1 Approximate Decision Tree

In this section the value of an empty sum is 0. Let $\varepsilon \in (0,1) \subset \mathbb{R}$ be a real number in the open interval $(0,1)$. Let $d, n \in \mathbb{Z}_{>0}$ be a pair of positive integers. Let $\sigma_1, \ldots, \sigma_d \in \mathbb{R}_{>0}$ be positive real numbers. Let $m_1, \ldots, m_d \in \mathbb{Z}_{>0}$ be positive integers. For $i \in [d]$, let

$$g_i = \left\{\sigma_i, (1+\varepsilon)\sigma_i, (1+\varepsilon)^2\sigma_i, \ldots, (1+\varepsilon)^m\sigma_i\right\}$$

and let $G = g_1 \times g_2 \times \cdots \times g_d$.

For $i \in [d]$ and $p \in \mathbb{R}^d$ we denote the i-th coordinate of p by $p[i]$. We may treat a point $p \in \mathbb{R}^d$ as a d-tuple by writing $p = (p[1], p[2], \ldots, p[d])$. For a pair of points $p, q \in \mathbb{R}^d$ we define

$$\gamma(q_1, q_2) := \left\{p \in \mathbb{R}^d \mid \forall i \in [d] : q_1[i] < p[i] \leq q_2[i]\right\}.$$

Let

$$A = \gamma((\sigma_1, \ldots, \sigma_d), ((1+\varepsilon)^m\sigma_1, (1+\varepsilon)^m, \ldots, (1+\varepsilon)^m\sigma_d)).$$

For a compact of points $P \subset A$ and an integer $i \in [d]$ we define

$$\begin{aligned}
\mu(P, i) &:= \frac{1}{|P|} \cdot \sum_{p \in P} p[i] \\
\alpha(P) &:= (\mu(P,1), \mu(P,2), \ldots, \mu(P,d)) \\
\beta(P) &:= \sum_{p \in P} \|p - \alpha(P)\|^2
\end{aligned}$$

For a point $p \in A$ we define

$$\begin{aligned}
\Phi &= \left\{(q_1, q_2) \in G^2 \mid \forall i \in [d] : q_1[i] < q_2[i]\right\} \\
\Lambda &:= \{\gamma(q_1, q_2) \mid q_1, q_2 \in \Phi\} \\
\lambda(p) &:= \{C \in \Lambda \mid p \in C\}
\end{aligned}$$

For a set of points $P \subseteq A$ an integer $i \in [d]$ and a real $t \in \mathbb{R}$ we define

$$\begin{aligned}
\mathcal{L}(P, i, t) &:= \{p \in P \mid p[i] < t\} \\
\mathcal{R}(P, i, t) &:= P \setminus \mathcal{L}(P, i, t)
\end{aligned}$$

For a positive integer h, *h-tree* is a tuple (t, i, L, R) where $i \in [d]$ is an integer, $t \in g_i$ is a coordinate and L, R are ℓ-tree and r-tree respectively for integers $0 \leq \ell, r < h$ such that either $\ell = h-1$ or $r = h-1$; 0-tree is an empty tuple. For h-tree (t, i, L, R) and a finite non-empty set of points $P \subset A$ we define

$$\begin{aligned} \text{cost}(P, ()) &:= \beta(P) \\ \text{cost}(P, (t, i, L, R)) &:= \text{cost}(\mathcal{L}(P, i, t), L) + \text{cost}(\mathcal{R}(P, i, t), R). \end{aligned}$$

Finally, we define the function $s : 2^A \times A \to \mathbb{R}$ to be

$$s(P, p) := \sum_{C \in \lambda(p)} \frac{\|\alpha(C \cap P) - p\|^2}{\text{cost}(C \cap P, ())}.$$

We will use the following algorithm in the next theorem.

Algorithm 1.1: $\text{Leaf}(p, T)$

comment: $p \in A$ and T is an h-tree

$\ell \leftarrow (\sigma_1, \sigma_2, \ldots, \sigma_d)$
$r \leftarrow ((1+\varepsilon)^m \sigma_1, (1+\varepsilon)^m \sigma_2, \ldots, (1+\varepsilon)^m \sigma_d)$
while $T \neq ()$

$$\textbf{do} \begin{cases} (t, i, L, R) \leftarrow T \\ \textbf{if } p[i] \leq t \\ \quad \textbf{then} \begin{cases} r[i] \leftarrow t \\ T \leftarrow L \end{cases} \\ \quad \textbf{else} \begin{cases} \ell[i] \leftarrow t \\ T \leftarrow R \end{cases} \end{cases}$$

return $(\gamma(\ell, r))$

Theorem 1. *For every positive integer h, for every set of points $P \subseteq A$, for every h-tree T and for every $p \in P$:*

$$\frac{\|\alpha(\text{Leaf}(p, T) \cap P) - p\|^2}{\text{cost}(P, T)} \leq s(P, p)$$

Proof. Let r, ℓ be the variables from Algorithm Leaf. These variables are points such that $r, \ell \in G$: the are initalized to be points in G and their coordinates are altered only to values from G.

Hence:

$$\text{Leaf}(p, T) = \gamma(\ell, r) \in \Lambda$$

On the other hand $p \in \textsc{Leaf}(p, T)$ since from the construction of Algorithm Leaf follows that for every $i \in [d] : \ell[i] \leq p[i] \leq r[i]$.

Let $C = \textsc{Leaf}(p, T)$. Since $C \in \Lambda$ and $p \in C$: $C \in \lambda(p)$.

Therefore, from the definition of $s(P, p)$ follows that

$$\frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(C \cap P, ())} \leq s(P, p)$$

On the other hand, from the definition of cost follows that

$$\mathrm{cost}(C \cap P, ()) \leq \mathrm{cost}(P, T)$$

and therefore

$$\frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(P, T)} \leq \frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(C \cap P, ())}.$$

Hence

$$\frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(P, T)} \leq s(P, p).$$

$\square$

Theorem 2. *For any finite set of points $P \subset A$:*

$$\sum_{p \in P} s(P, p) = O(m^{2d})$$

Proof. G is a cartesian product of d sets of cardinality m and therefore $|G| = m^d$. Therefore

$$|\Lambda| \leq \left|G^2\right| = \binom{m^d}{2} = O(m^{2d})$$

and therefore

$$\begin{aligned}\sum_{p \in P} s(P, p) &= \sum_{p \in P} \sum_{C \in \lambda(p)} \frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(C \cap P, ())} = \sum_{C \in \Lambda} \sum_{p \in P \cap C} \frac{\|\alpha(C \cap P) - p\|^2}{\mathrm{cost}(C \cap P, ())} \\ &= \sum_{C \in \Lambda} \frac{\mathrm{cost}(C \cap P, ())}{\mathrm{cost}(C \cap P, ())} = \sum_{C \in \Lambda} 1 = |\Lambda| = O(m^{2d}).\end{aligned}$$

$\square$