3.1.2

EE24BTECH11059 - Y Siddhanth

Question:
The coach of a cricket team buys 3 bats and 6 balls for ₹3900. Later, she buys another bat and 3 more balls of the same kind for ₹1300. Find the price of the ball and the bat using LU factorization.
Solution:
First, we rewrite the question as a system of linear equations.

$$x_1 \implies \text{bat} \tag{0.1}$$

$$x_2 \implies \text{ball} \tag{0.2}$$

$$3x_1 + 6x_2 = 3900 \tag{0.3}$$

$$x_1 + 3x_2 = 1300 \tag{0.4}$$

Now, converting into a matrix form, we get:

$$\begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 1300 \end{pmatrix} \tag{0.5}$$

$$\mathbf{A}x = \mathbf{b} \tag{0.6}$$

To solve the above equation, we have to apply LU - Factorization to matrix **A**.
We do so because,

$$A \to LU \tag{0.7}$$

$$L \to \text{Lower Triangular Matrix} \tag{0.8}$$

$$U \to \text{Upper Triangular Matrix} \tag{0.9}$$

Let $y = \mathbf{U}x$, then we can rewrite the above equation as,

$$\mathbf{A}x = \mathbf{b} \implies \mathbf{LU}x = \mathbf{b} \implies \mathbf{L}y = \mathbf{b} \tag{0.10}$$

Now, the above equation can be solved using front-substitution since **L** is lower triangular, thus we get the solution vector y.
Using this we solve for x in $y = \mathbf{U}x$ using back-substitution knowing that **U** is upper triangular. LU Factorizing **A**, we get:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \tag{0.11}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix} \tag{0.12}$$

$$\mathbf{U} = \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \tag{0.13}$$

The solution can now be obtained as:

$$\begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 1300 \end{pmatrix} \tag{0.14}$$

Solving for y, we clearly get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 0 \end{pmatrix} \tag{0.15}$$

Now, solving for x via back substitution,

$$\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 0 \end{pmatrix} \tag{0.16}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1300 \\ 0 \end{pmatrix} \tag{0.17}$$

Thus, the price of the bat (x_1) and the ball (x_2) are obtained.
The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:
Elements of the U Matrix:
For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{0.18}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{0.19}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{0.20}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{0.21}$$

This systematic approach ensures that the matrix A is decomposed into L and U without requiring row swaps, provided A is nonsingular.

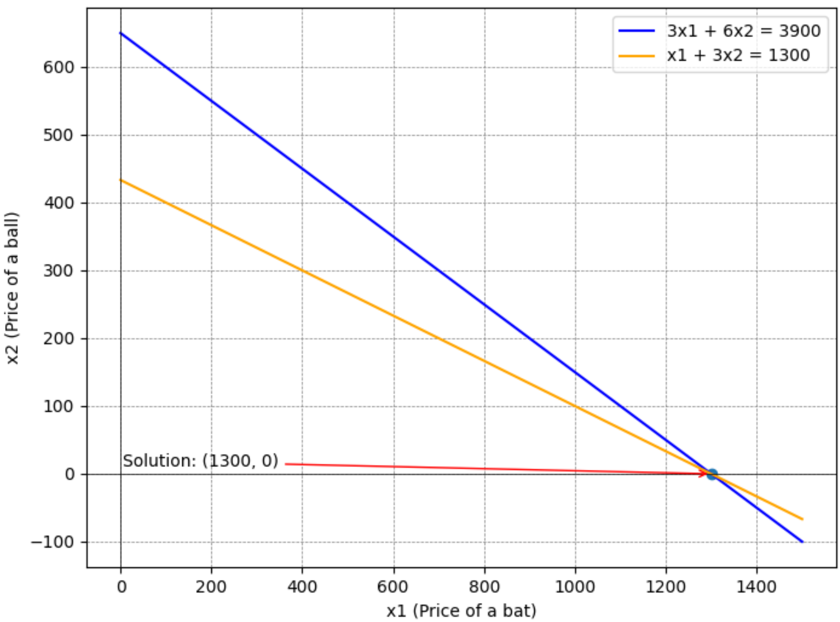

Fig. 0.1: Solution of the system of linear equations