STAT 311: Counting Rules

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Logistics

- Midterm on Friday
- Bring a calculator (that is not also your phone)
- Bring practice midterm to lab on Thursday

Counting Rules

There are often situations when I want to count the number of "outcomes"

- How many different outcomes can occur when I roll two dice?
- How many different outcomes can occur if I deal 2 cards from a deck of cards?
- How many different ways can I form a batting order out of 12 available players?
- How many different ways can I form a control and treatment group out of 300 individuals?

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This is generally a field of mathematics called combinatorics. However, it is very helpful in probability in cases when each of the outcomes are equally likely.

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Basic setup-

- We are selecting a single element from some set of elements
- We may repeat this process several times

Sampling with or without replacement

Suppose I am setting a schedule for where to eat today (3 meals). I have a set of 5 restaurants that I can choose from and will randomly select where to eat each meal

- Without replacement: If I have selected a restaurant for a previous meal, I cannot select it again
- With replacement: If I have selected a restaurant for a previous meal, I can select it again

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- Without replacement: Each meal location is dependent on previous meal locations
- With replacement: Each meal location is independent of previous meal locations

Sampling with replacement

Draw a tree for 5 restaurants

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If we are sampling with replacement a subset of k elements from a set of N elements, the total number of outcomes is

$$N^k$$

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Intuition: Each of the k elements in my subset has N possibilities, so I have N new branches at each level

Sampling without replacement

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Sampling without replacement

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Intuition: The first choice has N possibilities, but since I am sampling without replacement, the next choice only has $(N-1)$ choices, then $N-2$ and so on. Thus the number of new branches as each tier decreases by 1.

Examples

In Texas Hold 'Em poker, you receive two cards as your starting hand. If there are 52 total cards in a deck, how many different starting hands can you be dealt?

- Is this with or without replacement?
- What rule should I use?

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$$52 \times 51 = 2,652$$

Examples

I have 10 songs, and I want to randomly pick 5 songs for a playlist (I can listen to the same song multiple times)

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$$10^5 = 100,000$$

Examples

The number of playlists of 20 songs grows very large with the number of possible songs

Songs to Choose from	With Replacement	Without Replacement
21	2.8E+26	2.4E+18
30	3.5E+29	2.4E+25
50	9.5E+33	6.9E+31
100	1.0E+40	1.0E+39
200	1.0E+46	3.5E+45
500	9.5E+53	6.2E+53
1000	1.0E+60	8.1E+59

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The number of atoms on earth is roughly 10^{50}. The number of playlists grows even faster when the length of the playlist increases

What exactly am I considering an outcome

The total number of "outcomes" also depends on whether I care about the ordering or not

- Is $\{A\spadesuit, A\clubsuit\}$ a different hand than $\{A\clubsuit, A\spadesuit\}$?
- Does order matter in a batting lineup?

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- Is $\{A\spadesuit, A\clubsuit\}$ a different hand than $\{A\clubsuit, A\spadesuit\}$?
- Does order matter in a batting lineup?

- When order does not matter, we count **combinations**
- When order does matter, we count **permutations**

$\{A\spadesuit, A\clubsuit\}$ and $\{A\clubsuit, A\spadesuit\}$ are the same combination, but different permutations

Permutation Rule

When selecting a subset of k elements (without replacement) from a set of N elements, the number of permutations is

$$P_N^k = \frac{N!}{(N-k)!}$$

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Note this is what we implicitly assumed before

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Intuition: There are $k!$ permutations which map to the same combination. Thus, we take the total number of permutations, and divide by $k!$

Examples

What is the probability of being dealt 2 Aces if there are 4 aces in the deck?

- How many ways combinations of two aces can be formed from the 4 aces in the deck?
- $C_4^2 = \frac{4!}{2!2!} = 6$

Examples

What is the probability of being dealt 2 Aces if there are 4 aces in the deck?

- How many ways combinations of two aces can be formed from the 4 aces in the deck?
- $C_4^2 = \frac{4!}{2!2!} = 6$
- How many total combinations of two cards can be formed from the 52 cards
- $C_2^2 5 = \frac{52!}{50!2!} = 1326$

Because each combination is equally likely, then the probability of two aces is $6/1326 = 1\ /\ 221$

Partition Rule

Given a set of N individuals, how many ways can I divide them into m different subsets of $k_1, \ldots k_m$ size. Assume ordering does not matter and $k_1 + k_2 \ldots + k_m = N$

$$\frac{N!}{k_1!k_2!\ldots k_m!}$$

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Does this look familiar? When we were just selecting a combination, we can also think of that as dividing the group into two subsets, an "in" group and an "out" group.

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$$\frac{N!}{k_1!k_2!\dots k_m!}$$

Intuition: I have $\frac{N!}{(N-k_1)!k_1!}$ ways of assigning people to the first group, given that I've assigned the first group, then I have $\frac{(N-k_1)!}{(N-k_1-k_2)!k_2!}$ ways to assign the second group (conditional on the first group) and so on...

$$\frac{N!}{(N-k_1)!k_1!} \times \frac{(N-k_1)!}{(N-k_1-k_2)!k_2!} \times \frac{(N-k_1-k_2)!}{(N-k_1-k_2-k_3)!k_3!} \dots$$

Example

We are assigning a group of 30 volunteers to 4 different stations at a food bank. Station 1 needs 5 volunteers, Station 2 needs 10 volunteers, Station 3 needs 12 volunteers, Station 4 needs 3 volunteers. How many different ways can we assign everyone to a station?

$$\frac{30!}{5!10!12!3!} = 211,947,150,214,800$$

Drawing from different sets

Suppose instead of drawing from the same set of elements each time, I draw from a set of different size each time. The size of each set is N_1, N_2 $\ldots N_s$. If I draw one element from each set, then the number of permutations (or combinations, since they are the same here) is

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Intuition: Each tier has the same number of new branches as elements in the set you are drawing from. This is the exact same idea as drawing without replacement, except before we knew exactly how the set sizes changed

Example

I am trying to build an ice cream sundae, which requires the one of each of the following elements: Ice Cream, syrup, sprinkles, nuts

- 5 flavors of ice cream
- 3 types of syrups
- 2 types of sprinkles
- 3 types of nuts

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$$5 \times 3 \times 2 \times 3 = 90$$

Can we generalize this rule?

I am trying to build an ice cream sundae, which requires the one of each of the following elements: Ice Cream, syrup, sprinkles, nuts

- 5 flavors of ice cream; I must pick 3 flavors (without replacement)
- 3 types of syrups; I must pick 2 types (without replacement)
- 2 types of sprinkles; I must pick 1 type
- 3 types of nuts; I must pick 2 types (without replacement)

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$$C_5^3 \times C_3^2 \times C_2^1 \times C_3^2 = 10 \times 3 \times 2 \times 2 = 120$$