

MATH 8803: Representation Theory II

Frank Qiang
Instructor: Anton Zeitlin

Georgia Institute of Technology
Spring 2026

Contents

1 Jan. 12 — Introduction and Review	3
1.1 Review and Overview	3
1.2 Representations of Semisimple Lie Algebras	4
1.3 Representations of SL_n and GL_n	4
2 Jan. 14 — Applications of Schur-Weyl Duality	6
2.1 The Schur Functor	6
2.2 Invariant Theory	7
2.3 Weyl Character Formula for GL_n	8
2.4 Howe Duality	9
3 Jan. 21 — Minuscule Weights	10
3.1 Minuscule Weights	10
3.2 Applications of Minuscule Weights	11
4 Jan. 26 — Other Classical Lie Algebras	13
4.1 Applications of Minuscule Weights, Continued	13
4.2 Other Classical Lie Algebras	14
4.3 Representations of Symplectic Lie Algebras	14
4.4 Representations of Orthogonal Lie Algebras	15
5 Jan. 28 — Other Classical Lie Algebras, Part 2	16
5.1 More on Orthogonal Lie Algebras	16
5.2 Clifford Algebras	17
6 Feb. 2 — Duals, Maximal Weights, Exponents	19
6.1 Dual Representations	19
6.2 Maximal Weights	19
6.3 Principal \mathfrak{sl}_2 -Subalgebra and Exponents	20
6.4 Complex, Real, and Quaternionic Types	20
6.5 Review of Compact Lie Groups	21
7 Feb. 4 — Compact Groups	22
7.1 More on Exponents	22
7.2 Matrix Coefficients	22
7.3 Peter-Weyl Theorem	23
7.4 Introduction to Quantum Mechanics	24
8 Feb. 9 — Hydrogen Atom	25
8.1 Bound States of the Hydrogen Atom	25

8.2 Spin	27
8.3 Pauli Exclusion Principle	27
9 Feb. 11 — Real Forms	29
9.1 Automorphisms of Semisimple Lie Algebras	29
9.2 Real Forms of Semisimple Lie Algebras	29
10 Feb. 16 — Real Forms, Part 2	32
10.1 Compact Real Forms	32
10.2 Classification of Real Forms	33

Lecture 1

Jan. 12 — Introduction and Review

1.1 Review and Overview

Remark. Recall that we are interested in representations of Lie groups G , which is closely related to representations of Lie algebras \mathfrak{g} .

We are primarily interested in semisimple Lie algebras. In this case, we fix a *Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$, where $r = \dim \mathfrak{h}$ is called the *rank*. We have the Serre generators $\{h_i, e_i, f_i\}_{i=1}^r$ and relations

$$[h_i, e_j] = a_{i,j}e_j, \quad [h_i, f_j] = a_{i,j}f_j, \quad \text{ad}_{e_i}^{1-a_{i,j}} e_j = 0, \quad \text{ad}_{f_i}^{1-a_{i,j}} f_j = 0,$$

where $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Here $\{\alpha_i\} \subseteq \mathfrak{h}^*$ and we identify $\alpha_i^\vee \leftrightarrow h_i \in \mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where \mathfrak{n}_+ is generated by $\{e_i\}$ and \mathfrak{n}_- is generated by $\{f_i\}$. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

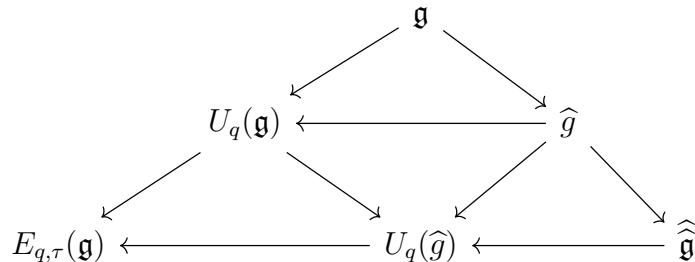
where $R = R_+ \sqcup R_-$. We have $R_+ \subseteq Q_+$ and $R_- \subseteq Q_-$, where $Q_+ = \{\sum_{i=1}^r n_i \alpha_i : n_i \geq 0\}$. If the $a_{i,j}$ are degenerate, then we can define $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}c$ is called the *central extension* and $d = t \frac{d}{dt}$. We can think of these as maps $S^1 \rightarrow \mathfrak{g}$.

We can also consider the universal enveloping algebra $U(\mathfrak{g})$, and the related object. $U_q(\mathfrak{g})$ We have an R -matrix $R_{V,W}$ for the representations $V \otimes W$ and $W \otimes V$, and we have the relation

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

in $V_1 \otimes V_2 \otimes V_3$. A main goal later in the course will be to relate the representations of $U_q(\mathfrak{g})$ and $\widehat{\mathfrak{g}}$.

In this case, we have the diagram:



The object $U_q(\widehat{\mathfrak{g}})$ is related to quantum integrable models of spin chain type (XXX and XXZ), and $E_{q,\tau}(\mathfrak{g})$ is the *elliptic quantum group* (XYZ).

1.2 Representations of Semisimple Lie Algebras

Remark. Recall the *Weyl group* $W = \{s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$. The *weight lattice* is

$$P = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in R\} = \bigoplus_i \mathbb{Z}\omega_i,$$

where ω_i are the fundamental weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

We can consider the *highest weight representation*. The *Verma module* is $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda$, where \mathbb{C}_λ is the 1-dimensional representation of $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ on which \mathfrak{h} acts by $\lambda(h)$. Then

$$P(M_\lambda) = \lambda - \mathbb{Q}_+,$$

and for each $\lambda \in \mathfrak{h}^*$, M_λ has a unique irreducible quotient L_λ . The *dominant integral weights* λ satisfy

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+, \quad 1 \leq i \leq r,$$

where $\lambda = \sum_{i=1}^r n_i \omega_i$ with $n_i \in \mathbb{Z}_+$.

Theorem 1.1. *The finite-dimensional irreps of \mathfrak{g} are classified up to isomorphism by $\lambda \in P_+$. Moreover, $P(V)$ is Weyl invariant, and for any $\mu \in P(V)$, $w \in W$,*

$$\dim L_\lambda[\mu] = \dim L_\lambda[w\mu].$$

Example 1.0.1. For $\mathfrak{g} = \mathfrak{sl}_2$, the dominant integral weights are $n \in \mathbb{Z}_{\geq 0}$, $L_n = V_n$, and the Weyl group W acts by reflection.

Remark (Weyl character formula). Let $\chi_V(g) = \text{tr}_V(g)$. We can represent $g \sim e^h$, where $h \in \mathfrak{h}$. Then

$$\chi_V(e^h) = \sum_{\mu \in P} (\dim V(\mu)) e^{\mu(h)}.$$

We can then formally define $\chi_V = \sum_{\mu \in P} (\dim V(\mu)) e^\mu$. The *Weyl character formula* is

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^\ell(w) e^{w(\lambda + \rho)}}{\Delta},$$

where $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w) w \rho}$ is the *Weyl denominator*. Here $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \omega_i$. The *Weyl dimension formula* is then

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

Recall the *Casimir operator* $\sum_{i=1}^{\dim \mathfrak{g}} x_i x^i \in U(\mathfrak{g})$, which acts by the scalar $(\lambda, \lambda + 2\rho)$.

1.3 Representations of SL_n and GL_n

Proposition 1.1. *For general simple \mathfrak{g} , let $\lambda = \sum_{i=1}^r m_i \omega_i$ be a dominant integral weight. Let $T_\lambda = \bigotimes_i L_{\omega_i}^{\otimes m_i}$ and $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$. Let V be the subrepresentation of T_λ generated by v . Then $V \cong L_\lambda$.*

Remark. For \mathfrak{sl}_n , we have $\lambda = \sum_{i=1}^{n-1} m_i \omega_i$. The Cartan subalgebra is

$$\mathfrak{h} = \mathbb{C}_0^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_1 + \dots + x_n = 0\}.$$

We have $\alpha_i^\vee = e_i - e_{i-1}$ and $\delta_{i,j} = (\omega_i, \alpha_j^\vee) = (\omega_i, e_j - e_{j+1})$, where $\omega_i = (1, \dots, 1, 0, \dots, 0)$ with i ones. We can associate λ with the partition

$$\lambda = (m_1 + \dots + m_{n-1}, m_2 + \dots + m_{n-1}, \dots, m_{n-1}, 0) = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0),$$

and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1}$. Note that L_{ω_1} is the defining representation, where $v_{\omega_1} = (1, 0, \dots, 0)^T = v_1$, where $\{v_1, \dots, v_n\}$ is a basis of the defining representation. Then we have that $L_{\omega_m} = \wedge^m V$ with highest weight $v_1 \wedge \dots \wedge v_m$. Here $e_i = E_{i,i+1}$. Then we see that $L_\lambda \subseteq \bigotimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$.

Remark. To move to GL_n , we can write

$$\mathrm{GL}_n(\mathbb{C}) = (\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})) / \mu_n,$$

where μ_n are the roots of unity embedded by $z \mapsto (z^{-1}, zI)$. We have a covering homomorphism

$$\begin{aligned} \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ (z, A) &\longmapsto zA. \end{aligned}$$

We need to determine the holomorphic representations of \mathbb{C}^\times . Its Lie algebra is spanned by h such that $e^{2\pi i h} = 1$. Within a representation, h acts by an operator H such that $e^{2\pi i H} = 1$. Thus all irreducible representations of \mathbb{C}^\times are of the form $\chi_N(z) = z^N$. So for $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$, we have $L_{\lambda, N} = \chi_N \otimes L_\lambda$.

Exercise 1.1. Show that if $L_{\lambda, N} = \chi_N \otimes L_\lambda$, then $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r .

Remark. Letting $m_n = r \geq 0$ in the above exercise, the representation $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ for \mathfrak{gl}_n corresponds to the partition $(m_1 + \dots + m_n, \dots, m_{n-1} + m_n, m_n)$.

Remark. For SL_n , the representation $\wedge^n V$ is trivial, but it is the determinant for GL_n . For GL_n , we also have χ^k and $(\chi^*)^k = \chi^{-k}$, these are called the *polynomial representations*.

Remark. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n$ be a partition with at most n parts. Then $|\lambda| = \sum_i \lambda_i$ is an eigenvalue of $1_n = \sum_{i=1}^n e_{i,i} \in \mathfrak{gl}_n$. We can realize λ as a Young diagram. Note that L_λ occurs in $V^{\otimes N}$, where V is the defining representation. We can decompose

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$. There is a natural action of S_N on $V^{\otimes N}$.

Theorem 1.2 (Schur-Weyl duality). *Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}(V^{\otimes N})$ and B be the image of $\mathbb{C}S_N$ in $\mathrm{End}(V^{\otimes N})$. Then*

1. *the centralizer of A is B and vice versa;*
2. *if λ has at most n parts, then the representation π_λ of B (and hence of S_N) is irreducible, and such representations are pairwise non-isomorphic;*
3. *if $\dim V \geq N$, then the π_λ exhaust all irreducible representations of S_N .*

Lecture 2

Jan. 14 — Applications of Schur-Weyl Duality

2.1 The Schur Functor

Remark. Let V be the defining representation for GL_n . Then

$$V^{\otimes N} = \bigoplus_{\lambda: |\lambda|=N} L_\lambda \otimes \pi_\lambda.$$

Recall that if $\lambda = (\lambda_1, \dots, \lambda_n)$, then we have

$$\lambda_1 = m_1 + \dots + m_n, \quad \lambda_2 = m_2 + \dots + m_n, \quad \dots, \quad \lambda_n = m_n.$$

Definition 2.1. Suppose we are given the partition λ of N . The *Schur functor* S^λ is given by

$$S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N})$$

for a vector space V . Note that this language, we have $V^{\otimes N} = \bigoplus_\lambda S^\lambda V \otimes \pi_\lambda$.

Example 2.1.1. Consider the following:

1. $S^{(n)}V = S^n V$, where (n) is the partition of n with a single part.
2. $S^{(1^n)}V = \wedge^n V$, where (1^n) is the partition of n with n parts equal to 1.
3. $V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_-$, where \mathbb{C}_2 acts trivially on \mathbb{C}_+ and by the sign on \mathbb{C}_- .
4. $V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$, where S_3 acts trivially on \mathbb{C}_+ and by sign on \mathbb{C}_- as before, and $\mathbb{C}^2 = \{(x, y, z) : x + y + z = 0\}$.

Note that $V \otimes V = S^2 V \oplus \wedge^2 V$, so $S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V$ and $\wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V$.

Remark. Let $\dim V = N$ and λ have k parts. Recall that by the Weyl dimension formula,

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}.$$

We have $R_+ = \{\alpha_{i,j} = e_i - e_j : i < j\}$ and $\rho = \sum_{i=1}^{N-1} \omega_i = (N-1, N-2, \dots, 1, 0)$ (recall that ω_i is i ones followed by zeros). Thus we see that

$$\dim S^\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i < k < j \leq N} \frac{\lambda_i + j - i}{j - i}.$$

We can rewrite the second product as

$$\prod_{1 \leq i < k < j \leq N} \frac{\lambda_i + j - i}{j - i} = \prod_{i=1}^k \frac{(N+1-i) \cdots (N+\lambda_i-i)}{(k+1-i) \cdots (k+\lambda_i-i)}.$$

Proposition 2.1. *We have $\dim S^\lambda V = P_\lambda(N)$, where P_λ is a polynomial of degree $|\lambda|$ with rational coefficients and integer roots. The roots of P_λ are all integers from the interval $[1 - \lambda_1, k - 1]$ (occurring with multiplicities).*

Example 2.1.2. Let $P_n(N)$ correspond to $S^n V$. Then $\lambda_1 = n$ and $k = 1$, and

$$P_n(N) = \dim S^n V = \binom{N+n-1}{n}.$$

Similarly, one can see that

$$P_{1^n}(N) = \dim \wedge^n V = \binom{N}{n}.$$

One can also consider $P_{(a,b)}(N)$ corresponding to partitions with two parts. The values $P_{(a,n)}(N)$ are called the Narayana numbers, which are of use in combinatorics.

2.2 Invariant Theory

Remark. Let V be a finite-dimensional vector space and $\{T_i\} \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ for $i = 1, \dots, k$. One would like to characterize *invariants* of such collections, i.e. polynomial functions $F(T_1, \dots, T_k)$ which are invariant under the action of $\mathrm{GL}(V)$.

One can think of such a tensor in $(V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$ as a vertex with m_i incoming edges and n_i outgoing edges. Then constructing invariants $\{T_i\}$ reduces to studying graphs where T_i corresponds to a vertex v_i of the graph Γ . This allows us to assign to a given graph an invariant function F_Γ .

Theorem 2.1. *The functions F_Γ for various Γ span the space of invariant functions.*

Proof. We can view an invariant as an invariant element of the space $\bigotimes_{i=1}^k ((V^*)^{\otimes m_i} \otimes V^{\otimes n_i})$, which we can view as $\mathrm{End}_{\mathrm{GL}(n)}(V^{\otimes M}, V^{\otimes N})$, where $M = \sum d_i m_i$ (the number of incoming edges) and $N = \sum d_i n_i$ (the number of outgoing edges). Note that this space is empty when $M \neq N$, and the statement follows by Schur-Weyl duality when $M = N$. \square

Example 2.1.3. Let $m_i = n_i = 1$. Then T_1, \dots, T_k are matrices. Then the graph Γ must look like a cycle, hence the invariants are all of the form

$$F_{j_1, \dots, j_r}(T_1, \dots, T_k) = \mathrm{tr}(T_{j_1} \cdots T_{j_r}).$$

Note that these invariants are asymptotically algebraically independent (when V is large enough). In particular, if $P(T_1, \dots, T_k) = 0$ in all dimensions, then $\mathrm{tr}(P(T_1, \dots, T_k) T_{k+1}) = 0$, which cannot be true as the trace decomposes in terms of the F_{j_1, \dots, j_r} . (However, note that $[X, Y] = 0$ for 1×1 matrices and $[Z, [X, Y]^2] = 0$ for 2×2 matrices.) This also implies the uniqueness of the μ_n in the BCH formula:

$$\log(\exp(x) \exp(y)) = \sum_{n \geq 1} \frac{\mu_n(x, y)}{n!}.$$

2.3 Weyl Character Formula for GL_n

Remark (Weyl character formula for GL_n). Recall that Weyl's character formula gives

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\sum_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}, \quad (*)$$

where the denominator is $\Delta = \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}) = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$. Letting $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$,

$$\Delta = e^\rho \prod_{\alpha \in R_+} (1 - e^{-\alpha}) = x_1^{n-1} x_2^{n-2} \cdots x_n^0 \prod_{i < j} (1 - x_j/x_i),$$

where $\rho = (n-1, n-2, \dots, 1, 0)$ and $x_i = e^{e_i}$ (e.g. $x_1 = e^{(1,0,\dots,0)}$). After multiplying we get that

$$\Delta = \prod_{i < j} (x_i - x_j).$$

On the other hand, using $\Delta = \prod_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$, we have

$$\Delta = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0.$$

Comparing these two formulas, we recover the formula for the Vandermonde determinant:

$$\det(\{x_j^{n-i}\}_{1 \leq i, j \leq n}) = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{n-1} \cdots x_{s(n)}^0 = \prod_{i < j} (x_i - x_j).$$

Now applying this to the numerator of (*), we have

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)} = \sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}.$$

Thus in total, the character χ_λ is given by

$$\chi_\lambda = \frac{\sum_{s \in S_n} \text{sign}(s) x_{s(1)}^{\lambda_1 + n - 1} \cdots x_{s(n)}^{\lambda_n + 0}}{\prod_{i < j} (x_i - x_j)} = \frac{\det(\{x_i^{\lambda_j + n - i}\})}{\prod_{i < j} (x_i - x_j)}.$$

These functions are known as the *Schur polynomials* $s_\lambda(x_1, \dots, x_n)$.

Example 2.1.4 (Character of $S^{(n)}V$). Using the above formula, we get the identity

$$s_{(m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} x_{j_1} \cdots x_{j_m} = h_m(x_1, \dots, x_m),$$

the m th complete symmetric function.

Example 2.1.5 (Character of $\lambda^n V$). Similarly, one gets the identity

$$s_{(1^m)}(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} x_{j_1} \cdots x_{j_m} = e_m(x_1, \dots, x_m),$$

the m th elementary symmetric function.

Example 2.1.6 (Trace in $V^{\otimes N}$). Consider $x \otimes \sigma$, where $x = \text{diag}(x_1, \dots, x_n)$ and σ has m_i cycles of length i . Then we have

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

By Schur-Weyl duality, we have that

$$\text{tr}_{V^{\otimes N}}(x \otimes \sigma) = \sum_{\lambda} \chi_{\lambda}(\sigma) s_{\lambda}(x) = \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Using the formula for the Schur polynomial, we get the identity

$$\sum_{\lambda} \chi_{\lambda}(\sigma) \det(\{x_i^{\lambda_j + N - j}\}) = \prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

Theorem 2.2 (Frobenius character formula). $\chi_{\lambda}(\sigma)$ is the coefficient of $x_1^{\lambda_1 + N - 1} \cdots x_N^{\lambda_N}$ in the polynomial

$$\prod_{i < j} (x_i - x_j) \prod_i (x_1^i + \cdots + x_n^i)^{m_i}.$$

2.4 Howe Duality

Remark. Fix V, W and consider $S^n(V \otimes W)$, which is a representation of $\text{GL}(V) \otimes \text{GL}(W)$.

Theorem 2.3 (Howe duality). We have a decomposition

$$S^n(V \otimes W) = \bigoplus_{\lambda: |\lambda|=n} S^{\lambda} V \otimes S^{\lambda} W.$$

Proof. We can write

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}.$$

Using Schur-Weyl duality for each part, we get that

$$\begin{aligned} S^n(V \otimes W) &= \left(\left(\bigoplus_{\lambda: |\lambda|=n} S^{\lambda} V \otimes \pi_{\lambda} \right) \otimes \left(\bigoplus_{\mu: |\mu|=n} S^{\mu} W \otimes \pi_{\mu} \right) \right)^{S_n} \\ &= \bigoplus_{\lambda, \mu: |\lambda|=|\mu|=n} S^{\lambda} V \otimes S^{\mu} W \otimes (\pi_{\lambda} \otimes \pi_{\mu})^{S_n}. \end{aligned}$$

Since $\pi_{\lambda} = \pi_{\lambda}^*$, by Schur's lemma we have $(\pi_{\lambda} \otimes \pi_{\mu})^{S_n} = \mathbb{C}^{\delta_{\lambda, \mu}}$. □

Corollary 2.3.1 (Cauchy identity). Let $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$. Then

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - zx_i y_j}.$$

Lecture 3

Jan. 21 — Minuscule Weights

3.1 Minuscule Weights

Remark. Let \mathfrak{g} be a simple complex Lie algebra.

Definition 3.1. A dominant integral weight ω for \mathfrak{g} is called *minuscule* if $\langle \omega, \beta \rangle \leq 1$ for every positive coroot β (equivalently, if $|\langle \omega, \alpha \rangle| \leq 1$ for any coroot β).

Example 3.1.1. Clearly $\omega = 0$ is minuscule.

Example 3.1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$ with fundamental weights $\{\omega_i\}_{i=1}^{n-1}$,¹ where

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0)$$

Let $\alpha_{i,j} = \alpha_{i,j}^\vee = e_i - e_j$. Note that $\langle \omega_i, e_j - e_k \rangle = 0$ when $j, k \leq i$ or $j, k > i$, and $\langle \omega_i, e_j - e_k \rangle = 1$ when $j \leq i < k$. So all of the ω_i are minuscule in this case.

Lemma 3.1. Every nonzero minuscule weight is fundamental.

Proof. Suppose ω is minuscule. Then there exists i with $\langle \omega, \alpha_i^\vee \rangle = 1$. Moreover, there can only be one such i , since if there were many, then $\langle \omega, \theta^\vee \rangle \geq 2$, where θ^\vee is the longest coroot (i.e. if $\theta = \sum_{m_i > 0} m_i \alpha_i$ is the longest root, then $\theta^\vee = \sum_{m_i > 0} m_i \alpha_i^\vee$). So ω is necessarily fundamental. \square

Example 3.1.3. For G_2 , F_4 , and F_8 , none of the fundamental weights are minuscule.

Lemma 3.2. A fundamental weight ω_i is minuscule if and only if $m_i = 1$ where $\theta^\vee = \sum_j m_j \alpha_j^\vee$.

Proof. By the minuscule condition, we know $m_i \leq 1$. If $m_i = 1$, then for any positive coroot $\beta = \sum n_j \alpha_j^\vee$ we have $n_j \leq m_j$, so $n_i \leq 1$. Thus $\langle \omega_i, \beta \rangle = n_i \leq 1$, so ω_i is minuscule. \square

Lemma 3.3. If $\omega \in Q$ with $|\langle \omega, \beta \rangle| \leq 1$ for all coroots β , then $\omega = 0$.

Proof. Assume to the contrary that $\omega = \sum_i m_i \alpha_i \neq 0$. We may assume that $\sum_i |m_i|$ is smallest possible. Then $0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i)$, since the form is positive definite. Thus there exists j such that m_j and $\langle \omega, \alpha_j^\vee \rangle$ have the same sign. By replacing ω with $-\omega$ if necessary, we may assume both are positive. Then $\langle \omega, \alpha_j^\vee \rangle = 1$. Consider the reflection $s_j(\omega) = \omega - \alpha_j = \sum_i m'_i \alpha_i$. So $m'_i = m_j - 1$ and $m'_i = m_i$. But then $\sum_i |m'_i| = \sum_i |m_i| - 1 < \sum_i |m_i|$, contradicting the minimality of ω . \square

¹Recall a *fundamental weight* is a weight ω_i such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for all simple coroots α_j^\vee .

Proposition 3.1. *The following conditions are equivalent:*

1. ω is minuscule;
2. all weights of L_ω belong to the Weyl orbit $W\omega$;
3. if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$, then $\lambda = \omega$.

Proof. (1 \Rightarrow 3) If $\omega = 0$, then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$ where $\rho = \sum_{i=1}^r \omega_i$, so $\lambda = 0$. Now let $\omega = \omega_i$ be minuscule. Then $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for $k \neq i$, then the problem reduces to a lower rank Dynkin diagram. So we can assume $m_k > 0$ for every $k \neq i$. Let β be a positive coroot, then

$$\langle \omega_i - \lambda, \beta \rangle = \langle \omega_i, \beta \rangle - \langle \lambda, \beta \rangle \leq \langle \omega_i, \beta \rangle \leq 1.$$

If α_j^\vee does not occur in β , then the above is ≤ 0 . In particular, we have $\langle \omega_i - \lambda, \alpha_j^\vee \rangle \leq 0$ for $j \neq i$. If we also have $\langle \omega_i - \lambda, \alpha_i^\vee \rangle \leq 0$, then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$. Otherwise, $\langle \omega_i - \lambda, \alpha_i^\vee \rangle = 1$. Then $m_j > 0$ for every j , so $\langle \omega_i - \lambda, \theta^\vee \rangle \geq 1$, since θ^\vee is a dominant coweight. Then $\langle \lambda, \theta^\vee \rangle \leq 0$, so we must have $\lambda = 0$ since θ^\vee contains all α_j^\vee with positive coefficients. But then $\omega_i \in Q$, which is impossible by Lemma 3.3.

(3 \Rightarrow 2) If μ is any weight of L_ω , then there exists $w \in W$ such that $\lambda = w\mu$ is dominant (since every orbit of W intersects the dominant chamber at exactly 1 point). Then $\omega - \lambda \in Q_+$, so $\lambda = \omega$, hence $\mu = w^{-1}\omega \in W\omega$.

(2 \Rightarrow 1) Suppose otherwise ω is not minuscule. Then $\langle \omega, \alpha^\vee \rangle > 1$ for some positive coroot α^\vee . Then

$$2(\omega, \alpha) > (\alpha, \alpha).$$

Note that $\omega - \alpha$ is a weight of L_ω (weight of $f_\alpha v_\omega$, where v_ω is a highest weight vector and $\{e_\alpha, f_\alpha, \alpha^\vee\}$ is an \mathfrak{sl}_2 -triple). But $\omega - \alpha$ is not W -conjugate to ω , since

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega)$$

but the pairing is W -invariant. Contradiction. □

Corollary 3.0.1. *If ω is minuscule, then $\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma$.*

3.2 Applications of Minuscule Weights

Proposition 3.2. $\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root \mathfrak{sl}_2 -subalgebra of \mathfrak{g} is the direct sum of 1-dimensional and 2-dimensional representations.

Proof. (\Rightarrow) Let ω be minuscule and $v \in L_\omega$ the highest weight vector (of weight $w\omega$) for $(\mathfrak{sl}_2)_\alpha$. Then

$$h_\alpha v = \langle w\omega, \alpha^\vee \rangle v = \langle \omega, w^{-1}\alpha^\vee \rangle v.$$

Then $h_\alpha v = 0$ or $h_\alpha v = v$, so the representation is 1-dimensional or 2-dimensional.

(\Leftarrow) Suppose ω is not minuscule. Then there exists $\alpha \in Q_+$ with $\langle \omega, \alpha^\vee \rangle = m > 1$. Let v_ω be a highest weight vector, then $h_\alpha v_\omega = \langle \omega, \alpha^\vee \rangle v_\omega$, which leads to a higher-dimensional \mathfrak{sl}_2 -representation. □

Corollary 3.0.2. *If ω is minuscule, then for every dominant integral weight λ of \mathfrak{g} , we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma}.$$

(It is assumed that if $\lambda + \gamma$ is not dominant, then $L_{\lambda+\gamma} = 0$.)

Proof. We know $\chi_\omega = \sum_{\mu \in W\omega} e^\mu$. Then we have

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\rho)+\mu}}{\Delta} = \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(\omega)} e^{w(\lambda+\gamma+\rho)}}{\Delta}$$

where Δ is the Weyl denominator. If $\lambda + \gamma \notin P_+$, then for some α_i^\vee , we get $\langle \lambda + \gamma, \alpha_i^\vee \rangle < 0$. But we know $\langle \gamma, \alpha_i^\vee \rangle \geq -1$, so $\langle \lambda + \gamma, \alpha_i^\vee \rangle = -1$. Thus $\langle \lambda + \gamma + \rho, \alpha_i^\vee \rangle = 0$, so for any $w\gamma$, the term $ws_i\gamma$ comes with the opposite sign. So we get that

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\Delta} = \sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \chi_{\lambda+\gamma},$$

which proves the desired result. \square

Example 3.1.4. For \mathfrak{sl}_2 , we have $L_1 \otimes L_m = L_{m+1} \oplus L_{m-1}$, which leads to the formula

$$L_m \otimes L_n = \bigoplus_{k=|m-n|}^{m+n} L_k$$

Example 3.1.5. Let $V = V_{\omega_1}$ be the defining representation for GL_n . Then

$$L_{\omega_1} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu,$$

where λ is a partition and $\lambda + \square$ denotes the set of partitions obtained by adding a single box to λ . For example, for $\lambda = (3, 3, 2, 1)$ we have

$$L_{\omega_1} \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

Similarly, for $\wedge^m V = L_{\omega_m}$ (where $\omega_m = (1, \dots, 1, 0, \dots, 0)$ with m ones), we have

$$L_{\omega_m} \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\square} L_\mu,$$

where we are allowed to add m boxes to λ in $\lambda + m\square$. For example,

$$\wedge^2 V \otimes S^{(3,1)}V = S^{(4,2)}V \oplus S^{(4,1,1)}V \oplus S^{(3,2,1)}V \oplus S^{(3,1,1,1)}V.$$

Lecture 4

Jan. 26 — Other Classical Lie Algebras

4.1 Applications of Minuscule Weights, Continued

Proposition 4.1. *We have the following:*

1. Let λ be a partition of N . Then $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.
2. Let μ be a partition of $N+1$. Then $\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \square} \pi_\lambda$.

Proof. (1) Let V be a vector space of sufficiently large dimension. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda, V^{\otimes(N+1)}) \cong \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes N} \otimes V) = V \otimes S^\lambda V.$$

Now by Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}\left(\bigoplus_{\mu \in \lambda + \square} \pi_\mu, V^{\otimes(N+1)}\right) = \bigoplus_{\mu \in \lambda + \square} S^\mu V.$$

Since $V \otimes S^\lambda V = \bigoplus_{\mu \in \lambda + \square} S^\mu V$, we conclude that $\mathbb{C}S_{N+1} \otimes_{S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \square} \pi_\mu$.

(2) This is left as an exercise. Use a different version of Frobenius reciprocity. \square

Definition 4.1. Let λ be a partition, and λ^\dagger be the *conjugate partition* (the one corresponding to the transposed diagram). For example, $(3, 3, 2, 1)^\dagger = (4, 3, 2)$.

Corollary 4.0.1. *Let \mathbb{C}_- be the sign representation of S_N . Then $\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}$.*

Proof. This is left as an exercise. The proof is by induction on $N = |\lambda|$. Let $C = \sum_{i < j} (i \ j)$, and note that its eigenvalues are the same as the Casimir operator of SL_N . \square

Proposition 4.2 (Skew Howe duality). *We have a decomposition $\wedge^n(V \otimes W) = \bigoplus_\lambda S^\lambda V \otimes S^{\lambda^\dagger} W$ (as $\mathrm{GL}(V) \otimes \mathrm{GL}(W)$ -modules).*

Proposition 4.3. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights, so the number of minuscule weights is equal to $\det A$, where A is the Cartan matrix.*

Proof. Let $C = a + Q \in P/Q$ be a coset. Let $\omega \in C \cap P_+$ be the element which minimizes $\langle \omega, \rho^\vee \rangle$. If λ is the dominant weight for L_ω , then $\lambda \in C \cap P_+$ implies that

$$(\lambda, \rho^\vee) \geq (\omega, \rho^\vee).$$

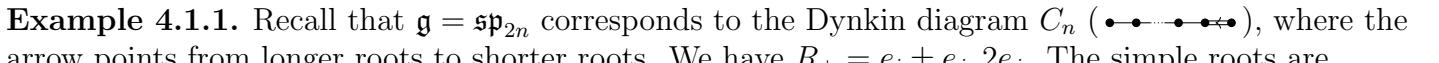
Thus $(\omega - \lambda, \rho^\vee) \leq 0$, so $\omega - \lambda \in Q_+$. Thus $\lambda = \omega$, so ω is minuscule. Now suppose $\omega_1, \omega_2 \in C$ are minuscule and $\omega_1 \neq \omega_2$ with $\omega_1 - \omega_2 \in Q$. By Lemma 3.3, we must have $\langle \omega_1 - \omega_2, \beta \rangle \geq 2$ for all coroots β . But then $\langle \omega_1, \beta \rangle = 1$ (which implies $\beta > 0$) and $\langle \omega_2, \beta \rangle = -1$ (which implies $\beta < 0$), a contradiction. \square

Remark. Let A be the Cartan matrix. For every root, we can write

$$\alpha_i = \sum_{j=1}^r A_{i,j} \omega_j.$$

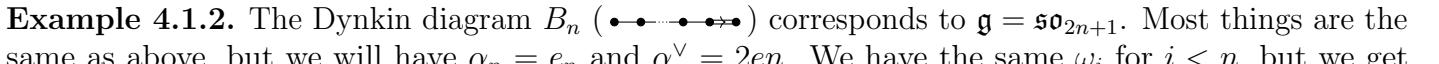
We have a covering map $\mathbb{R}^r/\Lambda_2 \rightarrow \mathbb{R}^r/\Lambda_1$, where $\Lambda_2 = P$ and $\Lambda_1 = Q$. Then $\det A$ is precisely the degree of this covering, which counts the number of cosets.

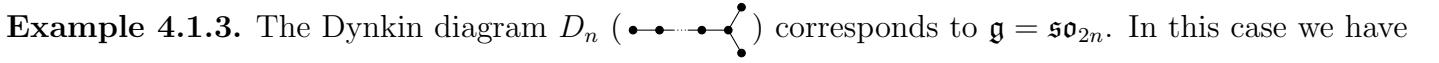
4.2 Other Classical Lie Algebras

Example 4.1.1. Recall that $\mathfrak{g} = \mathfrak{sp}_{2n}$ corresponds to the Dynkin diagram C_n (), where the arrow points from longer roots to shorter roots. We have $R_+ = e_i \pm e_j, 2e_j$. The simple roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n.$$

We have $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = e_n$, and $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $1 \leq i \leq n$.

Example 4.1.2. The Dynkin diagram B_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Most things are the same as above, but we will have $\alpha_n = e_n$ and $\alpha_n^\vee = 2e_n$. We have the same ω_i for $i < n$, but we get $\omega_n = (1/2, \dots, 1/2)$. We have $R_+ = e_i \pm e_j, e_i$.

Example 4.1.3. The Dynkin diagram D_n () corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. In this case we have $R_+ = e_i \pm e_j$, and simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-2} = e_{n-1}, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n.$$

We have $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (with i ones) for $i = 1, \dots, n-2$, but we get $\omega_{n-1} = (1/2, \dots, 1/2, 1/2)$ and $\omega_n = (1/2, \dots, 1/2, -1/2)$.

Remark. We have the following:

- For G_2, F_4, F_8 , we have $\det A = 1$ (here A is the Cartan matrix), so the only minuscule weight is 0.
- For B_n , we have $\det A = 2$ (the nontrivial minuscule weight is $(1/2, \dots, 1/2)$, and the representation has weights $(\pm 1/2, \dots, \pm 1/2)$ with all possible combinations of \pm and dimension 2^n).
- For D_n , we have $\det A = 4$. The minuscule weights are $\omega_1, \omega_{n-1}, \omega_n$. Here ω_1 is the $2n$ -dimensional defining representation. The other two are spin representations of dimension 2^{n-1} , with weights $(\pm 1/2, \dots, \pm 1/2)$, taking even or odd numbers of $-$ signs.

4.3 Representations of Symplectic Lie Algebras

Remark. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have the Dynkin diagram C_n and

$$\omega_i = (\underbrace{1, \dots, 1}_i \text{ ones}, 0, \dots, 0).$$

The elements of the Cartan subalgebra are given by $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. So $L_{\omega_1} = V$ (the defining representation) with highest weight e_1 . Note that $\wedge^2 V$ is not irreducible:

$$\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C},$$

where \mathbb{C} is the trivial representation spanned by $B^{-1} = \sum_i e_{i+n} \wedge e_i$ (note that B^{-1} is invariant under \mathfrak{sp}_{2n}). However, one can check that $\wedge_0^2 V$ is irreducible.

Now let us consider L_{ω_j} for $j \geq 2$. Let $B = \sum_i e_i^* \wedge e_{i+n}^*$. We have an operator

$$i_B : \wedge^{i+1} V \longrightarrow \wedge^{i-1} V,$$

and we can denote $\wedge_0^i V = \ker(i_B|_{\wedge^i V})$ (note that $i_B|_{\wedge^i V}$ is injective when $i \geq n$). The $\wedge_0^i V$ are irreducible for $i \leq n$, and one can check that these form all of the irreducible representations of \mathfrak{sp}_{2n} (compute their dimensions and compare them to the highest weight representations).

We can also define an operator

$$\begin{aligned} m_B : \wedge^{i-1} V &\longrightarrow \wedge^{i+1} V \\ u &\mapsto B^{-1} \wedge u. \end{aligned}$$

One can check that m_B and i_B together with h (acting as $i - n$ on $\wedge^i V$) form an \mathfrak{sl}_2 -triple. Then

$$\wedge V = \bigoplus_{i=0}^n L_{\omega_i} \otimes L_{n-j}$$

(where $\omega_0 = 0$ and L_{n-j} is the representation of \mathfrak{sl}_2 of weight $n - j$) as representations of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$.

4.4 Representations of Orthogonal Lie Algebras

Remark. First consider B_n , which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$. In this case, the Cartan subalgebra is given by elements of the form $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n, 0)$. Let V be the $(2n+1)$ -dimensional defining representation. Then for $1 \leq i \leq n-1$, the representation $\wedge^i V$ is irreducible (one can check this using the dimension formula) with highest weight

$$\omega_i = (\underbrace{1, \dots, 1}_{i \text{ ones}}, 0, \dots, 0).$$

On the other hand, $\wedge^n V$ is irreducible but not fundamental, with highest weight $(1, \dots, 1) = 2\omega_n$.

Now we consider the spin representation S (whose elements are called *spinors*). It has weights

$$(\pm 1/2, \pm 1/2, \dots, \pm 1/2)$$

(all possible combinations of \pm). The character of S is given by

$$\chi_S(x_1, \dots, x_n) = (x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}).$$

Remark. We will want to look at the Lie group $\text{Spin}_{2n+1}(\mathbb{C})$, the universal cover of $\text{SO}_{2n+1}(\mathbb{C})$. For $n = 1$, we have $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$. We will see that S is 2-dimensional, and $\pi_1(\text{SO}_3(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.

Lecture 5

Jan. 28 — Other Classical Lie Algebras, Part 2

5.1 More on Orthogonal Lie Algebras

Proposition 5.1. *For $n \geq 3$, we have $\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. There is a deformation retract from the surface X_n defined by $z_1^2 + \cdots + z_n^2 = 1$ in \mathbb{C}^n to the sphere $X_n^{\mathbb{R}} = X_n \cap \mathbb{R}^n$ defined by $x_1^2 + \cdots + x_n^2 = 1$ in \mathbb{R}^n : Let $\vec{z} = \vec{x} + i\vec{y} \in X_n$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, and note that $|\vec{z}|^2 = 1$ if and only if $|\vec{x}|^2 - |\vec{y}|^2 = 1$ and $\vec{x} \cdot \vec{y} = 0$. We also have

$$(\vec{x} + t i \vec{y})^2 = |\vec{x}|^2 - t^2 |\vec{y}|^2 = 1 + (1 - t^2) |\vec{y}|^2 \geq 1.$$

So we can define a homotopy $f_t : X_n \rightarrow X_n$ by

$$f_t(\vec{z}) = \frac{\vec{x} + t i \vec{y}}{\sqrt{|\vec{x}|^2 - t^2 |\vec{y}|^2}},$$

which satisfies $|f_t(z)|^2 = 1$, $f_1(z) = z$, and $f_0(z) \in X_n^{\mathbb{R}}$. Now observe that SO_n acts on X_n with fibers isomorphic to SO_{n-1} , so we have a long exact sequence

$$\pi_2(X_n) \longrightarrow \pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \longrightarrow \pi_1(\mathrm{SO}_n(\mathbb{C})) \longrightarrow \pi_1(X_n).$$

The first and last groups are trivial for $n \geq 4$, so we have that $\pi_1(\mathrm{SO}_{n-1}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_n(\mathbb{C}))$. Thus the result follows once one checks that $\pi_1(\mathrm{SO}_3(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$ (left as an exercise). \square

Remark. Now consider D_n , which corresponds to $\mathfrak{g} = \mathfrak{so}_{2n}$. Let $Q = \sum_{i=1}^n x_i x_{i+n}$. The elements of the Cartan subalgebra are given by $\text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. Let V be the $2n$ -dimensional defining representation, and consider $\wedge^i V$ for $1 \leq i \leq n$. We have $\wedge^i V$ is irreducible for $0 \leq i \leq n-1$, and $L_{\omega_i} = \wedge^i V$ for $1 \leq i \leq n-2$. Note that $L_{(1, \dots, 1, 0)}$ is irreducible but not fundamental. Letting

$$\omega_{n-1} = (1/2, \dots, 1/2, 1/2) \quad \text{and} \quad (1/2, \dots, 1/2, -1/2),$$

the corresponding $S_+ = L_{\omega_{n-1}}$ and $S_- = L_{\omega_n}$ are the spin representations. The characters are

$$\chi_{S_{\pm}} = ((x_1^{1/2} + x_1^{-1/2}) \cdots (x_n^{1/2} + x_n^{-1/2}))_{\pm},$$

where the \pm denotes an even or odd number of $-$ signs.

Example 5.0.1. We have $\text{Spin}_4 = \text{SL}_2 \times \text{SL}_2$, where factors correspond to S_+ and S_- . We have $\text{Spin}_5 = \text{Sp}_4$, where S is the 4-dimensional defining representation, and $\text{SO}_5 = \text{Sp}_4 / \{\pm 1\}$. We have $\text{Spin}_6 = \text{SL}_4$, where S_+, S_- are the 4-dimensional defining representation and its dual, and $\text{SO}_6 = \text{SL}_4 / \{\pm 1\}$.

Example 5.0.2. Let V be a finite-dimensional vector space, and consider $SV = \mathbb{C}[x_1, \dots, x_n]$, where x_1, \dots, x_n is an orthonormal basis. Denote $R^2 = \sum_{i=1}^n x_i^2 = S^2V$ and $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$. Then:

1. Find a first-order differential operator making $\{R^2, \Delta, \cdot\}$ an \mathfrak{sl}_2 -triple. Make sure that it commutes with the $\text{SO}(V)$ action.
2. Let $H_m \subseteq S^m V$ be the subspace of harmonic polynomials. Then

$$SV = \bigoplus_{m=0}^{\infty} H_m \otimes W_m,$$

where $H_m = L_{m\omega_1}$ is the irreducible representation of $\text{SO}(V)$, and W_m is the Verma module for \mathfrak{sl}_2 of highest weight m .

5.2 Clifford Algebras

Definition 5.1. Let V be a finite-dimensional vector space (over \mathbb{C}) and (\cdot, \cdot) a non-degenerate inner product on V . Give an associative algebra structure to V by

$$v^2 = \frac{1}{2}(v, v).$$

Such an algebra is called a *Clifford algebra*, and is denoted by $\text{Cl}(V)$.

Corollary 5.0.1. $ab + ba = (a + b)^2 - a^2 - b^2 = (a, b)$.

Example 5.1.1. The operators $i\partial/\partial x_i$ and $dx_i \wedge \cdot$ define a Clifford algebra.

Example 5.1.2. Let $e^i e^j + e^j e^i = \delta_{i,j}$. Then $D = \sum_{i=1}^n e^i \partial_i$ (the *Dirac operator*) satisfies $D^2 = \Delta$.

Theorem 5.1. The algebra $\text{Cl}(V)$ is isomorphic to $\text{Mat}_{2^n}(\mathbb{C})$ if $\dim V = 2n$ and to $\text{Mat}_{2^n}(\mathbb{C}) \oplus \text{Mat}_{2^n}(\mathbb{C})$ if $\dim V = 2n + 1$.

Proof. First consider the even case. Choose a basis $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$(a_i, a_j) = (b_i, b_j) = 0, \quad (a_i, b_j) = \delta_{i,j}, \quad a_i a_j + a_j a_i = 0, \quad b_i b_j + b_j b_i = 0, \quad b_i a_i + a_i b_i = 1.$$

Consider $\text{Cl}(V)$ -module $M = \wedge(a_1, \dots, a_n)$ (note that $\dim M = 2^n$) with action defined by

$$\rho(a_i)w = a_i w \quad \text{and} \quad \rho(b_i)w = \frac{\partial w}{\partial a_i}.$$

We have the relations

$$1 = \left[a_i, \frac{\partial}{\partial a_i} \right] = a_i \frac{\partial}{\partial a_i} + \frac{\partial}{\partial a_i} a_i \quad \text{and} \quad a_j \frac{\partial}{\partial a_i} = -\frac{\partial}{\partial a_i} a_j$$

for $i \neq j$. Let $c_{I,J} = a_{i_1} \cdots a_{i_k} b_{j_1} \cdots b_{j_m}$ for $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_m\}$. Check as an exercise that the $c_{I,J}$ are linearly independent, then $\rho : \text{Cl}(V) \rightarrow \text{End}(M)$ is an isomorphism.

If $\dim V = 2n + 1$, then we can pick an extra element z satisfying

$$(z, a_i) = (z, b_i) = 0 \quad \text{and} \quad (z, z) = 2,$$

with relations $za_i + a_iz = zb_i + b_iz = 0$ and $z^2 = 1$. Then $zw = \pm(-1)^{\deg w}wz$ for $w \in M_{\pm}$. \square

Remark. There is an embedding $\mathfrak{so}(V) \rightarrow \text{Cl}(V)$. Define a map

$$\begin{aligned} \xi : \wedge^2 V = \mathfrak{so}(V) &\longrightarrow \text{Cl}(V) \\ a \wedge b &\longmapsto \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a, b). \end{aligned}$$

One can check that $[\xi(a \wedge b), \xi(c \wedge d)] = \xi([a \wedge b, c \wedge d])$, so ξ is a homomorphism of Lie algebras. We have ξ^*M for even dimensional V and ξ^*M_{\pm} for odd dimensional V , and

$$\rho_{\xi^*M}(a) = \rho_M(\xi(a))$$

gives ξ^*M the structure of an $\mathfrak{so}(V)$ -representation (and similarly for ξ^*M_{\pm}). Notice that χ^*M is reducible:

$$\xi^*M = (\xi^*M)_0 \oplus (\xi^*M)_1$$

as representations, where the first factor corresponds to even degree and the second to odd degree.

Example 5.1.3. We have the following:

1. $(\xi^*M)_0 \cong S_+$ and $(\xi^*M)_1 \cong S_-$ for even dimensional V .
2. If $\dim V$ is odd, then χ^*M_{\pm} are both isomorphic to S .

Lecture 6

Feb. 2 — Duals, Maximal Weights, Exponents

6.1 Dual Representations

Remark. Let L_λ be the irreducible representation of highest weight λ . What is the highest weight of the dual representation L_λ^* ? Let w_0 be the maximal element in W .

Proposition 6.1. *We have $L_\lambda^* = L_{-w_0(\lambda)}$.*

Proof. Since λ is the highest weight in L_λ , for every weight μ in L_λ we have $\lambda - \mu \in Q_+$. So

$$Q_- \ni w_0(\lambda - \mu) = w_0(\lambda) - w_0(\mu),$$

so $w_0(\mu) - w_0(\lambda) \in Q_+$. Thus $w_0(\lambda) \leq w_0(\mu)$ for all $\mu \in L_\lambda$, so the length of w_0 is $|R_+|$. Thus $-w_0(\lambda)$ is the lowest weight of L_λ , which is the highest weight of L_λ^* . \square

Example 6.0.1. Since the length of w_0 is $|R_+|$, w_0 permutes the fundamental (co)weights and (co)roots, so w_0 is an automorphism of Dynkin diagrams. Note that that W acts on P/Q , and w_0 acts as inversion.

- The Dynkin diagrams $A_1, B_n, C_n, G_2, F_4, E_7, E_8$ have no automorphisms, so $L_\lambda^* = L_\lambda$ for these.
- For A_n with $n \geq 2$, we have $P/Q = \mathbb{Z}/n\mathbb{Z}$ (e.g. if V is the defining representation, then we have that $L_{\omega_1}^* = V^* = \wedge^{n-1}V = L_{\omega_{n-1}}$).
- For E_6 , we have $P/Q = \mathbb{Z}/3\mathbb{Z}$, where w_0 exchanges the two minuscule weights.
- For D_{2n+1} , we have $P/Q = \mathbb{Z}/4\mathbb{Z}$ and $S_+^* = S_-$. For D_{2n} , $P/Q = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $S_\pm^* = S_\pm$.

6.2 Maximal Weights

Definition 6.1. Let *maximal weight* of \mathfrak{g} , denoted θ , is the highest weight of the adjoint representation.

Example 6.1.1. If $\mathfrak{g} = \mathfrak{sl}_n$, then θ is the highest weight for $V^* \otimes V$ where V is the defining representation. Note that $V^* = \wedge^{n-1}V$, so the highest weight of $V^* \otimes V$ is $\theta = \omega_1 + \omega_{n-1}$. It is not fundamental.

Example 6.1.2. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $\mathfrak{g} = S^2V$ where V is the defining representation for \mathfrak{sp}_{2n} . Then $\theta = 2\omega_1$, which is also not fundamental.

Proposition 6.2. *For a simple Lie algebra with $\mathfrak{g} \neq \mathfrak{sl}_n, \mathfrak{sp}_{2n}$, the maximal weight θ is fundamental.*

Example 6.1.3. For \mathfrak{so}_N with $N \geq 7$ (type B or D), we have $\mathfrak{g} = \wedge^2V = L_{\omega_2}$.

6.3 Principal \mathfrak{sl}_2 -Subalgebra and Exponents

Definition 6.2. Let \mathfrak{g} be a simple Lie algebra and $\{e_i, f_i, h_i\}$ (where $h_i = \alpha_i^\vee$) be Chevalley generators. Let $e = \sum_{i=1}^r e_i$, and h such that $\alpha_i(h) = 2$ for all i (so $h = 2\rho^\vee$). Note that we have $[h, e] = 2e$ and $h = \sum_{i=1}^r (2\rho^\vee, \omega_i) \alpha_i^\vee$. Let $f = \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i$. Then $\{h, e, f\}$ spans the *principal \mathfrak{sl}_2 -subalgebra* of \mathfrak{g} .

Example 6.2.1. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then the restriction of the defining representation to the principal \mathfrak{sl}_2 is L_n , the irreducible representation of \mathfrak{sl}_2 of highest weight n .

Remark. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, so that $\mathfrak{g} = \sum \mathfrak{g}[2m]$ where m is the height of the corresponding root subspace (and $2m$ is the weight with respect to h). Note $\mathfrak{g}[0] = \mathfrak{h}$ and $\dim \mathfrak{g}[0] = r$. Let $r_m = \dim \mathfrak{g}[2m]$.

Definition 6.3. We say that m is an *exponent* of \mathfrak{g} if $r_m > r_{m+1}$. The *multiplicity* of an exponent m is $r_m - r_{m+1}$.

Remark. We have $r_0 = r$ and there are r exponents (counted with multiplicities) $m_1 \leq m_2 \leq \dots \leq m_r$. The roots of height 2 are given by $\alpha_i + \alpha_j$ (where i, j are connected in the Dynkin diagram). So $r_0 = r_1 = 1$ and $r_2 = r - 1$. Thus $m_1 = 1$ and $m_2 > 1$. We have

$$m_r = (\rho^\vee, \theta) = h_{\mathfrak{g}} - 1,$$

where θ is the highest root. We call $h_{\mathfrak{g}}$ the *Coxeter number* of \mathfrak{g} . Note that $\sum_{i=1}^r m_i = |R_+|$.

Proposition 6.3. The restriction of \mathfrak{g} to its principal \mathfrak{sl}_2 -subalgebra decomposes as $\bigoplus_{i=1}^r L_{2m_i+1}$.

Example 6.3.1. The exponents for \mathfrak{sl}_n are $1, 2, \dots, n-1$.

Definition 6.4. The *Coxeter number* of \mathfrak{g} is $h_{\mathfrak{g}} = \langle \theta, \rho^\vee \rangle + 1 = m_r + 1$, and the *dual Coxeter number* is

$$h_{\mathfrak{g}}^\vee = \langle \tilde{\theta}^\vee, \rho \rangle + 1,$$

where $\tilde{\theta}^\vee = 2\theta/(\theta, \theta)$. If we normalize $(\theta, \theta) = 2$, then $h_{\mathfrak{g}}^\vee = \frac{1}{2}(\theta, \theta + 2\rho)$, which is the eigenvalue of $\frac{1}{2}C$ (where C is the Casimir operator).

6.4 Complex, Real, and Quaternionic Types

Definition 6.5. Let G be a Lie group. An irreducible representation V of G or \mathfrak{g} is of *complex type* if $V \not\cong V^*$, *real type* if there exists a symmetric isomorphism $V \rightarrow V^*$ (i.e. a symmetric inner product for V), and *quaternionic (or symplectic) type* if the isomorphism is given through an anti-symmetric inner product.

Exercise 6.1. Let V be an irreducible representation of a finite group G . Show that $\text{End}_{\mathbb{R}G}(V)$ (i.e. $V \otimes V^*$) can only be one of three types:

- complex type if $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{C}$,
- real type if $\text{End}_{\mathbb{R}G}(V) \cong \text{Mat}_{2 \times 2}(\mathbb{R})$,
- quaternionic type if $\text{End}_{\mathbb{R}G}(V) \cong \mathbb{H}$.

Example 6.5.1. Let L_n be an irreducible representation of \mathfrak{sl}_2 . Then L_n is of real type for even n and of quaternionic type for odd n . Thus $L_n = S^n V$ where $V = L_1$ is 2-dimensional. The invariant form on $S^n V$ is $S^n B$, where B is a skew-symmetric invariant form on V .

Proposition 6.4. Assume $\lambda = -w_0(\lambda)$, so that the corresponding representation is of real or quaternionic type. Then L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.

Proof. The number $n = (2\rho^\vee, \lambda)$ is the eigenvalue of h (from the principal \mathfrak{sl}_2 -subalgebra) on the highest weight vector. Thus we have a decomposition

$$L_\lambda|_{\mathfrak{sl}_2} = L_n \oplus \bigoplus_{m < n} k_m L_m,$$

where L_n has multiplicity 1. One can determine the type based on L_n . □

6.5 Review of Compact Lie Groups

Remark. Let G be a real Lie group of dimension n . Then $\xi \in \wedge^n \mathfrak{g}^*$ gives a generating n -form ω , which is non-vanishing if ξ is non-vanishing. This gives rise to left- and right-invariant measures μ_L and μ_R on G , which are unique up to a constant. We say that G is *unimodular* if $\mu_L = \mu_R$ (up to constants).

When does $\mu_L = \mu_R$? For a 1-dimensional representation V of G , let $|V|$ be the representation of G on the same space where $\rho_{|V|}(g) = |\rho_V(g)|$ (where $\rho_V : G \rightarrow \text{Aut}(V) = \mathbb{R}^\times$).

Proposition 6.5. We have $\mu_L = \mu_R$ if and only if $|\wedge^n \mathfrak{g}^*|$ is a trivial representation of G .

Proof. We have $\mu_L = \mu_R$ if and only if the left-invariant form is right- or left-invariant up to a sign. This is equivalent to $\xi \in \wedge^n \mathfrak{g}^*$ being invariant up to a sign under the action of \mathfrak{g} . □

Proposition 6.6. A compact group is unimodular.

Proof. For compact groups, the representation $|\wedge^n \mathfrak{g}^*|$ gives a continuous homomorphism $G \rightarrow \mathbb{R}^+$, whose only compact subgroup is $\{1\}$. The result follows by Proposition 6.5. □

Proposition 6.7. Let V be an irreducible representation of G . Then V admits a G -invariant unitary structure.

Proof. Take any positive Hermitian form B on V , and define

$$B_{av}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w) dg.$$

This is well-defined and invariant by construction. □

Corollary 6.0.1 (Weyl unitary trick). Any finite-dimensional representation is completely reducible.

Proof. Write $V = W \oplus W^\perp$. If W is invariant, then so is W^\perp . □

Lecture 7

Feb. 4 — Compact Groups

7.1 More on Exponents

Theorem 7.1 (Chevalley's restriction theorem). *There is an isomorphism $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W$.*

Theorem 7.2 (Harish-Chandra theorem). *There is an isomorphism $\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\cong} \mathcal{Z}(U(\mathfrak{g}))$.*

Remark. Pick an ordering s_{i_1}, \dots, s_{i_r} of the simple roots. Then $c = s_{i_1} \cdots s_{i_r}$ is the *Coxeter element*, and $c^h = 1$ where h is the Coxeter number. Then the eigenvalues of c are ζ^{m_i+1} where $\zeta = e^{2\pi i/h}$ and the m_i are the exponents. Also note that $|W| = \prod_{i=1}^r (m_i + 1)$.

If e, f, h is the principal \mathfrak{sl}_2 -triple, then one can consider $e + \mathfrak{g}^f$ where $\mathfrak{g}^f = \ker \text{ad}_f$.

7.2 Matrix Coefficients

Remark. For the rest of this lecture, let G be a real compact group and V a finite-dimensional continuous complex representation of G .

Definition 7.1. A *matrix coefficient* of $\rho_V : G \rightarrow \text{GL}(V)$ is a function $G \rightarrow \mathbb{C}$ of the form

$$g \longmapsto \langle f, \rho_V(g)v \rangle$$

for some $v \in V$ and $f \in V^*$.

Proposition 7.1. *Matrix coefficients are smooth.*

Proof. Call $v \in V$ a smooth vector if $\langle f, \rho_V(g)v \rangle$ is smooth for all $f \in V^*$. It is obvious that such vectors form a subspace of V , call it $V_{\text{sm}} \subseteq V$. Fix $v \in V$ and $\phi : G \rightarrow \mathbb{C}$ smooth and with compact support. Let

$$w = w(\phi, v) = \int_G \phi(g) \rho_V(g)v \, dg.$$

We claim that w is smooth. We have

$$f(\rho(h)w) = f \left(\rho_V(h) \int_G \phi(g) \rho_V(g)v \, dg \right) = \int_G f(\phi(g) \rho_V(hg)v) \, dg = \int_G f(\phi(h^{-1}g) \rho_V(g)v) \, dg.$$

Differentiating under the integral sign and noting that $\phi(h^{-1}g)$ is smooth in h , we see that the above expression is smooth in h . Now choose a delta-like sequence ϕ_n with compact support around 1 so that

$$\int_G \phi_n(g) \, dg = 1.$$

Then $w_n = w(\phi_n, v) \rightarrow v$ and each w_n is smooth, so v is smooth. \square

Remark. Let V be an irreducible representation of G . Then:

1. V has an invariant positive-definite inner product which is unique up to scaling;
2. one can use an orthonormal basis v_1, \dots, v_n to define matrix coefficients:

$$\psi_{V,i,j}(g) = v_j^*(\rho_V(g)v_i) = (\rho_V(g)v_i, v_j)$$

(note that this definition is independent of normalization).

Theorem 7.3 (Orthonormality of matrix coefficients). *Let V, W be irreducible representations of G .*

1. *If V, W are not isomorphic, then*

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{W,k,\ell}(g) dg = 0.$$

2. *For $V = W$, we have*

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{V,k,\ell}(g) dg = \frac{\delta_{i,k} \delta_{j,\ell}}{\dim V}.$$

Proof. Let $\{v_i\}$ and $\{w_k\}$ be orthonormal bases for V and W , respectively. We have

$$\int_G \psi_{V,i,j}(g) \overline{\psi}_{W,k,\ell}(g) dg = \int_G ((\rho_V(g) \otimes \rho_{\overline{W}}(g))(v_i \otimes w_k), v_j \otimes w_\ell) dg$$

Define the operator

$$P = \int_G (\rho_V \otimes \rho_{\overline{W}})(g) dg = \int_G \rho_{V \otimes \overline{W}}(g) dg.$$

Since $\overline{W} \cong W^*$, we have $P : V \otimes W^* \rightarrow V \otimes W^*$. Thus

$$\text{Im } P \subseteq (V \otimes W^*)^G,$$

which is 0 if $V \not\cong W$. On the other hand, if $V \cong W$, then the only invariant is

$$\vec{u} = \sum_k (v_k \otimes \overline{v}_k),$$

so P is the orthogonal projection onto \vec{u} . Thus

$$P\vec{x} = \frac{(\vec{x}, \vec{u})}{(\vec{u}, \vec{u})} \vec{u},$$

so we have $(P(v_i \otimes w_k), v_j \otimes w_\ell) = \delta_{i,j} \delta_{k,\ell} / (\dim V)$. \square

7.3 Peter-Weyl Theorem

Theorem 7.4 (Peter-Weyl theorem). *The matrix coefficients $\psi_{V,i,j}$ form an orthogonal basis in $L^2(G)$.*

Remark. Let V be a finite-dimensional irrep of G . There is a natural inclusion

$$\begin{aligned} i_V : V^* &\longrightarrow \text{Hom}_G(V, L^2(G)), \\ f &\longmapsto [v \mapsto (\rho_{V^*}(\cdot)f)(v)]. \end{aligned}$$

We claim that i_V is also surjective. To see this, let $\phi \in \text{Hom}_G(V, L^2(G))$, i.e. an L^2 function left-invariant

under G . Thus we have that

$$\phi(x) = \rho_{V^*}(xg^{-1})\phi(g)$$

(after modifying ϕ on a set of measure zero). Setting $g = 1$, we get $\phi(x) = \rho_{V^*}(x)\phi(1)$, so we have

$$\xi : \bigoplus_{V \in \text{Irr}(G)} V \otimes V^* \cong \bigoplus_{V \in \text{Irr}(G)} V \otimes \text{Hom}_G(V, L^2(G)) \longrightarrow L^2(G),$$

an embedding of $(G \times G)$ -modules. Call the left-hand side $L^2_{\text{alg}}(G)$.

Theorem 7.5 (Peter-Weyl theorem, alternative). $L^2_{\text{alg}}(G)$ is dense in $L^2(G)$, i.e.

$$L^2(G) = \widehat{\bigoplus}_{V \in \text{Irr}(G)} V \otimes V^*.$$

Example 7.1.1. Let $G = S^1 = U(1)$. The irreducible representations of G are $\psi_n(\theta) = e^{in\theta}$. The $e^{in\theta}$ form a basis of $L^2(G) = L^2(S^1)$, where the norm is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta.$$

This is the usual Fourier series on S^1 . The Peter-Weyl theorem extends this to non-abelian groups.

Exercise 7.1. Let G be a compact group and H a closed subgroup of G .

1. Show that $L^2(G/H) = \widehat{\bigoplus}_{V \in \text{Irr}(G)} N_H(V)V$, where $N_H(V) = \dim V^H$ (the space of H -invariants).
2. Let $G = \text{SO}(3)$ and $H = \text{SO}(2)$. Then show that $L^2(G/H) = L^2(S^2) = \widehat{\bigoplus}_{m \geq 0} N_H(m)L_{2m}$, and that $N_H(m) = 1$ for every m .

7.4 Introduction to Quantum Mechanics

Remark. Let \mathcal{H} be a Hilbert space and H a self-adjoint operator on \mathcal{H} . The spectrum of H gives the *energy levels* of the system. The elements $\psi(x, y, z) \in L^2(\mathbb{R}^3)$ are called *wave functions*, and we assume that they are normalized so that $\|\psi\|_{L^2} = 1$. This is so that

$$|\psi(x, y, z)|^2 \Delta V$$

gives the probability of a quantum particle to be in the region ΔV .

In general, there is also a time dependence in the wave function ψ , so we have $\psi(x, y, z, t)$. The time dependence is governed by the Schrödinger equation:

$$i\partial_t \psi = H\psi.$$

One can solve this equation via separation of variables, and we can write

$$\psi(x, y, z, t) = \sum_N e^{-iE_N t} \psi_N(x, y, z),$$

where the ψ_N are eigenvectors satisfying $H\psi_N = E_N \psi_N$.

Example 7.1.2. For the hydrogen atom, we have

$$H = -\frac{1}{2}\Delta - \frac{1}{r},$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplacian and $r = \sqrt{x^2 + y^2 + z^2}$. The $\Delta/2$ is called the *kinetic part* of H , and the $1/r$ is called the *potential part* of H .

Lecture 8

Feb. 9 — Hydrogen Atom

8.1 Bound States of the Hydrogen Atom

Remark. We are looking for eigenvectors for $H = -\frac{1}{2}\Delta - \frac{1}{r}$, i.e. $\psi_N \in L^2(\mathbb{R}^3)$ such that $H\psi_N = E_N\psi_N$ with $E_N < 0$. We first write the Laplacian in spherical coordinates:

$$\begin{aligned}\Delta &= \Delta_r + \frac{1}{r}\Delta_{\text{sph}} \\ \Delta_r &= \partial_r^2 + \frac{2}{r}\partial_r \\ \Delta_{\text{sph}} &= \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta),\end{aligned}$$

where ϕ is the angle in the xy -plane and θ is the angle from the positive z -axis. Then we have

$$\partial_r^2\psi + \frac{2}{r}\partial_r\psi + \frac{2}{r}\psi + \frac{1}{r^2}\Delta_{\text{sph}}\psi = -2E\psi,$$

which is solved by $\psi(r, \vec{u}) = f(r)\xi(\vec{u})$ for $\vec{u} \in S^2$ satisfying

$$\begin{aligned}\Delta_{\text{sph}}\xi + \lambda\xi &= 0 \\ f''(r) + \frac{2}{r}f'(r) + \left(\frac{2}{r} - \frac{\lambda}{r^2} + 2E\right)f(r) &= 0.\end{aligned}\tag{*}$$

Note that (*) implies Δ_{sph} is rotationally invariant. By the Peter-Weyl theorem, we have

$$L^2(S^2) = \widehat{\bigoplus}_{\ell \geq 0} L_{2\ell},$$

where $S^2 = \text{SO}(3)/\text{SO}(2)$ and $L_{2\ell}$ are the irreps of $\text{SO}(3)$.

Let $Y_\ell^0 \subseteq L_{2\ell}$ be a vector of weight 0, which is invariant under $\text{SO}(2)$. Thus it depends only on θ . So we can write $Y_\ell^0(\theta) = P_\ell(\cos\theta)$, where P is a polynomial of degree ℓ . By orthogonality,

$$\int_{-1}^1 P_k(z)P_\ell(z) dz = 0, \quad k \neq n.$$

Thus we can write

$$-\lambda_e P_\ell(z) = \Delta_{\text{sph}} P_\ell(z) = \partial_z(1-z^2)\partial_z P_\ell(z).$$

From looking at the leading term we must have $\lambda_\ell = \ell(\ell+1)$.

Now take $Y_\ell^m \in L_{2\ell}$ for $-\ell \leq m \leq \ell$. Write $Y_\ell^m(\phi, \theta) = e^{im\phi} P_\ell^m(\cos \theta)$. So we have

$$\frac{-m^2}{1-z^2} P_\ell^m + \partial_z(1-z^2)\partial_z P_\ell^m + \ell(\ell+1)P_\ell^m = 0, \quad -\ell \leq m \leq \ell.$$

This equation has a unique solution (up to scaling) on $[-1, 1]$, given by

$$P_\ell^m = (1-z^2)^{m/2} \partial_z^{\ell+m} (1-z^2)^\ell.$$

Now we return to the radial equation:

$$f''(r) + \frac{2}{r} f'(r) + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + 2E \right) f(r) = 0.$$

Write $f(r) = r^\ell e^{-r/n} h(2r/n)$, where n is to be chosen later and h satisfies

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + \left(n - \ell - 1 + \frac{1}{4}(1+2En^2)\rho \right) h(\rho) = 0.$$

Now choose $n = 1/\sqrt{-2E}$, so that $E = -1/2n^2$. Then the above equation becomes

$$\rho h''(\rho) + (2\ell+2-\rho)h'(\rho) + (n - \ell - 1)h(\rho) = 0.$$

This equation is known as the *generalized Laguerre equation*. To get $\|\psi\|_{L^2}^2 < \infty$, we must have

$$\int_0^\infty \rho^{2\ell+2} e^{-\rho} |h(\rho)|^2 d\rho < \infty,$$

where the extra $+2$ in $\rho^{2\ell+2}$ comes from the Jacobian. Solutions around 0 behave like $\rho^s(1+o(1))$, so

$$s(s+2\ell+1) = 0.$$

Thus either $s = 0$ or $s = -2\ell - 1$.

First consider when $\ell = 0$. Then $s = -1$ and we have $\rho^{-1}(1+o(1))$, so $\psi \sim 1/r$ as $r \rightarrow 0$. Then

$$H\psi = E\psi + C\delta_0,$$

where δ_0 is the delta function at 0, so we do not get an eigenvector in this case.

Thus $s = -2\ell - 1$. Expanding $h(\rho)$ in a series and substituting, we get the recursive formula

$$h_n(\rho) = \sum_{k=0}^{\infty} \frac{(1+\ell-n) \cdots (k+\ell-n)}{(2\ell+2) \cdots (2\ell+1+k) k!} \rho^k.$$

This series converges, and we have

$$\lim_{\rho \rightarrow \infty} \frac{h_n(\rho)}{\rho} = 1$$

unless the series terminates. Thus $n - \ell - 1 \in \mathbb{Z}_{\geq 0}$, so we can write

$$h_n(\rho) = \sum_{k=0}^{n-\ell-1} \frac{(1+\ell-n) \cdots (k+\ell-n)}{(2\ell+2) \cdots (2\ell+1+k) k!} \rho^k = L_{n-\ell-1}^{2\ell+1}(\rho),$$

which is known as the *generalized Laguerre polynomial*:

$$L_N^\alpha(\rho) = \sum_{k=0}^N (-1)^N \frac{N \cdots (N-k+1)}{(\alpha+1) \cdots (\alpha+k)} \frac{\rho^k}{k!}.$$

Theorem 8.1. *The bound states (i.e. solutions to $H\psi = E\psi$ in $L^2(\mathbb{R}^3)$) of the hydrogen atom are*

$$\psi_{n,\ell,m}(r, \phi, \theta) = r^\ell e^{-r/n} L_{n-\ell-1}^{2\ell+1}(2r/n) Y_\ell^m(\theta, \phi),$$

where $n \in \mathbb{Z}_{>0}$, ℓ is an integer from $0, \dots, n-1$, $E_n = -1/2n^2$, and m is an integer between $-\ell, \dots, \ell$.

Remark. In Theorem 8.1, n is known as the *principal quantum number*, and ℓ is known as the *azimuthal quantum number*. Note that if $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\Delta_{\text{sph}}$, where iL_x, iL_y, iL_z are the generators of $\mathfrak{so}(3)$ satisfying $[L_{\{x}, L_y] = -iL_z]$, then $\vec{L}^2 = C = \ell(\ell+1)$ is the Casimir operator.

Corollary 8.1.1. *The space W_n of states with principal number n has dimension n^2 .*

Proof. This follows from $\sum_{\ell=0}^{n-1} (2\ell+1) = n^2$. □

Remark. Note that $\widehat{\bigoplus}_n W_n$ forms a proper, closed subspace $L_0^2(\mathbb{R}^3)$ of $L^2(\mathbb{R}^3)$. We need to find all φ with $(H\varphi, \varphi) \geq 0$ to reconstruct all of $L^2(\mathbb{R}^3)$. This corresponds to the continuous spectrum of H .

8.2 Spin

Remark. *Spin* is a kind of intrinsic angular momentum. Instead of just $L^2(\mathbb{R}^3)$, we should consider

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 = L^2(\mathbb{R}^3) \otimes L_1$$

to be the space of states for the hydrogen atom. We have

$$V_n = (L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}) \otimes L_1 = 2L_1 \oplus 2L_3 \oplus \cdots \oplus 2L_{2n-3} \oplus 2L_{2n-1},$$

so $\dim V_n = 2n^2$. We have an additional *spin operator* given by

$$S_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},$$

which acts on \mathbb{C}^2 in the standard basis e_+, e_- . Then we have

$$\psi_{n,\ell,m,+} = \psi_{n,\ell,m} \otimes e_+ \quad \text{and} \quad \psi_{n,\ell,m,-} = \psi_{n,\ell,m} \otimes e_-.$$

The *total spin* is $m+s$ (where s is the eigenvalue for S_z), which is either $m+1/2$, or $m-1/2$.

8.3 Pauli Exclusion Principle

Remark. The space $\wedge^k V_n$ corresponds to the space of states for k electrons at energy level n . Note that we must have $k \leq 2n^2$ to have $\wedge^k V_n \neq 0$, which gives the *Pauli exclusion principle*.

In the periodic table, one has *orbitals* s, p, d, f corresponding to $\ell = 0, 1, 2, 3$, respectively, written with coefficient n and with exponent k corresponding to the number of electrons in the orbital. For example, the element Ruthenium has

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^7 5s^1.$$

The periodic table is organized as follows: from left to right ordered by how many *valent* electrons (i.e. the number of electrons in the outermost orbital), and from top to bottom ordered by how many energy levels. For Ruthenium, it is on column 8 and row 5. The number is 44, for 44 total electrons.

Exercise 8.1. Let $\vec{r} = (x, y, z)$ and $\vec{p} = (-\partial_x, -i\partial_y, -i\partial_z)$ be the *position* and *momentum* operators. Let $\vec{L} = \vec{r} \times \vec{p}$ and $H = \frac{1}{2}\vec{p}^2 + U(r)$, where U is rotationally invariant. Show that:

1. The components $i\vec{L}$ are generators of the rotations on \mathbb{R}^3 , and $[\vec{L}, \vec{p}^2] = 0$.
2. $\vec{A}_0 = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})$ satisfies $[\vec{A}_0, \vec{p}^2] = 0$.
3. Let $A = \vec{A}_0 + \phi(r)\vec{r}$. There exists ϕ such that $[\vec{A}, H] = 0$ if and only if U is a *Coulomb potential* (i.e. $U(r) = \frac{C}{r} + D$), and in this case ϕ is completely determined.
4. (Hidden symmetry of the hydrogen atom) Use the commutation relations between \vec{A} and \vec{L} to define an action on $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3$, so that \vec{L} is the diagonal copy in this decomposition.
5. $W_n = L_{n-1} \boxtimes L_{n-1}$ as representations of $\mathfrak{so}_4 = \mathfrak{so}_3 \oplus \mathfrak{so}_3 = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Lecture 9

Feb. 11 — Real Forms

9.1 Automorphisms of Semisimple Lie Algebras

Remark. Recall that we can identify $\text{Aut}(\mathfrak{g})$ with a Lie group with Lie algebra \mathfrak{g} . The connected component of the identity $\text{Aut}^0(\mathfrak{g})$ (also known as the *adjoint group* G_{ad}) acts transitively on the set of Cartan subalgebras. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then there is a connected subgroup $H \subseteq G_{\text{ad}}$ which acts as 1 on \mathfrak{h} and as $e^{\alpha(x)}$ on g_α for $x \in \mathfrak{h}$. Then we have

$$\mathfrak{h}/2\pi i P^\vee \cong H,$$

and H is called a *maximal torus*.

Proposition 9.1. *The normalizer $N(H)$ of H in G_{ad} coincides with the stabilizer of \mathfrak{h} and contains H as a normal subgroup such that $N(H)/H = W$ (the Weyl group).*

Proof. Note that $\text{SL}_2(\mathbb{C})$ is simply connected, so for any simple root α_i there is a homomorphism

$$\eta_i : \text{SL}_2(\mathbb{C}) \longrightarrow G_{\text{ad}} = \tilde{G}/\mathcal{Z}(\tilde{G}).$$

Define $S_i = \eta_i \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$. Consider $w = s_{i_1} \cdots s_{i_n}$ and $\tilde{w} = S_{i_1} \cdots S_{i_n}$. Note that \tilde{w} acts on \mathfrak{h} acts w , and if $w = w_1 w_2$, then $\tilde{w} = \tilde{w}_1 \tilde{w}_2 h$ for some $h \in H$. To see the latter claim, note that h has the preserve the root decomposition, hence $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$. Thus $h = \exp(\sum_j b_j w_j^\vee) \in H$ (where $\langle w_j^\vee, \alpha_i \rangle = \delta_{j,i}$).

So \tilde{w} and H generate a subgroup $N \subseteq N(H)$ such that $N/H = W$. It remains to show that $N(H) = N$. Let $x \in N(H)$ and consider simple roots $\alpha'_i = x(\alpha_i)$. Then there exists $w \in W$ such that $w(\alpha'_i) = \alpha_{p(i)}$ for some permutation p . Then $\tilde{w}x(\alpha_i) = \alpha_{p(i)}$, so this is a Dynkin diagram automorphism. Now $\tilde{w}x$ is an element of a group, so the fundamental weights are fixed. Thus $p = \text{id}$. \square

Remark. Although $N(H)/H = W$, in general $N(H)$ is not a semidirect product of H and W .

Proposition 9.2. *The map $\xi : \text{Aut}(D) \times G_{\text{d}} \rightarrow \text{Aut}(\mathfrak{g})$ is an isomorphism.*

Proof. We have to show that ξ is surjective. Let $a \in \text{Aut}(\mathfrak{g})$. We can say that a preserves the Cartan subalgebra (if not, we can shift it by $g \in G_{\text{ad}}$). Multiplying by $\text{Aut}(D)N(H)$, we can make it act trivially on \mathfrak{h} and \mathfrak{g}_{α_i} . Then $a = 1$, so $a \in \text{Im } \xi$. \square

9.2 Real Forms of Semisimple Lie Algebras

Remark. Recall the Serre presentation for \mathfrak{g} , i.e. generators $\{h_i, f_i, e_i\}$ with certain relations. In this setting, everything was defined over \mathbb{Q} .

Definition 9.1. A semisimple Lie algebra is *split* if it admits a Chevalley-Serre basis over base field K .

Remark. Let L be a Galois extension of K ($\text{char } K = 0$), and assume that \mathfrak{g}_L is a split semisimple Lie algebra. We want to find \mathfrak{g} over K such that $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$. The problem is then to find a classification of all such \mathfrak{g} . Let $\Gamma = \text{Gal}(L/K)$. Define an action of Γ on \mathfrak{g}_L by

$$g(\lambda x) = g(\lambda)g(x), \quad x \in \mathfrak{g}_L, \lambda \in L, g \in \Gamma,$$

which is twisted linear. We can reconstruct \mathfrak{g} as the invariants \mathfrak{g}_L^Γ .

The simplest action of this kind is $\rho_0(g)$, which acts on scalars and preserves $\{h_i, e_i, f_i\}$. Any twisted linear action takes the form $\rho(g) = \eta(g)\rho_0(g)$ for some $\eta : \Gamma \rightarrow \text{Aut}(\mathfrak{g}_L)$. As ρ is a homomorphism,

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

and upon rearranging, we have

$$\eta(gh) = \eta(g)g(\eta(h)),$$

where $g(a) = \rho_0(g)a\rho_0(g)^{-1}$ for $a \in \text{Aut}(\mathfrak{g}_L)$. The above is called a *1-cocycle condition*.

Denote the Lie algebra associated to the cocycle η by \mathfrak{g}_η . When do we have $\mathfrak{g}_{\eta_1} \cong \mathfrak{g}_{\eta_2}$? This is the case when ρ_1 and ρ_2 are isomorphic, i.e. there exists $a \in \text{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$. Then

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

so $\eta_1(g) = a\eta_2(g)g(a)^{-1}$. Thus η_1 and η_2 are cohomologous cocycles.

Proposition 9.3. *The semisimple Lie algebras \mathfrak{g} over K which split over L (where L/K is Galois) are classified by $H^1(\Gamma, \text{Aut}(\mathfrak{g}_L))$, where $\Gamma = \text{Gal}(L/K)$.*

Remark. We will now specialize to $K = \mathbb{R}$, $L = \mathbb{C}$, where $\Gamma = \mathbb{Z}/2\mathbb{Z}$, generated by complex conjugation. We have $\text{Aut}(\mathfrak{g}_L) = \text{Aut}(D) \ltimes G_{\text{ad}}$. Since $\eta(1) = 1$, η is determined by $\eta(-1)$. The cocycle condition is

$$s\bar{s} = 1, \quad s = \eta(-1).$$

The corresponding Lie algebra (up to isomorphism) depends only on the cohomology class of s , where $s \mapsto as\bar{a}^{-1}$ for $a \in \text{Aut}(D)$.

Theorem 9.1. *The real semisimple Lie algebras whose complexification is \mathfrak{g} (i.e. the real forms of \mathfrak{g}) are classified by $s \in \text{Aut}(D) \ltimes G_{\text{ad}}$ such that $s\bar{s} = 1$, modulo the equivalence $s \mapsto as\bar{a}^{-1}$ for $a \in \text{Aut}(D)$.*

Remark. Note that complex conjugation acts trivially on $\text{Aut}(D)$.

Remark. Denote by $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \bar{x} = s(x)\}$ the real form corresponding to s . Denote by $\mathfrak{g}_{(1)}$ the split form consisting of real $x \in \mathfrak{g}$ (so that $x = \bar{x}$).

Alternatively, we can define an antilinear involution $\sigma_s(x) = \overline{s(x)}$. Then $\mathfrak{g}_{(s)}$ is the fixed point set of σ_s .

Remark. Note that s defines $s_0 \in \text{Aut}(D)$ with $s_0^2 = 1$.

Corollary 9.1.1. *The conjugacy class of s_0 is invariant under equivalences.*

Remark. Since s_0 permutes the connected components of the Dynkin diagram D , it preserves some and divides some into pairs. So every semisimple real Lie algebra is a direct sum of simple ones, and each simple one has either connected Dynkin diagram or consists of two identical components.

Remark. We now consider the case when D is connected and \mathfrak{g} is simple.

Definition 9.2. A real form $\mathfrak{g}_{(s)}$ of a complex simple Lie algebra is said to be *inner to* $\mathfrak{g}_{(s')}$ if $s' = gs$ up to equivalence, where $g \in G_{\text{ad}}$ (i.e. s, s' differ by an inner automorphism). The *inner class* of $\mathfrak{g}_{(s)}$ is the collection of all real forms inner to $\mathfrak{g}_{(s)}$. An *inner form* is a form inner to a split form. We call $\mathfrak{g}_{(s)}$ *quasi-split* if $s = s_0 \in \text{Aut}(D)$.

Corollary 9.1.2. *We have the following:*

1. Any real form is inner to a unique quasi-split form.
2. A real form which is both inner and quasi-split is split.

Example 9.2.1. Consider the *Cartan involution* τ defined by

$$\tau(h_j) = -h_j, \quad \tau(e_j) = -f_j, \quad \tau(f_j) = -e_j.$$

Then $\mathfrak{g}_{(\tau)} = \mathfrak{g}^c$ is called the *compact real form* of \mathfrak{g} .

Lecture 10

Feb. 16 — Real Forms, Part 2

10.1 Compact Real Forms

Proposition 10.1. *Let τ be the Cartan involution (defined in Example 9.2.1) and $\mathfrak{g}^c = \mathfrak{g}_{(\tau)}$. Then the Killing form of \mathfrak{g}^c is negative definite.*

Proof. We can write $\mathfrak{g}^c = (\mathfrak{h} \cap \mathfrak{g}^c) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. The Killing form is negative definite on $\mathfrak{h} \cap \mathfrak{g}^c$ since the inner product on a coroot lattice is positive definite. Thus it is negative definite on \mathfrak{g}^c , as $\{i\alpha_j^\vee\}$ is a basis for $\mathfrak{g}^c \cap \mathfrak{h}$. Now we need to show that it is negative definite on $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$. Note that for $\mathfrak{g} = \mathfrak{sl}_2$, we have a basis for \mathfrak{g}^c given by $ih, e - f, i(e + f)$, so $\mathfrak{g}^c = \mathfrak{su}(2)$. So the statement holds there. For general \mathfrak{g} , we have that S_i preserves \mathfrak{g}^c , since

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2).$$

So we have $\mathrm{Lie}(\mathrm{SU}(2)_i) \subseteq \mathfrak{g}^c$. For any $w \in W$, the lift \tilde{w} preserves \mathfrak{g}^c , so the restriction of the Killing form of \mathfrak{g}^c to $\mathfrak{g}^c \cap (\mathfrak{sl}_2)_\alpha$ is negative definite. \square

Remark. Consider $\mathrm{Aut}(\mathfrak{g}^c)$. Since the Killing form is negative definite, $\mathrm{Aut}(\mathfrak{g}^c)$ is a closed subgroup of $O(\mathfrak{g}^c)$, so it is compact. Moreover, this is a Lie group with Lie algebra \mathfrak{g}^c .

Corollary 10.0.1. $G_{\mathrm{ad}}^c = \mathrm{Aut}(\mathfrak{g}^c)^\circ$ is a connected, compact Lie group with Lie algebra \mathfrak{g}^c .

Example 10.0.1. We have the following:

1. For $\mathfrak{g} = \mathfrak{sl}_n$, we have $G_{\mathrm{ad}}^c = \mathrm{PSU}(n) = \mathrm{SU}(n)/\mu_n$, where μ_n is the n th roots of unity.
2. For $\mathfrak{g} = \mathfrak{so}_n$, we have $G_{\mathrm{ad}}^c = \mathrm{SO}(n)$ for odd n and $G_{\mathrm{ad}}^c = \mathrm{SO}(n)/\{\pm 1\}$ for even n .
3. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $G_{\mathrm{ad}}^c = \mathrm{U}(n, \mathbb{H})/\{\pm 1\}$. The group

$$\mathrm{U}(n, \mathbb{H}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$$

is called the *quaternionic unitary group*.

Example 10.0.2. We have the following:

1. Consider A_{n-1} , which corresponds to the split form $\mathfrak{sl}(n, \mathbb{R})$ and compact form $\mathfrak{su}(n)$. For $n > 2$, we have a quasi-split real form as follows: Let $s(A) = -JA^T J^{-1}$ where $J_{i,j} = (-1)^i \delta_{i,n+1-j}$. Then

$$e_i, f_i, h_i, \longmapsto e_{n+1-i}, f_{n+1-i}, h_{n+1-i}.$$

Note that J is a Hermitian or skew-Hermitian form of signature (p, p) with $n = 2p$ or of signature $(p+1, p), (p, p+1)$, which are isomorphic when $n = 2p+1$.

For $n = 2$, we have $\mathfrak{su}(1, 1) = \mathfrak{sl}(2, \mathbb{R})$ (and $\mathrm{PSU}(1, 1) = \mathrm{PSL}(2, \mathbb{R})$).¹

2. For type B_n , we have compact form $\mathfrak{so}(2n+1)$ and split form $\mathfrak{so}(n+1, n)$. There are no nontrivial automorphisms, so there are no non-split quasi-split forms.

Particular cases of interest are $\mathrm{SO}(3) \cong \mathrm{SU}(2)$ and $\mathrm{SO}^+(2, 1) = \mathrm{PSU}(1, 1) = \mathrm{PSL}(2, \mathbb{R})$.

3. For type C_n , we have the split form $\mathfrak{sp}(2n, \mathbb{R})$ and compact form $\mathfrak{u}(n, \mathbb{H})$. There are no non-split quasi-split real forms, as there are no nontrivial automorphisms of the Dynkin diagram.

Note that $B_2 = C_2$, so we have $\mathfrak{so}(3, 2) = \mathfrak{sp}_4(\mathbb{R})$ and $\mathfrak{so}(5) = \mathfrak{u}(2, \mathbb{H})$.

4. For type D_n , we have split form $\mathfrak{so}(n, n)$ and compact form $\mathfrak{so}(2n)$. For $n > 4$, there is a unique nontrivial involution, while for $n = 4$, we have $\mathrm{Aut}(D) = S_3$. However, there is still a unique non-split quasi-split form as there is only one nontrivial involution up to conjugation. Recall

$$A = -JA^TJ^{-1}, \quad J_{i,j} = \delta_{i,2n+1-j}.$$

Then the quasi-split form is given by $J \mapsto J' = gJ$, where g permutes e_n, e_{n+1} (which corresponds to α_{n-1}, α_n). The signature defined by J' is $(n+1, n-1)$, so the quasi-split form is $\mathfrak{so}(n+1, n-1)$.

Note that $D_2 = A_1 \oplus A_1$, so we have the following isomorphisms:

$$\begin{aligned} \mathfrak{so}(4) &= \mathfrak{su}(2) \oplus \mathfrak{su}(2), \\ \mathfrak{so}(2, 2) &= \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1), \\ \mathfrak{so}(3, 1) &= \mathfrak{sl}_2(\mathbb{C}), \end{aligned}$$

where in the last isomorphism we view $\mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra.

We also have $D_3 = A_3$, which gives the following isomorphisms:

$$\begin{aligned} \mathfrak{so}(6) &= \mathfrak{su}(4), \\ \mathfrak{so}(3, 3) &= \mathfrak{sl}_4(\mathbb{R}), \\ \mathfrak{so}(4, 2) &= \mathfrak{su}(2, 2). \end{aligned}$$

10.2 Classification of Real Forms

Remark. Write $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega = \sigma_\tau$ the Cartan antilinear involution (so that \mathfrak{g}^c is the fixed points of σ_τ). Another real structure on \mathfrak{g} is given by $\sigma = \omega \circ g$ for $g \in \mathrm{Aut}(\mathfrak{g})$, as

$$\sigma^2 = \omega \circ g \circ \omega \circ g = 1$$

(note that $\omega \circ \omega = \omega(g)$, and $\omega(g)g = 1$). Define

$$(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_{\omega(Y)}),$$

which is the Hermitian extension of the Killing form from \mathfrak{g}^c to \mathfrak{g} . Note that $\omega(g) = (\mathfrak{g}^\dagger)^{-1}$, where \mathfrak{g}^\dagger is the adjoint of g . Thus we see that g is self-adjoint.

¹Note that $\mathrm{PSU}(1, 1)$ is the group of automorphisms of the unit disk, and $\mathrm{PSL}(2, \mathbb{R})$ is the group of automorphisms of the upper half-plane. The isomorphism comes from the Cayley transform from the unit disk to the upper half-plane.

In particular, g is diagonalizable with real eigenvalues. So we can write

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma),$$

where $\mathfrak{g}(\gamma)$ is the eigenspace of \mathfrak{g} corresponding to eigenvalue γ . Note that

$$[\mathfrak{g}(\beta), \mathfrak{g}(\gamma)] = \mathfrak{g}(\beta\gamma).$$

Consider the operator $|g|^t$ for $t \in \mathbb{R}$, which acts on $\mathfrak{g}(\gamma)$ as $|\gamma|^t$. We can rewrite

$$|g|^t = \exp(t \log |g|) \in G_{\text{ad}},$$

which is a 1-parameter subgroup of G_{ad} . Then we can define $\theta := g|g|^{-1}$, and we have

$$\begin{cases} \theta \circ \omega = \omega \circ \theta, \\ \theta^2 = 1, \end{cases}$$

where the first identity follows from $(\theta^\dagger)^{-1} = \theta$. We have

$$\theta = |g|^{-1/2} g \omega(|g|^{1/2}).$$

We can assume $g = \theta$ with $\theta \circ \omega = \omega = \theta$, or equivalently, that $\theta \in \text{Aut}(\mathfrak{g}^c)$ and $\theta^2 = 1$. Thus θ has ± 1 eigenspaces. Note that θ' defines the same real form if and only if

$$\theta' = x\theta(\omega(x))^{-1}$$

for some $x \in \text{Aut}(\mathfrak{g})$. Then we have $x\theta(\omega(x))^{-1} = \omega(x)\theta x^{-1}$ (since $\theta'^2 = 1$, so $\theta'^{-1} = \theta'$). Let

$$z = (\omega(x))^{-1}x,$$

so that $\omega(z) = z^{-1}$. Then $\theta z = z^{-1}\theta$. Now note that $z = x^\dagger x$ is positive definite, so if $y = xz^{-1/2}$,

$$\omega(y) = \omega(x)z^{1/2} = (x^\dagger)^{-1}z^{1/2} = xz^{-1/2} = y.$$

Thus $y \in \text{Aut}(\mathfrak{g}^c)$, and we also have that

$$\theta' = x\theta\omega(x)^{-1} = x\theta zx^{-1} = xz^{-1/2}\theta z^{1/2}x^{-1} = y\theta y^{-1}.$$

Theorem 10.1. *The real forms of \mathfrak{g} are in one-to-one correspondence with the conjugacy classes of involutions $\theta \in \text{Aut}(\mathfrak{g}^c)$, where $\theta \mapsto \omega_\theta = \theta \circ \omega = \omega \circ \theta$.*

Remark. For $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, denote by \mathfrak{g}_θ the corresponding real form. For example,

$$\mathfrak{g}_1 = \mathfrak{g}^c = \mathfrak{g}_{(\tau)},$$

where the latter is our old notation using the split forms.

Remark. We now have a canonical (up to automorphisms of \mathfrak{g}^c) decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\theta = 1$ on \mathfrak{k} and $\theta = -1$ on \mathfrak{p} . Here \mathfrak{k} is a Lie subalgebra and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. For \mathfrak{g}^c itself, we have

$$\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c.$$

Moreover, we have $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, where $\mathfrak{p}_\theta = i\mathfrak{p}^c$ (a fixed point of $\sigma = \omega \circ \theta$ has to have an extra $-$ sign, which we can achieve by multiplying by i).

Exercise 10.1. Show that \mathfrak{k} is reductive but not necessarily semisimple.