

Affine Transformations

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Question: Plot the ellipse whose focus is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, directrix $x - y - 3 = 0$, and $e = \frac{1}{2}$. Plot the corresponding standard ellipse in the same graph using Affine Transformation.

Solution:

Variable	Description	Value
n	Normal of Directrix	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
c	c of Directrix	3
e	Eccentricity of conic	$\frac{1}{2}$
F	Focus of conic	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

TABLE 0: Variables Used

First we find the second order equation of the actual ellipse using the eccentricity definition.

Let $P \begin{pmatrix} x \\ y \end{pmatrix}$ be any point on the ellipse and PM be the perpendicular from P on the directrix. Then,

$$FP = e \times PM \quad (1.1)$$

$$FP = \frac{1}{2} \times PM \quad (1.2)$$

$$2FP = PM \quad (1.3)$$

$$4(FP)^2 = PM^2 \quad (1.4)$$

$$4[(x+1)^2 + (y-1)^2] = \left(\frac{x-y+3}{\sqrt{1^2 + (-1)^2}} \right)^2 \quad (1.5)$$

$$4[(x+1)^2 + (y-1)^2] = \left(\frac{x-y+3}{\sqrt{2}} \right)^2 \quad (1.6)$$

Upon simplification, we get the second order two variable conic equation to be,

$$\frac{7}{4}(x^2 + y^2) + \frac{1}{2}xy + \frac{5}{2}(-x + y) + \frac{7}{4} = 0 \quad (1.7)$$

Comparing (1.7) with $x^T \mathbf{V}x + 2\mathbf{u}^T x + f = 0$, we get:

$$\mathbf{V} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}, \quad (1.8)$$

$$\mathbf{u} = \frac{5}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.9)$$

$$f = \frac{7}{4} \quad (1.10)$$

For Affine Transformation, we have to spectral / eigen decompose the matrix \mathbf{V} .

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (1.11)$$

To find \mathbf{D} , we will have to find the eigen values of the matrix \mathbf{V} .

$$|\mathbf{V} - \lambda\mathbf{I}| = 0, \quad (1.12)$$

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} \frac{7}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} - \lambda \end{vmatrix} \quad (1.13)$$

$$\lambda_1, \lambda_2 = \frac{3}{2}, 2 \quad (1.14)$$

$$\therefore \mathbf{D} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad (1.15)$$

Note: In (1.15), the order of the eigen values that I have chosen is from smallest to biggest due to eccentricity calculation.

Now, to find \mathbf{P} which is the matrix containing the normalized eigen vectors. So we have to find the eigen-values.

$$\mathbf{V}\mathbf{p}'_1 = \lambda_1\mathbf{p}'_1 \quad (1.16)$$

$$\begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix} \mathbf{p}'_1 = \frac{3}{2} \mathbf{p}'_1 \quad (1.17)$$

$$(1.18)$$

Taking an augmented matrix,

$$\left(\begin{array}{cc|c} \frac{7}{4} - \lambda_1 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{7}{4} - \lambda_1 & 0 \end{array} \right) \mathbf{p}'_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.19)$$

$$\left(\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{array} \right) \xleftrightarrow{R_1 \leftarrow 4R_1} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad (1.20)$$

If we take $\mathbf{p}'_1 = \begin{pmatrix} p'_{11} \\ p'_{12} \end{pmatrix}$, and put in the above equation, we get,

$$\mathbf{p}'_1 = p'_{11} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \frac{3}{2} \mathbf{1}'_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.21)$$

$$\therefore \mathbf{p}'_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p}_1 = \frac{\mathbf{p}'_1}{\|\mathbf{p}'_1\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Similarly solving for \mathbf{p}'_2 , we get,

$$\mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.22)$$

$$(1.23)$$

$$\therefore \mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \mathbf{p}_2 = \frac{\mathbf{p}'_2}{\|\mathbf{p}'_2\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

By definition $\mathbf{P} = [\mathbf{p}_1 \ \mathbf{p}_2]$

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.24)$$

By comparing \mathbf{P} with a standard rotation matrix, $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ the angle of rotation of \mathbf{P} is $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ or $\frac{3\pi}{4}$.

Now, we will use Affine Transformations to get the standard ellipse equation.

$$\mathbf{y}^T \begin{pmatrix} \mathbf{D} \\ f_0 \end{pmatrix} \mathbf{y} = 1 \quad (1.25)$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (1.26)$$

$$(1.27)$$

Where

$$f_0 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = \frac{25}{4} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{7}{4} = \frac{25}{12} - \frac{7}{4} = \frac{1}{3} \quad (1.28)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = -\frac{5}{4} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{5}{6} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.29)$$

Finally, we get the standard form of the ellipse.

$$\mathbf{y}^T \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = \frac{1}{3} \quad (1.30)$$

Plotting the standard and the actual ellipse using python,

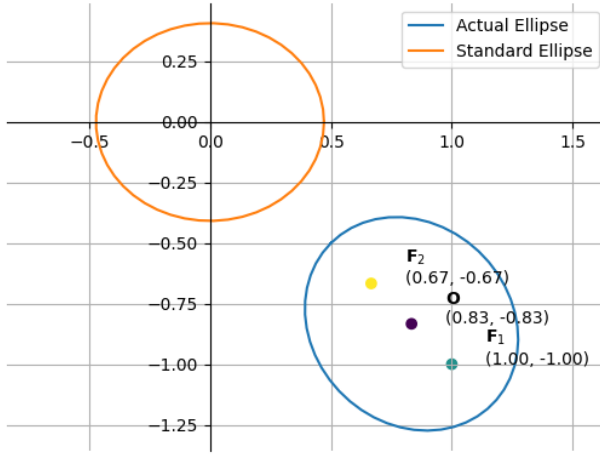


Fig. 1.1

To find the lengths of the semi-major and semi-minor axis,

$$a = \sqrt{\left(\frac{f_0}{\lambda_1}\right)} = \sqrt{\frac{2}{9}} \quad (1.31)$$

$$b = \sqrt{\left(\frac{f_0}{\lambda_2}\right)} = \sqrt{\frac{1}{6}} \quad (1.32)$$

Equations of the minor and major axis of the actual ellipse can be found with $p_i(x-c) = 0$, $i = 1, 2$ respectively

$$\text{Minor Axis} \equiv \left(-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right) x = 0 \quad (1.33)$$

$$\text{Major Axis} \equiv \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right) x = 0 \quad (1.34)$$

The Latus rectum of the ellipse can be found with,

$$l = 2 \frac{\sqrt{|f_0 \lambda_1|}}{\lambda_2} = \frac{1}{\sqrt{2}} \quad (1.35)$$

Code for Figure 1.1 can be found at:

codes/misc.py