9.9.3.17

EE24BTECH11059 - Yellanki Siddhanth

Question:
Using integration, find the area of the region enclosed by the line $y = \sqrt{3}x$ and semi-circle $y = \sqrt{4 - x^2}$ and x-axis in the first quadrant
Solution:

Variable	Description
$\mathbf{h}$	Point lying on the line
$\mathbf{m}$	Slope of line
$\mathbf{e}$	Eccentricity of conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
f	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition
$\mathbf{u}$	Vertex of conic with same directrix

TABLE 0

Line equation of form $\mathbf{x} = \mathbf{h} + k\mathbf{m}$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{0.1}$$

Equation of circle of form $\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ is

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -4, \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.2}$$

If a line intersects the conic, k value of intersecting point is given by,

$$k_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{0.3}$$

On substituting values of $\mathbf{u}, \mathbf{m}, \mathbf{h}, \mathbf{V}$ we get,

$$k = \pm 1 \tag{0.4}$$

Since we are considering the area only in first quadrant, we will only consider $k = +1$.
Thus point of intersection of line with circle is $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.
Area bound between the semi-circle and the line is,

$$\mathbf{S} = \int_0^1 x dx + \int_1^2 \sqrt{4 - x^2} dx \tag{0.5}$$

$$= \frac{1}{2} + \frac{2\pi}{3} - \frac{1}{2} = \frac{2\pi}{3} \tag{0.6}$$

Thus area between the line and the semi-circle is $S = \frac{2\pi}{3}$