GATE: BM - 28.2021

EE23BTECH11224 - Sri Krishna Prabhas Yadla*

Results and Derivations:
Let a function $y(x,t)$ be defined for all $t > 0$ and assumed to be bounded. Then we can apply the Laplace transform in t considering x as a parameter.

$$\mathcal{L}(y(x,t)) = \int_0^\infty e^{-st} y(x,t)\,dt \tag{1}$$
$$= Y(x,s) \tag{2}$$

Let $\frac{\partial y(x,t)}{\partial t}$ be $y_t(x,t)$ and $\frac{\partial y(x,t)}{\partial x}$ be $y_x(x,t)$, then

$$\mathcal{L}(y_t(x,t)) = \int_0^\infty e^{-st} y_t(x,t)\,dt \tag{3}$$
$$= e^{-st} y(x,t)\Big|_0^\infty + s\int_0^\infty e^{-st} y(x,t)\,dt \tag{4}$$
$$= sY(x,s) - y(x,0) \tag{5}$$
$$\mathcal{L}(y_x(x,t)) = \int_0^\infty e^{-st} y_x(x,t)\,dt \tag{6}$$
$$= \frac{d}{dx}\int_0^\infty e^{-st} y(x,t)\,dt \tag{7}$$
$$= \frac{dY(x,s)}{dx} \tag{8}$$

Question: Consider the following first order partial differential equation, also known as the transport equation

$$\frac{\partial y(x,t)}{\partial t} + 5\frac{\partial y(x,t)}{\partial x} = 0$$

with initial conditions given by $y(x,0) = \sin x$, $-\infty < x < \infty$. The value of $y(x,t)$ at $x = \pi$ and $t = \frac{\pi}{6}$ is _____.

(A) 1
(B) 2
(C) 0
(D) 0.5

(GATE BM 2021)

Solution:
From Laplace transforms (5) and (7), we get

$$sY(x,s) - y(x,0) + 5\frac{dY(x,s)}{dx} = 0 \tag{9}$$
$$\implies \frac{dY(x,s)}{dx} + \frac{s}{5}Y(x,s) = \frac{\sin x}{5} \tag{10}$$

$$e^{\frac{s}{5}x}Y(x,s) = \frac{1}{5}\int e^{\frac{s}{5}x}\sin x dx \tag{11}$$
$$= \frac{1}{s^2+25}e^{\frac{s}{5}x}(s\sin x - 5\cos x) + c \tag{12}$$
$$Y(x,s) = \frac{1}{s^2+25}(s\sin x - 5\cos x) + ce^{-\frac{s}{5}x} \tag{13}$$

$$\cos at \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+a^2} \tag{14}$$
$$\sin at \xleftrightarrow{\mathcal{L}} \frac{a}{s^2+a^2} \tag{15}$$

From Laplace transforms (14) and (15), we get

$$y(x,t) = ((\sin x\cos 5t - \cos x\sin 5t))\,u(t) + ce^{-\frac{s}{5}x}\delta(t) \tag{16}$$
$$= (\sin(x-5t))\,u(t) + ce^{-\frac{s}{5}x}\delta(t) \tag{17}$$
$$y(x,0) = \sin x + ce^{-\frac{s}{5}x}\delta(0) \tag{18}$$
$$\implies c = 0 \tag{19}$$
$$\therefore y(x,t) = (\sin(x-5t))\,u(t) \tag{20}$$
$$\implies y\left(\pi,\frac{\pi}{6}\right) = 0.5 \tag{21}$$

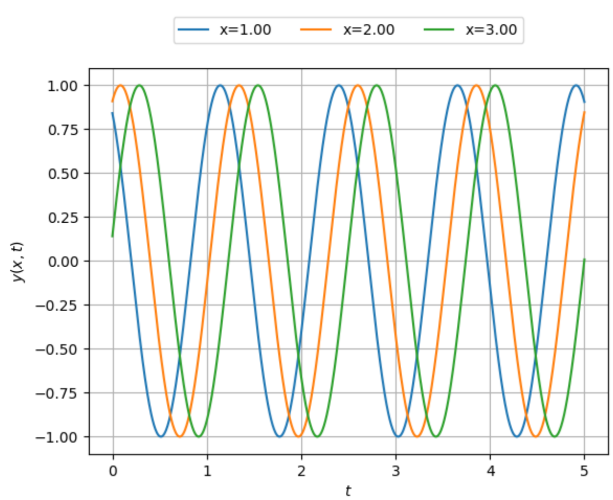

Fig. 1. Plot of $y(x,t)$