Affine Transformations

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Problem Statement

Plot the ellipse whose focus is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, directrix $x - y - 3 = 0$, and $e = \frac{1}{2}$.
Plot the corresponding standard ellipse in the same graph using Affine Transformation.

Method of Solving

First we find the second order equation of the actual ellipse using the eccentricity definition.

Let $P\begin{pmatrix} x \\ y \end{pmatrix}$ be any point on the ellipse and PM be the perpendicular from P on the directrix. Then,

$$FP = e \times PM \tag{3.1}$$

$$FP = \frac{1}{2} \times PM \tag{3.2}$$

$$2FP = PM \tag{3.3}$$

$$4(FP)^2 = PM^2 \tag{3.4}$$

$$4\left[(x+1)^2 + (y-1)^2\right] = \left(\frac{x-y+3}{\sqrt{1^2+(-1)^2}}\right)^2 \tag{3.5}$$

$$4\left[(x+1)^2 + (y-1)^2\right] = \left(\frac{x-y+3}{\sqrt{2}}\right)^2 \tag{3.6}$$

Method of Solving

Upon simplification, we get the second order two variable conic equation to be,

$$\frac{7}{4}\left(x^2+y^2\right)+\frac{1}{2}xy+\frac{5}{2}\left(-x+y\right)+\frac{7}{4}=0 \tag{3.7}$$

Comparing (3.7) with $x^\top \mathbf{V} x + 2\mathbf{u}^\top x + f = 0$, we get:

$$\mathbf{V} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}, \tag{3.8}$$

$$\mathbf{u} = \frac{5}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.9}$$

$$f = \frac{7}{4} \tag{3.10}$$

Method of Solving

For Affine Transformation, we have to spectral / eigen decompose the matrix $\mathbf{V}$.

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\top} \tag{3.11}$$

To find $\mathbf{D}$, we will have to find the eigen values of the matrix $\mathbf{V}$.

$$|\mathbf{V} - \lambda \mathbf{I}| = 0, \tag{3.12}$$

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{bmatrix} \frac{7}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} - \lambda \end{bmatrix} \tag{3.13}$$

$$\lambda_1, \lambda_2 = \frac{3}{2}, 2 \tag{3.14}$$

$$\therefore \mathbf{D} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix} \tag{3.15}$$

Method of Solving

Note: In (3.15), the order of the eigen values that I have chosen is from smallest to biggest due to eccentricity calculation.
Now, to find **P** which is the matrix containing the normalized eigen vectors. So we have to find the eigen-values.

$$\mathbf{V}\mathbf{p_1'} = \lambda_1 \mathbf{p_1'} \tag{3.16}$$

$$\begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix} \mathbf{p_1'} = \frac{3}{2}\mathbf{p_1'} \tag{3.17}$$

$$\tag{3.18}$$

Taking an augmented matrix,

$$\left(\begin{array}{cc|c} \frac{7}{4} - \lambda_1 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{7}{4} - \lambda_1 & 0 \end{array}\right) \mathbf{p_1'} \tag{3.19}$$

Method of Solving

$$\left(\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{array}\right) \xleftrightarrow{R_1 \leftarrow 4R_1} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{3.20}$$

If we take $\mathbf{p_1'} = \begin{pmatrix} p_{11}' \\ p_{12}' \end{pmatrix}$, and put in the above equation, we get,

$$\mathbf{p_1'} = p_{11}' \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p_1'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.21}$$

$$\therefore \mathbf{p_1'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies \mathbf{p_1} = \frac{\mathbf{p_1'}}{\|\mathbf{p_1'}\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Method of Solving

Similarly solving for $\mathbf{p_2'}$, we get,

$$\mathbf{p_2'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.22}$$

$$\tag{3.23}$$

$\therefore \mathbf{p_2'} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \mathbf{p_2} = \frac{\mathbf{p_2'}}{\|\mathbf{p_2'}\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

By definition $\mathbf{P} = [\mathbf{p_1} \ \mathbf{p_2}]$

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.24}$$

By comparing $\mathbf{P}$ with a standard rotation matrix, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ the angle of rotation of $\mathbf{P}$ is $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ or $\frac{3\pi}{4}$.

Affine Transformation

Now, we will use Affine Transformations to get the standard ellipse equation.

$$\mathbf{y}^{\mathsf{T}}\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \tag{3.25}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.26}$$

$$\tag{3.27}$$

Where

$$f_0 = \mathbf{u}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{u} - f = \frac{25}{4}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{7}{4} = \frac{25}{12} - \frac{7}{4} = \frac{1}{3} \tag{3.28}$$

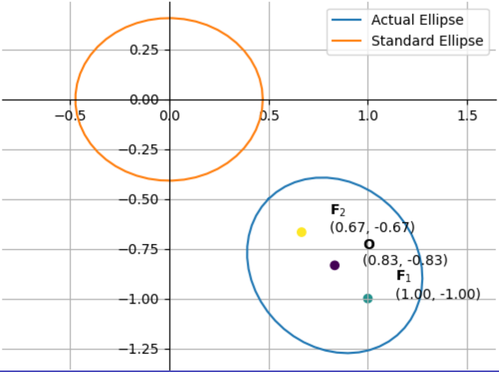
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = -\frac{5}{4}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{5}{6}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.29}$$

Affine Transformation

Finally, we get the standard form of the ellipse.

$$\mathbf{y}^{\top} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = \frac{1}{3} \tag{3.30}$$

Plotting the standard and the actual ellipse using python,



Affine Transformation

To find the lengths of the semi-major and semi-minor axis,

$$a = \sqrt{\left(\frac{f_0}{\lambda_1}\right)} = \sqrt{\frac{2}{9}} \tag{3.31}$$

$$b = \sqrt{\left(\frac{f_0}{\lambda_2}\right)} = \sqrt{\frac{1}{6}} \tag{3.32}$$

Equations of the minor and major axis of the actual ellipse can be found with $p_i(x - c) = 0,\ i = 1, 2$ respectively

$$\text{Minor Axis} \equiv \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} x = 0 \tag{3.33}$$

$$\text{Major Axis} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} x = 0 \tag{3.34}$$

The Latus rectum of the ellipse can be found with,

$$l = 2\frac{\sqrt{|f_0 \lambda_1|}}{\lambda_2} = \frac{1}{\sqrt{2}} \tag{3.35}$$