Infomation From a View of MLE Asymptotic Distribution

Denote $\mathcal{L}(x,\theta)$ as the likelihood function.
And let $l(x,\theta) = \ln(\mathcal{L}(x,\theta))$ as the log-likelihood.

1. $E[\frac{\partial l}{\partial \theta}] = 0$ $\quad \frac{\partial l}{\partial \theta} = \frac{1}{\mathcal{L}} \frac{\partial L}{\partial \theta}$ $\qquad$ Constant 1.

$$\int_X \frac{1}{L}\frac{\partial L}{\partial \theta} \cdot \mathcal{L}\, dx = \int_X \frac{\partial L}{\partial \theta} dx = \frac{\partial}{\partial \theta} \underline{\int_X L\, dx} = 0$$

2. $Var[\frac{\partial l}{\partial \theta}] = -E[\frac{\partial^2 l}{\partial \theta^2}] = R(\theta)$ $\quad$ Fisher Information

$$0 = \frac{\partial}{\partial \theta} E(\frac{\partial l}{\partial \theta}) = \frac{\partial}{\partial \theta}\int_X \frac{\partial l}{\partial \theta} L\, dx = \int_X \frac{\partial^2 l}{\partial \theta^2} L dx + \int_X \frac{\partial l}{\partial \theta}\frac{\partial L}{\partial \theta}\cdot\frac{1}{L}\cdot L dx$$
$$= -R(\theta) + Var[\frac{\partial l}{\partial \theta}]$$

3. Taylor expansion

$$\frac{\partial l}{\partial \theta}(X;\theta) = 0$$
$$= \underbrace{\frac{\partial l}{\partial \theta}(X;\theta_0)}_{n\beta_0} + (\theta-\theta_0)\underbrace{\frac{\partial^2 l}{\partial \theta^2}(x;\theta_0)}_{n\beta_1} + \frac{1}{2}(\theta-\theta_0)^2\frac{\partial^3 l}{\partial \theta^3}(x;\theta')$$

$\beta_0 \xrightarrow{P} 0$ From (1) $\qquad \beta_1 \xrightarrow{P} R(\theta)$ From (2)

$$n\beta_0 + (\theta-\theta_0)n\beta_1 + \frac{1}{2}(\theta-\theta_0)^2\frac{\partial^3 l}{\partial \theta^3}(\theta') = 0$$
$$\beta_0 + (\theta-\theta_0)\beta_1 + \frac{1}{2}\xi(\theta-\theta_0)^2\frac{\partial^3 l}{\partial \theta^3}(\theta_0) = 0$$
$$(\hat\theta-\theta_0)\left(\beta_1 + \frac{1}{2}\xi(\hat\theta-\theta_0)\frac{\partial^3 l}{\partial \theta^3}(\theta_0)\right) = -\beta_0$$
$$\sqrt{n}(\hat\theta-\theta_0)\underbrace{\left(\beta_1 + \frac{1}{2}\xi(\hat\theta-\theta_0)\frac{\partial^3 l}{\partial \theta^3}\right)}_{\downarrow\ R(\theta)} = \underbrace{\frac{1}{\sqrt{n}}\sum\frac{\partial l}{\partial \theta}(X_i,\theta_0)}_{\downarrow\ N(0,R(\theta))}$$

$$\sqrt{n}R(\theta)(\hat\theta-\theta_0) \xrightarrow{P} N(0, \mathcal{I})$$
$$\sqrt{n}(\hat\theta-\theta_0) \xrightarrow{P} N(0, \frac{1}{R(\theta)})$$