

# MATH 8803: Representation Theory II

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# Lecture 1

## Jan. 12 — Introduction and Review

### 1.1 Review and Overview

**Remark.** Recall that we are interested in representations of Lie groups  $G$ , which is closely related to representations of Lie algebras  $\mathfrak{g}$ .

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra*  $\mathfrak{h} \subseteq \mathfrak{g}$ , where  $r = \dim \mathfrak{h}$  is called the *rank*. We have the Serre generators  $\{h_i, e_i, f_i\}_{i=1}^r$  and relations

$$[h_i, e_j] = a_{i,j}e_j, \quad [h_i, f_j] = a_{i,j}f_j, \quad \text{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \text{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where  $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$  for  $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$ . Here  $\{\alpha_i\} \subseteq \mathfrak{h}^*$  and we identify  $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$ . Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where  $\mathfrak{n}_+$  is generated by  $\{e_i\}$  and  $\mathfrak{n}_-$  is generated by  $\{f_i\}$ . We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

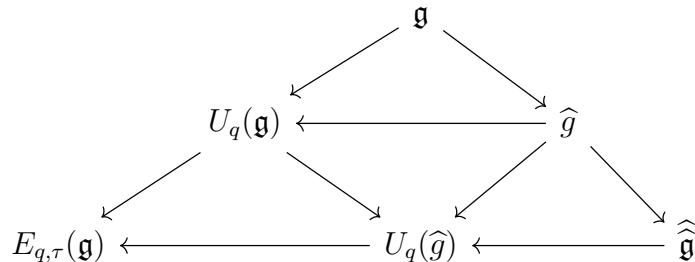
where  $R = R_+ \sqcup R_-$ . We have  $R_+ \subseteq Q_+$  and  $R_- \subseteq Q_-$ , where  $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$ . If the  $a_{i,j}$  are degenerate, then we can define  $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ , where  $\mathbb{C}c$  is called the *central extension* and  $d = t \frac{d}{dt}$ . We can think of these as maps  $S^1 \rightarrow \mathfrak{g}$ .

We can also consider the universal enveloping algebra  $U(\mathfrak{g})$ , and the related object.  $U_q(\mathfrak{g})$  We have an  $R$ -matrix  $R_{V,W}$  for the representations  $V \otimes W$  and  $W \otimes V$ , and we have the relation

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

in  $V_1 \otimes V_2 \otimes V_3$ . A main goal later in the course will be to relate the representations of  $U_q(\mathfrak{g})$  and  $\widehat{\mathfrak{g}}$ .

In this case, we have the diagram:



The object  $U_q(\widehat{\mathfrak{g}})$  is related to quantum integrable models of spin chain type (XXX and XXZ), and  $E_{q,\tau}(\mathfrak{g})$  is the *elliptic quantum group* (XYZ).

## 1.2 Representations of Semisimple Lie Algebras

**Remark.** Recall the *Weyl group*  $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$ . The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where  $\omega_i$  are the fundamental weights satisfying  $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ .

We can consider the *highest weight representation*. The *Verma module* is  $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$ , where  $\mathbb{C}_\lambda$  is the 1-dimensional representation of  $U(\mathfrak{h} \oplus \mathfrak{n}_+)$  on which  $\mathfrak{h}$  acts by  $\lambda(h)$ . Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each  $\lambda \in \mathfrak{h}^*$ ,  $M_\lambda$  has a unique irreducible quotient  $L_\lambda$ . The *dominant integral weights*  $\lambda$  satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \leq i \leq r,$$

where  $\lambda = \sum_{i=1}^r n_i \omega_i$  with  $n_i \in \mathbb{Z}_+$ .

**Theorem 1.1.** *The finite-dimensional irreps of  $\mathfrak{g}$  are classified up to isomorphism by  $\lambda \in P_+$ . Moreover,  $P(V)$  is Weyl invariant, and for any  $\mu \in P(V)$ ,  $w \in W$ ,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

**Example 1.0.1.** For  $\mathfrak{g} = \mathfrak{sl}_2$ , the dominant integral weights are  $n \in \mathbb{Z}_{\geq 0}$ ,  $L_n = V_n$ , and the Weyl group  $W$  acts by reflection.

**Remark** (Weyl character formula). Let  $\chi_V(g) = \text{tr}_V(g)$ . We can represent  $g \sim e^h$ , where  $h \in \mathfrak{h}$ . Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define  $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$ . The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^\ell(w) e^{w(\lambda + \rho)}}{\Delta},$$

where  $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w) w \rho}$  is the *Weyl denominator*. Here  $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$ . The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator*  $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$ , which acts by the scalar  $(\lambda, \lambda + 2\rho)$ .

## 1.3 Representations of $\text{SL}_n$ and $\text{GL}_n$

**Proposition 1.1.** *For general simple  $\mathfrak{g}$ , let  $\lambda = \sum_{i=1}^r m_i \omega_i$  be a dominant integral weight. Let  $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$  and  $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$ . Let  $V$  be the subrepresentation of  $T_\lambda$  generated by  $v$ . Then  $V \cong L_\lambda$ .*

**Remark.** For  $\mathfrak{sl}_n$ , we have  $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$ . The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 + \dots + x_n = 0\}.$$

We have  $\alpha_i^\vee = e_i - e_{i-1}$  and  $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$ , where  $\omega_i = (1, \dots, 1, 0, \dots, 0)$  with  $i$  ones. We can associate  $\lambda$  with the partition

$$\lambda = (m_1 + \dots + m_{n-1}, m_2 + \dots + m_{n-1}, \dots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0),$$

and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1}$ . Note that  $L_{\omega_1}$  is the defining representation, where  $v_{\omega_1} = (1, 0, \dots, 0)^T = v_1$ , where  $\{v_1, \dots, v_n\}$  is a basis of the defining representation. Then we have that  $L_{\omega_m} = \wedge^m V$  with highest weight  $v_1 \wedge \dots \wedge v_m$ . Here  $e_i = E_{i,i+1}$ . Then we see that  $L_\lambda \subseteq \bigotimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$ .

**Remark.** To move to  $\mathrm{GL}_n$ , we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})) / \mu_n,$$

where  $\mu_n$  are the roots of unity embedded by  $z \mapsto (z^{-1}, zI)$ . We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of  $\mathbb{C}^\times$ . Its Lie algebra is spanned by  $h$  such that  $e^{2\pi i h} = 1$ . Within a representation,  $h$  acts by an operator  $H$  such that  $e^{2\pi i H} = 1$ . Thus all irreducible representations of  $\mathbb{C}^\times$  are of the form  $\chi_N(z) = z^N$ . So for  $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$ , we have  $L_{\lambda, N} = \chi_N \otimes L_\lambda$ .

**Exercise 1.1.** Show that if  $L_{\lambda, N} = \chi_N \otimes L_\lambda$ , then  $N = nr + \sum_{i=1}^{n-1} \lambda_i$  for some integer  $r$ .

**Remark.** Letting  $m_n = r \geq 0$  in the above exercise, the representation  $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$  for  $\mathfrak{gl}_n$  corresponds to the partition  $(m_1 + \dots + m_n, \dots, m_{n-1} + m_n, m_n)$ .

**Remark.** For  $\mathrm{SL}_n$ , the representation  $\wedge^n V$  is trivial, but it is the determinant for  $\mathrm{GL}_n$ . For  $\mathrm{GL}_n$ , we also have  $\chi^k$  and  $(\chi^*)^k = \chi^{-k}$ , these are called the *polynomial representations*.

**Remark.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  with  $\lambda_1 \geq \dots \geq \lambda_n$  be a partition with at most  $n$  parts. Then  $|\lambda| = \sum_i \lambda_i$  is an eigenvalue of  $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$ . We can realize  $\lambda$  as a Young diagram. Note that  $L_\lambda$  occurs in  $V^{\otimes N}$ , where  $V$  is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda,$$

where  $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$ . There is a natural action of  $S_N$  on  $V^{\otimes N}$ .

**Theorem 1.2** (Schur-Weyl duality). *Let  $A$  be the image of  $U(\mathfrak{gl}_n)$  in  $\mathrm{End}(V^{\otimes N})$  and  $B$  be the image of  $\mathbb{C}S_N$  in  $\mathrm{End}(V^{\otimes N})$ . Then*

1. *the centralizer of  $A$  is  $B$  and vice versa;*
2. *if  $\lambda$  has at most  $n$  parts, then the representation  $\pi_\lambda$  of  $B$  (and hence of  $S_N$ ) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if  $\dim V \geq N$ , then the  $\pi_\lambda$  exhaust all irreducible representations of  $S_N$ .*

# Lecture 2

## Jan. 14 — Applications of Schur-Weyl Duality

### 2.1 The Schur Functor

**Remark.** Let  $V$  be the defining representation for  $\mathrm{GL}_n$ . Then

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda.$$

Recall that if  $\lambda = (\lambda_1, \dots, \lambda_n)$ , then we have

$$\lambda_1 = m_1 + \dots + m_n, \quad \lambda_2 = m_2 + \dots + m_n, \quad \dots, \quad \lambda_n = m_n.$$

**Definition 2.1.** Suppose we are given the partition  $\lambda$  of  $N$ . The *Schur functor*  $S^\lambda$  is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space  $V$ . Note that this language, we have  $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$ .

**Example 2.1.1.** Consider the following:

1.  $S^{(n)}V = S^n V$ , where  $(n)$  is the partition of  $n$  with a single part.
2.  $S^{(1^n)}V = \wedge^n V$ , where  $(1^n)$  is the partition of  $n$  with  $n$  parts equal to 1.
3.  $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$ , where  $\mathbb{C}_2$  acts trivially on  $\mathbb{C}_+$  and by the sign on  $\mathbb{C}_-$ .
4.  $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$ , where  $S_3$  acts trivially on  $\mathbb{C}_+$  and by sign on  $\mathbb{C}_-$  as before, and  $\mathbb{C}^2 = \{(x, y, z) : x + y + z = 0\}$ .

Note that  $V \otimes V = S^2 V \oplus \wedge^2 V$ , so  $S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V$  and  $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$ .

**Remark.** Let  $\dim V = N$  and  $\lambda$  have  $k$  parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

We have  $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$  and  $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \dots, 1, 0)$  (recall that  $\omega_i$  is  $i$  ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i < k < j \leq N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1 \leq i < k < j \leq N} \frac{\lambda_i + j - i}{j - i} = \prod_{i=1}^k \frac{(N+1-i) \cdots (N+\lambda_i-i)}{(k+1-i) \cdots (k+\lambda_i-i)}.$$

**Proposition 2.1.** *We have  $\dim S^\lambda V = P_\lambda(N)$ , where  $P_\lambda$  is a polynomial of degree  $|\lambda|$  with rational coefficients and integer roots. The roots of  $P_\lambda$  are all integers from the interval  $[1 - \lambda_1, k - 1]$  (occurring with multiplicities).*

**Example 2.1.2.** Let  $P_n(N)$  correspond to  $S^n V$ . Then  $\lambda_1 = n$  and  $k = 1$ , and

$$P_n(N) = \dim S^n V = \binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N) = \dim \wedge^n V = \binom{N}{n}.$$

One can also consider  $P_{(a,b)}(N)$  corresponding to partitions with two parts. The values  $P_{(a,n)}(N)$  are called the Narayana numbers, which are of use in combinatorics.

## 2.2 Invariant Theory

**Remark.** Let  $V$  be a finite-dimensional vector space and  $\{T_i\} \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$  for  $i = 1, \dots, k$ . One would like to characterize *invariants* of such collections, i.e. polynomial functions  $F(T_1, \dots, T_k)$  which are invariant under the action of  $\mathrm{GL}(V)$ .

One can think of such a tensor in  $(V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$  as a vertex with  $m_i$  incoming edges and  $n_i$  outgoing edges. Then constructing invariants  $\{T_i\}$  reduces to studying graphs where  $T_i$  corresponds to a vertex  $v_i$  of the graph  $\Gamma$ . This allows us to assign to a given graph an invariant function  $F_\Gamma$ .

**Theorem 2.1.** *The functions  $F_\Gamma$  for various  $\Gamma$  span the space of invariant functions.*

*Proof.* We can view an invariant as an invariant element of the space  $\bigotimes_{i=1}^k ((V^*)^{\otimes m_i} \otimes V^{\otimes n_i})$ , which we can view as  $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M}, V^{\otimes N})$ , where  $M = \sum d_i m_i$  (the number of incoming edges) and  $N = \sum d_i n_i$  (the number of outgoing edges). Note that this space is empty when  $M \neq N$ , and the statement follows by Schur-Weyl duality when  $M = N$ .  $\square$

**Example 2.1.3.** Let  $m_i = n_i = 1$ . Then  $T_1, \dots, T_k$  are matrices. Then the graph  $\Gamma$  must look like a cycle, hence the invariants are all of the form

$$F_{j_1, \dots, j_r}(T_1, \dots, T_k) = \mathrm{tr}(T_{j_1} \cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when  $V$  is large enough). In particular, if  $P(T_1, \dots, T_k) = 0$  in all dimensions, then  $\mathrm{tr}(P(T_1, \dots, T_k) T_{k+1}) = 0$ , which cannot be true as the trace decomposes in terms of the  $F_{j_1, \dots, j_r}$ . (However, note that  $[X, Y] = 0$  for  $1 \times 1$  matrices and  $[Z, [X, Y]^2] = 0$  for  $2 \times 2$  matrices.) This also implies the uniqueness of the  $\mu_n$  in the BCH formula:

$$\log(\exp(x) \exp(y)) = \sum_{n \geq 1} \frac{\mu_n(x, y)}{n!}.$$

## 2.3 Weyl Character Formula for $\mathrm{GL}_n$

**Remark** (Weyl character formula for  $\mathrm{GL}_n$ ). Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}, \quad (*)$$

where the denominator is  $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$ . Letting  $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ ,

$$\Delta = e^\rho \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \prod_{i < j} (1 - x_j/x_i),$$

where  $\rho = (n-1, n-2, \dots, 1, 0)$  and  $x_i = e^{e_i}$  (e.g.  $x_1 = e^{(1,0,\dots,0)}$ ). After multiplying we get that

$$\Delta = \prod_{i < j} (x_i - x_j).$$

On the other hand, using  $\Delta = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$ , we have

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1 \leq i, j \leq n}) = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0 = \prod_{i < j} (x_i - x_j).$$

Now applying this to the numerator of (\*), we have

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}.$$

Thus in total, the character  $\chi_\lambda$  is given by

$$\chi_\lambda = \frac{\sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}}{\prod_{i < j} (x_i - x_j)} = \frac{\det(\{x_i^{\lambda_j + n - i}\})}{\prod_{i < j} (x_i - x_j)}.$$

These functions are known as the *Schur polynomials*  $s_\lambda(x_1, \dots, x_n)$ .

**Example 2.1.4** (Character of  $S^{(n)}V$ ). Using the above formula, we get the identity

$$s_{(m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} x_{j_1} \cdots x_{j_m} = h_m(x_1, \dots, x_m),$$

the  $m$ th complete symmetric function.

**Example 2.1.5** (Character of  $\lambda^n V$ ). Similarly, one gets the identity

$$s_{(1^m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} x_{j_1} \cdots x_{j_m} = e_m(x_1, \dots, x_m),$$

the  $m$ th elementary symmetric function.

**Example 2.1.6** (Trace in  $V^{\otimes N}$ ). Consider  $x \otimes \sigma$ , where  $x = \text{diag}(x_1, \dots, x_n)$  and  $\sigma$  has  $m_i$  cycles of length  $i$ . Then we have

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_{\lambda} \chi_{\lambda}(\sigma) s_{\lambda}(x) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_{\lambda} \chi_{\lambda}(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

**Theorem 2.2** (Frobenius character formula).  $\chi_{\lambda}(\sigma)$  is the coefficient of  $x_1^{\lambda_1 + N - 1} \cdots x_N^{\lambda_N}$  in the polynomial

$$\prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

## 2.4 Howe Duality

**Remark.** Fix  $V, W$  and consider  $S^n(V \otimes W)$ , which is a representation of  $\text{GL}(V) \otimes \text{GL}(W)$ .

**Theorem 2.3** (Howe duality). We have a decomposition

$$S^n(V \otimes W) = \bigoplus_{\lambda: |\lambda|=n} S^{\lambda} V \otimes S^{\lambda} W.$$

*Proof.* We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left( \left( \bigoplus_{\lambda: |\lambda|=n} S^{\lambda} V \otimes \pi_{\lambda} \right) \otimes \left( \bigoplus_{\mu: |\mu|=n} S^{\mu} W \otimes \pi_{\mu} \right) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu: |\lambda|=|\mu|=n} S^{\lambda} V \otimes S^{\mu} W \otimes (\pi_{\lambda} \otimes \pi_{\mu})^{S_n}. \end{aligned}$$

Since  $\pi_{\lambda} = \pi_{\lambda}^*$ , by Schur's lemma we have  $(\pi_{\lambda} \otimes \pi_{\mu})^{S_n} = \mathbb{C}^{\delta_{\lambda, \mu}}$ . □

**Corollary 2.3.1** (Cauchy identity). Let  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_s)$ . Then

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - zx_i y_j}.$$

# Lecture 3

## Jan. 21 — Minuscule Weights

### 3.1 Minuscule Weights

**Remark.** Let  $\mathfrak{g}$  be a simple complex Lie algebra.

**Definition 3.1.** A dominant integral weight  $\omega$  for  $\mathfrak{g}$  is called *minuscule* if  $\langle \omega, \beta \rangle \leq 1$  for every positive coroot  $\beta$  (equivalently, if  $|\langle \omega, \alpha \rangle| \leq 1$  for any coroot  $\beta$ ).

**Example 3.1.1.** Clearly  $\omega = 0$  is minuscule.

**Example 3.1.2.** Let  $\mathfrak{g} = \mathfrak{sl}_n$  with fundamental weights  $\{\omega_i\}_{i=1}^{n-1}$ ,<sup>1</sup> where

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0)$$

Let  $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$ . Note that  $\langle \omega_i, e_j - e_k \rangle = 0$  when  $j, k \leq i$  or  $j, k > i$ , and  $\langle \omega_i, e_j - e_k \rangle = 1$  when  $j \leq i < k$ . So all of the  $\omega_i$  are minuscule in this case.

**Lemma 3.1.** Every nonzero minuscule weight is fundamental.

*Proof.* Suppose  $\omega$  is minuscule. Then there exists  $i$  with  $\langle \omega, \alpha_i^\vee \rangle = 1$ . Moreover, there can only be one such  $i$ , since if there were many, then  $\langle \omega, \theta^\vee \rangle \geq 2$ , where  $\theta^\vee$  is the longest coroot (i.e. if  $\theta = \sum_{m_i > 0} m_i \alpha_i$  is the longest root, then  $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$ ). So  $\omega$  is necessarily fundamental.  $\square$

**Example 3.1.3.** For  $G_2$ ,  $F_4$ , and  $F_8$ , none of the fundamental weights are minuscule.

**Lemma 3.2.** A fundamental weight  $\omega_i$  is minuscule if and only if  $m_i = 1$  where  $\theta^\vee = \sum_j m_j \alpha_j^\vee$ .

*Proof.* By the minuscule condition, we know  $m_i \leq 1$ . If  $m_i = 1$ , then for any positive coroot  $\beta = \sum n_j \alpha_j^\vee$  we have  $n_j \leq m_j$ , so  $n_i \leq 1$ . Thus  $\langle \omega_i, \beta \rangle = n_i \leq 1$ , so  $\omega_i$  is minuscule.  $\square$

**Lemma 3.3.** If  $\omega \in Q$  with  $|\langle \omega, \beta \rangle| \leq 1$  for all coroots  $\beta$ , then  $\omega = 0$ .

*Proof.* Assume to the contrary that  $\omega = \sum_i m_i \alpha_i \neq 0$ . We may assume that  $\sum_i |m_i|$  is smallest possible. Then  $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$ , since the form is positive definite. Thus there exists  $j$  such that  $m_j$  and  $\langle \omega, \alpha_j^\vee \rangle$  have the same sign. By replacing  $\omega$  with  $-\omega$  if necessary, we may assume both are positive. Then  $\langle \omega, \alpha_j^\vee \rangle = 1$ . Consider the reflection  $s_j(\omega) = \omega - \alpha_j = \sum_i m'_i \alpha_i$ . So  $m'_i = m_j - 1$  and  $m'_i = m_i$ . But then  $\sum_i |m'_i| = \sum_i |m_i| - 1 < \sum_i |m_i|$ , contradicting the minimality of  $\omega$ .  $\square$

---

<sup>1</sup>Recall a *fundamental weight* is a weight  $\omega_i$  such that  $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$  for all simple coroots  $\alpha_j^\vee$ .

**Proposition 3.1.** *The following conditions are equivalent:*

1.  $\omega$  is minuscule;
2. all weights of  $L_\omega$  belong to the Weyl orbit  $W\omega$ ;
3. if  $\lambda$  is a dominant integral weight such that  $\omega - \lambda \in Q_+$ , then  $\lambda = \omega$ .

*Proof.* (1  $\Rightarrow$  3) If  $\omega = 0$ , then  $-\lambda \in Q_+$ , so  $(\lambda, \rho) \leq 0$  where  $\rho = \sum_{i=1}^r \omega_i$ , so  $\lambda = 0$ . Now let  $\omega = \omega_i$  be minuscule. Then  $\omega_i - \lambda = \sum_k m_k \alpha_k$  with  $m_k \geq 0$ . If  $m_k = 0$  for  $k \neq i$ , then the problem reduces to a lower rank Dynkin diagram. So we can assume  $m_k > 0$  for every  $k \neq i$ . Let  $\beta$  be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \leq \langle \omega_i, \beta \rangle \leq 1.$$

If  $\alpha_j^\vee$  does not occur in  $\beta$ , then the above is  $\leq 0$ . In particular, we have  $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \leq 0$  for  $j \neq i$ . If we also have  $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \leq 0$ , then  $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$ , so  $\omega_i = \lambda$ . Otherwise,  $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$ . Then  $m_j > 0$  for every  $j$ , so  $\langle \omega_i - \lambda, \theta^\vee \rangle \geq 1$ , since  $\theta^\vee$  is a dominant coweight. Then  $\langle \lambda, \theta^\vee \rangle \leq 0$ , so we must have  $\lambda = 0$  since  $\theta^\vee$  contains all  $\alpha_j^\vee$  with positive coefficients. But then  $\omega_i \in Q$ , which is impossible by Lemma 3.3.

(3  $\Rightarrow$  2) If  $\mu$  is any weight of  $L_\omega$ , then there exists  $w \in W$  such that  $\lambda = w\mu$  is dominant (since every orbit of  $W$  intersects the dominant chamber at exactly 1 point). Then  $\omega - \lambda \in Q_+$ , so  $\lambda = \omega$ , hence  $\mu = w^{-1}\omega \in W\omega$ .

(2  $\Rightarrow$  1) Suppose otherwise  $\omega$  is not minuscule. Then  $\langle \omega, \alpha^\vee \rangle > 1$  for some positive coroot  $\alpha^\vee$ . Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that  $\omega - \alpha$  is a weight of  $L_\omega$  (weight of  $f_\alpha v_\omega$ , where  $v_\omega$  is a highest weight vector and  $\{e_\alpha, f_\alpha, \alpha^\vee\}$  is an  $\mathfrak{sl}_2$ -triple). But  $\omega - \alpha$  is not  $W$ -conjugate to  $\omega$ , since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is  $W$ -invariant. Contradiction. □

**Corollary 3.0.1.** *If  $\omega$  is minuscule, then  $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$ .*

## 3.2 Applications of Minuscule Weights

**Proposition 3.2.**  $\omega \in P_+$  is minuscule if and only if the restriction of  $L_\omega$  to any root  $\mathfrak{sl}_2$ -subalgebra of  $\mathfrak{g}$  is the direct sum of 1-dimensional and 2-dimensional representations.

*Proof.* ( $\Rightarrow$ ) Let  $\omega$  be minuscule and  $v \in L_\omega$  the highest weight vector (of weight  $w\omega$ ) for  $(\mathfrak{sl}_2)_\alpha$ . Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then  $h_\alpha v = 0$  or  $h_\alpha v = v$ , so the representation is 1-dimensional or 2-dimensional.

( $\Leftarrow$ ) Suppose  $\omega$  is not minuscule. Then there exists  $\alpha \in Q_+$  with  $\langle \omega, \alpha^\vee \rangle = m > 1$ . Let  $v_\omega$  be a highest weight vector, then  $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$ , which leads to a higher-dimensional  $\mathfrak{sl}_2$ -representation. □

**Corollary 3.0.2.** *If  $\omega$  is minuscule, then for every dominant integral weight  $\lambda$  of  $\mathfrak{g}$ , we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma}.$$

(It is assumed that if  $\lambda + \gamma$  is not dominant, then  $L_{\lambda+\gamma} = 0$ .)

*Proof.* We know  $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$ . Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where  $\Delta$  is the Weyl denominator. If  $\lambda + \gamma \notin P_+$ , then for some  $\alpha_i^\vee$ , we get  $\langle \lambda + \gamma, \alpha_i^\vee \rangle < 0$ . But we know  $\langle \gamma, \alpha_i^\vee \rangle \geq -1$ , so  $\langle \lambda + \gamma, \alpha_i^\vee \rangle = -1$ . Thus  $\langle \lambda + \gamma + \rho, \alpha_i^\vee \rangle = 0$ , so for any  $w\gamma$ , the term  $ws_i\gamma$  comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result.  $\square$

**Example 3.1.4.** For  $\mathfrak{sl}_2$ , we have  $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$ , which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

**Example 3.1.5.** Let  $V = V_{\omega_1}$  be the defining representation for  $\mathrm{GL}_n$ . Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where  $\lambda$  is a partition and  $\lambda + \square$  denotes the set of partitions obtained by adding a single box to  $\lambda$ . For example, for  $\lambda = (3, 3, 2, 1)$  we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

Similarly, for  $\wedge^m V = L_{\omega_m}$  (where  $\omega_m = (1, \dots, 1, 0, \dots, 0)$  with  $m$  ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add  $m$  boxes to  $\lambda$  in  $\lambda + m\square$ . For example,

$$\wedge^2 V \otimes S^{(3,1)}V = S^{(4,2)}V \oplus S^{(4,1,1)}V \oplus S^{(3,2,1)}V \oplus S^{(3,1,1,1)}V.$$

# Lecture 4

## Jan. 26 — Other Classical Lie Algebras

### 4.1 Applications of Minuscule Weights, Continued

**Proposition 4.1.** *We have the following:*

1. Let  $\lambda$  be a partition of  $N$ . Then  $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$ .
2. Let  $\mu$  be a partition of  $N+1$ . Then  $\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda$ .

*Proof.* (1) Let  $V$  be a vector space of sufficiently large dimension. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda, V^{\otimes(N+1)}) \cong \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N} \otimes V) = V \otimes S^\lambda V.$$

Now by Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}\left(\bigoplus_{\mu \in \lambda + \square} \pi_\mu, V^{\otimes(N+1)}\right) = \bigoplus_{\mu \in \lambda + \square} S^\mu V.$$

Since  $V \otimes S^\lambda V = \bigoplus_{\mu \in \lambda + \square} S^\mu V$ , we conclude that  $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$ .

(2) This is left as an exercise. Use a different version of Frobenius reciprocity.  $\square$

**Definition 4.1.** Let  $\lambda$  be a partition, and  $\lambda^\dagger$  be the *conjugate partition* (the one corresponding to the transposed diagram). For example,  $(3, 3, 2, 1)^\dagger = (4, 3, 2)$ .

**Corollary 4.0.1.** *Let  $\mathbb{C}_-$  be the sign representation of  $S_N$ . Then  $\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}$ .*

*Proof.* This is left as an exercise. The proof is by induction on  $N = |\lambda|$ . Let  $C = \sum_{i < j} (i \ j)$ , and note that its eigenvalues are the same as the Casimir operator of  $\mathrm{SL}_N$ .  $\square$

**Proposition 4.2** (Skew Howe duality). *We have a decomposition  $\wedge^n(V \otimes W) = \bigoplus_\lambda S^\lambda V \otimes S^{\lambda^\dagger} W$  (as  $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$ -modules).*

**Proposition 4.3.** *Every coset in  $P/Q$  contains a unique minuscule weight. This gives a bijection between  $P/Q$  and minuscule weights, so the number of minuscule weights is equal to  $\det A$ , where  $A$  is the Cartan matrix.*

*Proof.* Let  $C = a + Q \in P/Q$  be a coset. Let  $\omega \in C \cap P_+$  be the element which minimizes  $\langle \omega, \rho^\vee \rangle$ . If  $\lambda$  is the dominant weight for  $L_\omega$ , then  $\lambda \in C \cap P_+$  implies that

$$(\lambda, \rho^\vee) \geq (\omega, \rho^\vee).$$

Thus  $(\omega - \lambda, \rho^\vee) \leq 0$ , so  $\omega - \lambda \in Q_+$ . Thus  $\lambda = \omega$ , so  $\omega$  is minuscule. Now suppose  $\omega_1, \omega_2 \in C$  are minuscule and  $\omega_1 \neq \omega_2$  with  $\omega_1 - \omega_2 \in Q$ . By Lemma 3.3, we must have  $\langle \omega_1 - \omega_2, \beta \rangle \geq 2$  for all coroots  $\beta$ . But then  $\langle \omega_1, \beta \rangle = 1$  (which implies  $\beta > 0$ ) and  $\langle \omega_2, \beta \rangle = -1$  (which implies  $\beta < 0$ ), a contradiction.  $\square$

**Remark.** Let  $A$  be the Cartan matrix. For every root, we can write

$$\alpha_i = \sum_{j=1}^r A_{i,j} \omega_j.$$

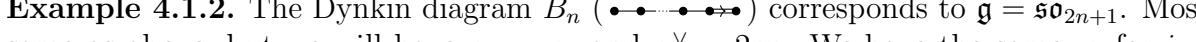
We have a covering map  $\mathbb{R}^r/\Lambda_2 \rightarrow \mathbb{R}^r/\Lambda_1$ , where  $\Lambda_2 = P$  and  $\Lambda_1 = Q$ . Then  $\det A$  is precisely the degree of this covering, which counts the number of cosets.

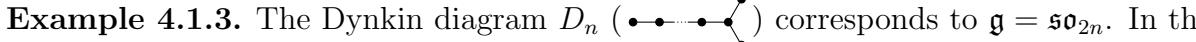
## 4.2 Other Classical Lie Algebras

**Example 4.1.1.** Recall that  $\mathfrak{g} = \mathfrak{sp}_{2n}$  corresponds to the Dynkin diagram  $C_n$  () , where the arrow points from longer roots to shorter roots. We have  $R_+ = e_i \pm e_j, 2e_j$ . The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

We have  $\alpha_i^\vee = \alpha_i$  for  $i \neq n$  and  $\alpha_n^\vee = e_n$ , and  $\omega_i = (1, \dots, 1, 0, \dots, 0)$  (with  $i$  ones) for  $1 \leq i \leq n$ .

**Example 4.1.2.** The Dynkin diagram  $B_n$  () corresponds to  $\mathfrak{g} = \mathfrak{so}_{2n+1}$ . Most things are the same as above, but we will have  $\alpha_n = e_n$  and  $\alpha_n^\vee = 2e_n$ . We have the same  $\omega_i$  for  $i < n$ , but we get  $\omega_n = (1/2, \dots, 1/2)$ . We have  $R_+ = e_i \pm e_j, e_i$ .

**Example 4.1.3.** The Dynkin diagram  $D_n$  () corresponds to  $\mathfrak{g} = \mathfrak{so}_{2n}$ . In this case we have  $R_+ = e_i \pm e_j$ , and simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-2} = e_{n-1}, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n.$$

We have  $\omega_i = (1, \dots, 1, 0, \dots, 0)$  (with  $i$  ones) for  $i = 1, \dots, n-2$ , but we get  $\omega_{n-1} = (1/2, \dots, 1/2, 1/2)$  and  $\omega_n = (1/2, \dots, 1/2, -1/2)$ .

**Remark.** We have the following:

- For  $G_2, F_4, F_8$ , we have  $\det A = 1$  (here  $A$  is the Cartan matrix), so the only minuscule weight is 0.
- For  $B_n$ , we have  $\det A = 2$  (the nontrivial minuscule weight is  $(1/2, \dots, 1/2)$ , and the representation has weights  $(\pm 1/2, \dots, \pm 1/2)$  with all possible combinations of  $\pm$  and dimension  $2^n$ ).
- For  $D_n$ , we have  $\det A = 4$ . The minuscule weights are  $\omega_1, \omega_{n-1}, \omega_n$ . Here  $\omega_1$  is the  $2n$ -dimensional defining representation. The other two are spin representations of dimension  $2^{n-1}$ , with weights  $(\pm 1/2, \dots, \pm 1/2)$ , taking even or odd numbers of  $-$  signs.

## 4.3 Representations of Symplectic Lie Algebras

**Remark.** For  $\mathfrak{g} = \mathfrak{sp}_{2n}$ , we have the Dynkin diagram  $C_n$  and

$$\omega_i = (\underbrace{1, \dots, 1}_i \text{ ones}, 0, \dots, 0).$$

The elements of the Cartan subalgebra are given by  $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$ . So  $L_{\omega_1} = V$  (the defining representation) with highest weight  $e_1$ . Note that  $\wedge^2 V$  is not irreducible:

$$\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C},$$

where  $\mathbb{C}$  is the trivial representation spanned by  $B^{-1} = \sum_i e_{i+n} \wedge e_i$  (note that  $B^{-1}$  is invariant under  $\mathfrak{sp}_{2n}$ ). However, one can check that  $\wedge_0^2 V$  is irreducible.

Now let us consider  $L_{\omega_j}$  for  $j \geq 2$ . Let  $B = \sum_i e_i^* \wedge e_{i+n}^*$ . We have an operator

$$i_B : \wedge^{i+1} V \longrightarrow \wedge^{i-1} V,$$

and we can denote  $\wedge_0^i V = \ker(i_B|_{\wedge^i V})$  (note that  $i_B|_{\wedge^i V}$  is injective when  $i \geq n$ ). The  $\wedge_0^i V$  are irreducible for  $i \leq n$ , and one can check that these form all of the irreducible representations of  $\mathfrak{sp}_{2n}$  (compute their dimensions and compare them to the highest weight representations).

We can also define an operator

$$\begin{aligned} m_B : \wedge^{i-1} V &\longrightarrow \wedge^{i+1} V \\ u &\mapsto B^{-1} \wedge u. \end{aligned}$$

One can check that  $m_B$  and  $i_B$  together with  $h$  (acting as  $i - n$  on  $\wedge^i V$ ) form an  $\mathfrak{sl}_2$ -triple. Then

$$\wedge V = \bigoplus_{i=0}^n L_{\omega_i} \otimes L_{n-j}$$

(where  $\omega_0 = 0$  and  $L_{n-j}$  is the representation of  $\mathfrak{sl}_2$  of weight  $n - j$ ) as representations of  $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$ .

## 4.4 Representations of Orthogonal Lie Algebras

**Remark.** First consider  $B_n$ , which corresponds to  $\mathfrak{g} = \mathfrak{so}_{2n+1}$ . Let  $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$ . In this case, the Cartan subalgebra is given by elements of the form  $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n, 0)$ . Let  $V$  be the  $(2n+1)$ -dimensional defining representation. Then for  $1 \leq i \leq n-1$ , the representation  $\wedge^i V$  is irreducible (one can check this using the dimension formula) with highest weight

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0).$$

On the other hand,  $\wedge^n V$  is irreducible but not fundamental, with highest weight  $(1, \dots, 1) = 2\omega_n$ .

Now we consider the spin representation  $S$  (whose elements are called *spinors*). It has weights

$$(\pm 1/2, \pm 1/2, \dots, \pm 1/2)$$

(all possible combinations of  $\pm$ ). The character of  $S$  is given by

$$\chi_S(x_1, \dots, x_n) = (x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}).$$

**Remark.** We will want to look at the Lie group  $\text{Spin}_{2n+1}(\mathbb{C})$ , the universal cover of  $\text{SO}_{2n+1}(\mathbb{C})$ . For  $n = 1$ , we have  $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ . We will see that  $S$  is 2-dimensional, and  $\pi_1(\text{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$ .

# Lecture 5

## Jan. 28 — Other Classical Lie Algebras, Part 2

### 5.1 More on Orthogonal Lie Algebras

**Proposition 5.1.** *For  $n \geq 3$ , we have  $\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$ .*

*Proof.* There is a deformation retract from the surface  $X_n$  defined by  $z_1^2 + \cdots + z_n^2 = 1$  in  $\mathbb{C}^n$  to the sphere  $X_n^\mathbb{R} = X_n \cap \mathbb{R}^n$  defined by  $x_1^2 + \cdots + x_n^2 = 1$  in  $\mathbb{R}^n$ : Let  $\vec{z} = \vec{x} + i\vec{y} \in X_n$  for  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , and note that  $|\vec{z}|^2 = 1$  if and only if  $|\vec{x}|^2 - |\vec{y}|^2 = 1$  and  $\vec{x} \cdot \vec{y} = 0$ . We also have

$$(\vec{x} + t i \vec{y})^2 = |\vec{x}|^2 - t^2 |\vec{y}|^2 = 1 + (1 - t^2) |\vec{y}|^2 \geq 1.$$

So we can define a homotopy  $f_t : X_n \rightarrow X_n$  by

$$f_t(\vec{z}) = \frac{\vec{x} + t i \vec{y}}{\sqrt{|\vec{x}|^2 - t^2 |\vec{y}|^2}},$$

which satisfies  $|f_t(z)|^2 = 1$ ,  $f_1(z) = z$ , and  $f_0(z) \in X_n^\mathbb{R}$ . Now observe that  $\mathrm{SO}_n$  acts on  $X_n$  with fibers isomorphic to  $\mathrm{SO}_{n-1}$ , so we have a long exact sequence

$$\pi_2(X_n) \longrightarrow \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \longrightarrow \pi_1(\mathrm{SO}_n(\mathbb{C})) \longrightarrow \pi_1(X_n).$$

The first and last groups are trivial for  $n \geq 4$ , so we have that  $\pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C}))$ . Thus the result follows once one checks that  $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$  (left as an exercise).  $\square$

**Remark.** Now consider  $D_n$ , which corresponds to  $\mathfrak{g} = \mathfrak{so}_{2n}$ . Let  $Q = \sum_{i=1}^n x_i x_{i+n}$ . The elements of the Cartan subalgebra are given by  $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$ . Let  $V$  be the  $2n$ -dimensional defining representation, and consider  $\wedge^i V$  for  $1 \leq i \leq n$ . We have  $\wedge^i V$  is irreducible for  $0 \leq i \leq n-1$ , and  $L_{\omega_i} = \wedge^i V$  for  $1 \leq i \leq n-2$ . Note that  $L_{(1, \dots, 1, 0)}$  is irreducible but not fundamental. Letting

$$\omega_{n-1} = (1/2, \dots, 1/2, 1/2) \quad \text{and} \quad (1/2, \dots, 1/2, -1/2),$$

the corresponding  $S_+ = L_{\omega_{n-1}}$  and  $S_- = L_{\omega_n}$  are the spin representations. The characters are

$$\chi_{S_\pm} = ((x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}))_\pm,$$

where the  $\pm$  denotes an even or odd number of  $-$  signs.

**Example 5.0.1.** We have  $\text{Spin}_4 = \text{SL}_2 \times \text{SL}_2$ , where factors correspond to  $S_+$  and  $S_-$ . We have  $\text{Spin}_5 = \text{Sp}_4$ , where  $S$  is the 4-dimensional defining representation, and  $\text{SO}_5 = \text{Sp}_4 / \{\pm 1\}$ . We have  $\text{Spin}_6 = \text{SL}_4$ , where  $S_+, S_-$  are the 4-dimensional defining representation and its dual, and  $\text{SO}_6 = \text{SL}_4 / \{\pm 1\}$ .

**Example 5.0.2.** Let  $V$  be a finite-dimensional vector space, and consider  $SV = \mathbb{C}[x_1, \dots, x_n]$ , where  $x_1, \dots, x_n$  is an orthonormal basis. Denote  $R^2 = \sum_{i=1}^n x_i^2 = S^2V$  and  $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$ . Then:

1. Find a first-order differential operator making  $\{R^2, \Delta, \cdot\}$  an  $\mathfrak{sl}_2$ -triple. Make sure that it commutes with the  $\text{SO}(V)$  action.
2. Let  $H_m \subseteq S^m V$  be the subspace of harmonic polynomials. Then

$$SV = \bigoplus_{m=0}^{\infty} H_m \otimes W_m,$$

where  $H_m = L_{m\omega_1}$  is the irreducible representation of  $\text{SO}(V)$ , and  $W_m$  is the Verma module for  $\mathfrak{sl}_2$  of highest weight  $m$ .

## 5.2 Clifford Algebras

**Definition 5.1.** Let  $V$  be a finite-dimensional vector space (over  $\mathbb{C}$ ) and  $(\cdot, \cdot)$  a non-degenerate inner product on  $V$ . Give an associative algebra structure to  $V$  by

$$v^2 = \frac{1}{2}(v, v).$$

Such an algebra is called a *Clifford algebra*, and is denoted by  $\text{Cl}(V)$ .

**Corollary 5.0.1.**  $ab + ba = (a + b)^2 - a^2 - b^2 = (a, b)$ .

**Example 5.1.1.** The operators  $i\partial/\partial x_i$  and  $dx_i \wedge \cdot$  define a Clifford algebra.

**Example 5.1.2.** Let  $e^i e^j + e^j e^i = \delta_{i,j}$ . Then  $D = \sum_{i=1}^n e^i \partial_i$  (the *Dirac operator*) satisfies  $D^2 = \Delta$ .

**Theorem 5.1.** The algebra  $\text{Cl}(V)$  is isomorphic to  $\text{Mat}_{2^n}(\mathbb{C})$  if  $\dim V = 2n$  and to  $\text{Mat}_{2^n}(\mathbb{C}) \oplus \text{Mat}_{2^n}(\mathbb{C})$  if  $\dim V = 2n + 1$ .

*Proof.* First consider the even case. Choose a basis  $a_1, \dots, a_n, b_1, \dots, b_n$  such that

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{i,j}, \quad a_i a_j + a_j a_i = 0, \quad b_i b_j + b_j b_i = 0, \quad b_i a_i + a_i b_i = 1.$$

Consider  $\text{Cl}(V)$ -module  $M = \wedge(a_1, \dots, a_n)$  (note that  $\dim M = 2^n$ ) with action defined by

$$\rho(a_i)w = a_i w \quad \text{and} \quad \rho(b_i)w = \frac{\partial w}{\partial a_i}.$$

We have the relations

$$1 = \left[ a_i, \frac{\partial}{\partial a_i} \right] = a_i \frac{\partial}{\partial a_i} + \frac{\partial}{\partial a_i} a_i \quad \text{and} \quad a_j \frac{\partial}{\partial a_i} = -\frac{\partial}{\partial a_i} a_j$$

for  $i \neq j$ . Let  $c_{I,J} = a_{i_1} \cdots a_{i_k} b_{j_1} \cdots b_{j_m}$  for  $I = \{i_1, \dots, i_k\}$  and  $J = \{j_1, \dots, j_m\}$ . Check as an exercise that the  $c_{I,J}$  are linearly independent, then  $\rho : \text{Cl}(V) \rightarrow \text{End}(M)$  is an isomorphism.

If  $\dim V = 2n + 1$ , then we can pick an extra element  $z$  satisfying

$$(z, a_i) = (z, b_i) = 0 \quad \text{and} \quad (z, z) = 2,$$

with relations  $za_i + a_iz = zb_i + b_iz = 0$  and  $z^2 = 1$ . Then  $zw = \pm(-1)^{\deg w}wz$  for  $w \in M_{\pm}$ .  $\square$

**Remark.** There is an embedding  $\mathfrak{so}(V) \rightarrow \text{Cl}(V)$ . Define a map

$$\begin{aligned} \xi : \wedge^2 V = \mathfrak{so}(V) &\longrightarrow \text{Cl}(V) \\ a \wedge b &\longmapsto \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a, b). \end{aligned}$$

One can check that  $[\xi(a \wedge b), \xi(c \wedge d)] = \xi([a \wedge b, c \wedge d])$ , so  $\xi$  is a homomorphism of Lie algebras. We have  $\xi^*M$  for even dimensional  $V$  and  $\xi^*M_{\pm}$  for odd dimensional  $V$ , and

$$\rho_{\xi^*M}(a) = \rho_M(\xi(a))$$

gives  $\xi^*M$  the structure of an  $\mathfrak{so}(V)$ -representation (and similarly for  $\xi^*M_{\pm}$ ). Notice that  $\chi^*M$  is reducible:

$$\xi^*M = (\xi^*M)_0 \oplus (\xi^*M)_1$$

as representations, where the first factor corresponds to even degree and the second to odd degree.

**Example 5.1.3.** We have the following:

1.  $(\xi^*M)_0 \cong S_+$  and  $(\xi^*M)_1 \cong S_-$  for even dimensional  $V$ .
2. If  $\dim V$  is odd, then  $\chi^*M_{\pm}$  are both isomorphic to  $S$ .

# Lecture 6

## Feb. 2 — Duals, Maximal Weights, Exponents

### 6.1 Dual Representations

**Remark.** Let  $L_\lambda$  be the irreducible representation of highest weight  $\lambda$ . What is the highest weight of the dual representation  $L_\lambda^*$ ? Let  $w_0$  be the maximal element in  $W$ .

**Proposition 6.1.** *We have  $L_\lambda^* = L_{-w_0(\lambda)}$ .*

*Proof.* Since  $\lambda$  is the highest weight in  $L_\lambda$ , for every weight  $\mu$  in  $L_\lambda$  we have  $\lambda - \mu \in Q_+$ . So

$$Q_- \ni w_0(\lambda - \mu) = w_0(\lambda) - w_0(\mu),$$

so  $w_0(\mu) - w_0(\lambda) \in Q_+$ . Thus  $w_0(\lambda) \leq w_0(\mu)$  for all  $\mu \in L_\lambda$ , so the length of  $w_0$  is  $|R_+|$ . Thus  $-w_0(\lambda)$  is the lowest weight of  $L_\lambda$ , which is the highest weight of  $L_\lambda^*$ .  $\square$

**Example 6.0.1.** Since the length of  $w_0$  is  $|R_+|$ ,  $w_0$  permutes the fundamental (co)weights and (co)roots, so  $w_0$  is an automorphism of Dynkin diagrams. Note that that  $W$  acts on  $P/Q$ , and  $w_0$  acts as inversion.

- The Dynkin diagrams  $A_1, B_n, C_n, G_2, F_4, E_7, E_8$  have no automorphisms, so  $L_\lambda^* = L_\lambda$  for these.
- For  $A_n$  with  $n \geq 2$ , we have  $P/Q = \mathbb{Z}/n\mathbb{Z}$  (e.g. if  $V$  is the defining representation, then we have that  $L_{\omega_1}^* = V^* = \wedge^{n-1}V = L_{\omega_{n-1}}$ ).
- For  $E_6$ , we have  $P/Q = \mathbb{Z}/3\mathbb{Z}$ , where  $w_0$  exchanges the two minuscule weights.
- For  $D_{2n+1}$ , we have  $P/Q = \mathbb{Z}/4\mathbb{Z}$  and  $S_+^* = S_-$ . For  $D_{2n}$ ,  $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , and  $S_\pm^* = S_\pm$ .

### 6.2 Maximal Weights

**Definition 6.1.** Let *maximal weight* of  $\mathfrak{g}$ , denoted  $\theta$ , is the highest weight of the adjoint representation.

**Example 6.1.1.** If  $\mathfrak{g} = \mathfrak{sl}_n$ , then  $\theta$  is the highest weight for  $V^* \otimes V$  where  $V$  is the defining representation. Note that  $V^* = \wedge^{n-1}V$ , so the highest weight of  $V^* \otimes V$  is  $\theta = \omega_1 + \omega_{n-1}$ . It is not fundamental.

**Example 6.1.2.** For  $\mathfrak{g} = \mathfrak{sp}_{2n}$ , we have  $\mathfrak{g} = S^2V$  where  $V$  is the defining representation for  $\mathfrak{sp}_{2n}$ . Then  $\theta = 2\omega_1$ , which is also not fundamental.

**Proposition 6.2.** *For a simple Lie algebra with  $\mathfrak{g} \neq \mathfrak{sl}_n, \mathfrak{sp}_{2n}$ , the maximal weight  $\theta$  is fundamental.*

**Example 6.1.3.** For  $\mathfrak{so}_N$  with  $N \geq 7$  (type  $B$  or  $D$ ), we have  $\mathfrak{g} = \wedge^2V = L_{\omega_2}$ .

## 6.3 Principal $\mathfrak{sl}_2$ -Subalgebra and Exponents

**Definition 6.2.** Let  $\mathfrak{g}$  be a simple Lie algebra and  $\{e_i, f_i, h_i\}$  (where  $h_i = \alpha_i^\vee$ ) be Chevalley generators. Let  $e = \sum_{i=1}^r e_i$ , and  $h$  such that  $\alpha_i(h) = 2$  for all  $i$  (so  $h = 2\rho^\vee$ ). Note that we have  $[h, e] = 2e$  and  $h = \sum_{i=1}^r (2\rho^\vee, \omega_i) \alpha_i^\vee$ . Let  $f = \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i$ . Then  $\{h, e, f\}$  spans the *principal  $\mathfrak{sl}_2$ -subalgebra* of  $\mathfrak{g}$ .

**Example 6.2.1.** Let  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ . Then the restriction of the defining representation to the principal  $\mathfrak{sl}_2$  is  $L_n$ , the irreducible representation of  $\mathfrak{sl}_2$  of highest weight  $n$ .

**Remark.** Let  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , so that  $\mathfrak{g} = \sum \mathfrak{g}[2m]$  where  $m$  is the height of the corresponding root subspace (and  $2m$  is the weight with respect to  $h$ ). Note  $\mathfrak{g}[0] = \mathfrak{h}$  and  $\dim \mathfrak{g}[0] = r$ . Let  $r_m = \dim \mathfrak{g}[2m]$ .

**Definition 6.3.** We say that  $m$  is an *exponent* of  $\mathfrak{g}$  if  $r_m > r_{m+1}$ . The *multiplicity* of an exponent  $m$  is  $r_m - r_{m+1}$ .

**Remark.** We have  $r_0 = r$  and there are  $r$  exponents (counted with multiplicities)  $m_1 \leq m_2 \leq \dots \leq m_r$ . The roots of height 2 are given by  $\alpha_i + \alpha_j$  (where  $i, j$  are connected in the Dynkin diagram). So  $r_0 = r_1 = 1$  and  $r_2 = r - 1$ . Thus  $m_1 = 1$  and  $m_2 > 1$ . We have

$$m_r = (\rho^\vee, \theta) = h_{\mathfrak{g}} - 1,$$

where  $\theta$  is the highest root. We call  $h_{\mathfrak{g}}$  the *Coxeter number* of  $\mathfrak{g}$ . Note that  $\sum_{i=1}^r m_i = |R_+|$ .

**Proposition 6.3.** The restriction of  $\mathfrak{g}$  to its principal  $\mathfrak{sl}_2$ -subalgebra decomposes as  $\bigoplus_{i=1}^r L_{2m_i+1}$ .

**Example 6.3.1.** The exponents for  $\mathfrak{sl}_n$  are  $1, 2, \dots, n-1$ .

**Definition 6.4.** The *Coxeter number* of  $\mathfrak{g}$  is  $h_{\mathfrak{g}} = \langle \theta, \rho^\vee \rangle + 1 = m_r + 1$ , and the *dual Coxeter number* is

$$h_{\mathfrak{g}}^\vee = \langle \tilde{\theta}^\vee, \rho \rangle + 1,$$

where  $\tilde{\theta}^\vee = 2\theta/(\theta, \theta)$ . If we normalize  $(\theta, \theta) = 2$ , then  $h_{\mathfrak{g}}^\vee = \frac{1}{2}(\theta, \theta + 2\rho)$ , which is the eigenvalue of  $\frac{1}{2}C$  (where  $C$  is the Casimir operator).

## 6.4 Complex, Real, and Quaternionic Types

**Definition 6.5.** Let  $G$  be a Lie group. An irreducible representation  $V$  of  $G$  or  $\mathfrak{g}$  is of *complex type* if  $V \not\cong V^*$ , *real type* if there exists a symmetric isomorphism  $V \rightarrow V^*$  (i.e. a symmetric inner product for  $V$ ), and *quaternionic (or symplectic) type* if the isomorphism is given through an anti-symmetric inner product.

**Exercise 6.1.** Let  $V$  be an irreducible representation of a finite group  $G$ . Show that  $\text{End}_{\mathbb{R}G}(V)$  (i.e.  $V \otimes V^*$ ) can only be one of three types:

- complex type if  $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{C}$ ,
- real type if  $\text{End}_{\mathbb{R}G}(V) \cong \text{Mat}_{2 \times 2}(\mathbb{R})$ ,
- quaternionic type if  $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{H}$ .

**Example 6.5.1.** Let  $L_n$  be an irreducible representation of  $\mathfrak{sl}_2$ . Then  $L_n$  is of real type for even  $n$  and of quaternionic type for odd  $n$ . Thus  $L_n = S^n V$  where  $V = L_1$  is 2-dimensional. The invariant form on  $S^n V$  is  $S^n B$ , where  $B$  is a skew-symmetric invariant form on  $V$ .

**Proposition 6.4.** Assume  $\lambda = -w_0(\lambda)$ , so that the corresponding representation is of real or quaternionic type. Then  $L_\lambda$  is of real type if  $(2\rho^\vee, \lambda)$  is even and of quaternionic type if it is odd.

*Proof.* The number  $n = (2\rho^\vee, \lambda)$  is the eigenvalue of  $h$  (from the principal  $\mathfrak{sl}_2$ -subalgebra) on the highest weight vector. Thus we have a decomposition

$$L_\lambda|_{\mathfrak{sl}_2} = L_n \oplus \bigoplus_{m < n} k_m L_m,$$

where  $L_n$  has multiplicity 1. One can determine the type based on  $L_n$ . □

## 6.5 Review of Compact Lie Groups

**Remark.** Let  $G$  be a real Lie group of dimension  $n$ . Then  $\xi \in \wedge^n \mathfrak{g}^*$  gives a generating  $n$ -form  $\omega$ , which is non-vanishing if  $\xi$  is non-vanishing. This gives rise to left- and right-invariant measures  $\mu_L$  and  $\mu_R$  on  $G$ , which are unique up to a constant. We say that  $G$  is *unimodular* if  $\mu_L = \mu_R$  (up to constants).

When does  $\mu_L = \mu_R$ ? For a 1-dimensional representation  $V$  of  $G$ , let  $|V|$  be the representation of  $G$  on the same space where  $\rho_{|V|}(g) = |\rho_V(g)|$  (where  $\rho_V : G \rightarrow \text{Aut}(V) = \mathbb{R}^\times$ ).

**Proposition 6.5.** We have  $\mu_L = \mu_R$  if and only if  $|\wedge^n \mathfrak{g}^*|$  is a trivial representation of  $G$ .

*Proof.* We have  $\mu_L = \mu_R$  if and only if the left-invariant form is right- or left-invariant up to a sign. This is equivalent to  $\xi \in \wedge^n \mathfrak{g}^*$  being invariant up to a sign under the action of  $\mathfrak{g}$ . □

**Proposition 6.6.** A compact group is unimodular.

*Proof.* For compact groups, the representation  $|\wedge^n \mathfrak{g}^*|$  gives a continuous homomorphism  $G \rightarrow \mathbb{R}^+$ , whose only compact subgroup is  $\{1\}$ . The result follows by Proposition 6.5. □

**Proposition 6.7.** Let  $V$  be an irreducible representation of  $G$ . Then  $V$  admits a  $G$ -invariant unitary structure.

*Proof.* Take any positive Hermitian form  $B$  on  $V$ , and define

$$B_{av}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w) dg.$$

This is well-defined and invariant by construction. □

**Corollary 6.0.1** (Weyl unitary trick). Any finite-dimensional representation is completely reducible.

*Proof.* Write  $V = W \oplus W^\perp$ . If  $W$  is invariant, then so is  $W^\perp$ . □

# Lecture 7

## Feb. 4 — Compact Groups

### 7.1 More on Exponents

**Theorem 7.1** (Chevalley's restriction theorem). *There is an isomorphism  $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W$ .*

**Theorem 7.2** (Harish-Chandra theorem). *There is an isomorphism  $\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\cong} \mathcal{Z}(U(\mathfrak{g}))$ .*

**Remark.** Pick an ordering  $s_{i_1}, \dots, s_{i_r}$  of the simple roots. Then  $c = s_{i_1} \cdots s_{i_r}$  is the *Coxeter element*, and  $c^h = 1$  where  $h$  is the Coxeter number. Then the eigenvalues of  $c$  are  $\zeta^{m_i+1}$  where  $\zeta = e^{2\pi i/h}$  and the  $m_i$  are the exponents. Also note that  $|W| = \prod_{i=1}^r (m_i + 1)$ .

If  $e, f, h$  is the principal  $\mathfrak{sl}_2$ -triple, then one can consider  $e + \mathfrak{g}^f$  where  $\mathfrak{g}^f = \ker \text{ad}_f$ .

### 7.2 Matrix Coefficients

**Remark.** For the rest of this lecture, let  $G$  be a real compact group and  $V$  a finite-dimensional continuous complex representation of  $G$ .

**Definition 7.1.** A *matrix coefficient* of  $\rho_V : G \rightarrow \text{GL}(V)$  is a function  $G \rightarrow \mathbb{C}$  of the form

$$g \longmapsto \langle f, \rho_V(g)v \rangle$$

for some  $v \in V$  and  $f \in V^*$ .

**Proposition 7.1.** *Matrix coefficients are smooth.*

*Proof.* Call  $v \in V$  a smooth vector if  $\langle f, \rho_V(g)v \rangle$  is smooth for all  $f \in V^*$ . It is obvious that such vectors form a subspace of  $V$ , call it  $V_{\text{sm}} \subseteq V$ . Fix  $v \in V$  and  $\phi : G \rightarrow \mathbb{C}$  smooth and with compact support. Let

$$w = w(\phi, v) = \int_G \phi(g) \rho_V(g)v \, dg.$$

We claim that  $w$  is smooth. We have

$$f(\rho(h)w) = f \left( \rho_V(h) \int_G \phi(g) \rho_V(g)v \, dg \right) = \int_G f(\phi(g) \rho_V(hg)v) \, dg = \int_G f(\phi(h^{-1}g) \rho_V(g)v) \, dg.$$

Differentiating under the integral sign and noting that  $\phi(h^{-1}g)$  is smooth in  $h$ , we see that the above expression is smooth in  $h$ . Now choose a delta-like sequence  $\phi_n$  with compact support around 1 so that

$$\int_G \phi_n(g) \, dg = 1.$$

Then  $w_n = w(\phi_n, v) \rightarrow v$  and each  $w_n$  is smooth, so  $v$  is smooth.  $\square$

**Remark.** Let  $V$  be an irreducible representation of  $G$ . Then:

1.  $V$  has an invariant positive-definite inner product which is unique up to scaling;
2. one can use an orthonormal basis  $v_1, \dots, v_n$  to define matrix coefficients:

$$\psi_{V,i,j}(g) = v_j^*(\rho_V(g)v_i) = (\rho_V(g)v_i, v_j)$$

(note that this definition is independent of normalization).

**Theorem 7.3** (Orthonormality of matrix coefficients). *Let  $V, W$  be irreducible representations of  $G$ .*

1. *If  $V, W$  are not isomorphic, then*

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{W,k,\ell}(g) dg = 0.$$

2. *For  $V = W$ , we have*

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{V,k,\ell}(g) dg = \frac{\delta_{i,k} \delta_{j,\ell}}{\dim V}.$$

*Proof.* Let  $\{v_i\}$  and  $\{w_k\}$  be orthonormal bases for  $V$  and  $W$ , respectively. We have

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{W,k,\ell}(g) dg = \int_G ((\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_i \otimes w_k), v_j \otimes w_\ell) dg$$

Define the operator

$$P = \int_G (\rho_V \otimes \rho_{\overline{W}})(g) dg = \int_G \rho_{V \otimes \overline{W}}(g) dg.$$

Since  $\overline{W} \cong W^*$ , we have  $P : V \otimes W^* \rightarrow V \otimes W^*$ . Thus

$$\text{Im } P \subseteq (V \otimes W^*)^G,$$

which is 0 if  $V \not\cong W$ . On the other hand, if  $V \cong W$ , then the only invariant is

$$\vec{u} = \sum_k (v_k \otimes \overline{v}_k),$$

so  $P$  is the orthogonal projection onto  $\vec{u}$ . Thus

$$P\vec{x} = \frac{(\vec{x}, \vec{u})}{(\vec{u}, \vec{u})} \vec{u},$$

so we have  $(P(v_i \otimes w_k), v_j \otimes w_\ell) = \delta_{i,j} \delta_{k,\ell} / (\dim V)$ .  $\square$

### 7.3 Peter-Weyl Theorem

**Theorem 7.4** (Peter-Weyl theorem). *The matrix coefficients  $\psi_{V,i,j}$  form an orthogonal basis in  $L^2(G)$ .*

**Remark.** Let  $V$  be a finite-dimensional irrep of  $G$ . There is a natural inclusion

$$\begin{aligned} i_V : V^* &\longrightarrow \text{Hom}_G(V, L^2(G)), \\ f &\longmapsto [v \mapsto (\rho_{V^*}(\cdot)f)(v)]. \end{aligned}$$

We claim that  $i_V$  is also surjective. To see this, let  $\phi \in \text{Hom}_G(V, L^2(G))$ , i.e. an  $L^2$  function left-invariant

under  $G$ . Thus we have that

$$\phi(x) = \rho_{V^*}(xg^{-1})\phi(g)$$

(after modifying  $\phi$  on a set of measure zero). Setting  $g = 1$ , we get  $\phi(x) = \rho_{V^*}(x)\phi(1)$ , so we have

$$\xi : \bigoplus_{V \in \text{Irr}(G)} V \otimes V^* \cong \bigoplus_{V \in \text{Irr}(G)} V \otimes \text{Hom}_G(V, L^2(G)) \longrightarrow L^2(G),$$

an embedding of  $(G \times G)$ -modules. Call the left-hand side  $L^2_{\text{alg}}(G)$ .

**Theorem 7.5** (Peter-Weyl theorem, alternative).  $L^2_{\text{alg}}(G)$  is dense in  $L^2(G)$ , i.e.

$$L^2(G) = \widehat{\bigoplus}_{V \in \text{Irr}(G)} V \otimes V^*.$$

**Example 7.1.1.** Let  $G = S^1 = U(1)$ . The irreducible representations of  $G$  are  $\psi_n(\theta) = e^{in\theta}$ . The  $e^{in\theta}$  form a basis of  $L^2(G) = L^2(S^1)$ , where the norm is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta.$$

This is the usual Fourier series on  $S^1$ . The Peter-Weyl theorem extends this to non-abelian groups.

**Exercise 7.1.** Let  $G$  be a compact group and  $H$  a closed subgroup of  $G$ .

1. Show that  $L^2(G/H) = \widehat{\bigoplus}_{V \in \text{Irr}(G)} N_H(V)V$ , where  $N_H(V) = \dim V^H$  (the space of  $H$ -invariants).
2. Let  $G = \text{SO}(3)$  and  $H = \text{SO}(2)$ . Then show that  $L^2(G/H) = L^2(S^2) = \widehat{\bigoplus}_{m \geq 0} N_H(m)L_{2m}$ , and that  $N_H(m) = 1$  for every  $m$ .

## 7.4 Introduction to Quantum Mechanics

**Remark.** Let  $\mathcal{H}$  be a Hilbert space and  $H$  a self-adjoint operator on  $\mathcal{H}$ . The spectrum of  $H$  gives the *energy levels* of the system. The elements  $\psi(x, y, z) \in L^2(\mathbb{R}^3)$  are called *wave functions*, and we assume that they are normalized so that  $\|\psi\|_{L^2} = 1$ . This is so that

$$|\psi(x, y, z)|^2 \Delta V$$

gives the probability of a quantum particle to be in the region  $\Delta V$ .

In general, there is also a time dependence in the wave function  $\psi$ , so we have  $\psi(x, y, z, t)$ . The time dependence is governed by the Schrödinger equation:

$$i\partial_t \psi = H\psi.$$

One can solve this equation via separation of variables, and we can write

$$\psi(x, y, z, t) = \sum_N e^{-iE_N t} \psi_N(x, y, z),$$

where the  $\psi_N$  are eigenvectors satisfying  $H\psi_N = E_N \psi_N$ .

**Example 7.1.2.** For the hydrogen atom, we have

$$H = -\frac{1}{2}\Delta - \frac{1}{r},$$

where  $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$  is the Laplacian and  $r = \sqrt{x^2 + y^2 + z^2}$ . The  $\Delta/2$  is called the *kinetic part* of  $H$ , and the  $1/r$  is called the *potential part* of  $H$ .

# Lecture 8

## Feb. 9 — Hydrogen Atom

### 8.1 Bound States of the Hydrogen Atom

**Remark.** We are looking for eigenvectors for  $H = -\frac{1}{2}\Delta - \frac{1}{r}$ , i.e.  $\psi_N \in L^2(\mathbb{R}^3)$  such that  $H\psi_N = E_N\psi_N$  with  $E_N < 0$ . We first write the Laplacian in spherical coordinates:

$$\begin{aligned}\Delta &= \Delta_r + \frac{1}{r}\Delta_{\text{sph}} \\ \Delta_r &= \partial_r^2 + \frac{2}{r}\partial_r \\ \Delta_{\text{sph}} &= \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta),\end{aligned}$$

where  $\phi$  is the angle in the  $xy$ -plane and  $\theta$  is the angle from the positive  $z$ -axis. Then we have

$$\partial_r^2\psi + \frac{2}{r}\partial_r\psi + \frac{2}{r}\psi + \frac{1}{r^2}\Delta_{\text{sph}}\psi = -2E\psi,$$

which is solved by  $\psi(r, \vec{u}) = f(r)\xi(\vec{u})$  for  $\vec{u} \in S^2$  satisfying

$$\begin{aligned}\Delta_{\text{sph}}\xi + \lambda\xi &= 0 \\ f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\lambda}{r^2} + 2E\right)f(r) &= 0.\end{aligned}\tag{*}$$

Note that (\*) implies  $\Delta_{\text{sph}}$  is rotationally invariant. By the Peter-Weyl theorem, we have

$$L^2(S^2) = \widehat{\bigoplus}_{\ell \geq 0} L_{2\ell},$$

where  $S^2 = \text{SO}(3)/\text{SO}(2)$  and  $L_{2\ell}$  are the irreps of  $\text{SO}(3)$ .

Let  $Y_\ell^0 \subseteq L_{2\ell}$  be a vector of weight 0, which is invariant under  $\text{SO}(2)$ . Thus it depends only on  $\theta$ . So we can write  $Y_\ell^0(\theta) = P_\ell(\cos\theta)$ , where  $P$  is a polynomial of degree  $\ell$ . By orthogonality,

$$\int_{-1}^1 P_k(z)P_\ell(z) dz = 0, \quad k \neq n.$$

Thus we can write

$$-\lambda_e P_\ell(z) = \Delta_{\text{sph}} P_\ell(z) = \partial_z(1-z^2)\partial_z P_\ell(z).$$

From looking at the leading term we must have  $\lambda_\ell = \ell(\ell+1)$ .

Now take  $Y_\ell^m \in L_{2\ell}$  for  $-\ell \leq m \leq \ell$ . Write  $Y_\ell^m(\phi, \theta) = e^{im\phi} P_\ell^m(\cos \theta)$ . So we have

$$\frac{-m^2}{1-z^2} P_\ell^m + \partial_z(1-z^2)\partial_z P_\ell^m + \ell(\ell+1)P_\ell^m = 0, \quad -\ell \leq m \leq \ell.$$

This equation has a unique solution (up to scaling) on  $[-1, 1]$ , given by

$$P_\ell^m = (1-z^2)^{m/2} \partial_z^{\ell+m} (1-z^2)^\ell.$$

Now we return to the radial equation:

$$f''(r) + \frac{2}{r} f'(r) + \left( \frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + 2E \right) f(r) = 0.$$

Write  $f(r) = r^\ell e^{-r/n} h(2r/n)$ , where  $n$  is to be chosen later and  $h$  satisfies

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + \left( n - \ell - 1 + \frac{1}{4}(1+2En^2)\rho \right) h(\rho) = 0.$$

Now choose  $n = 1/\sqrt{-2E}$ , so that  $E = -1/2n^2$ . Then the above equation becomes

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + (n - \ell - 1)h(\rho) = 0.$$

This equation is known as the *generalized Laguerre equation*. To get  $\|\psi\|_{L^2}^2 < \infty$ , we must have

$$\int_0^\infty \rho^{2\ell+2} e^{-\rho} |h(\rho)|^2 d\rho < \infty,$$

where the extra  $+2$  in  $\rho^{2\ell+2}$  comes from the Jacobian. Solutions around 0 behave like  $\rho^s(1+o(1))$ , so

$$s(s+2\ell+1) = 0.$$

Thus either  $s = 0$  or  $s = -2\ell - 1$ .

First consider when  $\ell = 0$ . Then  $s = -1$  and we have  $\rho^{-1}(1+o(1))$ , so  $\psi \sim 1/r$  as  $r \rightarrow 0$ . Then

$$H\psi = E\psi + C\delta_0,$$

where  $\delta_0$  is the delta function at 0, so we do not get an eigenvector in this case.

Thus  $s = -2\ell - 1$ . Expanding  $h(\rho)$  in a series and substituting, we get the recursive formula

$$h_n(\rho) = \sum_{k=0}^{\infty} \frac{(1+\ell-n) \cdots (k+\ell-n)}{(2\ell+2) \cdots (2\ell+1+k) k!} \rho^k.$$

This series converges, and we have

$$\lim_{\rho \rightarrow \infty} \frac{h_n(\rho)}{\rho} = 1$$

unless the series terminates. Thus  $n - \ell - 1 \in \mathbb{Z}_{\geq 0}$ , so we can write

$$h_n(\rho) = \sum_{k=0}^{n-\ell-1} \frac{(1+\ell-n) \cdots (k+\ell-n)}{(2\ell+2) \cdots (2\ell+1+k) k!} \rho^k = L_{n-\ell-1}^{2\ell+1}(\rho),$$

which is known as the *generalized Laguerre polynomial*:

$$L_N^\alpha(\rho) = \sum_{k=0}^N (-1)^N \frac{N \cdots (N-k+1)}{(\alpha+1) \cdots (\alpha+k)} \frac{\rho^k}{k!}.$$

**Theorem 8.1.** *The bound states (i.e. solutions to  $H\psi = E\psi$  in  $L^2(\mathbb{R}^3)$ ) of the hydrogen atom are*

$$\psi_{n,\ell,m}(r, \phi, \theta) = r^\ell e^{-r/n} L_{n-\ell-1}^{2\ell+1}(2r/n) Y_\ell^m(\theta, \phi),$$

where  $n \in \mathbb{Z}_{>0}$ ,  $\ell$  is an integer from  $0, \dots, n-1$ ,  $E_n = -1/2n^2$ , and  $m$  is an integer between  $-\ell, \dots, \ell$ .

**Remark.** In Theorem 8.1,  $n$  is known as the *principal quantum number*, and  $\ell$  is known as the *azimuthal quantum number*. Note that if  $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\Delta_{\text{sph}}$ , where  $iL_x, iL_y, iL_z$  are the generators of  $\mathfrak{so}(3)$  satisfying  $[L_{\{x}, L_y] = -iL_z]$ , then  $\vec{L}^2 = C = \ell(\ell+1)$  is the Casimir operator.

**Corollary 8.1.1.** *The space  $W_n$  of states with principal number  $n$  has dimension  $n^2$ .*

*Proof.* This follows from  $\sum_{\ell=0}^{n-1} (2\ell+1) = n^2$ . □

**Remark.** Note that  $\widehat{\bigoplus}_n W_n$  forms a proper, closed subspace  $L_0^2(\mathbb{R}^3)$  of  $L^2(\mathbb{R}^3)$ . We need to find all  $\varphi$  with  $(H\varphi, \varphi) \geq 0$  to reconstruct all of  $L^2(\mathbb{R}^3)$ . This corresponds to the continuous spectrum of  $H$ .

## 8.2 Spin

**Remark.** *Spin* is a kind of intrinsic angular momentum. Instead of just  $L^2(\mathbb{R}^3)$ , we should consider

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3) \otimes L_1$$

to be the space of states for the hydrogen atom. We have

$$V_n = (L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}) \otimes L_1 = 2L_1 \oplus 2L_3 \oplus \cdots \oplus 2L_{2n-3} \oplus 2L_{2n-1},$$

so  $\dim V_n = 2n^2$ . We have an additional *spin operator* given by

$$S_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

which acts on  $\mathbb{C}^2$  in the standard basis  $e_+, e_-$ . Then we have

$$\psi_{n,\ell,m,+} = \psi_{n,\ell,m} \otimes e_+ \quad \text{and} \quad \psi_{n,\ell,m,-} = \psi_{n,\ell,m} \otimes e_-.$$

The *total spin* is  $m+s$  (where  $s$  is the eigenvalue for  $S_z$ ), which is either  $m+1/2$ , or  $m-1/2$ .

## 8.3 Pauli Exclusion Principle

**Remark.** The space  $\wedge^k V_n$  corresponds to the space of states for  $k$  electrons at energy level  $n$ . Note that we must have  $k \leq 2n^2$  to have  $\wedge^k V_n \neq 0$ , which gives the *Pauli exclusion principle*.

In the periodic table, one has *orbitals*  $s, p, d, f$  corresponding to  $\ell = 0, 1, 2, 3$ , respectively, written with coefficient  $n$  and with exponent  $k$  corresponding to the number of electrons in the orbital. For example, the element Ruthenium has

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^7 5s^1.$$

The periodic table is organized as follows: from left to right ordered by how many *valent* electrons (i.e. the number of electrons in the outermost orbital), and from top to bottom ordered by how many energy levels. For Ruthenium, it is on column 8 and row 5. The number is 44, for 44 total electrons.

**Exercise 8.1.** Let  $\vec{r} = (x, y, z)$  and  $\vec{p} = (-\partial_x, -i\partial_y, -i\partial_z)$  be the *position* and *momentum* operators. Let  $\vec{L} = \vec{r} \times \vec{p}$  and  $H = \frac{1}{2}\vec{p}^2 + U(r)$ , where  $U$  is rotationally invariant. Show that:

1. The components  $i\vec{L}$  are generators of the rotations on  $\mathbb{R}^3$ , and  $[\vec{L}, \vec{p}^2] = 0$ .
2.  $\vec{A}_0 = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})$  satisfies  $[\vec{A}_0, \vec{p}^2] = 0$ .
3. Let  $A = \vec{A}_0 + \phi(r)\vec{r}$ . There exists  $\phi$  such that  $[\vec{A}, H] = 0$  if and only if  $U$  is a *Coulomb potential* (i.e.  $U(r) = \frac{C}{r} + D$ ), and in this case  $\phi$  is completely determined.
4. (Hidden symmetry of the hydrogen atom) Use the commutation relations between  $\vec{A}$  and  $\vec{L}$  to define an action on  $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3$ , so that  $\vec{L}$  is the diagonal copy in this decomposition.
5.  $W_n = L_{n-1} \boxtimes L_{n-1}$  as representations of  $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ .

# Lecture 9

## Feb. 11 — Real Forms

### 9.1 Automorphisms of Semisimple Lie Algebras

**Remark.** Recall that we can identify  $\text{Aut}(\mathfrak{g})$  with a Lie group with Lie algebra  $\mathfrak{g}$ . The connected component of the identity  $\text{Aut}^0(\mathfrak{g})$  (also known as the *adjoint group*  $G_{\text{ad}}$ ) acts transitively on the set of Cartan subalgebras. If  $\mathfrak{h} \subseteq \mathfrak{g}$  is a Cartan subalgebra, then there is a connected subgroup  $H \subseteq G_{\text{ad}}$  which acts as 1 on  $\mathfrak{h}$  and as  $e^{\alpha(x)}$  on  $g_\alpha$  for  $x \in \mathfrak{h}$ . Then we have

$$\mathfrak{h}/2\pi i P^\vee \cong H,$$

and  $H$  is called a *maximal torus*.

**Proposition 9.1.** *The normalizer  $N(H)$  of  $H$  in  $G_{\text{ad}}$  coincides with the stabilizer of  $\mathfrak{h}$  and contains  $H$  as a normal subgroup such that  $N(H)/H = W$  (the Weyl group).*

*Proof.* Note that  $\text{SL}_2(\mathbb{C})$  is simply connected, so for any simple root  $\alpha_i$  there is a homomorphism

$$\eta_i : \text{SL}_2(\mathbb{C}) \longrightarrow G_{\text{ad}} = \tilde{G}/\mathcal{Z}(\tilde{G}).$$

Define  $S_i = \eta_i \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ . Consider  $w = s_{i_1} \cdots s_{i_n}$  and  $\tilde{w} = S_{i_1} \cdots S_{i_n}$ . Note that  $\tilde{w}$  acts on  $\mathfrak{h}$  acts  $w$ , and if  $w = w_1 w_2$ , then  $\tilde{w} = \tilde{w}_1 \tilde{w}_2 h$  for some  $h \in H$ . To see the latter claim, note that  $h$  has the preserve the root decomposition, hence  $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$ . Thus  $h = \exp(\sum_j b_j w_j^\vee) \in H$  (where  $\langle w_j^\vee, \alpha_i \rangle = \delta_{j,i}$ ).

So  $\tilde{w}$  and  $H$  generate a subgroup  $N \subseteq N(H)$  such that  $N/H = W$ . It remains to show that  $N(H) = N$ . Let  $x \in N(H)$  and consider simple roots  $\alpha'_i = x(\alpha_i)$ . Then there exists  $w \in W$  such that  $w(\alpha'_i) = \alpha_{p(i)}$  for some permutation  $p$ . Then  $\tilde{w}x(\alpha_i) = \alpha_{p(i)}$ , so this is a Dynkin diagram automorphism. Now  $\tilde{w}x$  is an element of a group, so the fundamental weights are fixed. Thus  $p = \text{id}$ .  $\square$

**Remark.** Although  $N(H)/H = W$ , in general  $N(H)$  is not a semidirect product of  $H$  and  $W$ .

**Proposition 9.2.** *The map  $\xi : \text{Aut}(D) \times G_{\text{d}} \rightarrow \text{Aut}(\mathfrak{g})$  is an isomorphism.*

*Proof.* We have to show that  $\xi$  is surjective. Let  $a \in \text{Aut}(\mathfrak{g})$ . We can say that  $a$  preserves the Cartan subalgebra (if not, we can shift it by  $g \in G_{\text{ad}}$ ). Multiplying by  $\text{Aut}(D)N(H)$ , we can make it act trivially on  $\mathfrak{h}$  and  $\mathfrak{g}_{\alpha_i}$ . Then  $a = 1$ , so  $a \in \text{Im } \xi$ .  $\square$

### 9.2 Real Forms of Semisimple Lie Algebras

**Remark.** Recall the Serre presentation for  $\mathfrak{g}$ , i.e. generators  $\{h_i, f_i, e_i\}$  with certain relations. In this setting, everything was defined over  $\mathbb{Q}$ .

**Definition 9.1.** A semisimple Lie algebra is *split* if it admits a Chevalley-Serre basis over base field  $K$ .

**Remark.** Let  $L$  be a Galois extension of  $K$  ( $\text{char } K = 0$ ), and assume that  $\mathfrak{g}_L$  is a split semisimple Lie algebra. We want to find  $\mathfrak{g}$  over  $K$  such that  $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ . The problem is then to find a classification of all such  $\mathfrak{g}$ . Let  $\Gamma = \text{Gal}(L/K)$ . Define an action of  $\Gamma$  on  $\mathfrak{g}_L$  by

$$g(\lambda x) = g(\lambda)g(x), \quad x \in \mathfrak{g}_L, \lambda \in L, g \in \Gamma,$$

which is twisted linear. We can reconstruct  $\mathfrak{g}$  as the invariants  $\mathfrak{g}_L^\Gamma$ .

The simplest action of this kind is  $\rho_0(g)$ , which acts on scalars and preserves  $\{h_i, e_i, f_i\}$ . Any twisted linear action takes the form  $\rho(g) = \eta(g)\rho_0(g)$  for some  $\eta : \Gamma \rightarrow \text{Aut}(\mathfrak{g}_L)$ . As  $\rho$  is a homomorphism,

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

and upon rearranging, we have

$$\eta(gh) = \eta(g)g(\eta(h)),$$

where  $g(a) = \rho_0(g)a\rho_0(g)^{-1}$  for  $a \in \text{Aut}(\mathfrak{g}_L)$ . The above is called a *1-cocycle condition*.

Denote the Lie algebra associated to the cocycle  $\eta$  by  $\mathfrak{g}_\eta$ . When do we have  $\mathfrak{g}_{\eta_1} \cong \mathfrak{g}_{\eta_2}$ ? This is the case when  $\rho_1$  and  $\rho_2$  are isomorphic, i.e. there exists  $a \in \text{Aut}(\mathfrak{g}_L)$  such that  $\rho_1(g)a = a\rho_2(g)$ . Then

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

so  $\eta_1(g) = a\eta_2(g)g(a)^{-1}$ . Thus  $\eta_1$  and  $\eta_2$  are cohomologous cocycles.

**Proposition 9.3.** *The semisimple Lie algebras  $\mathfrak{g}$  over  $K$  which split over  $L$  (where  $L/K$  is Galois) are classified by  $H^1(\Gamma, \text{Aut}(\mathfrak{g}_L))$ , where  $\Gamma = \text{Gal}(L/K)$ .*

**Remark.** We will now specialize to  $K = \mathbb{R}$ ,  $L = \mathbb{C}$ , where  $\Gamma = \mathbb{Z}/2\mathbb{Z}$ , generated by complex conjugation. We have  $\text{Aut}(\mathfrak{g}_L) = \text{Aut}(D) \ltimes G_{\text{ad}}$ . Since  $\eta(1) = 1$ ,  $\eta$  is determined by  $\eta(-1)$ . The cocycle condition is

$$s\bar{s} = 1, \quad s = \eta(-1).$$

The corresponding Lie algebra (up to isomorphism) depends only on the cohomology class of  $s$ , where  $s \mapsto as\bar{a}^{-1}$  for  $a \in \text{Aut}(D)$ .

**Theorem 9.1.** *The real semisimple Lie algebras whose complexification is  $\mathfrak{g}$  (i.e. the real forms of  $\mathfrak{g}$ ) are classified by  $s \in \text{Aut}(D) \ltimes G_{\text{ad}}$  such that  $s\bar{s} = 1$ , modulo the equivalence  $s \mapsto as\bar{a}^{-1}$  for  $a \in \text{Aut}(D)$ .*

**Remark.** Note that complex conjugation acts trivially on  $\text{Aut}(D)$ .

**Remark.** Denote by  $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \bar{x} = s(x)\}$  the real form corresponding to  $s$ . Denote by  $\mathfrak{g}_{(1)}$  the split form consisting of real  $x \in \mathfrak{g}$  (so that  $x = \bar{x}$ ).

Alternatively, we can define an antilinear involution  $\sigma_s(x) = \overline{s(x)}$ . Then  $\mathfrak{g}_{(s)}$  is the fixed point set of  $\sigma_s$ .

**Remark.** Note that  $s$  defines  $s_0 \in \text{Aut}(D)$  with  $s_0^2 = 1$ .

**Corollary 9.1.1.** *The conjugacy class of  $s_0$  is invariant under equivalences.*

**Remark.** Since  $s_0$  permutes the connected components of the Dynkin diagram  $D$ , it preserves some and divides some into pairs. So every semisimple real Lie algebra is a direct sum of simple ones, and each simple one has either connected Dynkin diagram or consists of two identical components.

**Remark.** We now consider the case when  $D$  is connected and  $\mathfrak{g}$  is simple.

**Definition 9.2.** A real form  $\mathfrak{g}_{(s)}$  of a complex simple Lie algebra is said to be *inner to*  $\mathfrak{g}_{(s')}$  if  $s' = gs$  up to equivalence, where  $g \in G_{\text{ad}}$  (i.e.  $s, s'$  differ by an inner automorphism). The *inner class* of  $\mathfrak{g}_{(s)}$  is the collection of all real forms inner to  $\mathfrak{g}_{(s)}$ . An *inner form* is a form inner to a split form. We call  $\mathfrak{g}_{(s)}$  *quasi-split* if  $s = s_0 \in \text{Aut}(D)$ .

**Corollary 9.1.2.** *We have the following:*

1. *Any real form is inner to a unique quasi-split form.*
2. *A real form which is both inner and quasi-split is split.*

**Example 9.2.1.** Consider the *Cartan involution*  $\tau$  defined by

$$\tau(h_j) = -h_j, \quad \tau(e_j) = -f_j, \quad \tau(f_j) = -e_j.$$

Then  $\mathfrak{g}_{(\tau)} = \mathfrak{g}^c$  is called the *compact real form* of  $\mathfrak{g}$ .