

GATE: BM - 28.2021

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Results and Derivations:

Let a function $y(x, t)$ be defined for all $t > 0$ and assumed to be bounded. Then we can apply the Laplace transform in t considering x as a parameter.

$$\mathcal{L}(y(x, t)) = \int_0^\infty e^{-st} y(x, t) dt \quad (1)$$

$$= Y(x, s) \quad (2)$$

Let $\frac{\partial y(x, t)}{\partial t}$ be $y_t(x, t)$ and $\frac{\partial y(x, t)}{\partial x}$ be $y_x(x, t)$, then

$$\mathcal{L}(y_t(x, t)) = \int_0^\infty e^{-st} y_t(x, t) dt \quad (3)$$

$$= e^{-st} y(x, t) \Big|_0^\infty + s \int_0^\infty e^{-st} y(x, t) dt \quad (4)$$

$$= sY(x, s) - y(x, 0) \quad (5)$$

$$\mathcal{L}(y_x(x, t)) = \int_0^\infty e^{-st} y_x(x, t) dt \quad (6)$$

$$= \frac{d}{dx} \int_0^\infty e^{-st} y(x, t) dt \quad (7)$$

$$= \frac{dY(x, s)}{dx} \quad (8)$$

Question: Consider the following first order partial differential equation, also known as the transport equation

$$\frac{\partial y(x, t)}{\partial t} + 5 \frac{\partial y(x, t)}{\partial x} = 0$$

with initial conditions given by $y(x, 0) = \sin x$, $-\infty < x < \infty$. The value of $y(x, t)$ at $x = \pi$ and $t = \frac{\pi}{6}$ is

- _____.
 (A) 1
 (B) 2
 (C) 0
 (D) 0.5

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Solution:

From Laplace transforms (5) and (7), we get

$$sY(x, s) - y(x, 0) + 5 \frac{dY(x, s)}{dx} = 0 \quad (9)$$

$$\Rightarrow \frac{dY(x, s)}{dx} + \frac{s}{5} Y(x, s) = \frac{\sin x}{5} \quad (10)$$

$$e^{\frac{s}{5}x} Y(x, s) = \frac{1}{5} \int e^{\frac{s}{5}x} \sin x dx \quad (11)$$

$$= \frac{1}{s^2 + 25} e^{\frac{s}{5}x} (s \sin x - 5 \cos x) + c \quad (12)$$

$$Y(x, s) = \frac{1}{s^2 + 25} (s \sin x - 5 \cos x) + ce^{-\frac{s}{5}x} \quad (13)$$

$$\cos at \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + a^2} \quad (14)$$

$$\sin at \xleftrightarrow{\mathcal{L}} \frac{a}{s^2 + a^2} \quad (15)$$

From Laplace transforms (14) and (15), we get

$$y(x, t) = ((\sin x \cos 5t - \cos x \sin 5t)) u(t) + ce^{-\frac{s}{5}x} \delta(t) \quad (16)$$

$$= (\sin(x - 5t)) u(t) + ce^{-\frac{s}{5}x} \delta(t) \quad (17)$$

$$y(x, 0) = \sin x + ce^{-\frac{s}{5}x} \delta(0) \quad (18)$$

$$\Rightarrow c = 0 \quad (19)$$

$$\therefore y(x, t) = (\sin(x - 5t)) u(t) \quad (20)$$

$$\Rightarrow y\left(\pi, \frac{\pi}{6}\right) = 0.5 \quad (21)$$

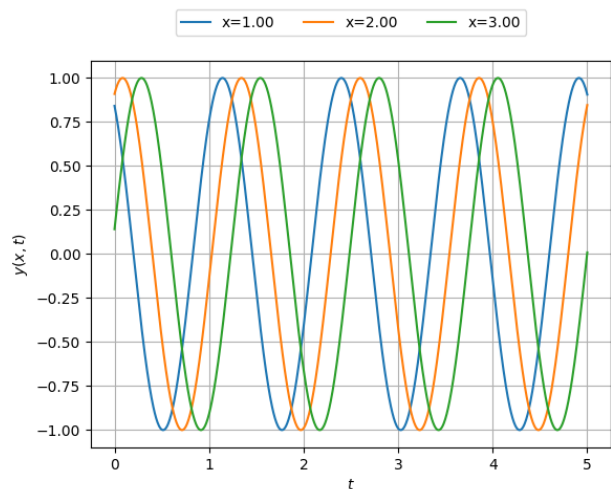


Fig. 1. Plot of $y(x, t)$