 1 + (1 - t^2)|\vec{y}|^2 \geq 1.$$

So we can define a homotopy $f_t : X_n \rightarrow X_n$ by

$$f_t(\vec{z}) = \frac{\vec{x} + ti\vec{y}}{\sqrt{|\vec{x}|^2 - t^2|\vec{y}|^2}},$$

which satisfies $|f_t(z)|^2 = 1$, $f_1(z) = z$, and $f_0(z) \in X_n^{\mathbb{R}}$. Now observe that SO_n acts on X_n with fibers isomorphic to SO_{n-1} , so we have a long exact sequence

$$\pi_2(X_n) \longrightarrow \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \longrightarrow \pi_1(\mathrm{SO}_n(\mathbb{C})) \longrightarrow \pi_1(X_n).$$

The first and last groups are trivial for $n \geq 4$, so we have that $\pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C}))$. Thus the result follows once one checks that $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$ (left as an exercise). \square

Remark. Now consider D_n , which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. Let $Q = \sum_{i=1}^n x_i x_{i+n}$. The elements of the Cartan subalgebra are given by $\mathrm{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. Let V be the $2n$ -dimensional defining representation, and consider $\wedge^i V$ for $1 \leq i \leq n$. We have $\wedge^i V$ is irreducible for $0 \leq i \leq n-1$, and $L_{\omega_i} = \wedge^i V$ for $1 \leq i \leq n-2$. Note that $L_{(1, \dots, 1, 0)}$ is irreducible but not fundamental. Letting

$$\omega_{n-1} = (1/2, \dots, 1/2, 1/2) \quad \text{and} \quad (1/2, \dots, 1/2, -1/2),$$

the corresponding $S_+ = L_{\omega_{n-1}}$ and $S_- = L_{\omega_n}$ are the spin representations. The characters are

$$\chi_{S_{\pm}} = ((x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}))_{\pm},$$

where the \pm denotes an even or odd number of $-$ signs.

Example 5.0.1. We have $\text{Spin}_4 = \text{SL}_2 \times \text{SL}_2$, where factors correspond to S_+ and S_- . We have $\text{Spin}_5 = \text{Sp}_4$, where S is the 4-dimensional defining representation, and $\text{SO}_5 = \text{Sp}_4 / \{\pm 1\}$. We have $\text{Spin}_6 = \text{SL}_4$, where S_+, S_- are the 4-dimensional defining representation and its dual, and $\text{SO}_6 = \text{SL}_4 / \{\pm 1\}$.

Example 5.0.2. Let V be a finite-dimensional vector space, and consider $SV = \mathbb{C}[x_1, \dots, x_n]$, where x_1, \dots, x_n is an orthonormal basis. Denote $R^2 = \sum_{i=1}^n x_i^2 = S^2 V$ and $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$. Then:

1. Find a first-order differential operator making $\{R^2, \Delta, \cdot\}$ an \mathfrak{sl}_2 -triple. Make sure that it commutes with the $\text{SO}(V)$ action.
2. Let $H_m \subseteq S^m V$ be the subspace of harmonic polynomials. Then

$$SV = \bigoplus_{m=0}^{\infty} H_m \otimes W_m,$$

where $H_m = L_{m\omega_1}$ is the irreducible representation of $\text{SO}(V)$, and W_m is the Verma module for \mathfrak{sl}_2 of highest weight m .

5.2 Clifford Algebras

Definition 5.1. Let V be a finite-dimensional vector space (over \mathbb{C}) and (\cdot, \cdot) a non-degenerate inner product on V . Give an associative algebra structure to V by

$$v^2 = \frac{1}{2}(v, v).$$

Such an algebra is called a *Clifford algebra*, and is denoted by $\text{Cl}(V)$.

Corollary 5.0.1. $ab + ba = (a + b)^2 - a^2 - b^2 = (a, b).$

Example 5.1.1. The operators $^i\partial/\partial x_i$ and $dx_i \wedge \cdot$ define a Clifford algebra.

Example 5.1.2. Let $e^i e^j + e^j e^i = \delta_{i,j}$. Then $D = \sum_{i=1}^n e^i \partial_i$ (the *Dirac operator*) satisfies $D^2 = \Delta$.

Theorem 5.1. The algebra $\text{Cl}(V)$ is isomorphic to $\text{Mat}_{2^n}(\mathbb{C})$ if $\dim V = 2n$ and to $\text{Mat}_{2^n}(\mathbb{C}) \oplus \text{Mat}_{2^n}(\mathbb{C})$ if $\dim V = 2n + 1$.

Proof. First consider the even case. Choose a basis $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{i,j}, \quad a_i a_j + a_j a_i = 0, \quad b_i b_j + b_j b_i = 0, \quad b_i a_i + a_i b_i = 1.$$

Consider $\text{Cl}(V)$ -module $M = \wedge(a_1, \dots, a_n)$ (note that $\dim M = 2^n$) with action defined by

$$\rho(a_i)w = a_i w \quad \text{and} \quad \rho(b_i)w = \frac{\partial w}{\partial a_i}.$$

We have the relations

$$1 = \left[a_i, \frac{\partial}{\partial a_i} \right] = a_i \frac{\partial}{\partial a_i} + \frac{\partial}{\partial a_i} a_i \quad \text{and} \quad a_j \frac{\partial}{\partial a_i} = -\frac{\partial}{\partial a_i} a_j$$

for $i \neq j$. Let $c_{I,J} = a_{i_1} \cdots a_{i_k} b_{j_1} \cdots b_{j_m}$ for $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_m\}$. Check as an exercise that the $c_{I,J}$ are linearly independent, then $\rho : \text{Cl}(V) \rightarrow \text{End}(M)$ is an isomorphism.

If $\dim V = 2n + 1$, then we can pick an extra element z satisfying

$$(z, a_i) = (z, b_i) = 0 \quad \text{and} \quad (z, z) = 2,$$

with relations $za_i + a_iz = zb_i + b_iz = 0$ and $z^2 = 1$. Then $zw = \pm(-1)^{\deg w}wz$ for $w \in M_{\pm}$. \square

Remark. There is an embedding $\mathfrak{so}(V) \rightarrow \text{Cl}(V)$. Define a map

$$\begin{aligned} \xi : \wedge^2 V = \mathfrak{so}(V) &\longrightarrow \text{Cl}(V) \\ a \wedge b &\longmapsto \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a, b). \end{aligned}$$

One can check that $[\xi(a \wedge b), \xi(c \wedge d)] = \xi([a \wedge b, c \wedge d])$, so ξ is a homomorphism of Lie algebras. We have ξ^*M for even dimensional V and ξ^*M_{\pm} for odd dimensional V , and

$$\rho_{\xi^*M}(a) = \rho_M(\xi(a))$$

gives ξ^*M the structure of an $\mathfrak{so}(V)$ -representation (and similarly for ξ^*M_{\pm} . Notice that χ^*M is reducible:

$$\xi^*M = (\xi^*M)_0 \oplus (\xi^*M)_1$$

as representations, where the first factor corresponds to even degree and the second to odd degree.

Example 5.1.3. We have the following:

1. $(\xi^*M)_0 \cong S_+$ and $(\xi^*M)_1 \cong S_-$ for even dimensional V .
2. If $\dim V$ is odd, then χ^*M_{\pm} are both isomorphic to S .

Lecture 6

Feb. 2 — Duals, Maximal Weights, Exponents

6.1 Dual Representations

Remark. Let L_λ be the irreducible representation of highest weight λ . What is the highest weight of the dual representation L_λ^* ? Let w_0 be the maximal element in W .

Proposition 6.1. *We have $L_\lambda^* = L_{-w_0(\lambda)}$.*

Proof. Since λ is the highest weight in L_λ , for every weight μ in L_λ we have $\lambda - \mu \in Q_+$. So

$$Q_- \ni w_0(\lambda - \mu) = w_0(\lambda) - w_0(\mu),$$

so $w_0(\mu) - w_0(\lambda) \in Q_+$. Thus $w_0(\lambda) \leq w_0(\mu)$ for all $\mu \in L_\lambda$, so the length of w_0 is $|R_+|$. Thus $-w_0(\lambda)$ is the lowest weight of L_λ , which is the highest weight of L_λ^* . \square

Example 6.0.1. Since the length of w_0 is $|R_+|$, w_0 permutes the fundamental (co)weights and (co)roots, so w_0 is an automorphism of Dynkin diagrams. Note that W acts on P/Q , and w_0 acts as inversion.

- The Dynkin diagrams $A_1, B_n, C_n, G_2, F_4, E_7, E_8$ have no automorphisms, so $L_\lambda^* = L_\lambda$ for these.
- For A_n with $n \geq 2$, we have $P/Q = \mathbb{Z}/n\mathbb{Z}$ (e.g. if V is the defining representation, then we have that $L_{\omega_1}^* = V^* = \wedge^{n-1}V = L_{\omega_{n-1}}$).
- For E_6 , we have $P/Q = \mathbb{Z}/3\mathbb{Z}$, where w_0 exchanges the two minuscule weights.
- For D_{2n+1} , we have $P/Q = \mathbb{Z}/4\mathbb{Z}$ and $S_+^* = S_-$. For D_{2n} , $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $S_\pm^* = S_\pm$.

6.2 Maximal Weights

Definition 6.1. Let *maximal weight* of \mathfrak{g} , denoted θ , is the highest weight of the adjoint representation.

Example 6.1.1. If $\mathfrak{g} = \mathfrak{sl}_n$, then θ is the highest weight for $V^* \otimes V$ where V is the defining representation. Note that $V^* = \wedge^{n-1}V$, so the highest weight of $V^* \otimes V$ is $\theta = \omega_1 + \omega_{n-1}$. It is not fundamental.

Example 6.1.2. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $\mathfrak{g} = S^2V$ where V is the defining representation for \mathfrak{sp}_{2n} . Then $\theta = 2\omega_1$, which is also not fundamental.

Proposition 6.2. *For a simple Lie algebra with $\mathfrak{g} \neq \mathfrak{sl}_n, \mathfrak{sp}_{2n}$, the maximal weight θ is fundamental.*

Example 6.1.3. For \mathfrak{so}_N with $N \geq 7$ (type B or D), we have $\mathfrak{g} = \wedge^2V = L_{\omega_2}$.

6.3 Principal \mathfrak{sl}_2 -Subalgebra

Definition 6.2. Let \mathfrak{g} be a simple Lie algebra and $\{e_i, f_i, h_i\}$ (where $h_i = \alpha_i^\vee$) be Chevalley generators. Let $e = \sum_{i=1}^r e_i$, and h such that $\alpha_i(h) = 2$ for all i (so $h = 2\rho^\vee$). Note that we have $[h, e] = 2e$ and $h = \sum_{i=1}^r (2\rho^\vee, \omega_i) \alpha_i^\vee$. Let $f = \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i$. Then $\{h, e, f\}$ spans the *principal \mathfrak{sl}_2 -subalgebra* of \mathfrak{g} .

Example 6.2.1. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then the restriction of the defining representation to the principal \mathfrak{sl}_2 is L_n , the irreducible representation of \mathfrak{sl}_2 of highest weight n .

Remark. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, so that $\mathfrak{g} = \sum \mathfrak{g}[2m]$ where m is the height of the corresponding root subspace (and $2m$ is the weight with respect to h). Note $\mathfrak{g}[0] = \mathfrak{h}$ and $\dim \mathfrak{g}[0] = r$. Let $r_m = \dim \mathfrak{g}[2m]$.

Definition 6.3. We say that m is an *exponent* of \mathfrak{g} if $r_m > r_{m+1}$. The *multiplicity* of an exponent m is $r_m - r_{m+1}$.

Remark. We have $r_0 = r$ and there are r exponents (counted with multiplicities) $m_1 \leq m_2 \leq \dots \leq m_r$. The roots of height 2 are given by $\alpha_i + \alpha_j$ (where i, j are connected in the in the Dynkin diagram). So $r_0 = r_1 = 1$ and $r_2 = r - 1$. Thus $m_1 = 1$ and $m_2 > 1$. We have

$$m_r = (\rho^\vee, \theta) = h_{\mathfrak{g}} - 1,$$

where θ is the highest root. We call $h_{\mathfrak{g}}$ the *Coxeter number* of \mathfrak{g} . Note that $\sum_{i=1}^r m_i = |R_+|$.

Proposition 6.3. The restriction of \mathfrak{g} to its principal \mathfrak{sl}_2 -subalgebra decomposes as $\bigoplus_{i=1}^r L_{2m_i+1}$.

Example 6.3.1. The exponents for \mathfrak{sl}_n are $1, 2, \dots, n-1$.

Definition 6.4. The *Coxeter number* of \mathfrak{g} is $h_{\mathfrak{g}} = \langle \theta, \rho^\vee \rangle + 1 = m_r + 1$, and the *dual Coxeter number* is

$$h_{\mathfrak{g}}^\vee = \langle \tilde{\theta}^\vee, \rho \rangle + 1,$$

where $\tilde{\theta}^\vee = 2\theta/(\theta, \theta)$. If we normalize $(\theta, \theta) = 2$, then $h_{\mathfrak{g}}^\vee = \frac{1}{2}(\theta, \theta + 2\rho)$, which is the eigenvalue of $\frac{1}{2}C$ (where C is the Casimir operator).

6.4 Complex, Real, and Quaternionic Types

Definition 6.5. Let G be a Lie group. An irreducible representation V of G or \mathfrak{g} is of *complex type* if $V \not\cong V^*$, *real type* if there exists a symmetric isomorphism $V \rightarrow V^*$ (i.e. a symmetric inner product for V), and *quaternionic (or symplectic) type* if the isomorphism is given through an anti-symmetric inner product.

Exercise 6.1. Let V be an irreducible representation of a finite group G . Show that $\text{End}_{\mathbb{R}G}(V)$ (i.e. $V \otimes V^*$) can only be one of three types:

- complex type if $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{C}$,
- real type if $\text{End}_{\mathbb{R}G}(V) \cong \text{Mat}_{2 \times 2}(\mathbb{R})$,
- quaternionic type if $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{H}$.

Example 6.5.1. Let L_n be an irreducible representation of \mathfrak{sl}_2 . Then L_n is of real type for even n and of quaternionic type for odd n . Thus $L_n = S^n V$ where $V = L_1$ is 2-dimensional. The invariant form on $S^n V$ is $S^n B$, where B is a skew-symmetric invariant form on V .

Proposition 6.4. *Assume $\lambda = -w_0(\lambda)$, so that the corresponding representation is of real or quaternionic type. Then L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.*

Proof. The number $n = (2\rho^\vee, \lambda)$ is the eigenvalue of h (from the principal \mathfrak{sl}_2 -subalgebra) on the highest weight vector. Thus we have a decomposition

$$L_\lambda|_{\mathfrak{sl}_2} = L_n \oplus \bigoplus_{m < n} k_m L_m,$$

where L_n has multiplicity 1. One can determine the type based on L_n . □

6.5 Review of Compact Lie Groups

Remark. Let G be a real Lie group of dimension n . Then $\xi \in \wedge^n \mathfrak{g}^*$ gives a generating n -form ω , which is non-vanishing if ξ is non-vanishing. This gives rise to left- and right-invariant measures μ_L and μ_R on G , which are unique up to a constant. We say that G is *unimodular* if $\mu_L = \mu_R$ (up to constants).

When does $\mu_L = \mu_R$? For a 1-dimensional representation V of G , let $|V|$ be the representation of G on the same space where $\rho_{|V|}(g) = |\rho_V(g)|$ (where $\rho_V : G \rightarrow \text{Aut}(V) = \mathbb{R}^\times$).

Proposition 6.5. *We have $\mu_L = \mu_R$ if and only if $|\wedge^n \mathfrak{g}^*|$ is a trivial representation of G .*

Proof. We have $\mu_L = \mu_R$ if and only if the left-invariant form is right- or left-invariant up to a sign. This is equivalent to $\xi \in \wedge^n \mathfrak{g}^*$ being invariant up to a sign under the action of \mathfrak{g} . □

Proposition 6.6. *A compact group is unimodular.*

Proof. For compact groups, the representation $|\wedge^n \mathfrak{g}^*|$ gives a continuous homomorphism $G \rightarrow \mathbb{R}^+$, whose only compact subgroup is $\{1\}$. The result follows by Proposition 6.5. □

Proposition 6.7. *Let V be an irreducible representation of G . Then V admits a G -invariant unitary structure.*

Proof. Take any positive Hermitian form B on V , and define

$$B_{\text{av}}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w) dg.$$

This is well-defined and invariant by construction. □

Corollary 6.0.1 (Weyl unitary trick). *Any finite-dimensional representation is completely reducible.*

Proof. Write $V = W \oplus W^\perp$. If W is invariant, then so is W^\perp . □