GATE: BM - 28.2021

EE23BTECH11224 - Sri Krishna Prabhas Yadla*

Question: Consider the following first order partial differential equation, also known as the transport equation

$$\frac{\partial y(x,t)}{\partial t} + 5\frac{\partial y(x,t)}{\partial x} = 0$$

with initial conditions given by $y(x,0) = \sin x, -\infty < x < \infty$. The value of $y(x,t)$ at $x = \pi$ and $t = \frac{\pi}{6}$ is ____.

(A) 1
(B) 2
(C) 0
(D) 0.5

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Solution:

$$\frac{\partial y(x,t)}{\partial t} \overset{\mathcal{L}}{\longleftrightarrow} sY(x,s) - y(x,0) \quad (1)$$

$$\frac{\partial y(x,t)}{\partial x} \overset{\mathcal{L}}{\longleftrightarrow} \frac{dY(x,s)}{dx} \quad (2)$$

From Laplace transforms (1) and (2), we get

$$sY(x,s) - y(x,0) + 5\frac{dY(x,s)}{dx} = 0 \quad (3)$$

$$\implies \frac{dY(x,s)}{dx} + \frac{s}{5}Y(x,s) = \frac{\sin x}{5} \quad (4)$$

$$e^{\frac{s}{5}x}Y(x,s) = \frac{1}{5}\int e^{\frac{s}{5}x}\sin x dx \quad (5)$$

$$= \frac{1}{s^2+25}e^{\frac{s}{5}x}(s\sin x - 5\cos x) + c \quad (6)$$

$$Y(x,s) = \frac{1}{s^2+25}(s\sin x - 5\cos x) + ce^{-\frac{s}{5}x} \quad (7)$$

$$\cos at \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+a^2} \quad (8)$$

$$\sin at \overset{\mathcal{L}}{\longleftrightarrow} \frac{a}{s^2+a^2} \quad (9)$$

From Laplace transforms (8) and (9), we get

$$y(x,t) = ((\sin x\cos 5t - \cos x\sin 5t))u(t) + ce^{-\frac{s}{5}x}\delta(t) \quad (10)$$

$$= (\sin(x-5t))u(t) + ce^{-\frac{s}{5}x}\delta(t) \quad (11)$$

$$y(x,0) = \sin x + ce^{-\frac{s}{5}x}\delta(0) \quad (12)$$

$$\implies c = 0 \quad (13)$$

$$\therefore y(x,t) = (\sin(x-5t))u(t) \quad (14)$$

$$\implies y\left(\pi,\frac{\pi}{6}\right) = 0.5 \quad (15)$$

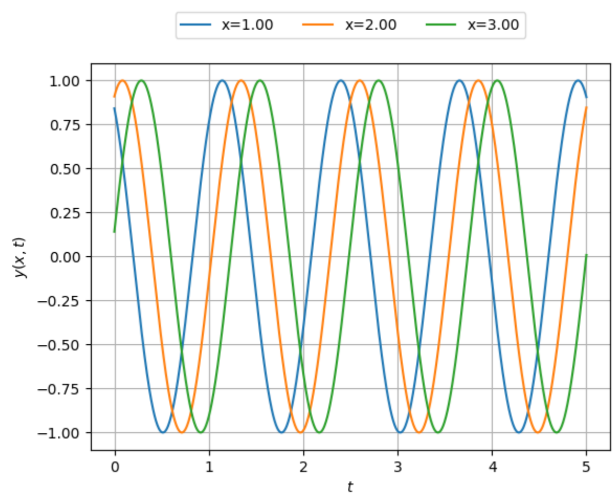

Fig. 1. Plot of $y(x,t)$