Multipole Expansion of Electric Fields in FEKO at Far Field

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March 23, 2020

In Feko electric field strength is represented in terms of finite expansion of spherical vector harmonics, commonly referred to as multipole expansion:

$$E(r,\theta,\phi) = \sum_{c=3}^{4}\sum_{s=1}^{2}\sum_{n=1}^{N_{max}}\sum_{m=-n}^{n} Q_{smn}^{(c)}\vec{F}_{smn}^{(c)}(r,\theta,\phi) \tag{1}$$

Feko is using the c=3 to describe inward propagating waves, and c=4 for outward propagating waves. The modes described with c=3 radiate towards r=0, while the modes with c=4 radiate from r=0 towards r$\to\infty$. The index s=1 represents Transverse electric (TE) modes, for instance where there is no electric field in the direction of wave propagation. Likewise, s=2 is used to described Transverse magnetic (TM) modes with no magnetic field present in the direction of propagation.
Feko Far Fields are defined as following:

$$E_\theta = E_\theta^{FF}\frac{e^{-j\beta R}}{R} \tag{2}$$

$$E_\phi = E_\phi^{FF}\frac{e^{-j\beta R}}{R} \tag{3}$$

The spherical modes have units of $\sqrt{W}$. For purely transmitting antennas, c=4. The spherical wave functions in the Far Field corresponding to spherical modes are described as following:

$$\vec{F}_{1mn}^{FF} = \sqrt{\frac{Z_f}{2\pi}}\frac{1}{\sqrt{n(n+1)}}\left(-\frac{m}{|m|}\right)^m j^{n+1}e^{jm\phi}\left(j\frac{m\hat{P}_n^{|m|}(\cos\theta)}{\sin\theta}\hat{\theta} - \frac{\partial}{\partial\theta}[\hat{P}_n^{|m|}(\cos\theta)]\hat{\phi}\right) \tag{4}$$

$$\vec{F}^{FF}_{2mn} = \sqrt{\frac{Z_f}{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left(-\frac{m}{|m|}\right)^m j^n e^{jm\phi} \left(j\frac{m\hat{P}_n^{|m|}(\cos\theta)}{\sin\theta}\hat{\phi} + \frac{\partial}{\partial\theta}[\hat{P}_n^{|m|}(\cos\theta)]\hat{\theta}\right) \tag{5}$$

Note that the spherical waves used above have no radial dependence, and j^{n+1} and j^n appear due to spherical Hankel functions of the second kind at Far Field, for instance,

$$h^{(2)}(\beta r) \approx j^{n+1}\frac{e^{-j\beta r}}{\beta r} \tag{6}$$

$$\frac{1}{\beta r}\frac{\partial}{\partial(\beta r)}[\beta r h^{(2)}(\beta r)] \approx -j^{n+2}\frac{e^{-j\beta r}}{\beta r} = j^n\frac{e^{-j\beta r}}{\beta r} \tag{7}$$

The term $\frac{e^{-j\beta r}}{\beta r}$ vanishes in both cases due to definitions given in (2) and (3).
Finally, breaking down the above functions into θ- and ϕ- dependent components, the following expressions are obtained:

$$\Gamma^{\theta}_{mn} = \sqrt{\frac{Z_f}{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left(-\frac{m}{|m|}\right)^m j^n e^{jm\phi} \left(-Q_{1mn}\frac{m\hat{P}_n^{|m|}(\cos\theta)}{\sin\theta} + Q_{2mn}\frac{\partial}{\partial\theta}[\hat{P}_n^{|m|}(\cos\theta)]\right) \tag{8}$$

$$\Gamma^{\phi}_{mn} = \sqrt{\frac{Z_f}{2\pi}} \frac{1}{\sqrt{n(n+1)}} \left(-\frac{m}{|m|}\right)^m j^{n+1} e^{jm\phi} \left(Q_{2mn}\frac{m\hat{P}_n^{|m|}(\cos\theta)}{\sin\theta} - Q_{1mn}\frac{\partial}{\partial\theta}[\hat{P}_n^{|m|}(\cos\theta)]\right) \tag{9}$$

$$\vec{E}^{FF}(\theta,\phi) = \sum_{n=1}^{N_{max}}\sum_{m=-n}^{n}\left[\Gamma^{\theta}_{mn}\hat{\theta} + \Gamma^{\phi}_{mn}\hat{\phi}\right] \tag{10}$$

The term $\hat{P}_n^{|m|}(\cos\theta)$ represents the normalised Legendre polynomial over a unit circle without Condon-Shortley Phase, following Feko's convention.
Taweetham Limpanuparb, Josh Milthorpe (2014) in their paper "Associated Legendre Polynomials and Spherical Harmonics Computation for Chemistry Applications" describe the mechanism for generation of normalised Associated Legendre polynomials.
P. Li and L. Jiang in their paper "The Far Field Transformation for the Antenna Modelling Based on Spherical Electric Field Measurements" describe the asymptotic case for $m = 1$ and $\cos(\theta) = 1$. It should be noted that their formula produced incorrect result when taking upper summation index as $\lfloor n/2 \rfloor$ and should be replaced

with $\lceil n/2 \rceil$ instead. The overall formula for $m = 1$:

$$\frac{\hat{P}_n^m(\cos\theta)}{\sin\theta} = -\frac{1}{2^n}\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\sum_{k=0}^{\lceil n/2 \rceil}(-1)^k\frac{(n-k+1)_{(n-k)}}{k!(n-2k-1)!} \tag{11}$$

where $(n-k+1)_{(n-k)}$ denotes Pochhammer symbol, or reduced rising factorial. Computation of factorial fractions represented via Pochhammer symbol delays overflow problem to higher values of n and m. Note, the sign of the expression is negative at $x \to 0^+$ and positive at $x \to 0^-$.
Update:
According to Wolfram Alpha, the expression above reduces to

$$\frac{\hat{P}_n^m(\cos\theta)}{\sin\theta}|_{\theta\to 0^+} = -\frac{1}{2}\sqrt{\frac{n(2n+1)(n+1)}{2}} \tag{12}$$

I confirmed that both expressions give identical results at this limit.
Finally, using recurrence relation for the first derivative of Associated Legendre polynomials, we obtain

$$\frac{\partial}{\partial\theta}\left[P_n^m(\cos\theta)\right] = -(n+1)\cos\theta\frac{P_n^m(\cos\theta)}{\sin\theta} + (n+1-m)\frac{P_{n+1}^m(\cos\theta)}{\sin\theta} \tag{13}$$

We know that

$$\hat{P}_n^m(\cos\theta) = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta) \tag{14}$$

Rearranging:

$$\sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}}\hat{P}_n^m(\cos\theta) = P_n^m(\cos\theta) \tag{15}$$

Substituting (14) into L.H.S. of (12):

$$LHS = \frac{\partial}{\partial\theta}\left(\sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}}\hat{P}_n^m(\cos\theta)\right) = \sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}}\frac{\partial}{\partial\theta}\left[\hat{P}_n^m(\cos\theta)\right] \tag{16}$$

Substituting (14) into R.H.S. of (12):

$$RHS = -(n+1)\cos\theta\sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}}\frac{\hat{P}_n^m(\cos\theta)}{\sin\theta} + (n+1-m)\sqrt{\frac{2}{2(n+1)+1}\frac{(n+m+1)!}{(n-m+1)!}}\frac{\hat{P}_{n+1}^m(\cos\theta)}{\sin\theta} \tag{17}$$

Equating and simplifying R.H.S. and L.H.S.:

$$\frac{\partial}{\partial\theta}\left[\hat{P}_n^m(\cos\theta)\right] = -(n+1)\cos\theta\frac{\hat{P}_n^m(\cos\theta)}{\sin\theta} + \sqrt{\frac{(2n+1)(n+m+1)(n-m+1)}{(2n+3)}}\frac{\hat{P}_{n+1}^m(\cos\theta)}{\sin\theta} \tag{18}$$

$$\forall n \geq m$$

The above method permits computation of the first derivative of associated Legendre polynomials with high accuracy up to 45-th order of m. After this, the factorial function leads to overflow or underflow. To overcome this challenge, one can increase the standard precision on MATLAB or Python code beyond 64-bit double precision format, using extended precision method. Using $Decimal = 100$, I was able to obtain Legendre Derivatives and up to more than 200-th order of m. The method was proven to be a robust and efficient alternative to finite difference.