Risk premium

Definition

With the initial wealth w, the risk premium (R) with respect to a random return a is defined implicitly as

$$\mathbb{E}[U(w+a)] = U(w + \mathbb{E}[a] - R).$$

Note that the risk premium is defined relative to the expected wealth $w + \mathbb{E}[a]$, which is a risk-free wealth. With R, we say

$$\text{A decision maker is} \begin{cases} \text{risk averse} & \text{if } R > 0 \\ \text{risk neutral} & \text{if } R = 0 \\ \text{risk loving} & \text{if } R < 0. \end{cases}$$

Characterization

We have two different characterizations of R:

$$R = w + \mathbb{E}[a] - U^{-1}(\mathbb{E}[U(w+a)]) \tag{1}$$

$$\approx -0.5\frac{u''}{u'}\mathrm{Var}(a). \tag{2}$$

(1) is obtained by simple rearrangement of the definition, while (2) is an approximation and clearly expresses that R is proportional to $\mathrm{Var}(a)$. Note that given $u' > 0$ assumed and $\mathrm{Var}(a) > 0$, the sign of R is the opposite sign of u''.

Proof of the proportionality

Since it is somewhat difficult to follow the proof provided in the text (Chavas, 2004, p.36), here is another derivation of (2). First, cosider both first and second order Taylor approximations of $U(x)$ around $w + \mathbb{E}[a]$,

$$U(x) \approx u + u' \cdot (x - (w + \mathbb{E}[a])) \tag{3}$$

$$U(x) \approx u + u' \cdot (x - (w + \mathbb{E}[a])) + 0.5u'' \cdot (x - (w + \mathbb{E}[a]))^2. \tag{4}$$

Note that u,u', and u'' denote just three numbers (not variables or functions):

$$u = U[w + \mathbb{E}[a]],\ u' = U'[w + \mathbb{E}[a]],\ u'' = U''[w + \mathbb{E}[a]].$$

Next, substitute $x = w + \mathbb{E}[a] - R$ in (3),

$$U(w + \mathbb{E}[a] - R) \approx u + u' \cdot (w + \mathbb{E}[a] - R - (w + \mathbb{E}[a])) \tag{5}$$

$$= u + u' \cdot (-R) \tag{6}$$

$$= u - u' \cdot R. \tag{7}$$

Also, substitute $x = w + a$ in (4),

$$U(w+a) \approx u + u' \cdot (w + a - (w + \mathbb{E}[a])) + 0.5u'' \cdot (w + a - (w + \mathbb{E}[a]))^2 \tag{8}$$

$$= u + u' \cdot (a - \mathbb{E}[a]) + 0.5u'' \cdot (a - \mathbb{E}[a])^2. \tag{9}$$

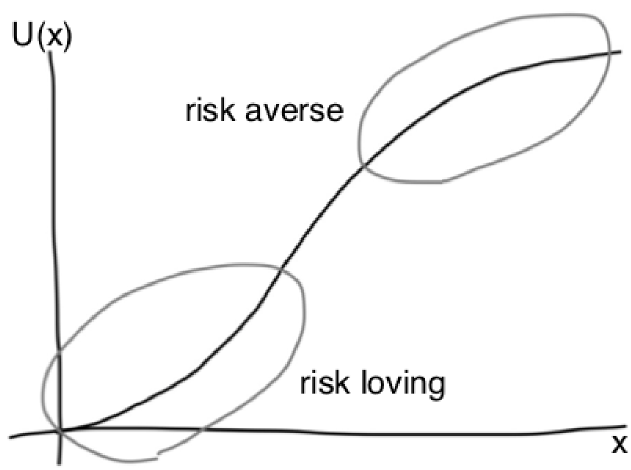
Then, take an expectation of both sides of (9),

$$\begin{aligned}
\mathbb{E}[U(w+a)] &\approx \mathbb{E}[u + u' \cdot (a - \mathbb{E}[a]) + 0.5u'' \cdot (a - \mathbb{E}[a])^2] && (10)\\
&= \mathbb{E}[u] + \mathbb{E}[u' \cdot (a - \mathbb{E}[a])] + \mathbb{E}[0.5u'' \cdot (a - \mathbb{E}[a])^2] && (11)\\
&= u + u' \cdot \mathbb{E}[(a - \mathbb{E}[a])] + 0.5u'' \cdot \mathbb{E}[(a - \mathbb{E}[a])^2] && (12)\\
&= u + 0.5u'' \cdot \mathrm{Var}(a). && (13)
\end{aligned}$$

Since $\mathbb{E}[U(w+a)] = U(w + \mathbb{E}[a] - R)$ by definition, combining (7) and (13), we obtain (2). □

Notes

- By the nature of the Taylor approximation, (2) is only valid in the neighborhood of $w + \mathbb{E}[a]$.
- Risk premium is specific to a particular random return. That is, even for the same individual (with fixed $U(\cdot)$), different random returns a_1 and a_2 usually result in different risk premiums R_1 and R_2.
- In general, it is quite possible for an individual to exhibit both risk aversion and risk loving depending on wealth and random return. For example, I enjoy little excitement of gambling in a region of low stake, while I do not take risks between \$1 million and \$5 millions.



- It is convenient to assume that an individual is risk averse no matter what or equivalently U is concave, i.e. $U''(x) < 0$ for all x.

Risk aversion

There has been a lot of evidence that most people are risk averse. As seen in (2), it turns out convenient to give a name to $-u''/u'$.

Arrow-Pratt coefficient of absolute risk aversion

$$r(x) = -\frac{U''(x)}{U'(x)} = -\frac{d\log(U'(x))}{dx}$$

In (2) we evaluate $r(x)$ at $x = w + \mathbb{E}[a]$, i.e. $r(w + \mathbb{E}[a]) = u''/u'$, so given the random return a, it is a function of the wealth level w.

DARA: $r(x)$ is decreasing in x	$\Leftrightarrow$	R is decreasing in x
CARA: $r(x)$ is constant in x	$\Leftrightarrow$	R is constant in x
IARA: $r(x)$ is increasing in x	$\Leftrightarrow$	R is increasing in x

Remember that $r(x)$ is a property of the utility function, whereas R in addition reflects a particular random return in the form of $\mathrm{Var}(a)$.

CARA

Risk preferences are said to exhibit constant absolute risk aversion (CARA) when $r(x)$ does not depend on wealth, i.e. $r(x) = r$ for all x.

Recall that we may re-write the utility function in terms of r (Chavas, 2004, p.38),

$$U(x) = c_1 \int e^{-\int r} + c_2$$

where c_1, c_2 are constant and $c_1 > 0$. Note that this is an indefinite integral, so although r is constant and $U(x)$ does not look like a function of x, it is still. Due to the uniqueness under an affine transformation, we may simplify it:

$$\begin{aligned} U(x) &= \int e^{-\int r} \\ &= \begin{cases} -k_1 e^{-rx} + k_2 & r > 0 \\ k_1 x + k_2 & r = 0 \\ k_1 e^{-rx} + k_2 & r < 0 \end{cases} \end{aligned}$$

where k_1, k_2 are constant and $k_1 > 0$. Applying an affine transformation again,

$$U(x) = \begin{cases} -e^{-rx} & r > 0 \\ x & r = 0 \\ e^{-rx} & r < 0 \end{cases}$$

This is particularly useful when a random return has a normal distribution, $a \sim \mathcal{N}(A, V)$, because the expected utility has a nice expression. When $r > 0$, for example,

$$\begin{aligned}
\mathbb{E}[U(a)] &= \int_{-\infty}^{\infty} -e^{-rx} \cdot \frac{1}{\sqrt{2\pi V}} \exp\left[-\frac{1}{2V}(x-A)^2\right] dx \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} \exp\left[-\frac{1}{2V}(x-A)^2 - rx\right] dx \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} \exp\left[-\frac{1}{2V}(x-A+rV)^2 - Ar + \frac{r^2 V}{2}\right] dx \\
&= -\exp\left[-Ar + \frac{r^2 V}{2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V}} \exp\left[-\frac{1}{2V}(x-(A-rV))^2\right] dx \\
&= -\exp\left[-r\left(A - \frac{r}{2}V\right)\right]
\end{aligned}$$

As you can see, the larger $A - rV/2$, the higher $\mathbb{E}[U(a)]$. The implication is that if all of our choices $a_1, a_2, \ldots$ have normal distributions $a_1 \sim \mathcal{N}(A_1, V_1), a_2 \sim \mathcal{N}(A_2, V_2), \ldots$, we should choose the one with the largest $A - rV/2$.

Another implication is that we have an exact expression of (2), i.e.

$$R = -0.5 \frac{u''}{u'} \mathrm{Var}(a) = \frac{r}{2} V.$$

Notice that the approximation ($\approx$) has become the equality ($=$). A proof is simply to plug everything in the definition and rearrange it.

$$\begin{aligned}
& \mathbb{E}[U(w+a)] = U(w + \mathbb{E}[a] - R) \\
\Leftrightarrow\ & -\exp\left[-r\left((w+A) - \frac{r}{2}V\right)\right] = -\exp\left[-r\left(w + A - R\right)\right] \\
\Leftrightarrow\ & -r\left(w + A - \frac{r}{2}V\right) = -r\left(w + A - R\right) \\
\Leftrightarrow\ & R = \frac{r}{2}V
\end{aligned}$$

$\square$

Later in the class, we will learn mean-variance analysis (Ch 6), where we make another simplifying assumption of $\mathbb{E}[U(w+a)] = W(A, V)$, just a function of mean and variance of a. Here, we have seen that such a convenient framework automatically arises with CARA and normality assuptions.

DARA

There is empirical evidence that most individuals exhibit DARA risk preferences. We know the equivalence between risk aversion, concavity of U, and $U'' < 0$. So, intuitively, if risk aversion decreases, it must be the case that U becomes "less concave" or U'' increases (i.e. $|U''|$ decreases). Formally,

$$\frac{dr(x)}{dx} = -\frac{d}{dx}\frac{U''(x)}{U'(x)} = -\frac{U'''U' - U''U''}{(U')^2} = -\frac{U'''}{U'} + \left(\frac{U''}{U'}\right)^2.$$

With $U' > 0$ (non-satiation) and $U'' < 0$ (risk aversion), we have

$$\text{DARA} \;\Leftrightarrow\; \frac{dr(x)}{dx} \leq 0 \;\Leftrightarrow\; -\frac{U'''}{U'} + \left(\frac{U''}{U'}\right)^2 \leq 0 \;\Rightarrow\; U''' > 0.$$

The first two equivalences are by definition, all of which implies $U''' > 0$.