Affine Transformations

EE24BTECH11059 - Y SIDDHANTH

Question: Plot the ellipse whose focus is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, directrix $x - y - 3 = 0$, and $e = \frac{1}{2}$. Plot the corresponding standard ellipse in the same graph using Affine Transformation.

Solution:

Variable	Description	Value
$\mathbf{n}$	Normal of Directrix	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\mathbf{c}$	c of Directrix	3
$\mathbf{e}$	Eccentricity of conic	$\frac{1}{2}$
$\mathbf{F}$	Focus of conic	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\mathbf{I}$	Identity matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
f	A variable in the conic equation	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$

TABLE 0: Variables Used

First we find the second order equation of the actual ellipse using the eccentricity definition.
Let $P\begin{pmatrix} x \\ y \end{pmatrix}$ be any point on the ellipse and PM be the perpendicular from P on the directrix. Then,

$$FP = e \times PM \tag{1.1}$$

$$FP = \frac{1}{2} \times PM \tag{1.2}$$

$$2FP = PM \tag{1.3}$$

$$4(FP)^2 = PM^2 \tag{1.4}$$

$$4\left[(x+1)^2 + (y-1)^2\right] = \left(\frac{x-y+3}{\sqrt{1^2 + (-1)^2}}\right)^2 \tag{1.5}$$

$$4\left[(x+1)^2 + (y-1)^2\right] = \left(\frac{x-y+3}{\sqrt{2}}\right)^2 \tag{1.6}$$

Upon simplification, we get the second order two variable conic equation to be,

$$\frac{7}{4}\left(x^2 + y^2\right) + \frac{1}{2}xy + \frac{5}{2}(-x+y) + \frac{7}{4} = 0 \tag{1.7}$$

Comparing (1.7) with $x^\top \mathbf{V} x + 2\mathbf{u}^\top x + f = 0$, we get:

$$\mathbf{V} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}, \tag{1.8}$$

$$\mathbf{u} = \frac{5}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.9}$$

$$f = \frac{7}{4} \tag{1.10}$$

For Affine Transformation, we have to spectral / eigen decompose the matrix $\mathbf{V}$.

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{1.11}$$

To find $\mathbf{D}$, we will have to find the eigen values of the matrx $\mathbf{V}$.

$$|\mathbf{V} - \lambda \mathbf{I}| = 0, \tag{1.12}$$

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} \frac{7}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} - \lambda \end{vmatrix} \tag{1.13}$$

$$\lambda_1, \lambda_2 = \frac{3}{2}, 2 \tag{1.14}$$

$$\therefore \mathbf{D} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}, \tag{1.15}$$

Note: In (1.15), the order of the eigen values that I have chosen is from smallest to biggest due to eccentricity calculation

Now, to find $\mathbf{P}$ which is the matrix containing the normalized eigen vectors. So we have to find the eigen-values.

$$\mathbf{V}\mathbf{p_1'} = \lambda_1 \mathbf{p_1'} \tag{1.16}$$

$$\begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}\mathbf{p_1'} = \frac{3}{2}\mathbf{p_2'} \tag{1.17}$$

$$\tag{1.18}$$

Taking an augmented matrix,

$$\left(\begin{array}{cc|c} \frac{7}{4} - \lambda_1 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{7}{4} - \lambda_1 & 0 \end{array}\right)\mathbf{p_1'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.19}$$

$$\left(\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{array}\right) \xleftrightarrow{R_1 \leftarrow 4R_1} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{1.20}$$

If we take $\mathbf{p_1'} = \begin{pmatrix} p'_{11} \\ p'_{12} \end{pmatrix}$, and put in the above equation, we get,

$$\mathbf{p_1'} = p'_{11}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \frac{3}{2}\mathbf{1_2'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.21}$$

$\therefore \mathbf{p_1'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p_1} = \frac{\mathbf{p_1'}}{||\mathbf{p_1'}||} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
Similarly solving for $\mathbf{p_2'}$, we get,

$$\mathbf{p_2'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.22}$$

$$\tag{1.23}$$

$\therefore \mathbf{p_2'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \mathbf{p_2} = \frac{\mathbf{p_2'}}{||\mathbf{p_2'}||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
By definition $\mathbf{P} = \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix}$

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.24}$$

By comparing $\mathbf{P}$ with a standard rotation matrix, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ the angle of rotation of $\mathbf{P}$ is $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ or $\frac{3\pi}{4}$.

Now, we will use Affine Transformations to get the standard ellipse equation.

$$\mathbf{y}^{\mathrm{T}}\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \tag{1.25}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{1.26}$$

$$\tag{1.27}$$

Where

$$f_0 = \mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f = \frac{25}{4}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{7}{4} = \frac{25}{12} - \frac{7}{4} = \frac{1}{3} \tag{1.28}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\frac{5}{4}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{5}{6}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.29}$$

Finally, we get the standard form of the ellipse.

$$\mathbf{y}^{\mathrm{T}}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}\mathbf{y} = \frac{1}{3} \tag{1.30}$$

Plotting the standard and the actual ellipse using python,

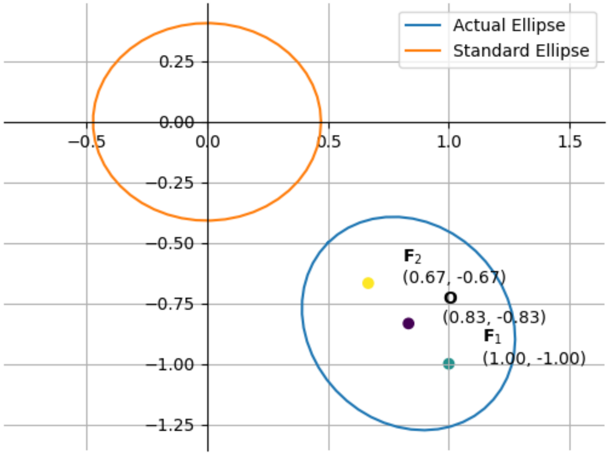

Fig. 1.1

To find the lengths of the semi-major and semi-minor axis,

$$a = \sqrt{\left(\frac{f_0}{\lambda_1}\right)} = \sqrt{\frac{2}{9}} \tag{1.31}$$

$$b = \sqrt{\left(\frac{f_0}{\lambda_2}\right)} = \sqrt{\frac{1}{6}} \tag{1.32}$$

Equations of the minor and major axis of the actual ellipse can be found with $p_i(x-c) = 0$, $i = 1, 2$ respectively

$$\text{Minor Axis} \equiv \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} x = 0 \tag{1.33}$$

$$\text{Major Axis} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} x = 0 \tag{1.34}$$

The Latus rectum of the ellipse can be found with,

$$l = 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} = \frac{1}{\sqrt{2}} \tag{1.35}$$

Code for Figure 1.1 can be found at:

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codes/misc.py
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