# Solving the 1D Euler equations using the Discontinuous Galerkin Method

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## Euler equations in 1D

The one-dimensional Euler equations (in $x$) can be written as (in conservative form) -

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{1}$$

where the state variable $\mathbf{u}$ and the flux vector $\mathbf{F}$ are given as :

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} \tag{2}$$

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u\left(\rho E + p\right) \end{bmatrix} \tag{3}$$

With the pressure $p$ defined as

$$p = (\gamma - 1)\left(\rho E - \frac{\rho u^2}{2}\right) \tag{4}$$

$\rho$, $u$ and $E$ refer to the mass density, velocity (in the x-direction) and total energy of the fluid. We assume $\gamma = 1.4$ for air. Also important is the use of the sound speed $a$ defined as

$$a = \sqrt{\frac{\gamma p}{\rho}} \tag{5}$$

If we define a matrix $\mathbf{A}$ such that

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \tag{6}$$

Then it follows if Eq 1 is hyperbolic that $\mathbf{A}$ is *diagonalizable* / it can be written in the form

$$\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1} \tag{7}$$

Where the eigenvalues for the 1D Euler equations are $\mathbf{\Lambda} = \mathrm{diag}\left(u - a, u, u + a\right)$ and the corresponding right eigenvector matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 \\ u - a & u & u + a \\ H - ua & \frac{u^2}{2} & H + ua \end{bmatrix} \tag{8}$$

$H$ is defined as -

$$H = E + \frac{p}{\rho} \tag{9}$$

Furthermore if $\mathbf{F} = \mathbf{A}\mathbf{u}$ (where $\mathbf{A}$ may be a function of $\mathbf{u}$) then the system of equations is termed *homogenous*, a property that is useful for methods such as flux-vector splitting. The Euler equations indeed are homogeneous.

# Approximate fluxes at Riemann interfaces

A hyperbolic system of equations gives rise to solutions in the form of travelling waves. The Riemann problem is described as follows. Given two states $\mathbf{u}_R$ and $\mathbf{u}_L$ cleanly separated at $t = 0$, to find the state $\mathbf{u}$ of the system at a time $t > 0$. It is a common feature of finite-volume methods that the fluxes at cell interfaces be calculated through the solution of a Riemann problem at that interface, either exactly or approximately. Exact solutions for the Euler equations can be determined, but are generally costly to implement in practical CFD codes. In most cases, an approximate solution may suffice. Although the flux calculation at the interfaces may be exact, the order of accuracy may still be $\mathcal{O}(\Delta x)$ since we are representing the solution in a cell using cell-averages i.e. as a zeroth order polynomial.

In a setting where the left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$ are known, we can represent the flux at the interface $\mathbf{F}^*$ as

$$\mathbf{F}^* = f(\mathbf{u}_L, \mathbf{u}_R) \tag{10}$$

Some examples for approximating $\mathbf{F}^*$ are given below -

## Gudonov's method

Gudonov's method gives the exact solution to the Riemann problem. Consider two states $L$ and $R$ separated at $t = 0$ with the interface located at $x = 0$. The solution for the Riemann problem is of the form sketched in Fig 1.

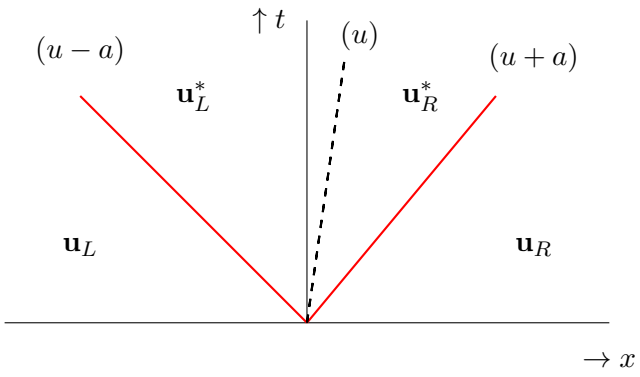


**Figure 1:** Solution to the 1D Riemann problem for the Euler equations. The right and left going waves can be rarefactions or shocks whereas the middle wave is a contact discontinuity.

It has been established that the pressure and velocity remain constant across the contact discontinuity i.e. $p^*_L = p^*_R = p^*$ and $u^*_L = u^*_R = u^*$. Hence our unknowns are - $(\rho^*_L, \rho^*_R, p^*, u^*)$. The following algorithm may be used to calculate these quantities -

The algorithm described above can be used to reconstruct the exact solution at any time $t > 0$, but since we are only interested at the flux at $x = 0$ we identify that

$$\mathbf{F}^*(x = 0) = \begin{cases} \mathbf{F}(\mathbf{u}^*_L) & \text{if } u^* > 0 \\ \mathbf{F}(\mathbf{u}^*_R) & \text{if } u^* < 0 \end{cases} \tag{11}$$

Also note that while the contact discontinuity lies in a region with velocity $u^*$, the right/left going waves do not necessarily correspond to $u^* + a$ and $u^* - a$ respectively. If either wave is a rarefaction wave then the velocity across the rarefaction changes and is determined by the states surrounding the rarefaction.

### Lax-Friedrich's flux

$$\mathbf{F}^* = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{\Delta t}{2\Delta x}\left(\mathbf{u}_R - \mathbf{u}_L\right) \tag{12}$$

### Rusanov flux

$$\mathbf{F}^* = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{s_{\max}}{2}\left(\mathbf{u}_R - \mathbf{u}_L\right) \tag{13}$$

$$s_{\max} = \max\left(a_L + |u_L|, a_R + |u_R|\right) \tag{14}$$

### Steger-Warming Flux Vector Splitting

$$\mathbf{F}^* = \mathbf{F}_L^+ + \mathbf{F}_R^- \tag{15}$$

$$\mathbf{F}^\pm = \mathbf{A}^\pm \mathbf{u} = \mathbf{R}\mathbf{\Lambda}^\pm \mathbf{R}^{-1}\mathbf{u} \tag{16}$$

$$\mathbf{\Lambda}^\pm = \lambda_i^\pm = \frac{\lambda_i \pm |\lambda_i|}{2} \tag{17}$$

### van Leer Flux Vector Splitting

$$\mathbf{F}^* = \mathbf{F}_L^+ + \mathbf{F}_R^- \tag{18}$$

$$\mathbf{F}^\pm = \pm\frac{1}{4}\rho a\left(1 \pm M^2\right)\begin{bmatrix} 1 \\ \frac{2a}{\gamma}\left(\frac{\gamma-1}{2}M \pm 1\right) \\ \frac{2a^2}{\gamma^2-1}\left(\frac{\gamma-1}{2}M \pm 1\right)^2 \end{bmatrix} \tag{19}$$

$$M = \frac{|u|}{a} \tag{20}$$

### Roe flux

$$\mathbf{F}^* = \frac{1}{2}\left(\mathbf{F}_L + \mathbf{F}_R\right) - \frac{1}{2}\left[\mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}\right]_{\mathbf{u}=\hat{\mathbf{u}}}\left(\mathbf{u}_R - \mathbf{u}_L\right) \tag{21}$$

$$\hat{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{22}$$

$$\hat{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{23}$$

**HLLE flux**

## Discontinuous Galerkin Method

Take Eq 1. Assume it has $s$ number of conservation laws corresponding to $s$ variables. We seek to find weak solutions to this pde such that for a function $\phi$,

$$\int_{\Omega} \phi \left( \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} \right) dx = 0 \tag{24}$$

In DG method we assume the solution at element $k$ to take the form

$$\mathbf{u}_k = \sum_{j=1}^{p+1} \mathbf{U}_{k,j}(t)\, \phi_j(x) \tag{25}$$

Where $\phi_j(x)$ are the basis functions we use to represent the solution in element $k$ and $\mathbf{U}_{k,j}$ are the expansion coefficients for element $k$ for the $j$th order polynomial. One can see that there are a total of $(p+1)$ basis functions with $p$ being the order of accuracy. Rewriting the weak form of the pde for one element $k$,

$$\int_{\Omega_k} \phi_i \left( \phi_j \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} \right) dx = 0 \tag{26}$$

Which can be written as

$$\int_{\Omega_k} \phi_i \phi_j \frac{\partial \mathbf{U}}{\partial t}\, dx - \int_{\Omega_k} \frac{\partial \phi_i}{\partial x} \mathbf{F}\, dx + [\phi \mathbf{F}]_{x_{k-1/2}}^{x_{k+1/2}} = 0 \tag{27}$$

Or in matrix form,

$$\mathbf{M}_k \frac{\partial \mathbf{U}_k}{\partial t} + \boldsymbol{\mathcal{R}}_k = 0 \tag{28}$$

In Eq 28 we denote all terms by the subscript $k$ to emphasize that the following equation holds true for each element. We will omit the $k$ subscript from now on. We can further reduce the memory by writing this equation for a particular conservation law e.g. $s$ (note that the mass matrix remains the same for every conservation law),

$$\mathbf{M} \frac{\partial \mathbf{U}_s}{\partial t} + \boldsymbol{\mathcal{R}}_s = 0 \tag{29}$$

Where,

$$\mathbf{M} \equiv M_{ij} = \int_{\Omega} \phi_i \phi_j\, dx \tag{30}$$

And we can split the residual $\boldsymbol{\mathcal{R}}$ into $\boldsymbol{\mathcal{R}}_1$ and $\boldsymbol{\mathcal{R}}_2$ where

$$\boldsymbol{\mathcal{R}}_{1,s} = \mathcal{R}_{1,s,i} = \int_{\Omega_k} \frac{\partial \phi_i}{\partial x} F_s\, dx \tag{31}$$

$$\boldsymbol{\mathcal{R}}_{2,s} = \mathcal{R}_{2,s,i} = [\phi_i F_s]_{k+1/2} - [\phi_i F_s]_{k-1/2} \tag{32}$$

Quantities which require integration over the entire cell need to make use of quadrature rules to preserve accuracy. Flux values at the interfaces i.e. $x_{k+1/2}$ and $x_{k-1/2}$ are obtained through the solution of the Riemann problem as in the finite-volume method.

One thing to note however is that the integrals can be calculated using change of variables in reference space. We construct a reference element $E \in [-1, 1]$ and map every physical element $K$ to it. The mapping $f : E \to K \in \mathbb{R}^1$ can be written as (for e.g. for a location $x$ in element $k$)

$$x\left(\xi\right)|_k = x_{k-1/2} + \frac{\xi + 1}{2}\left(x_{k+1/2} - x_{k-1/2}\right) \tag{33}$$

Which gives the jacobian

$$\frac{dx}{d\xi} = \frac{\Delta x_k}{2} \tag{34}$$

Where $\Delta x_k$ is the size in physical space of element $k$. Using change of variables we can convert Eq 27 as

$$\int_{-1}^{1} \phi_i \phi_j \frac{\partial \mathbf{U}_j}{\partial t}\left(\frac{dx}{d\xi}\right) d\xi + \int_{-1}^{1} \frac{d\phi_i}{d\xi}\mathbf{F}\, d\xi + [\phi \mathbf{F}]_{\xi=-1}^{\xi=1} = 0 \tag{35}$$

Note that the jacobian is not needed for the calculation of the second term as it is negated by its inverse (atleast for 1D).