8.2.5

EE24BTECH11059 - Y Siddhanth

Question:
Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$, and $x = 4$.
Solution:
Theoretical Solution:
First, we find the intersection points, clearly x = 4 is the intersection point for 2 sides.

$$-3x + y = 1 \tag{0.1}$$
$$-2x + y = 1 \tag{0.2}$$

Taking an augmented matrix to solve the above equations,

$$\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \tag{0.3}$$

Using row operations to reduce it,

$$\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - \frac{2}{3}R_1]{} \begin{pmatrix} -3 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \xrightarrow[R_1 \to -\frac{1}{3}R_1]{R_2 \to 3R_2} \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{0.4}$$

Thus, the intersection of the lines $(-3x + y = 1, -2x + y = 1)$ is $(0, 1)$
To find the area, we integrate the difference between the two lines from their intersection point at $x = 0$ to $x = 4$:

$$Area = \int_0^4 ((3x + 1) - (2x + 1))dx \tag{0.5}$$
$$= \int_0^4 x dx \tag{0.6}$$
$$= \left[\frac{1}{2}x^2\right]_0^4 \tag{0.7}$$
$$= \frac{1}{2}(4^2) - \frac{1}{2}(0^2) \tag{0.8}$$
$$= 8 \tag{0.9}$$

Thus, the area of the triangular region is 8 square units.
Numerical Solution:
The numerical solution for this question can be found using the trapezoidal rule (0.10).

$$A = \int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right] \tag{0.10}$$

Here, $h = \frac{b-a}{n}$, and n is the number of nodes where x_j and a, b are called nodes.

$$f(x) = (3x+1) - (2x+1) = x \tag{0.11}$$

$$A = \int_a^b x\,dx \tag{0.12}$$

$$\approx \frac{h}{2}\left[a + 2x_1 + 2x_2 + \cdots + 2x_{n-1} + b\right] \tag{0.13}$$

Clearly, a, x_j, b are in an A.P, where $x_i = a + ih$. Thus,

$$A \approx \frac{h}{2}\left[a + (2a+h) + (2a+4h) + \cdots + (2a + 2(n-1)h) + (a+nh)\right] \tag{0.14}$$

$$\approx \frac{h}{2}\left[2na + (n-1)nh\right] \tag{0.15}$$

Apply definition of h and values of a, b to the above equation, we get

$$A \approx \frac{4}{2n}\left[2n(0) + (n-1)(4)\right] \tag{0.16}$$

$$A \approx \frac{8(n-1)}{n} \tag{0.17}$$

Clearly, as n gets larger or the number of nodes increase, the value of area approaches the theoretical value. Taking n as 100000,

$$A \approx 7.99992 \tag{0.18}$$

$$A \approx 8 \tag{0.19}$$

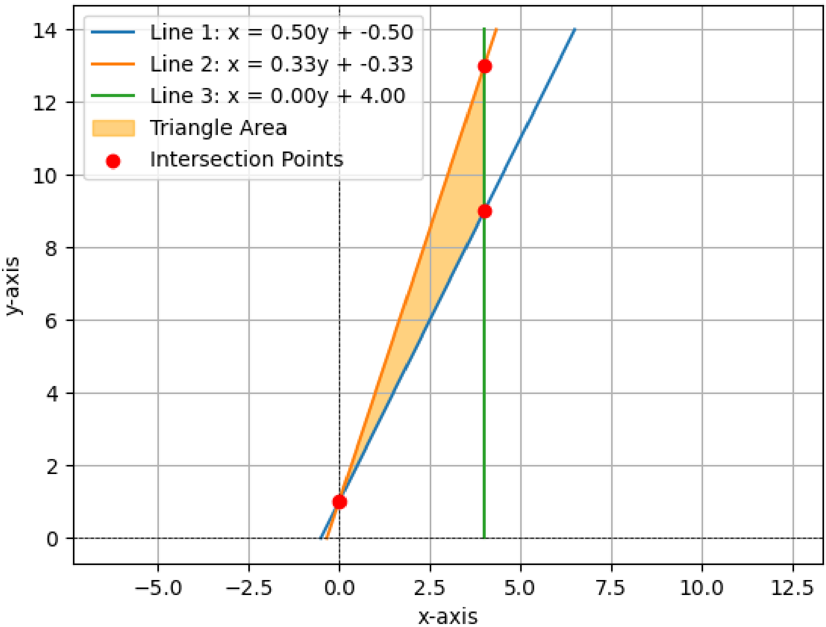

Fig. 0.1: Comparison between the Theoretical solution and Numerical solution