Affine Transformations

EE24BTECH11059 - Y SIDDHANTH

Question: Plot the ellipse whose focus is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, directrix $x-y-3=0$, and $e=\frac{1}{2}$. Plot the corresponding standard ellipse in the same graph using Affine Transformation.

Solution:

Variable	**Description**	**Value**
$\mathbf{n}^\top x = c$	Directrix Equation	$\begin{pmatrix} 1 & -1 \end{pmatrix} x = 3$
e	Eccentricity of conic	$\frac{1}{2}$
F	Focus of conic	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
I	Identity matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
f	A variable in the conic equation	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$

TABLE 0: Variables Used

First we find the V, u, f for the actual ellipse using the following formulas:

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top, \tag{1.1}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{1.2}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{1.3}$$

Upon solving, we get:

$$\mathbf{V} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}, \tag{1.4}$$

$$\mathbf{u} = \frac{5}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.5}$$

$$f = \frac{7}{4} \tag{1.6}$$

For Affine Transformation, we have to spectral / eigen decompose the matrix $\mathbf{V}$.

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{1.7}$$

To find $\mathbf{D}$, we will have to find the eigen values of the matrx $\mathbf{V}$.

$$|\mathbf{V} - \lambda \mathbf{I}| = 0, \tag{1.8}$$

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} \frac{7}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} - \lambda \end{vmatrix} \tag{1.9}$$

$$\lambda_1, \lambda_2 = 2, \frac{3}{2} \tag{1.10}$$

$$\therefore \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \tag{1.11}$$

Note: In (1.11), the order of the eigen values that I have chosen is random but once a certain order has been set for the eigen values, then that order must be followed throughout the procedure.

Now, to find **P** which is the matrix containing the normalised eigen vectors. So we have to find the eigen-values.

$$\mathbf{V}\mathbf{p_2'} = \lambda_2 \mathbf{p_2'} \tag{1.12}$$

$$\begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix} \mathbf{p_2'} = 2\mathbf{p_2'} \tag{1.13}$$

$$\tag{1.14}$$

Taking an augmented matrix,

$$\left(\begin{array}{cc|c} \frac{7}{4} - \lambda_2 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{7}{4} - \lambda_2 & 0 \end{array} \right) \mathbf{p_2'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.15}$$

$$\left(\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & 2 \\ \frac{1}{4} & \frac{1}{4} & 2 \end{array} \right) \xleftrightarrow{R_1 \leftarrow 4R_1} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \tag{1.16}$$

If we take $\mathbf{p_2'} = \begin{pmatrix} p_{21}' \\ p_{22}' \end{pmatrix}$, and put in the above equation, we get,

$$\mathbf{p_2'} = p_{21}' \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p_2'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.17}$$

$\therefore \mathbf{p_2'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p_2} = \frac{\mathbf{p_2'}}{||\mathbf{p_2'}||} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Similarly solving for $\mathbf{p_1'}$, we get,

$$\mathbf{p_1'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.18}$$

$$\tag{1.19}$$

$\therefore \mathbf{p_1'} = \begin{pmatrix}1\\1\end{pmatrix} \implies \mathbf{p_1} = \frac{\mathbf{p_1'}}{||\mathbf{p_1'}||} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$
By definition $\mathbf{P} = \begin{bmatrix}\mathbf{p_1} & \mathbf{p_2}\end{bmatrix}$

$$\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \tag{1.20}$$

Now, we will use Affine Transformations to get the standard ellipse equation.

$$\mathbf{y}^{\mathrm{T}}\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \tag{1.21}$$
$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{1.22}$$
$$\tag{1.23}$$

Where

$$f_0 = \mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f = \frac{25}{4}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}\frac{7}{12} & \frac{-1}{12}\\\frac{7}{12} & \frac{-1}{12}\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{25}{12} \tag{1.24}$$
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\frac{5}{4}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}\frac{7}{12} & \frac{-1}{12}\\\frac{7}{12} & \frac{-1}{12}\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{5}{6}\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.25}$$

Finally, we get the standard form of the ellipse.

$$\mathbf{y}^{\mathrm{T}}\begin{pmatrix}2 & 0\\0 & \frac{3}{2}\end{pmatrix}\mathbf{y} = \frac{1}{3} \tag{1.26}$$

Plotting the standard and the actual ellipse using python,

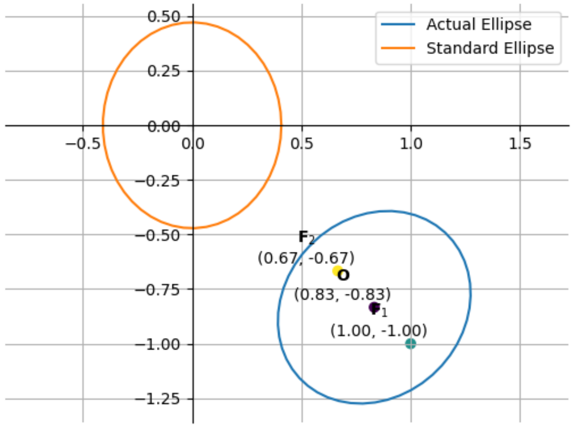

Fig. 1.1

Code for Figure 1.1 can be found at:

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codes/misc.py
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