

8.2.5

EE24BTECH11059 - Y Siddhanth

Question:

Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.

Solution:

Theoretical Solution:

First, we find the intersection points, clearly $x = 4$ is the intersection point for 2 sides.

$$-3x + y = 1 \quad (0.1)$$

$$-2x + y = 1 \quad (0.2)$$

Taking an augmented matrix to solve the above equations,

$$\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \quad (0.3)$$

Using row operations to reduce it,

$$\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - \frac{2}{3}R_1} \begin{pmatrix} -3 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \xrightarrow[R_1 \rightarrow -\frac{1}{3}R_1]{R_2 \rightarrow 3R_2} \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (0.4)$$

Thus, the intersection of the lines $(-3x + y = 1, -2x + y = 1)$ is $(0, 1)$

To find the area, we integrate the difference between the two lines from their intersection point at $x = 0$ to $x = 4$:

$$Area = \int_0^4 ((3x + 1) - (2x + 1))dx \quad (0.5)$$

$$= \int_0^4 x dx \quad (0.6)$$

$$= \left[\frac{1}{2}x^2 \right]_0^4 \quad (0.7)$$

$$= \frac{1}{2}(4^2) - \frac{1}{2}(0^2) \quad (0.8)$$

$$= 8 \quad (0.9)$$

Thus, the area of the triangular region is 8 square units.

Numerical Solution:

The numerical solution for this differential equation can be found using the trapezoidal

rule (0.10).

$$A = \int_a^b f(x) dx \approx \frac{h}{2} [f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)] \quad (0.10)$$

Here, $h = \frac{b-a}{n}$, and n is the number of nodes where x_j and a, b are called nodes.

$$f(x) = (3x + 1) - (2x + 1) = x \quad (0.11)$$

$$A = \int_a^b x dx \quad (0.12)$$

$$\approx \frac{h}{2} [a + 2x_1 + 2x_2 + \cdots + 2x_{n-1} + b] \quad (0.13)$$

Clearly, a, x_j, b are in an A.P, where $x_i = a + ih$. Thus,

$$A \approx \frac{h}{2} [a + (2a + h) + (2a + 4h) + \cdots + (2a + 2(n-1)h) + (a + nh)] \quad (0.14)$$

$$\approx \frac{h}{2} [2na + (n-1)nh] \quad (0.15)$$

Apply definition of h and values of a, b to the above equation, we get

$$A \approx \frac{4}{2n} [2n(0) + (n-1)(4)] \quad (0.16)$$

$$A \approx \frac{8(n-1)}{n} \quad (0.17)$$

Clearly, as n gets larger or the number of nodes increase, the value of area approaches the theoretical value. Taking n as 100000,

$$A \approx 7.99992 \quad (0.18)$$

$$A \approx 8 \quad (0.19)$$

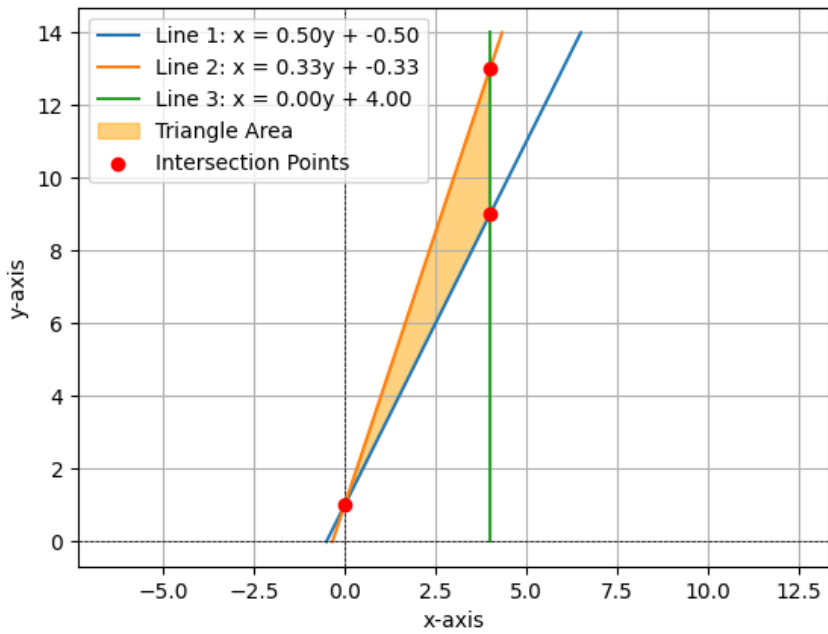


Fig. 0.1: Comparison between the Theoretical solution and Numerical solution