A Solution to Big-data Dynamic Portfolio Selection Problem with No-shorting Constraints

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***Abstract*— This paper gives a solution to continuous-time mean-variance portfolio selection problem with no-shorting constraints when the dimension of data is high. In the situations where the portfolio size is large, the theoretical result can not be used because the estimation of Σ^{-1} is difficult. We provide a computing method which applies the thought of l_1-constrained least square regression to realize the theoretical portfolio, and discuss its convergence from the aspects of theory and practice. We prove its theoretical convergence, also show that a better behavior of the portfolio needs us to estimate β_1 precisely, so that it could be applied well in practice.**

***keywords*: short-selling prohibition, high-dimensional portfolio selection, continuous-time mean-variance portfolio, constrained l_1 minimization, sparse portfolio**

I. Introduction

After Harry Markovitz's leading work (1952) [1] of mean-variance portfolio selection, which aims to get an expected value of terminal wealth and minimize its variance, different kinds of portfolio selection problems were studied during the last several decades, including multi-period [2]–[5], continuous [6]–[8] and constrained [9]–[12] portfolio selection problems. Among the problems, the portfolio selection problem with no-shorting constraints is meaningful and worth studying for some time or as an aspect of considering investing strategy.

Extending Markovitz's idea to the multi-period or continuous time settings is difficult and could not be done for a long time, though. In 2000, Li and Ng [13], Zhou and Li [14] got the solution of multi-period and continuous mean-variance portfolio (MVP) problems, separately, using embedding techniques to deal with the non-separable variance operator. The paper (Zhou and Li, 2000) is very meaningful, as it provides a framework of stochastic linear-quadratic (LQ) optimal control, which developed fast in those years [15]–[18]. After that, the MV frameworks in many problems, including cardinality [19], [20], cone [21] and no-shorting [22] constrained problems were studied by Duan Li and other researchers.

In 2002, Li, Zhou, and Lim [23] got the result of dynamic MV portfolio selection problem under no-shorting constraints. They firstly introduced the Lagrange dual method to transfer the continuous MV portfolio selection problem into an unconstrained stochastic LQ control problem, and with the help of viscosity theory, they got the analytical solution of the value function (wealth of the risk assets). They proved that it is a viscosity solution, and the corresponding μ^* is the optimal control. The method of Lagrange dual is simple, which leaded to many applications to get the optimal portfolio, including a multi-period model considering bankruptcy risk [24], portfolio selection in a continuous incomplete market with no-shorting constraints [25], a continuous asset-liability management problem with random market parameters [26].

Although the analytical solution is obtained, its not easy to solve in the experience. Because of the existence of the term Σ^{-1}, the solution can rarely give a reasonable and robust portfolio. In fact, even for the MVP problem without constraints, the huge, accumulated errors of estimation will lead to the failure of the strategy, whose performance could even be worse than equally weighted portfolios [27], [28].

In 2017, Chiu, Pun and Wong [27] gave a new aspect of solving the high-dimensional problem based on the thought of l_1-constrained least-squares regression proposed by Fan et al. (2012) [29]. They took the continuous MVP problem as an example. In calculating the optimal portfolio, one can substitute $\Sigma^{-1}\beta$ with an approximate solution, which is to set a constraint $||\Sigma\eta - \beta||_\infty \leq \lambda$, and minimize $||\eta||_1$ to get a sparse portfolio, where λ can be chosen using cross validation to get a larger Sharpe ratio. The method gets a good approximation of $\Sigma^{-1}\beta$ through a linear program, so it can be called as a linear programming optimal (LPO) approach. This is a whole new way of thinking of Σ^{-1}, for it allows a little error to get a more stable and useful portfolio. Also, it is natural to try to use the thought to explore the portfolio selection with no-shorting constraints.

This paper aims to use the thought of Chiu, Pun, and Wong (2017) to provide a method to calculate the μ^* in the portfolio proposed by Li, Zhou, and Lim (2002). The efficiency from the aspect of asymptotic properties can be proved theoretically. However, because of the error of estimating the result of a quadratic programming problem in the strategy, it is hard to be used in practice, especially when p is large, and a lower λ can not be used. Lastly, some examples of simulation and empirical study are provided. Section 2 reviews the dynamic portfolio selection problem with no-shorting constraints and its solution. Section 3 gives the estimation and the corresponded properties of the terminal wealth for a simpler strategy. Section 4 introduces the newly proposed no shorting LPO (NSLPO) approach and discusses its convergence. Section 5 gives the simulation of discussing the error of the result of the quadratic programming, and examples of considering the two strategies (LPO and NSLPO) with strict or flexible no-shorting constraints. Section 6 concludes the article.

II. PRELIMINARIES

Consider a market with p risky assets (stocks) and one risk-free asset (bond) which are traded continuously. The uncertainty is generated by a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}\}_{t\geq 0}$, is generated by a p-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_p(t))'$. The price of the bond is according to:

$$\begin{cases} dP_0(t) = rP_0(t)dt \\ P_0(0) = p_0 > 0, \end{cases}$$

where r is the constant risk-free interest rate. And the prices of the risky assets are generated by:

$$\begin{cases} dP_i(t) = P_i(t)[\mu_i dt + \sigma_i dW(t)] \\ P_i(0) = p_i > 0, \end{cases}$$

where $\mu_i > 0$ and σ_i are the corresponded appreciation rate and the volatility, separately.

Suppose a person has wealth $X_0 > 0$ firstly, and his wealth at time t is $X(t)$, $t = 1, \ldots, n$. The wealth of risky assets is denoted as $u(t) = (u_1(t), \ldots, u_p(t))'$, where $u_i(t) \geq 0, i = 1, \ldots, p$. But the wealth of risk-free bond can be negative, which means we can borrow money from bank if we need. If we assume the portfolio is self-financing, and do not consider the transaction costs, $X(t)$ follows:

$$\begin{cases} dX(t) = [rX(t) + u(t)'\beta]d(t) + u(t)'\sigma dW(t)] \\ X(0) = x_0 > 0, \end{cases} \tag{1}$$

where $\beta = (\mu_1 - r, \ldots, \mu_p - r)$ are the excess returns. $\sigma = (\sigma_1', \ldots, \sigma_p')' \in \mathbb{R}^{p\times p}$ is the volatility matrix, and $\Sigma = \sigma\sigma'$ is the variance-covariance matrix of the stocks. Notice that we keep β and Σ constant here for simplicity of proving.

The mean-variance portfolio selection problem under constraints of no-shorting is:

$$\min\ Var\ X^u(T)$$

$$subject\ to \begin{cases} E[X^u(T)] = z \\ u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^p_+), \\ (X(\cdot), u(\cdot))\ satisfy\ (1) \end{cases}$$

The strategy proposed by Li, Zhou, and Lim (2002) is:

$$u(t, x) = \begin{cases} -\Sigma^{-1}\beta_1[X - e^{-r(T-t)}\gamma_z], \\ if\ X \leq e^{-r(T-t)}\gamma_z \\ 0, \\ if\ X > e^{-r(T-t)}\gamma_z \end{cases} \tag{2}$$

$$\beta_1 = \beta + \underset{\pi\in[0,+\infty)^m}{\arg\min}\ (\pi + \beta)'\Sigma^{-1}(\pi + \beta)$$

$$\gamma_z = (ze^{\Theta T} - x_0e^{rT})/(e^{\Theta T} - 1)$$

$$\Theta = \beta_1'\Sigma^{-1}\beta_1$$

It is a viscosity solution to the original problem, where $u(t) = 0$ when $X > e^{-r(T-t)}\gamma_z$ ensures the non-negativity of elements in u. If we forget the no-shorting constraints, and always use the first result, we can get a new strategy:

$$u_1(t, x) = -\Sigma^{-1}\beta_1[X - e^{-r(T-t)}\gamma_z] \tag{3}$$

where the notations are the same as that in (2). Noted that it's similar to the result of MVP. With the known parameters, the expectation, variance and profit-making probability (PMP) of the new strategy are:

$$E[X_1^u(T)] = z + (z - x_0e^{rT})\frac{1 - e^{(\Theta-\beta'\Sigma^{-1}\beta_1)T}}{e^{\Theta T} - 1}$$

$$\sigma^u_{1,T} = (z - x_0e^{rT})\frac{\sqrt{e^{\Theta T} - 1}}{e^{\beta'\Sigma^{-1}\beta_1 T} - 1}$$

$$F_1 = P(X^u(T) \geq x_0e^{rT}) = \Phi(\sqrt{T}\frac{\beta'\Sigma^{-1}\beta_1}{\sqrt{\Theta}} + \frac{\sqrt{\Theta T}}{2})$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Here Σ^{-1} is theoretically non-negative defined, and elements in $\Sigma^{-1}\beta_1$ should be no less than 0 ensured by the properties of the solution to the no-shorting problem, so $\Theta - \beta'\Sigma^{-1}\beta_1 = (\beta_1 - \beta)'\Sigma^{-1}\beta_1 \geq 0$. Recall that the expectation, standard deviation and PMP of MVP are: z, $(z - x_0e^{rT})/\sqrt{e^{\Theta T} - 1}$ and $\Phi(\frac{3\sqrt{\Theta T}}{2})$. For the new strategy (3), the expectation is less than z, while the standard deviation is more, and the PMP is less than that of Markovitz's portfolio. Finally, if the value of $(\beta'\Sigma^{-1}\beta_1 + \frac{1}{2}\Theta)$ is negative, the PMP will be even less than $\frac{1}{2}$. We can see that because of the existence of the constraints, the performance of the strategy is not as good as that of MVP.

III. ESTIMATION AND TERMINAL WEALTH

For a sample of daily log returns of p risky assets and n observations: $D = r^{(l)} = (r_1^{(l)}, \ldots, r_p^{(l)})', 1 \leq l \leq n$, if we observe daily, the time step is $\delta t = 1/252$ for 252 trading days each year. The maximum likelihood estimators (MLEs) are:

$$\hat{\beta} = \frac{1}{\delta t}\bar{r} + \frac{1}{2}diag(\hat{\Sigma}) - r\mathbf{1}_p$$

$$\hat{\Sigma} = \frac{1}{n\delta t}\sum_{l=1}^{n}(r^{(l)} - \bar{r})(r^{(l)} - \bar{r})'$$

where $\bar{r} = \frac{1}{n}\sum_{l=1}^{n} r^{(l)}$, $\mathbf{1}_p = (1, \ldots, 1)' \in \mathbb{R}^p$. The plug-in version of the strategy (2) is:

$$u(t, X) = \begin{cases} -\hat{\Sigma}^{-1}\hat{\beta}_1[X - e^{-r(T-t)}\hat{\gamma}_z], & X \leq e^{-r(T-t)}\hat{\gamma}_z \\ 0, & X > e^{-r(T-t)}\hat{\gamma}_z \end{cases}$$

$$\hat{\beta}_1 = \hat{\beta} + \underset{\pi\in[0,+\infty)^m}{\arg\min}\ (\pi + \hat{\beta})'\hat{\Sigma}^{-1}(\pi + \hat{\beta})$$

$$\hat{\gamma}_z = (ze^{\hat{\Theta}T} - x_0e^{rT})/(e^{\hat{\Theta}T} - 1)$$

$$\hat{\Theta} = \hat{\beta}_1'\hat{\Sigma}^{-1}\hat{\beta}_1,$$

Ignore the no-shorting constraints, and consider strategy (3). After substituting the estimated new strategy (3) into the wealth process (1) and solving the partial differential equation, we get the realized terminal wealth for the new strategy:

$$X_1(T) = \hat{\gamma}_z + (x_0e^{rT} - \hat{\gamma}_z)\ exp\{(-\beta'\hat{\Sigma}^{-1}\hat{\beta}_1$$

$$-\frac{\hat{\beta}_1'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\beta}_1}{2})T - \hat{\beta}_1'\hat{\Sigma}^{-1}\sigma W_T\}$$

The realized mean, standard deviation and PMP are calculated:

$$E[X_1^{\hat{u}}(T)] = z + (z - x_0e^{rT})\frac{1 - e^{(\hat{\Theta}-\beta'\hat{\Sigma}^{-1}\hat{\beta}_1)T}}{e^{\hat{\Theta}T} - 1}$$

$$\sigma_{1,T}^{\hat{u}} = (z - x_0e^{rT})\frac{\sqrt{e^{\hat{\beta}_1'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\beta}_1 T} - 1}}{e^{\beta'\hat{\Sigma}^{-1}\hat{\beta}_1 T} - 1}$$

$$\hat{F}_1 = \Phi(\sqrt{T}\frac{\beta'\hat{\Sigma}^{-1}\hat{\beta}_1 + \frac{1}{2}\hat{\beta}_1'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\beta}_1}{\sqrt{\hat{\beta}_1'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\beta}_1}})$$

IV. The NSLPO Method

In this problem, the main challenge is how to calculate $\hat{\Sigma}^{-1}\hat{\beta}_1$, which determines the final portfolio we get. We can see that only the value of $\hat{\Sigma}^{-1}\hat{\beta}_1$ matters, which means if we could find a good approximation of it, we could get the good $\bar{\pi}$ and u^*. The method using the framework (Chiu, Pun, and Wong, 2017) is:

$$\tilde{\eta} = \arg\min_{\eta\in\mathbb{R}^p}\{|\eta|_1 \ subject \ to \ |\hat{\Sigma}\eta - \pi - \hat{\beta}|_\infty \le \lambda\},$$

and $\hat{\beta}_1$ is obtained from:

$$\hat{\beta}_1 = \hat{\beta} + \arg\min_{\pi\in[0,+\infty)^m} (\pi + \hat{\beta})'\tilde{\eta}$$

After the optimization, we can get the optimized estimation $\hat{\eta}$ and $\hat{\Theta}$, and the corresponded NSLPO strategy is:

$$u(t,X) = \begin{cases} -\hat{\eta}[X - e^{-r(T-t)}\hat{\gamma}_z], & if \ X \le e^{-r(T-t)}\hat{\gamma}_z \\ 0 & if \ X > e^{-r(T-t)}\hat{\gamma}_z \end{cases}$$

$$\hat{\gamma}_z = (ze^{\hat{\Theta}T} - x_0e^{rT})/(e^{\hat{\Theta}T} - 1)$$

$$\hat{\Theta} = \hat{\beta}_1'\hat{\eta}$$

In the last part of the section, we want to prove its convergence. Assume the following regular condition, which is often used in high-dimensional statistics:

$$\log p \le n, \max_{1\le i\le p}\sigma_{ii}^0 \le M \tag{4}$$

$$and \ \beta'\Sigma^{-1}\beta \ge K \ for \ some \ M, K > 0.$$

Also we have the distribution condition: For any $x > 0$ and $|\delta| \le 1$,

$$\frac{\Phi(x+\delta)}{\Phi(x)} - 1 \le c_1|\delta|(|x|+1)e^{c_2|x\delta|} \tag{5}$$

for some positive constants c_1, c_2, which are independent of x, δ.

Ignore the no-shorting constraint, and consider the new strategy (3) firstly. For $\beta_1 = \arg\min_{x_i\ge\beta_i, i=1,\ldots,p} x'\Sigma^{-1}x$ and $\eta = \Omega\beta_1$, denote $a = |\hat{\Sigma} - \Sigma|_\infty$, $b_1 = |\hat{\beta}_1 - \beta_1|_\infty$. We get the following lemma:

Lemma 1: Suppose that the historical daily return data satisfy (1), and (4) holds. We have:

$$|\hat{\Sigma}\eta - \hat{\beta}_1|_\infty \le Ca + b_1$$

for a large constant C, where $\hat{\beta}_1$ and $\hat{\Sigma}$ are the same as above, and $\eta = \Omega\beta_1$ is the background effective parametric vector.

The following proving is similar as proves of Theorem 3 (Chiu, Pun, and Wong, 2017). Lemma 1 tells us that we can realize NSLPO by choosing a proper λ. Denote $\lambda = C_1a + C_2b_1$ and $d = (\lambda + b_1)|\eta|_1 + a|\eta|_1^2$. We can get the results: $|\eta - \hat{\eta}|_\infty \le |\hat{\Sigma}|_\infty(|\hat{\Sigma}^{-1}\eta - \hat{\beta}|_\infty + |\hat{\Sigma}^{-1}\hat{\eta} - \hat{\beta}|_\infty) \le C\lambda$, $|\Theta - \hat{\Theta}| \le (2\lambda + 3b_1)|\eta|_1$ and $|\hat{\eta}'\Sigma\hat{\eta} - \Theta| \le Cd$.

For the new strategy's realized mean, standard deviation and PMP:

$$|E[X_1^{\hat{u}}(T)] - E[X_1^u(T)]| = |(z - x_0e^{rT})(\frac{1 - e^{(\hat{\Theta}-\beta'\hat{\eta})T}}{e^{\hat{\Theta}T} - 1} - \frac{1 - e^{(\Theta-\beta'\eta)T}}{e^{\Theta T} - 1})|$$
$$\le C[|e^{-\hat{\Theta}\beta'\eta T^2} - e^{-\Theta\beta'\hat{\eta}T^2}| + \frac{|e^{\Theta T} - e^{\hat{\Theta}T}|}{e^{\Theta\hat{\Theta}T^2}} + \frac{|e^{(\Theta-\beta'\eta)T} - e^{(\hat{\Theta}-\beta'\hat{\eta})T}|}{e^{\Theta\hat{\Theta}T^2}}]$$
$$\le \mathcal{O}_{\mathbb{P}}(d),$$

$$|(\sigma_{1,T}^{\hat{u}})^2 - (\sigma_{1,T}^u)^2| = (z - x_0e^{rT})^2|\frac{e^{\hat{\eta}'\Sigma\hat{\eta}T} - 1}{(e^{\beta'\hat{\eta}T} - 1)^2} - \frac{e^{\Theta T} - 1}{(e^{\beta'\eta T} - 1)^2}|$$
$$\le \frac{C}{e^{4\beta'\hat{\eta}\beta'\eta T^2}}[|e^{2\beta'\eta\hat{\eta}'\Sigma\hat{\eta}T^3} - e^{2\beta'\hat{\eta}\Theta T^3}| + 2|e^{\beta'\eta\hat{\eta}'\Sigma\hat{\eta}T^2} - e^{\beta'\hat{\eta}\Theta T^2}|$$
$$+ |e^{\hat{\Theta}T} - e^{\Theta T}| + |e^{2\beta'\eta T^2} - e^{2\beta'\hat{\eta}T^2}| + 2|e^{\beta'\eta T} - e^{\beta'\hat{\eta}T}|] \le \mathcal{O}_{\mathbb{P}}(d)$$

$$|\frac{\beta'\hat{\eta} + \frac{1}{2}\hat{\eta}'\Sigma\hat{\eta}}{\sqrt{\hat{\eta}'\Sigma\hat{\eta}}} - \frac{\beta'\eta}{\sqrt{\Theta}} - \frac{\sqrt{\Theta}}{2}| \le |\frac{\beta'\hat{\eta}}{\sqrt{\hat{\eta}'\Sigma\hat{\eta}}} - \frac{\beta'\eta}{\sqrt{\Theta}}| + \frac{1}{2}|\sqrt{\hat{\eta}'\Sigma\hat{\eta}} - \sqrt{\Theta}| \le \mathcal{O}_{\mathbb{P}}(d)$$

Use the result of 5, we get $|\frac{\hat{F}}{F} - 1| \le Cde^{Cd}$.

Now return to the original strategy (2). Consider a sample of realized daily returns r^* during $[0,T]$. If $X \le e^{-r(T-t)}\hat{\gamma}_z$ all the time, it's the same as strategy (3). If $X > e^{-r(T-t)}\hat{\gamma}_z$ at one day, $r^* = (r_1^*, r_2^*)$, where the last element in r_1^* is greater than δtr, and $r_2^* = (\delta tr, \ldots, \delta tr)$. In this case the mean and PMP will be greater.

For the standard deviation, because we only concern the risk that the terminal wealth is lower than the goal, we can define the semi-variance: $\sigma_{-,X}^2 = E(\min\{X,\mu\} - \mu)^2$ for terminal wealth with expectation μ. As the original strategy will stop investing stocks when it achieves the goal, it has less chances to get a terminal wealth which is lower than its expectation, which means $(\sigma_{-,T}^{\hat{u}})^2 \le (\sigma_{-,1,T}^{\hat{u}})^2 \le (\sigma_{1,T}^{\hat{u}})^2 \le (\sigma_{1,T}^u)^2 + \mathcal{O}_{\mathbb{P}}(d)$.

Finally, we get the theorem:

Theorem 1: Define a, b_1, λ, d as above. Suppose that condition (4) holds, we have:

$$E[X^{\hat{u}}(T)] \ge E[X_1^u(T)] + \mathcal{O}_{\mathbb{P}}(d)$$
$$(\sigma_{-,T}^{\hat{u}})^2 \le (\sigma_{1,T}^u)^2 + \mathcal{O}_{\mathbb{P}}(d)$$
$$\hat{F} \ge F_1(1 - \mathcal{O}_{\mathbb{P}}(de^d))$$

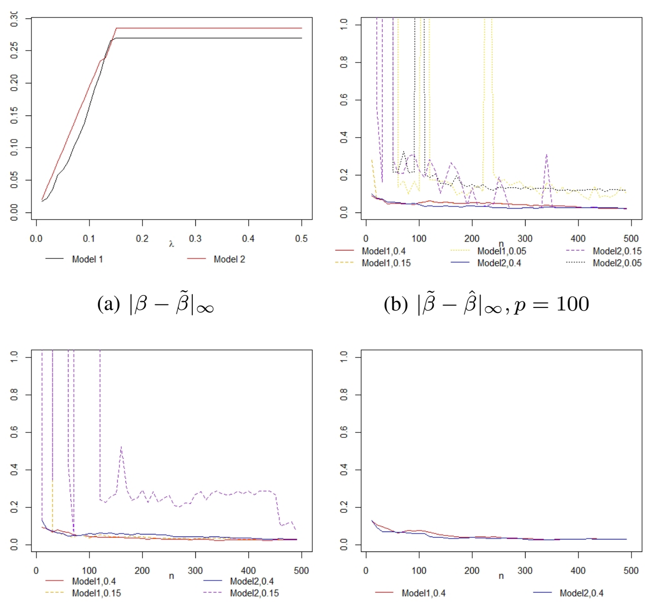
We can see that the theoretical convergence is good, however, the results depend on how close we can estimate Σ and β_1, which is actually difficult in practice. For $\hat{\beta}_1$, the result by NSLPO depends strongly on λ, while in practice, λ is difficult to be small for computing reasons. In the next section, we would like to investigate the convergence of $\hat{\beta}_1$ with method of simulation, and give examples that use the 2 strategies in practice.

V. Simulation and Empirical Studies

A. Simulation

In this part, we simulate the log daily returns with $p = 100, 200$ or 500 stocks. In each test, we calculate $\hat{\beta}$, $\hat{\Sigma}$ and $\hat{\beta}_1$ using $n = 1, \dots, 500$ samples. The theoretical daily return is obtained from a normal distribution $N(\beta + r - \frac{1}{2}diag(\Sigma), \Sigma)$. Choose $r = 0.2$, $\beta = (0.3, \dots, 0.3, 0, \dots, 0)'$, where number of 0.3 $p_1 = 20$, and $\Sigma = 0.2^2\Gamma$, which is from the two models:
Model 1:$\gamma_{ij} = 0.9^{|i-j|}, 1 \leq i, j \leq p$;
Model 2:$\gamma_{ii} = 1, \gamma_{ij} = 0.5$ for $i \neq j$.

Notice that $b_1 \leq |\beta - \tilde{\beta}|_\infty + |\tilde{\beta} - \hat{\beta}|_\infty$, where $\tilde{\beta}$ is got by NSLPO with theoretical β and Σ. Analyze the two items separately, we get the results shown in figure 1.



(a) $|\beta - \tilde{\beta}|_\infty$ (b) $|\tilde{\beta} - \hat{\beta}|_\infty, p = 100$

(c) $|\tilde{\beta} - \hat{\beta}|_\infty, p = 200$ (d) $|\tilde{\beta} - \hat{\beta}|_\infty, p = 500$

Fig. 1: Simulation results, where in the first figure, the curves when $p = 100, 200$ or 500 are the same for both models, and $0.4, 0.15, 0.05$ are value of lambdas in the legends of the last 3 figures.

We can see that the error $|\tilde{\beta} - \hat{\beta}|_\infty$ can be controlled when n becomes large for different p and λ in the two models, although it converges slower when p is large. However, for $|\beta - \tilde{\beta}|_\infty$, it depends strongly on λ. For both models, it is only when λ is really small that one can get a better result. In practice, the situation could be so complex that a small λ like 0.01 cannot be chosen because of the computing costs and time, and we need to deal with the trade-off between accuracy and costs.

B. Empirical Study

In this part, we choose the historical data of S&P stocks in two years: 2013 and 2015, which represents a rising and shock market, separately. A downward market is not studied, because the best strategy is to put all the money into the bank in this situation. We downloaded the data from Yahoo!Finance from January 1, 2012 to December 21, 2015. After deleting the stocks with incomplete data, our data set contains 462 stocks.

Using an approximately 1-year window, we estimate the expectation and variance-covariance matrix of daily returns with the data during the past 250 days. Before we invest following the strategy quarterly, we choose the value of risk-free interest rate $r(0.03)$ and the quarterly terminal value $z(x_0 \times e^{0.05})$, where x_0 is the wealth at the beginning of each quarter, and at the beginning of the year, one has a wealth of 1×10^6.

In the process, we have found that choice of λ is strongly related with the final results, by affecting the gross-exposure number and percentage of stocks used in the investment. For example, if the percentage is too large, he has to face the risk by investing too much money, especially when the number is small. To avoid this, we choose the λ by maximizing the Sharpe ratio of last year in each method, so that we can compare the cases in a more realistic way.

1) Market with Strict No-shorting Constraints: Some countries including China will limit short-selling, especially at the time when financial crisis happens. In this situation, we need to think of some methods to conduct us to invest. There are 4 methods: buying the S&P 500 index directly, MVP solution using LPO approach, no-shorting solution by computing Σ^{-1} directly and using LPO approach, which are named as S&P500, LPO, Base, NSLPO, separately. Noted that we need to change the negative ones in the portfolio with 0s, as the constraints are strict. From the following charts, we can draw some conclusions:

- The two methods can not always achieve the goal at the end of each quarter, which is consistent with the theoretical findings. Also "Base" is too unstable to be used, as it puts too much money in the stock market.
- The standard deviations and Sharpe ratios are good for both methods, even when the market is downward. And the maximum draw-downs are controlled perfectly, which shows the effectiveness of the strategies.
- In most time, the speed of achieving the goal, standard deviation, Sharpe ratio and maximum draw-down of LPO is more superior, when it invests more money in stocks. And in practice, LPO is more applied and easier to realize.

2) A Strategy Considering No-shorting Constraints: Another aspect is that theres no constraints in the market, but we are interested in whats the difference if we use a no-shorting strategy. The methods are: buying the S&P 500 index directly, MVP solution using LPO approach, no-shorting solution using LPO approach, which are named as "S&P500", "LPO", "NSLPO", separately. Note that NSLPO

TABLE I: Parameters of methods in a strict market, where the 4 numbers represent mean, standard deviation, Sharpe ratio and maximum draw down of the daily returns, separately.

	2013.1				2013.2			
S&P500	0.34	0.10	3.05	0.03	0.15	0.14	0.91	0.06
NSLPO	0.18	0.05	2.95	0.01	0.13	0.07	1.50	0.02
LPO	0.17	0.05	3.00	0.01	0.19	0.06	2.52	0.02
	2013.3				2013.4			
S&P500	0.20	0.09	1.86	0.05	0.35	0.10	3.11	0.02
NSLPO	0.20	0.06	3.11	0.01	0.19	0.11	1.55	0.04
LPO	0.20	0.04	4.75	0.01	0.19	0.14	1.19	0.05
	2015.1				2015.2			
S&P500	0.02	0.14	-0.07	0.04	0.02	0.10	-0.06	0.03
NSLPO	0.22	0.06	3.33	0.01	0.10	0.10	0.77	0.04
LPO	0.20	0.07	2.63	0.02	0.14	0.10	1.11	0.04
	2015.3				2015.4			
S&P500	-0.31	0.21	-1.57	0.12	0.24	0.15	1.45	0.05
NSLPO	0.12	0.12	0.85	0.04	0.14	0.08	1.46	0.03
LPO	0.20	0.09	1.83	0.02	0.16	0.06	2.12	0.02

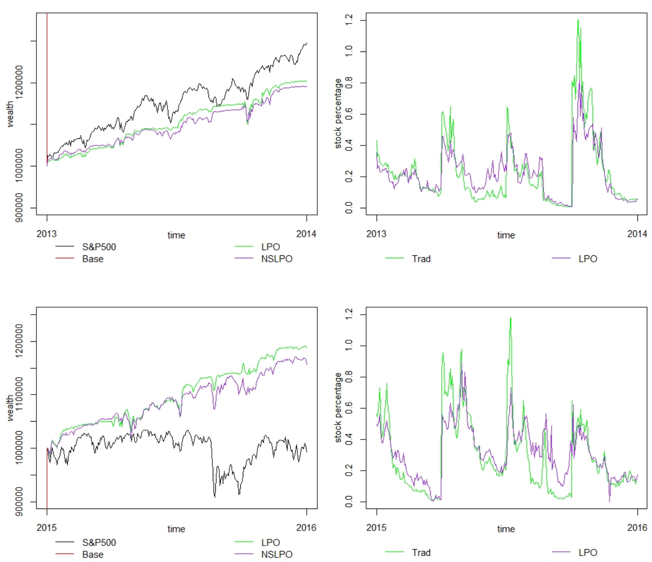


Fig. 2: Behaviors of Methods in a Strict Market

approach do not allow short-selling strictly, so that the comparison is meaningful. We can draw some conclusions:

- "LPO" can always achieve the goal precisely and very fast. But at the beginning of each period, it tends to invest so much money in the market, that when the market is rising, it has some chances to suffer a loss.
- In a rising market, a "NSLPO" method is meaningful, because it avoids the risk of shorting, so that it behaves better (standard deviation, Sharpe ratio, maximum draw-down, etc.), which can be useful in practice.

VI. CONCLUSIONS

In this paper, we analyze the viscosity solution of the portfolio problem under no-shorting constraints and calculate the theoretical and realized statistics of terminal wealth. We propose a new calculation method (NSLPO) to implement

TABLE II: Parameters of methods in a flexible market, where the 4 numbers represent mean, standard deviation, Sharpe ratio and maximum draw-down of the daily return, separately.

	2013.1				2013.2			
S&P500	0.34	0.10	3.05	0.03	0.15	0.14	0.91	0.06
NSLPO	0.18	0.05	2.95	0.01	0.13	0.07	1.50	0.02
LPO	0.19	0.13	1.30	0.06	0.20	0.06	3.05	0.01
	2013.3				2013.4			
S&P500	0.20	0.09	1.86	0.05	0.35	0.10	3.11	0.02
NSLPO	0.20	0.06	3.11	0.01	0.19	0.11	1.55	0.04
LPO	0.20	0.09	1.96	0.03	0.20	0.24	0.73	0.08
	2015.1				2015.2			
S&P500	0.02	0.14	-0.07	0.04	0.02	0.10	-0.06	0.03
NSLPO	0.22	0.06	3.33	0.01	0.10	0.10	0.77	0.04
LPO	0.20	0.04	4.96	0.00	0.20	0.37	0.47	0.09
	2015.3				2015.4			
S&P500	-0.31	0.21	-1.57	0.12	0.24	0.15	1.45	0.05
NSLPO	0.12	0.12	0.85	0.04	0.14	0.08	1.46	0.03
LPO	0.20	0.04	4.00	0.01	0.20	4.50	0.04	0.91

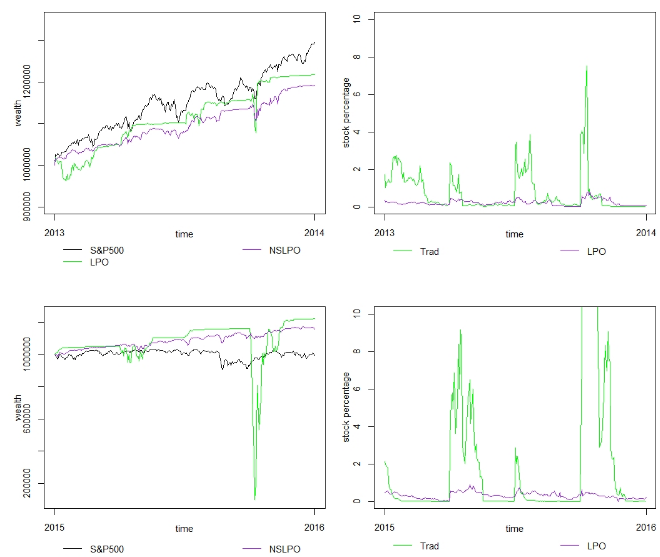


Fig. 3: Behaviors of Methods in a Flexible Market

strategies in high-dimensional data, theoretically prove the convergence properties of the strategy, and find the key factor affecting convergence.

After that, we use the methods of simulation and empirical research to study the convergence properties and actual effectiveness of NSLPO, and compare the results with LPO in the market with and without strict no-shorting restrictions. We can conclude that NSLPO has a good convergence property theoretically, but in practice it is more difficult to apply due to the inaccurate estimation of β_1. Future work may be to find a better method to compute β_1 efficiently and precisely, so that one can apply the strategy in practice.

ACKNOWLEDGMENT

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