# CS663 Assignment 4

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## Question 5

$C$ is $d \times d$ symmetric matrix and hence by spectral theorem it has $d$ orthogonal eigenvectors with real eigenvalues. Let the eigen decomposition of $C$ equal $UDU^T$ where $U = [u_1|u_2|u_3|...|u_d]$ where $u_i$ is the $d \times 1$ eigenvector corresponding to the $i^{th}$ largest eigen value and and $D$ is the diagonal matrix whose $D(i,i) = e_i$ where $e_i$ is the $i^{th}$ largest eigen value and rest entries are zero. Since all $u_i$ are orthonormal and there are $d$ of them it implies any vector in $R^d$ can be written as a linear combination of them i.e. $f = \sum_{i=1}^{d} c_i u_i$ where $c_i \in R$. Thus

$$f^T C f = f^T U D U^T f = f^T U D (f^T U)^T$$

and using $f = \sum_{i=1}^{d} c_i u_i$ we know that

$$f^T U = [f^T u_1 | f^T u_2 | ... | f^T u_d] = [c_1|c_2|c_3|...|c_d]$$

thus combining both equations we have

$$f^T C f = [c_1|c_2|c_3|...|c_d] D [c_1|c_2|c_3|...|c_d]^T = \sum_{i=1}^{d} e_i c_i^2$$

Since $f \perp u_1$ as given in question, $c_1 = 0$ and hence $f^T C f = 0 + \sum_2^d e_i c_i^2$. Without loss of generality we can assume $f$ to be a unit vector, which implies $\sum_{i=2}^{d} c_i^2 = 1$ and hence to maximize $\sum_2^d e_i c_i^2$ we choose $c_i = 1$ for that value of $i$ for which $e_i$ is maximum, and zero on rest.

This is because $\sum_2^d e_i c_i^2$ is a linear combination of $\{e_i\}_{i=1}^d$ with non-negative weights ($c_i^2 \geq 0 \ \forall i$) that sum up to one, and hence it is also a convex combination of $\{e_i\}_{i=1}^d$. Thus, it will always lie within the range $[\min e_i, \max e_i]$.

Thus we get $c_2 = 1$, and rest $c_i = 0$ (since it is given that all eigen values are distinct and $rank(C) > 2$, only $c_2$ should be 1). Hence $f = u_2$ or the eigenvector with the second largest eigenvalue, which we set out to prove.