

# Tightening Faster Model Checking of First-Order Graph Properties, via Tarski's Calculus of Relations

YN: Tentative

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## Abstract

This is a SUPER-STRONG-PAPER and brings a NEW ERA to the field of FO.

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## 1 Introduction

(Background)

- $\mathbb{FO}^k$  モデル検査: in  $\mathcal{O}(\|\varphi\| \cdot n^k)$  time (see, *e.g.*, [19, Proposition 3.1])
- Williams' algorithm: in  $\mathcal{O}(2^{\|\varphi\|} \cdot n^\omega)$  time for  $\mathbb{FO}^3$  [21, Corollary 1.3][9, Theorem 7].  
(★:  $\mathcal{O}(2^{\|\varphi\|} \cdot n^{k-3+\omega})$  time for  $\mathbb{FO}^k$  は正しい?)
- CoR モデル検査: in  $\mathcal{O}(\|\varphi\| \cdot n^\omega)$  time.

(Contribution)

- Williams' algorithm (for  $\mathbb{FO}^3$  and for  $\mathbb{FO}^k$  in PNF)  $\approx$   $\mathbb{FO}^3$ -to-CoR translation.
- For  $\mathbb{FO}^k$ , for any  $\varepsilon > 0$ , there is no algorithm in  $\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$  time, under SETH.
  - There is no polynomial-time translation from  $\mathbb{FO}^3$  to CoR, under SETH.
- $\text{poly}(\|\varphi\|) \cdot n^{3-\varepsilon}$  time algorithms for (parameterized) fragments of  $\mathbb{FO}^3$ .
  - “**quantifier-width**”
  - A dichotomy *w.r.t.* signatures

...

YN: TODO: (Below is wip.)

The *model checking problem* is the following problem:

given both a sentence  $\varphi$  and a structure  $\mathfrak{A}$ , does  $\mathfrak{A} \models \varphi$  hold?

Unfortunately, the *model checking problem* for  $\mathbb{FO}$  is complete in PSPACE. Nevertheless, when the number of variables is fixed, the *model checking problem* for  $\mathbb{FO}^k$  can be decided in  $\mathcal{O}(\|\varphi\| \cdot n^k)$  time by a naive bottom-up evaluation algorithm (see, *e.g.*, [19, Proposition 3.1]), where  $\|\varphi\|$  is the length of  $\varphi$  and  $n$  is the number of vertices in  $\mathfrak{A}$ . A natural question is

Can the complexity be improved from  $\mathcal{O}(\|\varphi\| \cdot n^k)$  time?

Williams [21] gave a positive answer to this question. He showed that when  $\varphi$  is *fixed*, the following holds:



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- 25 ■ For every  $k \geq 3$ , the model checking problem for  $\mathbb{FO}^k$  sentences<sup>1</sup> can be decided in  
 26  $\mathcal{O}(n^{k-3+\omega})$  time [21, Corollary 1.3][9, Theorem 7], where  $\omega$  is the exponent of matrix  
 27 multiplication.
- 28 ■ For some  $k \geq 3$ ,<sup>2</sup> if the model checking problem for  $\mathbb{FO}^k$  sentences<sup>3</sup> can be decided in  
 29  $\mathcal{O}(n^{k-1-\varepsilon})$  time for some  $\varepsilon > 0$ , then the Strong Exponential Time Hypothesis (SETH)  
 30 is false [21, Corollary 1.4].

31 However when  $\varphi$  is not fixed, Williams' algorithm [21] for  $\mathbb{FO}^3$  sentences is  $\mathcal{O}(2^{\|\varphi\|} \cdot n^\omega)$   
 32 time. The exponential blowup *w.r.t.*  $\|\varphi\|$  is due to that the algorithm requires a transformation  
 33 to the disjunctive normal form (DNF).

34 ...

35 In this paper, we first revisit Williams' algorithm. We give a perspective from *Tarski's*  
 36 *calculus of relations* (henceforth, *CoR*) [17]. ... We can translate  $\mathbb{FO}^3$  sentences into CoR  
 37 sentences in  $\mathcal{O}(2^{\|\varphi\|})$  time. The original translation is given in [18]. An  $\mathcal{O}(2^{\|\varphi\|})$  time  
 38 translation is given in [15]; an implementation of this translation is given in [2].<sup>4</sup> For CoR  
 39 sentences, the model checking problem can be decided in  $\mathcal{O}(\|\varphi\| \cdot n^\omega)$  time by a naive  
 40 bottom-up evaluation algorithm, where we use the *matrix multiplication* algorithm for the  
 41 relational composition ( $;$ ). Combining them, we have that the model checking problem for  
 42  $\mathbb{FO}^3$  sentences can be decided in  $\mathcal{O}(2^{\|\varphi\|} \cdot n^\omega)$  time.

43 Now, ...

YN: TODO: Parameterized complexity (*e.g.*, [7, 6, 5, 8]).  
 Parameterized complexity for space complexity [3]

44

## 45 2 Preliminaries

46 We write  $\mathbb{Z}$  for the set of all integers.

47 We write  $[l, r]$  for the set  $\{i \in \mathbb{Z} \mid l \leq i \leq r\}$ .

48 A *relational signature* (henceforth, *signature*)  $\sigma$  is ...

49 A *structure*  $\mathfrak{A}$  over a signature  $\sigma$  is a tuple  $\langle |\mathfrak{A}|, \{R^{\mathfrak{A}}\}_{R \in \sigma} \rangle$ , where its *universe*  $|\mathfrak{A}|$  is a  
 50 finite<sup>5</sup> set of *vertices* and each  $R^{\mathfrak{A}}$  is a binary relation on  $|\mathfrak{A}|$ .

51 A *sentence* is a formula without free variables.

52 For a formula  $\varphi$  and a *valuation*  $\mathbf{v}$ , we write  $\mathfrak{A}, \mathbf{v} \models \varphi$  to denote that  $\varphi$  is true at  $\mathfrak{A}$  under  
 53  $\mathbf{v}$ . For a sentence  $\varphi$ , we write  $\mathfrak{A} \models \varphi$  to denote that  $\varphi$  is true at  $\mathfrak{A}$ .

54 We write  $\mathbb{FO}^k$  to denote the set of formulas of First-order predicate logic with  $k$ -variables.

<sup>1</sup> Precisely, Williams [21] shows for  $\mathbb{FO}^k$  sentences in *prenex normal form* (so, the number of occurrences of *quantifiers* is at most  $k$ ) [21, Corollary 1.3]. Later, Gao and Impagliazzo [9, Theorem 7] claims that Williams' algorithm can be extended for general  $\mathbb{FO}^k$  sentences.

YN: ( $\not\propto 3$ )

<sup>2</sup> When  $k \leq 3$ , it is almost clear that the model checking problem cannot be decided in  $\mathcal{O}(n^{k-1-\varepsilon})$ , as the input adjacent matrix is given by  $\Theta(n^2)$  cells. See Proposition 14 for a more precise proof.

<sup>3</sup> This claims holds even for  $\mathbb{FO}^k$  sentences in  $\exists^*\forall$ -prenex normal form.

<sup>4</sup> Williams' algorithm takes almost the same steps as in the translation in [15]... (TODO: revise the sentences)

<sup>5</sup> In this paper, we are only interested in finite structures.

## 55 Important Remark (Unary and Binary Predicates)

56 We allow unary and binary predicates in our  $\mathbb{FO}^3$ . Therefore, we do not allow terms involving  
57 ternary predicates such as  $\forall x. \exists y. \forall z. P(x, y, z)$ .<sup>6</sup>

58 On this remark, we can identify each model  $\mathcal{M}$  as color graphs as follows:

- 59 ■ Each element corresponds a node (vertex).
- 60 ■ Each unary predicate  $U_i$  means “color- $i$ ” node.
- 61 ■ Each binary predicate  $P_j$  is “color- $j$ ” edge.
- 62 ■ For example, on two nodes  $x, y$ , a term  $(U_i(x) \wedge U_j(y) \wedge P_k(x, y))$  means that  $x$  has the  
63 color  $i$ ,  $y$  has the color  $j$ , and there is a color  $k$  edge from  $x$  to  $y$ .

## 64 Quantifier width

Quantifier widthという謎の指標を導入します。  
与えられた式の部分項  $\forall/\exists x.(\dots)$  について  $\dots$  部分に「直接」出てくるquantifierの  
個数を、この部分項のquantifier widthと呼ぶことにします。  
たとえば、次の部分項

$$\forall x. \left( \left( \underline{\forall y. P(y)} \right) \vee \left( \underline{\forall z. (\forall x. P(z, x))} \wedge Q(z) \right) \right)$$

については、内部に3つのquantifierの出現（下線）があるのですが、直接の出現は赤  
くした2つだけなので、この部分項のquantifier widthは「2」です。  
そして、式のquantifier widthというのは、最大のquantifier widthを持つ部分項におけ  
る、その数として定義します。

## 66 Tarski's Calculus of Relations (CoR)

67 CoR **terms** are generated by the following grammar:

$$\begin{aligned} 68 \quad t, s &::= a \mid t + s \mid t \cap s \mid t^- \\ 69 \quad &\mid t ; s \mid t \uparrow s \mid t^\pi. \end{aligned}$$

70 ► **Proposition 1.** *The model checking problem for **atomic** CoR formulas is in  $\mathcal{O}(\|\varphi\| \cdot n^\omega)$*   
71 *time.*

72 **Proof Sketch.** By a naive bottom-up evaluation algorithm. Here, we use the **matrix multi-**  
73 **plication** algorithm for the **relational composition** ( $;$ ). ◀

## 74 SETH

75 ► **Conjecture 2** (The **Strong Exponential Time Hypothesis** (**SETH**) [11, 12]). *For every  $\delta < 1$ ,*  
76 *there exists an integer  $k$  such that **k-CNF-SAT** with  $n$  variables cannot be solved in  $\mathcal{O}(2^{\delta n})$*   
77 *time.*

78 **SETH** implies the following hypothesis.

<sup>6</sup> For each unary predicate  $U$ , we can simulate it by a diagonal binary predicate  $B_U(x, y) := x = y \wedge U(x)$ .  
However, in our later construction, we use some unary predicates; thus, we also allow unary predicates  
explicitly.

YN: たとえば [20] で  
は、 $\mathcal{O}(2^{\delta n})$  でなく  
 $2^{\delta n} \text{poly}(n)$  を用いて  
いるが、  
予想としては同値 (の  
はず。  $\text{poly}(n) < \mathcal{O}(2^{\delta n})$  なの)  
。  
 $\mathcal{O}(2^n \text{poly}(n))$  のアル  
ゴリズムが存在するという  
fact に合わせている?

YN: However, it is  
not known whether  
the two conjectures  
are equivalent. [4,  
Previous Work]

79 ► **Conjecture 3** (e.g., in [16]). *For every  $\delta < 1$ , CNF-SAT with  $n$  variables and  $m$  clauses*  
 80 *cannot be solved in  $2^{\delta n} \cdot \text{poly}(m)$  time.*

### 81 **3 Faster FO3 model checking and the FO3-to-CoR translation**

82 YN: TODO: Give an outline of the translation [15].

83 For instance, when  $\varphi$  is the following  $\mathbb{FO}^3$  sentence

$$84 \quad \exists y.((P(x, y) \vee Q(y, z)) \wedge R(x, y)) \wedge S(x, z),$$

85 we can translate  $\varphi$  into the CoR formula as follows:

$$\begin{aligned} 86 \quad & \exists y.((P(x, y) \vee Q(y, z)) \wedge R(x, y)) \wedge S(x, z) \\ 87 \quad & \rightsquigarrow \exists y.(P(x, y) \wedge R(x, y) \wedge S(x, z)) \vee (Q(y, z) \wedge R(x, y) \wedge S(x, z)) \quad (\text{taking the DNF}) \\ 88 \quad & \rightsquigarrow ((\exists y.P(x, y) \wedge R(x, y)) \wedge S(x, z)) \vee ((\exists y.Q(y, z) \wedge R(x, y)) \wedge S(x, z)) \\ & \quad \quad \quad (\text{pushing quantifiers}) \\ 89 \quad & \rightsquigarrow (((P \cap R) ; \top)(x, z) \wedge S(x, z)) \vee ((R ; Q)(x, z) \wedge S(x, z)) \\ & \quad \quad \quad (\text{translating into CoR (intermediate)}) \\ 90 \quad & \rightsquigarrow (((P \cap R) ; \top) \cap S) + ((R ; Q) \cap S)(x, z). \quad (\text{translating into CoR}) \end{aligned}$$

91 ► **Proposition 4** ([15]). *There is an  $\mathcal{O}(2^{\|\varphi\|})$ -time translation from an  $\mathbb{FO}^3$  formula  $\varphi$  with*  
 92 *at most two free variables into an **atomic** CoR formula **semantically equivalent** to  $\varphi$ .*

93 Hence, combining with Proposition 1, we have obtained Williams' result for  $k = 3$ , via  
 94 the  $\mathbb{FO}^3$ -to-CoR translation.

95 ► **Theorem 5.** *The model checking problem for  $\mathbb{FO}^3$  sentences can be decided in  $\mathcal{O}(2^{\|\varphi\|} \cdot n^\omega)$*   
 96 *time.*

97 **Proof.** Given an  $\mathbb{FO}^3$  sentence  $\varphi$ , we can translate  $\varphi$  into an **atomic** CoR formula  $\varphi'$  in  
 98  $\mathcal{O}(2^{\|\varphi\|})$  time (Proposition 4). The **size** of  $\varphi'$  is also  $\mathcal{O}(2^{\|\varphi\|})$ . Hence, by Proposition 1, we  
 99 have obtained an  $\mathcal{O}(2^{\|\varphi\|} \cdot n^\omega)$  time algorithm. ◀

100 By Proposition 1 and Theorem 5, when the input formula is given by CoR instead of  $\mathbb{FO}^3$ ,  
 101 the model checking problem is exponentially more efficiently solvable.

### 102 **4 Complexity gap between FO3 and CoR**

103 A natural question arising from Theorem 5 is whether there is a faster algorithm without  
 104 the exponential blowup with respect to the length of the input sentence. Namely,

105 Can we give an  $\text{poly}(\|\varphi\|) \cdot n^{3-\varepsilon}$  time algorithm for the model checking problem for  $\mathbb{FO}^3$ ?

106 In this section, we answer to this question negatively, assuming SETH, as follows.

107 ► **Theorem 6.** *Under SETH, for any  $k \geq 1$  and any  $\varepsilon > 0$ ,<sup>7</sup> the model checking problem for*  
 108  *$\mathbb{FO}^k$  sentences cannot be decided in  $\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$  time.*

<sup>7</sup> When  $k \leq 2$ , the model checking problem cannot be decided in  $\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$  time (even when  $\varphi$  is fixed), as the input adjacent matrix is given by  $\Theta(n^2)$  cells (Proposition 14, for a more precise proof).

109 Theorem 6 is a direct consequence of the following lemma.

110 ► **Lemma 7.** *Suppose that there are an integer  $k \geq 1$ , a function  $f$ , and an  $\varepsilon > 0$  such that*  
 111 *the model checking problem for  $\mathbb{FO}^k$  sentences is solvable in  $\mathcal{O}(f(\|\varphi\|) \cdot n^{k-\varepsilon})$  time. Then*  
 112 *CNF-SAT is in  $\mathcal{O}(f(\|\varphi\|) \cdot 2^{n(1-\varepsilon/k)})$  time, where  $\|\varphi\|$  is the length of the CNF  $\varphi$  and  $n$  is*  
 113 *the number of variables in  $\varphi$ .*

114 **Proof.** Let  $\varphi$  be a CNF with  $n$  variables. *W.l.o.g.*, assume that  $k$  divides  $n$ . Let  $p_1, \dots, p_n$   
 115 be variables in  $\varphi$ . We define  $\mathfrak{A}$  as the structure with  $2^{n/k}$  vertices and unary relation symbols  
 116  $P_1, \dots, P_{n/k}$  such that for every distinct pair  $\langle v, w \rangle$  of vertices in  $\mathfrak{A}$ , there is some unary  
 117 relation symbol  $P_i$  such that  $v \in P_i^{\mathfrak{A}}$  and  $w \notin P_i^{\mathfrak{A}}$  hold or  $v \notin P_i^{\mathfrak{A}}$  and  $w \in P_i^{\mathfrak{A}}$  hold. We also  
 118 define  $\psi$  as the following  $\mathbb{FO}^k$  sentence:

$$119 \quad \exists z_1 \dots \exists z_k. \varphi',$$

120 where  $\varphi'$  is the  $\varphi$  in which each variable  $p_{kq+r}$  (where  $0 \leq q < n/k$  and  $1 \leq r \leq k$ ) has been  
 121 replaced with the atomic formula  $P_q(z_r)$ . For instance, when  $\varphi$  is the following CNF with 6  
 122 variables  $(p_1 \vee p_3) \wedge (p_5 \vee p_2) \wedge (p_4 \vee p_6)$ , the  $\mathbb{FO}^3$  sentence  $\psi$  is given as follows:

$$123 \quad \exists z_1. \exists z_2. \exists z_3. (P_1(z_1) \vee P_1(z_3)) \wedge (P_2(z_2) \vee P_1(z_2)) \wedge (P_2(z_1) \vee P_2(z_3)).$$

124 For each valuation  $\mathbf{v}$  of  $\varphi$ , let  $\mathbf{v}'$  be the (unique) valuation so that for all  $0 \leq q < n/k$  and  
 125  $1 \leq r \leq k$ ,  $\mathbf{v}(p_{kq+r})$  is true iff  $\mathbf{v}'(z_r) \in P_q^{\mathfrak{A}}$ . Then,  $\varphi$  is true at  $\mathbf{v}$  iff  $\mathfrak{A}, \mathbf{v}' \models \varphi'$ . As the  
 126 map above is a bijection, we have that  $\varphi$  is satisfiable iff  $\mathfrak{A} \models \psi$ . Thus, by applying the  
 127 assumption, CNF-SAT is in  $\mathcal{O}(f(\|\varphi\|) \cdot (2^{n/k})^{k-\varepsilon})$  time. ◀

128 YN: TODO: Add a note on succinctness.

129 Moreover, the result above still holds even over the signature with exactly one binary  
 130 relation symbol.

YN: Theorem 8 uses non-PNF formulas.

131 ► **Theorem 8.** *Under SETH, for any  $k \geq 1$  and any  $\varepsilon > 0$ ,<sup>8</sup> the model checking problem for*  
 132  *$\mathbb{FO}^k$  sentences with exactly one binary relation symbol  $E$  cannot be decided in  $\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$*   
 133 *time.*

134 Theorem 8 is a direct consequence of the following lemma.

135 ► **Lemma 9.** *Suppose that there is an integer  $k \geq 2$  and an  $\varepsilon > 0$  such that the model*  
 136 *checking problem for  $\mathbb{FO}^k$  sentences with exactly one binary relation symbol  $E$  is solvable in*  
 137  *$\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$  time. Then CNF-SAT is in  $\text{poly}(\|\varphi\|) \cdot 2^{n(1-\varepsilon/k)}$  time, where  $\|\varphi\|$  is the*  
 138 *length of the CNF  $\varphi$  and  $n$  is the number of variables in  $\varphi$ .*

139 **Proof.** Let  $\varphi$  be a CNF with  $n$  variables. *W.l.o.g.*, assume that  $k$  divides  $n$ . Let  $p_1, \dots, p_n$   
 140 be variables in  $\varphi$ . We define the structure  $\mathfrak{A}$  as follows:

- 141 ■  $|\mathfrak{A}| := (\{V\} \times [0, 2^{n/k} - 1]) \uplus (\{E\} \times [1, n/k])$ ; we write  $v^V$  for  $\langle V, v \rangle$  (similarly for  $v^E$ ).
- 142 ■  $E^{\mathfrak{A}} := \{ \langle v^V, w^E \rangle \mid v \in [0, 2^{n/k} - 1], w \in [1, n/k], \text{ and the } w\text{-th bit of } v \text{ is } 1 \} \cup \{ \langle w^E, (w + 1)^E \rangle \mid w \in [1, n - 1] \} \cup \{ \langle n^E, n^E \rangle \}$ .

<sup>8</sup> When  $k \leq 2$ , the model checking problem cannot be decided in  $\text{poly}(\|\varphi\|) \cdot n^{k-\varepsilon}$  time (even when  $\varphi$  is fixed), as the input adjacent matrix is given by  $\Theta(n^2)$  cells (Proposition 14, for a more precise proof).

144 We also define  $\psi$  as the following  $\mathbb{FO}^k$  sentence:

$$145 \quad \exists z_1 \dots \exists z_k. \varphi',$$

146 where  $\varphi'$  is the  $\varphi$  in which each variable  $p_{kq+r}$  (where  $0 \leq q < n/k$  and  $1 \leq r \leq k$ ) has been  
147 replaced with the  $\mathbb{FO}^2$  formula  $X_q(z_r)$  defined as follows:

$$148 \quad X_q(z) := \begin{cases} E(z, z) & \text{if } q = 0, \\ \exists z'. E(z, z') \wedge X_{q-1}(z') & \text{if } q > 0 \text{ where } z' \text{ is a variable not equal to } z. \end{cases}$$

149 Note that  $X_q(z_r)$  is **semantically equivalent** to the CoR formula  $(E^q(E \cap I))^d(z_r, z_r)$ .

150 For instance when  $k = 3$ , the variable  $p_{2 \cdot 1 + 3}$  is transformed to the following formula:

$$151 \quad \exists z'. E(z_3, z') \wedge (\exists z''. E(z', z'') \wedge E(z'', z'')).$$

152 For each **valuation**  $\mathbf{v}$  of  $\varphi$ , we can take a **valuation**  $\mathbf{v}'$  of  $\varphi'$  so that for all  $0 \leq q < n/k$  and  
153  $1 \leq r \leq k$ ,  $\mathbf{v}(p_{kq+r})$  is true iff  $\mathbf{v}' \models X_q(z_r)$ . Then,  $\varphi$  is true at  $\mathbf{v}$  iff  $\mathbf{v}' \models \varphi'$ . Conversely,  
154 for each **valuation**  $\mathbf{v}'$  of  $\varphi'$ , we can take a **valuation**  $\mathbf{v}$  of  $\varphi$  so that  $\varphi$  is true at  $\mathbf{v}$  iff  $\mathbf{v}' \models \varphi'$ .  
155 Thus, we have that  $\varphi$  is **satisfiable** iff  $\mathbf{v}' \models \psi$ . Thus, by applying the assumption, CNF-SAT is  
156 in  $\text{poly}(n/k \cdot \|\varphi\|) \cdot (2^{n/k} + n/k)^{k-\varepsilon}$  time. As  $n \leq \|\varphi\|$ , this implies  $\text{poly}(\|\varphi\|) \cdot 2^{n(1-\varepsilon/k)}$ . ◀

157 **YN: TODO: Add a dichotomy ( $\times \in 3$ ) in some section.**

## 158 5 FPT on F03

159 We write  $\mathbb{FO}_{k,w}^3$  to denote the set of formulas of  $\mathbb{FO}^3$  with  $k$ -predicates and  $w$ -quantifier-width.

160 ► **Theorem 10** ( $\mathbb{FO}_{k,w}^3$  model checkig is FPT).

161 *Let  $\varphi : \mathbb{FO}_{k,w}^3$ .*

162 *For a given model (graph)  $\mathcal{G}$ , we can decide if  $\mathcal{G} \models \varphi$  in  $O(2^{k+w} \cdot |\varphi| \cdot N^\omega)$  where*

163 **■**  *$N$  is the number of vertices in  $\mathcal{G}$ ; and*

164 **■**  *$\omega$  is a complexity constant for the boolean matrix multiplication (BMM): i.e.,  $O(N^\omega)$ -time  
165 is the time complexity for BMM of two  $(N, N)$  matrices.<sup>9</sup>*

166 ► **Corollary 11.** *If we fix the number of predicates  $k$  and width  $w$ , the model checking problem  
167  $\mathcal{G} \models \varphi$  in  $O(|\varphi| \cdot N^\omega)$  time-complexity where  $N$  is the number of nodes in  $\mathcal{G}$ .*

$k$ をfixしない、素朴な  $\{(x, y, z)\}$  構築アルゴリズムだと  $O(|\varphi| \cdot N^3)$  になって、その  
設定だと Theorem 10 は 係数 が大きくなりすぎて遅いということを、意味のある形  
で言及した方が良い。

168  
169 以降、Theorem 10 の証明をやっていきます。

### 170 5.1 Proof of Theorem 10

171 Here we use  $P, Q, R \dots$  to denote binary predicates and  $U, \dots$  for unary predicates.

172 For a given graph (model)  $\mathcal{G}$ , we write  $N_{\mathcal{G}}$  or simply  $N$  to denote the number of vertices  
173 of  $\mathcal{G}$ .

<sup>9</sup> 何らかの論文をciteして、今の記録がどれくらい書いておくと分かりやすい

## 174 Our idea: Quantifier elimination (Simple case)

175 Our query evaluation strategy is evaluating the innermost (quantifier-free formula) to the  
176 outermost along with quantifier eliminating.

Let us consider the following situation:

$$Qx \cdots (Qy \cdots (Qz \cdots (Qx \cdots (Qy \cdots (\exists x.(P(y, x) \wedge R(x, z)))))))$$

To eliminate  $\exists z.P(y, x) \wedge R(x, z)$ , we introduce a new predicate defined as follows:

$$S(x, y) := \exists z.P(x, z) \wedge Q(z, y).$$

177 It is worth noting that  $S$  is just the composed predicate of  $P$  and  $Q$ . So, we write  $P \odot Q$   
178 instead of  $S$ .

Using this predicate, we can rewrite the above one to the following one:

$$Qx \cdots (Qy \cdots (Qz \cdots (Qx \cdots (Qy \cdots (P \odot R)(y, z))))))$$

179

180 Continuing this process, if we eventually enter  $\forall x.\exists y.T(x, y)$ :

- 181 1. building a unary predicate  $T_y(x) := \exists y.T(x, y)$  in  $O(N^2)$ -time, we rewrite it to  $\forall x.T_y(x)$ .
- 182 2. we then simply evaluate  $\forall x.T_y(x)$  as  $\bigwedge_{1 \leq i \leq N} T_y(i)$  in  $O(N)$ -time.

## 183 Time-complexity of basic operations

184 Here we enumerate the time-complexity of our basic operations, which will be used later:

- 185 ■ Building a new predicate  $Q(x, y) := P(x, y) \bowtie R(x, y)$  ( $\bowtie \in \{\wedge, \vee\}$ ) needs  $O(N^2)$ -time.
- 186 ■ Transposing a predicate  $P^T(x, y) := P(y, x)$  needs  $O(N^2)$ -time.
- 187 ■ Evaluating unary predicates  $Qx.P(x)$  ( $Q \in \{\forall, \exists\}$ ) needs  $O(N)$ -time.
- 188 ■ Building a new predicate  $Q(x) := Qy.P(x, y)$  ( $Q \in \{\exists, \forall\}$ ) needs  $O(N^2)$ -time.

189 So, the remaining operator is predicate composing  $(P \odot Q)(x, y) := \exists z.P(x, z) \wedge Q(z, y)$   
190 from predicates  $P$  and  $Q$ . Although a very simple matrix multiplication requires  $O(N^3)$ -time,  
191 we can build such  $P \odot Q$  using fast matrix multiplication algorithms.

192 ► **Proposition 12 (Fact).** *From two predicates  $P$  and  $Q$ , we can build the composited predicate*  
193  *$P \odot Q$  in  $O(N^\omega)$ -time.*

## 194 5.2 Quantifier elimination: General case

195 If we encounter terms of the very restricted form like  $\exists z.P(x, z) \wedge Q(z, y)$ , we can replace it  
196 by  $(P \odot Q)(x, y)$  with eliminating the quantifier occurrence  $\exists z$ .

In other words, if we encounter more general form of  $\exists z \cdots$ , we need to translate it to some adequate form for quantifier elimination. Let us consider the following term:

$$\exists z. \left( (P(x, z) \vee P(y, z) \vee P(x, y)) \wedge (Q(x, z) \vee Q(x, y)) \wedge (R(y, z) \vee R(x, y)) \right).$$

197 We cannot directly apply our quantifier elimination strategy because  $\exists$ -quantifiers do not  
198 distribute on  $\wedge$ : i.e.,  $(\exists z.\varphi_1 \wedge \varphi_2) \neq (\exists z.\varphi_1) \wedge (\exists z.\varphi_2)$ .

To apply our quantifier elimination procedure, DNF is an adequate form. However, as well-known, converting-to-DNF translation generates an exponential-large expression in the worst case. For our example, we first translate the first  $(\dots) \wedge (\dots)$  subterm as follows:

$$\begin{aligned} & (P(x, z) \vee P(y, z) \vee P(x, y)) \wedge (Q(x, z) \vee Q(x, y)) \\ \Rightarrow & (P(x, z) \wedge Q(x, z)) \vee (P(y, z) \wedge Q(x, z)) \vee (P(x, y) \wedge Q(x, z)) \\ & \vee (P(x, z) \wedge Q(x, y)) \vee (P(y, z) \wedge Q(x, y)) \vee (P(x, y) \wedge Q(x, y)). \end{aligned}$$

199 We continue to normalize this term with  $R(y, z) \vee R(x, y)$  and it generates DNF with 12  
200 bases of the form  $P(\_, \_) \wedge Q(\_, \_)$ .

201 Since our goal is to design an  $O(F(k, w) \cdot |\varphi| \cdot N^\omega)$ -time algorithm, we cannot adopt  
202 converting-to-DNF transformations.

### 203 5.3 Tseytin-like (?) transformation for General Cases

204 As we have seen above, explicit translations to DNF does not work well since it may explode  
205 given expression to an exponential size.

206 We here develop a translation inspired by Tseytin's transformation.<sup>10</sup>

実際はインスパイアされたという程でもないんですが、とはいえよく知られた構成  
な気もするので、何かciteできるものがあればしたいという感じ。

207

Let us revisit the above example:

$$\exists z. \left( (P(x, z) \vee P(y, z) \vee P(x, y)) \wedge (Q(x, z) \vee Q(x, y)) \wedge (R(y, z) \vee R(x, y)) \right).$$

First, we generate all possible *assignments* to predicates. For our example, we have the following assignments:

	$P(x, z)$	$P(y, z)$	$P(x, y)$	$Q(x, z)$	$Q(x, y)$	$R(y, z)$	$R(x, y)$
$\alpha_1$	0	0	0	0	0	0	0
$\alpha_2$	0	0	0	0	0	0	1
$\alpha_3$	0	0	0	0	0	1	0
$\vdots$							
$\alpha$	1	0	1	0	1	0	0
$\vdots$							
$\alpha_\star$	1	1	1	1	1	1	1

208 Let  $\mathcal{A}$  be the set of all assignments.

209 Let  $\llbracket \Psi \rrbracket_\alpha$  be a boolean value obtained by evaluationg  $\Psi$  under an assignment  $\alpha$ .

210 For our example expression  $\psi$ ,  $\llbracket \psi \rrbracket_\alpha = 0$  using  $\alpha$  appearing in the above table.

211 If we change  $\alpha$  as  $\alpha(R(y, z)) \leftarrow 1$ , then  $\llbracket \psi \rrbracket_\alpha = 1$ .

212 Introducing assignment leads to the following useful proposition.

► **Proposition 13.**

$$\Psi(x, y, z) \iff \bigvee_{\alpha \in \mathcal{A}} (\alpha \wedge \llbracket \Psi \rrbracket_\alpha).$$

<sup>10</sup>[https://en.wikipedia.org/wiki/Tseytin\\_transformation](https://en.wikipedia.org/wiki/Tseytin_transformation)



*Epecially, let  $\mathcal{V} = \{\alpha \in \mathcal{A} : \llbracket \Psi \rrbracket_\alpha\}$ , then we have:*

$$\Psi(x, y, z) \iff \bigvee_{\alpha \in \mathcal{V}} \alpha(x, y, z). \quad (\star)$$

By the definition of assignments,  $\alpha$  takes the following form:

$$\alpha(x, y, z) \equiv P_1(x, y) \wedge P_1(x, z) \wedge P_2(y, z) \wedge \cdots \wedge P_k(x, z).$$

It means that the right-hand side of  $(\star)$  takes DNF. Therefore,

$$\exists z. \Psi(x, y, z) \iff (\exists z. \alpha_1(x, y, z)) \vee (\exists z. \alpha_2(x, y, z)) \vee \cdots \vee (\exists z. \alpha_{|\mathcal{V}|}(x, y, z)).$$

213 It suffices to eliminating an  $\exists$ -quantifier from  $\exists z. \alpha(x, y, z)$  for an assignment  $\alpha$ .

We first divide  $\alpha(x, y, z)$  into two parts  $\alpha_z$  and  $\alpha_{-z}$  as follows:

$$\alpha(x, y, z) \equiv \underbrace{(P_1(x, z) \wedge P_2(z, x) \wedge P_1(y, z) \wedge \cdots)}_{\alpha_z: z\text{-involving-terms}} \wedge \underbrace{(P_1(x, y) \wedge P_1(y, x) \wedge P_2 \wedge \cdots)}_{\alpha_{-z}: z\text{-non-involving-terms}}$$

We now have the following:

$$(\exists z. \alpha(x, y, z)) \iff (\alpha_{-z}(x, y) \wedge (\exists z. \alpha_z(x, y, z))).$$

214 Finally, we transform  $\alpha_z$  in the following steps (order):

- 215 1. If  $\alpha_z$  has a term  $P(z, x)$ , we replace it with  $\tilde{P}(x, z)$  where  $\tilde{P}(a, b) = P(b, a)$ .
- 216 2. Similarly, we change terms of the form  $P(y, z)$  to  $\tilde{P}(z, y)$ .
3. At this point,

$$\alpha_z(x, y, z) \equiv \underbrace{(P_1(x, z) \wedge P_3(x, z) \wedge P_4(x, z) \wedge \cdots)}_{\Psi_x} \wedge \underbrace{(P_2(z, y) \wedge P_3(z, y) \wedge P_5(z, y) \wedge \cdots)}_{\Psi_y}.$$

- 217 4. Let  $\Psi_x$  (resp.  $\Psi_y$ ) be the set of predicates appearing in the above former (resp. latter) part.
- 218 5. Let us introduce two new predicates:

$$\pi(x, z) := \bigwedge_{P \in \Psi_x} P(x, z), \quad \lambda(z, y) := \bigwedge_{Q \in \Psi_y} Q(z, y).$$

6. We reach our goal:

$$\alpha_z(x, y, z) = \pi(x, z) \wedge \lambda(z, y).$$

Using the final form, we obtain the following:

$$\begin{aligned} (\exists z. \alpha(x, y, z)) &\iff \left( \alpha_{-z}(x, y) \wedge (\exists z. \pi(x, z) \wedge \lambda(z, y)) \right) \\ &\iff \left( \alpha_{-z}(x, y) \wedge (\pi \odot \lambda)(x, y) \right) \\ &\iff (\alpha_{-z} \wedge (\pi \odot \lambda))(x, y) \end{aligned}$$

219 **5.4 Considering Complexity**

220 To eliminate an  $\exists$ -quantifier from an assignment  $\alpha$ , we introduce a new single predicate  
 221  $\alpha_{-z} \wedge (\pi \odot \lambda)$ .

Now we have a question: To eliminate the  $\exists$ -quantifier from  $\exists z. \Psi(x, y, z)$ , from the following characterization

$$\left( \exists z. \Psi(x, y, z) \right) \iff \left( \exists z. \bigvee_{\alpha \in \mathcal{V}} \alpha(x, y, z) \right) \iff \bigvee_{\alpha \in \mathcal{V}} \left( \exists z. \alpha(x, y, z) \right) \iff \bigvee_{\alpha \in \mathcal{V}} (\alpha_{-z} \wedge (\pi \odot \lambda))(x, y)$$

can we avoid to introduce  $O(2^k)$  predicates if we have  $k$ -predicates?? Please recall that each assignment is a choice from  $9k$ -literals from  $\mathcal{L}$  where

$$\mathcal{L} = \{Q(a, b) : a, b \in \{x, y, z\}, Q \in \{P_1, P_2, \dots, P_k\}\}.$$

**Yes. We can avoid introducing a large number of predicates** because we just need to introduce the following single (but large) predicate:

$$(\alpha_{-z}^1 \wedge (\pi^1 \odot \lambda^1)) \vee (\alpha_{-z}^2 \wedge (\pi^2 \odot \lambda^2)) \vee \dots \vee (\alpha_{-z}^n \wedge (\pi^n \odot \lambda^n))$$

222 for  $\mathcal{V} = \{\alpha^1, \alpha^2, \dots, \alpha^n\}$ .

Let us consider what happens when repeatedly eliminating quantifiers from the innermost to the outermost:

$$\begin{aligned} & \exists y. \left( (\exists z. \dots) \bowtie (\exists z. \dots) \bowtie \dots \bowtie (\exists z. \dots) \right) \\ & \Rightarrow \\ & \exists y. \left( \begin{array}{l} \text{Q. how many predicates appear in this scope?} \\ \text{A. } k + w \text{ where } w \text{ is the quantifier width of } \psi \end{array} \right) \\ & \Rightarrow \\ & \text{Eliminating } (\exists y.) \text{ introduces a single new predicate.} \end{aligned}$$

223 Since the required number of quantifier elimination (= the quantifier depth) is bounded  
 224 by the size of a given expression  $|\psi|$ , we obtain the following our theorem in the end.

225 ► **Theorem 10 (Restatement).** Let  $\varphi : \mathbb{FO}_{k,w}^3$ .

226 For a given model (graph)  $\mathcal{G}$ , we can decide if  $\mathcal{G} \models \varphi$  in  $O(2^{k+w} \cdot |\varphi| \cdot N^w)$ .

227 **6 FO2 model checking**

$\mathbb{FO}^2$ の論理式  $\psi$  と、グラフ  $\mathcal{G}$  について  $\mathcal{G} \models \psi$  のモデル検査問題は  $O(|\psi| \cdot |\mathcal{G}|)$  で解ける気がします。ただし、 $|\mathcal{G}|$ は頂点数ではなく、述語（つまり辺）の定義の記述長も合わせたものです。面白そうなら書いても良いかと思うのですが、 $\mathbb{FO}^3$ の話だけにしようが、話がぼやけないならなくても良いかなという気がします。

Dense graphについては  $(x, y)$  の値の集合を構築する方法で  $O(|\psi| \cdot N^2)$  になります。この  $N$  は  $\mathcal{G}$  の頂点数の方です。ただ、sparse graphの時には  $N^2$  は  $|\mathcal{G}|$  について線形ではないので、そこを改良したいという感じです。

アイディアは、 $\mathbb{FO}^3$ の時と比べると、 $\exists y. P(x, y) \wedge Q(y, x) \simeq (P \odot Q)$  の計算について  $(P \odot Q)(x, x)$  の diagonal な形しか要求しないので、行列積を全部やる必要が多分なさそうとかそういう感じです。

Sparse graphの話なので、もしかすると Impagliazzo たちがやっているかもしれません。

YN: [10] Lemma 9.1 Base Case は  $O(|\psi| \cdot |\mathcal{G}|)$  ??

## 7 Conclusion

### References

- 1 Karl Bringmann, Nick Fischer, and Marvin Künnemann. A fine-grained analogue of schaefer's theorem in P: Dichotomy of  $\exists^k\forall$ -quantified first-order graph properties. In *CCC*, volume 137 of *LIPICs*, page 31:1–31:27. Schloss Dagstuhl, 2019. doi:10.4230/LIPICs.CCC.2019.31.
- 2 Anthony Brogni and Sebastiaan J. C. Joosten. Translating three-variable first-order predicate logic to relation algebra, implemented using Z3, 2025. doi:10.48550/arXiv.2308.02513.
- 3 Yijia Chen, Michael Elberfeld, and Moritz Müller. The parameterized space complexity of model-checking bounded variable first-order logic. *Logical Methods in Computer Science*, Volume 15, Issue 3, 2019. doi:10.23638/LMCS-15(3:31)2019.
- 4 Marek Cygan, Holger Dell, Daniel Lokshtanov, Dániel Marx, Jesper Nederlof, Yoshio Okamoto, Ramamohan Paturi, Saket Saurabh, and Magnus Wahlström. On problems as hard as CNF-SAT. *ACM Trans. Algorithms*, 12(3):41:1–41:24, 2016. doi:10.1145/2925416.
- 5 Joerg Flum and Martin Grohe. Model-checking problems as a basis for parameterized intractability. *Logical Methods in Computer Science*, Volume 1, Issue 1, 2005. doi:10.2168/LMCS-1(1:2)2005.
- 6 Jörg Flum and Martin Grohe. Fixed-parameter tractability, definability, and model-checking. *SIAM Journal on Computing*, 31(1):113–145, 2001. doi:10.1137/S0097539799360768.
- 7 Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. doi:10.1007/3-540-29953-X.
- 8 Markus Frick and Martin Grohe. The complexity of first-order and monadic second-order logic revisited. *Annals of Pure and Applied Logic*, 130(1-3):3–31, 2004. doi:10.1016/J.APAL.2004.01.007.
- 9 Jiawei Gao and Russell Impagliazzo. The fine-grained complexity of strengthenings of first-order logic, 2019. URL: <https://eccc.weizmann.ac.il/report/2019/009/>.
- 10 Jiawei Gao, Russell Impagliazzo, Antonina Kolokolova, and Ryan Williams. Completeness for first-order properties on sparse structures with algorithmic applications. *ACM Trans. Algorithms*, 15(2):23:1–23:35, 2018. doi:10.1145/3196275.
- 11 Russell Impagliazzo and Ramamohan Paturi. On the complexity of  $k$ -SAT. *Journal of Computer and System Sciences*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727.
- 12 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001. doi:10.1006/jcss.2001.1774.
- 13 François Le Gall, Harumichi Nishimura, and Seiichiro Tani. Quantum algorithms for finding constant-sized sub-hypergraphs. *Theoretical Computer Science*, 609:569–582, 2016. doi:10.1016/j.tcs.2015.10.006.
- 14 Andrea Lincoln, Virginia Vassilevska Williams, and Ryan Williams. Tight hardness for shortest cycles and paths in sparse graphs. In *SODA*, pages 1236–1252. SIAM, 2018. doi:10.1137/1.9781611975031.80.
- 15 Yoshiki Nakamura. Expressive power and succinctness of the positive calculus of binary relations. *Journal of Logical and Algebraic Methods in Programming*, 127:100760, 2022. doi:10.1016/j.jlamp.2022.100760.
- 16 Mihai Pătraşcu and Ryan Williams. On the possibility of faster SAT algorithms. In *SODA*, Proceedings, pages 1065–1075. SIAM, 2010. doi:10.1137/1.9781611973075.86.
- 17 Alfred Tarski. On the calculus of relations. *The Journal of Symbolic Logic*, 6(3):73–89, 1941. doi:10.2307/2268577.
- 18 Alfred Tarski and Steven Givant. *A Formalization of Set Theory without Variables*, volume 41. American Mathematical Society, 1987. doi:10.1090/coll/041.
- 19 Moshe Y. Vardi. On the complexity of bounded-variable queries (extended abstract). In *PODS*, page 266–276. ACM, 1995. doi:10.1145/212433.212474.

- 280 **20** Virginia Vassilevska Williams. Hardness of easy problems: Basing hardness on popular  
281 conjectures such as the strong exponential time hypothesis. In *IPEC*, volume 43 of *LIPICs*,  
282 page 17–29. Schloss Dagstuhl, 2015. doi:10.4230/LIPICs.IPEC.2015.17.
- 283 **21** Ryan Williams. Faster decision of first-order graph properties. In *CSL-LICS*, page 1–6. ACM,  
284 2014. doi:10.1145/2603088.2603121.

## A Tips

### A.1 Trivial Results

► **Proposition 14** (Almost trivial fact). *Let  $\varphi_0$  be the following  $\text{FO}^k$  sentence:*

$$\exists x_1 \dots \exists x_k. R(x_1, \dots, x_k).$$

*Then for any  $\varepsilon > 0$ , the model checking problem where the sentence is fixed to  $\varphi_0$  cannot be decided in  $\mathcal{O}(n^{k-\varepsilon})$  time (without any assumptions).*

**Proof.** (Below, we suppose that we can access to at most one cell in each step.) Towards a contradiction, assume that for some  $M$ , there is an  $Mn^{k-\varepsilon}$  time RAM model  $\mathcal{M}$  solving the model checking problem (for sufficiently large  $n$ ). Then, a sufficiently large  $n_0 > n$  satisfies  $Mn_0^{k-\varepsilon} < n_0^k$ .

Let  $\mathfrak{A}$  be the structure with  $n_0$  vertices such that  $R^{\mathfrak{A}} = \emptyset$ . We consider the run  $\rho$  of  $\mathcal{M}$  when the input is  $\mathfrak{A}$ . In the run  $\rho$  (of time at most  $Mn^{k-\varepsilon}$ ), as  $Mn_0^{k-\varepsilon} < n_0^k$ , there is an input cell  $c$  for asserting  $\langle v_1, \dots, v_k \rangle \in R^{\mathfrak{A}}$  such that  $\rho$  does not access  $c$ . We then define  $\mathfrak{B}$  as the structure  $\mathfrak{A}$  in which  $R^{\mathfrak{B}}$  has been replaced with the singleton set  $\{\langle v_1, \dots, v_k \rangle\}$ . When  $\mathfrak{B}$  is input, the run is the same as  $\rho$ , as the cell  $c$  is not accessed in  $\rho$ . Hence, in  $\mathcal{M}$ , the output of  $\mathfrak{B}$  is also the same as that of  $\mathfrak{A}$ , reaching a contradiction. (When  $\mathfrak{A}$  is input, the output should be “no”. When  $\mathfrak{B}$  is input, the output should be “yes”.) ◀

YN: 同様の議論がある論文/本からの引用などで簡単に済ませたい。  
とくに 3 項関係記号をもつ  $\text{FO}^3$  は、 $\mathcal{O}(n^{3-\varepsilon})$  で解けない (行列積で解けるのは 2 項関係記号の時)。

► **Proposition 15.** *The model checking problem for  $\text{FO}^k$  with only unary relation symbols (without equality) is solvable in  $\mathcal{O}(n)$  time.*

**Proof Sketch.** In this case, by taking the DNF, we can transform each subformula of the form  $\exists x. \varphi$  into the following form:  $\exists x. \bigwedge_i P_i(x)$ . Thus, each subformula has at most one free variable. Hence, by the naive evaluation algorithm, we have obtained  $\mathcal{O}(n)$  time. ◀

### A.2 Summary

Problem / max. arity $k$	1	2	3
PNF $\text{FO}^3$ (data)	$\mathcal{O}(n)$ (Proposition 15)	$\mathcal{O}(n^\omega)$ [21]	$\mathcal{O}(n^3)$ (trivial)
$\text{FO}^3$ (data)	no $\mathcal{O}(n^{k-\varepsilon})$ (Proposition 14)		
PNF $\text{FO}^3$ (combined)	$\mathcal{O}(\ \varphi\  \cdot n^3)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^3)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^3)$ (trivial)
$\text{FO}^3$ (combined)	$\mathcal{O}(2^{\ \varphi\ } \cdot n)$ (Proposition 15)	$\mathcal{O}(2^{\ \varphi\ } \cdot n^\omega)$ [21, 15]	
	no poly $(\ \varphi\ ) \cdot n^{3-\varepsilon}$ , under SETH (Theorem 6)		

(以下おまけ)

Problem / max. arity $k$	1	2	3	4
PNF $\text{FO}^4$ (data)	$\mathcal{O}(n)$ (Proposition 15)	$\mathcal{O}(n^{1+\omega})$ [21, Cor. 1.3] (?? for $\text{FO}^4$ )	$\mathcal{O}(n^{\omega_3})$	$\mathcal{O}(n^4)$ (trivial)
$\text{FO}^4$ (data)	no $\mathcal{O}(n^{1-\varepsilon})$ (Proposition 14)	$\mathcal{O}(n^{\omega_3})$	no $\mathcal{O}(n^{k-\varepsilon})$ (Proposition 14)	
PNF $\text{FO}^4$ (combined)	$\mathcal{O}(\ \varphi\  \cdot n^4)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^4)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^4)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^4)$ (trivial)
$\text{FO}^4$ (combined)	$\mathcal{O}(2^{\ \varphi\ } \cdot n)$ (Proposition 15)	$\mathcal{O}(2^{\ \varphi\ } \cdot n^{1+\omega})$ [21, Cor. 1.3] (?? for $\text{FO}^4$ )	$\mathcal{O}(2^{\ \varphi\ } \cdot n^{\omega_3})$	
		$\mathcal{O}(2^{\ \varphi\ } \cdot n^{\omega_3})$		
	no poly $(\ \varphi\ ) \cdot n^{4-\varepsilon}$ , under SETH (Theorem 6)			

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315

Problem / max. arity $k$	1	2	3	4	5
PNF $\mathbb{FO}^5$ (data)	$\mathcal{O}(n)$ (Proposition 15)	$\mathcal{O}(n^{2+\omega})$ [21, Cor. 1.3] (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(n^{1+\omega_3})$ (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(n^{\omega_4})$	$\mathcal{O}(n^{1+\omega_3})$ [21, Cor. 1.3] (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(n^{\omega_4})$	$\mathcal{O}(n^{\omega_4})$	$\mathcal{O}(n^5)$ (trivial)
$\mathbb{FO}^5$ (data)	no $\mathcal{O}(n^{1-\varepsilon})$ (Proposition 14)	no $\mathcal{O}(n^{1-\varepsilon})$ under SETH [21, Cor. 1.4]		no $\mathcal{O}(n^{1-\varepsilon})$ (Proposition 14)	
PNF $\mathbb{FO}^5$ (combined)	$\mathcal{O}(\ \varphi\  \cdot n^5)$ (trivial)	$\mathcal{O}(\ \varphi\  \cdot n^5)$ (trivial) $\mathcal{O}(2^{\ \varphi\ } \cdot n^{2+\omega})$ [21, Cor. 1.3] (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(2^{\ \varphi\ } \cdot n^{1+\omega_3})$ (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(2^{\ \varphi\ } \cdot n^{\omega_4})$	$\mathcal{O}(\ \varphi\  \cdot n^5)$ (trivial) $\mathcal{O}(n^{1+\omega_3})$ (?? for $\mathbb{FO}^5$ ) $\mathcal{O}(2^{\ \varphi\ } \cdot n^{\omega_4})$	$\mathcal{O}(\ \varphi\  \cdot n^5)$ (trivial) $\mathcal{O}(2^{\ \varphi\ } \cdot n^{\omega_4})$	$\mathcal{O}(\ \varphi\  \cdot n^5)$ (trivial)
$\mathbb{FO}^5$ (combined)	$\mathcal{O}(2^{\ \varphi\ } \cdot n)$ (Proposition 15)	no poly $(\ \varphi\ ) \cdot n^{\omega-\varepsilon}$ , under SETH (Theorem 6)			

YN:

■ ?? は主張が不明/怪しい箇所（メモ2参照）

- 主張が成り立たなそうであることを言うには、  
Binary  $\mathbb{FO}^4$  is in  $\mathcal{O}(n^{4-\varepsilon})$  time  $\implies \omega_3 < 4$  を言えればよさそう。

► **Observation 16.** \*  $\omega_2 = \omega < 2.38 (< 3)$ .

\*  $\omega_3 < 4$  is wished [21] but open.

4-clique in a 3-uniform hypergraph を含む難しい問題のようです。Cf. HyperClique Conjecture

- もしくは、もう少し精密化して今回の設定に合った行列積の問題に一般化する。  
入力を  $[n]$  上の任意有限個の  $m$  項関係として、 $k$  引数関係  $A_1, \dots, A_k$  は入力の  $m$  項関係から  $\mathbb{FO}^{k+1}$ -definable であるとする。この時の“一般化行列積”

$$D[i_1, \dots, i_k] = \bigvee_{\ell=1}^n \bigwedge_{j=1}^k A_j[i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k, \ell]$$

の時間計算量を  $\mathcal{O}(n^{\omega_{k,m}})$  とする。

\* Fact:  $\omega_{k,k} = \omega_k$ 。とくに  $\omega_{2,2} = \omega$ 。

( $\omega_{k,k} \leq \omega_k$ :  $\mathbb{FO}^{k+1}$  model checking が  $\mathcal{O}(n^{\omega_k})$  より。  $\omega_{k,k} \geq \omega_k$ :  $A_1, \dots, A_k$  が入力になる場合を考える)

\* Fact:  $\omega_{k,1} = k$  (成分ごと独立なので)

\* Fact:  $k = \omega_{k,1} \leq \omega_{k,2} \leq \dots \leq \omega_{k,k} = \omega_k \leq k+1$

\* Fact:  $\omega_k = k+1$ , under  $k$ -uniform HyperClique Hypothesis [1], for  $k \geq 3$

$\rightsquigarrow$  (??)  $\omega_{k,2} = k+1$ , under “Strong”  $k$ -uniform HyperClique Hypothesis, for  $k \geq 3$

みたいなことを言っておいても許される??

We then have:

► **Proposition 17.** Binary  $\mathbb{FO}^4$  is in  $\mathcal{O}(n^c)$  time  $\iff \omega_{3,2} \leq c$ .

**Proof.**  $\Leftarrow$ : By an analog of the FO3-to-CoR translation.  $\Rightarrow$ : Every matrix multiplication problem w.r.t.  $\omega_{3,2}$  can be expressed as a model checking problem where the sentence is fixed to a binary  $\mathbb{FO}^4$  sentence. ◀

► **Corollary 18.** If binary  $\mathbb{FO}^4$  is in  $\mathcal{O}(n^{1+\omega})$  time, then  $\omega_{3,2} \leq 1 + \omega$ . Hence,  $\omega_{3,2} < 4$ .

$\rightsquigarrow$  “Strong”  $k$ -uniform HyperClique Hypothesis に反する

$\rightsquigarrow \omega_{3,2}$  の問題の形で書けるが  $\mathcal{O}(n^{4-\varepsilon})$  が open な問題はあるでしょうか?

■ 赤の箇所は未証明

316

## B memo

## 1. 2025/06/25 メールより

あとは、FO3 が  $O(2^{\delta\|\varphi\|}n^\omega)$  time ( $\delta < 1$ ) で解けないことを示せるか? という問題も考えられそうです。(これが示せば  $O(2^{\delta\|\varphi\|})$  time の FO3-to-CoR の変換がないこともいえます)

(Open Problem)

## 2. 2025/07/04 メールより

YN:

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[21, Theorem 1.3] の大枠の方針は、CoR変換のコアの部分と似た感じで、 $\exists y.P(\vec{x}, y) \wedge Q(y, \vec{z})$  の形に持って行って、\*最も内側の\*  $\exists$  に対して行列積するというものです。ここで  $\vec{x}$  と  $\vec{z}$  には以下のような条件を課しておきます。

- 共通の変数を持たない
- 次元はおおよそ  $(k-1)/2$  ずつに分割されている

この方針を一般に拡張しようとすると、たとえば

$$\exists y.P(x_1, x_2, y) \wedge Q(x_2, x_3, y) \wedge R(x_3, x_2, y)$$

(ただし、各  $P(x_1, x_2, y)$  はたとえば  $\exists z.P'(x_1, z) \wedge P'(x_2, z) \wedge P'(z, y)$  のような式)

みたいな式をどうやって行列積にしますか? という問題が発生しそうです。

自由変数が高々2の場合 (FO3の場合) には、外側の量子子の自由変数を除くと自由変数が高々1つなので、綺麗な分割ができますが、そうでなかった場合には、自由変数間の依存が生じるので、行列積に持っていけないのでは? という気がしています。# 逆に言うとこの辺りの計算量をもう少し厳密に考える価値がありそう?

★  $k \geq 4$  の時に以下は本当に成り立つか? (cf. [9, Theorem 7])

- the model checking problem for  $\mathbb{FO}^k$  sentences in ~~prenex normal form~~ can be decided in  $n^{k-1+o(1)}$  time for  $k \geq 9$ .
- the model checking problem for  $\mathbb{FO}^k$  sentences in ~~prenex normal form~~ can be decided in  $O(n^{k-3+\omega})$  time for  $k \geq 3$ .
  - 定数置き換えの方針で、たとえば上の論理式で  $x_3$  を定数としても  $P(x_1, x_2, y)$  によって行列積にならない (cf. Proposition 14)
  - 下の“関連”の方針から高々  $k-1$  項関係であれば  $O(n^{\omega_{k-1}})$  time は言えるはず。
    - \* “ $k$ 次元行列積”をもつ “ $k$ -ary CoR” を考えると同じ議論。
    - \* したがって、 $\omega_{k-1} < k$  の仮説 (?) のもと  $O(n^{k-\varepsilon})$  time for some  $\varepsilon > 0$ .

関連 ( $\mathbb{FO}^4$  が3項関係をもつ場合、 $O(2^{\|\varphi\|}n^{\omega_3})$  time。ただし  $\omega_k$  は以下の“ $k$ 次元行列積”の時間計算量 ( $k-1 \leq \omega_k \leq k$ ) ( $\omega_2 = \omega$ 。  $\omega_k < 1$  を満たすかどうかは  $k \geq 3$  で open (?) )

- [21, Open Problem 2 (p.5)]

“What about for vocabularies with ternary relations? ... We wish to compute the following 3D matrix in  $n^{4-\varepsilon}$  time, for some  $\varepsilon > 0$ :

$$D[i, j, k] = \sum_{\ell=1}^n A[i, j, \ell] \cdot B[j, k, \ell] \cdot C[k, i, \ell].$$

... would allow us to more efficiently find (for example) a 4-clique in a 3-uniform hypergraph, which can be expressed as a first-order sentence over a ternary relation.”

- [13]

“... no efficient classical algorithm for 4-clique finding has been discovered so far.”

- [14]

“the generalized matrix product related to finding hypercliques in  $k$ -uniform hypergraphs cannot be sped up with a Strassen like technique”

## 323 3. 2025/6/27 メールより

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(補足) 結果を組み合わせると、以下の dichotomy を示せそうです (関係記号の個数, 定数記号の個数, 等号の有無を考えています)

【冠頭標準形の FO3 の場合】

- 一般 (\*) :  $\text{poly}(|\psi|)N^{3-\varepsilon}$  だと SETH に反する ← ② と その修正 (\*)  
厳密には、1 引数関係記号を  $\omega$  個含む場合 もしくは 定数記号を  $\omega$  個含む場合 (等号あり) もしくは 2 引数関係記号 1 個以上含むかつ定数記号を  $\omega$  個含む場合
- 関係記号の個数  $k$  固定 :  $O(|\psi|N^\omega)$  ← 上里さんの方針
- 1 引数関係記号  $k$  個 かつ 定数記号  $\omega$  個 (等号なし) :  $O(|\psi|N^\omega)$  ← 上里さんの方針 (+ 定数記号を先に除去)

【一般の FO3 の場合】

- 一般 (\*) :  $\text{poly}(|\psi|)N^{3-\varepsilon}$  だと SETH に反する ← ② と その修正 (\*)  
厳密には、1 引数関係記号を  $\omega$  個含む場合 もしくは 定数記号を  $\omega$  個含む場合 (等号あり) もしくは 2 引数関係記号 1 個以上含む場合
- 1 引数関係記号  $k$  個 かつ 定数記号  $\omega$  個 (等号なし) :  $O(|\psi|N^\omega)$  ← 上里さんの方針 (+ 定数記号を先に除去 + DNF 変換して量子子内側入れの後の論理式  $\exists x \bigwedge_i \alpha_i$  (各  $\alpha_i$  は  $P(x)$  or  $\neg P(x)$ ) が有限通り)
- 1 引数関係記号  $k$  個 かつ 定数記号  $k'$  個 (等号あり) :  $O(|\psi|N^\omega)$  ← 上里さんの方針 (+ DNF 変換して量子子内側入れの後の論理式  $\exists x \bigwedge_i \alpha_i$  (各  $\alpha_i$  は  $P(x), \neg P(x), \neg x = y, \neg x = z, \text{ or } \neg x = c$ ) が有限通り)

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↪ 「定数記号を  $\omega$  個含む場合 (等号あり)」は結論自体誤り。<sup>11</sup>

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↪ ひとまず定数記号を考えず、関係記号の個数と等号の有無を考えるなら以下

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【冠頭標準形の FO3 の場合】

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- 一般 (\*) :  $\text{poly}(|\psi|)N^{3-\varepsilon}$  だと SETH に反する ← ② と その修正 (\*) 厳密には、1 引数関係記号を  $\omega$  個含む場合 (等号なし)

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- 関係記号の個数  $k$  固定 (等号あり) :  $O(|\psi|N^\omega)$  ← 上里さんの方針

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【一般の FO3 の場合】

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- 一般 (\*) :  $\text{poly}(|\psi|)N^{3-\varepsilon}$  だと SETH に反する ← ② と その修正 (\*) 厳密には、1 引数関係記号を  $\omega$  個含む場合 (等号なし) もしくは 2 引数関係記号 1 個以上含む場合 (等号なし)

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- 1 引数関係記号  $k$  個 (等号あり) :  $O(|\psi|N^\omega)$  ← 上里さんの方針 (+ DNF 変換して量子子内側入れの後の論理式  $\exists x \bigwedge_i \alpha_i$  (各  $\alpha_i$  は  $P(x), \neg P(x), \neg x = y, \neg x = z$ ) が有限通り)

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<sup>11</sup> この時、各変数の割当は、定数が指す箇所  $\|\psi\|$  通り + 定数が指さない場合 1 通りのみを考えれば十分なので  $\text{poly}(\|\psi\|)$  time。 (構造のサイズに依存しない)