NCERT: 10.3.1.2

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Problem Statement

The coach of a cricket team buys 3 bats and 6 balls for Rs.3900. Later, she buys another bat and 3 more balls of the same kind for Rs.1300. Find the price of the ball and the bat using LU factorization.

Converting to Matrix Form

First, we rewrite the question as a system of linear equations.

$$x_1 \implies \text{bat} \tag{3.1}$$
$$x_2 \implies \text{ball} \tag{3.2}$$

$$3x_1 + 6x_2 = 3900 \tag{3.3}$$
$$x_1 + 3x_2 = 1300 \tag{3.4}$$

Now, converting into a matrix form, we get:

$$\begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 1300 \end{pmatrix} \tag{3.5}$$
$$\mathbf{A}x = \mathbf{b} \tag{3.6}$$

LU-Decomposition

To solve the above equation, we have to apply LU - Factorization to matrix **A**.
We do so because,

$$A \rightarrow LU \tag{3.7}$$
$$L \rightarrow \text{Lower Triangular Matrix} \tag{3.8}$$
$$U \rightarrow \text{Upper Triangular Matrix} \tag{3.9}$$

Let $y = \mathbf{U}x$, then we can rewrite the above equation as,

$$\mathbf{A}x = \mathbf{b} \implies \mathbf{LU}x = \mathbf{b} \implies \mathbf{L}y = \mathbf{b} \tag{3.10}$$

Now, the above equation can be solved using front-substitution since **L** is lower triangular, thus we get the solution vector y.

LU-Decomposition

Using this we solve for x in $y = \mathbf{U}x$ using back-substitution knowing that $\mathbf{U}$ is upper triangular. LU Factorizing $\mathbf{A}$, we get:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \tag{3.11}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix} \tag{3.12}$$

$$\mathbf{U} = \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \tag{3.13}$$

The solution can now be obtained as:

$$\begin{pmatrix} 1 & 0 \\ 0.33 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 1300 \end{pmatrix} \tag{3.14}$$

Solving for y, we clearly get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 0 \end{pmatrix} \tag{3.15}$$

LU-Decomposition

Now, solving for x via back substitution,

$$\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3900 \\ 0 \end{pmatrix} \tag{3.16}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1300 \\ 0 \end{pmatrix} \tag{3.17}$$

Thus, the price of the bat (x_1) and the ball (x_2) are obtained.

Doolittle's Algrorithm

The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:

Elements of the U Matrix:

For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 1, \tag{3.18}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 1. \tag{3.19}$$

Elements of the L Matrix:

For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 1, \tag{3.20}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 1. \tag{3.21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{3.22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \tag{3.23}$$