Transversality

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1 Basics

Definition. Let $f : M \to N$ be a smooth map from a manifold M with or without boundary to a manifold N. Let S be an embedded submanifold of N. We say that f is **transversal** to S if

$$df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$$

for all $p \in f^{-1}(S)$. ❖

Theorem 1.1. [GP10, pg. 28] *Let $f : M \to N$ be a smooth map from a smooth manifold M to a smooth manifold N. If f is transversal to an embedded submanifold S of N, then $f^{-1}(S)$ is an embedded submanifold of M. Furthermore, the codimension of $f^{-1}(S)$ in M is same as the codimension of S in N.*

Theorem 1.2. [GP10, pg. 60] *Let $f : X \to N$ be a smooth map from a smooth manifold X with boundary to a smooth manifold N. Let S be an embedded submanifold of N such that $f : X \to N$ and $\partial f : \partial X \to N$ are both transversal to S. Then $f^{-1}(S)$ is an embedded submanifold with boundary of X. Further, $\partial(f^{-1}(S)) = f^{-1}(S) \cap \partial X$, and the codimension of $f^{-1}(S)$ in X is same as the codimension of S in N.*

2 The Transversality Theorem

Theorem 2.1. ***The Transversality Theorem.*** *Let M, A and N be smooth manifolds, and $F : M \times A \to N$ be a smooth map. If F is transversal to an embedded submanifold S of N, then the map $F_a : M \to N$ is transversal to S for almost all $a \in A$, where F_a is the map $M \to N$ which sends $p \in M$ to $F(p, a)$.*

Proof. We know by Theorem 1.1 that $W := F^{-1}(S)$ is a submanifold of $M \times A$. Let $\pi_A : M \times A \to A$ and $\pi_M : M \times A \to M$ be the natural projections, and let $\rho : W \to A$ denote the restriction of π_A to W. We will show that whenever $a \in A$ is a regular value ρ, then F_a is transversal to S.

Claim. If a is a regular value of ρ, and $(p,a) \in W$, then $T_{(p,a)}(M \times A) = T_{(p,a)}W + T_{(p,a)}(M \times \{a\})$.

Proof. Let $v \in T_{(p,a)}(M \times A)$ be arbitrary. Let $v_A = d\pi_A|_{(p,a)}(v)$ and $v_M = d\pi_M|_{(p,a)}(v)$. Then $v = v_A + v_M$. The assumption that a is a regular value of ρ implies that there is $w \in T_{(p,a)}W$ such that $d\rho_{(p,a)}(w) = v_A$. But $d\rho_{(p,a)}(w) = d\pi_A|_{(p,a)}(w)$. Therefore $v = d\pi_A|_{(p,a)}(w) + v_M$. Split $w = w_A + w_M$, just like we had split v, and note that $d\pi_A|_{(p,a)}(w_M) = 0$ and $d\pi_A|_{(p,a)}(w_A) = w_A$. Thus we have

$$v = w_A + v_M = (w_A + w_M) + (v_M - w_M) = w + (v_M - w_M)$$

But $v_M - w_M \in T_{(p,a)}(M \times \{a\})$, and the claim is proved.

Now from the fact that F is transversal to S, for any $(p,a) \in F^{-1}(S)$, we have

$$dF_{(p,a)}(T_{(p,a)}(M \times A)) + T_sS = T_sN$$

where $s = F(p,a)$. This gives

$$dF_{(p,a)}(T_{(p,a)}W) + dF_{(p,a)}(T_{(p,a)}(M \times \{a\})) + T_sS = T_sN$$

where we have used the claim. Now since $F(W) \subseteq S$, we have $dF_{(p,a)}(T_{(p,a)}W) \subseteq T_sS$, which leads to

$$dF_{(p,a)}(T_{(p,a)}(M \times \{a\})) + T_sS = T_sN$$

But $dF_{(p,a)}(T_{(p,a)}(M \times \{a\})) = dF_a|_p(T_pM)$. Thus

$$dF_a|_p(T_pM) + T_sS = T_sN$$

This shows that F_a is transversal to S. Now by Sard's theorem, almost all $a \in A$ are regular values of ρ, and thus we have F_a is transversal to S for almost all $a \in A$, and we are done. ■

By a similar reasoning as in the above, we can prove

Theorem 2.2. ***The Transversality Theorem (Boundary Version).*** *Let X be a smooth manifold with boundary, and A and N be smooth manifolds, and $F : X \times A \to N$ be a smooth map. If both F and ∂F are transversal to an embedded submanifold S of N, then the maps $F_a : X \to N$ and ∂F_a are transversal to S for almost all $a \in A$*[1]

Corollary 2.3. ***General Position Lemma.*** *Let M and S be smooth submanifolds of $\mathbf{R}^n$. Then for almost all $a \in \mathbf{R}^n$, we have that the manifold $M_a := \{p + a : \ p \in M\}$ is transversal to S.*

Proof. Consider the map $F : M \times \mathbf{R}^n \to \mathbf{R}^n$ defined as $F(p,a) = p + a$. Then F is a submersion, and is hence transversal to S. Thus, by Theorem 2.1, we have $F_a : M \to \mathbf{R}^n$ is transversal to S for almost all $a \in \mathbf{R}^n$. This is same as saying that M_a is transversal to F_a for almost all $a \in \mathbf{R}^n$ and we are done. ■

3 Transversality and Homotopy

Theorem 3.1. ***ε-Neighborhood Theorem.*** *Let N be a compact manifold in $\mathbf{R}^n$ and let $\varepsilon > 0$. Let N^ε be the set of all the points in $\mathbf{R}^n$ which are at a distance less than ε from N. If ε is sufficiently small, then there is a submersion $\pi : N^\varepsilon \to N$ such that π restricts to the identity on N.*

Proof. Let $\rho : E \to N$ denote the normal bundle of N in $\mathbf{R}^n$. Define a map $f : E \to \mathbf{R}^n$ as $f(y,v) = y+v$ for all $(y,v) \in E$. It is clear that df is an isomorphism at each point $(y,0) \in E$. Further, f maps $N \times \{0\}$ diffeomorphically onto N. Thus, by the inverse function theorem, there is a neighborhood of $N \times \{0\}$ in E which maps, under f, diffeomorphically onto a neighborhood of N in $\mathbf{R}^n$. By compactness of N, we

[1]Here F_a is the map $X \to N$ which sends $p \in X$ to $F(p,a)$ and similarly for ∂F_a.

can choose this neighborhood to be of 'uniform thickness'.[2] So there exists $\varepsilon > 0$ small enough such that a neighborhood U of $N \times \{0\}$ in E maps diffeomorphically onto N^ε. Write h to denote $f|_U : U \to N^\varepsilon$. Since $\rho : U \to N \times \{0\}$ is a submersion, we have $\rho \circ h^{-1} : N^\varepsilon \to N \times \{0\}$ is also a submersion. Identifying $N \times \{0\}$ with N, we see that π is the required submersion. ■

Theorem 3.2. ***Transversality Homotopy Theorem.*** *Let $f : M \to N$ be a smooth map between smooth manifolds and S be an embedded submanifold of N. Then there is a smooth map $g : M \to N$ homotopic to f such that $g \pitchfork S$.*

Proof. We may assume that N is embedded in $\mathbf{R}^n$. Since N is compact, by the ε-Neighborhood Theorem there is $\varepsilon > 0$ small enough such that $\pi : N^\varepsilon \to N$ is a smooth submersion which restricts to the identity on N. Let B be the unit ball in $\mathbf{R}^n$, and define a map $F : X \times B \to N$ as $F(x, b) = \pi(f(x) + \varepsilon b)$.

Since F is the composite of two submersions, we see that F itself is a submersion, and is therefore transversal to S. Thus, by Transversality Theorem, there is a $b \in B$ such that $g : M \to N$ defined as $g(x) = F(x, b)$ is transversal to S. Finally, the map $H : X \times I \to N$ defined as $H(x, t) = F(x, tb)$ is a homotopy between f and g and we are done. ■

Lemma 3.3. *Let $f : X \to N$ be a smooth map, where X is a smooth manifold with boundary and N is a smooth manifold. Let S be a closed embedded submanifold of N. Then the set of points $x \in X$ where f is transversal to S is an open set of X.*

Proof. Let $x \in X$ be such that f is transversal to S at x. There are two cases. Assume first that $x \notin f^{-1}(S)$. Since S is closed in N, $f^{-1}(S)$ is closed in X, and thus there is a neighborhood of x in X which avoids $f^{-1}(S)$, and f is vacuously transversal to S on this neighborhood.

Now assume that $x \in f^{-1}(S)$. Consider a chart $(V\psi)$ on N centered at $f(x)$ such that ψ maps $S \cap V$ to a slice in $\mathbf{R}^n$, where n is the dimension of N. Compose ψ by an appropriate projection π which collapses this slice to a point. Thus $\pi \circ \psi \circ f$ is a submersion at x, and therefore it remains a submersion in a neighborhood U of x. It is easy to see that f is transversal to S on U, and we are done. ■

Theorem 3.4. ***Transversality Homotopy Theorem (Boundary Version).*** *Let $f : X \to N$ be a smooth map from a smooth manifold with boundary X to a smooth manifold N. Assume N is compact. Let S be a closed embedded submanifold of N. If $\partial f : \partial X \to N$ is transversal to S, then there is a smooth map $g : X \to N$ homotopic to f such that $g \pitchfork S$ and $\partial g = \partial f$.*

Proof. By Lemma 3.3, we see that there is a neighborhood U of ∂X in X such that $f|_U$ is transversal to S. Let $\gamma : X \to [0, 1]$ be a smooth map which is identically 1 outside U and is identically 0 in a neighborhood of ∂X. Define $\tau : X \to [0, 1]$ by setting $\tau = \gamma^2$.

We may assume that N is embedded in $\mathbf{R}^n$. Since N is compact, there is $\varepsilon > 0$ small enough such that $\pi : N^\varepsilon \to N$ is a smooth submersion. Let B be the unit ball in $\mathbf{R}^n$, and define a map $F : X \times B \to N$ as $F(x, b) = \pi(f(x) + \varepsilon b)$. Further define $G : X \times B \to N$ as $G(x, b) = F(x, \tau(x)b)$.

We show that G is transversal to S. Let $(x, b) \in G^{-1}(S)$. If $\tau(x) \neq 0$, then G is a submersion at (x, b), because the map $B \to N$ defined as $b \mapsto G(x, \tau(x)b)$ is a submersion, being the composite of the submersions $b \mapsto \tau(x)b : B \to B$ and $b \mapsto F(x, b) : B \to Y$. Thus G is transversal to S at (x, b). So we may assume $\tau(x) = 0$. Thus $d\tau_x = 0$.[3] Define $\mu : X \times B \to X \times B$ as $\mu(x, b) = (x, \tau(x)b)$. Then we have $d\mu_{(x,b)}(u, v) = (v, \tau(x)w + d\tau_x(v)b)$. By the chain rule applied to $F \circ \mu$, we have $dG_{(x,b)}(v, w) = dF_{(x,0)}(v, 0)$, which is same as $df_x(v)$. But since $\tau(x) = 0$, we have $x \in U$, and since $f|_U$ is transversal to S, we have conclude that G is transversal to S at (x, b). This completes the proof that G is transversal to S.

Therefore, in particular, G restricted to $(X \times S) \setminus \partial(X \times S)$ is transversal to S. Note that $(X \times S) \setminus \partial(X \times S) = (X \setminus \partial X) \times S$. Thus, by Theorem 2.1, there is a $b \in B$ such that $g : X \setminus \partial X \to N$ defined as $g(x) = G(x, b)$ is transversal to S. Consider the extension $\tilde{g} : X \to N$ of g defined as $\tilde{g}(x) = G(x, b)$ for

[2]This makes sense because the normal bundle is naturally equipped with a Riemannian metric.
[3]This is the reason to consider the square of γ.

all $x \in X$. Since τ vanishes identically at ∂X, we see that $\partial \tilde{g} = \partial f$. Thus $\tilde{g}$ is transversal to S. The map $H : X \times I \to N$ defined as $H(x,t) = G(x, tb)$ is a homotopy between f and g and we are done. ■

4 Intersection Number Mod 2

Let M and N be smooth manifolds and S be a submanifold of N. We say that M and S are of **complementary dimension** if $\dim M + \dim S = \dim N$. Now assume that S is closed in N, M is compact, and let $f : M \to N$ be a smooth map which is transversal to S. Then $f^{-1}(S)$ is a 0-dimensional closed submanifold of M, and is hence finite. We write $I_2(f,S)$ to denote $|f^{-1}(S)|$ mod 2. We call $I_2(f,S)$ the **mod 2 intersection number** of f with S.

Theorem 4.1. *Let $f_0, f_1 : M \to N$ be smooth maps between smooth manifolds, both transversal to a closed submanifold S of N. Assume M is compact. If f_0 and f_1 are homotopic then $I_2(f_0,S) = I_2(f_1,S)$.*

Proof. Let $F : M \times I \to N$ be a homotopy between f_0 and f_1. Note that ∂F is transversal to S. By the Transversality Homotopy Theorem (boundary version), we have a map $G : M \times I \to N$ homotopic to F such that G is transversal to S and $\partial G = \partial F$. Thus, by Theorem 1.2, $G^{-1}(S)$ is a compact 1-manifold K of $M \times I$ with boundary, such that

$$\partial K = G^{-1}(S) \cap \partial(M \times I) = (f_0^{-1}(S) \times \{0\}) \cup (f_1^{-1}(S) \times \{1\})$$

By classification of 1-manifolds, if $G^{-1}(S)$ has k-components, then the cardinality of $\partial G^{-1}(S)$ is $2k$, which is even. Therefore $|f_0^{-1}(S)| \equiv |f_1^{-1}(S)| \pmod 2$. ■

The above theorem allows us to define the mod 2 intersection number of an arbitrary smooth map $f : M \to N$ with S, where M is compact and S is a closed submanifold of N. For by the Transversality Homotopy Theorem, there is a smooth map $g : M \to N$ homotopic to f which is transversal to S. We define $I_2(f,S) := I_2(g,S)$. The above theorem guarantees that this is well defined.

Corollary 4.2. *Let $f_0, f_1 : M \to N$ be any two homotopic maps from a compact smooth manifold M to a smooth manifold N. Let S be a closed submanifold of N. Then $I_2(f_0,S) = I_2(f_1,S)$.*

Theorem 4.3. ***Boundary Theorem.*** *Let $f : M \to N$ be a smooth map between smooth manifolds and assume that M is the boundary of some compact manifold X. If f can be extended smoothly to all of X then $I_2(f,S) = 0$ for any closed submanifold S in N of dimension complementary to M.*

Proof. Let $F : X \to N$ be an extension of f. Let S be a closed embedded submanifold of N of complementary dimension to M. By the Transversality Theorem (Boundary Version), there is a map $G : X \to N$ homotopic to F such that both G and $g := \partial G$ are transversal to S. Thus we have f is homotopic to g, so we need to show that $I_2(g,S) = 0$. But by Theorem 1.2, we have $G^{-1}(S)$ is a 1-dimensional submanifold with boundary in X whose boundary is same as $G^{-1}(S) \cap M$, which is same as $g^{-1}(S)$. But the boundary of $G^{-1}(S)$ has an even number of points, and thus $I_2(g,S) = 0$, and we are done. ■

5 Degree Mod 2

Theorem 5.1. *Let M and N be smooth manifolds where M is compact and N is connected. Let $f : M \to N$ be a smooth map. If $\dim M = \dim N$, then $I_2(f, \{q\})$ is same for all $q \in N$. This common value is termed as the **mod 2 degree** of f, and we denote it by $\deg_2(f)$.*

Proof. Assume $\dim M = \dim N > 1$. Let q and q' be two distinct points in N, and let S be the image of an embedding of a circle in N which passes through both q and q'. Let $g : M \to N$ be a map homotopic to f which is transversal to S, and well as both q and q'. Since M and N are of the same dimension, by Theorem 1.1, $g^{-1}(S)$ is an embedded submanifold of M of dimension 1. Thus $g^{-1}(S)$ is a disjoint union of finitely many submanifolds of M, each diffeomorphic to S^1. This reduces the problem to the case where both M and N are S^1, in which case the proof is easy. An alternate proof can be found in [GP10, pg. 80]. ■

Theorem 5.2. *Homotopic maps have the same mod two degree.*

Proof. Immediate from Theorem 4.1. ■

Theorem 5.3. *Let M and N be smooth manifolds of the same dimension, where M is compact and N is connected. Assume that M is the boundary of a manifold X. Let $f : M \to N$ be a smooth map. If f is smoothly extendible to all of X, then $\deg_2(f) = 0$.*

Proof. Immediate from the Boundary Theorem. ■

Corollary 5.4. *S^1 is not simply-connected.*

Theorem 5.5. *No compact manifold other than the one point space is contractible.*

Proof. Let M be a compact connected manifold of dimension at least 1. We want to show that M is not contractible. Suppose not. Fix a point $p \in M$. The contractibility of M implied that the identity map $\mathrm{Id} : M \to M$ is homotopic to the constant map $c : M \to M$ whose image if $\{p\}$. Let $q \in M$ be different from p. Clearly, both the identity map and the constant map c are transversal to $\{q\}$. Thus we have $\deg_2(\mathrm{Id}) = \deg_2(c)$. But $\deg_2(\mathrm{Id}) = I_2(\mathrm{Id}, \{q\}) = 1$, and $\deg_2(c) = I_2(c, \{q\}) = 0$. Thus we arrive at a contradiction, and therefore we must have the manifold M is not contractible. ■

Note. For any questions or comments please write to me at `khetan@math.tifr.res.in`

References

[GP10] Victor Guillemin and Alan Pollack. *Differential Topology.* AMS Chelsea Publishing, Providance, RI, 2010.