IEOR 4102 lec 10

Intro to point processes,
renewal processes,
Elementary Renewal theorem

Poisson processes

A simple point process

is a sequence of points $\{t_n : n \geq 1\}$

$0 < t_1 < t_2 < \cdots$ $\quad \left(\underline{t_0 \overset{\text{def}}{=} 0} \right)$

$$\lim_{n \to \infty} t_n = \infty$$

$\psi = \{t_n : n \geq 1\}$

interarrival times

$X_1 = t_1$

$X_2 = t_2 - t_1$

$\vdots$

$X_n = t_n - t_{n-1}, \quad n \geq 1$

$t_n = X_1 + \cdots + X_n$

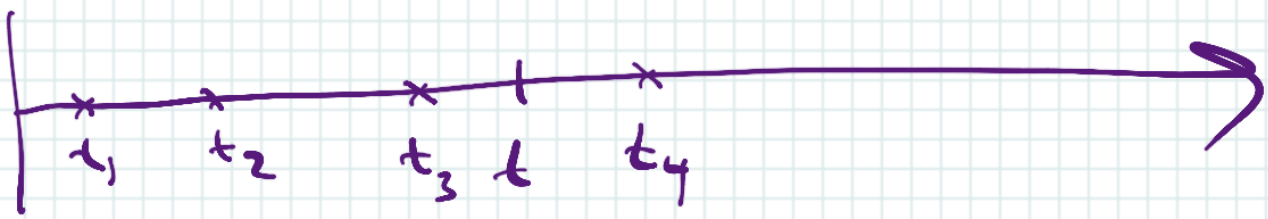
The counting process

$$\{N(t) : t \geq 0\}$$

$N(t) = \#$ pts in the time interval $(0, t]$

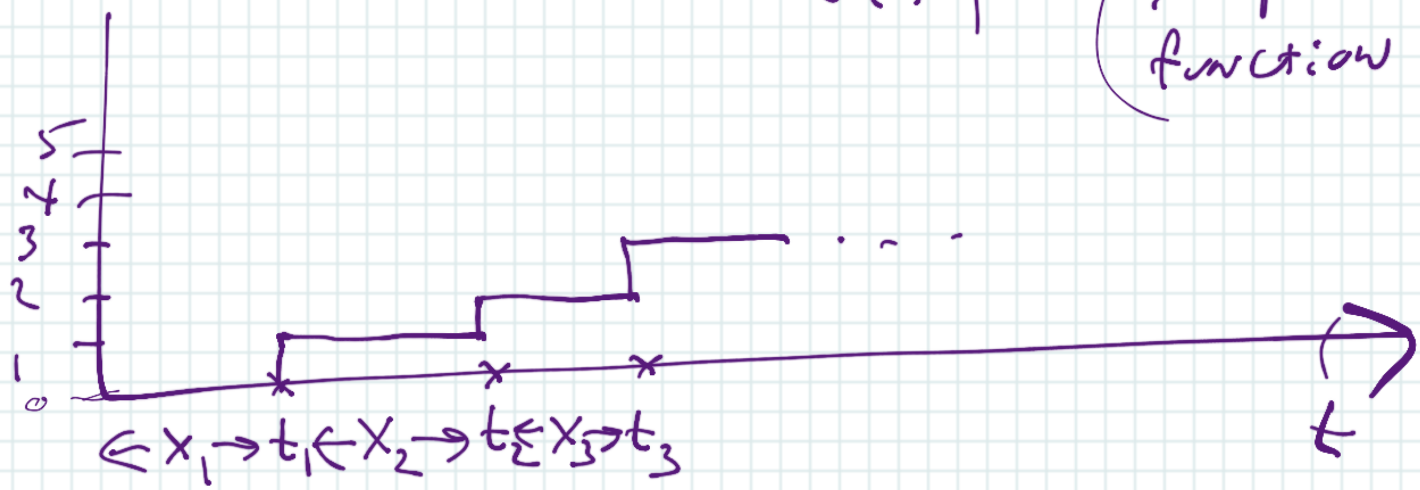
$N(0) = 0$

$N(t) = \max\{n \geq 1 : t_n \leq t\}$



$N(t) = 3$

$N(t)$ (Step function)



$$\{N(t)=0\} = \{t_1 > t\} = \{X_1 > t\}$$

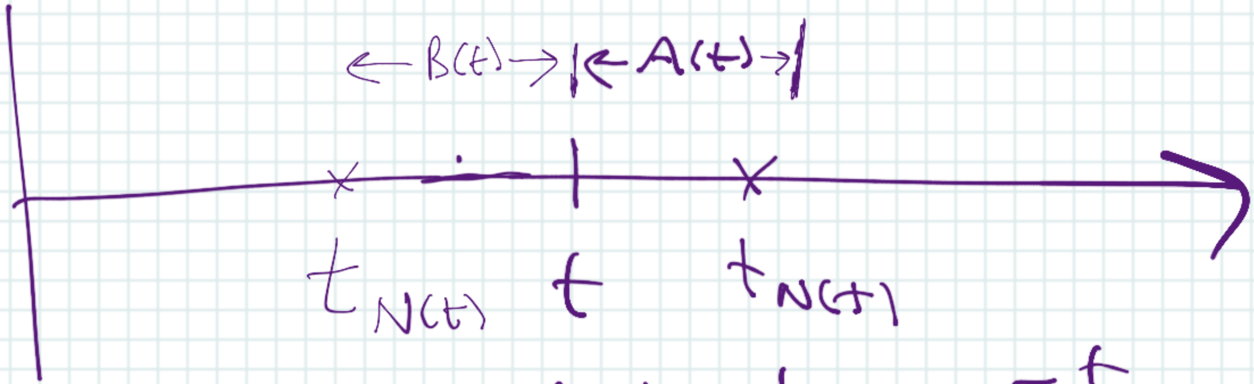
$$\{N(t)=n\} = \{t_n \leq t,\ t_{n+1} > t\}$$

$$\{N(t) \geq n\} = \{t_n \leq t\}$$

Modeling

1) times at which your phone calls come in
2) times at which jobs are sent to a printer
3) times that subways arrive to a station
4) times at which a claim comes in for an insurance company

etc.



$$A(t) = t_{N(t)+1} - t$$

$$B(t) = t - t_{N(t)}$$

when $\{t_n\}$ $\left(\{X_n\}\right)$ are random variables

we call $t = \{t_n\}$ a random point process

$$\mathbb{P}(N(t)=0) = \mathbb{P}(t_1 > t) = \mathbb{P}(X_1 > t)$$

$$\boxed{\mathbb{P}(N(t)=n)} = \mathbb{P}(t_n \leq t,\ t_{n+1} > t)$$

$$= \mathbb{P}(X_1 + \cdots + X_n \leq t,\ X_1 + \cdots + X_{n+1} > t)$$

$E(N(t)) = ?$ $\quad Var(N(t)) = ?$

A **renewal process** is a random point process $\psi = \{t_n\}$ in which the interarrival times $\{X_n\}$ are iid with interarrival time

dist. $F(x) = \mathbb{P}(X \leq x) \quad x \geq 0$

$\left(0 < E(X) < \infty\right)$ we assume $\quad \lambda \overset{\text{def}}{=} \frac{1}{E(X)}$ "rate"

Elementary renewal Thm:

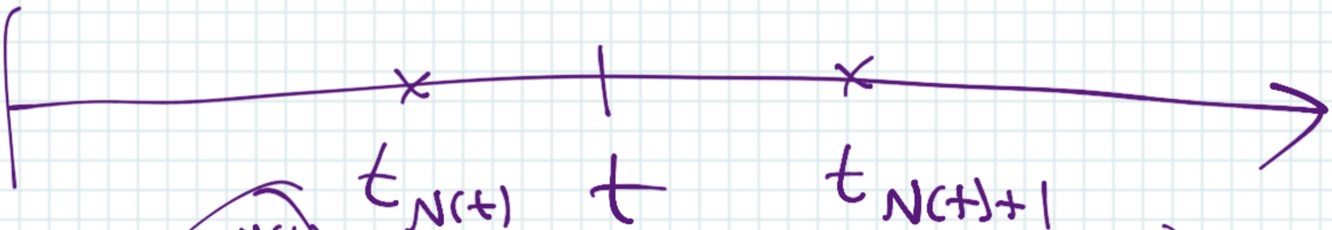
For a renewal process

a) $\lim_{t \to \infty} \frac{N(t)}{t} = \lambda$ wp 1

long-run rate at which points occur

b) $\lim_{t \to \infty} \frac{E(N(t))}{t} = \lambda$

Proof of (a) $t_n = \overbrace{X_1 + \cdots + X_n}^{iid}$



$$t_{N(t)} = \sum_{j=1}^{N(t)} X_j \qquad \boxed{t_{N(t)} \le t \le t_{N(t)+1}} \qquad t_{N(t)+1} = \sum_{j=1}^{N(t)+1} X_j$$

$$\Rightarrow \frac{t_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{t_{N(t)+1}}{N(t)}$$

$$L(t) = \frac{1}{N(t)} \sum_{j=1}^{N(t)} X_j \le \frac{t}{N(t)} \le \frac{1}{N(t)} \sum_{j=1}^{N(t)+1} X_j = R(t)$$

$$\lim_{t\to\infty} L(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j = E(X) = \frac{1}{\lambda} \quad \text{wp1} \quad \textbf{SLLN}$$

$$\lim_{t\to\infty} R(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n+1} X_j \underset{\text{wp1}}{=} E(X) = \frac{1}{\lambda}$$

$$\Rightarrow \lim_{t\to\infty} \frac{t}{N(t)} = \frac{1}{\lambda} \quad \text{wp1} \rightarrow \boxed{\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \quad \text{wp1}}$$

$\mathbb{P}(N(t) = 5) = ?$

we would need more information about $F(x)$

We now will consider the very special case of a renewal process for which

$$F(x) = \mathbb{P}(X \le x) = 1 - e^{-\lambda x}, \quad x \ge 0$$

$$E(X) = \frac{1}{\lambda} \qquad \exp(\lambda) \text{ dist}$$

Called a Poisson Process at rate λ

$t_n = X_1 + \cdots + X_n$

has an Erlang(n, λ)

dist.

X_i is called the

$1 \le i \le n$ i^{th} "phase"

Erlang(n,λ) is a special case

of a <u>gamma (n,λ)</u> dist.

$$E(t_n) = E(X_1 + \cdots + X_n)$$
$$= n\,E(X) = \frac{n}{\lambda}, \quad n \geq 1$$

why is this called a "Poisson" process?

Because: for every fixed $t > 0$, $N(t)$ has a Poisson dist. with mean $E(N(t)) = \lambda t = Var(N(t))$

$$\mathbb{P}(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k \geq 0$$

$$\mathbb{P}(N(t) = 0) = e^{-\lambda t}$$

$$\mathbb{P}(t_1 > t)$$

$$= \mathbb{P}(X_1 > t)$$

For all $s \geq 0, \quad t \geq 0$

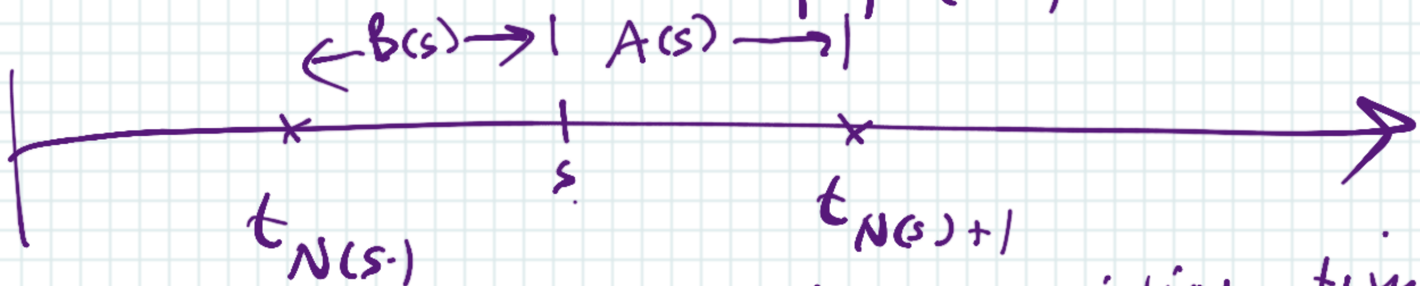
$N(s+t) - N(s) = $ # arrivals in $(s, s+t]$

length interval

$\stackrel{\text{dist}}{=} N(t)$

$$\mathbb{P}(N(s+t)-N(s)=k)$$
$$= \mathbb{P}(N(t)=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k \geq 0$$

imagine subways arriving according to a $PP(\lambda)$



$A(s) =$ your waiting time

By the memoryless property of the exponential dist.

$A(s) \sim \exp(\lambda)$

and is independent of $B(s)$;

independent of the past

$\{N(u) : 0 \leq u < s\}$

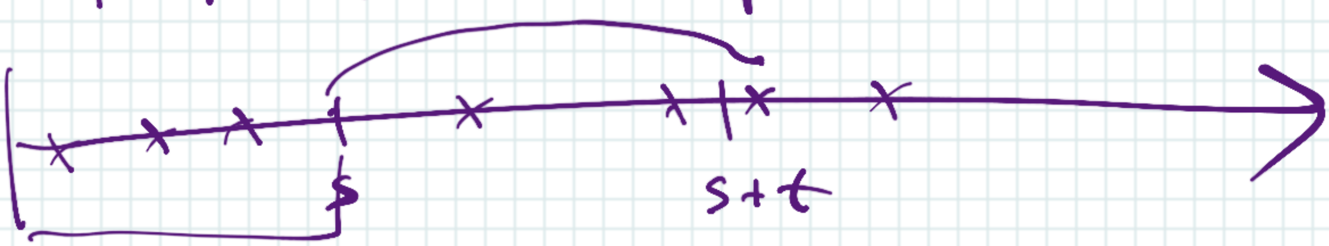
$\mathbb{P}(A(s) > t) = e^{-\lambda t}, \quad t \geq 0$

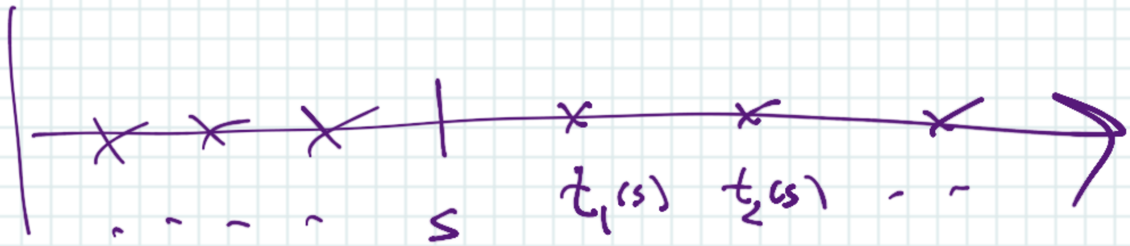
For <u>any</u> s

$\Rightarrow$ $N(s+t) - N(s)$ is independent of $N(s)$

$\overset{dist}{=} N(t)$ for all s

the $PP(\lambda)$ process from time s onwards is yet again $PP(\lambda)$ independent of the past





$$t_1(s) = t_{N(s)+1} - s$$
$$t_2(s) = t_{N(s)+2} - s$$

we consider the $PP(\lambda)$ starting from time s

$$\{t_n(s) : n \geq 1\} \overset{?}{\sim} PP(\lambda)$$

independent of $\{N(u) : 0 \leq u \leq s\}$

$$\underline{N(s+t)} = N(s) + \underbrace{\underline{(N(s+t) - N(s))}}_{\text{independent}} \overset{\text{dist.}}{=} \underline{N(t)}$$

Future

Present state

Markov property holds

Continuous-time Markov chain

$(N(t))$ for a $PP(\lambda)$
has both stationary
and independent increments :

1) Stationary increments Definition
For all $s \geq 0, t \geq 0$
$N(s+t) - N(s)$ has a dist.
that only depends on the length t
of the interval $(s, s+t]$

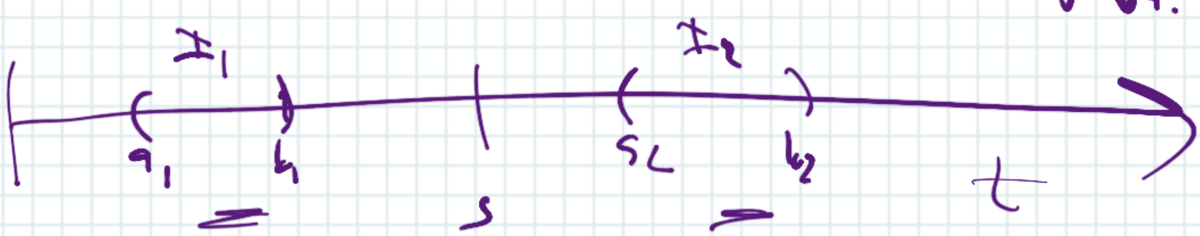
$$\overset{dist}{=} N(t)$$

2) Independent increments Definition:

$I = (a, b]$ $\quad 0 \le a < b$ an interval.

Let $N(I) = N(b) - N(a)$ = # pts that fall in $I = (a, b)$

If I_1 and I_2 are non-overlapping intervals then $N(I_1)$ and $N(I_2)$ are independent r.v.s.



For a $PP(\lambda)$

$N(t)$ has a Poisson dist. with mean λt.

and by stationary increments we thus have $N(s+t) - N(t) \xrightarrow{\text{dist.}} N(t)$ for all $s \geq 0$

$$\mathbb{P}(N(s+t) - N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \geq 0$$

Examples

Cars arrive to the GW Bridge

$\sim PP(\lambda) \qquad \lambda = 1000/\text{hr}$

a) $E(\#\text{ arrivals between } 2 \text{ \& } 3 \text{ pm})$

$$= E(N(3)-N(2)) = E(N(1)) \quad \text{stationary increments}$$

$$= \lambda \cdot 1 = \lambda$$

b) $E(\underset{\text{pm}}{N(3)-N(2)} \mid \underset{\text{AM}}{N(10)-N(9)} = 700) = 1000$

$$= E(N(3)-N(2)) \quad \text{independent increments}$$

$$= E(N(1)) = 1000$$

c) $\mathbb{P}\left(\begin{array}{l}\text{No arrivals} \\ \text{in a given 15 minute} \\ \text{interval}\end{array}\right)$

15 min $= \frac{1}{4}$ hr

$$\mathbb{P}(N(\tfrac{1}{4})=0) = e^{-\lambda \cdot \frac{1}{4}} = e^{-250}$$

$$\shortparallel$$

$$\mathbb{P}(t_1 > \tfrac{1}{4})$$

$$= \mathbb{P}(X_1 > \tfrac{1}{4}) = e^{-\lambda \cdot \frac{1}{4}} = e^{-250}$$

Fact: The PP(λ) process is the only simple point process with both stationary and independent increments

So if you have a simple point process and it has both stationary & independent increments then it is a Poisson Process with $\lambda = E(N(1))$

Proof involves proving that

(X_n) are iid exponential rvs.

Lets show that X_1 is exponential

via showing it has the memoryless property

$$\mathbb{P}(X_1 > s+t) \stackrel{?}{=} \mathbb{P}(X_1>s)\,\mathbb{P}(X_1>t)$$

$$\begin{aligned}
\mathbb{P}(N(s+t)=0) &= \mathbb{P}(N(s)=0,\ N(s+t)-N(s)=0) \\
&= \mathbb{P}(N(s)=0)\,\mathbb{P}(N(s+t)-N(s)=0) && \text{independent increments} \\
&= \mathbb{P}(N(s)=0)\,\mathbb{P}(N(t)=0) && \text{stationary increments} \\
&= \mathbb{P}(X_1>s)\,\mathbb{P}(X_1>t) \quad \checkmark
\end{aligned}$$

We continue using independent & stationary increments sequentially to show (induction) that X_{n+1} has the same exponential dist. as X_n and is independent of $X_1, \dots, X_n$

$\Rightarrow \{X_n\}$ iid $\exp(\lambda)$ some $\lambda > 0$.

Subway Example

Consider 96th / Broadway subway station to go downtown.

$(N_1(t))$ 1) Locals (L) (1) $\sim PP(\lambda_1)$ $\lambda_1 = 7/hr$

$(N_2(t))$ 2) Express (2,3) (E) $\sim PP(\lambda_2)$ $\lambda_2 = 3/hr$

independent

You arrive

a) You decide to take 1st express

$$\mathbb{P}(\text{wait} > 10 \text{ min}) = \mathbb{P}(N_2(\tfrac{1}{6}) = 0)$$

$$= e^{-\lambda_2 \cdot \frac{1}{6}} = e^{-\frac{1}{2}} \approx .61$$

b) You decide to take which ever arrives 1st

$E(\text{waiting time}) = E(\min(X_1(1), X_1(2)))$

$= E(Z)$

independent $X_1(1) \sim \exp(\lambda_1)$ = time until first L

$X_1(2) \sim \exp(\lambda_2)$ = time until first E

$= \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{10}$

= 6 minutes

$E(\text{waiting time} \mid \text{local was first})$

$E(Z \mid X_1(1) < X_1(2)) = E(Z) =$ 6 min also.