Differential Geometry & GR

Chap 1. Introduction to topological spaces

1.01

$X \to$ set $\quad x \in X \quad A \subset X$ (subset) $\quad$ set $\Rightarrow \{\quad\}$

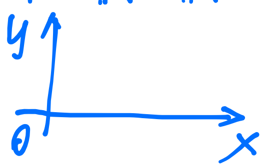
$\to$ element $\quad$ ex.: $X = \{3, 72\}$

$\mathbb{R}$, ex. $X = \{x \in \mathbb{R} \mid x > 9\}$ real numbers which are bigger than 9

Suppose two set X, Y $\Rightarrow X \times Y \equiv \{(x, y) \mid x \in X, y \in Y\}$

Suppose $X = \mathbb{R} = Y \Rightarrow X \times Y = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ $\quad \mathbb{R} \Rightarrow \mathbb{R}^1$ ("r" one)

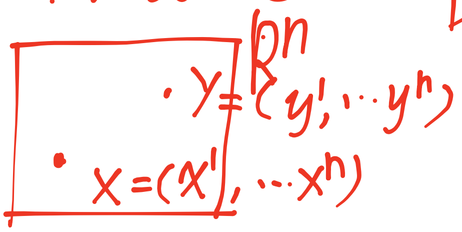
$\mathbb{R}^2$ is a plane



$\mathbb{R}^n \to$ natural coordinates (自然坐标)

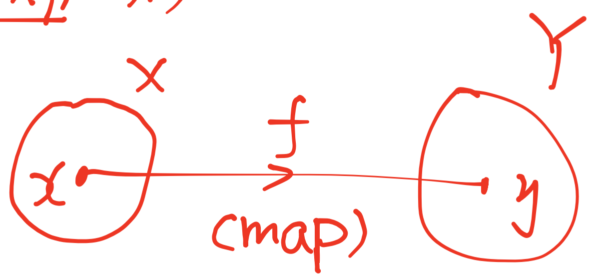
$\to$ set of coordinates

Distance



$$D \equiv |y - x| = \sqrt{\sum_{i=1}^{n} (y^i - x^i)^2}$$

$f: X \to Y$



f is a map from set X to set Y

y is called the image of x, x is the inverse image of y, $y = f(x)$: called "'f' of 'x'"

$f: x \mapsto y$ (element) is also good.

Some special maps

① 1→1 (单射) $\qquad f: \mathbb{R} \to \mathbb{R}$ (一元函数)

② onto (满射)

$\mathbb{R} \to \mathbb{R}$

$f: \mathbb{R}^n \to \mathbb{R}^m$ (m个n元函数)

For $f: \mathbb{R} \to \mathbb{R}$

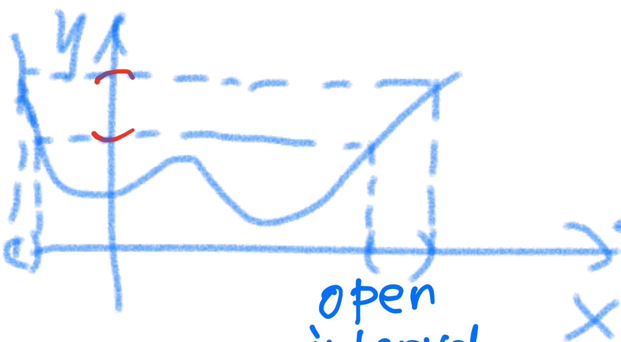
③ C^0 => Continuous

defination: $f: \mathbb{R} \to \mathbb{R}$ is C^0 at $x \in \mathbb{R}$ ($\varepsilon - \delta$) if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. (使得) $|x' - x| < \delta \Rightarrow |f(x') - f(x)| < \varepsilon$

rewrite ⟹

$\boxed{\text{$f: \mathbb{R} \to \mathbb{R}$ is C^0 if Y的任一开区间的 "inverse image" is X开区间之并}}$

visual picture

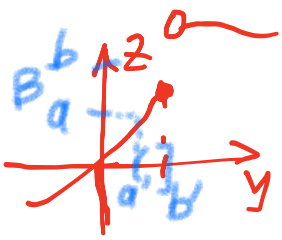


if $f: x \mapsto y$ is 1-1, $\exists f^{-1}: y \mapsto x$, For ,

if we consider a subset B in Y, "Inverse Image" of B is $f^{-1}[B] \equiv \{x \in X \mid f(x) \in B\}$

1.02

discontinuous examples



B = (a, b) inverse image is (a', b'], shows that new defination is right

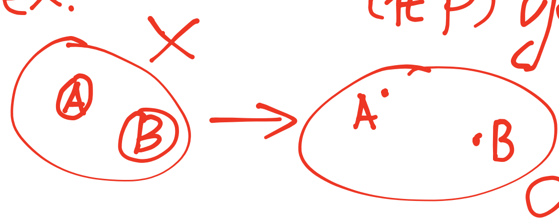
"开区间之并" ═══ open subset in $\mathbb{R}^1$

$\begin{cases} \text{open subset} \\ \text{"非开子集"} \begin{cases} (\ ] \\ [\ ) \\ [\ ] \end{cases} \end{cases}$ only two kinds.

=> So For other sets, how to define "open subset"?

So topology is a subject which is equaled to find a way to the "open subset" in a set.

ex. (在P) $\mathcal{P}$



A, B, ... are subsets of X
if we define a subset of $\mathcal{P}$ $\mathcal{T}$, which include A. Any element in $\mathcal{T}$ is "open subset", so that we choose one Topology.

$\mathcal{T}$ should have three natures

(a) $X, \phi \in \mathcal{T}$ (b) $O_i \in \mathcal{T}, i=1,2,\cdots,n \Rightarrow \bigcap_{i=1}^{n} O_i \in \mathcal{T}$ (n isn't infinate) (c) $O_\alpha \in \mathcal{T}, \forall \alpha, \Rightarrow \bigcup_\alpha O_\alpha \in \mathcal{T}$ (α can be infinate)

Topological space

Defination: $(X, \mathcal{T})$ So $(X, \mathcal{T}) \neq (X, \mathcal{T}')$

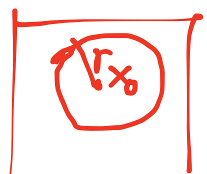
ex. ① Discrete Topology: $\mathcal{T}_1$ includes all subsets of X

② Coherent Topology: $\mathcal{T}_2 \equiv \{X, \phi\}$

$\mathcal{T}_1$ has the most open subsets while $\mathcal{T}_2$ has the least

$X = \mathbb{R}^n$

Usual Topology ※



Firstly, explain "open ball", suppose $\mathbb{R}^n$

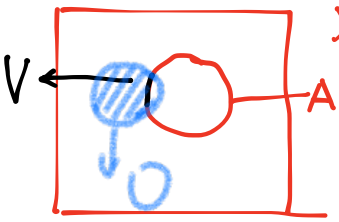
$B(x_0, r) \equiv \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$

So Usual Topology $\mathcal{T}_u \equiv \{$可表为开球之并的子集$\}$

$(X,\mathscr{T})$ $V\subset A\subset X$ How to define a "Topological subspace"

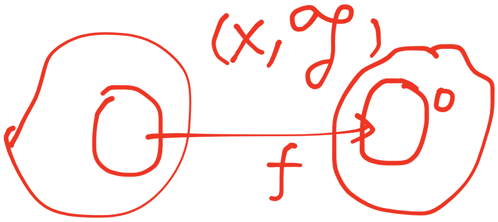
$(A,\mathscr{S})$ $\mathscr{S}\equiv\{V\subset A \mid V\in\mathscr{T}\}$ ✗ (if $A\notin\mathscr{T}$, $V=A$, this defination is against "(a)": $A,\varnothing\in\mathscr{S}$

$$\mathscr{S}\equiv\{V\subset A \mid \exists O\in\mathscr{T} \text{ s.t. } O\cap A=V\}$$



$X=\mathbb{R}^2$ is usual Topology, make a circle $A\subset X$, $A\notin\mathscr{T}_u$. choose V is a line around A, so, $\mathscr{S}$ can be a nice induced topology. and $(A,\mathscr{S})$ is a topological subspace.

$(X,\mathscr{T})$ $(Y,\mathscr{S})$ $f: X\to Y$ is C^0 if $O\in\mathscr{S}$



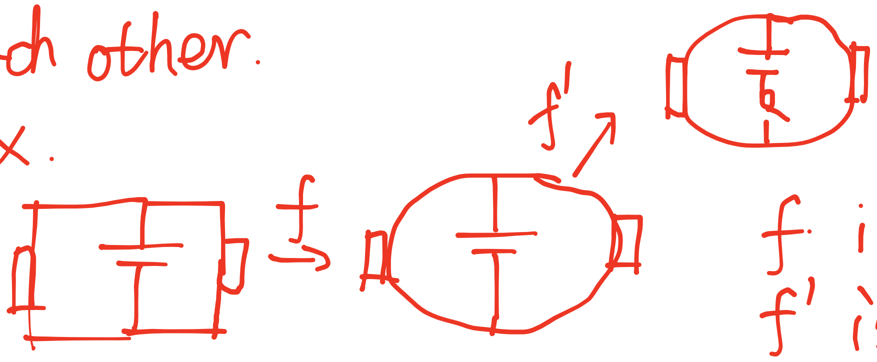
$\Rightarrow f^{-1}[O]\in\mathscr{T}$

if ① f is one to one ② f, f^{-1} is C^0

$\Rightarrow$ f is homeomorphism 同胚

$(X,\mathscr{T})$ and $(Y,\mathscr{S})$ is homeomorphic to each other.

ex.



f is homeomorphism
f' is not homeomorphism

Homework: T5, T8, T9, T10

1-03 Chap. 2 Manifolds and Tensor Fields

§2.1 Differentiable manifolds (微分流形)

Supports 3 sets

so map m is called "$g \circ f$"

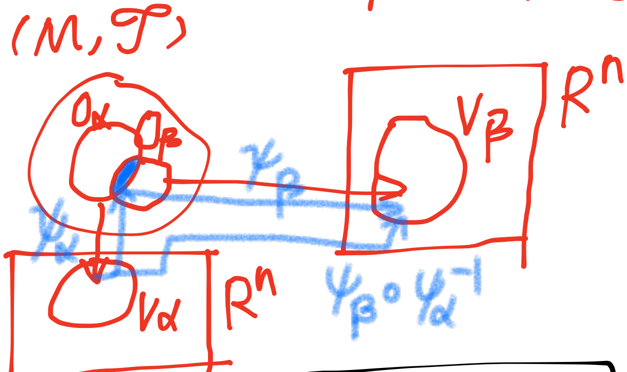
Manifold is important in Physics because we always meet a space which has "background".

A n-dimension manifold is like $\mathbb{R}^n$ locally (局部地)

Def 1. if M has a open cover $\{O_\alpha\}$, ($O_\alpha \in \mathscr{T}$, $M = \bigcup_\alpha O_\alpha$)

The $\{O_\alpha\}$ has 2 features

(a) $\forall O_\alpha\ \exists$ homeo $\psi_\alpha : O_\alpha \to V_\alpha$, V_α is an open subset in $\mathbb{R}^n$



(b) if $O_\alpha \cap O_\beta \neq \phi$, then $\psi_\beta \circ \psi_\alpha^{-1}$ is $\boxed{C^\infty}$ (对交集部分补充要求)

C^∞ 在多元函数微积分中已定义过，而 $\psi_\beta \circ \psi_\alpha^{-1}$ is from $\mathbb{R}^n$ to $\mathbb{R}^n$，可以套用该定义

(b) is called $\boxed{\text{compatibility}}$ condition

Then M can be called a n-dimetional differential manifold.

the coordinate of An element in O_α can be shown by an element in V_α, which is called the coordinate in map ψ_α.

Specially, For the blue part in $(M, \mathscr{T})$, it has two kinds of coordinates $\vec{x}(x_1, x_2, \cdots x_n)$, $\vec{x}'(x_1', x_2', \cdots x_n')$。

So $x_1' = \phi^1(x_1, x_2, \cdots x_n)$ $x_2' = \phi^2(x_1, x_2, \cdots x_n)$

$\cdots$ $x_n' = \phi^n(x_1, x_2, \cdots x_n)$

$\Rightarrow \vec{x'} = A\vec{x}$ A为坐标变换矩阵

这是坐标变换的底层逻辑根源

and, (O_α, ψ_α) is called coodinate system in maths. this is also called "chart" (图)

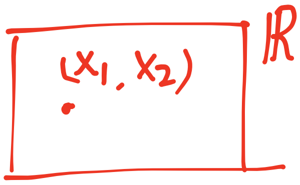
$\{(O_\alpha, \psi_\alpha)\}$ is called "atlas" (图册)

manifold

atlas (恒等映射)

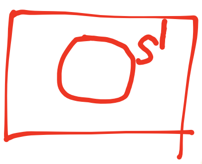
Example 1. $M = \mathbb{R}^2$ $\{(O_1, \psi_1)\}$, $O_1 = \mathbb{R}^2$, $\psi_1 \equiv f: O_1 \to O_1$

This is "trival manifold"



sphere 球面 (指表面)

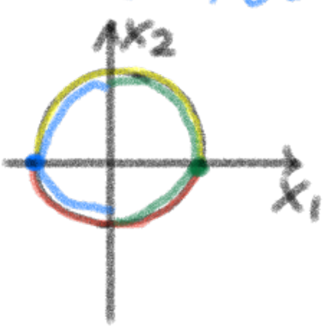
1-04 1. S^1



atlas of S^1 $\{$ $\}$

4 charts: ① 上半圆周 ② 下半圆周

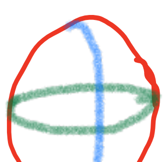
③ 左半平面 ④ 右半平面



对①, ②, ψ是将曲线"映射"在X_1轴上 (相容性证明是习题)

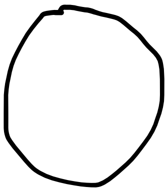
③, ④ 映在X_2轴上

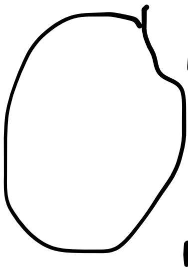
3. S^2 (二维球面)



6 charts, Like the example 2

(左右 上下 前后 现映射到 Disc disk)

对比 $(X,\mathcal{T})$  $(X,\mathcal{T}')$

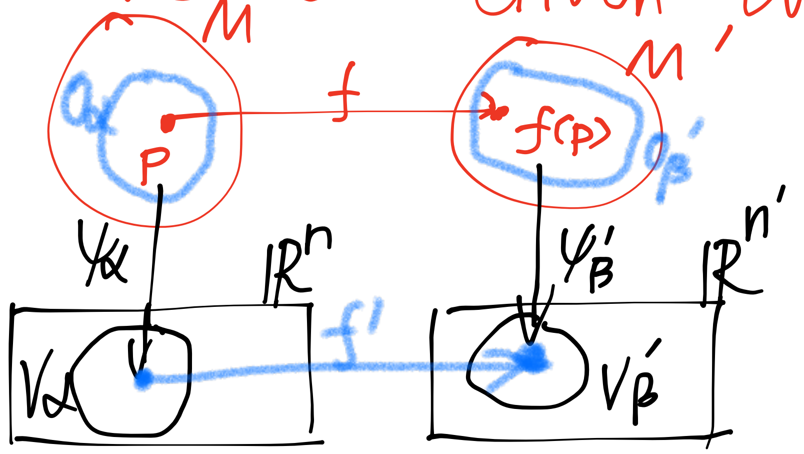


$M \to \{(O_\alpha, \psi_\alpha)\}$ 甲

$\searrow \{(O'_\alpha, \psi'_\alpha)\}$ 乙

if $O_\alpha \cap O'_\beta \neq \phi$, it might not be compatible, so two manifolds are different, so manifold shows M's differencial structure(微分结构). However, if they are compatible, $\{(O_\alpha, \psi_\alpha); (O'_\beta, \psi'_\beta)\}$ is also an atlas of M. So when we say "manifold" of M, we always refer to the "max" atlas. (最大图册)

下面定义"C^r" Given two manifolds



Like the picture on the left, $\because \psi_\alpha, \psi'_\beta$ are homeo $\therefore f'$ is $C^r \Longleftrightarrow f$ is C^r. f' is C^r is easy to understand.

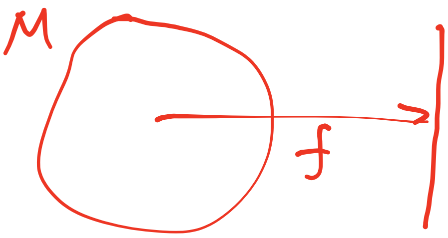
So $f: M \xrightarrow{C^r} M'$ has a defination.

Next problem: How to define a "Homeomorphism" between two manifolds? (which is called diffeomorphism)

类比 Homeomorphism: $f: M \to M'$

① one to one, onto ② $f: M \to M'$ is C^∞ ③ same dimension

Consider a map $f: M \to \mathbb{R}$. This map is called function or (Scalar field)

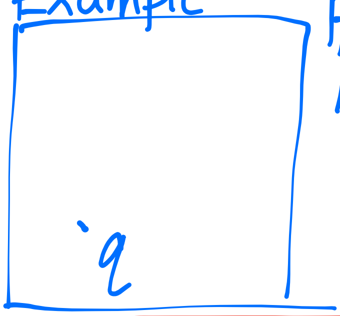


$$f: M \to \mathbb{R} \quad \begin{cases} \xrightarrow{\{x^\mu\}} F(x^1, x^2, \cdots, x^n) \equiv F(x) \equiv f(x) \\ \xrightarrow{\{x'^\mu\}} F'(x'^1, \cdots x'^n) \equiv F'(x) \equiv f(x) \end{cases}$$

so if you call f "function", you should give a $\{x^\mu\}$ (coordinate system), But scalar field don't need.

So $f: M \to \mathbb{R}$ is absolute, but F/F' is relative

Example $\mathbb{R}^3$ consider potential of q



$M \equiv \mathbb{R}^3 - \{q\}$

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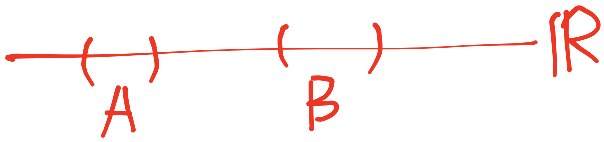
closed subset:

if $-C$ is open, C is closed subset

$X \in \mathcal{T}$, $-X = \phi \in \mathcal{T}$ so X, ϕ are closed subsets



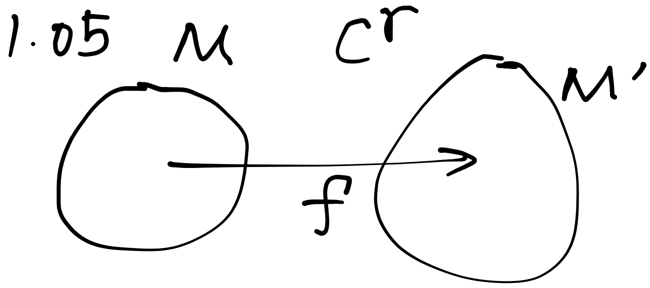
令 $X \equiv A \cup B \subset \mathbb{R}$

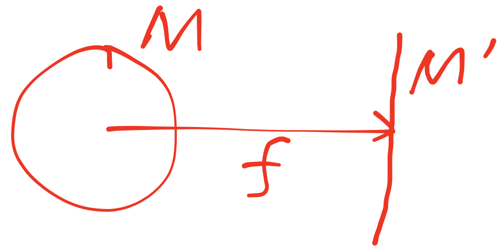
so A, B are also closed subsets

A Topological space is connected if only X and ϕ are both open and closed

Homework Chapter 2, T_1

1.05

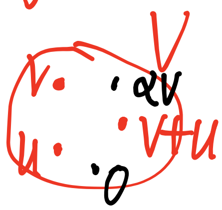




scalar field

§2.2 Tangent vector fields

Vector space: Support a set V, To $\forall v, u \in V$,



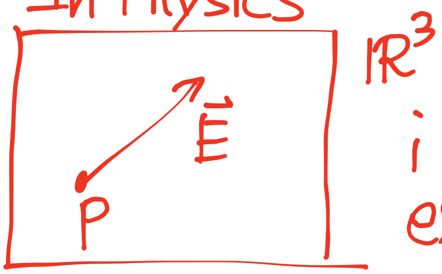
$u+v \in V$, and for $\alpha \in \mathbb{R}$, $\alpha v \in V$ and $0 \in V$

... (7 conditions) → 卡式积

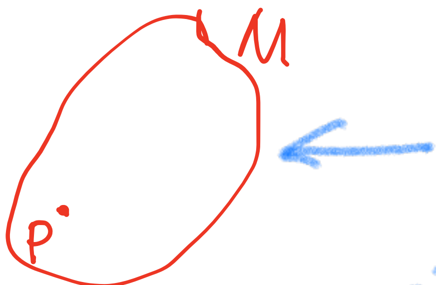
So we can see $V \times V \longrightarrow V$ (加法运算)

and $\mathbb{R} \times V \longrightarrow V$ (数乘运算)

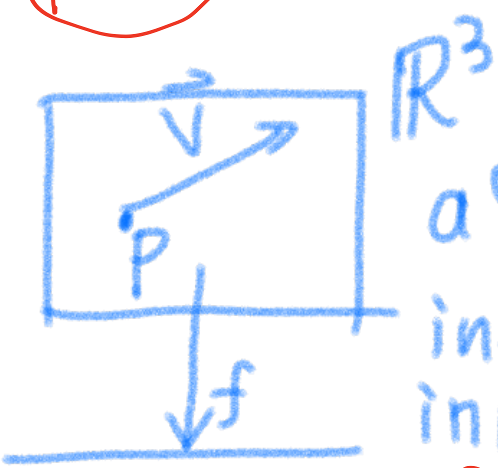
In Physics



is a vector, which is a special example in Maths



How to define a vector space in a manifold M?

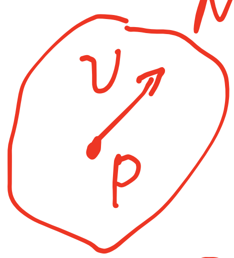


For manifold $\mathbb{R}^3$, define a $\mathcal{F}_{\mathbb{R}^3}$, which is a set including all the $\boxed{C^\infty}$ Scalar field in $\mathbb{R}^3$

For $f \in \mathcal{F}_{\mathbb{R}^3}$, $\vec{v}$ is a map, $\vec{v}: \mathcal{F}_{\mathbb{R}^3} \longrightarrow \mathbb{R}$

(就是对任意一个f在$\vec{v}$方向求方向导数

So



$v: \mathscr{F}_M \to \mathbb{R}^1$

satisfying: ⓐ $\forall f, g \in \mathscr{F}_M,\ \alpha, \beta \in \mathbb{R}^1$

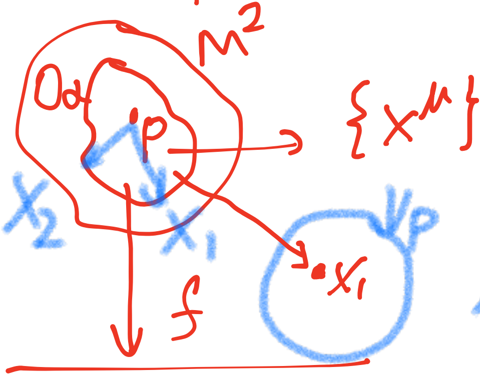
$V(\alpha f + \beta g) = \alpha V(f) + \beta V(g)$ (导数的线性)

$\because (fg) \in \mathscr{F}_M,\ (fg)|_p \equiv f|_p \cdot g|_p$ ⓑ $V(fg) = f|_p V(g) + g|_p V(f)$

(导数莱布尼兹法则)

This is a def of vector in manifold M

Example



$$X_1(f) \equiv \left.\frac{\partial F(x^1, x^2)}{\partial x^1}\right|_p \in \mathbb{R}$$

$$\vdots$$

$$X_\mu(f) \equiv \left.\frac{\partial F(x)}{\partial x^\mu}\right|_p, \forall f \in \mathscr{F}_M$$

proof Claim2: All the vectors at p in a manifold M form a vector space V_p and $\dim V_p = \dim M$

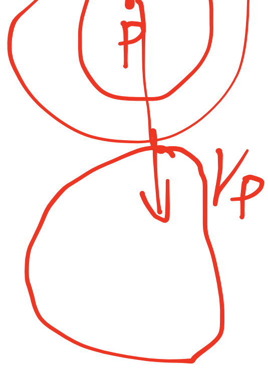
pf: (A) (定义线性) for $\forall v, u \in V_p$ $(v+u)(f) \equiv v(f) + u(f)$ $(\alpha v)(f) \equiv \alpha \cdot v(f)$ $0(f) \equiv 0 \in \mathbb{R}$

(B) (证同维)

1) 构造基底 选刚才的$\{V_\mu\}$做基矢

下面证 $V_1, V_2, \cdots V_n$ 线性无关

假设 $\sum_{\mu=1}^{n} V_\mu \alpha^\mu = 0 \equiv V_\mu \alpha^\mu$, 且$\alpha^\mu \neq 0$



$\nu \in \{1, 2, \cdots, \mu\}$

$$\Rightarrow (\alpha^\mu V_\mu) x^\nu = 0(x^\nu) = 0$$

$$\hookrightarrow \alpha^\mu \cdot (V_\mu(x^\nu)) = 0 \quad ①$$

$$\because V_\mu = \frac{\partial F(x)}{\partial x^\mu} \quad \text{(由上example)}$$

$$\therefore ① \rightarrow \alpha^\mu \cdot \underbrace{\frac{\partial x^\nu}{\partial x^\mu}}_{=\delta^\nu_\mu} = 0 \quad \Rightarrow \alpha^\mu \delta^\nu_\mu = 0$$

$\therefore \alpha^\nu = 0$ 矛盾, So $\{V_\mu\}$ are linearly independent.

So we proof that $\dim V_p \geq n$

(c) pf $\dim V_p \leq n$

$\Leftrightarrow \forall\, v \in V_p$, can show $V = a^\mu V_\mu$ (Wald, 1984)

(略)

How to get a^ν? ($\nu \in \{1, 2, \cdots n\}$)

$$\boxed{V = a^\mu V_\mu} \Rightarrow V(x^\nu) = (a^\mu V_\mu)(x^\nu) = a^\mu \delta^\nu_\mu = a^\nu$$

$$\therefore a^\nu = V x^\nu$$

So V_μ is basis vector, $\{V_\mu\}$ is a coordinate basis.

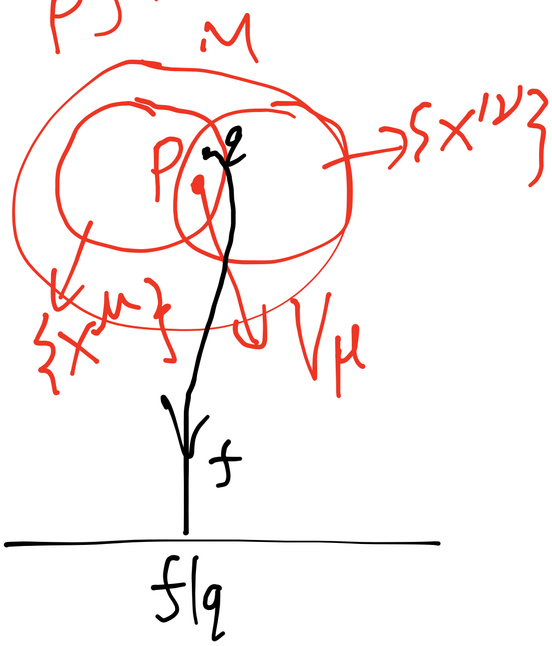
Support 2 coordinate systems

$$v = a^\mu V_\mu = a'^\nu V'_\nu$$

claim 3: $a'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\Big|_p a^{\mu}$ (指标平衡)

This is vector transformation compression.

pf:



$$V_\mu(f) = \frac{\partial f(x)}{\partial x^{\mu}}\Big|_p \quad —— ①$$

$$V'_\nu(f) = \frac{\partial f'(x)}{\partial x'^{\nu}}\Big|_p \quad —— ②$$

$$\therefore f|_q = f(x(q)) = f'(x'(q))$$

$$\Longrightarrow f(x) = f'(x')$$ (标量场不随参考系变化)

and $x'^{\mu} = x'^{\mu}(x)$

$$\therefore f(x) = f'(x'(x))$$

$$\therefore V_\mu(f) \equiv \frac{\partial f'(x'(x))}{\partial x^{\mu}}\Big|_p = \left(\frac{\partial f'(x')}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)\Big|_p$$ (复合求导)

由② $V_\mu(f) = \underbrace{V'_\nu(f)}_{\text{Real}} \underbrace{\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right)\Big|_p}_{\text{Real}}$ (由数乘定义)

$$= \left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\Big|_p V'_\nu\right)(f)$$

$$\therefore v = a^{\mu} V_\mu = \boxed{a'^{\nu} V'_\nu}$$

$$= \boxed{a^{\mu}\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\Big|_p V'_\nu\right)}$$

$$\therefore a'^{\nu} = a^{\mu}\left(\frac{\partial x'^{\nu}}{\partial x^{\mu}}\Big|_p\right)$$

Curve

How to define a curve in manifold M

Define a map $c : I \to M$

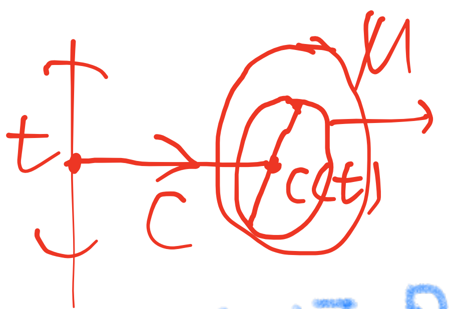
We call this map "curve" usually

t is a parameter on I (参数)

So although $c'(t')$ and $c(t)$ have same picture, they are not same.

Parametrization (重参数化)

$\Longleftrightarrow \exists \alpha : I \to I'$ s.t. $c = c' \circ \alpha$



(coordinate system) $\longrightarrow \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{pmatrix} \Rightarrow x^\mu = x^\mu(t)$

这便是参数方程(曲线参数式)

(But coordinate system is unnecessary it just makes the expression easier)