LDA (Fisher Linear Discriminant) ⇛

Example ⇛

Consider a 2-D dataset.

$C_1 \Rightarrow X_1 = (x_1, x_2) \Rightarrow \{(4,2), (2,4), (2,3), (3,6), (4,4)\}$

$C_2 \Rightarrow X_2 = (x_1, x_2) \Rightarrow \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

Step 1 ⇛ Compute within-class Scatter Matrix (S_w)

$$\boxed{S_w = S_1 + S_2}$$

S_1 is the covariance matrix for the class C_1 &

S_2 is the covariance matrix for the class C_2.

∴ Covariance matrix is calculated as,

$$S = \sum_{x \in C} (x - \mu)(x - \mu)^T$$, Here, μ is mean of class elements

$$\therefore S_1 = \sum_{x \in C_1} (x - \mu_1)(x - \mu_2)^T$$

$$\& \quad S_2 = \sum_{x \in C_2} (x - \mu_2)(x - \mu_2)^T$$

∴ ① ~~Give~~ Mean of each class is as follows,

$$\mu_1 = \frac{1}{N_1} \sum_{x \in C_1} x = \frac{1}{5}\left[\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix} + \begin{pmatrix}4\\4\end{pmatrix}\right]$$

$$\mu_1 = \frac{1}{5}\begin{bmatrix}15\\19\end{bmatrix} = \begin{bmatrix}3\\3.8\end{bmatrix}\begin{array}{l} x_1 \Rightarrow \mu_{1x_1} \\ x_2 \Rightarrow \mu_{1x_2} \end{array}$$

Similary,

$$\mu_2 = \frac{1}{N_2} \sum_{x \in C_2} x = \frac{1}{5} \left[\begin{pmatrix} 9 \\ 10 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \end{pmatrix} + \begin{pmatrix} 9 \\ 5 \end{pmatrix} + \begin{pmatrix} 8 \\ 7 \end{pmatrix} + \begin{pmatrix} 10 \\ 8 \end{pmatrix} \right]$$

$$\mu_2 = \frac{1}{5} \begin{bmatrix} 42 \\ 38 \end{bmatrix} = \begin{bmatrix} 8.4 \\ 7.6 \end{bmatrix} \begin{matrix} x_1 \Rightarrow \mu_{2x_1} \\ x_2 \Rightarrow \mu_{2x_2} \end{matrix}$$

3

2) The format of covariance matrix for class C_1 is,

$$S_1 = \begin{bmatrix} \overset{\rightarrow var(x_1)}{Cov(x_1, x_1)} & Cov(x_1, x_2) \\ Cov(x_2, x_1) & \underset{\hookrightarrow va(x_2)}{Cov(x_2, x_2)} \end{bmatrix}$$

$$Cov(x_1, x_1) = \frac{1}{N-1} \sum_{x_1 \in C_1} (x_1 - \mu_{x_1})^2$$

$$= \frac{1}{4} \left[(4-3)^2 + (2-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 \right]$$

$$= \frac{1}{4} \left[1+1+1+0+1 \right] = \frac{4}{4} = \underline{\underline{1}}$$

$$\boxed{Cov(x_1, x_1) = 1}$$

$$Cov(x_1, x_2) = \frac{1}{N-1} \sum_{(x_1 \in C_1, x_2)} (x_1 - \mu_{x_1})(x_2 - \mu_{x_2})$$

$$= \frac{1}{4} \Big[(4-3)(2-3.8) + (2-3)(4-3.8) + (2-3)(3-3.8) + (3-3)(6-3.8) + (4-3)(4-3.8) \Big]$$

$$= \frac{1}{4} \Big[(1)(-1.8) + (-1)(0.2) + (-1)(-0.8) + (0)(2.2) + (1)(0.2) \Big]$$

$$= \frac{1}{4} \left[-1.8 + (-0.2) + (0.8) + (0) + (0.2) \right]$$

$$= \frac{1}{4} \left[-1 \right]$$

$$\therefore \boxed{Cov(x_1, x_2) = -0.25}$$

$$\text{Cov}(x_2, x_1) = \frac{1}{N-1} \sum_{(x_1, x_2) \in C_1} (x_2 - \mu_{x_2})(x_1 - \mu_{x_1})$$

$$= \frac{1}{4}\Big[(2-3.8)(4-3) + (4-3.8)(2-3) + (2-3)(3-3.8) + (3-3)(6-3.8) + (4-3)(4-3.8)\Big]$$

$$= \frac{1}{4}[-1]$$

$$\boxed{\text{Cov}(x_2, x_1) = -0.25}$$

$$\text{Cov}(x_2, x_2) = \frac{1}{N-1} \sum_{x_2 \in C} (x_2 - \mu_{x_2})^2$$

$$= \frac{1}{4}\Big[(2-3.8)^2 + (4-3.8)^2 + (3-3.8)^2 + (6-3.8)^2 + (4-3.8)^2\Big]$$

$$= \frac{1}{4}\Big[(-1.8)^2 + (0.2)^2 + (-0.8)^2 + (2.2)^2 + (+0.2)^2\Big]$$

$$= \frac{1}{4}\Big[3.24 + 0.04 + 0.64 + 4.84 + 0.04\Big]$$

$$= \frac{1}{4}[8.8] = 2.2$$

$$\boxed{\text{Cov}(x_2, x_2) = 2.2}$$

$$\therefore \; S_1 = \begin{bmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{bmatrix}$$

Similary, using μ_2 & class C_2 sample points

$$S_2 = \begin{bmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{bmatrix}$$

$\therefore$ Within-class Scatter matrix is,

$$S_W = S_1 + S_2 = \begin{bmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{bmatrix} + \begin{bmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{bmatrix}$$

$$\boxed{S_W = \begin{bmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{bmatrix}}$$

Step 2: Between class Scatter Matrix is,

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right] \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]^T$$

$$S_B = \begin{bmatrix} -5.4 \\ -3.8 \end{bmatrix} \begin{bmatrix} -5.4 & -3.8 \end{bmatrix}$$

$$\boxed{S_B = \begin{bmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{bmatrix}}$$

Step 3: LDA projection vector.

We find projection vector using eigen ~~values~~ vectors having largest eigen value.

$$S_W^{-1} S_B \, v = \lambda \, v \quad \sim (\mathrm{I})$$

Here, λ is eigen values.

'v' is eigen vector.

For finding eigen values,

$$\left| S_W^{-1} S_B - \lambda I \right| = 0 \quad \sim (\mathrm{II}) \qquad \left(\because \left| A - \lambda I \right| = 0\right)$$

First we require S_W^{-1},

Inverse of 2×2 matrix is calculated as,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \quad S_W^{-1} = \begin{bmatrix} \overset{a}{3.3} & \overset{b}{-0.3} \\ \overset{c}{-0.3} & \overset{d}{5.5} \end{bmatrix}^{-1} = \frac{1}{(3.3)(5.5) - (-0.3)(-0.3)} \begin{bmatrix} 5.5 & -0.3 \\ -0.3 & 3.3 \end{bmatrix}$$

$$= \frac{1}{18.15 - 0.09} \begin{bmatrix} 5.5 & -0.3 \\ -0.3 & 3.3 \end{bmatrix}$$

$$= \frac{1}{18.06} \begin{bmatrix} 5.5 & -0.3 \\ -0.3 & 3.3 \end{bmatrix}$$

$$\therefore \quad \boxed{S_W^{-1} = \begin{bmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{bmatrix}}$$

As per equation (II),

$$| S_W^{-1} S_B - \lambda I | = 0$$

$$\left| \begin{bmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{bmatrix} \begin{bmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 9.2213 & 6.489 \\ 4.2339 & 2.9794 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 9.2213-\lambda & 6.489 \\ 4.2339 & 2.9794-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow (9.2213-\lambda)(2.9794-\lambda) - 6.489 \times 4.2339 = 0$$

$$\Rightarrow \cancel{\lambda^2 - 12.2007\lambda = 0}$$

$$\cancel{27.4738} - 9.2213\lambda - 2.9794\lambda$$

$$+\lambda^2 - \cancel{27.473} = 0$$

$$\lambda^2 - 12.2007\lambda = 0$$

$$\lambda(\lambda - 12.2007) = 0$$

$$\boxed{\lambda_1 = 0, \ \lambda_2 = 12.2007}$$

For 2x2 determinant,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

if equation is in form of,

$$ax^2 + bx + c = 0$$

then calculate 'x' using,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

~~Step~~

We will eigen vector using $\lambda_2 = 12.2007$

For eigen vectors, we will consider,

$$(S_W^{-1} S_B - \lambda_2 I) V = 0$$

$$\left(\begin{bmatrix} 9.2213 & 6.489 \\ 4.2339 & 2.9794 \end{bmatrix} - \overset{\lambda_2}{\cancel{12.2007}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 9.2213 - \lambda_2 & 6.489 \\ 4.2339 & 2.9794 - \lambda_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$\therefore (9.2213 - \lambda_2) V_1 + 6.489 V_2 = 0 \sim (\text{III})$$

$$(4.2339 V_1) + (2.9794 - \lambda) V_2 = 0 \sim (\text{IV})$$

Consider, eq^n (III) for calculating eigen vectors,

$$\therefore (9.2213 - \lambda_2) V_1 \cancel{\circ} + 6.489 V_2 = 0$$

$$(9.2213 - \lambda_2) V_1 = -6.489 V_2$$

$$\frac{V_1}{-6.489} = \frac{V_2}{(9.2213 - \lambda_2)} = t$$

when, $t = 1$ $\quad \frac{V_1}{-6.489} = 1 \Rightarrow V_1 = -6.489$

$$V_2 = 9.2213 - \lambda_2 = 9.2213 - 12.007 = -2.9794$$

We always used in normalised eigen vectors,

$$V_{(norm)} = \begin{bmatrix} \dfrac{v_1}{\sqrt{v_1^2 + v_2^2}} \\ \dfrac{v_2}{\sqrt{v_1^2 + v_2^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-6.489}{\sqrt{(-6.489)^2 + (-2.9794)^2}} \\ \dfrac{-2.9794}{\sqrt{(-6.489)^2 + (-2.9794)^2}} \end{bmatrix}$$

$$= \begin{bmatrix} -0.9088 \\ -0.4173 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.9088 \\ 0.4173 \end{bmatrix}$$

Another easy way,

$$V = S_w^{-1}(\mu_1 - \mu_2) = \begin{bmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{bmatrix}^{-1} \begin{bmatrix} -5.4 \\ -3.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{bmatrix} \begin{bmatrix} -5.4 \\ -3.8 \end{bmatrix}$$

$$\boxed{V_1 = \begin{array}{cc} \begin{bmatrix} 0.9088 \\ 0.4173 \end{bmatrix} & \begin{array}{c} v_1 \\ v_2 \end{array} \end{array}}$$

Step 4: Now we will calculate the score for reduced dimension. $\text{Score} = x_1 \cdot v_1 + x_2 \cdot v_2$

Class	x_1	x_2	Score
C_1	4	2	4.46
C_1	2	4	3.48
C_1	2	3	3.06
C_1	3	6	5.23
C_1	4	4	5.30
C_2	9	10	8.17
C_2	6	8	8.79
C_2	9	5	10.26
C_2	8	7	10.19
C_2	10	8	12.42