

9.ex.24

EE24BTECH11059 - Y Siddhanth

Question:

Find the solution of the given differential equation

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

Solution:

Theoretical Solution:

To apply \mathcal{L} -Transform to the above equation, we define:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} = s^2Y(s) - sy(0) - y'(0) \quad (0.1)$$

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = sY(s) - y(0) \quad (0.2)$$

$$\mathcal{L}\{y(x)\} = Y(s) \quad (0.3)$$

Taking the \mathcal{L} -Transform of the equation:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 2a\mathcal{L}\left\{\frac{dy}{dx}\right\} + (a^2 + b^2)\mathcal{L}\{y\} = 0 \quad (0.4)$$

$$(s^2Y(s) - sy(0) - y'(0)) - 2a(sY(s) - y(0)) + (a^2 + b^2)Y(s) = 0 \quad (0.5)$$

$$(s^2 - 2as + (a^2 + b^2))Y(s) = (sy(0) + y'(0) - 2ay(0)) \quad (0.6)$$

Thus, $Y(s)$ can be written as,

$$Y(s) = \frac{(sy(0) + y'(0) - 2ay(0))}{(s^2 - 2as + (a^2 + b^2))} \quad (0.7)$$

$$(0.8)$$

We get,

$$Y(s) = \frac{sy(0) + y'(0) - 2ay(0)}{(s - a)^2 + b^2} \quad (0.9)$$

$$(0.10)$$

To apply partial fractions, we can rewrite the above as:

$$Y(s) = \frac{A(s - a) + B}{(s - a)^2 + b^2}, \quad (0.11)$$

Using the inverse Laplace transform formulas:

$$\mathcal{L}^{-1} \left\{ \frac{s-a}{(s-a)^2 + b^2} \right\} = e^{ax} \cos(bx)u(x), \quad (0.12)$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-a)^2 + b^2} \right\} = \frac{1}{b} e^{ax} \sin(bx)u(x), \quad (0.13)$$

Thus, general solution is then:

$$y(x) = e^{ax}(C_1 \cos(bx) + C_2 \sin(bx))u(x), \quad (0.14)$$

Taking initial conditions as $(0, 1)$, $\frac{dy}{dx} = 1$ and $a = 1, b = 1$, we get

$$y(x) = e^x \cos(x)u(x) \quad (0.15)$$

Numerical Solution:

We have to apply the trapezoidal rule,

$$J = \int_a^b f(x) dx \quad (0.16)$$

$$\approx h \left(\frac{1}{2} f(a) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2} f(b) \right) \quad (0.17)$$

$$(0.18)$$

Discretizing the steps using trapezoidal rule for $y'' = f(x, y, y') = 2y' - 2y$ gives us

$$y'_{n+1} = y'_n + \frac{h}{2} (f(x_n, y_n, y'_n) + f(x_{n+1}, y_{n+1}, y'_{n+1})) \quad (0.19)$$

$$y'_{n+1} = y'_n + \frac{h}{2} (2y'_{n+1} - 2y_{n+1} + 2y'_n - 2y_n) \quad (0.20)$$

$$y_{n+1} = y_n + \frac{h}{2} (y'_{n+1} + y'_n) \quad (0.21)$$

$$(0.22)$$

Solving (0.20), (0.24),

$$y'_{n+1} = y'_n \left(\frac{1+h-\frac{h^2}{2}}{1-h+\frac{h^2}{2}} \right) - \left(\frac{2h}{1-h+\frac{h^2}{2}} \right) y_n \quad (0.23)$$

$$y_{n+1} = y_n + \frac{h}{2} \left(y'_n \left(\frac{1+h-\frac{h^2}{2}}{1-h+\frac{h^2}{2}} \right) - \left(\frac{2h}{1-h+\frac{h^2}{2}} \right) y_n + y'_n \right) \quad (0.24)$$

$$(0.25)$$

By applying the above 2 equations iteratively, we can plot the curve.

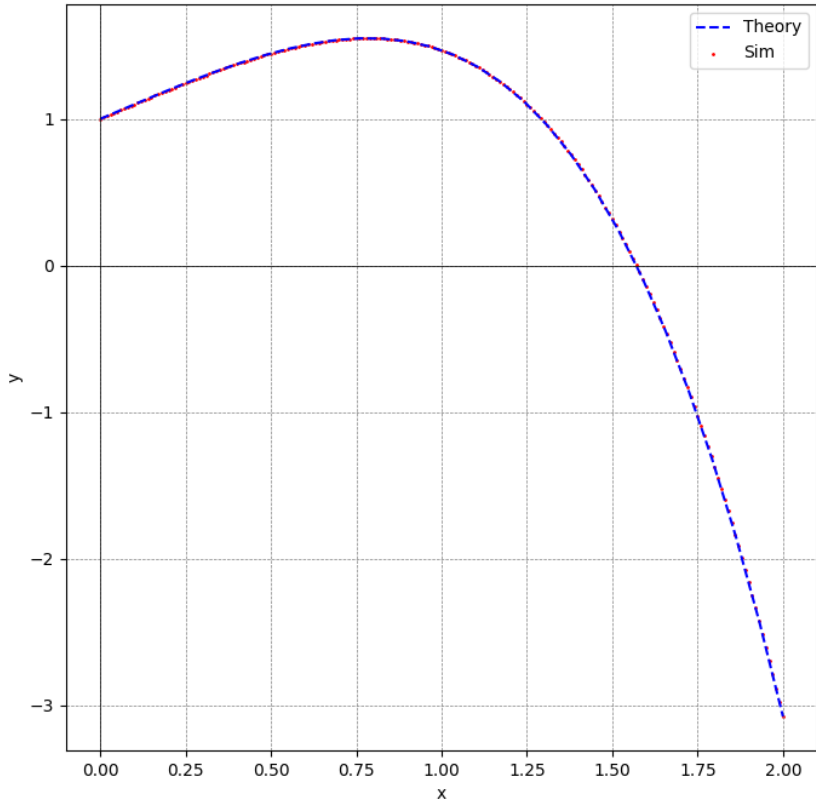


Fig. 0.1: Comparison between the Theoretical solution and Numerical solution