Classification of Stochastic Process

For a fixed time $t=t_1$, the term $X(t_1)$ is a simple r.v, ~~the prob. of the event $[X(t_1) \le x_1]$ gives the CDF of the r.v $X(t_1)$, denoted by~~ that describes the state of the process at time t_1. For a fixed x_1, the probability of the event $[X(t_1) \le x_1]$ gives the CDF of the r.v $X(t_1)$, denoted by

$$F(x_1; t_1) = F(x_1) = P[X(t_1) \le x_1].$$

$F(x_1; t_1)$ is known as the I-order distr. of the process $\{X(t) \mid t \ge 0\}$.

Given two time instants t_1 and t_2, $X(t_1)$ and $X(t_2)$ are two r.vs on the same prob. space. Their joint distr. is known as the II-order distr. of the process and is given by

$$F(x_1, x_2; t_1, t_2) = P[X(t_1) \le x_1, X(t_2) \le x_2]$$

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In general, We define the nth-order distr. of the stochastic ~~order~~ process $X(t), t \in T$ by

$$F(x;t) = P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n] \quad -(1)$$

for all $x = (x_1, \ldots, x_n) \in R^n$ and $t = (t_1, t_2, \ldots, t_n) \in T^n$ such that $t_1 < t_2 < \ldots < t_n$. Such a complete description of a process is no small task.

For instance, the nth-order joint distribution fn. is often found to be invarient under shifts of the time origin. Such a process is said to be a <u>strict-sense stationary stochastic process</u>.

<u>Defn: (Strictly Stochastic Process)</u>.

A stochastic process $\{X(t) | t \in T\}$ is said to be stationary in the strict sense if for $n \geq 1$, its n-order joint CDF satisfies the condition:

$$F(x; t) = F(x; t+\tau)$$

for all vectors $x \in \mathbb{R}^n$ and $t \in T$ and all scalars τ such that

$t_i + \tau \in T$.

$t+\tau \Rightarrow$ scalar τ is added to all components of vector t.

We let $\mu(t) = E[X(t)]$ denote the time-dependent mean of the stochastic process.

$\mu(t) =$ ensemble average of the S.P.

Defn. (Independent process).

A S.P. $\{X(t) \mid t \in T\}$ is said to be an <u>independent process</u> provided its nth-order joint distr. satisfies the condition:

$$F(x;t) = \prod_{i=1}^{n} F(x_i;t_i)$$

$$= \prod_{i=1}^{n} P[X(t_i) \le x_i] \quad -(2)$$

Defn. (Renewal process)

A renewal process is defined as a discrete-time independent process $\{X_n | n=1,2,\ldots\}$ where $X_1, X_2, \ldots$ are i.i.d r.v.

Ex. Consider a system in which the repair after a failure is performed, require negligible time. Now, the times b/w successive failures might be i.i.d r.v $\{X_n | n=1,2,\ldots\}$ of a renewal process.

If we force to consider some dependence among these r.Vs, then it is Markov dependence.

Defn (Markov Process)

A s.p. $\{X(t) \mid t \in T\}$ is called a **Markov process** if for any $t_0 < t_1 < t_2 < \cdots < t_n < t$, the conditional distr. of $X(t)$ for given values of $X(t_0), X(t_1), \ldots, X(t_n)$ depends only on $X(t_n)$:

$$P[X(t) \le x \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0]$$

$$= P[X(t) \le x \mid X(t_n) = x_n] \quad -(3)$$

This defn. applies to Markov processes with continuous state ~~processes~~ space. Discrete-state Markov processes.

Stationary Random processes

Stationarity refers to time invariance of some, or all, of the statistics of a random process, such as mean, autocorrelation, n-th order distr.

Two types of stationarity:
① Strict sense (SSS) and ② wide sense (WSS).

Auto Correlation (Serial Correlation)

is a correlation of a signal with a delayed copy of itself as a fn. of delay. Informally, it is the similarity b/w observations as a fn. of the time lag b/w them. Used for finding repeated patterns.

The autocorrelation of a SP is the pearson correlation b/w values of the process at different times, as a fn. of the two times or of the time lag. Let $\{X_t\}$ be a sp, and t be the point in time (discrete or continuous). Then X_t is the value produced by a given run of the process at time t. Suppose that the process has mean μ_t and variance σ_t^2 at time t.

Then, the auto-correlation fn. b/w times t_1 and t_2 is:

$$R(t_1, t_2) = E[X_{t_1} \bar{X}_{t_2}]$$

bar = complex conjugation.

Correlation: (or dependence)

is any statistical relationship, whether casual or not, b/w two r.v. Ex: Corr. b/w physical statures of parents and their offsprings, Corr. b/w the price of a goods and the quantity of consumers willing to purchase, that can be depicted in a demand-curve.

Correlation Coefficient $\rho_{X,Y}$ b/w two r.Vs X and Y with μ_X and μ_Y and σ_X and σ_Y is

$$\rho_{X,Y} = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

Covariance: measure of the joint variability of 2 r.vs. If the 2 r.Vs tend to show similar behavior, the Covariance is +ve, else if they tend to show opposite behavior, the Covariance is -ve.

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY] - E[X]E[Y]$$

A r.p. $X(t)$ or X_n is said to be SSS if all its first order distributions are time invariant, i.e, the joint CDFs of

$X(t_1), X(t_2), \ldots, X(t_n)$ and

$X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau)$

are ~~all~~ the same for all n, all $t_1, t_2, \ldots, t_n$ and all time shifts τ.

So, for a SSS process, the I-order distr. is independent of t, and II-order distr. - the distr. of any two samples $X(t_1)$ and $X(t_2)$ - depends only on $\gamma = t_2 - t_1$.

A measure of dependence among the random variables of a sp is provided by its autocorrelation fn. R, defined by

$$R(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

$$\text{cov}[X(t_1), X(t_2)] = R(t_1, t_2) - \mu(t_1) \cdot \mu(t_2).$$

Defn (Wide-Sense Stationary processes)

A sp is considered WSS if.

1. $\mu(t) = E[X(t)]$ is independent of t.
2. $R(t_1, t_2) = R(0, t_2 - t_1) = R(\gamma)$.
3. $R(0) = E[X^2(t)] < \infty$.

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or, A r.p $X(t)$ is said to be WSS if its mean and autocorrelation are time invarient, i.e,

1. $E[X(t)] = \mu$
2. $R(t_1, t_2)$ is a fn. only of the time difference $t_2 - t_1$.

Since $R_X(t_1, t_2) = R_X(t_2, t_1)$, for any WSS process $X(t)$, $R_X(t_1, t_2)$ is a fn. only of $|t_2 - t_1|$

Cleary, SSS $\Rightarrow$ WSS, but conversly not necessarily true.

Ex: Let

$$X(t) = \begin{cases} +\sin t \\ -\sin t \\ +\cos t \\ -\cos t \end{cases} \text{ with prob} = 1/4$$

* $E[X(t)] = 0$ and $R_X(t_1, t_2) = \frac{1}{2}\cos(t_2 - t_1)$, thus $X(t)$ is WSS.

* But $X(0)$ and $X(\frac{\pi}{4})$ do not have the same pmf (different ranges), so the I-order pmf is not stationary.