Rethinking Early Mathematics: What *Is* Research-Based Curriculum for Young Children?

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How many times have you heard, "Our mathematics curriculum is based on research"? Have all these curricula (including early childhood education programs, educational software, teaching strategies, etc.) been similar? Have they all been effective? Many who publish a curriculum, or write or speak about a teaching approach, claim their approach is based on research, even though they vary widely (Battista and Clements 2000; Clements 2007; Senk and Thompson 2003). Such claims are often vacuous, including general statements about what "the research says." Unfortunately, such overuse of the phrase "research-based" undermines attempts to create a shared research foundation for the development of, and informed choices about, classroom curricula (National Research Council 2002, 2004).

We believe that researchers and practitioners can work together to ameliorate this situation and develop, evaluate, and use valid research-based approaches. To support such collaborative activity, we have developed two major conceptual tools. The first is a set of *learning trajectories* that describe how children learn major topics in mathematics and how teachers can support that learning. Based upon studies in fields ranging from cognitive and developmental psychology to early childhood and mathematics education, these guide the creation of standards, curricula, and teaching strategies. They also are at the core of the second conceptual tool, a framework for developing curricula and teaching strategies. This framework describes criteria and procedures for creating scientifically-based curricula.

Learning Trajectories

Research-based learning trajectories are tools educators can use to improve mathematics learning and teaching (Simon 1995). Our learning trajectories are based on

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evidence that children generally follow natural paths—sequences of increasingly sophisticated levels of thinking—as they learn mathematics topics (Clements and Sarama 2009; Sarama and Clements 2009b). These sequences can be described as *developmental progressions*. When teachers understand these developmental progressions, and use them in selecting and sequencing instructional activities, they can build more effective mathematics learning environments.

A complete learning trajectory has three parts: a goal, a developmental progression, and instructional activities. To attain a certain mathematical competence within a given domain (the goal), children typically learn each successive level of thinking (the developmental progression), aided by activities (instructional tasks) designed to build the mental actions-on-objects that enable thinking at each higher level (Clements and Sarama 2004b).

Early Addition and Subtraction: An Example

The main *goal* of the counting-based addition and subtraction learning trajectory is that children learn to solve different types of arithmetic problems (Carpenter et al. 1988) and develop accuracy and eventually fluency with arithmetic combinations. The second component of the learning trajectory is the *developmental progression*, which describes a typical counting-based trajectory children follow in developing understanding and skill in arithmetic. The left column in Fig. 1 describes several levels of thinking in the learning trajectory and provides examples of children's behavior for each level. The right column provides examples of *instructional tasks*, matched to each of the levels of thinking in the developmental progression. These tasks are designed to help children learn the ideas and skills needed to achieve that level of thinking. However, instructional tasks are always examples—many tasks and approaches to teaching are possible. Therefore, curriculum developers and teachers should translate developmental progressions and instructional tasks for specific cultural, school, and individual contexts. That is, to re-think mathematics education, we must also re-consider the cultural and sociopolitical contexts children experience (Wager and Carpenter 2012, discuss these issues at length). Thus, there is no single or "ideal" developmental progression, and thus learning trajectory. The following presents just one example.

Summarizing, learning trajectories describe the goals of learning, the developmental progression through which children pass, and the learning activities in which children might profitably engage. They are based on research first because the *sources* of the developmental progressions are extensive research reviews and empirical work (Sarama and Clements in press). They are also research-based because whenever possible, the instructional tasks are guided by this same empirical work and by classroom-based research and the wisdom of expert teacher practice. Although it is beyond the scope of this chapter to present this body of research (see Sarama and Clements 2009b), along with the complex, cognitive actions-on-objects that underlie all the example behaviors in Fig. 1, we will provide one illustration of

***Goal:* Children solve different types of arithmetic problems and develop accurate and eventually fluent competencies with arithmetic combinations**

Developmental Progression (including Example Behaviors for each Level of Thinking)	*Instructional Tasks*
Nonverbal +/− Adds and subtracts very small collections nonverbally. Shown 1 object then 1 object going under a cover, identifies or makes a set of 2 objects to "match."	"Blocks in the Box": Children play a game in which, for example, 2 blocks then 1 block go into a box, and try to "guess" how many are in the box. The cover is taken off and the blocks counted to check.
Small Number +/− Finds sums for joining problems up to $3+2$ by counting-all with objects. Asked, "You have 2 balls and get 1 more. How many in all?" counts out 2, then counts out 1 more, then counts the total.	"Word Problems (Join result unknown or separate, result unknown (take-away) problems, numbers < 5)": "You have 2 balls and get 1 more. How many in all?" "Finger Word Problems": Tell children to solve simple addition problems with their fingers. Use very small numbers. Children should place their hands in their laps between each problem. To solve the problems above, guide children in showing 3 fingers on one hand and 2 fingers on the other and reiterate: How many is that altogether? Ask children how they got their answer and repeat with other problems.

Fig. 1 Samples from the Learning Trajectory for Counting-based Arithmetic (addition and subtraction, adapted from Clements and Sarama 2009, 2012; Sarama and Clements 2009b)*

both the cognitive actions-on-objects that underlie the levels of thinking and how different trajectories grow not in isolation, but interactively.

Consider learning a critical competence for early arithmetic—counting on, used especially at the *Counting Strategies* level in Fig. 1. Children need to develop competencies from three learning trajectories to learn to count on meaningfully. Two provide support: (1) counting (Fuson 1988) and (2) subitizing, the quick recognition of the number in small sets without counting (e.g., Antell and Keating 1983; Kobayashi et al. 2004). (These two learning trajectories are described in Clements and Sarama 2009; Sarama and Clements 2009b.) The third, of course, is the arithmetic learning trajectory from Fig. 1.

Find Result +/−
Finds sums for joining (you had 3 apples and get 3 more, how many do you have in all?) and part-part-whole (there are 6 girls and 5 boys on the playground, how many children were there in all?) problems *by direct modeling, counting-all, with objects.*
Solves take-away problems by separating with objects.

Asked, "You have 2 red balls and 3 blue balls. How many in all?" counts out 2 red, then counts out 3 blue, then counts all the balls.
Asked, "You have 5 balls and give 2 to Tom. How many do you have left?" counts out 5 balls, then takes away 2, and then counts remaining 3.

"Word Problems": Children solving all the above problems types using manipulatives or their fingers to represent objects.
For Separate, result unknown (take-away),

"You have 5 balls and give 2 to Tom. How many do you have left?" Children might counts out 5 balls, then takes away 2, and then counts remaining 3.

For Part-part-whole, whole unknown problems, they might solve

"You have 2 red balls and 3 blue balls. How many in all?"

"Places Scenes (Addition—Part-part-whole, whole unknown problems)": Children play with a toy on a background scene and combine groups. For example, they might place 4 tyrannosaurus rexes and 5 apatosauruses on the paper and then count all 9 to see how many dinosaurs they have in all.

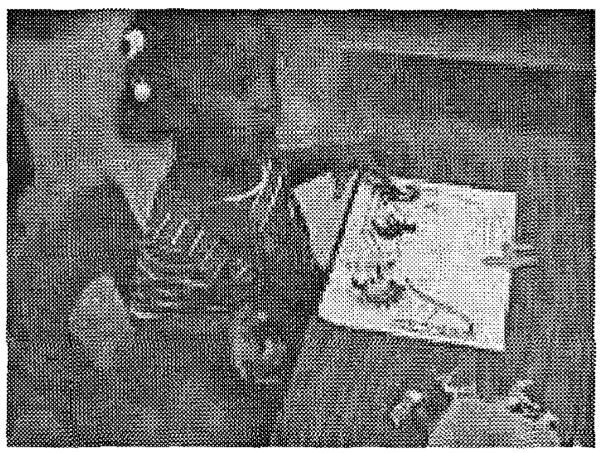


Fig. 1 (Continued)

From the counting learning trajectory, children learn to count forward starting with any number. Then, they learn to understand explicitly and apply the idea that each number in the counting sequence includes *the number before, hierarchically.* That is, 5 includes 4, which includes 3, and so forth. For example, 3-year-old Abby could count up past 20, but always had to start at one. Asked to start at four, she hesitated, then said, "One, two three four, five...." Her teacher played informal games with her such as placing a couple of blocks in a box, asking, "How many are in the box now?", adding one, and repeating the question. She also had Abby work on the computer activities in Fig. 2a. In *Build Stairs 2*, children slowly count, clicking on the next number. In the next level, they have to see that 4 comes after 3.

Counting Strategies +/− Finds sums for joining (you had 8 apples and get 3 more...) and part-part-whole (6 girls and 5 boys...) problems with finger patterns and/or by counting on.

Counting-on. "How much is 4 and 3 more?" "Four... five, six, seven [uses rhythmic or finger pattern to keep track]. Seven!"

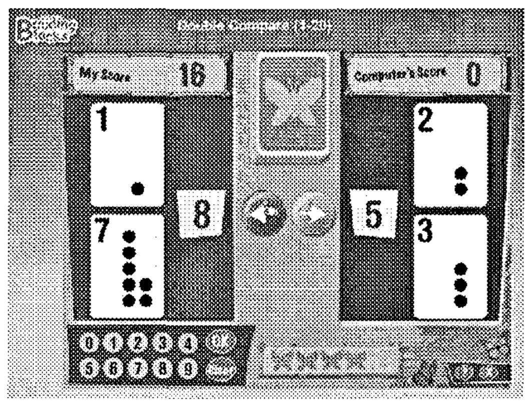
Counting-up-to May solve missing addend (3 + _ = 7) or compare problems by counting up; e.g., counts "4, 5, 6, 7" while putting up fingers; and then counts or recognizes the 4 fingers raised.

Asked, "You have 6 balls. How many more would you need to have 8?" says, "Six, seven [puts up first finger], eight [puts up second finger]. Two!"

"How Many Now?": Have the children count objects as you place them in a box. Ask, "How many are in the box now?" Add one, repeating the question, then check the children's responses by counting all the objects. Repeat, checking occasionally. When children are ready, sometimes add two, and eventually more, objects.

Variations: Place coins in a coffee can. Declare that a given number of objects are in the can. Then have the children close their eyes and count on by listening as additional objects are dropped in.

"Double Compare." Students compare sums of cards to determine which sum is greater. Encourage children to use more sophisticated strategies, such as counting on.



"Easy as Pie": Students add two numerals to find a total number (sums of one through ten), and then move forward a corresponding number of spaces on a game board. The game encourages children to count on from the larger number (e.g., to add 3 + 4, they would count "four ... 5, 6, 7!")

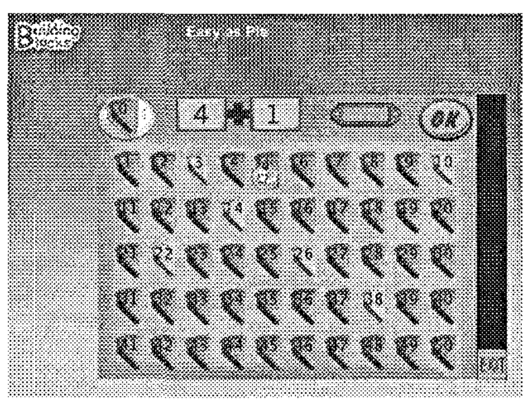


Fig. 1 (Continued)

Deriver +/− Uses flexible strategies and derived combinations (e.g., "7 + 7 is 14, so 7 + 8 is 15) to solve all types of problems. Includes Break-Apart-to-Make-Ten (BAMT). Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. Asked, "What's 7 plus 8?" thinks: 7 + 8 —> 7 + [7 + 1] —> [7+7]+1 = 14+1 = 15. Or, using BAMT, thinks, 8 + 2 = 10, so separate 7 into 2 and 5, add 2 and 8 to make 10, then add 5 more, 15.	(The BAMT strategy is taught here.) "Tic-Tac-Total": Draw a tic-tac-toe board and write the numbers 1 to 10. Players take turns crossing out one of the numbers and writing it in the board. Whoever makes 15 first wins. "21": Play cards, where Ace is worth either 1 or 11 and 2 to 10 are worth their values. Dealer gives everyone 2 cards, including herself. On each round, each player, if sum is less than 21, can request another card, or "hold." If any new card makes the sum more than 21, the player is out. Continue until everyone "holds." The player whose sum is closest to 21 wins.
Problem Solver +/− Solves all types of problems, with flexible strategies and known combinations.	"Word Problems (all types of problem structures for single-digit problems)"

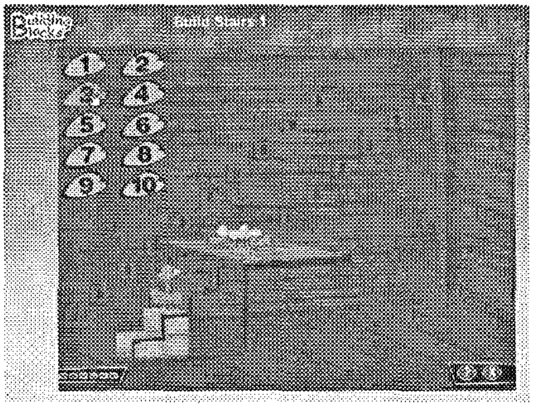
* Note that these counting-based strategies are only one of the paths to arithmetic (see Chap. 6 in each of two companion books, Clements and Sarama 2009; Sarama and Clements 2009b)

Fig. 1 (Continued)

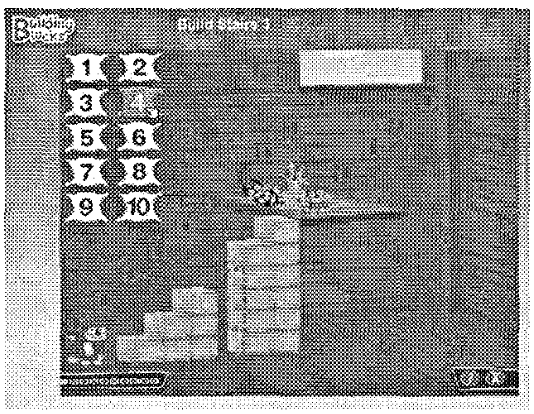
Both activities benefit from computer technology. Young children find computer interactions motivating, especially in narrative contexts in which they help animals (Sarama and Clements 2002). The computer manipulatives are just as easy for them to use, and often provide better supports for learning (Sarama and Clements 2009a). Finally, children receive immediate, patient feedback on their actions (Clements and Sarama 2002). After these experiences, Abby could start at any number up to 10 and count forward or backward.

From the subitizing learning trajectory, children learn to quickly recognize the number in visual sets, such as a triangle pattern of blocks or a spatial pattern of three fingers. Importantly, they also learn *rhythmic patterns*. For example, they learn the rhythm of three ("Doo—Day—Doo" or hearing three taps, etc.). In the same time period as she learned the counting skills, Abby engaged in a series of activities that developed her ability to subitize small numbers. Figure 2b illustrates the type of activity that helped Abby become fluent in this ability. The activity "Snapshots"

a. Counting from any number
"Build Stairs 2": Students identify the appropriate stacks of unit cubes to fill in a series of staircase steps.

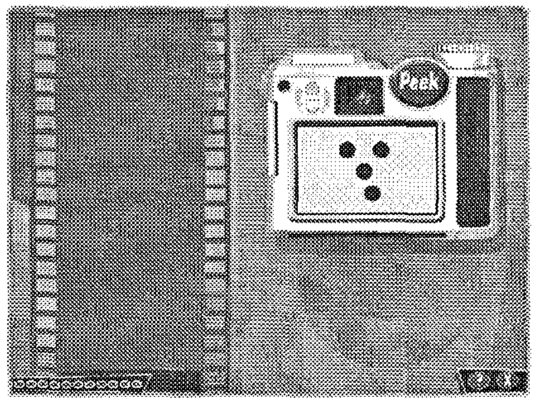


"Build Stairs 3": Students identify the numeral that represents a missing number in a sequence.

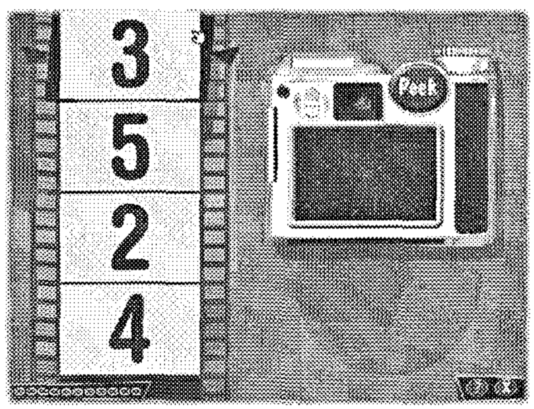


b. Subitizing
In "Snapshots": (b-1) Children are shown an arrangement of dots for 2 seconds. (b-2) They are then asked to click on the corresponding numeral. They can "peek" for 2 more seconds if necessary. (b-3) They are given feedback verbally and by seeing the dots again.

b-1



b-2



b-3

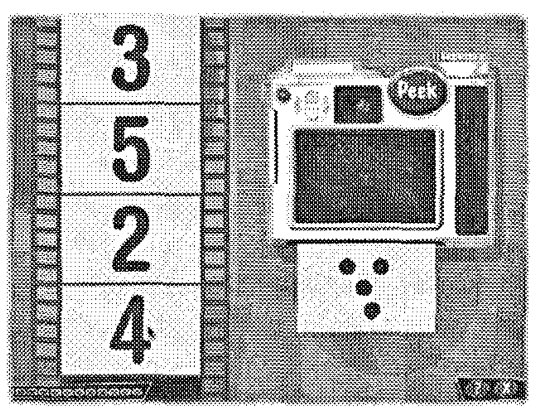
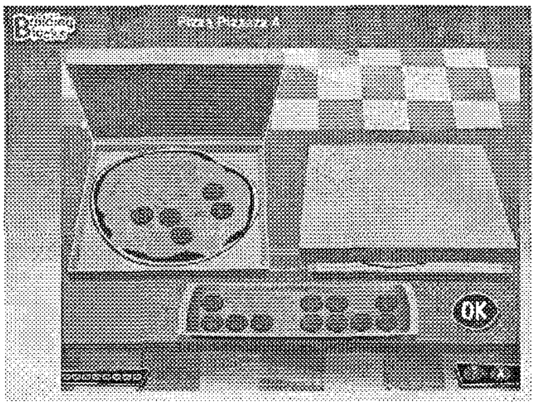


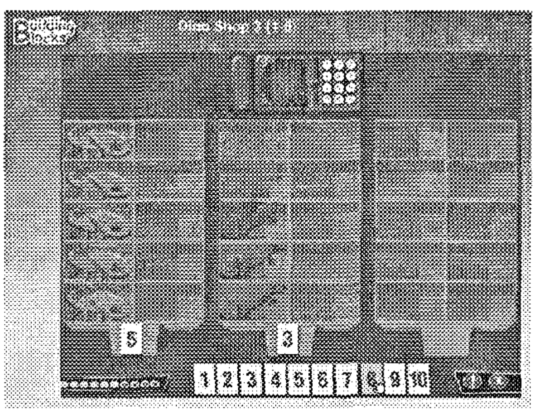
Fig. 2 Teaching the levels of thinking from different learning trajectories that help children learn to count on

c. The main addition and subtraction learning trajectory

(c-1) “Pizza Pazzazz 4”: Students add and subtract numbers up to totals of 3, (with objects shown, but then hidden) matching target amounts.



(c-2) “Dinosaur Shop 3”: Customers at the shop asks students to combine their two orders and add the contents of two boxes of toy dinosaurs (number frames) and click a target numeral that represents the sum.



(c-3) “Bright Idea”: Students are given a numeral and a frame with dots. They count on from this numeral to identify the total amount, and then move forward a corresponding number of spaces on a game board.

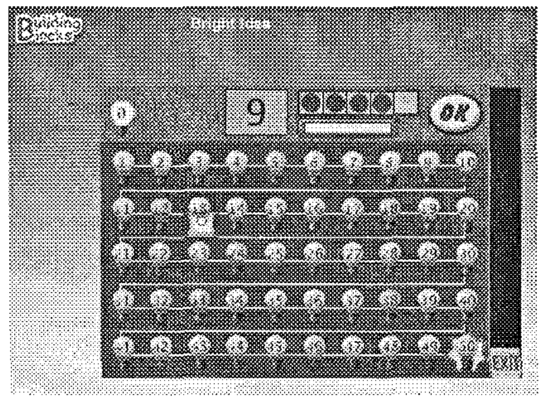


Fig. 2 (Continued)

from *Building Blocks* (Clements and Sarama 2007a) moves along a learning trajectory from the smallest numbers (1–2) to slightly larger sets (3–5) and also from matching exact dot arrangements to different dot arrangements to matching dots to numerals, as shown in Fig. 2b. Children can ask for a “Peek” to see the set again before giving their response (but only once more).

From the addition and subtraction learning trajectory, children learn to interpret additive situations mathematically, such as interpreting a real-world problem as a "part-part-whole" situation. Examining a small part of the developmental progression, at the earliest level of thinking (see *Nonverbal* +/− in Fig. 1), children use initial bootstrapping abilities (inchoate premathematical and general cognitive competencies and predispositions at birth or soon thereafter) and intuitive competencies based on mental images of very small sets. Children later learn to use counting to determine the number in each part and in the whole, originally needing to directly model the situation, using one object for each element in the problem and counting each part and the whole starting from "one" each time (see the subsequent two levels, *Small Number* +/− and *Find Result* +/−). Abby had quickly worked through the *Nonverbal* +/− level activity, "Pizza Pazzazz 4" (Fig. 2-c-1) up to the *Find Result* +/− activity "Dinosaur Shop 3." Generally, she counted all objects. That is, she counted out 5 green toy dinosaurs, then 3 red dinosaurs, then counted them all, starting at 1, clicking on the "8," thereby showing she understood the task.

Next, through the constructive synthesis of the levels of thinking from these three learning trajectories, counting, subitizing, and addition and subtraction, children learn to solve problems such as, "You have three blue blocks and seven red blocks. How many blocks do you have in all?" by modeling the problem situation and counting on Carpenter and Moser (1984). They understand that these numbers are two parts and that they need to find the whole. They also understand that the order of numbers does not matter in addition. They know, in practice, that the sum is the number that results by starting at the first number and counting on a number of iterations equal to the second number. They can use counting to solve this, starting by saying "seven..." because they understand that word can stand for the counting acts from 1 to 7 (because 7 includes 6, and 6 includes 5...). The elongated pronunciation may be substituting for counting the initial set one-by-one. It is *as if* they counted a set of 7 items. Finally, they know *how many more* to count because they use the subitized rhythm of three, so they then say, "*eight, nine, ten*!"

To develop this level of thinking, Abby engaged in many activities. Illustrated in Fig. 2-c-3, "Bright Idea" is a game in which, for the first time in a series of similar board games, not all quantities were represented by sets of dots. Instead, one of the addends is represented by a numeral (and a large single-digit at that), which research (Siegler and Jenkins 1989) shows encouraged Abby to count on: "9, 10, 11, 12, 13!"

Abby learned more quickly than most, but the *Building Blocks Software*'s (Clements and Sarama 2007a) automatic movement along the learning trajectory supported her learning and illustrates the great potential children have to learn mathematics. By four years of age, Abby was given five train engines. She walked in one day with three of them. Her father said, "Where's the other ones?" "I lost them," she admitted. "How many are missing?" he asked. "I have 1, 2, 3. So [pointing in the air] four, five...two are missing, four and five. [pause] No! I want these to be [pointing at the three engines] one, three, and five. So, two and four are missing. Still two missing, but they're numbers two and four." Abby thought about counting and numbers—at least small numbers—abstractly. She could assign 1, 2, and 3 to the three engines, or 1, 3, and 5! Moreover, she could count the numbers. That is, she applied counting... to counting numbers.

Reflection: How Learning Trajectories Require Rethinking Early Mathematics

Developing and implementing learning trajectories such as these has several implications for reconceptualizing early childhood mathematics education, some of which may be more apparent than others. We mention just a few.[1]

- *Rethinking goals.* Counting and arithmetic are standard curriculum content. However, *subitizing* has too often been ignored. Although it is the first-developing numerical competence (e.g., Antell and Keating 1983), the lack of attention to it may result in children *regressing* in their subitizing skills (Wright 1994). Not only is it a valuable competence itself, but also the brief discussion of arithmetic showed how it supports later learning of other topics. As a second example, goals for children have often been thought of mostly as procedural skills. The levels of thinking presented within learning trajectories combine conceptual knowledge, skills, and problem-solving competencies.
- *Rethinking curricular sequences.* A typical traditional sequence of instruction is teaching counting in kindergarten and then introducing addition and subtraction the next year (first grade or Year 1). In contrast, research underlying the learning trajectories indicates that counting and arithmetic begin in the first years of life (e.g., Kobayashi et al. 2004; Wynn 1992), and develop in parallel, gradually becoming increasingly intertwined and connected (e.g., Baroody 2004; Fuson 1988, 2004).
- *Rethinking goals for specific age children.* Children may benefit from working on topics previously thought too difficult for their age (National Research Council 2009). Examples include the incorporation of much larger numbers, activities involving challenging reasoning, and rich geometry (e.g., symmetry, composition, motions, notions to which we return) (Carpenter et al. 1988; Clements and Sarama 2009; Sarama and Clements 2009b; Zvonkin 2010). Although Abby was an exceptional learner, her thinking makes it clear that mathematical goals need to be reconsidered, as they often underestimate what young children can learn.
- *Rethinking curriculum and teaching strategies.* Recognition of the sequence of levels of *thinking* (as opposed to simple accumulation of facts and skills) implies a different view of curriculum and teaching (Fuson et al. 2000). Further, available research can sometimes give quite specific guidance on teaching strategies.

Let's examine two examples of such specific guidance for early arithmetic, addressing two critical points in the learning trajectory, learning to count on and learning arithmetic combinations. Regarding the former, most children can invent counting on in environments in which children's inventions and discussions of strategies

[1]These are not limited to recent work on learning trajectories, of course. They have been raised by other projects, such as cognitively-guided instruction (Carpenter and Franke 2004) on which our notion of learning trajectories are based (for a discussion of these roots, see Sarama and Clements in press).

are encouraged. However, for a variety of reasons, some individuals may have difficulty learning counting on. The teaching sequence in Fig. 3 has proven effective (El'konin and Davydov 1975; Fuson 1992). These understandings and skills are reinforced with additional problems and focused questions.

Besides carefully addressing necessary ideas and subskills, this instructional activity is successful because it promotes *psychological curtailment* (Clements and Burns 2000; Krutetskii 1976). Curtailment is an encapsulation process in which one mental activity gradually "stands in for" another mental activity. Children must learn that it is not necessary to enumerate each element of the first set. The teacher explains this, then demonstrates by naming the number of that set with an elongated number word and a sweeping gesture of the hand before passing on to the second addend. El'konin and Davydov (1975) claim that such abbreviated actions are not eliminated but are transferred to the position of actions which are considered as if they were carried out and are thus "implicit." The sweeping movement gives rise to a "mental plan" by which addition is performed, because only in this movement does the child begin to view the group as a unit. The child becomes aware of addition as distinct from counting. This construction of counting on must be based on physically present objects. Then, through introspection (considering the basis of one's own ways of acting), the object set is transformed into a symbol (El'konin and Davydov 1975).

Our second example of instructional activities supported by specific research evidence is found in the next level in Fig. 1, *Deriver* $+/-$. The goal is to build fluency with basic combinations while maintaining understanding. Teaching the BAMT Strategy actually consists of a series of instructional activities involving several interrelated learning trajectories (from Murata 2004). BAMT stands for Break-Apart-to-Make-Ten. Before lessons on BAMT, children work on several related learning trajectories. They develop solid knowledge of numerals and counting (i.e., move along the counting learning trajectory). This includes the number structure for teen numbers as $10 +$ another number, which is more straightforward in Asian languages than English ("thirteen" is "ten and three"—note that U.S. teachers must be particularly attentive to this competence). They learn to solve addition and subtraction of numbers with totals less than 10, often chunking numbers into 5 (e.g., 7 as 5-plus-2) and using visual models.

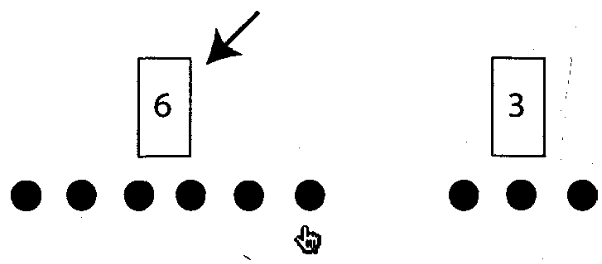
With these levels of thinking established, children develop several levels of thinking within the composition/decomposition developmental progression. For example, they work on "break-apart partners" of numbers less than or equal to 10. They solve addition and subtraction problems involving teen numbers using the 10s structure (e.g., $10 + 2 = 12$; $18 - 8 = 10$), and addition and subtraction with three addends using 10s (e.g., $4 + 6 + 3 = 10 + 3 = 13$; $15 - 5 - 9 = 10 - 9 = 1$).

Teachers then introduce problems such as $9 + 6$. They first elicit, value, and discuss child-invented strategies (such as counting on) and encourage children to use these strategies to solve a variety of problems. Only then do they proceed to the use of BAMT. They provide supports to connect visual and symbolic representations of quantities. In the example $9 + 4$, they show 9 counters (or fingers) and 4 counters, then move one counter from the group of four to make a group of ten. Next, they

1. Make sure the child can count starting at numbers other than one (see the "Counter from N ($N+1, N-1$)" level in the counting learning trajectory, Clements and Sarama 2009; Sarama and Clements 2009b).
2. Try larger numbers: several such as $22+1$, then problems such as $1+18$. Although smaller numbers are beneficial to children in many situations, here, the child's desire to find a simpler way can often motivate the invention and use of counting on.
3. If these fail, teach individuals or small groups each component of counting on as follows.

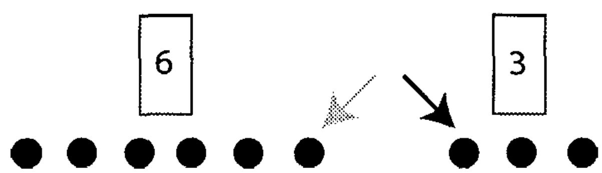
Lay out the problem $(6+3)$ with numeral cards. Count out objects into a line below each card.

Ask child to count out a set of 6. When they reach the sixth object, point to numeral card and say, "See this is six also. It tells how many dots there are here" (gesture around all 6 dots).



Solve another problem. If the child counts the first set starting with one again, interrupt them sooner and ask what number they will say when they get to the last object in the first set. Emphasize it will be the same as the numeral card.

Point to the last dot and say (using $6+3$ again for this example) "See, there are six here, so this one (exaggerated jump from last object in the first set to first object in the second set) gets the number *seven*."



Repeat with new problems. As necessary, interrupt child's counting of the first set with questions: "How many are here (first set)? So this (last of first) gets what number? ("Six!") And what number for this one"? ("Seven!")

Fig. 3 Teaching counting on skills to children who need assistance to use counting on, or do not spontaneously invent this strategy

The line slants between the numbers, indicating that we need to find a partner for 9 to make 10.	Four is separated into two partners, 1 and 3.	The ring shows how the numbers combine to make 10.	Ten and three are shown to add to 13.

Fig. 4 *Teaching BAMT* (modified from Murata 2004)

highlight the three left in the group. Then children are reminded that the 9 and 1 made 10. Last, children see 10 counters and 3 counters and think ten-three, or count on "ten-one, ten-two, ten-three." Later, representational drawings serve this role, in a sequence such as shown in Fig. 4.

Teachers spend many lessons ensuring children's understanding and skill using the BAMT strategy. Children are asked why the strategy works and what its advantages are. Extensive use of BAMT to solve problems helps children develop fluency.

Not all instructional tasks are as specific as these just outlined. In many cases, the instructional tasks presented with the learning trajectories are simply illustrations of the kind of effective activities that would be appropriate to reach a certain level of thinking. For example, the problems suggested for each level should be changed for different children, but the *type* of problem is important.

A final observation regarding our Fig. 1 discussions is that learning trajectories promote learning skills and concepts together, as mentioned previously (see "Rethinking goals"). Learning skills before developing understanding can lead to learning difficulties (Baroody 2004; Fuson 2004; Kilpatrick et al. 2001; Sophian 2004; Steffe 2004). Further, effective curricula and teaching often build on children's thinking, provide opportunities for both invention and practice, and ask children to explain their various strategies (Hiebert 1999). Such programs facilitate conceptual growth and higher-order thinking without sacrificing skill learning. Effective teachers also consistently integrate real-world situations, problem solving, and mathematical content (Fuson 2004). Making connections to real-life situations also enhances children's knowledge and positive beliefs about mathematics (Perlmutter et al. 1997). Thus, a critical task for teachers is to adapt activities such as those in Fig. 1 so that they are relevant and appropriate to their own students.

Rethinking Curriculum Development: What Is a Research-Based Curriculum?

Children's progress through learning trajectories is profoundly influenced by their first educational experiences. Indeed, "the early grades may be precisely the time that schools have their strongest effects" (Alexander and Entwisle 1988). Research also suggests that early childhood classrooms underestimate children's ability to learn mathematics and are too often ill suited to help them learn due to lack of knowledge of the variety of learning trajectories. Thus, children actually *regress* on some math skills during kindergarten (Wright 1994). For example, perhaps because they do not understand the idea and importance of subitizing, kindergarten teachers often told children who had already subitized a small collection correctly, to "*Count* them!" thus undermining children's use of a valuable practice. We need more structured, sophisticated, better-developed and well-sequenced mathematics in early childhood education. How do we do that well? How do we avoid the problem mentioned in the introduction: that it is difficult to know how to evaluate if a curriculum truly is "based on research"?

A Framework for Research-Based Curricula

Based on a review of research and expert practice (Clements 2008), we constructed and tested a framework for the construct of research-based curricula. Our "Curriculum Research Framework" (CRF, Clements 2007) rejects the sole use of commercially-oriented "market research" and "research-to-practice" strategies. Although included in the CRF, such strategies alone are inadequate. For example, research-to-practice strategies are flawed in their presumptions because they employ one-way translations of research results, are insensitive to changing goals in the content area, and are unable to contribute to a revision of the theory and knowledge. Such knowledge building is—alongside the development of a scientifically-based, effective curriculum—a critical objective of a scientific curriculum research program. Indeed, a valid scientific curriculum development program should address two basic questions—about effects and conditions—in three domains-practice, policy, and theory. For example, such a program should address the practical question of whether the curriculum is effective in helping children achieve specific learning goals, but also under what conditions it is effective. Theoretically, the research program should also address why it is effective and why certain sets of conditions decrease or increase the curriculum's effectiveness.

To address all these issues, the CRF includes three broad categories of research and development work, within which there are ten phases through which a curriculum should be subjected to warrant the claim that it is based on research. The three categories are: (1) reviewing existing research (A Priori Foundations), (2) building models of children's thinking and learning in a domain (Learning Trajectories), and (3) appraising the effectiveness and general worth of the result (Evaluation, both formative, leading to revisions, and summative, to determine the effects of the completed curriculum). The categories and phases within them are outlined in Table 1. The categories are described in the leftmost column. The questions addressed are provided in the middle column, and the specific methodologies to address these questions within each phase are described in the rightmost column.

Table 1 *Categories and Phases of the* Curriculum Research Framework (*adapted from Clements 2007*)

Categories	Questions Asked	Phases
A Priori Foundations: In variants of the research-to-practice model, extant research is reviewed and implications for the nascent curriculum development effort drawn.	What is already known that can be applied to the anticipated curriculum?	Established review procedures and content analyses are employed to gather knowledge concerning the specific subject matter content, including the role it would play in students' development (*phase 1*); general issues concerning psychology, education, and systemic change (*phase 2*); and pedagogy, including the effectiveness of certain types of activities (*phase 3*).
Learning Trajectories: Activities are structured in accordance with empirically-based models of children's thinking and learning in the targeted subject-matter domain.	How might the curriculum be constructed to be consistent with models of students' thinking and learning?	In *phase 4,* the nature and content of activities is based on models of children's mathematical thinking and learning. Specific learning trajectories are built for each major topic.
Evaluation: In these phases, empirical evidence is collected to evaluate the curriculum, realized in some form. The goal is to evaluate the appeal, usability, and effectiveness of an instantiation of the curriculum.	How can market share for the curriculum be maximized?	*Phase 5* focuses on marketability, using strategies such as gathering information about mandated educational objectives and surveys of consumers.
	Is the curriculum usable by, and effective with, various student groups and teachers?	Formative phase*s 6 to 8* seek to understand the meaning that students and teachers give to the curriculum objects and activities in progressively expanding social contexts; for example, the usability and effectiveness of specific components and characteristics of the curriculum as implemented by a teacher who is familiar with the materials with individuals or small groups (*phase 6*) and whole classes (*phase 7*) and, later, by a diverse group of teachers (*phase 8*). The curriculum is altered based on empirical results, with the focus expanding to include aspects of support for teachers.

The first curriculum to be developed using the Curriculum Research Framework (CRF) was Building Blocks (Clements and Sarama 2003, 2007b, 2013), a NSF-funded PreK to grade 2 mathematics research and curriculum development project that was one of the first to develop materials that comprehensively address recent standards for early mathematics education for all children (e.g., Clements and Conference Working Group 2004; National Council of Teachers of Mathematics 2000,

Table 1 (Continued)

Categories	Questions Asked	Phases
	What is the effectiveness (e.g., in affecting teaching practices and ultimately student learning) of the curriculum, now in its complete form, as it is implemented in realistic contexts?	Summative phases 9 and 10 both use randomized field trials and differ from each other most markedly on the characteristic of scale. They both examine the fidelity or enactment, and sustainability, of the curriculum when implemented on a small (*phase 9)* or large (*phase 10)* scale, with *phase 10* also investigating the critical contextual and implementation variables that influence its effectiveness. Experimental or carefully planned quasi-experimental designs, incorporating observational measures and surveys, are useful for generating political and public support, as well as for their research advantages. In addition, qualitative approaches continue to be useful for dealing with the complexity and indeterminateness of educational activity.

2006). We will illustrate the CRF by giving concrete descriptions of how the phases were enacted in the development of the Building Blocks preschool curriculum.

A Priori Foundations

The first category includes three variants of the research-to-practice strategy, in which existing research is reviewed and implications for the nascent curriculum development effort are drawn.

(1) In *General A Priori Foundation*, developers review broad philosophies, theories, and empirical results on learning and teaching. Based on theory and research on early childhood learning and teaching (e.g., National Research Council 2001), we determined that *Building Blocks*' basic approach would be finding the mathematics in, and developing mathematics from, children's activity. That is, we wanted to "mathematize" everyday activities, such as puzzles, songs, moving, and building. For example, teachers might help children mathematize moving their bodies in many ways. Children might count their steps as they walk. They might also move in patterns: step, step, hop; step, step, hop.... They might do both, counting as they walk, "one, *two*, three, *four*, five, *six*,...". These examples show that mathematizing means representing and elaborating everyday activities mathematically. Children create models of everyday situations with mathematical objects, such as numbers and shapes; mathematical actions, such as counting or transforming shapes; and their structural relationships—and use those understandings to solve problems. They learn to increasingly see the world through mathematical lenses.

(2) In *Subject Matter A Priori Foundation*, developers review research and consult with experts to identify topics that make a substantive contribution to children's mathematical development, are generative in children's development of future mathematical understanding, and are interesting to children. We determined the topics that fit those criteria by considering what mathematics is culturally valued and empirical research on what constituted the core ideas and skill areas of mathematics for young children (Baroody 2004; Clements and Battista 1992; Clements and Conference Working Group 2004; Fuson 1997). We then organized for the development of learning trajectories in the domains of number (counting, subitizing, sequencing, arithmetic), geometry (matching, naming, building and combining shapes), patterning, and measurement.

(3) In *Pedagogical A Priori Foundation*, developers review empirical findings on making activities educationally effective—motivating and efficacious—to create general guidelines for the generation of activities. As an example, research using computer software with young children (Clements et al. 1993; Clements and Swaminathan 1995; Steffe and Wiegel 1994) showed that preschoolers can use computers effectively and that software can be made more effective by employing animation, children's voices, and clear feedback. Although such software is only a small component of the Building Blocks curriculum, it makes a significant contribution, because research was used in its development, giving the developers information on how to make the software targeted and effective.

Learning Trajectories

In the second category, developers structure activities in accordance with theoretically- and empirically-based models of children's thinking in the targeted subject-matter domain. This phase involves the creation of research-based learning trajectories. Figure 1 illustrated a part of our arithmetic learning trajectory. We turn to one in geometry so we do not give the misimpression that learning trajectories only apply to numerical domains.

When we were working with kindergartners, one of us (Sarama) observed that several children followed a similar progression in choosing and combining shapes (e.g., rhombi or equilateral triangles) to make another shape (e.g., to cover a hexagon as in Fig. 5) (Sarama et al. 1996). Initially, they merely appreciated how one pattern block could be made using other pattern blocks, but their efforts to cover a hexagon with other pattern blocks was by trial-and-error. Later, they explicitly recognized the hexagon could be made with 2 trapezoids, followed by other combinations. Sarama reviewed the behaviors of all the kindergarten children. She found several similar sequences and noted that, throughout the study, children's development appeared to move from placing shapes separately to considering shapes in combination; from manipulation- and perception-bound strategies to the formation of mental images (e.g., decomposing shapes imagistically); from trial and error to intentional and deliberate action and eventually to the prediction of succeeding placements of shapes;

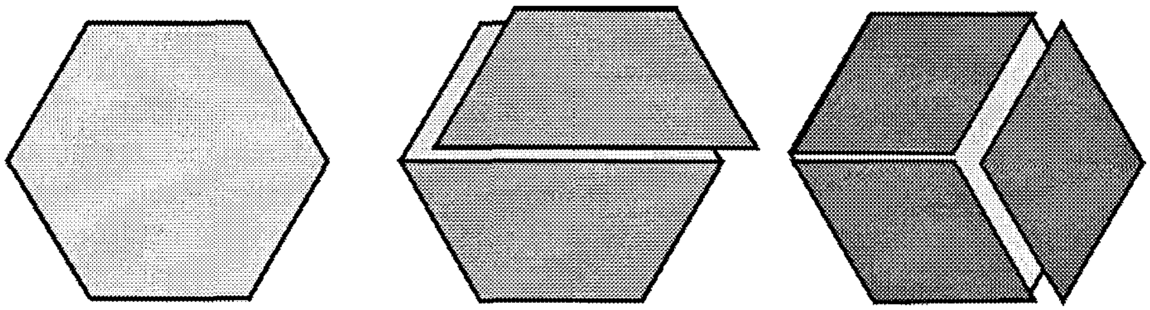


Fig. 5 How children might cover a pattern block hexagon with other pattern block shapes

and from consideration of visual "wholes" to a consideration of side length, and, eventually, angles.

Based on these observations, we wrote a tentative set of levels of thinking, the first draft of the developmental progression, then revised it as we studied the results of other researchers (Mansfield and Scott 1990; Sales 1994; Vurpillot 1976). At this point, we involved several additional teachers, because any learning trajectory should "speak" to practitioners as well as researchers. Eight volunteers who were helping us develop the *Building Blocks* curriculum worked with us for six months testing and refining activities (for the complete story, see Clements et al. 2004). Their case studies indicated that the work of about four-fifths of the children studied was consistent with the developmental progression. Finally, we conducted a study with David Wilson, utilizing 72 randomly selected children from pre-K to grade 2. Analyses again indicated child progress consistent with the developmental progression (Clements et al. 2004).

In this way, through cycles of curriculum revision and observations, we created the complete learning trajectory, including the developmental progression and a set of instructional tasks, which include on- and off-computer puzzles, that appeared to facilitate growth for children at different points along the trajectory (as embodied now in the *Building Blocks* concrete puzzles). Figure 6 illustrates several early levels of the learning trajectory.

Again, computer technology makes a substantive contribution. First, the *Building Blocks Software* (Clements and Sarama 2007a) moves children forward (or backward) along the learning trajectory automatically based on children's performance. Second, the tasks are designed to fit the trajectory precisely. For example, children's work is initially scaffolded by the inclusion of internal line segments in most cases, but these are faded in subsequent puzzles. Third, analyses of children's responses are often superior to situations using physical manipulatives. For example, children will often place physical shapes so that they cover a puzzle but also "hang over" outside of the puzzle—and they and their teachers rarely notice this. Computers detect every error. Fourth, when such errors are detected, the feedback is immediate and in some ways superior. For example, the shapes placed by the child can be made translucent, clearly showing the mismatch between the child's solution and the actual puzzle. In these ways, work with computers provides a unique and substantial contribution to children's learning.

Goal: **Children compose geometric shapes intentionally to create a superordinate shape, building understanding of part-whole relationships as well as the properties of the original and composite shapes.**

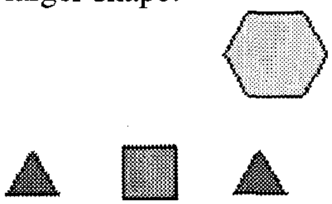
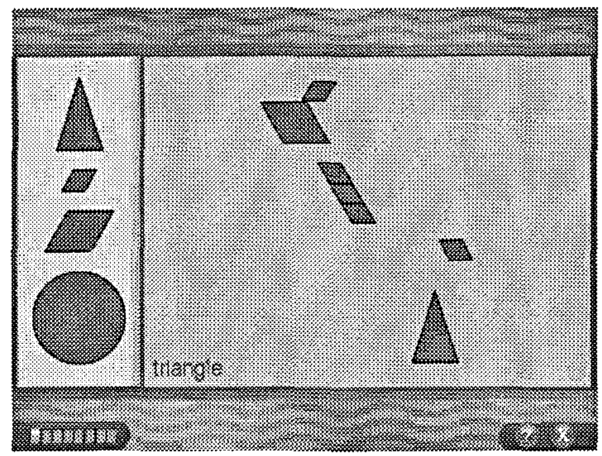
Developmental Progression	*Instructional Tasks*
Pre-Composer Manipulates shapes as individuals, but is unable to combine them to compose a larger shape. 	This level is not an instructional goal. However, several preparatory activities may orient 2- to 4-year-old children to the task, and move them toward the next levels that do represent (some) competence. In "Shape Pictures," children play with pattern blocks and Shape Sets, often making simple pictures. In "Mystery Toys" children match shapes, but the *result* of their work is a pictured made up of other shapes—a demonstration of composition. 

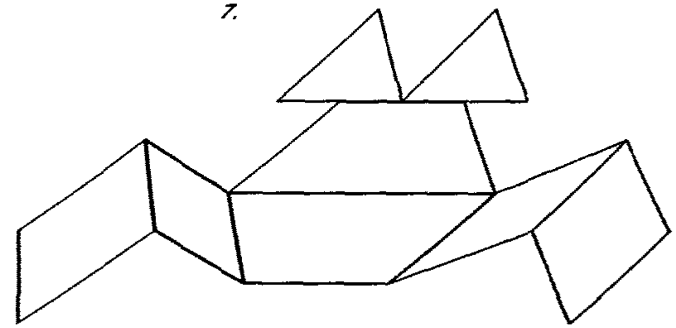
Fig. 6 Samples from the Learning Trajectory for the Composition of 2D Shapes (adapted from Clements and Sarama 2009; Sarama and Clements 2009b)

...ce Assembler *Makes pictures ...ich each shape represents a ...e (e.g., one shape for ...t) and shapes touch. ...tern Block ...nd error.*

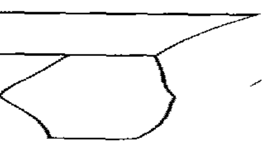
In the first "Pattern Block Puzzles" tasks, each shape is not only outlined, but touches other shapes only at a point, making the matching as easy as possible. Children merely match pattern blocks to the outlines. Then, the puzzles moved to those that combine shapes by matching their sides, but still mainly serve separate roles.

Pattern Block Puzzles

7.

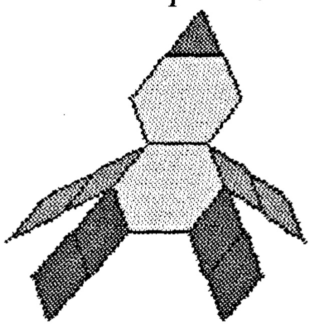


... "Pattern Block Puzzles" at this level start with those where several shapes are combined ...e one "part," but internal lines are still available.

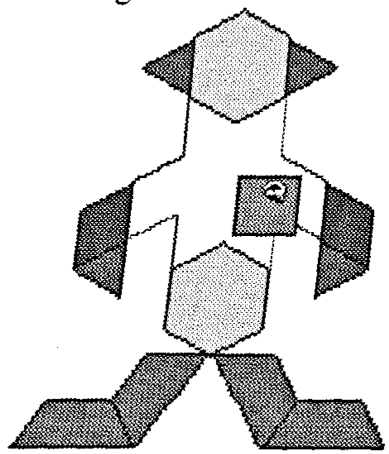


Fills "easy" "Pattern Block Puzzles" that suggest the placement of each shape (but note below that they child is trying to put a square in the puzzle where its right angles will not fit).

Make a picture



Later puzzles in the sequence require combining shapes to fill one or more regions, without the guidance of internal line segments.



"Piece Puzzler 3" is a similar computer activity. In the first tasks, children must concatenate shapes, but are helped with internal line segments in most cases; these internal segments are faded in subsequent puzzles.

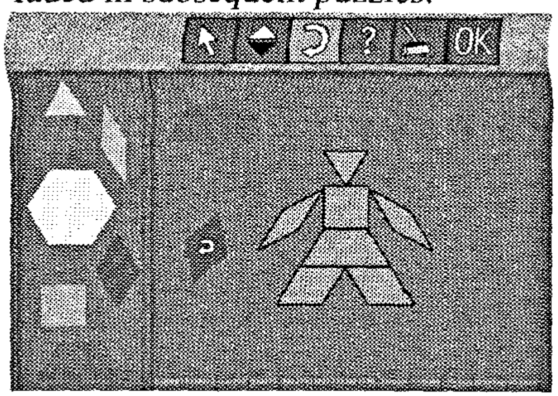
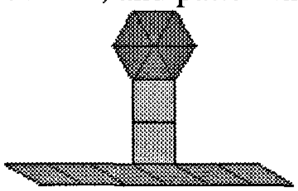


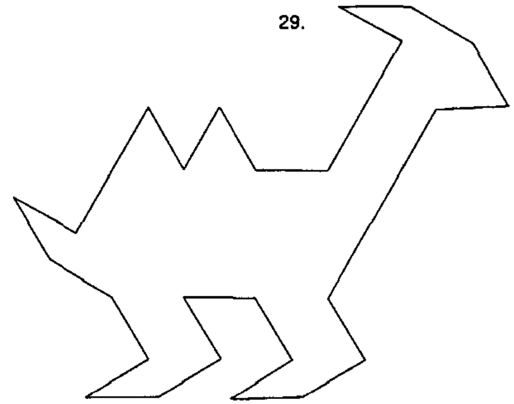
Fig. 6 (Continued)

Shape Composer. Composes shapes with anticipation ("I know what will fit!"). Chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. In the "Pattern Block Puzzles" below, all angles are correct, and patterning is evident.



The "Pattern Block Puzzles" and "Piece Puzzler" activities have no internal guidelines and larger areas; therefore, children must compose shapes accurately.

Pattern Block Puzzles



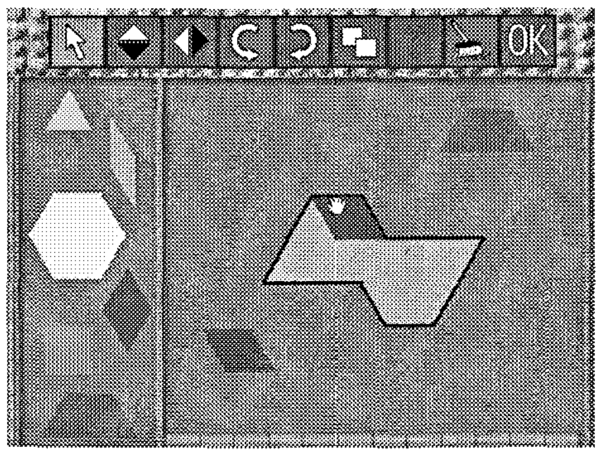


Fig. 6 (Continued)

Evaluation

In the third category of the CRF, developers collect empirical evidence to evaluate the appeal, usability, and effectiveness of a version of the curriculum. Past phase (5) *Market Research* is (6) *Formative Research: Small Group*, in which developers conduct pilot tests with individuals or small groups on components (e.g., a particular activity, game, or software environment) or sections of the curriculum. Although teachers are involved in all phases of research and development, the process of curricular enactment is emphasized in the next two phases. Studies with a teacher who participated in the development of the materials in phase (7) *Formative Research: Single Classroom*, and then teachers newly introduced to the materials in phase (8) *Formative Research: Multiple Classrooms*, provide information about the usability of the curriculum and requirements for professional development and support materials. We conducted multiple case studies at each of these three phases (e.g., Clements and Sarama 2004a; Sarama 2004), revising the curriculum multiple times, including two distinct published versions (Clements and Sarama 2003, 2007c).

In the last two phases, (9) *Summative Research: Small Scale* and (10) *Summative Research: Large Scale*, developers evaluate what can actually be achieved with typical teachers under realistic circumstances. To avoid the misconstrual that the CRF privileges scientific research of a limited nature, note that *the first 8 phases involve only qualitative research and the last two combine quantitative and qualitative research.* The CRF uses a wide range of methods, omitting no genre of educational research.

An initial phase-9 summary research project (Clements and Sarama 2007d) yielded effect sizes between 1 and 2 (standard deviation units). However, this study only involved 4 classrooms. Thus, we moved to phase 10, which also uses randomized trials, which provide the most efficient and least biased designs to assess causal relationships (Cook 2002), where the curriculum is implemented in a greater number of classrooms, with more diversity, and less ideal conditions. In a larger study (Clements and Sarama 2008), we randomly assigned 36 classrooms to one of three conditions. The experimental group used Building Blocks (Clements and Sarama 2007b). The comparison group used a different preschool mathematics curriculum—the same as we previously used in the Preschool Curriculum Evaluation Research (Preschool Curriculum Evaluation Research Consortium 2008) research (mainly Klein et al. 2002). The control used their schools' existing curriculum ("business as usual"). Two observational measures indicated that the curricula were implemented with fidelity and that the experimental condition had significant positive effects on classrooms' mathematics environment and teaching. From the beginning to end of the school year, the experimental group score increased significantly more than the comparison group score (effect size, .47) and the control group score (effect size, 1.07). Focused early mathematical interventions, especially those based on a comprehensive model of developing and evaluating research-based curricula, can increase the quality of the mathematics environment and teaching, and can help preschoolers develop a foundation of informal mathematics knowledge (Clements and Sarama 2008). We believe that these positive effects, even when compared to another curriculum supported equivalently, were due to *Building Blocks'* development within the CRF and especially the core use of learning trajectories.

Conclusion

Through our collaboration with teachers, administrators, and other researchers, we believe we have developed and evaluated truly research-based approaches. The two major conceptual tools, sets of learning trajectories and the Curriculum Research Framework, have shown their effectiveness in a number of studies. We hope others test whether these and other similar tools (see Maloney et al. in press) contribute to a scientific base for early mathematics education.

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