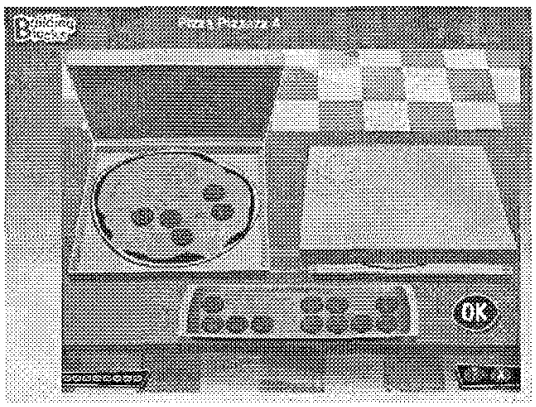
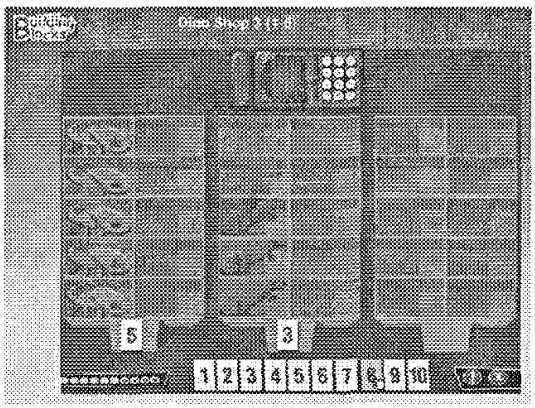
c. The main addition and subtraction learning trajectory

(c-1) "Pizza Pazzazz 4": Students add and subtract numbers up to totals of 3, (with objects shown, but then hidden) matching target amounts.



(c-2) "Dinosaur Shop 3": Customers at the shop asks students to combine their two orders and add the contents of two boxes of toy dinosaurs (number frames) and click a target numeral that represents the sum.



(c-3) "Bright Idea": Students are given a numeral and a frame with dots. They count on from this numeral to identify the total amount, and then move forward a corresponding number of spaces on a game board.

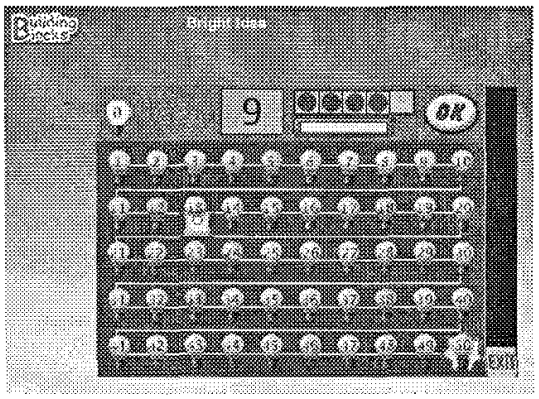


Fig. 2 (Continued)

from *Building Blocks* (Clements and Sarama 2007a) moves along a learning trajectory from the smallest numbers (1–2) to slightly larger sets (3–5) and also from matching exact dot arrangements to different dot arrangements to matching dots to numerals, as shown in Fig. 2b. Children can ask for a "Peek" to see the set again before giving their response (but only once more).

From the addition and subtraction learning trajectory, children learn to interpret additive situations mathematically, such as interpreting a real-world problem as a

"part-part-whole" situation. Examining a small part of the developmental progression, at the earliest level of thinking (see *Nonverbal* +/− in Fig. 1), children use initial bootstrapping abilities (inchoate premathematical and general cognitive competencies and predispositions at birth or soon thereafter) and intuitive competencies based on mental images of very small sets. Children later learn to use counting to determine the number in each part and in the whole, originally needing to directly model the situation, using one object for each element in the problem and counting each part and the whole starting from "one" each time (see the subsequent two levels, *Small Number* +/− and *Find Result* +/−). Abby had quickly worked through the *Nonverbal* +/− level activity, "Pizza Pazzazz 4" (Fig. 2-c-1) up to the *Find Result* +/− activity "Dinosaur Shop 3." Generally, she counted all objects. That is, she counted out 5 green toy dinosaurs, then 3 red dinosaurs, then counted them all, starting at 1, clicking on the "8," thereby showing she understood the task.

Next, through the constructive synthesis of the levels of thinking from these three learning trajectories, counting, subitizing, and addition and subtraction, children learn to solve problems such as, "You have three blue blocks and seven red blocks. How many blocks do you have in all?" by modeling the problem situation and counting on Carpenter and Moser (1984). They understand that these numbers are two parts and that they need to find the whole. They also understand that the order of numbers does not matter in addition. They know, in practice, that the sum is the number that results by starting at the first number and counting on a number of iterations equal to the second number. They can use counting to solve this, starting by saying "seven..." because they understand that word can stand for the counting acts from 1 to 7 (because 7 includes 6, and 6 includes 5...). The elongated pronunciation may be substituting for counting the initial set one-by-one. It is *as if* they counted a set of 7 items. Finally, they know *how many more* to count because they use the subitized rhythm of three, so they then say, "*eight, nine, ten*!"

To develop this level of thinking, Abby engaged in many activities. Illustrated in Fig. 2-c-3, "Bright Idea" is a game in which, for the first time in a series of similar board games, not all quantities were represented by sets of dots. Instead, one of the addends is represented by a numeral (and a large single-digit at that), which research (Siegler and Jenkins 1989) shows encouraged Abby to count on: "9, 10, 11, 12, 13!"

Abby learned more quickly than most, but the *Building Blocks Software*'s (Clements and Sarama 2007a) automatic movement along the learning trajectory supported her learning and illustrates the great potential children have to learn mathematics. By four years of age, Abby was given five train engines. She walked in one day with three of them. Her father said, "Where's the other ones?" "I lost them," she admitted. "How many are missing?" he asked. "I have 1, 2, 3. So [pointing in the air] four, five...two are missing, four and five. [pause] No! I want these to be [pointing at the three engines] one, three, and five. So, two and four are missing. Still two missing, but they're numbers two and four." Abby thought about counting and numbers—at least small numbers—abstractly. She could assign 1, 2, and 3 to the three engines, or 1, 3, and 5! Moreover, she could count the numbers. That is, she applied counting... to counting numbers.