Predicate Calculus

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Introduction

http://www.youtube.com/watch?v=3PycZtfns_U
http://www.youtube.com/watch?v=eWMtUDJQfYs

Functions

A function is a mapping, or relationship between elements from two sets.
A function maps values in one set (the domain) to values in the other (the range). By definition, a function maps each element in its domain to at most one element in its range.
$f : A \rightarrow B$: function f with domain as the set A and range as the set B.

Functions

$sqrt : R^+ \longrightarrow R^+$
$sqrt.16 = 4$

$max2 : Z \times Z \longrightarrow Z$
$max2.(4, 1) = 4$
Or use currying.

Function currying

$f.x.y \equiv (f.x).y$
The result of applying f to x must be a function, which is then applied to y.

Redefine $max2(a, b)$ as $max.a.b$
What is the type of max?
Can you graph $max.4$?

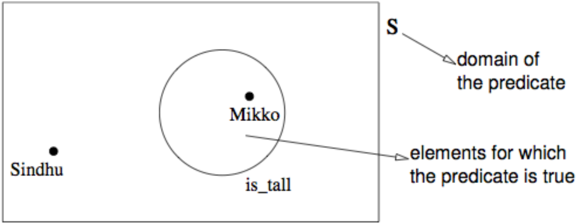
Booleans

Boolean refers to set $\{True, False\}$
Some operations on booleans:

- conjunction ($\wedge$, pronounced and)
- disjunction ($\vee$, pronounced or)
- negation ($\neg$, pronounced not)
- equivalence ($\equiv$, pronounced equivals)
- implication ($\Rightarrow$, pronounced implies)

Predicates

A predicate is function whose range is boolean.
$P : S \longrightarrow \text{boolean}$

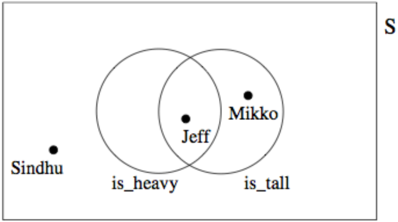


Predicates

In computer science, the domain of a predicate is quite often the state space of a program. For example, in the context of a program with two variables, x and y, both integers, we might write expressions such as $even.x$, $prime.y$, and $x + y = 3$.

These are all boolean expressions that are either true or false for each point in the state space.

Lifting



$\wedge$: boolean X boolean $\longrightarrow$ boolean
$\wedge$: predicate X predicate $\longrightarrow$ predicate

(isTall $\wedge$ isHeavy).Jeff = ?
(isTall $\wedge$ isHeavy).Mikko = ?
(isTall $\wedge$ isHeavy).Sindhu = ?

Everywhere brackets

(isTall $\equiv$ is Heavy) is a predicate, and can be evaluated to T or F for each point in the domain space.

But what if we wanted to talk about the equivalence of the two predicates themselves and state the claim “Being tall is the same as being heavy” ? (everywhere is implicit).

We denote this as: [isTall $\equiv$ isHeavy]

Predicate Calculus

Equivalence

$sqrt.16 = 4 = 22$ is called chaining and is a short-hand for writing: $(\text{sqrt}.16 = 4) \wedge (4 = 22)$

Chaining does not apply to $\equiv$. Evaluate the following: $\text{false} \equiv \text{true} \equiv \text{false}$

Equivalence

Axiom 1. Associativity of $\equiv$:
$[((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))]$

Axiom 2. Commutativity of $\equiv$:
$[X \equiv Y \equiv Y \equiv X]$

Axiom 3. Definition of true : $[Y \equiv Y \equiv \mathit{true}]$

Disjunction

$\vee$ binds more tightly than $\equiv$

Axiom 4. Associativity of $\vee$.
$[X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z]$

Axiom 5. Commutativity of $\vee$.
$[X \vee Y \equiv Y \vee X]$

Axiom 6. Idempotence of $\vee$.
$[X \vee X \equiv X]$

Axiom 7. Distribution of $\vee$ over $\equiv$.
$[X \vee (Y \equiv Z) \equiv (X \vee Y) \equiv (X \vee Z)]$

Proof format

Consider a proof of $[A \equiv B]$:

$$
\begin{array}{l}
A \\
\equiv \{ \text{ reason why } [A \equiv C] \ \} \\
C \\
\equiv \{ \text{ reason why } [C \equiv B] \ \} \\
B \\
\quad \square
\end{array}
$$

Proof example

Theorem 1. true is the zero of $\vee$.

$[X \vee \mathit{true} \equiv \mathit{true}]$
Proof.
$X \vee \mathit{true}$
$\equiv$ { definition of true }
$X \vee (Y \equiv Y)$
$\equiv$ { distribution of $\vee$ over $\equiv$ }
$(X \vee Y) \equiv (X \vee Y)$
$\equiv$ { definition of true, with Y as $X \vee Y$ }
true

Conjunction

$\wedge$ has the same binding as $\vee$

Axiom 10. Golden rule (also definition of $\wedge$).
$[X \vee Y \equiv X \equiv Y \equiv X \wedge Y]$

Implication

$\Rightarrow$ has lower binding than $\vee$ and higher than $\equiv$

Axiom 11. Definition of $\Rightarrow$.
$[X \vee Y \equiv Y \equiv X \Rightarrow Y]$

Proof example

Theorem 2. true is the identity of $\wedge$.
$[X \wedge true \equiv X]$

Proof example

Theorem 2. $[X \wedge true \equiv X]$

Proof.

$X \wedge true$

$\equiv$ { golden rule }

$X \vee true \equiv true \equiv X$

$\equiv$ { Theorem 1 }

$true \equiv true \equiv X$

$\equiv$ { definition of true, with Y as true }

$true \equiv X$

$\equiv$ { definition of true, with Y as X }

X

Negation

Axiom 12. Law of Excluded Middle.
$[\neg X \vee X]$

Axiom 13. Distribution of $\neg$ over $\equiv$.
$[\neg(X \equiv Y) \equiv X \equiv \neg Y]$

Axiom 14. Definition of false.
$[\neg \mathit{true} \equiv \mathit{false}]$

Quantification

Syntax

A quantification has the form: $(Qi : r.i : t.i)$

- Q is the "operator"
- i is the "bound variable",
- $r.i$ is the "range", and
- $t.i$ is the "term".

For a bound variable of type T, the range is a predicate on T, the term is an expression of some type D, and the quantifier an operation of type $D \times D \rightarrow D$.

Interpretation

If the set of values for which $r.i$ holds is $\{i0, i1, i2, \ldots, iN\}$
$uQt.i0Qt.i1Qt.i2Q \ldots Qt.iN$
where u is the identity element of the operator Q.

$(*i : 1 \leq i \leq 4 : i)$

$(\wedge n : n \geq 0 : even.n)$

Common quantifiers

operator	quantification symbol
$\wedge$	$\forall$
$\vee$	$\exists$
$+$	$\sum$
$\times$	$\prod$

Empty range

Axiom 16. Empty range.
$(Qi : false : t.i) = u$

Example $\prod_{i=4}^{1} i$