STAT 311: Homework 4
Due: Jul 25, in class

Name:

The material covered includes CH 7 from U+H and lecture 10. In general, rounding to 2 digits is sufficient. You may find the following probability rules from the lecture to be helpful-

- Complement rule (Not):[$P(A) = 1 - P(A^c)$
- Union Rule (Or):$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Intersection Rule (and): $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
- Two events are independent if and only if: $P(A|B) = P(A)$ and $P(B|A) = P(B)$
- Bayes rule: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$

Settlers of Catan is a board game in which players try "to build settlements on the board using 'resource' cards[1]." The board consists of 5 types of resources hexagons (brick, lumber, ore, wool and grain) which are each assigned a number between 2 and 12. Each player has settlements/cities which they place on the board. Every turn, a player rolls two dice and if the sum of the dice match a number on a resource hexagon adjacent to any of your cities, you receives that resource. This homework will deal with calculating probabilities relating to the game.

1 On a roll: Basic Probabilities

1. *When rolling two dice, there are 36 outcomes (ie, you could get a (1,1) or a (1, 2) or (2, 1) or ...). Assuming the outcome of each dice is independent and each die has an equal probability of rolling each value 1-6, what is the probability of each outcome?*

   Since there are 36 outcomes, and they all have equal probability, then each outcome has a probability of 1/36.

2. *If I am only interested in the sum of the two dice, how many outcomes are in the sample space? List each of the outcomes in the entire sample space below.*

   The outcomes of the sum of the two dice are now $2, 3, 4, \ldots 12$.

3. *Assuming the outcome of each dice is independent, what is the probability of each outcome (when only considering the sum)? (Hint: Think about how many outcomes in question 1 result in each outcome you listed in question 2)*

   The outcomes of the sum of the two dice are now $2, 3, 4, \ldots 12$. However, each does not have equal probability, because there are multiple ways of arriving at each sum. In the table below, the inner cells show the sum of the two dice, and the rows and columns represent the result of dice 1 and 2 respectively. Because each inner cell is one of the 36 outcomes mentioned in question 1, each of the inner cells has equal probability. So for the probability of a result X, we can just count up the number of times it appears in the table.

[1] http://www.wsj.com/articles/the-packers-of-catan-green-bays-board-game-obsession-1421346102

Dice 1 $\downarrow$ \ Dice 2 $\rightarrow$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

This is essentially an application of the union rule. Since each of the dice roll permutations is disjoint, I can just add up the probabilities.

This gives us the following probabilities

x	2	3	4	5	6	7	8	9	10	11	12
P(Sum = x)	0.03	0.06	0.08	0.11	0.14	0.17	0.14	0.11	0.08	0.06	0.03

4. *Using the probabilities you calculated in question 3, what is the probability that the sum of the dice is less than or equal to 5? Write that in probability notation.*

$$P(\text{Sum} \leq 5 = P(\text{Sum} = 2) + P(\text{Sum} = 3) + P(\text{Sum} = 4) + P(\text{Sum} = 5) = 10/36$$

5. *Given that the first dice shows 4, what is the probability that the sum is 8? Write that in probability notation.*

$$P(\text{Sum} = 8|\text{Dice } 1 = 4) = P(\text{Dice } 2 = 4|\text{Dice } 1 = 4) = P(\text{Dice } 2 = 4) = 1/6$$

2 Bored Game: more complicated probabilities

1. *In Catan, if someone rolls a 7, you have to give up half the resources you already own if you have more than 7 resource cards. Suppose you currently have 9 resource cards. What is the probability of losing half your cards? What is the probability of not not losing half your cards?*

$$P(\text{Lose half your cards}) = P(\text{Sum} = 7) = 6/36$$

$$P(\text{Not Lose half your cards}) = 1 - P(\text{Not Lose half your cards}) = 30/36$$

2. *What is the probability of avoiding a 7 twice in a row? What assumption did you make to arrive at your answer?*

$$\begin{aligned} P(\text{No 7 in two rolls}) &= P(\text{Not 7 on 1st roll} \bigcap \text{Not 7 on 2nd roll}) \\ &= P(\text{Not 7 on 1st roll})P(\text{Not 7 on 2nd roll}) \\ &= 30/36 \times 30/36 = 25/36 \end{aligned}$$

Note that we had to assume that the rolls are independent, which is reasonable

3. *If there are 4 players, what is the probability that at least 1 player will roll a 7 in this round (four dice rolls)? Hint: Calculating this directly is tough. What is the complement of at least 1 player rolling a 7? Can we calculate this more easily and then back out the answer?*

$$\begin{aligned} P(\text{At least 1 7 in 4 rolls}) &= 1 - P(\text{No 7s in 4 rolls} \\ &= 1 - (30/36)^4 = 625/1296 \end{aligned}$$

4. *If you just want to maximize the number of cards you get, would you rather have a city with adjacent numbers of* $\{4, 8, 2\}$ *or* $\{5, 9, 10\}$*?*

$$P(4 \bigcup 8 \bigcup 2) = 3/36 + 5/36 + 1/36 = 8/36$$

$$P(5 \bigcup 9 \bigcup 10) = 4/36 + 4/36 + 3/36 = 11/36$$

So you would rather have 5,9,10

5. *Suppose you have cities which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(6); sheep(2); ore (11). What is the probability that you receive lumber?*

$$P(Lumber) = P(5 \bigcup 6) = 4/36 + 5/36 = 1/4$$

6. *Suppose you have cities which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(6); sheep(2); ore (11). What is the probability that you receive wheat?*

$$P(Lumber) = P(10 \bigcup 6) = 5/36 + 3/36 = 2/9$$

7. *Suppose you have cities which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(6); sheep(2); ore (11). What is the probability that you receive wheat or lumber? Use the union rule.*

$$P(\text{Lumber} \bigcup \text{Wheat}) = P(\text{Lumber}) + P(\text{Wheat}) - P(\text{Lumber} \bigcap \text{Wheat}) = 1/4 + 2/9 - 5/36 = 1/3$$

8. *Suppose you have cities which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(6); sheep(2); ore (11). What is the probability that you receive wheat and lumber?*

$$P(\text{Lumber} \bigcap \text{Wheat}) = P(6) = 5/36$$

9. *Suppose you have cities which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(6); sheep(2); ore (11). Are receiving lumber and receiving wheat independent events? Show mathematically why or why not.*

$$P(\text{Lumber}|\text{Wheat}) = \frac{P(\text{Lumber} \bigcap \text{Wheat})}{P(\text{Wheat})} = (5/36)/(8/36) = 5/8$$

This is not equal to the marginal probability of lumber (1/4), so the events are dependent.

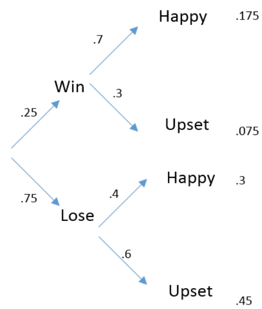
3 All about that Bayes

1. *Suppose you have settlements which is adjacent to the following resource hexagons (and corresponding numbers): lumber (5); lumber (6); wheat (10); wheat(8); sheep(2); ore (11). After you roll, I don't see what the outcome was, but I see that you recieve a single lumber card. Using Bayes rule, what would I surmise about the probability that you had rolled a 5? Hint: Since you recieved a lumber card, what events should I condition on?*

$$\begin{aligned} P(\text{Sum} = 5|\text{Lumber}) &= P(\text{Sum} = 5|5 \bigcup 6) \\ &= \frac{P(5 \bigcap (5 \bigcup 6))}{P(5 \bigcup 6)} \\ &= \frac{P(5)}{P(5 \bigcup 6)} \\ &= \frac{(4/36)}{(4/36) + (5/36)} = 4/9 \end{aligned}$$

2. *Suppose that you have a .25 chance of winning the game. If you lose, there is a .6 chance that you will be upset and if you win there is a .3 chance that you will be upset. What is the probability that you are upset after the game? Draw a probability tree to calculate this value.*

   We can see from the probability tree that the chance you are upset after the game is $.45 + .075 = .525$



3. *Your friend sees in class the next day that you are visibly upset. Since your friend know Bayes rule, what value would they assign to the probability that you lost. Write this in probability notation, and then give the numeric value*

$$P(\text{Lost}|\text{Upset}) = P(\text{Lost} \bigcap \text{Upset})/P(\text{Upset}) = .45/.525 = .86$$