My introduction

I've been wanting uniform, clean, and editable math notes for a time now. It's the third attempt.

As much as this helped me, I hope it helps others.

Best of luck,
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Chapter 1

Introduction

A transformation is a mathematical operation, which transforms mathematical expressions into another equivalent simple form. The Laplace transform enables us to solve differential equations using algebraic methods. Laplace transform is a mathematical tool that can be used to solve many problems in Science and engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1790, in his work on probability theory This technique became very popular when the Heaveside function was applied to the solution of ordinary differential equations in electrical Engineering problems.

Many kinds of transformation exist, but Laplace transform and fourier transform are the most well known. The Laplace transform is related to fourier transform, but whereas the fourier transform expresses a function or signal as a series of mode of vibrations, the Laplace transform resolves a function into its moments.

Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In Physics and Engineering, it is used for the analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation from the time domain in which inputs and outputs are functions of time, to the frequency domain, where the same inputs and outputs are functions of complex angular frequency in radius per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system or synthesizing a new system based on a set of specifications. The Laplace transform belongs to the family of integral transforms. The solutions to mechanical or electrical problems involving discontinuous force function are obtained easily by Laplace transforms.

[1]

1.1 Definition

Let $f(t)$ be a functions of the variable t which is defined for all positive values of t. Let s be the real constant.

[1] This Chapter is borrowed from Here

If the integral $\int_0^\infty e^{-st}f(t)dt$ exist and is equal to F(s) , then F(s) is called the Laplace transform of $f(t)$ and is denoted by the symbol $\mathcal{L}\{f(t)\}$.

$$\text{i.e } \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$$

The Laplace Transform of $f(t)$ is said to exist if the integral converges for some values of s, otherwise it does not exist. Here the operator L is called the Laplace transform operator which transforms the functions $f(t)$ into F(s) .

$$\text{Remark: } \lim_{s\to\infty} F(s) = 0$$

Chapter 2

Properties

In my experience, I am unable to recall numerous operator properties just by looking at them. This also applied to derivatives, limits, and integrals, which is how I ended up not pursuing mathematics for a few years. The solution i found about this matter was to engage my self with the process of proofing the idea. This very helpful because I learned how I can effectively apply these mathematics tool to achieve the desired result.

I have found that simply looking at many operator attributes does not help me remember them. This also held true for integrals, derivatives, and limits, which is how I decided to put mathematics on hold for a few years. My answer to this challenge was to put myself through the idea proofing. This was really beneficial since I discovered how to use these mathematical tools efficiently to get the intended outcome.

This is significant in this case as well because a lot of problems demand creative problem-solving, and partly remembering the proof helps one recall the concept.

2.1 Linear Properties

Linear property of Laplace Transform.

1. $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

2. $\mathcal{L}\{Kf(t)\} = K\mathcal{L}\{f(t)\}$

proof: By the definition of $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

$$\begin{aligned}
\mathcal{L}\{f(t) \pm g(t)\} &= \int_0^\infty e^{-st}[f(t) \pm g(t)]dt \\
&= \int_0^\infty e^{-st}f(t)dt \pm \int_0^\infty e^{-st}g(t)dt \\
&= \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}\left\{Kf(t)\right\} &= \int_0^\infty e^{-st}Kf(t)dt \\
&= K\int_0^\infty e^{-st}f(t)dt \\
&= K\mathcal{L}\left\{f(t)\right\}
\end{aligned}
$$

2.2 Famous Laplaces

2.2.1 $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

Prove that $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$ where $s+a > 0$ or $s > -a$ Proof: By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

$$
\begin{aligned}
\mathcal{L}\{e^{-at}\} &= \int_0^\infty e^{-st}\cdot e^{-at}dt \\
&= \int_0^\infty e^{-t(s+a)}dt \\
&= \left[\frac{-e^{-t(s+a)}}{s+a}\right]_0^\infty \\
&= \frac{-1}{s+a}\left[e^{-\infty} - e^0\right]
\end{aligned}
$$

Hence

$$
\mathcal{L}[e^{-at}] = \frac{1}{s+a} \tag{2.1}
$$

2.2.2 $\mathcal{L}\{\cos{(at)}\} = \dfrac{s}{s^2+a^2}$

Prove that $\mathcal{L}\{\cos{(at)}\} = \dfrac{s}{s^2+a^2}$ By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

$$
\begin{aligned}
\mathcal{L}\{\cos{(at)}\} &= \int_0^\infty e^{-st}f(t)dt \\
&= \int_0^\infty e^{-st}\cos{(at)}\,dt \\
&= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty \\
&= 0 - \frac{1}{s^2+a^2}(-s) \\
&= \frac{s}{s^2+a^2}
\end{aligned}
$$

Hence

$$
\mathcal{L}\{\cos{(at)}\} = \frac{s}{s^2+a^2} \tag{2.2}
$$

2.2.3 $\mathcal{L}\{t^n\} = \frac{n!}{s^n} \cdot \frac{1}{s}$

$$
\begin{aligned}
\mathcal{L}(t^n) &= \int_0^\infty e^{-st} t^n dt \\
&= \left[(t^n) \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty n t^{n-1} (\frac{e^{-st}}{-s}) dt \\
&= (0 - 0) + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt \\
&= \frac{n}{s} \mathcal{L}(t^{n-1}) \\
&= \frac{n}{s} \cdot \frac{n-1}{s} \mathcal{L}(t^{n-2}) \\
&= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{3}{s} \cdot \frac{2}{s} \cdot \frac{1}{s} \cdot L(1) \\
&= \frac{n!}{s^n} L[1] = \frac{n!}{s^n} \cdot \frac{1}{s}
\end{aligned}
$$