1-25 §3.5 Intrinsic curvature & extrinsic curvature

内禀 外

$R_{abc}{}^{d}$

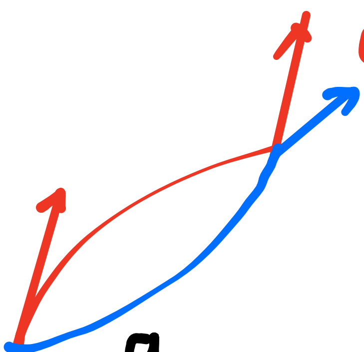
在$(\mathbb{R}^3,\delta_{ab})$空间中，曲面是"弯"的，这种弯曲叫作 extrinsic curvature

而$R_{abc}{}^{d}$是 intrinsic curvature

对$R_{abc}{}^{d}\neq 0$空间

(1) $(\nabla_a\nabla_b-\nabla_b\nabla_a)\omega_c=R_{abc}{}^{d}\omega_d$

(2) the translation of vector depand on the curve

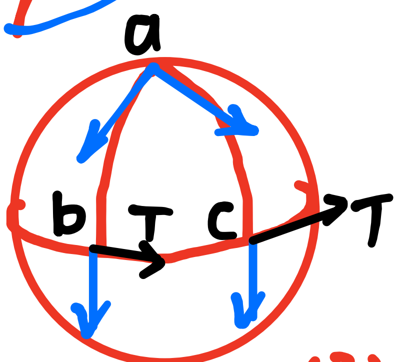


counter example.

从 $a\to b\to c\to a$

都沿 geodesic 移动，保内积

但从$b\to a$与从$c\to a$有很大差距，



(3) ∃初始平行，后来不平行的测地线

但对$(\mathbb{R}^n,\delta_{ab})$ $\because \partial_a\partial_b=\partial_b\partial_a$

$T \downarrow ac$ 故 $R_{abc}{}^{d} \equiv 0$.

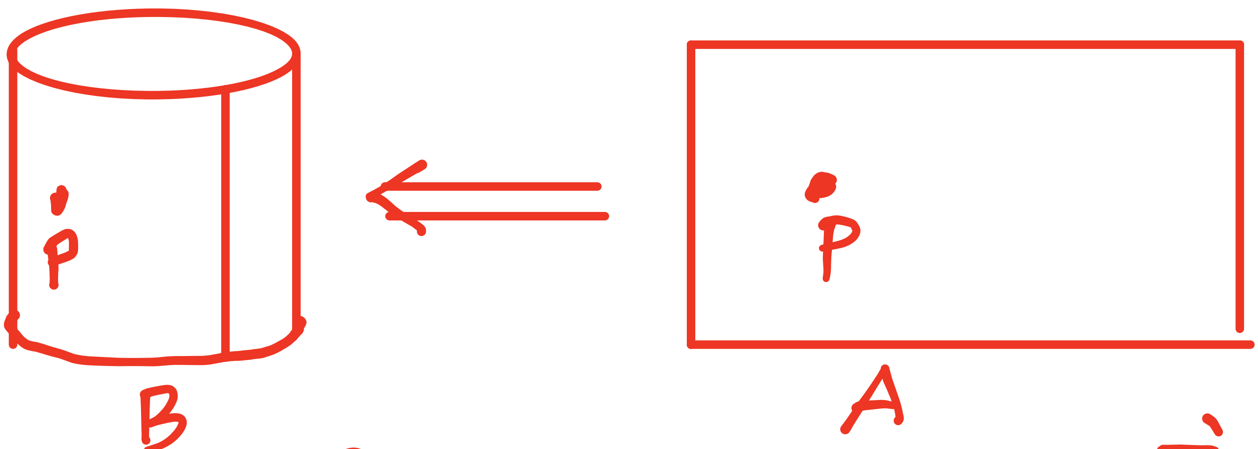
故应有

① 初始平行的 geodesics 一定不会相交

② the translation of vector doesn't depand on the curve

对 extrinsic curvature，定义在 §14，但直观上就是从系统外宏观地看一个面是否弯曲

Ex



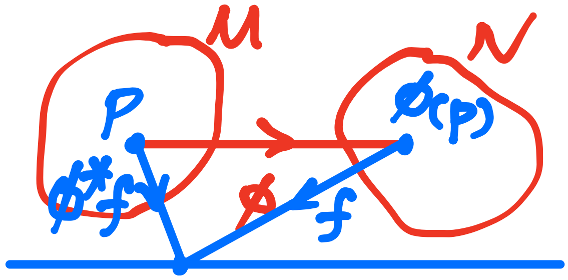
从 $A \to B$ ex-curvature：从0变为某非零值

in-curvature：恒为0

这是内禀曲率与外曲率区别

Chap4 Lie derivatives, Killing fields & hyper-surfaces. 超曲面

§4.1 maps of manifolds



$\mathscr{F}_M(k,l)$

$\mathscr{F}_M(0,0) \equiv \mathscr{F}_M$

given $\phi: M \to N$ (only C^∞)

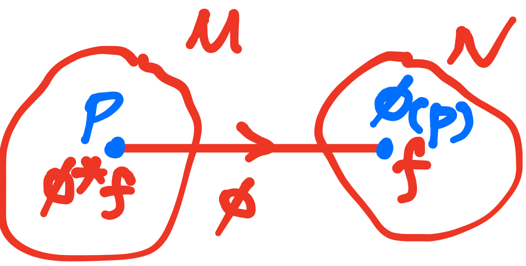
ϕ can induce a number of maps

1: $\phi^*: \mathscr{F}_N(0,0) \longrightarrow \mathscr{F}_M(0,0)$

def of ϕ^*: $\forall f \in \mathscr{F}_N$, def $(\phi^* f) \in \mathscr{F}_M$ by

$(\phi^* f)|_p := f|_{\phi(p)}$, 如上图示

i.e., $\phi^* f = f \circ \phi$. ϕ^*叫作 pull back map



Vector is a map from Scalar field to a number

2: $\phi_*: V_p \longrightarrow V_{\phi(p)}$

$\forall v \in V_p$, def $\phi_*(v) \in V_{\phi(p)}$ by

$(\phi_* v)(f) := v(\phi^* f) \quad \forall f \in \mathscr{F}_N$

ϕ_* is called push forward map.

3. $\phi^*: \mathscr{F}_N(0,l) \longrightarrow \mathscr{F}_M(0,l)$

$\forall T \in \mathscr{F}_N(0,l)$, def $\phi^* T \in \mathscr{F}_M(0,l)$

by $(\phi^* T)_{a_1 \dots a_l}|_p (v_1)^{a_1} \cdots (v_l)^{a_l} := T_{a_1 \cdots a_l}|_{\phi(p)} (\phi_* v_1)^{a_1} \cdots (\phi_* v_n)^{a_n}$

$\forall p \in M, (v_1)^{a_1} \dots (v_n)^{a_n}$

4. $\phi_* : \mathscr{T}_{V_p}(k,0) \longrightarrow \mathscr{T}_{V_{\phi(p)}}(k,0)$,
$\forall T \in \mathscr{T}_{V_p}(k,0)$, def $\phi_* T \in \mathscr{T}_{V_{\phi(p)}}(k,0)$ by
$(\phi_* T)^{a_1 \cdots a_k} (\omega^1)_{a_1} (\omega^2)_{a_2} \cdots (\omega^k)_{a_k} := T^{a_1 \cdots a_k} (\phi^* \omega^1)_{a_1} \cdots (\phi^* \omega^k)_{a_k}$. $\forall (\omega^1)_{a_1} (\omega^2)_{a_2} \cdots (\omega^k)_{a_k} \in V^*_{\phi(p)}$

We can see 1, 3 are from field to field, but 2, 4 are from a point to a point. why?

suppose that ? $\phi_* : \mathscr{F}_M(1,0) \xrightarrow{?} \mathscr{F}_N(1,0)$,
$\forall T \in \mathscr{F}_M(1,0)$, def $(\phi_* T)^a|_q \omega_a := T^a|_?$

Because ϕ is not one-one or onto
namely may not have ϕ^{-1},

But if ϕ is diffeomorphism, we can get $\phi_* : \mathscr{F}_M(k,0) \longrightarrow \mathscr{F}_N(k,0)$
(Because ϕ^{-1} is well defined.
又由于 3. 就可以

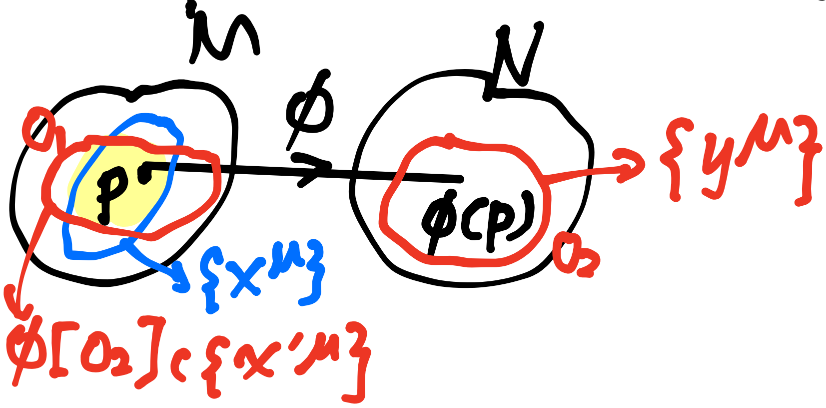
$\mathscr{F}_N(0,l) \longrightarrow \mathscr{F}_M(0,l)$

又 ϕ 可逆

we can get $\boxed{\phi_*: \mathscr{F}_M(k,l) \longrightarrow \mathscr{F}_N(k,l)}$

同样 $\phi^*: \mathscr{F}_N(k,l) \longrightarrow \mathscr{F}_M(k,l)$

premise : ϕ is diffeomorphism.



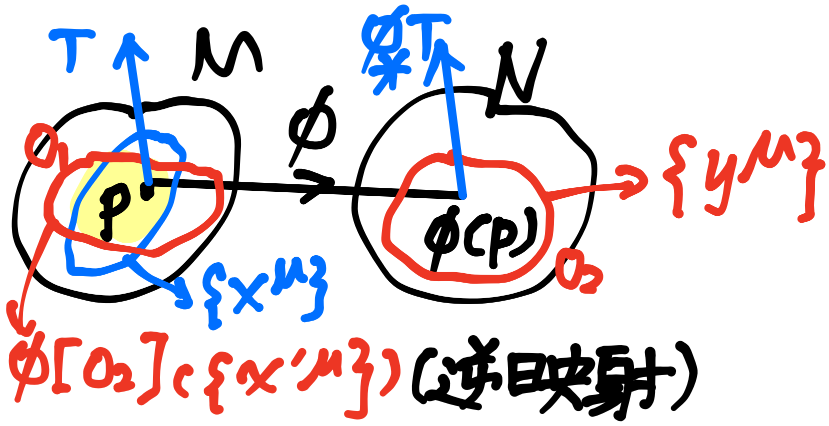
$x'^\mu(p) := y^\mu(\phi(p))$

在intersection中 有 $\{x^\mu\} \xmapsto{\phi} \{x'^\mu\}$

2 viewpoints

① active viewpoint $p \to \phi(p) \Rightarrow T \to \phi_* T$

② passive viewpoint $\phi: M \to N \Rightarrow \{x^\mu\} \mapsto \{x'^\mu\}$
(T不变)



(逆映射)

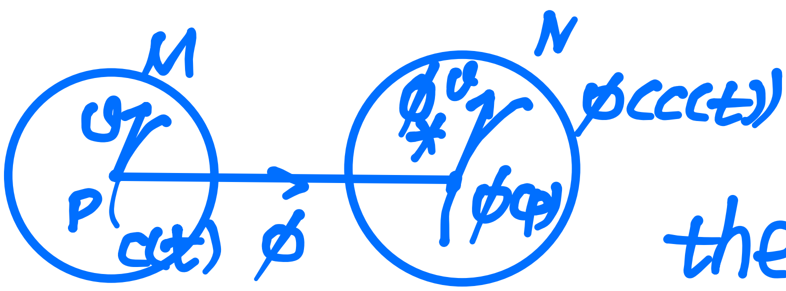
$\{y^\mu\}$

Claim $(\phi_* T)^{\mu}{}_{\nu}|_{\phi(p)} = T'^{\mu}{}_{\nu}|_p$

新点的新张量在老坐标系下值等于老点的老张量在新坐标系下的值

即主被动等价。（证明是习题）

Claim 2



v is Tangent, the $\phi_* v$ is a tangent

（习题）

即切线的像的切矢等于切线切矢的像

像之切 = 切之像

证明上述结论可用如下论断

$\{x'^\mu(p)\} \mapsto \{y^\mu(\phi(p))\}$,

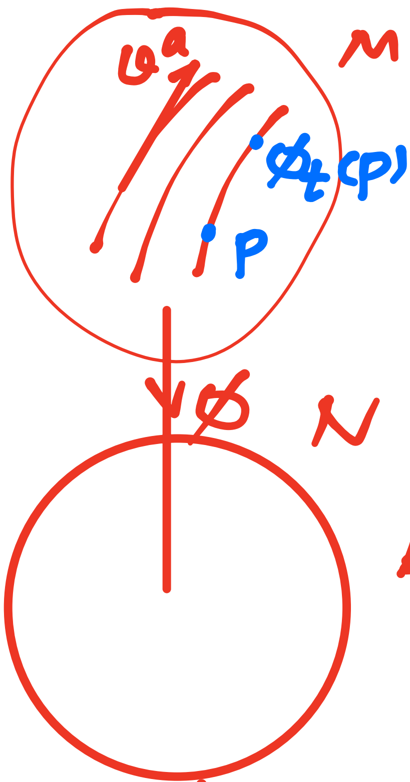
那么 $\{x'(p)\}$ 中一条 coordinate line 应与 $\{y^\mu(\phi(p))\}$ 中对应同一位置，二者相等

$$\phi_*\left[\left(\frac{\partial}{\partial x'^\mu}\right)^a\Big|_p\right] = \left(\frac{\partial}{\partial y^\mu}\right)^a$$

又由于 $dx'^\mu \frac{\partial}{\partial x'^\nu} = \delta^\mu{}_\nu$

可知 $\phi_*[(dx'^\mu)_a|_p] = (dy^\mu)^a\Big|_{\phi(p)}$

§4.2 Lie derivative



$\{\phi_t \mid t \in \mathbb{R}\}$, $\phi_t : M \to M$

（单参微分同胚群）

However, we can define $\phi_t : M \to N$, $t \in \mathbb{R}$

EX: $T^a{}_b$

$\phi^* : \mathscr{F}_N(k,l) \longrightarrow \mathscr{F}_M(k,l)$

考虑 $I = \frac{1}{t}(\phi_t^* T^a{}_b - T^a{}_b)|_p$

if $t \to 0$, $\phi_t^* \to E$（单位元）

I is called Lie derivative of $T^a{}_b$ with respect to v^a 　T 关于 v 的李导数

$$\mathscr{L}_v T^a{}_b := \lim_{t\to 0} \frac{1}{t}(\phi_t^* T^a{}_b - T^a{}_b)|_p$$

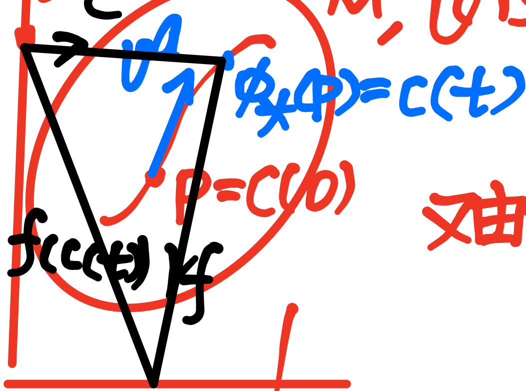
Claim 1

$$\mathscr{L}_v f = v(f)$$

<u>pf</u> $\mathscr{L}_v f|_p \equiv \lim \frac{1}{t}(\phi_t^* f - f)|_p = \lim \frac{1}{t}[$

M, v^a is vector field

M, v is vector field

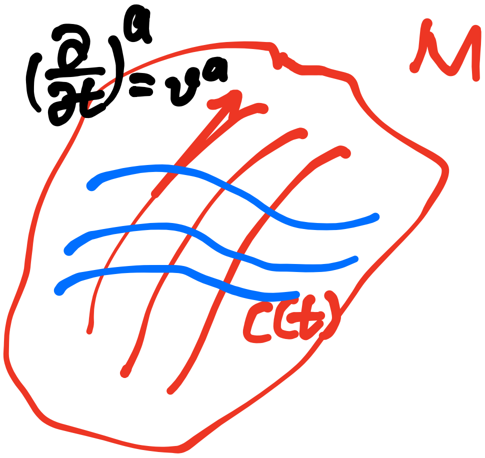


对于 $\phi^* f|_p \equiv f|_{\phi(p)}$

$$\mathcal{L}_v f|_p \equiv \lim \frac{1}{t}(\phi_t^* f - f)|_p = \lim \frac{1}{t}[f|_{\phi_t(p)} - f|_p]$$

$$= \lim_{t\to 0} \frac{1}{t}[f(c(t)) - f(c(0))]$$

$$= \underbrace{\frac{d}{dt}}_{} f(c(t))\Big|_{t=0} = v(f)$$ (Tangent vector)



$\{x^1, x^2\}$

将 x^1 选为 t

即 $v^a = (\frac{\partial}{\partial t})^a = (\frac{\partial}{\partial x^1})^a$,

再选一系列斜截线（与 $c(t)$ 不平行）

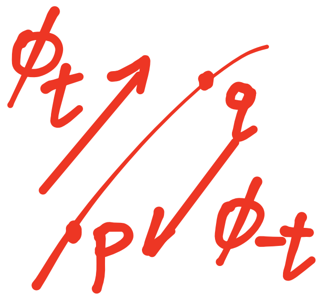
以同样方法由斜截线可定义出 x^2

<u>Claim2</u> $(\mathcal{L}_v T)^\mu{}_\nu = \frac{\partial T^\mu{}_\nu}{\partial x^1}$

pf: $(\mathcal{L}_v T)^\mu{}_\nu = \lim_{t\to 0} \frac{1}{t}[(\phi_t^* T)^\mu{}_\nu|_p - T^\mu{}_\nu|_p]$

$\phi_t^* = (\phi_t^{-1})_*$ 记为 ϕ_{-t*}

$\therefore (\phi_t^* T)^\mu{}_\nu|_p = (\phi_{-t*} T)^\mu{}_\nu|_p$



$q = \phi_t(p)$, so for ϕ_{-t} , q is the old one and p is the new one $\therefore$ "

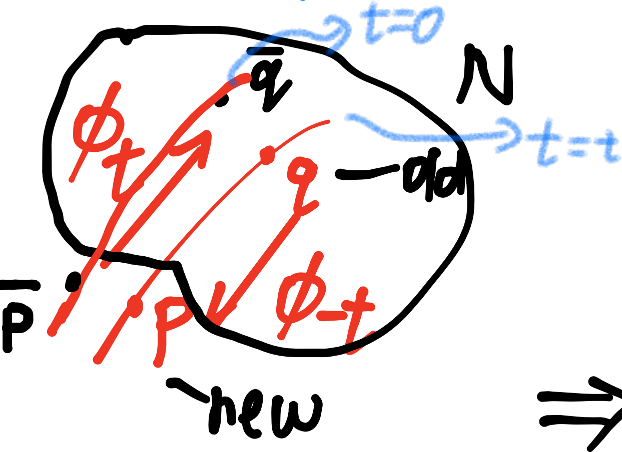
$\therefore (\phi_{-t*} T)^\mu{}_\nu|_p$ is "新新老"

它等于"老老新"

↓新系

$$(\phi_t^* T)^\mu{}_\nu|_p = (\phi_{-t*} T)^\mu{}_\nu|_p = T'^\mu{}_\nu|_q$$

$$= \left(\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma\right)\Big|_q$$



只改变t

$$\begin{cases} x'^1(\bar{q}) \equiv x^1(\bar{p}) = x^1(\bar{q}) - t \\ x'^2(\bar{q}) \equiv x^2(\bar{p}) = x^2(\bar{q}) \end{cases}$$

$$\Rightarrow \begin{cases} x'^1 = x^1 - t, \\ x'^2 = x^2 \end{cases}$$

$$\therefore \left(\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu} T^\rho{}_\sigma\right)\Big|_q = \left(\delta^\mu_\rho \delta^\sigma_\nu T^\rho{}_\sigma\right)\Big|_q$$

$\therefore \left(\frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} T^{\rho}{}_{\sigma}\right)\Big|_q = \cdots$

$= T^{\mu}{}_{\nu}|_q$

$\therefore$

$(\mathcal{L}_v T)^{\mu}{}_{\nu} = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^* T)^{\mu}{}_{\nu}|_p - T^{\mu}{}_{\nu}|_p\right]$

$= \lim_{t\to 0} \frac{1}{t}\left[T^{\mu}{}_{\nu}|_q - T^{\mu}{}_{\nu}|_p\right] = \frac{\partial T^{\mu}{}_{\nu}}{\partial x^1}\Big|_p$ □

Claim 3 $\mathcal{L}_v u^a = [v,u] = v^b \nabla_b u^a$

$- u^b \nabla_b v^a$

pf $[v,u]^{\mu} = (dx^{\mu})_a [v,u]^a = (dx^{\mu})_a$

$(v^b \partial_b u^a - u^b \underbrace{\partial_b v^a}_{=0})$

$= v^b \partial_b u^{\mu}$

$= v(u^{\mu})$

$= \frac{\partial u^{\mu}}{\partial x^1} = \mathcal{L}_v u$

$\because \left(\frac{\partial}{\partial x^1}\right)^a = v^a$ (坐标基矢)

$\therefore \partial_b v^a = 0$ 同坐标下对基矢求导为0

claim 4

$$\boxed{\mathcal{L}_v \omega_a = v^b \nabla_b \omega_a + \omega_b \nabla_a v^b}$$

pf : $(\mathcal{L}_v \omega_a)_{\mu} = \left(\frac{\partial}{\partial x_{\mu}}\right)^a (v^b \partial_b \omega_a + \omega_b$

$\nabla_a v^b)$

$$= \left(\frac{\partial}{\partial x^\mu}\right)^a v^b \partial_b \omega_a = v^b \partial_b \omega_\mu$$

$$= \frac{\partial \omega_\mu}{\partial x^1} = (\mathcal{L}_v \omega_a)_\mu$$

Generally, For $T \in \mathscr{T}(k,l)$

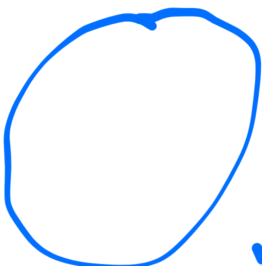
$$\mathcal{L}_v T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$$

$$= v^c \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_k} - \sum_{i=1}^{k} T^{a_1 \cdots c \cdots a_k}{}_{b_1 \cdots b_l} \nabla_c v^{a_i}$$

$$+ \sum_{j=1}^{l} T^{a_1 \cdots a_k}{}_{b_1 \cdots c \cdots b_l} \nabla_{b_j} v^c$$

口诀：上负下正，负缩一二，正缩一三

§4.3 Killing vector field



(M, g_{ab}) suppose we have a metric

$$\phi : M \to M$$

if $\phi^* g_{ab} = g_{ab}$, ϕ is not only diffeomorphism, but also called isometry (等度规)

$v^a \mapsto \{\phi_t \mid t \in \mathbb{R}\}$, $\phi_t : M \to M$ is a diff

However, on a subset in $\mathscr{F}_M(1,0)$

$\mathscr{F}_M(1,0)$ we can induce a group from ξ^a

namely $\xi^a \longmapsto \{\phi_t \mid \cdots\}$ $\phi_t : M \to M$ is iso

(单参等度规群)

So that ξ^a is called Killing field

$\therefore\ \mathcal{L}_\xi g_{ab} = \lim \frac{1}{t}(\phi^* g_{ab} - g_{ab}) = 0$

So ξ^a is Killing $\Longleftrightarrow \mathcal{L}_\xi g_{ab} = 0$

由上式

$$0 = \mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + g_{cb} \nabla_a \xi^c + g_{ac} \nabla_b \xi^c =$$

choose ∇_a s.t. $\nabla_c g_{ab} = 0$

$$= g_{cb} \nabla_a \xi^c + g_{ac} \nabla_b \xi^c$$

$$= \nabla_a \xi_b + \nabla_b \xi_a = 2\nabla_{(a}\xi_{b)}$$

So

ξ^a is Killing $\Longleftrightarrow \mathcal{L}_\xi g_{ab} = 0 \Leftrightarrow \nabla_{(a}\xi_{b)} = 0$

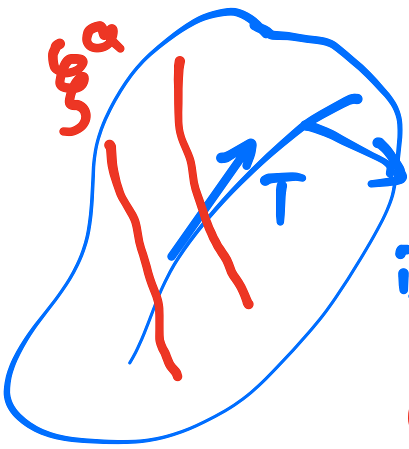
$\Longleftrightarrow \nabla_a \xi_b = \nabla_{[a}\xi_{b]}$

Claim $\exists \{x^\mu\}$ s.t. $\frac{\partial g_{\mu\nu}}{\cdots} = 0 \Longrightarrow \left(\frac{\partial}{\cdots}\right)^a$ is

Claim $\exists\{x^\mu\}$ s.t. $\frac{\partial g_{\mu\nu}}{\partial x^1} = 0 \Rightarrow \left(\frac{\partial}{\partial x^1}\right)$ is killing

pf: $(\mathcal{L}_{\frac{\partial}{\partial x^1}} g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^1} = 0$

so $\frac{\partial}{\partial x^1}$ is Killing



(M, g_{ab})

$\gamma(t)$ is geodesic

claim: $T^b\nabla_b(\xi_a T^a) = 0$

namely $\xi_a T^a = \text{const}$

pf: $T^b\nabla_b(\xi_a T^a)$

$= (T^b\nabla_b\xi_a)T^a + \xi_a\underbrace{(T^b\nabla_b T^a)}_{\text{Geodesic, equals 0}}$

$= T^aT^b\nabla_b\xi_a \qquad \because \nabla_b\xi_a = \nabla_{[b}\xi_{a]}$ ~~Killing's~~ def

$= T^aT^b\nabla_{[b}\xi_{a]}$, 又由于 T^aT^b 和 T^bT^a 等价

$\Rightarrow T^{(a}T^{b)}\nabla_{[b}\xi_{a]}$

$= 0$ (异括号归零)

[illegible] $T^aT^b\nabla\xi \quad T^bT^a\nabla\xi$ (哑指标)

or $T^aT^b\nabla_b\xi_a = T^bT^a\nabla_a\xi_b$ (∵哑指标)

又 $T^aT^b = T^bT^a$, $\nabla_b\xi_a = -\nabla_a\xi_b$

so $2T^aT^b\nabla_b\xi_a = 0$

Killing field's characters

if ξ^a, η^a is killing

① $\alpha\xi^a + \beta\eta^a \in \mathscr{K}$

$\mathscr{K}$ is a vector space (easy to proof)

$\dim\mathscr{K} \leq \frac{n(n+1)}{2}$, $\dim M = n$

$\phi_t^* g_{ab} = g_{ab}$

Generally $\dim\mathscr{K}$ 越大 symmetry 越好. 若等于 $\frac{n(n+1)}{2}$，称为最高对称性空间

EX:

ex1: $(\mathbb{R}^2, \delta_{ab})$

取笛卡尔坐标系 $ds^2 = dx^2 + dy^2$

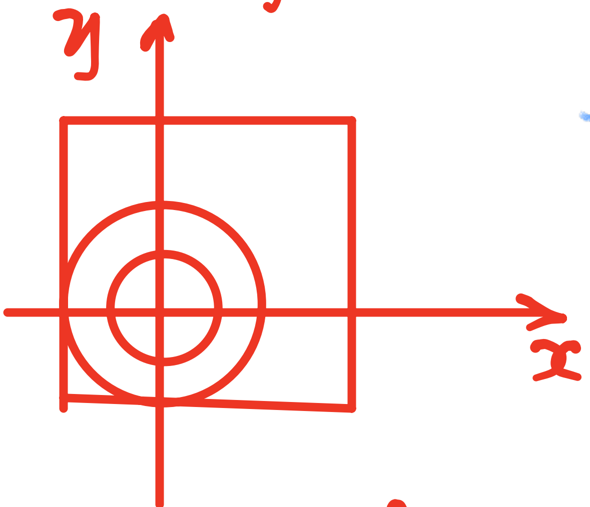
$$g_{11}=1 \quad g_{22}=1 \quad g_{12}=0$$

$$\therefore \frac{\partial g_{\mu\nu}}{\partial x}=\frac{\partial g_{\mu\nu}}{\partial y}=0 \qquad \therefore \xi_1=\frac{\partial}{\partial x},\ \xi_2=\frac{\partial}{\partial x_2}$$

$$\{x,y\}\mapsto\{r,\varphi\},\ x=r\cos\varphi,\ y=r\sin\varphi$$

$$\because ds^2=dr^2+r^2d\varphi^2$$

$$\therefore \frac{\partial g_{\mu\nu}}{\partial\varphi}=0 \qquad \therefore \xi_3=\frac{\partial}{\partial\varphi}=\boxed{-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}}$$



故$(\mathbb{R}^2,\delta_{ab})$对称性最高
(2个平移，1个旋转)

EX2

$(\mathbb{R}^3,\delta_{ab})$

3个平移 $\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}$

3个旋转

$$-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}$$

EX3 $(\mathbb{R}^2,\eta_{ab})$,

$$ds^2=-dt^2+dx^2$$

2个平移 同上

1个超转

$\{x,t\} \mapsto \{\psi,\eta\}$, $x=\psi\,\mathrm{ch}\,\eta$, $t=\psi\,\mathrm{sh}\,\eta$

$\therefore\ ds^2 = d\psi^2 - \psi^2 d\eta^2$

$\therefore\ \dfrac{\partial \eta_{ab}}{\partial \eta} = 0$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$$

(boost 的转动)

对比

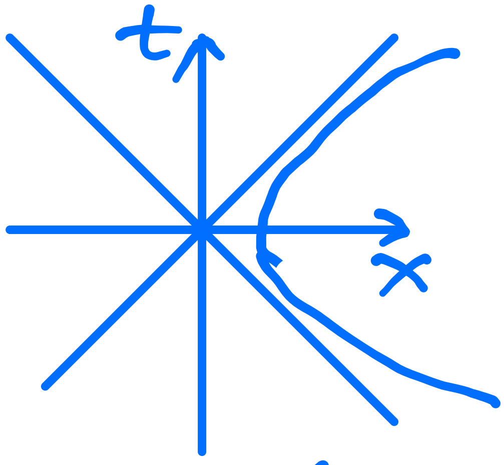
$x^2+y^2=r^2 \xRightarrow{\frac{\partial}{\partial \varphi}\text{的积分曲线}}$ 同心圆

$x^2-t^2=\psi^2 \xRightarrow[\frac{\partial}{\partial \eta}\text{的积分曲线}]{}$ 双曲线（同渐近线）

主动观点

boost $\Longleftrightarrow$ Lorentz transformation

↳ Killing field ⇒ 单参微分同胚群 ↗ 被动观点（坐标变换）



EX4 : $(\mathbb{R}^4, \eta_{ab})$ $\qquad \frac{4\times5}{2}=10$

① 4个平移（同上）

② 3个空间分量中的 rotation

Such as $\{t, x, y, z\} \mapsto \{t, r, \varphi, z\}$

it includes C_3^2 kinds

③ 3个 t 分量与任一空间分量的 boost

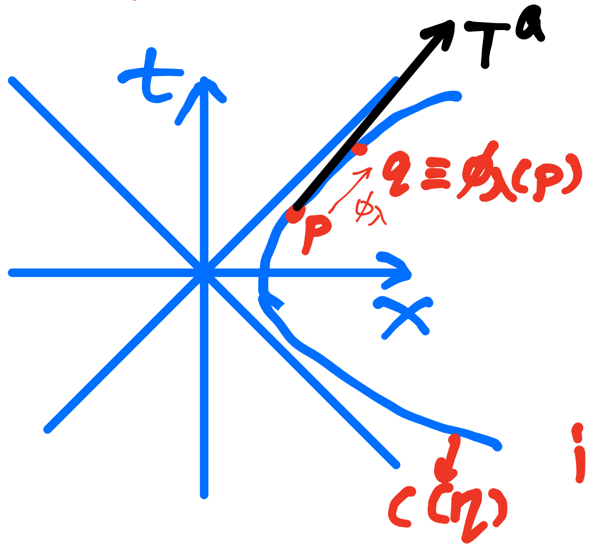
Such as $\{t, x, y, z\} \mapsto \{\psi, \eta, y, z\}$

共10种

For boost

$\frac{\partial}{\partial \eta} \longmapsto \{\phi_\lambda \mid \lambda \in \mathbb{R}\}$ (isometry)

$\phi_\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



How to proof ϕ_λ is Lorentz transform? namely pf that

$$\left\{ \begin{array}{l} x^\mu|_p \\ \downarrow \\ x'^\mu|_p \equiv x^\mu|_q \end{array} \right.$$

is L- transform

$\underline{x^\mu(\eta)}$

$\because T^a = x\left(\frac{\partial}{\partial t}\right)^a + t\left(\frac{\partial}{\partial x}\right)^a = \frac{\partial}{\partial \eta}$

$\{ T^\mu = (x, t)$ ，同时地

$$T^\mu = \left(\frac{dt}{d\eta}, \frac{dx}{d\eta}\right)$$

$$\Rightarrow \frac{dt}{d\eta} = x, \quad \frac{dx}{d\eta} = t$$

$$\therefore \begin{cases} x(\eta) = x_p \operatorname{ch}\eta + t_p \operatorname{sh}\eta \\ t(\eta) = x_p \operatorname{sh}\eta + t_p \operatorname{ch}\eta \end{cases}$$

可以验证

$$\frac{dt}{d\eta} = x \quad \frac{d^2t}{d\eta^2} = \frac{dx}{d\eta} = t \Rightarrow \frac{d^2t}{d\eta^2} - t = 0$$

故 $t = A\operatorname{ch}\eta + B\operatorname{sh}\eta$

$$x = \frac{dt}{d\eta} = A\operatorname{sh}\eta + B\operatorname{ch}\eta$$

又代入初态 $\eta=0$ 时 $x(0)=x(p)$, $t(0)=t(p)$

知 $A=t_p$, $B=x_p$

$\therefore x'_p \equiv x_q$ （注意，ϕ_λ 中 λ 是不动参数）

$= x(\lambda) = x_p \operatorname{ch}\lambda + t_p \operatorname{sh}\lambda$

$t'_p \equiv t_q$

$= t(\lambda) = x_p \operatorname{sh}\lambda + t_p \operatorname{ch}\lambda$

$$\Longrightarrow \begin{cases} x' = x\operatorname{ch}\lambda + t\operatorname{sh}\lambda \\ t' = x\operatorname{sh}\lambda + t\operatorname{ch}\lambda \end{cases}$$

（这段有循环论述之嫌，η本身是“猜出”的双曲换元）

$$\Rightarrow \begin{cases} x' = \operatorname{ch}\lambda\,(x + t\operatorname{th}\lambda) \\ t' = \operatorname{ch}\lambda\,(t + x\operatorname{th}\lambda) \end{cases}$$

令 $v \equiv \operatorname{th}\lambda$, $\operatorname{ch}\lambda = \frac{1}{\sqrt{1-v^2}} \equiv \gamma$

$$\therefore \begin{cases} x' = \gamma(x+vt) \\ t' = \gamma(t+vx) \end{cases}$$

此即 Lorentz transformation

洛 boost 洛

$\{t, x\} \mapsto \{t', x'\}$

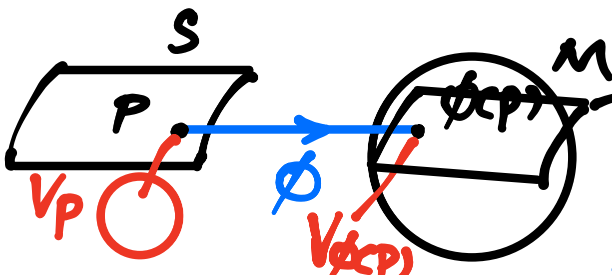
$\{t', x'\}$ is also a Lorentz system

For rotation, it is also right

So claim 洛 $\xmapsto{\text{isometry}}$ 洛

pf is on the book

§4.4 超曲面



$\to \phi[S]$

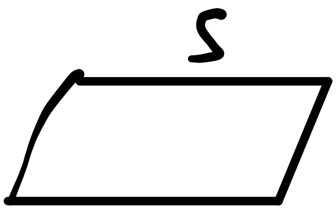
$\dim S \leq \dim M$

Def: $\phi: S \to M$ is called an imbedding if ϕ is

① C^∞, ② 1-1 and ③ $\phi_*: V_p \to V_{\phi(p)}$ is nondegenerate $\forall p \in S$

so $\phi: S \longrightarrow \phi[S]$ is also right

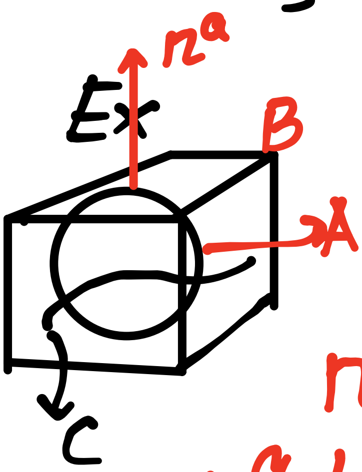
and ϕ is diffeomorphism to each other



ϕ or $\phi[S]$ is called imbedding submanifold, if $\dim S = \dim M - 1$

S is called hypersurface (超曲面)

Ex



$\dim A = 2 = \dim B - 1$

So A is hypersurface

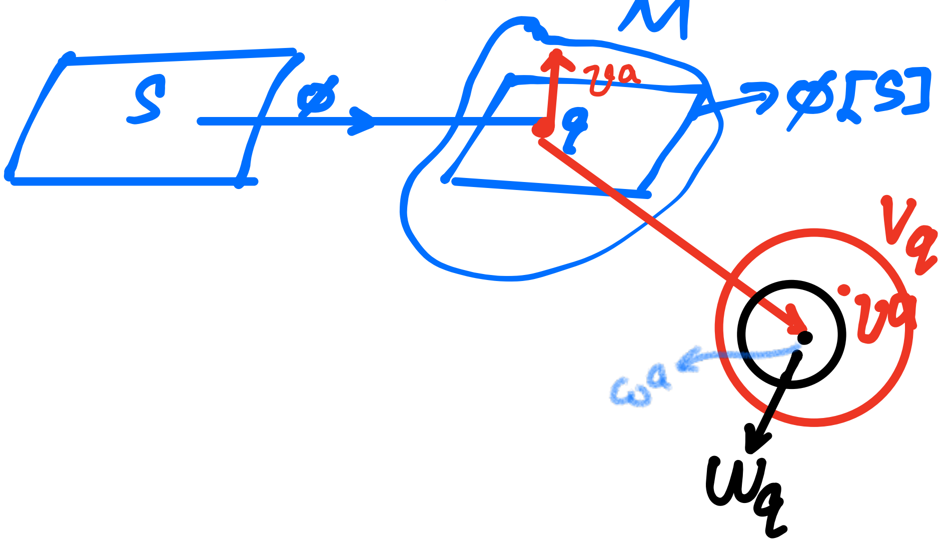
n^a : normal vector (法矢)

n^a 存在，且在只有长度因素的条件下唯一

But for C : $\dim C = 1 = \dim B - 2$

we cannot find such great n^a

imbedding



① V_q, vector space of q on M

② W_q, vector space of q on $\phi[S]$

Suppose $\phi[S]$ is hypersurface, How to get the normal vector

But we cannot define "normal"

If we don't have metric duel vector

However, we can define normal covector

def: $n_a w^a = 0$, $\forall w^a \in W_q$

因为缩并不需要度规

先证 n_a 的存在性

① suppose the basis of W_q is $\{(e_2)^a, \dots, (e_n)^a\}$

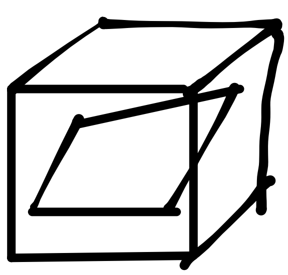
and find a vector in V_q but don't in W_q as the $(e_1)^a$, of course they can be normal, 令 $n_a = (e_1)^{a*} = (e^1)_a$

$\therefore n_a (e_\tau)^a = (e^1)_a (e_\tau)^a = \delta_{1\tau} = 0 \quad \tau = 2, 3, \dots$

唯一性则与法向量同，可乘一常系数

(pf is on book)

Ex



$f_1 = ax+by+c=0$ is a hypersurface (平面)

$f_2 = x^2+y^2+z^2-a^2=0$ is also a hypersurface

However, if $a \to 0$, that is not right, if $a=0$.

$$df\Big|_{f=0} = 2(x+z+y)\Big|_{f=0} = 0$$

故如果 df 在你想定义的面上为0，则给不出超曲面，反之则

可得出

so $df|_{f=0} \neq 0$ or $\nabla_a f|_{f=0} \neq 0$

claim : $\nabla_a f|_p = n_a|_p$ (normal covector)

pf : $\boxed{w^a \nabla_a f = w(f)} = \frac{\partial}{\partial t}(f) = \frac{\partial(0)}{\partial t} = 0$

求导的定义　　Tangent vector

and if we have metric g_{ab}

we can define normal vector

$v_a \equiv w^b g_{ab}$

suppose $g_{ab} = \eta_{ab}$, n_a can be an element in W_q, iff n_a is null vector (类光矢量)

(也就是 n_a 和自己正交, 正交与类光性相合)

Claim $n^a \in W_q \iff n_a n^a = 0$

pf $(\Rightarrow)$ $n^a w_a = 0 \Longrightarrow n_a n^a = 0$

$\cap$
W_q

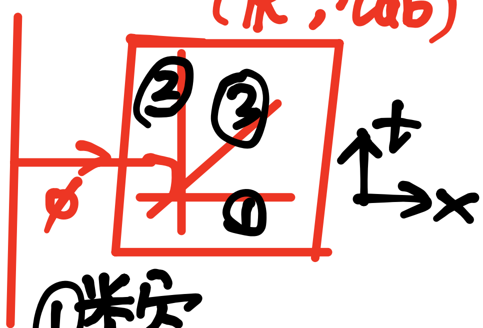
W_q

$(\Leftarrow)$ 由于余法矢存在性内的证明，n^a 的2.3，…，n分量均在 W_q 的基底下，且1分量为 $n^a=(e_1)^a$

$\therefore n^1 = n^a (e^1)_a = n^a n_a = 0$，故1分量为0，$n^a \in W_q$

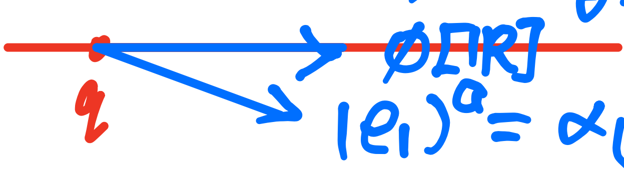
$(\mathbb{R}^2, \eta_{ab})$

EX:



①类空

$(e_2)^a = (\frac{\partial}{\partial x})^a$



$(e_1)^a = \alpha (\frac{\partial}{\partial t})^a + \beta (\frac{\partial}{\partial x})^a, \ \alpha \neq 0$

取 duel vector $(e^1)_a = (dt)_a \alpha^{-1} = n_a$

$\therefore n^a = g^{ab} n_b = \alpha^{-1} g^{ab} (dt)_b = -\alpha^{-1} (\frac{\partial}{\partial t})^a$

③类光

取 $(e_2)^a = (\frac{\partial}{\partial t})^a + (\frac{\partial}{\partial x})^a$

$(e_1)^a = \alpha (\frac{\partial}{\partial t})^a + \beta (\frac{\partial}{\partial x})^a \quad (\alpha \neq \beta)$

则 duel vector

$\boxed{n_a = (e^1)_a = (\alpha - \beta)^{-1} [(dt)_a - (dx)_a]}$ ×

$\therefore n^a = g^{ab} n_b = (\alpha-\beta)^{-1} [g^{ab} (dt)_b - g^{ab} (dx)_b]$

$= (\alpha-\beta)^{-1} [(\frac{\partial}{\partial t})^a + (\frac{\partial}{\partial x})^a]$

$= (\alpha-\beta)^{-1}(e_2)^{a} \in W_q$