

# Affine Transformations

EE24BTECH11059 - Y SIDDHANTH

**Question:** Plot the ellipse whose focus is  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , directrix  $x - y - 3 = 0$ , and  $e = \frac{1}{2}$ . Plot the corresponding standard ellipse in the same graph using Affine Transformation.

**Solution:**

Variable	Description	Value
$\mathbf{n}^\top x = c$	Directrix Equation	$\begin{pmatrix} 1 & -1 \end{pmatrix} x = 3$
$e$	Eccentricity of conic	$\frac{1}{2}$
$\mathbf{F}$	Focus of conic	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\mathbf{I}$	Identity matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$f$	A variable in the conic equation	$\ \mathbf{n}\ ^2 \ \mathbf{F}\ ^2 - c^2 e^2$

TABLE 0: Variables Used

First we find the  $V, u, f$  for the actual ellipse using the following formulas:

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (1.1)$$

$$\mathbf{u} = c e^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (1.2)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (1.3)$$

Upon solving, we get:

$$\mathbf{V} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix}, \quad (1.4)$$

$$\mathbf{u} = \frac{5}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.5)$$

$$f = \frac{7}{4} \quad (1.6)$$

For Affine Transformation, we have to spectral / eigen decompose the matrix  $\mathbf{V}$ .

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^\top \quad (1.7)$$

To find  $\mathbf{D}$ , we will have to find the eigen values of the matrix  $\mathbf{V}$ .

$$|\mathbf{V} - \lambda \mathbf{I}| = 0, \quad (1.8)$$

$$|\mathbf{V} - \lambda \mathbf{I}| = \begin{vmatrix} \frac{7}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} - \lambda \end{vmatrix} \quad (1.9)$$

$$\lambda_1, \lambda_2 = \frac{3}{2}, 2 \quad (1.10)$$

$$\therefore \mathbf{D} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad (1.11)$$

Note: In (??), the order of the eigen values that I have chosen is from smallest to biggest due to eccentricity calculation

Now, to find  $\mathbf{P}$  which is the matrix containing the normalized eigen vectors. So we have to find the eigen-values.

$$\mathbf{V}\mathbf{p}'_1 = \lambda_1 \mathbf{p}'_1 \quad (1.12)$$

$$\begin{pmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{pmatrix} \mathbf{p}'_1 = \frac{3}{2} \mathbf{p}'_2 \quad (1.13)$$

$$(1.14)$$

Taking an augmented matrix,

$$\left( \begin{array}{cc|c} \frac{7}{4} - \lambda_1 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{7}{4} - \lambda_1 & 0 \end{array} \right) \mathbf{p}'_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.15)$$

$$\left( \begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{array} \right) \xleftrightarrow{R_1 \leftarrow 4R_1} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad (1.16)$$

If we take  $\mathbf{p}'_1 = \begin{pmatrix} p'_{11} \\ p'_{12} \end{pmatrix}$ , and put in the above equation, we get,

$$\mathbf{p}'_1 = p'_{11} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \frac{3}{2} \mathbf{1}'_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.17)$$

$$\therefore \mathbf{p}'_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{p}_1 = \frac{\mathbf{p}'_1}{\|\mathbf{p}'_1\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Similarly solving for  $\mathbf{p}'_2$ , we get,

$$\mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.18)$$

$$(1.19)$$

$$\therefore \mathbf{p}'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{p}_2 = \frac{\mathbf{p}'_2}{\|\mathbf{p}'_2\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

By definition  $\mathbf{P} = [\mathbf{p}_1 \ \mathbf{p}_2]$

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.20)$$

Now, we will use Affine Transformations to get the standard ellipse equation.

$$\mathbf{y}^T \begin{pmatrix} \mathbf{D} \\ f_0 \end{pmatrix} \mathbf{y} = 1 \quad (1.21)$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (1.22)$$

$$(1.23)$$

Where

$$f_0 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = \frac{25}{4} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{7}{4} = \frac{25}{12} - \frac{7}{4} = \frac{1}{3} \quad (1.24)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = -\frac{5}{4} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{12} & \frac{-1}{12} \\ \frac{-1}{12} & \frac{7}{12} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{5}{6} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.25)$$

Finally, we get the standard form of the ellipse.

$$\mathbf{y}^T \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = \frac{1}{3} \quad (1.26)$$

Plotting the standard and the actual ellipse using python,

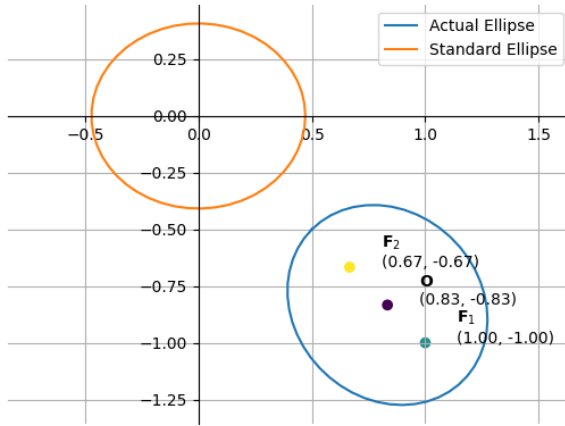


Fig. 1.1

Code for Figure ?? can be found at:

codes/misc.py