9.ex.24

EE24BTECH11059 - Y Siddhanth

Question:
Find the solution of the given differential equation

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

.
Solution:
Theoretical Solution:
To apply $\mathcal{L}$-Transform to the above equation, we define:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} = s^2Y(s) - sy(0) - y'(0) \tag{0.1}$$

$$\mathcal{L}\left\{\frac{dy}{dx}\right\} = sY(s) - y(0) \tag{0.2}$$

$$\mathcal{L}\{y(x)\} = Y(s) \tag{0.3}$$

Taking the $\mathcal{L}$-Transform of the equation:

$$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 2a\mathcal{L}\left\{\frac{dy}{dx}\right\} + (a^2 + b^2)\mathcal{L}\{y\} = 0 \tag{0.4}$$

$$(s^2Y(s) - sy(0) - y'(0)) - 2a(sY(s) - y(0)) + (a^2 + b^2)Y(s) = 0 \tag{0.5}$$

$$(s^2 - 2as + (a^2 + b^2))Y(s) = (sy(0) + y'(0)) - 2ay(0)) \tag{0.6}$$

Thus, $Y(s)$ can be written as,

$$Y(s) = \frac{(sy(0) + y'(0)) - 2ay(0))}{(s^2 - 2as + (a^2 + b^2))} \tag{0.7}$$

$$\tag{0.8}$$

We get,

$$Y(s) = \frac{sy(0) + y'(0) - 2ay(0)}{(s-a)^2 + b^2} \tag{0.9}$$

$$\tag{0.10}$$

To apply partial fractions, we can rewrite the above as:

$$Y(s) = \frac{A(s-a) + B}{(s-a)^2 + b^2} \tag{0.11}$$

Using the inverse Laplace transform formulas:

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{ax}\cos(bx)u(x), \tag{0.12}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{ax}\sin(bx)u(x), \tag{0.13}$$

Thus, general solution is then:

$$y(x) = e^{ax}(C_1\cos(bx) + C_2\sin(bx))u(x), \tag{0.14}$$

Taking initial conditions as $(0, 1)$, $\frac{dy}{dx} = 1$ and $a = 1, b = 1$, we get

$$y(x) = e^{x}\cos(x)u(x) \tag{0.15}$$

Numerical Solution:
We have to apply the trapezoidal rule,

$$J = \int_a^b f(x)\,dx \tag{0.16}$$

$$\approx h\left(\frac{1}{2}f(a) + f(x_1) + f(x_2)\cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \tag{0.17}$$

$$\tag{0.18}$$

Discretizing the steps using trapezoidal rule for $y'' = f(x, y, y') = 2y' - 2y$ gives us

$$y'_{n+1} = y'_n + \frac{h}{2}\left(f(x_n, y_n, y'_n) + f(x_{n+1}, y_{n+1}, y'_{n+1})\right) \tag{0.19}$$

$$y'_{n+1} = y'_n + \frac{h}{2}\left(2y'_{n+1} - 2y_{n+1} + 2y'_n - 2y_n\right) \tag{0.20}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(y'_{n+1} + y'_n\right) \tag{0.21}$$

$$\tag{0.22}$$

Solving (0.20), (0.24),

$$y'_{n+1} = y'_n\left(\frac{1+h-\frac{h^2}{2}}{1-h+\frac{h^2}{2}}\right) - \left(\frac{2h}{1-h+\frac{h^2}{2}}\right)y_n \tag{0.23}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(y'_n\left(\frac{1+h-\frac{h^2}{2}}{1-h+\frac{h^2}{2}}\right) - \left(\frac{2h}{1-h+\frac{h^2}{2}}\right)y_n + y'_n\right) \tag{0.24}$$

$$\tag{0.25}$$

By applying the above 2 equations iteratively, we can plot the curve.
Alternatively, we can using the bilinear transform on (0.11) to find a more accurate

difference equation.

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \tag{0.26}$$

$$Y(s) = \frac{s-1}{(s-1)^2+1} \tag{0.27}$$

$$Y(z) = \frac{\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} - 1}{(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} - 1)^2 + 1} \tag{0.28}$$

$$\tag{0.29}$$

Simplifying it,we get:

$$Y(z) = \frac{2\left(\left(h - \frac{h^2}{2}\right) - h^2 z^{-1} + \left(-h - \frac{h^2}{2}\right) z^{-2}\right)}{\left((2-h)^2 + h^2\right) + 2z^{-1}\left(2h^2 - 4\right) + \left((2+h)^2 + h^2\right) z^{-2}} \tag{0.30}$$

$$\left((2-h)^2 + h^2\right) Y(z) + 2z^{-1} Y(z) \left(2h^2 - 4\right) + \left((2+h)^2 + h^2\right) z^{-2} Y(z) \tag{0.31}$$

$$= 2\left(\left(h - \frac{h^2}{2}\right) - h^2 z^{-1} + \left(-h - \frac{h^2}{2}\right) z^{-2}\right) \tag{0.32}$$

Applying inverse-Z transform, we get

$$y_n = -2y_{n-1} \cdot \frac{2h^2 - 4}{(2-h)^2 + h^2} - y_{n-2} \cdot \frac{(2+h)^2 + h^2}{(2-h)^2 + h^2} \tag{0.33}$$

$$+ \frac{2\left(\left(h - \frac{h^2}{2}\right)\delta[n] - h^2\delta[n-1] + \left(-h - \frac{h^2}{2}\right)\delta[n-2]\right)}{(2-h)^2 + h^2} \tag{0.34}$$

Plotting Bilinear Transform(Sim2) and Trapezoidal Rule(Sim1), we get

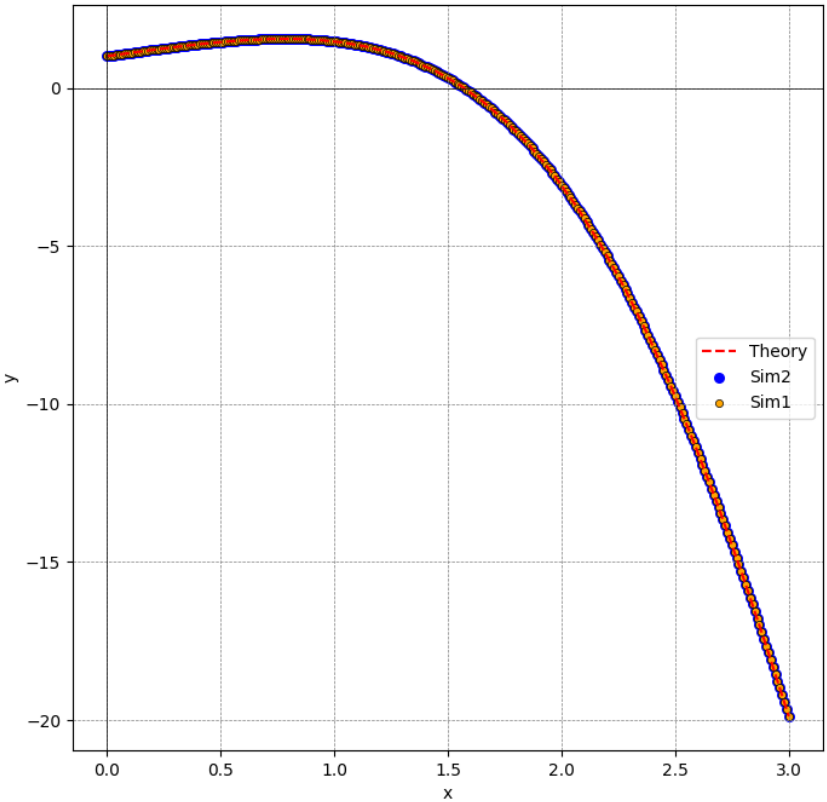

Fig. 0.1: Comparison between the Theoretical solution and Numerical solution