MATH 8803: Representation Theory II

Frank Qiang
Instructor: Anton Zeitlin

Georgia Institute of Technology
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Contents

Lecture 1

Jan. 12 — Introduction and Review

1.1 Review and Overview

Remark. Recall that we are interested in representations of Lie groups G, which is closely related to representations of Lie algebras $\mathfrak{g}$.

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$, where $r = \dim \mathfrak{h}$ is called the *rank*. We have the Serre generators $\{h_i, e_i, f_i\}_{i=1}^r$ and relations

$$[h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = a_{i,j} f_j, \quad \mathrm{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \mathrm{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Here $\{\alpha_i\} \subseteq \mathfrak{h}^*$ and we identify $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{n}_+$ is generated by $\{e_i\}$ and $\mathfrak{n}_-$ is generated by $\{f_i\}$. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R = R_+ \sqcup R_-$. We have $R_+ \subseteq Q_+$ and $R_- \subseteq Q_-$, where $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$. If the $a_{i,j}$ are degenerate, then we can define $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}c$ is called the *central extension* and $d = t\frac{d}{dt}$. We can think of these as maps $S^1 \to \mathfrak{g}$.

We can also consider the universal enveloping algebra $U(\mathfrak{g})$, and the related object. $U_q(\mathfrak{g})$ We have an R-matrix $R_{V,W}$ for the representations $V \otimes W$ and $W \otimes V$, and we have the relation

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}$$

in $V_1 \otimes V_2 \otimes V_3$. A main goal later in the course will be to relate the representations of $U_q(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.

In this case, we have the diagram:

$$\begin{tikzcd}
 & & \mathfrak{g} \arrow[dl] \arrow[dr] & & \\
 & U_q(\mathfrak{g}) \arrow[dl] \arrow[dr] & & \widehat{g} \arrow[ll] \arrow[dl] \arrow[dr] & \\
E_{q,\tau}(\mathfrak{g}) & & U_q(\widehat{g}) \arrow[ll] & & \widehat{\widehat{\mathfrak{g}}} \arrow[ll]
\end{tikzcd}$$

The object $U_q(\widehat{g})$ is related to quantum integrable models of spin chain type (XXX and XXZ), and $E_{q,\tau}(\mathfrak{g})$ is the *elliptic quantum group* (XYZ).

1.2 Representations of Semisimple Lie Algebras

Remark. Recall the *Weyl group* $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$. The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where ω_i are the fundamental weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We can consider the *highest weight representation.* The *Verma module* is $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ on which $\mathfrak{h}$ acts by $\lambda(h)$. Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each $\lambda \in \mathfrak{h}^*$, M_λ has a unique irreducible quotient L_λ. The *dominant integral weights* λ satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \le i \le r,$$

where $\lambda = \sum_{i=1}^r n_i \omega_i$ with $n_i \in \mathbb{Z}_+$.

Theorem 1.1. *The finite-dimensional irreps of $\mathfrak{g}$ are classified up to isomorphism by $\lambda \in P_+$. Moreover, $P(V)$ is Weyl invariant, and for any $\mu \in P(V)$, $w \in W$,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

Example 1.0.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the dominant integral weights are $n \in \mathbb{Z}_{\ge 0}$, $L_n = V_n$, and the Weyl group W acts by reflection.

Remark (Weyl character formula)**.** Let $\chi_V(g) = \mathrm{tr}_V(g)$. We can represent $g \sim e^h$, where $h \in \mathfrak{h}$. Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$. The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^\ell (w) e^{w(\lambda+\rho)}}{\Delta},$$

where $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w) w\rho}$ is the *Weyl denominator*. Here $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$. The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator* $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$, which acts by the scalar $(\lambda, \lambda + 2\rho)$.

1.3 Representations of SL_n and GL_n

Proposition 1.1. *For general simple $\mathfrak{g}$, let $\lambda = \sum_{i=1}^r m_i \omega_i$ be a dominant integral weight. Let $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$ and $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$. Let V be the subrepresentation of T_λ generated by v. Then $V \cong L_\lambda$.*

Remark. For $\mathfrak{sl}_n$, we have $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$. The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 + \cdots + x_n = 0\}.$$

We have $\alpha_i^\vee = e_i - e_{i-1}$ and $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$, where $\omega_i = (1, \ldots, 1, 0, \ldots, 0)$ with i ones. We can associate λ with the partition

$$\lambda = (m_1 + \cdots + m_{n-1}, m_2 + \cdots + m_{n-1}, \ldots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, 0),$$

and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{n-1}$. Note that L_{ω_1} is the defining representation, where $v_{\omega_1} = (1, 0, \ldots, 0)^T = v_1$, where $\{v_1, \ldots, v_n\}$ is a basis of the defining representation. Then we have that $L_{\omega_m} = \wedge^m V$ with highest weight $v_1 \wedge \cdots \wedge v_m$. Here $e_i = E_{i,i+1}$. Then we see that $L_\lambda \subseteq \bigotimes_{i=1}^{n-1}(\wedge^i V)^{\otimes m_i}$.

Remark. To move to GL_n, we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))/\mu_n,$$

where μ_n are the roots of unity embedded by $z \mapsto (z^{-1}, zI)$. We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of $\mathbb{C}^\times$. Its Lie algebra is spanned by h such that $e^{2\pi i h} = 1$. Within a representation, h acts by an operator H such that $e^{2\pi i H} = 1$. Thus all irreducible representations of $\mathbb{C}^\times$ are of the form $\chi_N(z) = z^N$. So for $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$, we have $L_{\lambda,N} = \chi_N \otimes L_\lambda$.

Exercise 1.1. Show that if $L_{\lambda,N} = \chi_N \otimes L_\lambda$, then $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r.

Remark. Letting $m_n = r \geq 0$ in the above exercise, the representation $L_{\lambda, nm_n + \sum_{i=1}^{n-1}\lambda_i}$ for $\mathfrak{gl}_n$ corresponds to the partition $(m_1 + \cdots + m_n, \ldots m_{n-1} + m_n, m_n)$.

Remark. For SL_n, the representation $\wedge^n V$ is trivial, but it is the determinant for GL_n. For GL_n, we also have χ^k and $(\chi^*)^k = \chi^{-k}$, these are called the *polynomial representations*.

Remark. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \geq \cdots \geq \lambda_n$ be a partition with at most n parts. Then $|\lambda| = \sum_i \lambda_i$ is an eigenvalue of $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$. We can realize λ as a Young diagram. Note that L_λ occurs in $V^{\otimes N}$, where V is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda : |\lambda| = N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$. There is a natural action of S_N on $V^{\otimes N}$.

Theorem 1.2 (Schur-Weyl duality)**.** *Let A be the image of $U(\mathfrak{gl}_n)$ in* $\mathrm{End}(V^{\otimes N})$ *and B be the image of $\mathbb{C}S_N$ in* $\mathrm{End}(V^{\otimes N})$. *Then*

1. *the centralizer of A is B and vice versa;*
2. *if λ has at most n parts, then the representation π_λ of B (and hence of S_N) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if $\dim V \geq N$, then the π_λ exhaust all irreducible representations of S_N.*

Lecture 2

Jan. 14 — Applications of Schur-Weyl Duality

2.1 The Schur Functor

Remark. Let V be the defining representation for GL_n. Then

$$V^{\otimes N} = \bigoplus_{\lambda : |\lambda| = N} L_\lambda \otimes \pi_\lambda.$$

Recall that if $\lambda = (\lambda_1, \dots, \lambda_n)$, then we have

$$\lambda_1 = m_1 + \cdots + m_n, \quad \lambda_2 = m_2 + \cdots + m_n, \quad \dots, \quad \lambda_n = m_n.$$

Definition 2.1. Suppose we are given the partition λ of N. The *Schur functor* S^λ is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space V. Note that this language, we have $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$.

Example 2.1.1. Consider the following:

1. $S^{(n)}V = S^n V$, where (n) is the partition of n with a single part.
2. $S^{(1^n)}V = \wedge^n V$, where (1^n) is the partition of n with n parts equal to 1.
3. $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$, where $\mathbb{C}_2$ acts trivially on $\mathbb{C}_+$ and by the sign on $\mathbb{C}_-$.
4. $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$, where S_3 acts trivially on $\mathbb{C}_+$ and by sign on $\mathbb{C}_-$ as before, and $\mathbb{C}^2 = \{(x, y, z) : x + y + z = 0\}$.

   Note that $V \otimes V = S^2 V \oplus \wedge^2 V$, so $S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V$ and $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$.

Remark. Let $\dim V = N$ and λ have k parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

We have $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$ and $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \dots, 1, 0)$ (recall that ω_i is i ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \le i < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \le i < k < j \le N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1\le<k<j\le N} \frac{\lambda_i + j - i}{j-i} = \prod_{i=1}^{k} \frac{(N+1-i)\cdots(N+\lambda_i - i)}{(k+1-i)\cdots(k+\lambda_i-i)}.$$

Proposition 2.1. *We have* $\dim S^\lambda V = P_\lambda(N)$*, where* P_λ *is a polynomial of degree* $|\lambda|$ *with rational coefficients and integer roots. The roots of* P_λ *are all integers from the interval* $[1-\lambda_1, k-1]$ *(occurring with multiplicities).*

Example 2.1.2. Let $P_n(N)$ correspond to $S^n V$. Then $\lambda_1 = n$ and $k = 1$, and

$$P_n(N) = \dim S^n V = \binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N) = \dim \wedge^n V = \binom{N}{n}.$$

One can also consider $P_{(a,b)}(N)$ corresponding to partitions with two parts. The values $P_{(a,n)}(N)$ are called the Narayana numbers, which are of use in combinatorics.

2.2 Invariant Theory

Remark. Let V be a finite-dimensional vector space and $\{T_i\} \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ for $i = 1, \ldots, k$. One would like to characterize *invariants* of such collections, i.e. polynomial functions $F(T_1, \ldots, T_k)$ which are invariant under the action of $\mathrm{GL}(V)$.

One can think of such a tensor in $(V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ as a vertex with m_i incoming edges and n_i outgoing edges. Then constructing invariants $\{T_i\}$ reduces to studying graphs where T_i corresponds to a vertex v_i of the graph Γ. This allows us to assign to a given graph an invariant function F_Γ.

Theorem 2.1. *The functions* F_Γ *for various* Γ *span the space of invariant functions.*

Proof. We can view an invariant as an invariant element of the space $\bigotimes_{i=1}^k((V^*)^{\otimes m_i} \otimes V^{\otimes n_i})$, which we can view as $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M}, V^{\otimes N})$, where $M = \sum d_i m_i$ (the number of incoming edges) and $N = \sum d_i n_i$ (the number of outgoing edges). Note that this space is empty when $M \neq N$, and the statement follows by Schur-Weyl duality when $M = N$. □

Example 2.1.3. Let $m_i = n_i = 1$. Then $T_1, \ldots, T_k$ are matrices. Then the graph Γ must look like a cycle, hence the invariants are all of the form

$$F_{j_1,\ldots,j_r}(T_1, \ldots, T_k) = \mathrm{tr}(T_{j_1} \cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when V is large enough). In particular, if $P(T_1, \ldots, T_k) = 0$ in all dimensions, then $\mathrm{tr}(P(T_1, \ldots, T_k)T_{k+1}) = 0$, which cannot be true as the trace decomposes in terms of the $F_{j_1,\ldots,j_r}$. (However, note that $[X, Y] = 0$ for 1×1 matrices and $[Z, [X, Y]^2] = 0$ for 2×2 matrices.) This also implies the uniqueness of the μ_n in the BCH formula:

$$\log(\exp(x)\exp(y)) = \sum_{n\ge1} \frac{\mu_n(x,y)}{n!}.$$

2.3 Weyl Character Formula for GL_n

Remark (Weyl character formula for GL_n)**.** Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})}, \tag{$*$}$$

where the denominator is $\Delta = \prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2}) = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$. Letting $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$,

$$\Delta = e^\rho \prod_{\alpha\in R_+}(1-e^{-\alpha}) = x_1^{n-1}x_2^{n-2}\cdots x_n^0\prod_{i<j}(1-x_j/x_i),$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$ and $x_i = e^{e_i}$ (e.g. $x_1 = e^{(1,0,\ldots,0)}$). After multiplying we get that

$$\Delta = \prod_{i<j}(x_i - x_j).$$

On the other hand, using $\Delta = \prod_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$, we have

$$\Delta = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1\le i,j\le n}) = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{n-1}\cdots x_{s(n)}^0 = \prod_{i<j}(x_i-x_j).$$

Now applying this to the numerator of $(*)$, we have

$$\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)} = \sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}.$$

Thus in total, the character χ_λ is given by

$$\chi_\lambda = \frac{\sum_{s\in S_n}\mathrm{sign}(s)x_{s(1)}^{\lambda_1+n-1}\cdots x_{s(n)}^{\lambda_n+0}}{\prod_{i<j}(x_i-x_j)} = \frac{\det(\{x_i^{\lambda_j+n-i}\})}{\prod_{i<j}(x_i-x_j)}.$$

These functions are known as the *Schur polynomials* $s_\lambda(x_1,\ldots,x_n)$.

Example 2.1.4 (Character of $S^{(n)}V$)**.** Using the above formula, we get the identity

$$s_{(m)}(x_1,\ldots,x_n) = \sum_{1\le j_1\le\cdots\le j_m\le n}x_{j_1}\cdots x_{j_m} = h_m(x_1,\ldots,x_m),$$

the mth complete symmetric function.

Example 2.1.5 (Character of $\lambda^n V$)**.** Similarly, one gets the identity

$$s_{(1^m)}(x_1,\ldots,x_n) = \sum_{1\le j_1<\cdots<j_m\le n}x_{j_1}\cdots x_{j_m} = e_m(x_1,\ldots,x_m),$$

the mth elementary symmetric function.

Example 2.1.6 (Trace in $V^{\otimes N}$)**.** Consider $x \otimes \sigma$, where $x = \mathrm{diag}(x_1, \ldots, x_n)$ and σ has m_i cycles of length i. Then we have

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\mathrm{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_\lambda \chi_\lambda(\sigma) s_\lambda(x) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_\lambda \chi_\lambda(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i<j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Theorem 2.2 (Frobenius character formula)**.** *$\chi_\lambda(\sigma)$ is the coefficient of $x_1^{\lambda_1 + N - 1} \cdots x_N^{\lambda_N}$ in the polynomial*

$$\prod_{i<j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

2.4 Howe Duality

Remark. Fix V, W and consider $S^n(V \otimes W)$, which is a representation of $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$.

Theorem 2.3 (Howe duality)**.** *We have a decomposition*

$$S^n(V \otimes W) = \bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes S^\lambda W.$$

Proof. We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left(\Big(\bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes \pi_\lambda \Big) \otimes \Big(\bigoplus_{\mu : |\mu| = n} S^\mu W \otimes \pi_\mu \Big) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu : |\lambda| = |\mu| = n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}. \end{aligned}$$

Since $\pi_\lambda = \pi_\lambda^*$, by Schur's lemma we have $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathbb{C}^{\delta_{\lambda,\mu}}$. $\square$

Corollary 2.3.1 (Cauchy identity)**.** *Let $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_s)$. Then*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_i}.$$

Lecture 3

Jan. 21 — Minuscule Weights

3.1 Minuscule Weights

Remark. Let $\mathfrak{g}$ be a simple complex Lie algebra.

Definition 3.1. A dominant integral weight ω for $\mathfrak{g}$ is called *minuscule* if $\langle \omega, \beta \rangle \leq 1$ for every positive coroot β (equivalently, if $|\langle \omega, \alpha \rangle| \leq 1$ for any coroot β).

Example 3.1.1. Clearly $\omega = 0$ is minuscule.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$ with fundamental weights $\{\omega_i\}_{i=1}^{n-1}$,[1] where

$$\omega_i = (\underbrace{1, \ldots, 1}_{i \text{ ones}}, 0, \ldots, 0)$$

Let $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$. Note that $\langle \omega_i, e_j - e_k \rangle = 0$ when $j, k \leq i$ or $j, k > i$, and $\langle \omega_i, e_j - e_k \rangle = 1$ when $j \leq i < k$. So all of the ω_i are minuscule in this case.

Lemma 3.1. *Every nonzero minuscule weight is fundamental.*

Proof. Suppose ω is minuscule. Then there exists i with $\langle \omega, \alpha_i^\vee \rangle = 1$. Moreover, there can only be one such i, since if there were many, then $\langle \omega, \theta^\vee \rangle \geq 2$, where $\theta^\vee$ is the longest coroot (i.e. if $\theta = \sum_{m_i > 0} m_i \alpha_i$ is the longest root, then $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$). So ω is necessarily fundamental. $\square$

Example 3.1.3. For G_2, F_4, and F_8, none of the fundamental weights are minuscule.

Lemma 3.2. *A fundamental weight ω_i is minuscule if and only if $m_i = 1$ where $\theta^\vee = \sum_j m_j \alpha_j^\vee$.*

Proof. By the minuscule condition, we know $m_i \leq 1$. If $m_i = 1$, then for any positive coroot $\beta = \sum n_j \alpha_j^\vee$ we have $n_j \leq m_j$, so $n_i \leq 1$. Thus $\langle \omega_i, \beta \rangle = n_i \leq 1$, so ω_i is minuscule. $\square$

Lemma 3.3. *If $\omega \in Q$ with $|\langle \omega, \beta \rangle| \leq 1$ for all coroots β, then $\omega = 0$.*

Proof. Assume to the contrary that $\omega = \sum_i \alpha_i \neq 0$. We may assume that $\sum_i |m_i|$ is smallest possible. Then $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$, since the form is positive definite. Thus there exists j such that m_j and $\langle \omega, \alpha_j^\vee \rangle$ have the same sign. By replacing ω with $-\omega$ if necessary, we may assume both are positive. Then $\langle \omega, \alpha_j^\vee \rangle = 1$. Consider the reflection $s_j(\omega) = \omega - \alpha_j = \sum_i m_i' \alpha_i$. So $m_i' = m_j - 1$ and $m_i' = m_i$. But then $\sum_i |m_i'| = \sum_i |m_i| - 1 < \sum_i |m_i|$, contradicting the minimality of ω. $\square$

[1] Recall a *fundamental weight* is a weight ω_i such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all simple coroots $\alpha_j^\vee$.

Proposition 3.1. *The following conditions are equivalent:*

1. ω *is minuscule;*

2. *all weights of* L_ω *belong to the Weyl orbit* $W\omega$*;*

3. *if* λ *is a dominant integral weight such that* $\omega - \lambda \in Q_+$*, then* $\lambda = \omega$.

Proof. ($1 \Rightarrow 3$) If $\omega = 0$, then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$ where $\rho = \sum_{i=1}^r \omega_i$, so $\lambda = 0$. Now let $\omega = \omega_i$ be minuscule. Then $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for $k \neq i$, then the problem reduces to a lower rank Dynkin diagram. So we can assume $m_k > 0$ for every $k \neq i$. Let β be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \leq \langle \omega_i, \beta \rangle \leq 1.$$

If $\alpha_i^\vee$ does not occur in β, then the above is ≤ 0. In particular, we have $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \leq 0$ for $j \neq i$. If we also have $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \leq 0$, then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$. Otherwise, $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$. Then $m_j > 0$ for every j, so $\langle \omega_i - \lambda, \theta^\vee \rangle \geq 1$, since $\theta^\vee$ is a dominant coweight. Then $\langle \lambda, \theta^\vee \rangle \leq 0$, so we must have $\lambda = 0$ since $\theta^\vee$ contains all $\alpha_j^\vee$ with positive coefficients. But then $\omega_i \in Q$, which is impossible by Lemma 3.3.

($3 \Rightarrow 2$) If μ is any weight of L_ω, then there exists $w \in W$ such that $\lambda = w\mu$ is dominant (since every orbit of W intersects the dominant chamber at exactly 1 point). Then $\omega - \lambda \in Q_+$, so $\lambda = \omega$, hence $\mu = w^{-1}\omega \in W\omega$.

($2 \Rightarrow 1$) Suppose otherwise ω is not minuscule. Then $\langle \omega, \alpha^\vee \rangle > 1$ for some positive coroot $\alpha^\vee$. Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that $\omega - \alpha$ is a weight of L_ω (weight of $f_\alpha v_\omega$, where v_ω is a highest weight vector and $\{e_\alpha, f_\alpha, \alpha^\vee\}$ is an $\mathfrak{sl}_2$-triple). But $\omega - \alpha$ is not W-conjugate to ω, since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is W-invariant. Contradiction. □

Corollary 3.0.1. *If* ω *is minuscule, then* $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$.

3.2 Applications of Minuscule Weights

Proposition 3.2. $\omega \in P_+$ *is minuscule if and only if the restriction of* L_ω *to any root* $\mathfrak{sl}_2$*-subalgebra of* $\mathfrak{g}$ *is the direct sum of* 1*-dimensional and* 2*-dimensional representations.*

Proof. ($\Rightarrow$) Let ω be minuscule and $v \in L_\omega$ the highest weight vector (of weight $w\omega$) for $(\mathfrak{sl}_2)_\alpha$. Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then $h_\alpha v = 0$ or $h_\alpha v = v$, so the representation is 1-dimensional or 2-dimensional.

($\Leftarrow$) Suppose ω is not minuscule. Then there exists $\alpha \in Q_+$ with $\langle \omega, \alpha^\vee \rangle = m > 1$. Let v_ω be a highest weight vector, then $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$, which leads to a higher-dimensional $\mathfrak{sl}_2$-representation. □

Corollary 3.0.2. *If ω is minuscule, then for every dominant integral weight λ of $\mathfrak{g}$, we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma}.$$

(It is assumed that if $\lambda+\gamma$ is not dominant, then $L_{\lambda+\gamma} = 0$.)

Proof. We know $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$. Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where Δ is the Weyl denominator. If $\lambda+\gamma \notin P_+$, then for some $\alpha_i^\vee$, we get $\langle \lambda+\gamma, \alpha_i^\vee \rangle < 0$. But we know $\langle \gamma, \alpha_i^\vee \rangle \geq -1$, so $\langle \lambda+\gamma, \alpha_i^\vee \rangle = -1$. Thus $\langle \lambda+\gamma+\rho, \alpha_i^\vee \rangle = 0$, so for any $w\gamma$, the term $ws_i\gamma$ comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega \,:\, \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result. □

Example 3.1.4. For $\mathfrak{sl}_2$, we have $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$, which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

Example 3.1.5. Let $V = V_{\omega_1}$ be the defining representation for GL_n. Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where λ is a partition and $\lambda + \square$ denotes the set of partitions obtained by adding a single box to λ. For example, for $\lambda = (3,3,2,1)$ we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)} V = S^{(4,3,2,1)} V \oplus S^{(3,3,3,1)} V \oplus S^{(3,3,2,2)} V \oplus S^{(3,3,2,1,1)} V.$$

Similarly, for $\wedge^m V = L_{\omega_m}$ (where $\omega_m = (1, \ldots, 1, 0, \ldots, 0)$ with m ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add m boxes to λ in $\lambda + m\square$. For example,

$$\wedge^2 V \otimes S^{(3,1)} V = S^{(4,2)} V \oplus S^{(4,1,1)} V \oplus S^{(3,2,1)} V \oplus S^{(3,1,1,1)} V.$$

Lecture 4

Jan. 26 — Other Classical Lie Algebras

4.1 Applications of Minuscule Weights, Continued

Proposition 4.1. *We have the following:*

1. *Let λ be a partition of N. Then $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.*
2. *Let μ be a partition of $N+1$. Then $\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda$.*

Proof. (1) Let V be a vector space of sufficiently large dimension. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda, V^{\otimes(N+1)}) \cong \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N} \otimes V) = V \otimes S^\lambda V.$$

Now by Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}\Big(\bigoplus_{\mu \in \lambda + \square} \pi_\mu, V^{\otimes(N+1)}\Big) = \bigoplus_{\mu \in \lambda + \square} S^\mu V.$$

Since $V \otimes S^\lambda V = \bigoplus_{\mu \in \lambda + \square} S^\mu V$, we conclude that $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.

(2) This is left as an exercise. Use a different version of Frobenius reciprocity. $\square$

Definition 4.1. Let λ be a partition, and $\lambda^\dagger$ be the *conjugate partition* (the one corresponding to the transposed diagram). For example, $(3,3,2,1)^\dagger = (4,3,2)$.

Corollary 4.0.1. *Let $\mathbb{C}_-$ be the sign representation of S_N. Then $\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}$.*

Proof. This is left as an exercise. The proof is by induction on $N = |\lambda|$. Let $C = \sum_{i<j} (i\ j)$, and note that its eigenvalues are the same as the Casimir operator of SL_N. $\square$

Proposition 4.2 (Skew Howe duality)**.** *We have a decomposition $\wedge^n(V \otimes W) = \bigoplus_\lambda S^\lambda V \otimes S^{\lambda^\dagger} W$ (as $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$-modules).*

Proposition 4.3. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights, so the number of minuscule weights is equal to* $\det A$*, where A is the Cartan matrix.*

Proof. Let $C = a + Q \in P/Q$ be a coset. Let $\omega \in C \cap P_+$ be the element which minimizes $\langle \omega, \rho^\vee \rangle$. If λ is the dominant weight for L_ω, then $\lambda \in C \cap P_+$ implies that

$$(\lambda, \rho^\vee) \geq (\omega, \rho^\vee).$$

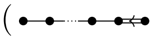
Thus $(\omega - \lambda, \rho^\vee) \le 0$, so $\omega - \lambda \in Q_+$. Thus $\lambda = \omega$, so ω is minuscule. Now suppose $\omega_1, \omega_2 \in C$ are minuscule and $\omega_1 \neq \omega_2$ with $\omega_1 - \omega_2 \in Q$. By Lemma 3.3, we must have $\langle \omega_1 - \omega_2, \beta\rangle \ge 2$ for all coroots β. But then $\langle \omega_1, \beta\rangle = 1$ (which implies $\beta > 0$) and $\langle \omega_2, \beta \rangle = -1$ (which implies $\beta < 0$), a contradiction. $\square$

Remark. Let A be the Cartan matrix. For every root, we can write

$$\alpha_i = \sum_{j=1}^{r} A_{i,j}\omega_j.$$

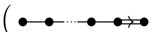
We have a covering map $\mathbb{R}^r/\Lambda_2 \to \mathbb{R}^r/\Lambda_1$, where $\Lambda_2 = P$ and $\Lambda_1 = Q$. Then $\det A$ is precisely the degree of this covering, which counts the number of cosets.

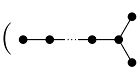
4.2 Other Classical Lie Algebras

Example 4.1.1. Recall that $\mathfrak{g} = \mathfrak{sp}_{2n}$ corresponds to the Dynkin diagram C_n (), where the arrow points from longer roots to shorter roots. We have $R_+ = e_i \pm e_j, 2e_j$. The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

We have $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = e_n$, and $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $1 \le i \le n$.

Example 4.1.2. The Dynkin diagram B_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Most things are the same as above, but we will have $\alpha_n = e_n$ and $\alpha_n^\vee = 2en$. We have the same ω_i for $i < n$, but we get $\omega_n = (1/2, \dots, 1/2)$. We have $R_+ = e_i \pm e_j, e_i$.

Example 4.1.3. The Dynkin diagram D_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. In this case we have $R_+ = e_i \pm e_j$, and simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-2} = e_{n-1}, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n.$$

We have $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $i = 1, \dots, n-2$, but we get $\omega_{n-1} = (1/2, \dots, 1/2, 1/2)$ and $\omega_n = (1/2, \dots, 1/2, -1/2)$.

Remark. We have the following:

- For G_2, F_4, F_8, we have $\det A = 1$ (here A is the Cartan matrix), so the only minuscule weight is 0.
- For B_n, we have $\det A = 2$ (the nontrivial minuscule weight is $(1/2, \dots, 1/2)$, and the representation has weights $(\pm 1/2, \dots, \pm 1/2)$ with all possible combinations of $\pm$ and dimension 2^n).
- For D_n, we have $\det A = 4$. The minuscule weights are $\omega_1, \omega_{n-1}, \omega_n$. Here ω_1 is the $2n$-dimensional defining representation. The other two are spin representations of dimension 2^{n-1}, with weights $(\pm 1/2, \dots, \pm 1/2)$, taking even or odd numbers of $-$ signs.

4.3 Representations of Symplectic Lie Algebras

Remark. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have the Dynkin diagram C_n and

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0).$$

The elements of the Cartan subalgebra are given by $\mathrm{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. So $L_{\omega_1} = V$ (the defining representation) with highest weight e_1. Note that $\wedge^2 V$ is not irreducible:

$$\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C},$$

where $\mathbb{C}$ is the trivial representation spanned by $B^{-1} = \sum_i e_{i+n} \wedge e_i$ (note that B^{-1} is invariant under $\mathfrak{sp}_{2n}$. However, one can check that $\wedge_0^2 V$ is irreducible.

Now let us consider L_{ω_j} for $j \geq 2$. Let $B = \sum_i e_i^* \wedge e_{i+n}^*$. We have an operator

$$i_B : \wedge^{i+1} V \longrightarrow \wedge^{i-1} V,$$

and we can denote $\wedge_0^i V = \ker(i_B|_{\wedge^i V})$ (note that $i_B|_{\wedge^i V}$ is injective when $i \geq n$). The $\wedge_0^i V$ are irreducible for $i \leq n$, and one can check that these form all of the irreducible representations of $\mathfrak{sp}_{2n}$ (compute their dimensions and compare them to the highest weight representations).

We can also define an operator

$$\begin{aligned} m_B : \wedge^{i-1} V &\longrightarrow \wedge^{i+1} V \\ u &\mapsto B^{-1} \wedge u. \end{aligned}$$

One can check that m_B and i_B together with h (acting as $i - n$ on $\wedge^i V$) form an $\mathfrak{sl}_2$-triple. Then

$$\wedge V = \bigoplus_{i=0}^{n} L_{\omega_i} \otimes L_{n-j}$$

(where $\omega_0 = 0$ and L_{n-j} is the representation of $\mathfrak{sl}_2$ of weight $n - j$) as representations of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$.

4.4 Representations of Orthogonal Lie Algebras

Remark. First consider B_n, which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$. In this case, the Cartan subalgebra is given by elements of the form $\mathrm{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n, 0)$. Let V be the $(2n+1)$-dimensional defining representation. Then for $1 \leq i \leq n-1$, the representation $\wedge^i V$ is irreducible (one can check this using the dimension formula) with highest weight

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0).$$

On the other hand, $\wedge^n V$ is irreducible but not fundamental, with highest weight $(1, \dots, 1) = 2\omega_n$.

Now we consider the spin representation S (whose elements are called *spinors*). It has weights

$$(\pm 1/2, \pm 1/2, \dots, \pm 1/2)$$

(all possible combinations of $\pm$). The character of S is given by

$$\chi_S(x_1, \dots, x_n) = (x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}).$$

Remark. We will want to look at the Lie group $\mathrm{Spin}_{2n+1}(\mathbb{C})$, the universal cover of $\mathrm{SO}_{2n+1}(\mathbb{C})$. For $n = 1$, we have $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$. We will see that S is 2-dimensional, and $\pi_1(\mathrm{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.

Lecture 5

Jan. 28 — Other Classical Lie Algebras, Part 2

5.1 More on Orthogonal Lie Algebras

Proposition 5.1. *For $n \geq 3$, we have $\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. There is a deformation retract from the surface X_n defined by $z_1^2 + \cdots + z_n^2 = 1$ in $\mathbb{C}^n$ to the sphere $X_n^{\mathbb{R}} = X_n \cap \mathbb{R}^n$ defined by $x_1^2 + \cdots + x_n^2 = 1$ in $\mathbb{R}^n$: Let $\vec{z} = \vec{x} + i\vec{y} \in X_n$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, and note that $|\vec{z}|^2 = 1$ if and only if $|\vec{x}|^2 - |\vec{y}|^2 = 1$ and $\vec{x} \cdot \vec{y} = 0$. We also have

$$(\vec{x} + ti\vec{y})^2 = |\vec{x}|^2 - t^2|\vec{y}|^2 = 1 + (1 - t^2)|\vec{y}|^2 \geq 1.$$

So we can define a homotopy $f_t : X_n \to X_n$ by

$$f_t(\vec{z}) = \frac{\vec{x} + ti\vec{y}}{\sqrt{|\vec{x}|^2 - t^2|\vec{y}|^2}},$$

which satisfies $|f_t(z)|^2 = 1$, $f_1(z) = z$, and $f_0(z) \in X_n^{\mathbb{R}}$. Now observe that SO_n acts on X_n with fibers isomorphic to SO_{n-1}, so we have a long exact sequence

$$\pi_2(X_n) \longrightarrow \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \longrightarrow \pi_1(\mathrm{SO}_n(\mathbb{C})) \longrightarrow \pi_1(X_n).$$

The first and last groups are trivial for $n \geq 4$, so we have that $\pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C}))$. Thus the result follows once one checks that $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$ (left as an exercise). □

Remark. Now consider D_n, which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. Let $Q = \sum_{i=1}^n x_i x_{i+n}$. The elements of the Cartan subalgebra are given by $\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)$. Let V be the $2n$-dimensional defining representation, and consider $\wedge^i V$ for $1 \leq i \leq n$. We have $\wedge^i V$ is irreducible for $0 \leq i \leq n-1$, and $L_{\omega_i} = \wedge^i V$ for $1 \leq n-2$. Note that $L_{(1,\ldots,1,0)}$ is irreducible but not fundamental. Letting

$$\omega_{n-1} = (1/2, \ldots, 1/2, 1/2) \quad \text{and} \quad (1/2, \ldots, 1/2, -1/2),$$

the corresponding $S_+ = L_{\omega_{n-1}}$ and $S_- = L_{\omega_n}$ are the spin representations. The characters are

$$\chi_{S_\pm} = ((x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}))_\pm,$$

where the $\pm$ denotes an even or odd number of $-$ signs.

Example 5.0.1. We have $\mathrm{Spin}_4 = \mathrm{SL}_2 \times \mathrm{SL}_2$, where factors correspond to S_+ and S_-. We have $\mathrm{Spin}_5 = \mathrm{Sp}_4$, where S is the 4-dimensional defining representation, and $\mathrm{SO}_5 = \mathrm{Sp}_4/\{\pm 1\}$. We have $\mathrm{Spin}_6 = \mathrm{SL}_4$, where S_+, S_- are the 4-dimensional defining representation and its dual, and $\mathrm{SO}_6 = \mathrm{SL}_4/\{\pm 1\}$.

Example 5.0.2. Let V be a finite-dimensional vector space, and consider $SV = \mathbb{C}[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is an orthonormal basis. Denote $R^2 = \sum_{i=1}^n x_i^2 = S^2V$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. Then:

1. Find a first-order differential operator making $\{R^2, \Delta, \cdot\}$ an $\mathfrak{sl}_2$-triple. Make sure that it commutes with the $\mathrm{SO}(V)$ action.
2. Let $H_m \subseteq S^mV$ be the subspace of harmonic polynomials. Then

$$SV = \bigoplus_{m=0}^{\infty} H_m \otimes W_m,$$

where $H_m = L_{m\omega_1}$ is the irreducible representation of $\mathrm{SO}(V)$, and W_m is the Verma module for $\mathfrak{sl}_2$ of highest weight m.

5.2 Clifford Algebras

Definition 5.1. Let V be a finite-dimensional vector space (over $\mathbb{C}$) and $(\cdot,\cdot)$ a non-degenerate inner product on V. Give an associative algebra structure to V by

$$v^2 = \frac{1}{2}(v, v).$$

Such an algebra is called a *Clifford algebra*, and is denoted by $\mathrm{Cl}(V)$.

Corollary 5.0.1. $ab + ba = (a+b)^2 - a^2 - b^2 = (a, b)$.

Example 5.1.1. The operators ${}^i\partial/\partial x_i$ and $dx_i \wedge \cdot$ define a Clifford algebra.

Example 5.1.2. Let $e^ie^j + e^je^i = \delta_{i,j}$. Then $D = \sum_{i=1}^n e^i\partial_i$ (the *Dirac operator*) satisfies $D^2 = \Delta$.

Theorem 5.1. *The algebra* $\mathrm{Cl}(V)$ *is isomorphic to* $\mathrm{Mat}_{2^n}(\mathbb{C})$ *if* $\dim V = 2n$ *and to* $\mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C})$ *if* $\dim V = 2n+1$.

Proof. First consider the even case. Choose a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{i,j}, \quad a_ia_j + a_ja_i = 0, \quad b_ib_j + b_jb_i = 0, \quad b_ia_i + a_ib_i = 1.$$

Consider $\mathrm{Cl}(V)$-module $M = \wedge(a_1, \ldots, a_n)$ (note that $\dim M = 2^n$) with action defined by

$$\rho(a_i)w = a_iw \quad \text{and} \quad \rho(b_i)w = \frac{\partial w}{\partial a_i}.$$

We have the relations

$$1 = \left[a_i, \frac{\partial}{\partial a_i}\right] = a_i\frac{\partial}{\partial a_i} + \frac{\partial}{\partial a_i}a_i \quad \text{and} \quad a_j\frac{\partial}{\partial a_i} = -\frac{\partial}{\partial a_i}a_j$$

for $i \neq j$. Let $c_{I,J} = a_{i_1} \cdots a_{i_k} b_{j_1} \cdots b_{j_m}$ for $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_m\}$. Check as an exercise that the $c_{I,J}$ are linearly independent, then $\rho : \mathrm{Cl}(V) \to \mathrm{End}(M)$ is an isomorphism.

If $\dim V = 2n+1$, then we can pick an extra element z satisfying

$$(z,a_i) = (z,b_i) = 0 \quad \text{and} \quad (z,z) = 2,$$

with relations $za_i + a_i z = zb_i + b_i z = 0$ and $z^2 = 1$. Then $zw = \pm(-1)^{\deg w} wz$ for $w \in M_\pm$. $\square$

Remark. There is an embedding $\mathfrak{so}(V) \to \mathrm{Cl}(V)$. Define a map

$$\begin{aligned} \xi : \wedge^2 V = \mathfrak{so}(V) &\longrightarrow \mathrm{Cl}(V) \\ a \wedge b &\longmapsto \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a,b). \end{aligned}$$

One can check that $[\xi(a \wedge b), \xi(c \wedge d)] = \xi([a \wedge b, c \wedge d])$, so ξ is a homomorphism of Lie algebras. We have $\xi^* M$ for even dimensional V and $\xi^* M_\pm$ for odd dimensional V, and

$$\rho_{\xi^* M}(a) = \rho_M(\xi(a))$$

gives $\xi^* M$ the structure of an $\mathfrak{so}(V)$-representation (and similarly for $\xi^* M_\pm$. Notice that $\chi^* M$ is reducible:

$$\xi^* M = (\xi^* M)_0 \oplus (\xi^* M)_1$$

as representations, where the first factor corresponds to even degree and the second to odd degree.

Example 5.1.3. We have the following:

1. $(\xi^* M)_0 \cong S_+$ and $(\xi^* M)_1 \cong S_-$ for even dimensional V.
2. If $\dim V$ is odd, then $\chi^* M_\pm$ are both isomorphic to S.

Lecture 6

Feb. 2 — Duals, Maximal Weights, Exponents

6.1 Dual Representations

Remark. Let L_λ be the irreducible representation of highest weight λ. What is the highest weight of the dual representation L_λ^*? Let w_0 be the maximal element in W.

Proposition 6.1. *We have* $L_\lambda^* = L_{-w_0(\lambda)}$.

Proof. Since λ is the highest weight in L_λ, for every weight μ in L_λ we have $\lambda - \mu \in Q_+$. So

$$Q_- \ni w_0(\lambda - \mu) = w_0(\lambda) - w_0(\mu),$$

so $w_0(\mu) - w_0(\lambda) \in Q_+$. Thus $w_0(\lambda) \leq w_0(\mu)$ for all $\mu \in L_\lambda$, so the length of w_0 is $|R_+|$. Thus $-w_0(\lambda)$ is the lowest weight of L_λ, which is the highest weight of L_λ^*. □

Example 6.0.1. Since the length of w_0 is $|R_+|$, w_0 permutes the fundamental (co)weights and (co)roots, so w_0 is an automorphism of Dynkin diagrams. Note that that W acts on P/Q, and w_0 acts as inversion.

- The Dynkin diagrams $A_1, B_n, C_n, G_2, F_4, E_7, E_8$ have no automorphisms, so $L_\lambda^* = L_\lambda$ for these.
- For A_n with $n \geq 2$, we have $P/Q = \mathbb{Z}/n\mathbb{Z}$ (e.g. if V is the defining representation, then we have that $L_{\omega_1}^* = V^* = \wedge^{n-1} V = L_{\omega_{n-1}}$).
- For E_6, we have $P/Q = \mathbb{Z}/3\mathbb{Z}$, where w_0 exchanges the two minuscule weights.
- For D_{2n+1}, we have $P/Q = \mathbb{Z}/4\mathbb{Z}$ and $S_+^* = S_-$. For D_{2n}, $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $S_\pm^* = S_\pm$.

6.2 Maximal Weights

Definition 6.1. Let *maximal weight* of $\mathfrak{g}$, denoted θ, is the highest weight of the adjoint representation.

Example 6.1.1. If $\mathfrak{g} = \mathfrak{sl}_n$, then θ is the highest weight for $V^* \otimes V$ where V is the defining representation. Note that $V^* = \wedge^{n-1} V$, so the highest weight of $V^* \otimes V$ is $\theta = \omega_1 + \omega_{n-1}$. It is not fundamental.

Example 6.1.2. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have$\mathfrak{g} = S^2 V$ where V is the defining representation for $\mathfrak{sp}_{2n}$. Then $\theta = 2\omega_1$, which is also not fundamental.

Proposition 6.2. *For a simple Lie algebra with* $\mathfrak{g} \neq \mathfrak{sl}_n, \mathfrak{sp}_{2n}$*, the maximal weight* θ *is fundamental.*

Example 6.1.3. For $\mathfrak{so}_N$ with $N \geq 7$ (type B or D), we have $\mathfrak{g} = \wedge^2 V = L_{\omega_2}$.

6.3 Principal $\mathfrak{sl}_2$-Subalgebra and Exponents

Definition 6.2. Let $\mathfrak{g}$ be a simple Lie algebra and $\{e_i, f_i, h_i\}$ (where $h_i = \alpha_i^\vee$) be Chevalley generators. Let $e = \sum_{i=1}^r e_i$, and h such that $\alpha_i(h) = 2$ for all i (so $h = 2\rho^\vee$). Note that we have $[h, e] = 2e$ and $h = \sum_{i=1}^r (2\rho^\vee, \omega_i)\alpha_i^\vee$. Let $f = \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i$. Then $\{h, e, f\}$ spans the *principal $\mathfrak{sl}_2$-subalgebra* of $\mathfrak{g}$.

Example 6.2.1. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then the restriction of the defining representation to the principal $\mathfrak{sl}_2$ is L_n, the irreducible representation of $\mathfrak{sl}_2$ of highest weight n.

Remark. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, so that $\mathfrak{g} = \sum \mathfrak{g}[2m]$ where m is the height of the corresponding root subspace (and $2m$ is the weight with respect to h). Note $\mathfrak{g}[0] = \mathfrak{h}$ and $\dim \mathfrak{g}[0] = r$. Let $r_m = \dim \mathfrak{g}[2m]$.

Definition 6.3. We say that m is an *exponent* of $\mathfrak{g}$ if $r_m > r_{m+1}$. The *multiplicity* of an exponent m is $r_m - r_{m+1}$.

Remark. We have $r_0 = r$ and there are r exponents (counted with multiplicities) $m_1 \le m_2 \le \cdots \le m_r$. The roots of height 2 are given by $\alpha_i + \alpha_j$ (where i, j are connected in the in the Dynkin diagram). So $r_0 = r_1 = 1$ and $r_2 = r - 1$. Thus $m_1 = 1$ and $m_2 > 1$. We have

$$m_r = (\rho^\vee, \theta) = h_{\mathfrak{g}} - 1,$$

where θ is the highest root. We call $h_{\mathfrak{g}}$ the *Coxeter number* of $\mathfrak{g}$. Note that $\sum_{i=1}^r m_i = |R_+|$.

Proposition 6.3. *The restriction of $\mathfrak{g}$ to its principal $\mathfrak{sl}_2$-subalgebra decomposes as $\bigoplus_{i=1}^r L_{2m_i+1}$.*

Example 6.3.1. The exponents for $\mathfrak{sl}_n$ are $1, 2, \ldots, n-1$.

Definition 6.4. The *Coxeter number* of $\mathfrak{g}$ is $h_{\mathfrak{g}} = \langle \theta, \rho^\vee \rangle + 1 = m_r + 1$, and the *dual Coxeter number* is

$$h_{\mathfrak{g}}^\vee = \langle \widetilde{\theta}^\vee, \rho \rangle + 1,$$

where $\widetilde{\theta}^\vee = 2\theta/(\theta, \theta)$. If we normalize $(\theta, \theta) = 2$, then $h_{\mathfrak{g}}^\vee = \frac{1}{2}(\theta, \theta + 2\rho)$, which is the eigenvalue of $\frac{1}{2}C$ (where C is the Casimir operator).

6.4 Complex, Real, and Quaternionic Types

Definition 6.5. Let G be a Lie group. An irreducible representation V of G or $\mathfrak{g}$ is of *complex type* if $V \not\cong V^*$, *real type* if there exists a symmetric isomorphism $V \to V^*$ (i.e. a symmetric inner product for V), and *quaternionic (or symplectic) type* if the isomorphism is given through an anti-symmetric inner product.

Exercise 6.1. Let V be an irreducible representation of a finite group G. Show that $\mathrm{End}_{\mathbb{R}G}(V)$ (i.e. $V \otimes V^*$) can only be one of three types:

- complex type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathbb{C}$,
- real type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$,
- quaternionic type if $\mathrm{End}_{\mathbb{R}G}(V) \cong \mathbb{H}$.

Example 6.5.1. Let L_n be an irreducible representation of $\mathfrak{sl}_2$. Then L_n is of real type for even n and of quaternionic type for odd n. Thus $L_n = S^n V$ where $V = L_1$ is 2-dimensional. The invariant form on $S^n V$ is $S^n B$, where B is a skew-symmetric invariant form on V.

Proposition 6.4. *Assume $\lambda = -w_0(\lambda)$, so that the corresponding representation is of real or quaternionic type. Then L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.*

Proof. The number $n = (2\rho^\vee, \lambda)$ is the eigenvalue of h (from the principal $\mathfrak{sl}_2$-subalgebra) on the highest weight vector. Thus we have a decomposition

$$L_\lambda|_{\mathfrak{sl}_2} = L_n \oplus \bigoplus_{m<n} k_m L_m,$$

where L_n has multiplicity 1. One can determine the type based on L_n. □

6.5 Review of Compact Lie Groups

Remark. Let G be a real Lie group of dimension n. Then $\xi \in \wedge^n \mathfrak{g}^*$ gives a generating n-form ω, which is non-vanishing if ξ is non-vanishing. This gives rise to left- and right-invariant measures μ_L and μ_R on G, which are unique up to a constant. We say that G is *unimodular* if $\mu_L = \mu_R$ (up to constants).

When does $\mu_L = \mu_R$? For a 1-dimensional representation V of G, let $|V|$ be the representation of G on the same space where $\rho_{|V|}(g) = |\rho_V(g)|$ (where $\rho_V : G \to \mathrm{Aut}(V) = \mathbb{R}^\times$).

Proposition 6.5. *We have $\mu_L = \mu_R$ if and only if $|\wedge^n \mathfrak{g}^*|$ is a trivial representation of G.*

Proof. We have $\mu_L = \mu_R$ if and only if the left-invariant form is right- or left-invariant up to a sign. This is equivalent to $\xi \in \wedge^n \mathfrak{g}^*$ being invariant up to a sign under the action of $\mathfrak{g}$. □

Proposition 6.6. *A compact group is unimodular.*

Proof. For compact groups, the representation $|\wedge^n \mathfrak{g}^*|$ gives a continuous homomorphism $G \to \mathbb{R}^+$, whose only compact subgroup is $\{1\}$. The result follows by Proposition 6.5. □

Proposition 6.7. *Let V be an irreducible representation of G. Then V admits a G-invariant unitary structure.*

Proof. Take any positive Hermitian form B on V, and define

$$B_{\mathrm{av}}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w)\, dg.$$

This is well-defined and invariant by construction. □

Corollary 6.0.1 (Weyl unitary trick)**.** *Any finite-dimensional representation is completely reducible.*

Proof. Write $V = W \oplus W^\perp$. If W is invariant, then so is $W^\perp$. □

Lecture 7

Feb. 4 — Compact Groups

7.1 More on Exponents

Theorem 7.1 (Chevalley's restriction theorem). *There is an isomorphism* $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W$.

Theorem 7.2 (Harish-Chandra theorem). *There is an isomorphism* $\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\cong} \mathcal{Z}(U(\mathfrak{g}))$.

Remark. Pick an ordering $s_{i_1}, \ldots, s_{i_r}$ of the simple roots. Then $c = s_{i_1} \cdots s_{i_r}$ is the *Coxeter element*, and $c^h = 1$ where h is the Coxeter number. Then the eigenvalues of c are ζ^{m_i+1} where $\zeta = e^{2\pi i/h}$ and the m_i are the exponents. Also note that $|W| = \prod_{i=1}^r (m_i + 1)$.

If e, f, h is the principal $\mathfrak{sl}_2$-triple, then one can consider $e + \mathfrak{g}^f$ where $\mathfrak{g}^f = \ker \operatorname{ad}_f$.

7.2 Matrix Coefficients

Remark. For the rest of this lecture, let G be a real compact group and V a finite-dimensional continuous complex representation of G.

Definition 7.1. A *matrix coefficient* of $\rho_V : G \to \mathrm{GL}(V)$ is a function $G \to \mathbb{C}$ of the form

$$g \longmapsto \langle f, \rho_V(g)v \rangle$$

for some $v \in V$ and $f \in V^*$.

Proposition 7.1. *Matrix coefficients are smooth.*

Proof. Call $v \in V$ a smooth vector if $\langle f, \rho_V(g)v \rangle$ is smooth for all $f \in V^*$. It is obvious that such vectors form a subspace of V, call it $V_{\mathrm{sm}} \subseteq V$. Fix $v \in V$ and $\phi : G \to \mathbb{C}$ smooth and with compact support. Let

$$w = w(\phi, v) = \int_G \phi(g)\rho_V(g)v \, dg.$$

We claim that w is smooth. We have

$$f(\rho(h)w) = f\left(\rho_V(h) \int_G \phi(g)\rho_V(g)v \, dg\right) = \int_G f(\phi(g)\rho_V(hg)v) \, dg = \int_G f(\phi(h^{-1}g)\rho_V(g)v) \, dg.$$

Differentiating under the integral sign and noting that $\phi(h^{-1}g)$ is smooth in h, we see that the above expression is smooth in h. Now choose a delta-like sequence ϕ_n with compact support around 1 so that

$$\int_G \phi_n(g) \, dg = 1.$$

Then $w_n = w(\phi_n, v) \to v$ and each w_n is smooth, so v is smooth. $\square$

Remark. Let V be an irreducible representation of G. Then:

1. V has an invariant positive-definite inner product which is unique up to scaling;
2. one can use an orthonormal basis $v_1, \dots, v_n$ to define matrix coefficients:
$$\psi_{V,i,j}(g) = v_j^*(\rho_V(g)v_i) = (\rho_V(g)v_i, v_j)$$
(note that this definition is independent of normalization).

Theorem 7.3 (Orthonormality of matrix coefficients)**.** *Let V, W be irreducible representations of G.*

1. *If V, W are not isomorphic, then*
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{W,k,\ell}(g)\, dg = 0.$$
2. *For $V = W$, we have*
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{V,k,\ell}(g)\, dg = \frac{\delta_{i,k}\delta_{j,\ell}}{\dim V}.$$

Proof. Let $\{v_i\}$ and $\{w_k\}$ be orthonormal bases for V and W, respectively. We have
$$\int_G \psi_{V,i,j}(g)\overline{\psi}_{W,k,\ell}(g)\, dg = \int_G ((\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_i \otimes w_k), v_j \otimes w_\ell)\, dg$$
Define the operator
$$P = \int_G (\rho_V \otimes \rho_{\overline{W}})(g)\, dg = \int_G \rho_{V \otimes \overline{W}}(g)\, dg.$$
Since $\overline{W} \cong W^*$, we have $P : V \otimes W^* \to V \otimes W^*$. Thus
$$\operatorname{Im} P \subseteq (V \otimes W^*)^G,$$
which is 0 if $V \not\cong W$. On the other hand, if $V \cong W$, then the only invariant is
$$\vec{u} = \sum_k (v_k \otimes \overline{v}_k),$$
so P is the orthogonal projection onto $\vec{u}$. Thus
$$P\vec{x} = \frac{(\vec{x}, \vec{u})}{(\vec{u}, \vec{u})}\vec{u},$$
so we have $(P(v_i \otimes w_k), v_j \otimes w_\ell) = \delta_{i,j}\delta_{k,\ell}/(\dim V)$. $\square$

7.3 Peter-Weyl Theorem

Theorem 7.4 (Peter-Weyl theorem)**.** *The matrix coefficients $\psi_{V,i,j}$ form an orthogonal basis in $L^2(G)$.*

Remark. Let V be a finite-dimensional irrep of G. There is a natural inclusion
$$\begin{aligned} i_V : V^* &\hookrightarrow \operatorname{Hom}_G(V, L^2(G)), \\ f &\longmapsto [v \mapsto (\rho_{V^*}(\cdot)f)(v)]. \end{aligned}$$
We claim that i_V is also surjective. To see this, let $\phi \in \operatorname{Hom}_G(V, L^2(G))$, i.e. an L^2 function left-invariant

under G. Thus we have that

$$\phi(x) = \rho_{V^*}(xg^{-1})\phi(g)$$

(after modifying ϕ on a set of measure zero). Setting $g = 1$, we get $\phi(x) = \rho_{V^*}(x)\phi(1)$, so we have

$$\xi : \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^* \cong \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes \mathrm{Hom}_G(V, L^2(G)) \hookrightarrow L^2(G),$$

an embedding of $(G \times G)$-modules. Call the left-hand side $L^2_{\mathrm{alg}}(G)$.

Theorem 7.5 (Peter-Weyl theorem, alternative)**.** *$L^2_{\mathrm{alg}}(G)$ is dense in $L^2(G)$, i.e.*

$$L^2(G) = \widehat{\bigoplus}_{V \in \mathrm{Irr}(G)} V \otimes V^*.$$

Example 7.1.1. Let $G = S^1 = U(1)$. The irreducible representations of G are $\psi_n(\theta) = e^{in\theta}$. The $e^{in\theta}$ form a basis of $L^2(G) = L^2(S^1)$, where the norm is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 \, d\theta.$$

This is the usual Fourier series on S^1. The Peter-Weyl theorem extends this to non-abelian groups.

Exercise 7.1. Let G be a compact group and H a closed subgroup of G.

1. Show that $L^2(G/H) = \widehat{\bigoplus}_{V \in \mathrm{Irr}(G)} N_H(V)V$, where $N_H(V) = \dim V^H$ (the space of H-invariants).

2. Let $G = \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$. Then show that $L^2(G/H) = L^2(S^2) = \widehat{\bigoplus}_{m \geq 0} N_H(m) L_{2m}$, and that $N_H(m) = 1$ for every m.

7.4 Introduction to Quantum Mechanics

Remark. Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. The spectrum of H gives the *energy levels* of the system. The elements $\psi(x, y, z) \in L^2(\mathbb{R}^3)$ are called *wave functions*, and we assume that they are normalized so that $\|\psi\|_{L^2} = 1$. This is so that

$$|\psi(x, y, z)|^2 \Delta V$$

gives the probability of a quantum particle to be in the region ΔV.

In general, there is also a time dependence in the wave function ψ, so we have $\psi(x, y, z, t)$. The time dependence is governed by the Schroedinger equation:

$$i\partial_t \psi = H\psi.$$

One can solve this equation via separation of variables, and we can write

$$\psi(x, y, z, t) = \sum_N e^{-iE_N t} \psi_N(x, y, z),$$

where the ψ_N are eigenvectors satisfying $H\psi_N = E_N \psi_N$.

Example 7.1.2. For the hydrogen atom, we have

$$H = -\frac{1}{2}\Delta - \frac{1}{r},$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian and $r = \sqrt{x^2 + y^2 + z^2}$. The $\Delta/2$ is called the *kinetic part* of H, and the $1/r$ is called the *potential part* of H.

Lecture 8

Feb. 9 — Hydrogen Atom

8.1 Bound States of the Hydrogen Atom

Remark. We are looking for eigenvectors for $H = -\frac{1}{2}\Delta - \frac{1}{r}$, i.e. $\psi_N \in L^2(\mathbb{R}^3)$ such that $H\psi_N = E_N\psi_N$ with $E_N < 0$. We first write the Laplacian in spherical coordinates:

$$\begin{aligned}\Delta &= \Delta_r + \frac{1}{r}\Delta_{\text{sph}} \\ \Delta_r &= \partial_r^2 + \frac{2}{r}\partial_r \\ \Delta_{\text{sph}} &= \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\,\cdot\,),\end{aligned}$$

where ϕ is the angle in the xy-plane and θ is the angle from the positive z-axis. Then we have

$$\partial_r^2\psi + \frac{2}{r}\partial_r\psi + \frac{2}{r}\psi + \frac{1}{r^2}\Delta_{\text{sph}}\psi = -2E\psi,$$

which is solved by $\psi(r, \vec{u}) = f(r)\xi(\vec{u})$ for $\vec{u} \in S^2$ satisfying

$$\Delta_{\text{sph}}\xi + \lambda\xi = 0 \tag{$*$}$$

$$f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\lambda}{r^2} + 2E\right)f(r) = 0.$$

Note that $(*)$ implies Δ_{sph} is rotationally invariant. By the Peter-Weyl theorem, we have

$$L^2(S^2) = \widehat{\bigoplus}_{\ell \geq 0} L_{2\ell},$$

where $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ and $L_{2\ell}$ are the irreps of $\mathrm{SO}(3)$.

Let $Y_\ell^0 \subseteq L_{2\ell}$ be a vector of weight 0, which is invariant under $\mathrm{SO}(2)$. Thus it depends only on θ. So we can write $Y_\ell^0(\theta) = P_\ell(\cos\theta)$, where P is a polynomial of degree ℓ. By orthogonality,

$$\int_{-1}^{1} P_k(z)P_\ell(z)\,dz = 0, \quad k \neq n.$$

Thus we can write

$$-\lambda_e P_\ell(z) = \Delta_{\text{sph}}P_\ell(z) = \partial_z(1 - z^2)\partial_z P_\ell(z).$$

From looking at the leading term we must have $\lambda_\ell = \ell(\ell + 1)$.

Now take $Y_\ell^m \in L_{2\ell}$ for $-\ell \le m \le \ell$. Write $Y_\ell^m(\phi,\theta) = e^{im\phi}P_\ell^m(\cos\theta)$. So we have

$$\frac{-m^2}{1-z^2}P_\ell^m + \partial_z(1-z^2)\partial_z P_\ell^m + \ell(\ell+1)P_\ell^m = 0, \quad -\ell \le m \le \ell.$$

This equation has a unique solution (up to scaling) on $[-1,1]$, given by

$$P_\ell^m = (1-z^2)^{m/2}\partial_z^{\ell+m}(1-z^2)^\ell.$$

Now we return to the radial equation:

$$f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + 2E\right)f(r) = 0.$$

Write $f(r) = r^\ell e^{-r/n}h(2r/n)$, where n is to be chosen later and h satisfies

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + \left(n-\ell-1+\frac{1}{4}(1+2En^2)\rho\right)h(\rho) = 0.$$

Now choose $n = 1/\sqrt{-2E}$, so that $E = -1/2n^2$. Then the above equation becomes

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + (n-\ell-1)h(\rho) = 0.$$

This equation is known as the *generalized Laguerre equation*. To get $\|\psi\|_{L^2}^2 < \infty$, we must have

$$\int_0^\infty \rho^{2\ell+2}e^{-\rho}|h(\rho)|^2\,d\rho < \infty,$$

where the extra $+2$ in $\rho^{2\ell+2}$ comes from the Jacobian. Solutions around 0 behave like $\rho^s(1+o(1))$, so

$$s(s+2\ell+1) = 0.$$

Thus either $s=0$ or $s=-2\ell-1$.

First consider when $\ell = 0$. Then $s=-1$ and we have $\rho^{-1}(1+o(1))$, so $\psi \sim 1/r$ as $r \to 0$. Then

$$H\psi = E\psi + C\delta_0,$$

where δ_0 is the delta function at 0, so we do not get an eigenvector in this case.

Thus $s = -2\ell-1$. Expanding $h(\rho)$ in a series and substituting, we get the recursive formula

$$h_n(\rho) = \sum_{k=0}^\infty \frac{(1+\ell-n)\cdots(k+\ell-n)}{(2\ell+2)\cdots(2\ell+1+k)k!}\rho^k.$$

This series converges, and we have

$$\lim_{\rho\to\infty}\frac{h_n(\rho)}{\rho} = 1$$

unless the series terminates. Thus $n-\ell-1 \in \mathbb{Z}_{\ge 0}$, so we can write

$$h_n(\rho) = \sum_{k=0}^{n-\ell-1}\frac{(1+\ell-n)\cdots(k+\ell-n)}{(2\ell+2)\cdots(2\ell+1+k)k!}\rho^k = L_{n-\ell-1}^{2\ell+1}(\rho),$$

which is known as the *generalized Laguerre polynomial*:

$$L_N^\alpha(\rho) = \sum_{k=0}^N (-1)^N\frac{N\cdots(N-k+1)}{(\alpha+1)\cdots(\alpha+k)}\frac{\rho^k}{k!}.$$

Theorem 8.1. *The bound states (i.e. solutions to $H\psi = E\psi$ in $L^2(\mathbb{R}^3)$) of the hydrogen atom are*

$$\psi_{n,\ell,m}(r,\phi,\theta) = r^\ell e^{-r/n} L^{2\ell+1}_{n-\ell-1}(2r/n) Y^m_\ell(\theta,\phi),$$

where $n \in \mathbb{Z}_{>0}$, ℓ is an integer from $0,\dots,n-1$, $E_n = -1/2n^2$, and m is an integer between $-\ell,\dots,\ell$.

Remark. In Theorem 8.1, n is known as the *principal quantum number*, and ℓ is known as the *azimuthal quantum number*. Note that if $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\Delta_{\text{sph}}$, where iL_x, iL_y, iL_z are the generators of $\mathfrak{so}(3)$ satisfying $[L_{\{x}, L_y] = -iL_{z\}}$, then $\vec{L}^2 = C = \ell(\ell+1)$ is the Casimir operator.

Corollary 8.1.1. *The space W_n of states with principal number n has dimension n^2.*

Proof. This follows from $\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$. □

Remark. Note that $\widehat{\bigoplus}_n W_n$ forms a proper, closed subspace $L_0^2(\mathbb{R}^3)$ of $L^2(\mathbb{R}^3)$. We need to find all φ with $(H\varphi,\varphi) \geq 0$ to reconstruct all of $L^2(\mathbb{R}^3)$. This corresponds to the continuous spectrum of H.

8.2 Spin

Remark. *Spin* is a kind of intrinsic angular momentum. Instead of just $L^2(\mathbb{R}^3)$, we should consider

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3) \otimes L_1$$

to be the space of states for the hydrogen atom. We have

$$V_n = (L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}) \otimes L_1 = 2L_1 \oplus 2L_3 \oplus \cdots \oplus 2L_{2n-3} \oplus 2L_{2n-1},$$

so $\dim V_n = 2n^2$. We have an additional *spin operator* given by

$$S_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

which acts on $\mathbb{C}^2$ in the standard basis e_+, e_-. Then we have

$$\psi_{n,\ell,m,+} = \psi_{n,\ell,m} \otimes e_+ \quad \text{and} \quad \psi_{n,\ell,m,-} = \psi_{n,\ell,m} \otimes e_-.$$

The *total spin* is $m + s$ (where s is the eigenvalue for S_z), which is either $m + 1/2$, or $m - 1/2$.

8.3 Pauli Exclusion Principle

Remark. The space $\wedge^k V_n$ corresponds to the space of states for k electrons at energy level n. Note that we must have $k \leq 2n^2$ to have $\wedge^k V_n \neq 0$, which gives the *Pauli exclusion principle*.

In the periodic table, one has *orbitals* s, p, d, f corresponding to $\ell = 0, 1, 2, 3$, respectively, written with coefficient n and with exponent k corresponding to the number of electrons in the orbital. For example, the element Ruthenium has

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^7 5s^1.$$

The periodic table is organized as follows: from left to right ordered by how many *valent* electrons (i.e. the number of electrons in the outermost orbital), and from top to bottom ordered by how many energy levels. For Ruthenium, it is on column 8 and row 5. The number is 44, for 44 total electrons.

Exercise 8.1. Let $\vec{r} = (x, y, z)$ and $\vec{p} = (-\partial_x, -i\partial_y, -i\partial_z)$ be the *position* and *momentum* operators. Let $\vec{L} = \vec{r} \times \vec{p}$ and $H = \frac{1}{2}\vec{p}^2 + U(r)$, where U is rotationally invariant. Show that:

1. The components $i\vec{L}$ are generators of the rotations on $\mathbb{R}^3$, and $[\vec{L}, \vec{p}^2] = 0$.
2. $\vec{A}_0 = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})$ satisfies $[\vec{A}_0, \vec{p}^2] = 0$.
3. Let $A = \vec{A}_0 + \phi(r)\vec{r}$. There exists ϕ such that $[\vec{A}, H] = 0$ if and only if U is a *Columb potential* (i.e. $U(r) = \frac{C}{r} + D$), and in this case ϕ is completely determined.
4. (Hidden symmetry of the hydrogen atom) Use the commutation relations between $\vec{A}$ and $\vec{L}$ to define an action on $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3$, so that $\vec{L}$ is the diagonal copy in this decomposition.
5. $W_n = L_{n-1} \boxtimes L_{n-1}$ as representations of $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Lecture 9

Feb. 11 — Real Forms

9.1 Automorphisms of Semisimple Lie Algebras

Remark. Recall that we can identify $\mathrm{Aut}(\mathfrak{g})$ with a Lie group with Lie algebra $\mathfrak{g}$. The connected component of the identity $\mathrm{Aut}^0(\mathfrak{g})$ (also known as the *adjoint group* G_{ad}) acts transitively on the set of Cartan subalgebras. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then there is a connected subgroup $H \subseteq G_{\mathrm{ad}}$ which acts as 1 on $\mathfrak{h}$ and as $e^{\alpha(x)}$ on g_α for $x \in \mathfrak{h}$. Then we have

$$\mathfrak{h}/2\pi i P^\vee \cong H,$$

and H is called a *maximal torus.*

Proposition 9.1. *The normalizer $N(H)$ of H in G_{ad} coincides with the stabilizer of $\mathfrak{h}$ and contains H as a normal subgroup such that $N(H)/H = W$ (the Weyl group).*

Proof. Note that $\mathrm{SL}_2(\mathbb{C})$ is simply connected, so for any simple root α_i there is a homomorphism

$$\eta_i : \mathrm{SL}_2(\mathbb{C}) \longrightarrow G_{\mathrm{ad}} = \widetilde{G}/\mathcal{Z}(\widetilde{G}).$$

Define $S_i = \eta_i \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$. Consider $w = s_{i_1} \cdots s_{i_n}$ and $\widetilde{w} = S_{i_1} \cdots S_{i_n}$. Note that $\widetilde{w}$ acts on $\mathfrak{h}$ acts w, and if $w = w_1 w_2$, then $\widetilde{w} = \widetilde{w}_1 \widetilde{w}_2 h$ for some $h \in H$. To see the latter claim, note that h has the preserve the root decomposition, hence $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$. Thus $h = \exp(\sum_j b_j w_j^\vee) \in H$ (where $\langle w_j^\vee, \alpha_i \rangle = \delta_{j,i}$).

So $\widetilde{w}$ and H generate a subgroup $N \subseteq N(H)$ such that $N/H = W$. It remains to show that $N(H) = N$. Let $x \in N(H)$ and consider simple roots $\alpha_i' = x(\alpha_i)$. Then there exists $w \in W$ such that $w(\alpha_i') = \alpha_{p(i)}$ for some permutation p. Then $\widetilde{w}x(\alpha_i) = \alpha_{p(i)}$, so this is a Dynkin diagram automorphism. Now $\widetilde{\omega}x$ is an element of a group, so the fundamental weights are fixed. Thus $p = \mathrm{id}$. $\square$

Remark. Although $N(H)/H = W$, in general $N(H)$ is not a semidirect product of H and W.

Proposition 9.2. *The map $\xi : \mathrm{Aut}(D) \ltimes G_{\mathrm{d}} \to \mathrm{Aut}(\mathfrak{g})$ is an isomorphism.*

Proof. We have to show that ξ is surjective. Let $a \in \mathrm{Aut}(\mathfrak{g})$. We can say that a preserves the Cartan subalgebra (if not, we can shift it by $g \in G_{\mathrm{ad}}$). Multiplying by $\mathrm{Aut}(D)N(H)$, we can make it act trivially on $\mathfrak{h}$ and $\mathfrak{g}_{\alpha_i}$. Then $a = 1$, so $a \in \mathrm{Im}\,\xi$. $\square$

9.2 Real Forms of Semisimple Lie Algebras

Remark. Recall the Serre presentation for $\mathfrak{g}$, i.e. generators $\{h_i, f_i, e_i\}$ with certain relations. In this setting, everything was defined over $\mathbb{Q}$.

Definition 9.1. A semisimple Lie algebra is *split* if it admits a Chevalley-Serre basis over base field K.

Remark. Let L be a Galois extension of K ($\operatorname{char} K = 0$), and assume that $\mathfrak{g}_L$ is a split semisimple Lie algebra. We want to find $\mathfrak{g}$ over K such that $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$. The problem is then to find a classification of all such $\mathfrak{g}$. Let $\Gamma = \operatorname{Gal}(L/K)$. Define an action of Γ on $\mathfrak{g}_L$ by

$$g(\lambda x) = g(\lambda)g(x), \quad x \in \mathfrak{g}_L, \lambda \in L, g \in \Gamma,$$

which is twisted linear. We can reconstruct $\mathfrak{g}$ as the invariants $\mathfrak{g}_L^\Gamma$.

The simplest action of this kind is $\rho_0(g)$, which acts on scalars and preserves $\{h_i, e_i, f_i\}$. Any twisted linear action takes the form $\rho(g) = \eta(g)\rho_0(g)$ for some $\eta : \Gamma \to \operatorname{Aut}(\mathfrak{g}_L)$. As ρ is a homomorphism,

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

and upon rearranging, we have

$$\eta(gh) = \eta(g)g(\eta(h)),$$

where $g(a) = \rho_0(g)a\rho_0(g)^{-1}$ for $a \in \operatorname{Aut}(\mathfrak{g}_L)$. The above is called a 1-*cocycle condition*.

Denote the Lie algebra associated to the cocycle η by $\mathfrak{g}_\eta$. When do we have $\mathfrak{g}_{\eta_1} \cong \mathfrak{g}_{\eta_2}$? This is the case when ρ_1 and ρ_2 are isomorphic, i.e. there exists $a \in \operatorname{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$. Then

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

so $\eta_1(g) = a\eta_2(g)g(a)^{-1}$. Thus η_1 and η_2 are cohomologous cocycles.

Proposition 9.3. *The semisimple Lie algebras $\mathfrak{g}$ over K which split over L (where L/K is Galois) are classified by $H^1(\Gamma, \operatorname{Aut}(\mathfrak{g}_L))$, where $\Gamma = \operatorname{Gal}(L/K)$.*

Remark. We will now specialize to $K = \mathbb{R}$, $L = \mathbb{C}$, where $\Gamma = \mathbb{Z}/2\mathbb{Z}$, generated by complex conjugation. We have $\operatorname{Aut}(\mathfrak{g}_L) = \operatorname{Aut}(D) \ltimes G_{\mathrm{ad}}$. Since $\eta(1) = 1$, η is determined by $\eta(-1)$. The cocycle condition is

$$s\overline{s} = 1, \quad s = \eta(-1).$$

The corresponding Lie algebra (up to isomorphism) depends only on the cohomology class of s, where $s \to as\overline{a}^{-1}$ for $a \in \operatorname{Aut}(D)$.

Theorem 9.1. *The real semisimple Lie algebras whose complexification is $\mathfrak{g}$ (i.e. the real forms of $\mathfrak{g}$) are classified by $s \in \operatorname{Aut}(D) \ltimes G_{\mathrm{ad}}$ such that $s\overline{s} = 1$, modulo the equivalence $s \to as\overline{a}^{-1}$ for $a \in \operatorname{Aut}(D)$.*

Remark. Note that complex conjugation acts trivially on $\operatorname{Aut}(D)$.

Remark. Denote by $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \overline{x} = s(x)\}$ the real form corresponding to s. Denote by $\mathfrak{g}_{(1)}$ the split form consisting of real $x \in \mathfrak{g}$ (so that $x = \overline{x}$).

Alternatively, we can define an antilinear involution $\sigma_s(x) = \overline{s(x)}$. Then $\mathfrak{g}_{(s)}$ is the fixed point set of σ_s.

Remark. Note that s defines $s_0 \in \operatorname{Aut}(D)$ with $s_0^2 = 1$.

Corollary 9.1.1. *The conjugacy class of s_0 is invariant under equivalences.*

Remark. Since s_0 permutes the connected components of the Dynkin diagram D, it preserves some and divides some into pairs. So every semisimple real Lie algebra is a direct sum of simple ones, and each simple one has either connected Dynkin diagram or consists of two identical components.

Remark. We now consider the case when D is connected and $\mathfrak{g}$ is simple.

Definition 9.2. A real form $\mathfrak{g}_{(s)}$ of a complex simple Lie algebra is said to be *inner to* $\mathfrak{g}_{(s')}$ if $s' = gs$ up to equivalence, where $g \in G_{\mathrm{ad}}$ (i.e. s, s' differ by an inner automorphism). The *inner class* of $\mathfrak{g}_{(s)}$ is the collection of all real forms inner to $\mathfrak{g}_{(s)}$. An *inner form* is a form inner to a split form. We call $\mathfrak{g}_{(s)}$ *quasi-split* if $s = s_0 \in \mathrm{Aut}(D)$.

Corollary 9.1.2. *We have the following:*

1. *Any real form is inner to a unique quasi-split form.*
2. *A real form which is both inner and quasi-split is split.*

Example 9.2.1. Consider the *Cartan involution* τ defined by

$$\tau(h_j) = -h_j, \quad \tau(e_j) = -f_j, \quad \tau(f_j) = -e_j.$$

Then $\mathfrak{g}_{(\tau)} = \mathfrak{g}^c$ is called the *compact real form* of $\mathfrak{g}$.

Lecture 10

Feb. 16 — Real Forms, Part 2

10.1 Compact Real Forms

Proposition 10.1. *Let τ be the Cartan involution (defined in Example 9.2.1) and $\mathfrak{g}^c = \mathfrak{g}_{(\tau)}$. Then the Killing form of $\mathfrak{g}^c$ is negative definite.*

Proof. We can write $\mathfrak{g}^c = (\mathfrak{h} \cap \mathfrak{g}^c) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. The Killing form is negative definite on $\mathfrak{h} \cap \mathfrak{g}^c$ since the inner product on a coroot lattice is positive definite. Thus it is negative definite on $\mathfrak{g}^c$, as $\{i\alpha_j^\vee\}$ is a basis for $\mathfrak{g}^c \cap \mathfrak{h}$. Now we need to show that it is negative definite on $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. Note that for $\mathfrak{g} = \mathfrak{sl}_2$, we have a basis for $\mathfrak{g}^c$ given by ih, $e - f$, $i(e+f)$, so $\mathfrak{g}^c = \mathfrak{su}(2)$. So the statement holds there. For general $\mathfrak{g}$, we have that S_i preserves $\mathfrak{g}^c$, since

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2).$$

So we have $\mathrm{Lie}(\mathrm{SU}(2)_i) \subseteq \mathfrak{g}^c$. For any $w \in W$, the lift $\widetilde{w}$ preserves $\mathfrak{g}^c$, so the restriction of the Killing form of $\mathfrak{g}^c$ to $\mathfrak{g}^c \cap (\mathfrak{sl}_2)_\alpha$ is negative definite. □

Remark. Consider $\mathrm{Aut}(\mathfrak{g}^c)$. Since the Killing form is negative definite, $\mathrm{Aut}(\mathfrak{g}^c)$ is a closed subgroup of $\mathrm{O}(\mathfrak{g}^c)$, so it is compact. Moreover, this is a Lie group with Lie algebra $\mathfrak{g}^c$.

Corollary 10.0.1. $G^c_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}^c)^\circ$ *is a connected, compact Lie group with Lie algebra* $\mathfrak{g}^c$.

Example 10.0.1. We have the following:

1. For $\mathfrak{g} = \mathfrak{sl}_n$, we have $G^c_{\mathrm{ad}} = \mathrm{PSU}(n) = \mathrm{SU}(n)/\mu_n$, where μ_n is the nth roots of unity.
2. For $\mathfrak{g} = \mathfrak{so}_n$, we have $G^c_{\mathrm{ad}} = \mathrm{SO}(n)$ for odd n and $G^c_{\mathrm{ad}} = \mathrm{SO}(n)/\{\pm 1\}$ for even n.
3. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $G^c_{\mathrm{ad}} = \mathrm{U}(n, \mathbb{H})/\{\pm 1\}$. The group

$$\mathrm{U}(n, \mathbb{H}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$$

is called the *quaternionic unitary group.*

Example 10.0.2. We have the following:

1. Consider A_{n-1}, which corresponds to the split form $\mathfrak{sl}(n, \mathbb{R})$ and compact form $\mathfrak{su}(n)$. For $n > 2$, we have a quasi-split real form as follows: Let $s(A) = -JA^T J^{-1}$ where $J_{i,j} = (-1)^i \delta_{i,n+1-j}$. Then

$$e_i, f_i, h_i, \longmapsto e_{n+1-i}, f_{n+1-i}, h_{n+1-i}.$$

Note that J is a Hermitian or skew-Hermitian form of signature (p,p) with $n = 2p$ or of signature $(p+1,p), (p,p+1)$, which are isomorphic when $n = 2p+1$.

For $n = 2$, we have $\mathfrak{su}(1,1) = \mathfrak{sl}(2,\mathbb{R})$ (and $\mathrm{PSU}(1,1) = \mathrm{PSL}(2,\mathbb{R})$).[1]

2. For type B_n, we have compact form $\mathfrak{so}(2n+1)$ and split form $\mathfrak{so}(n+1,n)$. There are no nontrivial automorphisms, so there are no non-split quasi-split forms.

   Particular cases of interest are $\mathrm{SO}(3) \cong \mathrm{SU}(2)$ and $\mathrm{SO}^+(2,1) = \mathrm{PSU}(1,1) = \mathrm{PSL}(2,\mathbb{R})$.

3. For type C_n, we have the split form $\mathfrak{sp}(2n,\mathbb{R})$ and compact form $\mathfrak{u}(n,\mathbb{H})$. There are no non-split quasi-split reals forms, as there are no nontrivial automorphisms of the Dynkin diagram.

   Note that $B_2 = C_2$, so we have $\mathfrak{so}(3,2) = \mathfrak{sp}_4(\mathbb{R})$ and $\mathfrak{so}(5) = \mathfrak{u}(2,\mathbb{H})$.

4. For type D_n, we have split form $\mathfrak{so}(n,n)$ and compact form $\mathfrak{so}(2n)$. For $n > 4$, there is a unique nontrivial involution, while for $n = 4$, we have $\mathrm{Aut}(D) = S_3$. However, there is still a unique non-split quasi-split form as there is only one nontrivial involution up to conjugation. Recall

   $$A = -JA^T J^{-1}, \quad J_{i,j} = \delta_{i,2n+1-j}.$$

   Then the quasi-split form is given by $J \mapsto J' = gJ$, where g permutes e_n, e_{n+1} (which corresponds to α_{n-1}, α_n). The signature defined by J' is $(n+1, n-1)$, so the quasi-split form is $\mathfrak{so}(n+1,n-1)$.

   Note that $D_2 = A_1 \oplus A_1$, so we have the following isomorphisms:

   $$\begin{aligned} \mathfrak{so}(4) &= \mathfrak{su}(2) \oplus \mathfrak{su}(2), \\ \mathfrak{so}(2,2) &= \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1), \\ \mathfrak{so}(3,1) &= \mathfrak{sl}_2(\mathbb{C}), \end{aligned}$$

   where in the last isomorphism we view $\mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra.

   We also have $D_3 = A_3$, which gives the following isomorphisms:

   $$\begin{aligned} \mathfrak{so}(6) &= \mathfrak{su}(4), \\ \mathfrak{so}(3,3) &= \mathfrak{sl}_4(\mathbb{R}), \\ \mathfrak{so}(4,2) &= \mathfrak{su}(2,2). \end{aligned}$$

10.2 Classification of Real Forms

Remark. Write $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega = \sigma_\tau$ the Cartan antilinear involution (so that $\mathfrak{g}^c$ is the fixed points of σ_τ). Another real structure on $\mathfrak{g}$ is given by $\sigma = \omega \circ g$ for $g \in \mathrm{Aut}(\mathfrak{g})$, as

$$\sigma^2 = \omega \circ g \circ \omega \circ g = 1$$

(note that $\omega \circ \omega = \omega(g)$, and $\omega(g)g = 1$). Define

$$(X,Y) = \mathrm{tr}(\mathrm{ad}_X \, \mathrm{ad}_{\omega(Y)}),$$

which is the Hermitian extension of the Killing form from $\mathfrak{g}^c$ to $\mathfrak{g}$. Note that $\omega(g) = (\mathfrak{g}^\dagger)^{-1}$, where $g^\dagger$ is the adjoint of g. Thus we see that g is self-adjoint.

[1]Note that $\mathrm{PSU}(1,1)$ is the group of automorphisms of the unit disk, and $\mathrm{PSL}(2,\mathbb{R})$ is the group of automorphisms of the upper half-plane. The isomorphism comes from the Cayley transform from the unit disk to the upper half-plane.

In particular, g is diagonalizable with real eigenvalues. So we can write

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma),$$

where $\mathfrak{g}(\gamma)$ is the eigenspace of $\mathfrak{g}$ corresponding to eigenvalue γ. Note that

$$[\mathfrak{g}(\beta), \mathfrak{g}(\gamma)] = \mathfrak{g}(\beta\gamma).$$

Consider the operator $|g|^t$ for $t \in \mathbb{R}$, which acts on $\mathfrak{g}(\gamma)$ as $|\gamma|^t$. We can rewrite

$$|g|^t = \exp(t \log |g|) \in G_{\mathrm{ad}},$$

which is a 1-parameter subgroup of G_{ad}. Then we can define $\theta := g|g|^{-1}$, and we have

$$\begin{cases} \theta \circ \omega = \omega \circ \theta, \\ \theta^2 = 1, \end{cases}$$

where the first identity follows from $(\theta^\dagger)^{-1} = \theta$. We have

$$\theta = |g|^{-1/2} g \omega(|g|^{1/2}).$$

We can assume $g = \theta$ with $\theta \circ \omega = \omega = \theta$, or equivalently, that $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$ and $\theta^2 = 1$. Thus θ has ± 1 eigenspaces. Note that θ' defines the same real form if and only if

$$\theta' = x\theta(\omega(x))^{-1}$$

for some $x \in \mathrm{Aut}(\mathfrak{g})$. Then we have $x\theta(\omega(x))^{-1} = \omega(x)\theta x^{-1}$ (since $\theta'^2 = 1$, so $\theta'^{-1} = \theta'$). Let

$$z = (\omega(x))^{-1} x,$$

so that $\omega(z) = z^{-1}$. Then $\theta z = z^{-1}\theta$. Now note that $z = x^\dagger x$ is positive definite, so if $y = xz^{-1/2}$,

$$\omega(y) = \omega(x) z^{1/2} = (x^\dagger)^{-1} z^{1/2} = xz^{-1/2} = y.$$

Thus $y \in \mathrm{Aut}(\mathfrak{g}^c)$, and we also have that

$$\theta' = x\theta\omega(x)^{-1} = x\theta z x^{-1} = xz^{-1/2}\theta z^{1/2} x^{-1} = y\theta y^{-1}.$$

Theorem 10.1. *The real forms of* $\mathfrak{g}$ *are in one-to-one correspondence with the conjugacy classes of involutions* $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$*, where* $\theta \mapsto \omega_\theta = \theta \circ \omega = \omega \circ \theta$*.*

Remark. For $\theta : \mathfrak{g} \to \mathfrak{g}$, denote by $\mathfrak{g}_\theta$ the corresponding real form. For example,

$$\mathfrak{g}_1 = \mathfrak{g}^c = \mathfrak{g}_{(\tau)},$$

where the latter is our old notation using the split forms.

Remark. We now have a canonical (up to automorphisms of $\mathfrak{g}^c$) decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. Here $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. For $\mathfrak{g}^c$ itself, we have

$$\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c.$$

Moreover, we have $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, where $\mathfrak{p}_\theta = i\mathfrak{p}^c$ (a fixed point of $\sigma = \omega \circ \theta$ has to have an extra $-$ sign, which we can achieve by multiplying by i).

Exercise 10.1. Show that $\mathfrak{k}$ is reductive but not necessarily semisimple.

Lecture 11

Feb. 18 — Real Forms, Part 3

11.1 Classification of Real Forms, Continued

Proposition 11.1. *There exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ which is invariant under θ and such that $\mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra in $\mathfrak{k}$.*

Proof. Consider a generic element $t \in \mathfrak{k}^c$. It is regular and semisimple. Consider $\mathfrak{h}_+^c$, the centralizer of t in $\mathfrak{k}^c$. Then necessarily $\mathfrak{h}_+ := \mathfrak{h}_+^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra in $\mathfrak{k}$. Let $\mathfrak{h}_-^c$ be a maximal subspace of $\mathfrak{p}^c$ so that $\mathfrak{h}^c = \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ is a commutative subalgebra of $\mathfrak{g}^c$. Then we claim that $\mathfrak{h} = \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ consists of the semisimple elements, and all elements in $\mathfrak{g}^c$ are anti-self-adjoint operators. If $z \in \mathfrak{g}$ commutes with $\mathfrak{h}$, then

$$z = z_+ + z_-, \quad z_+ \in \mathfrak{k}, z_- \in \mathfrak{p},$$

where $z_\pm$ commute with $\mathfrak{h}$. Then $z_+ \in \mathfrak{h}_+$, and

$$z_- = x + iy, \quad x, y \in \mathfrak{p}^c,$$

where x, y commute with $\mathfrak{h}$. Then $x, y \in \mathfrak{h}_-^c$ (by the definition of $\mathfrak{h}_-^c$), so $z \in \mathfrak{h}$. □

Corollary 11.0.1. *We have $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where $\theta = 1$ on $\mathfrak{h}_+$ and $\theta = -1$ on $\mathfrak{h}_-$.*

Lemma 11.1. *There are no coroots of $\mathfrak{g}$ in $\mathfrak{h}_-$.*

Proof. Suppose otherwise that $\alpha^\vee \in \mathfrak{h}_-$. Then $\theta(\alpha^\vee) = -\alpha^\vee$, so

$$\theta(e_\alpha) = e_{-\alpha} \quad \text{and} \quad \theta(e_{-\alpha}) = e_\alpha$$

for some $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$. Then $x = e_\alpha + e_{-\alpha}$ satisfies $\theta(x) = x$, so $x \in \mathfrak{k}$. But $x \notin \mathfrak{h}_+$ (since $x \perp \mathfrak{h}_+$). Thus $[\mathfrak{h}_+, x] = 0$ since α vanishes on $\mathfrak{h}_+$, a contradiction as $\mathfrak{h}_+$ is a maximal commutative subalgebra of $\mathfrak{k}$. □

Remark. Pick a generic element $t \in \mathfrak{h}_+$, which is regular in $\mathfrak{g}$. Choose t so that

$$\mathrm{Re}(t, \alpha^\vee) \neq 0$$

for all $\alpha^\vee$ of $\mathfrak{g}$. Then we can define a *polarization* on R by

$$R_+ = \{\alpha \in R : \mathrm{Re}(t, \alpha^\vee) > 0\}$$

which satisfies $\theta(R_+) = R_+$. Now $\{\theta(i) : i \in D\}$ gives the action of θ: If $\theta = i$, then

$$\theta(e_i) = \pm e_i, \quad \theta(h_i) = h_i, \quad \theta(f_i) = \pm f_i.$$

Otherwise, if $\theta(i) \neq i$, then we can choose generators $h_i, e_i, e_{\theta(i)}, f_i, f_{\theta(i)}, h_{\theta(i)}$ such that

$$\theta(x_i) = x_{\theta(i)}, \quad x = e, f, h.$$

We can then construct *markings* on the Dynkin diagram as follows:

- Connect vertices i and $\theta(i)$ if $\theta(i) \neq i$.
- Mark a vertex i as white if $\theta(e_i) = e_i$.
- Mark a vertex i as black if $\theta(e_i) = -e_i$.

This is the *Vogan diagram* associated to the Dynkin diagram. Note that $e_i \in P$ is a non-compact root.

Exercise 11.1. Showing the following:

1. The signature of the Killing form of g_θ is $(\dim \mathfrak{p}, \dim \mathfrak{k})$. Moreover, the Killing form is negative definite if and only if $\theta = 1$, i.e. $\mathfrak{g} = \mathfrak{g}^c$.
2. For a split real form, $\dim \mathfrak{k} = |R_+|$.
3. Show that for any real form in a compact inner class, $\operatorname{rank}(\mathfrak{k}) = \operatorname{rank}(\mathfrak{g})$.

11.2 Real Forms of Classical Lie Algebras

Example 11.0.1. We have the following real forms of the classical Lie algebras:

1. Type A_{n-1}, compact inner class.

   Let θ be the inner automorphism element of $\mathrm{PSU}(n)$ of order 2. Let $g \in \mathrm{U}(n)$ such that $g^2 = 1$. Then $\theta(x) = gxg^{-1}$, so $g = \mathrm{id}_p \otimes (-\mathrm{id}_q)$ with $p + q = n$. Thus

   $$\mathfrak{g}_\theta = \mathfrak{su}(p, q) \quad \text{and} \quad \mathfrak{k} = \mathfrak{gl}_p \oplus \mathfrak{sl}_q.$$

   For $n = 2$, we have $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$, with $\mathfrak{k} = \mathfrak{gl}_1$.

2. Type A_{n-1}, split inner class.

   If n is odd (so all vertices are divided into connected pairs), then

   $$\mathfrak{g}_\theta = \mathfrak{sl}_n(\mathbb{R}).$$

   If n is even, then there is 1 stable vertex (which is either black or white). In these cases we either have $\mathfrak{k} = \mathfrak{sp}_{2k}$ (in which case $\mathfrak{g}_\theta = \mathfrak{sl}(k, \mathbb{H})$) or $\mathfrak{k} = \mathfrak{so}_{2k}$ (which is just the split form $\mathfrak{sl}_n(\mathbb{R})$).

3. Type B_n (i.e. $\mathfrak{so}_{2n+1}$).

   Let θ be the inner automorphism of order ≤ 2. We can write

   $$\theta = \mathrm{id}_{2p+1} \oplus (-\mathrm{id}_{2q})$$

   where $p + q = n$. The real forms are $\mathfrak{so}(2p + 1, 2q)$, with $\mathfrak{k} = \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2q}$.

4. Type C_n.

   We can have $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ such that $g^2 = 1$ or $g^2 = -1$. The adjoint compact group is

   $$(\mathrm{Sp}(2n) \cap U(n))/\pm 1.$$

If $g^2 = 1$, then the eigenspace with eigenvalue 1 has dimension $2p$, and the eigenspace for -1 has dimension $2q$ (where $p + q = n$). Assume that $p \geq q$ (otherwise take $g \mapsto -g$). Then we have

$$\mathfrak{sp}(2p, 2q) = \mathfrak{sp}_{2n} \cap \mathfrak{u}(p, q) = \mathfrak{u}(p, q, \mathbb{H}).$$

In this case, we have $\mathfrak{k} = \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q}$.

If $g^2 = -1$, then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$. In this case, $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$, as for any $(w, w) = w \cdot \overline{w}$ in $\mathbb{C}^n$,

$$\operatorname{Im}(w, w) = i \operatorname{Im}(w \wedge \overline{w})$$

defines a symplectic form on $\mathbb{R}^{2n}$. Thus we can view $\mathrm{U}(n) \subseteq \mathrm{Sp}_{2n}(\mathbb{R})$.

5. Type D_n, compact inner class.

   In this case, θ is given by $g \in \mathrm{SO}(2n)$ with $g^2 = \pm 1$.

   If $g^2 = 1$, then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$ (where $\dim V(1) = 2p$ and $\dim V(-1) = 2q$ with $p + q = n$). Note that $\det(g) = 1$, so the eigenspaces are even-dimensional. Then

   $$\mathfrak{g}_\theta = \mathfrak{so}(2p, 2q) \quad \text{and} \quad \mathfrak{k} = \mathfrak{so}(2p) \oplus \mathfrak{so}(2q).$$

   If $g^2 = -1$, then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$, so $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. In this case, we have

   $$\mathfrak{g}_\theta = \mathfrak{so}^*(2n),$$

   which is known as the *quaternionic orthogonal Lie algebra*.

6. Type D_n, the other inner class.

   In this case, θ is given by $g \in \mathrm{O}(2n)$ with $\det(g) = -1$ and $g^2 = \pm 1$.

   Note that if $g^2 = -1$, then $\det(g) = 1$, which is a contradiction. So we can only have $g^2 = 1$. Then

   $$\mathbb{C}^{2n} = V(1) \oplus V(-1),$$

   where $\dim V(1) = 2p + 1$ and $\dim V(-1) = 2q + 1$. Here $\mathfrak{k} = \mathfrak{so}(2p + 1) \oplus \mathfrak{so}(2q + 1)$.

11.3 More on Compact Groups

Exercise 11.2. Show that if K^c is a compact Lie group, then $\mathfrak{k} = \mathrm{Lie}_{\mathbb{C}}(K^c)$ is a reductive Lie algebra.

Example 11.0.2. Let $G_{\mathrm{ad}} = \mathrm{Aut}^0(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$, and let G_{ad}^c be its compact form. Consider the following product:

$$(S^1)^r \times G_{\mathrm{ad}}^c.$$

We will see that any Lie algebra of a compact group is isomorphic to a Lie algebra of such a product. We can also consider covering spaces, i.e. what is $\pi_1(G_{\mathrm{ad}}^c)$?