

## CS663 Assignment 4

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### Question 6

(a)

$P = A^T A \implies y^T P y = (y^T A^T)(y^T A^T)^T = v v^T = \sum_{i=1}^{|v|} v_i^2 \geq 0$  where  $v = y^T A^T$  and  $|v|$  is number of elements in the vector  $v$  (note that  $v$  is a row vector).

Similarly,

$Q = A A^T \implies z^T Q z = (z^T A)(z^T A)^T = v v^T = \sum_{i=1}^{|v|} v_i^2 \geq 0$  where  $v = z^T A$  and  $|v|$  is number of elements in the vector  $v$ .

These properties of  $P$  and  $Q$  are true for any arbitrary  $y$  and  $z$ . Choose  $y$  and  $z$  to be the eigen vectors of  $P$  and  $Q$  with eigenvalues  $\lambda_1$  and  $\lambda_2$  respectively. Substituting, we get

$$y^T P y = y^T (\lambda_1 y) = \lambda_1 (y^T y) = \lambda_1 \geq 0$$

$$z^T Q z = z^T (\lambda_2 z) = \lambda_2 (z^T z) = \lambda_2 \geq 0$$

Hence,  $P$  and  $Q$  will always have non-negative eigenvalues (since the eigen vectors were chosen arbitrarily).

(b)

$$P u = \lambda u \implies A^T A u = \lambda u$$

Pre-multiplying  $A$  on both sides of the equation yields

$$A A^T (A u) = \lambda (A u)$$

Hence by definition of eigenvectors  $A u$  is an eigenvector of  $A A^T$  (or  $Q$ ) with eigenvalue  $\lambda$ . Similarly,

$$Q v = \mu v \implies A A^T v = \mu v$$

Pre-multiplying  $A^T$  on both sides of the equation yields

$$A^T A (A^T v) = \mu (A^T v)$$

Hence by definition of eigen vectors  $A^T v$  is eigen vector of  $A^T A$  (or  $P$ ) with eigenvalue  $\mu$ . Keeping in mind the validity of multiplication operation in matrices, the number of elements in the vectors  $u, v$  will be  $n$  (dimension is  $n \times 1$ ) and  $m$  (dimension is  $m \times 1$ ) respectively.

(c)

Let  $k = \|A^T v_i\|_2$  where  $k \in R^+$ . Since  $v_i$  is an eigen vector of  $Q$ , by definition of eigen vectors we have  $Q v_i = \lambda v_i$  for some  $\lambda \in R^+$  (since by part (a) we know that the eigenvalues are non-negative for  $Q$ , and  $\lambda$  is the eigenvalue for eigen vector  $v_i$ ). Thus

$$A u_i = \frac{A A^T v_i}{k} = \frac{Q v_i}{k} = \frac{\lambda}{k} v_i$$

Thus,  $\gamma_i = \frac{\lambda}{k} \in R^+$  as the numerator is non-negative (shown above) and the denominator is positive ( $L_2$  norm of any vector is positive). Thus, we have shown the existence of a non-negative real  $\gamma_i$  by finding it out using the corresponding eigenvalue.

(d)

It is given in the question that  $u_i$ 's are orthonormal among themselves and  $v_i$ 's are orthonormal among themselves. Also,  $U = [v_1|v_2|\dots|v_m]$  and  $V = [u_1|u_2|\dots|u_m]$ . We solve backwards.  $A = U\Gamma V^T \iff AV = U\Gamma$  (post multiplying with  $V$  and making use of the fact that  $V$  is a orthonormal matrix as all column vectors are orthonormal to each other). Now we show separately that LHS and RHS are equal to the same matrix and hence are in turn equal to one another (which would imply that  $A = U\Gamma V^T$ )

$$AV = A[u_1|u_2|\dots|u_m] = [Au_1|Au_2|\dots|Au_m] = [\gamma_1 v_1|\gamma_2 v_2|\dots|\gamma_m v_m]$$

$$U\Gamma = U[x_1|x_2|\dots|x_m]$$

where  $x_i$  is the  $i^{th}$  column vector of  $\Gamma$ . By definition of  $\Gamma$  (diagonal matrix), only  $i^{th}$  entry of  $x_i$  is non-zero and equal to  $\gamma_i$ . Now we can simplify it as  $U\Gamma = [Ux_1|Ux_2|\dots|Ux_m]$ ;  $Ux_i$  is equal to  $\gamma_i$  times the  $i^{th}$  column of  $U$  which is  $v_i$ . Thus,  $U\Gamma = [\gamma_1 v_1|\gamma_2 v_2|\dots|\gamma_m v_m]$ . Hence both RHS and LHS have been shown to be equal, thereby establishing the required equality.