Problems:- GD

1) If the probability that a target is destroyed on any shot is 0.5, what is the probability that it would be destroyed on the 6^{th} attemp?

Soln) $P(X=6) = (0.5)(1-0.5)^{6-1}$ where $x = 6$, $p = 0.5$, $q = 1-p = 0.5$.

$P(X=6) = 0.015625$ //

[Formula used :- pmf of GD
$P(X=x) = pq^{x-1}$; $x = 1, 2 \ldots$]

2) Suppose that a trainee solder shoots a target in an independent fashion. If the probability that the target is shot is 0.8.

(a) what is the probability that the target would be hit on the 6^{th} attempt?

(b) what is the probability that it takes him less than 5 shots?

(c) what is the probability that it takes him an even no of shots?

(a) [formula used -
pmf of GD

$P(X=x) = pq^{x-1}$; $x = 1, 2 \ldots$]

given $p = 0.8$ $q = 1-p = 0.2$

$P(X=6) = (0.8)(0.2)^{6-1}$

$P(X=6) = 0.000256$

(b) $P(X<5) = ?$

$P(X<5) = 1 - [P(X=5) + P(X=6)]$

$= 1 - [(0.8)(0.2)^4 + (0.8)(0.2)^5]$

$= 1 - [0.00128 + 0.000256]$

$= 1 - [0.001536]$

$P(X<5) = 0.998464$ // [Probability that it takes him less than 5 shots]

(c) The probability that takes him an even no of shots is

$= P(X=2) + P(X=4) + P(X=6) + \ldots$

$= (0.8)(0.2) + (0.8)(0.2)^3 + (0.8)(0.2)^5 + \ldots]$

$= (0.8)(0.2)[1 + (0.2)^2 + (0.2)^4 + \ldots]$

$= (0.8)(0.2)[1 + 0.04 + (0.04)^2 + \ldots]$

$= (0.8)(0.2)[1 - 0.04]^{-1}$

$= (0.8)(0.2)(0.96)^{-1}$

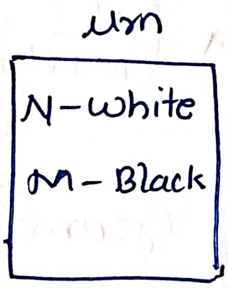
$= 0.16667$ //

34 An urn contains N white & M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that:-

(a) Exactly n draws are needed &

(b) At least k draws are needed?

Soln)



Since balls are replaced, the trials are independent & the probability of success (a black ball is drawn) remains same.

If we let X denote the no of draws needed to select a black ball, then X is a geometric R.V. with $p = \frac{M}{(M+N)}$ & $q = \left(\frac{N}{M+N}\right)$

Hence

(a) $$P[X=n] = \left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)^{n-1}$$

$$= \frac{MN^{(n-1)}}{(M+N)^{n}} \quad //$$

(b) $P[X \geq K] = \sum_{n=K}^{\infty} (pq^{n-1})$

$$= \sum_{n=k}^{\infty} \left(\frac{M}{M+N}\right)\left(\frac{N}{M+N}\right)^{n-1}$$

$$= \frac{M}{(M+N)} \sum_{n=k}^{\infty} \left(\frac{N}{M+N}\right)^{n-1}$$

$$= \frac{M}{M+N}\left[\left(\frac{N}{M+N}\right)^{K-1} + \left(\frac{N}{M+N}\right)^{K} + \left(\frac{N}{M+N}\right)^{K+1} + \dots\right]$$

$$= \frac{M}{(M+N)}\left(\frac{N}{M+N}\right)^{K-1}\left[\underbrace{1 + \left(\frac{N}{M+N}\right) + \left(\frac{N}{M+N}\right)^{2} + \dots}\right]$$

$\therefore$

Geometric progression summation $\rightarrow \frac{a}{1-r}$

where $a = 1$ & $r = \frac{N}{M+N}$

$$P[X \geq K] = \frac{M(N)^{K-1}}{(M+N)^{K}}\left[\frac{1}{\left(1-\frac{N}{M+N}\right)}\right]$$

$$= \frac{M(N)^{K-1}}{(M+N)^{K}} \cdot \frac{(M+N)}{M}$$

$$= \left(\frac{N}{M+N}\right)^{K-1}$$

Problems on Exponential distribution :-

Assume that the length of a phone call in minutes is an exponential random variable X with parameter $\lambda = \frac{1}{10}$. If a customer arrives at a phone booth just before you arrive, find the probability that you have to wait :-

(a) Less than 5 minutes.

(b) greater than 10 minutes

(c) b/w 5 & 10 minutes

(d) mean & variance.

Soln)

(a) CDF :-

$$F = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

$$P(X \leq x) = 1 - e^{-\lambda x}.$$

$$P(X < 5) = 1 - e^{-0.1 \times 5} = 0.3935 \quad [\lambda = 0.1]$$

(b) $1 - P(X \leq 10)$

$$1 - [1 - e^{-0.1 \times 10}] = 0.3679.$$

(c) $\boxed{P(5 < x < 10) = \int_5^{10} f(x)\,dx.}$

or

$P(X<10) - P(X>5)$

$= [1 - e^{-0.1\times10}] - [1 - e^{-0.1\times5}]$

$= 0.6321 - 0.3935$

$= 0.2386$

(d) Mean $= 1/\lambda$. Variance $= \dfrac{1}{\lambda^2} = \mu^2$.

mean $= 10$ Variance $= 100$.

Problems on Uniform distribution:-

(1) Arrival of customers at a certain checkout counter follow a Poisson distribution. It is known that, during a given 30-minute period, one customer arrived at the counter. Find the probability that the customer arrived during the last 5 minutes of the 30-minute period.

Sol^n) The actual time of arrival follows a UD over the interval of (0, 30). If X denotes the arrival time, then

$$P(25 \le X \le 30) = \int_{25}^{30} f(x)\, dx \qquad \left[f(x) = \frac{1}{b-a} \right]$$

$$P(25 \le X \le 30) = \int_{25}^{30} \frac{1}{30}\, dx = \frac{30-25}{30} = \frac{1}{6} = 0.1666$$

(2) Electric trains on a certain line run every half an hour b/w mid-night & 6 in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least 20 minutes?

Sol^n) Let the R.V X denote the waiting time in minutes for the next train. Given that a man arrives at the station at random, X is UD on (0, 30) with density

$$f(x) = \begin{cases} 1/30 & 0 < x < 30 \\ 0 & x \geq 30 \end{cases}$$

Thus the probability that he has to wait for atleast 20 minutes is

$$P(X \geq 20) = \int_{20}^{30} f(x)\,dx = \int_{20}^{30} \left(\frac{1}{30-0}\right) dx = \int_{20}^{30} \frac{1}{30}\,dx$$

$$= \frac{[x]_{20}^{30}}{30} = \frac{10}{30} = \frac{1}{3} = 0.3333$$

3) The time (in minutes) passenger must wait for a computer plane in a main railway station is UD on the interval [0, 60], what is the probability that a passenger waits

(a) Less than 20 minutes

(b) More than 40 minutes.

Soln Since 20, 40 ∈ [0, 60], therefore the CDF, with $a=0$ & $b=60$ is

$$F(x) = \frac{x-0}{60-0} = \frac{x}{60}.$$

(a) $P(X \leq 20) = F(20) = \frac{20}{60} = \frac{1}{3} = 0.3333$

(b) $P(X > 40) = 1 - P(x \leq 40) = 1 - \frac{40}{60} = \frac{1}{3} = 0.3333$

4) If X is uniformly distributed over (0, 10), find the probability that

(a) $X < 2$

(b) $X > 2$

(c) $3 < X < 9$

Soln) Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$ then $\int_{0}^{10} \frac{1}{10}\,dx = 1$

(a) $P(X < 2) = \int_{0}^{2} \frac{1}{10}\,dx = \frac{1}{5} = 0.2$

(b) $P(X > 2) = \int_{8}^{10} \frac{1}{10}\,dx = \frac{1}{5} = 0.2$

(c) $P(3 < X < 9) = \int_{3}^{9} \frac{1}{10}\,dx = 3/5 = 0.6$

Problems on Reliability & failure Rate.

1) The failure rate of a certain component is $h(t) = \lambda_0 t$, where $\lambda_0 > 0$ is a given constant.

Determine the reliability $R(t)$, of the component.

Repeat for $h(t) = \lambda_0 t^{1/2}$

Sol) (a) → $R(t) = e^{-H(t)}$ where $H(t) = \int_0^t h(t)\,dt$.

$$H(t) = \int_0^t [\lambda_0 t]\,dt.$$

$$H(t) = \frac{\lambda_0 t^2}{2}$$ → put it in eqn $R(t)$

$$\boxed{R(t) = e^{-\lambda_0 (t^2/2)}}$$

(b) →

$$h(t) = \lambda_0 t^{1/2}$$

$$H(t) = \int_0^t [\lambda_0 t^{1/2}]\,dt$$

$$= \lambda_0 \int_0^t [t^{1/2}]\,dt$$

$$= \lambda_0 \frac{t^{1/2+1}}{(3/2)}$$

$$H(t) = \frac{2}{3} \lambda_0 t^{3/2}$$

put $x(t)$ value in $R(t)$ eqn

$$R(t) = e^{-\left(\frac{2\lambda_0}{3} t^{3/2}\right)}$$ // ans.

27 The failure rate of a computer system for onboard control of a space vehicle is estimated to be the following f^n of time :-

$$h(t) = \alpha \mu t^{\alpha-1} + \beta \gamma t^{\beta-1}$$

Derive an expression for the reliability $R(t)$ of the system. Plot $h(t)$ & $R(t)$ as f^ns of time with parameter values $\alpha = \frac{1}{4}$, $\beta = \frac{1}{7}$, $\mu = 0.0004$, $\gamma = 0.0007$

Soln) given $\alpha = 1/4$ $\quad \mu = 0.0004$

$\beta = 1/7$ $\quad \gamma = 0.0007$

$$h(t) = 0.0001\, t^{-3/4} + 0.0001\, t^{-6/7}.$$

Let $c = 0.0001$

wkt $\quad H(t) = \int_0^t h(t)\,dt$

$$= c \int_0^t \left[t^{-3/4} + t^{-6/7}\right] dt.$$

$$= c \left[\frac{t^{-3/4+1}}{(1/4)} + \frac{t^{-6/7+1}}{(1/7)}\right]$$

$$= c\left[4 t^{1/4} + 6 t^{1/7}\right]$$

$$H(t) = 0.0004\, t^{1/4} + 0.0007\, t^{1/7}$$

$$R(t) = e^{-[c_1 t^{1/4} + c_2 t^{1/7}]}$$ where

$c_1 = 0.0004$ &

$c_2 = 0.0007$ //

<u>Graph :-</u> $h(t)$ vs t & $f(t)$ vs t.

Problems on 2-D R.Vs.

1) The Joint Probability mass function (pmf) of X & Y is given the table below :-

Compute the following

a) $P(X \leq 1)$

b) $P(Y \leq 3)$

c) $P(X \leq 1, Y \leq 3)$

d) $P(X \leq 1 \mid Y \leq 3)$

e) $P(Y \leq 3 \mid X \leq 1)$

f) $P(X+Y \leq 4)$

	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Sol^n)

	1	2	3	4	5	6	$P_X(x) = f(x)$
0	0 $P(0,1)$	0 $P(0,2)$	$1/32$ $P(0,3)$	$2/32$ $P(0,4)$	$2/32$ $P(0,5)$	$3/32$ $P(0,6)$	$P(X=0) = 8/32$
1	$1/16$ $P(1,1)$	$1/16$ $P(1,2)$	$1/8$ $P(1,3)$	$1/8$ $P(1,4)$	$1/8$ $P(1,5)$	$1/8$ $P(1,6)$	$P(X=1) = \frac{20}{32}$
2	$1/32$ $P(2,1)$	$1/32$ $P(2,2)$	$1/64$ $P(2,3)$	$1/64$ $P(2,4)$	0 $P(2,5)$	$2/64$ $P(2,6)$	$P(X=2) = 4/32$
$P_Y(x) = f(y)$	$P(Y=1) = 3/32$	$P(Y=2) = 3/32$	$P(Y=3) = 11/64$	$P(Y=4) = 13/64$	$P(Y=5) = 6/32$	$P(Y=6) = 16/64$	1

(a) $P(X \leq 1) = P(X=0) + P(X=1)$

$= \frac{8}{32} + \frac{20}{32} = \frac{28}{32} = 0.875$ //

(b) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$.

$= 0.359375$ //

(c) $P(X \leq 1, Y \leq 3) = P(0,1) + P(0,2) + P(0,3) + P(1,1)$

$+ P(1,2) + P(1,3)$

$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$= \frac{1+4+4}{32} = \frac{9}{32} = 0.28125$ //

(d) $P(X \leq 1 \mid Y \leq 3) = \dfrac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}$

$= \dfrac{9/32}{23/64} = \dfrac{18}{23} = 0.7826$ //

~~(f) P(X+Y≤4)~~

(e) $P(Y \leq 3 \mid X \leq 1) = \dfrac{P(X \leq 1, Y \leq 3)}{P(X \leq 1)}$

$= \dfrac{9/32}{28/32} = \dfrac{9}{28} = 0.32143$ //.

(f) $P(X+Y \leq 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4)$
$+ P(1,1) + P(1,2) + P(1,3)$
$+ P(2,1) + P(2,2)$

$P(X+Y \leq 4) = 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$
$+ \frac{1}{32} + \frac{1}{32}$

$= \dfrac{13}{32} = 0.40625$ //.

Q22) The following table represents the Joint probability distribution of the discrete RV (X, Y)

(a) Evaluate the marginal distribution of X & Y

(b) Find the conditional distribution of X given Y = 2

(c) Find the conditional distribution of Y given X = 3

	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

Sol^n)

	1	2	3	
1	1/12	1/6	0	1/4
2	0	1/9	1/5	14/45
3	1/18	1/4	2/15	79/180
	5/36	19/36	1/3	1

?

3. The joint pdf of two random variables X and Y is given by

$$f(x,y)=\begin{cases}\frac{1}{8}x(x-y) & 0<x<2; -x<y<x\\ 0 & otherwise\end{cases}$$

Find $f(y|x)$.

Solution: From the definition of the conditional probability density function of Y from equation 8.11, we have

$f_{XY}(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}$ where $f_X(x)$ is the marginal density function of X.

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy$$

$$=\int_{-x}^{x}\frac{1}{8}x(x-y)dy$$

$$=\frac{1}{8}\left(x^2y-x\frac{y^2}{2}\right)_{-x}^{x}=\frac{1}{8}\left(x^3-\frac{x^3}{2}+x^3-\frac{x^3}{2}\right)=\frac{x^3}{8}, 0<x<2$$

$$f_{XY}(y|x)=\frac{\frac{1}{8}x(x-y)}{\frac{x^3}{8}}, 0<x<2; -x<y<x$$

$$f(y|x)=\begin{cases}\frac{x-y}{x^2} & -x<y<x\\ 0 & otherwise\end{cases}$$

4. The joint pdf of the random variable (X, Y) is given by

$f(x,y) = Kxye^{-(x^2+y^2)}; x > 0, y > 0.$

Find the value of K and prove that X and Y are independent.

Solution: Here the range space is the entire first quadrant of the xy-plane **By the** property of the joint pdf, we have

$\int_{x>0}\int_{y>0} Kxye^{-(x^2+y^2)}dxdy = 1$

$K\int_0^\infty\int_0^\infty ye^{-y^2}\cdot xe^{-x^2}dxdy = 1$

Put $x^2 = t$; so $2xdx = dt$

Then $\frac{K}{2}\int_0^\infty ye^{-y^2}\cdot\int_0^\infty e^{-t}dtdy = 1$

$\frac{K}{2}\int_0^\infty ye^{-y^2}\left(\frac{e^{-t}}{-1}\right)_0^\infty dy = 1$

$\frac{K}{2}\int_0^\infty ye^{-y^2}(1)dy = 1$

Put $y^2 = v$; so $2ydy = dv$

Then $\frac{K}{2}\cdot\frac{1}{2}\int_0^\infty e^{-v}dv = 1$

$\frac{K}{2}\cdot\frac{1}{2}\left(\frac{e^{-v}}{-1}\right)_0^\infty = 1$

$\frac{K}{2}\cdot\frac{1}{2}(1) = 1$

$\frac{K}{4} = 1 \Rightarrow K = 4$

To prove that X and Y are independent, we should show that $f(x,y) = f_X(x) f_Y(y)$ from equation 8.15. So we need to compute the marginal density of X and Y, i.e. $f_Y(x)$ and $f_Y(y)$

$$
\begin{aligned}
f_X(x) &= \int_0^\infty f(x,y)dy \\
&= 4x \int_0^\infty ye^{-(x^2+y^2)}dy \\
&= 4xe^{-x^2} \int_0^\infty ye^{-y^2} dy \\
&= 4xe^{-x^2} \int_0^\infty e^{-t} \frac{1}{2} dt \\
&= 2xe^{-x^2} \left[\frac{e^{-t}}{-1}\right]_0^\infty \\
&= 2xe^{-x^2}, x > 0
\end{aligned}
$$

Similarly

$$
\begin{aligned}
f_Y(y) &= \int_0^{\infty} f(x,y)dx \\
&= 4y\int_0^{\infty} ye^{-(x^2+y^2)}dx \\
&= 4ye^{-y^2}\int_0^{\infty} xe^{-x^2}dx \\
&= 4ye^{-y^2}\int_0^{\infty} e^{-t}\frac{1}{2}dt \\
&= 2ye^{-y^2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} \\
&= 2ye^{-y^2}, y > 0
\end{aligned}
$$

Now, $f_X(x)\cdot f_Y(y) = 2xe^{-x^2}\cdot 2ye^{-y^2} = 4xye^{-(x^2+y^2)} = f_{XY}(x,y)$. **Hence, X and Y** are independent.