POL 213 – Spring 2024

Lecture 6

Binary Choice Models II - Interpreting the Logit Model

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Regression with Binary Dependent Variables

Last time...

- Discussed motivations for Logistic and Probit regression
- Demonstrated how linear probability model has several undesired features
- Derived the logistic regression model
- Showed how logistic regression results can be interpreted as (log) odds ratios

This time...

- Practice with logistic regression in R.
- Interpretation of logistic regression results and odds ratios.
- Working with interaction model specifications.

Practice with Logistic Regression

Please use the following files:

`Field.r` and `field_201202.txt`

The data and example code are from Scott MacKenzie, who kindly shared the teaching materials. The data reflect a 2012 Field Poll on support for same sex marriage, conducted among 515 California residents.

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Interpreting the Logit Model

The logit model is nonlinear in probabilities but linear in the log-odds; how do we interpret α and β?

- Example: Support for same sex marriage is the binary response variable. The predictors are: age, party affiliation (-1,0,1) and born-again Christian.

```
> summary(mydata)
same_sex           age              party              born_christian
Min.   :0.0000   Min.   :1.000   Min.   :-1.00000   Min.   :0.0000
1st Qu.:0.0000   1st Qu.:3.000   1st Qu.:-1.00000   1st Qu.:0.0000
Median :1.0000   Median :4.000   Median : 0.00000   Median :0.0000
Mean   :0.5586   Mean   :4.044   Mean   : 0.02092   Mean   :0.3033
3rd Qu.:1.0000   3rd Qu.:6.000   3rd Qu.: 1.00000   3rd Qu.:1.0000
Max.   :1.0000   Max.   :6.000   Max.   : 1.00000   Max.   :1.0000
```

Interpreting the Logit Model

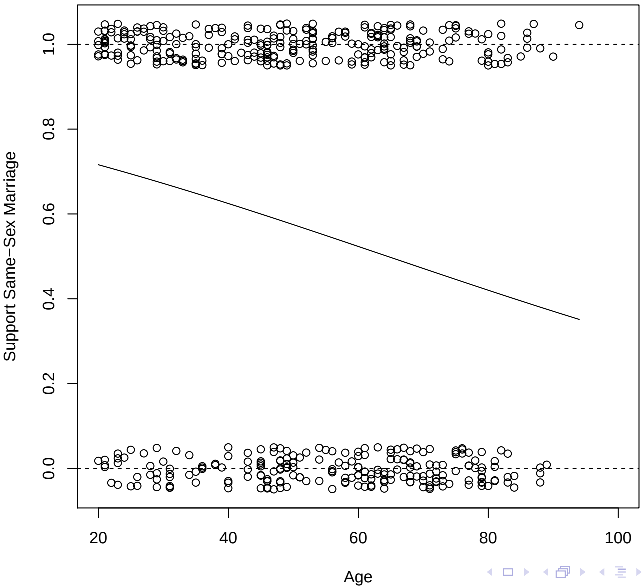
- Plot Support as a function of Age

```
# Scatterplot of Support for Same-sex  Marriage against Age

plot(jitter(same_sex, .25) ~ age, mydata,  xlab="Age", xlim=c(20,100)
ylab="Support Same-Sex Marriage")
abline(h = 1, lty = 2)
abline(h = 0, lty = 2)
```

- We can estimate the logit model for π_i, i.e. $\Pr(Y_i = yes)$

$$\pi_i = \Lambda(\alpha + \beta X_i) = \frac{1}{1 + exp(-\alpha + \beta X_i)}$$

where support is a function of the respondent's age.


Support Same−Sex Marriage
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
100
Age

Interpreting the Logit Model

- β is negative and indicates the effect of the log-odds of a 1-year increase in age.
  - When β is negative (positive), the curve descends (ascends)
  - The rate of change increases as $|\beta|$ increases
- Since the logit is S-shaped in probabilities, the rate of change in π_i depends on x. We can write slope = $\beta(\pi_i(1-\pi_i))$

```
-0.004 = -0.020776(0.35*0.65)
-0.005 = -0.020776(0.56*0.44)
-0.004 = -0.020776(0.72*0.28)
```

```
> logit.field <- glm(same_sex   age, mydata, family = binomial)

> summary(logit.field)


Call: glm(formula = same_sex   age, family = binomial, data = mydata)


Deviance Residuals:
Min 1Q Median 3Q Max
-1.587 -1.200 0.854 1.069 1.446


Coefficients:
Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.340342 0.291319 4.601 4.21e-06 ***
age -0.020776 0.005166 -4.022 5.77e-05 ***


Null deviance: 654.43 on 476 degrees of freedom
Residual deviance: 637.65 on 475 degrees of freedom
AIC: 641.65


Number of Fisher Scoring iterations: 4
```

Interpreting the Logit Model

- Since β is negative, the estimated probability is smaller at greater ages

$$\frac{\exp(1.3403 - 0.0207 * 20)}{1 + \exp(1.3403 - 0.0207 * 20)} = 0.716$$

$$\frac{\exp(1.3403 - 0.0207 * 94)}{1 + \exp(1.3403 - 0.0207 * 94)} = 0.351$$

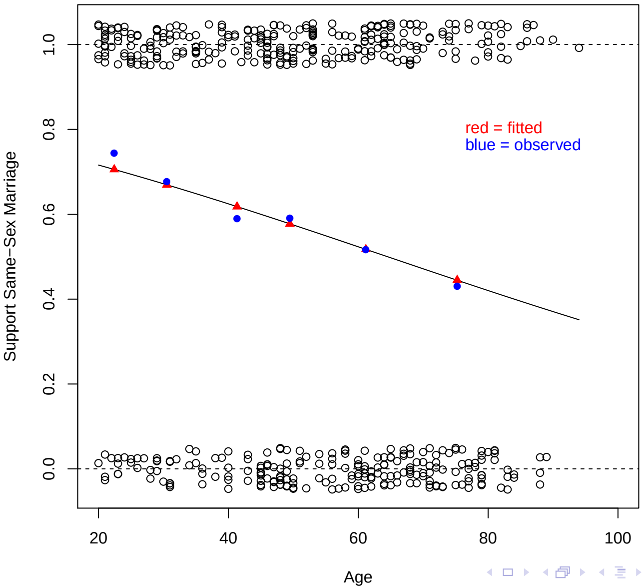
- The rate of change is greatest when $\pi_i = 0.50$; the age (i.e., median effective level) this occurs is $\dfrac{-\alpha}{\beta}$

$$-\frac{\alpha}{\beta} = -\frac{1.3403}{0.0207} = 64.514$$

$$\frac{\exp(1.3403 - 0.0207 * 64.5)}{1 + \exp(1.3403 - 0.0207 * 64.5)} = 0.50$$

Interpreting the Logit Model

- Plotting y against x when y takes only 0 and 1 values can provide only limited information about whether a logit model is reasonable
- By grouping age values into categories and plotting proportions, we can better assess the trend
- We can also plot the fitted values (grouped or not) and compare; the fit here looks adequate.


red = fitted
blue = observed
Support Same−Sex Marriage
Age
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
100

Interpreting the Logit Model

- One advantage of the logit model is that we can interpret the coefficients about as easily as those we get from OLS.
- Recall:

$$\log \frac{\pi_i}{1-\pi_i} = \alpha + \beta X_i$$

- This means the odds of a success versus failure (i.e. a respondent supports versus opposes same-sex marriage) is $\exp(\alpha + \beta X_i)$
  - For example, the odds ratio that a respondent at the median of age supports same-sex marriage is $\exp(1.3403 - 0.0207 * 54) = 1.2702$
- The odds multiply by $\exp(\beta)$ or $\exp(-0.0207) = 0.979$ for each 1-year decrease in age; that is, there is 2.1% decrease in the odds
  - For example, a 53-yr old, $\pi_i = 0.56$ with odds $\dfrac{(0.56)}{(0.44)} = 1.27$
  - For a 54-yr old, $\pi_i = 0.55$ with odds $\dfrac{(0.55)}{(0.45)} = 1.24$; $\dfrac{1.24}{1.27} = 0.979$

Interpreting the Logit Model

- Using the output from the logit model, we can calculate Wald statistics and confidence intervals for α and β

```
> z_intercept <- (1.340342 - 0) / 0.291319; z_intercept
[1] 4.600943
> c(1.340342-1.96*0.291319, 1.340342+1.96*0.291319)
[1] 0.7693568 1.9113272
>
> z_beta <- (-0.020776 - 0) / 0.005166; z_beta
[1] -4.02168
> c(-0.020776-1.96*0.005166, -0.020776+1.96*0.005166)
[1] -0.03090136 -0.01065064
```

- The corresponding statistics from the likelihood-ratio test

```
> g_statusquo <- 2*(-318.8259 - -327.2174); g_statusquo
[1] 16.783

> confint(logit.field)
Waiting for profiling to be done...
2.5 %      97.5 %
(Intercept)  0.77703500  1.92077448
age         -0.03103374 -0.01075556
```

Interpreting the Logit Model

- We can also calculate confidence intervals for estimated probabilities of support at given values of age

```
> logit_53 <- 1.340342 - 0.020776*53; logit_53
[1] 0.239214

> vcov(logit.field)
(Intercept)              age
(Intercept)  0.084866 -0.0014245
age         -0.001424  0.0000266

> var_logit_53 <- 0.084866 + (53)^2*0.0000266 + 2*53*-0.001424; var_l
[1] 0.008817552
> c(logit_53 + 1.96*(var_logit_53)^.5, logit_53 - 1.96*(var_logit_53)
[1] 0.42326157 0.05516643
> logit_53_lo <- exp(0.05516643) / (1 + exp(0.05516643)); logit_53_lo
[1] 0.5137881
> logit_53_hi <- exp(0.42326157) / (1 + exp(0.42326157)); logit_53_hi
[1] 0.6042635
```

- The probability of supporting same-sex marriage for a 53-yr old is 0.559 with confidence interval (0.513, 0.604)

Interpreting the Logit Model

Expanding the logit model to multiple predictors, including categorical predictors is easy.

- For categorical predictors we need c-1 indicator variables where c is the number of categories

The logit model with age, party and born-again predictors is:

$$logit(\Pr(Y = 1)) = \alpha + \beta_1 IND + \beta_2 GOP + \beta_3 Born + \beta_4 age$$

Interpreting the Logit Model

```
> logit.field2 <- glm(same_sex ~ independent + republican + born_christian
mydata, family = binomial(link=logit))

glm(formula = same_sex ~ independent + republican + born_christian +
age, family = binomial(link = logit), data = mydata)

Deviance Residuals:
Min         1Q   Median       3Q      Max
-2.0052  -1.0155   0.5916   0.8237   2.0532

Coefficients:
Estimate Std. Error z value Pr(>|z|)
(Intercept)     2.225929   0.376157   5.918 3.27e-09 ***
independent    -0.404662   0.316698  -1.278    0.201
republican     -1.257992   0.230950  -5.447 5.12e-08 ***
born_christian -1.639462   0.232279  -7.058 1.69e-12 ***
age            -0.016335   0.005944  -2.748    0.006 **
---
(Dispersion parameter for binomial family taken to be 1)

Null deviance: 654.43  on 476  degrees of freedom
Residual deviance: 539.65  on 472  degrees of freedom
AIC: 549.65
```

Interpreting the Logit Model

The model assumes that the effects of age are constant across partisanship and born-again status. The implied logits are:

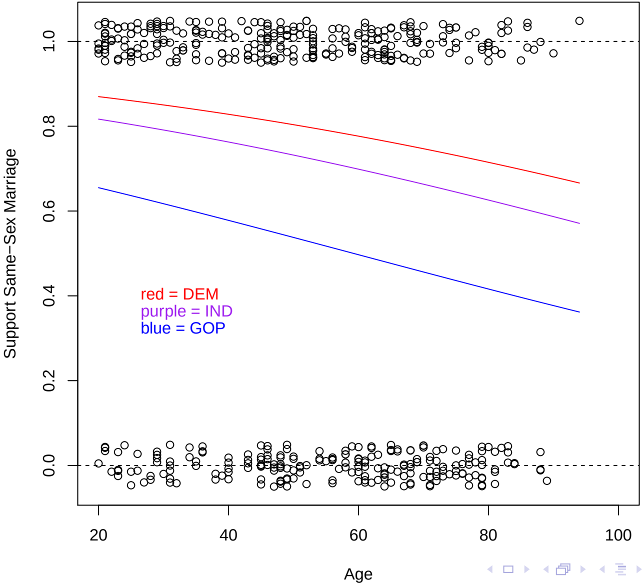
Party	Born Again	Logit
Dem	No	α
Dem	Yes	$\alpha + \beta_3$
Indept	No	$\alpha + \beta_1$
Indept	Yes	$\alpha + \beta_1 + \beta_3$
Rep	No	$\alpha + \beta_2$
Rep	Yes	$\alpha + \beta_2 + \beta_3$

- For example, the logit for a 53-yr old born-again Republican is $logit(\Pr(y = 1|GOP = 1, born = 1)) = 2.225 - 1.258 - 1.639 - 0.016 * 53 = -1.537$
- The odds ratio for such a person is $\exp(-1.537) = 0.215$; $\pi_i = 0.177$

Interpreting the Logit Model

- We can plot the fitted lines for each party assuming Born=0.
  - Since β_1 and β_2 are negative, lines for INDs and GOPs are left of the line for DEMs
- At fixed values of age and born again status, the effect on the logit of changing from DEM (GOP = 0) to GOP (GOP=1) is

  $$[\alpha + \beta_2(1) + \beta_3(born) + \beta_4(age)] \\ - [\alpha + \beta_2(0) + \beta_3(born) + \beta_4(age)] = \beta_2$$


Support Same−Sex Marriage
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
100
Age
red = DEM
purple = IND
blue = GOP

Interpreting the Logit Model

- By itself, the estimate for β_2 (or any categorical predictor) is irrelevant; the estimate only makes sense when compared to the estimate for another category
- $\beta_2 = -1.258$ signifies that the conditional odds ratio between being GOP and support are $\exp(-1.258) = 0.284$; that is, the odds for GOPs are 0.284 times the odds for DEMS.
- Similarly, the odds of support among born again Christians are $\exp(-1.639) = 0.194$ times the odds of those not born again.

Interpreting the Logit Model

- Equivalently, we can take the inverse of 0.284 to get the odds for DEMs – the odds are 3.52 times those of GOPs
- Verify that if you re-estimated the logit model with DEM as a predictor and GOP as the excluded category, you would recover this value for β_2
- Similarly, those who are **not** born again are $\frac{1}{0.194} = 5.15$ times more likely that born again christians to support same-sex marriage

Interpreting the Logit Model

- Treating independents as a separate category does not seem to add much to the model. Can we simply by excluding it or treaty party as continuous?
- The continuous predictor has a small standard error, showing strong evidence of an effect.
- Compare the fit of this simple model against the more complex one with two terms:

```
g_statusquo <- 2*(-269.8239 - -270.1203)
g_statusquo
[1] 0.5928
```

- The simple model seems good.

Interpreting the Logit Model

```
> logit.field3 <- glm(same_sex   party + born_christian + age, mydata,
family = binomial(link=logit))

glm(formula = same_sex   party + born_christian + age,
family = binomial(link = logit), data = mydata)

Deviance Residuals:
Min 1Q Median 3Q Max
-2.0419 -1.0119 0.5732 0.8467 2.0571

Coefficients:
Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.708971 0.330556 5.170 2.34e-07 ***
party -0.630857 0.115964 -5.440 5.32e-08 ***
born_christian -1.655363 0.231573 -7.148 8.78e-13 ***
age -0.017628 0.005702 -3.091 0.00199 **

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 654.43 on 476 degrees of freedom
Residual deviance: 540.24 on 473 degrees of freedom
AIC: 548.24
```

Interpreting the Logit Model

- We can use similar procedures to test for interactions. Consider whether the effect of age differs for GOP and others.
- We can estimate a model with GOP*Age and compare against a simpler model with only GOP

$$logit(\Pr(Y = 1)) = \alpha + \beta_1(GOP) + \beta_2(born) + \beta_3(age) + \beta_4(GOP * age)$$

- The effect of the interaction is not statistically significant. For reasons we will discuss, this is not determinative.

Interpreting the Logit Model

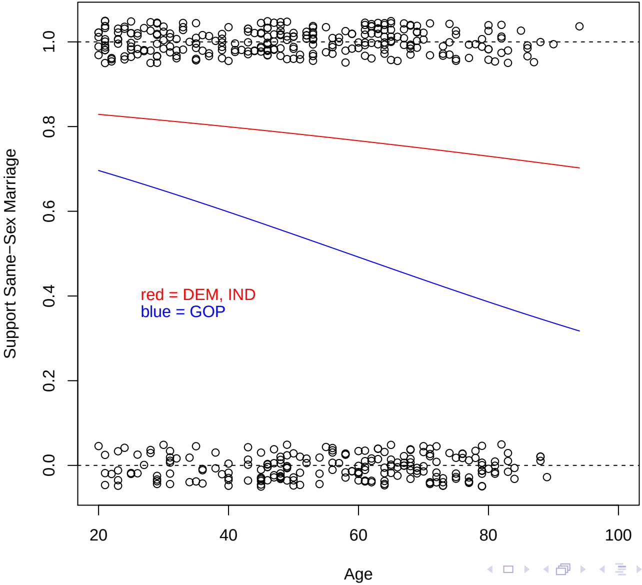
```
glm(formula = same_sex   republican + born_christian + age + republican *
age,
family = binomial(link = logit), data = mydata)
Coefficients:
Estimate Std. Error z value Pr(>|z|)
(Intercept) 1.771855 0.409882 4.323 1.54e-05 ***
republican -0.510810 0.668003 -0.765 0.444
born_christian -1.630770 0.231213 -7.053 1.75e-12 ***
age -0.009713 0.007554 -1.286 0.198
republican:age -0.011849 0.011756 -1.008 0.314

Null deviance: 654.43 on 476 degrees of freedom
Residual deviance: 540.24 on 472 degrees of freedom
AIC: 550.24
```

Interpreting the Logit Model

- We can get a better idea about this interaction by plotting the fitted lines for GOPs and others
- The slope of age for GOPs is noticeably steeper

```
> lrtest(logit.field5, logit.field4)

Likelihood ratio test

Model 1: same_sex ~ republican + born_christian + age + republican *

Model 2: same_sex ~ republican + born_christian + age

#Df  LogLik Df  Chisq Pr(>Chisq)

1   5 -270.12

2   4 -270.63 -1 1.0206     0.3124
```

- The LR test does not offer strong evidence of an interaction


red = DEM, IND
blue = GOP
Support Same−Sex Marriage
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
100
Age

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Additional Quantities of Interest

- One method of assessment is to examine the signs of coefficients and determine their statistical significance.
- This is no more difficult in binary choice than for OLS models
  - A positive (negative) coefficient indicates that increases (decreases) in X lead to increases (decreases) in $\Pr(Y = 1)$
  - The ratio of β to its SE is a z-score that can be used for hypothesis testing.
- In the previous model, we observed that age was not a significant predictor (for Democrats); however, since we have an interaction, the effects of age are conditional.
  - We can use the `glht()` command in R to assess whether Age and GOP* Age are simultaneously 0

Additional Quantities of Interest

```
> summary(glht(logit.field5, linfct = c("age + republican:age = 0")))

Simultaneous Tests for General Linear Hypotheses

Fit: glm(formula = same_sex ~ republican + born_christian + age +

republican * age, family = binomial(link = logit), data = mydata)

Linear Hypotheses:

Estimate Std. Error z value Pr(>|z|)

age + republican:age == 0 -0.021563   0.009008  -2.394   0.0167 *
```

For GOPs, age is significant and the effect is twice as large as for DEMS.

Additional Quantities of Interest

- Simply focusing on the signs and significance of coefficients, however, is a poor method of interpreting and evaluating a binary choice (or any other) model.
- Typically, we are interested in whether and by how much a change in X changes the outcome of interest, i.e., $\Pr(Y_i = 1)$
- Assessing the effects of such changes is more involved than for OLS; the effects of our predictors are linear in the latent variable ξ_i but not in Y_i
- We also know that the net effect of a change in X depends critically on the values of other predictors and parameters in the model.
  - For example, the difference in the probability of support between GOPs and others is $0.61 - 0.80 = -0.19$ for a 37 year old.
  - For a 67-year old, the difference is $0.45 - 0.75 = -0.30$

Additional Quantities of Interest

- Using the formula for calculating probabilities, we can summarize the effect of an indicator variable by calculating estimated probability at its two values:

$$\Delta \Pr(Y_i = 1)_{XA \rightarrow XB} = \frac{exp(\alpha + \beta X_A + ... + \beta X_K)}{1 + exp(\alpha + \beta X_A + ... + \beta X_K)} - \frac{exp(\alpha + \beta X_B + ... + \beta X_K)}{1 + exp(\alpha + \beta X_B + ... + \beta X_K)}$$

- For continuous variables, it makes sense to calculate estimated probabilities at particular values, e.g. lower and upper quartiles, because the min and max reflect outliers.

Variable	β	**se**	**Comparison**	$\Delta \Pr(Y_i = 1)$
Intercept	1.772	0.410		
GOP	-0.511	0.668	$(1, 0)$ *at* X_{med}	0.53 - 0.78 = -0.25
Born Again	-1.631	0.231	$(1, 0)$ *at* X_{med}	0.41 - 0.78 = -037
Age	-0.010	0.008	(UQ, LQ) *at* $GOP = 0$	0.80 - 0.75 = 0.05
Age * GOP	-0.012	0.012	(UQ, LQ) *at* $GOP = 1$	0.61 - 0.45 = 0.15

Additional Quantities of Interest

- In addition to estimating point predictions for relevant profiles, we can use the `predict()` command to obtain predicted probabilities and standard errors.
- Process: we generate new dataset of hypothetical cases (i.e. profiles). Start with a baseline of non-GOP non-born again respondents whose ages range from 20 to 94. The hypothetical cases should be plausible (possible).

```
basedata = data.frame(age=x, republican=rep(0, length(x)), born_chris
length(x)))
```

- Next, calculate predicted probabilities and standard errors using the `glm` object `logit.field5`.

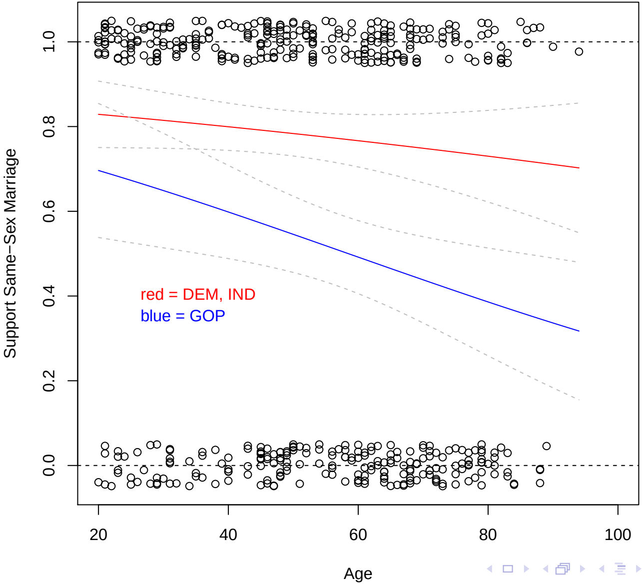
```
> base_field5 <- predict(logit.field5, newdata=basedata, type="respon
se.fit=TRUE)
```

- Overlay scatterplot with predictions and confidence intervals for GOP and others.

Additional Quantities of Interest

```
> lines(x, base_field5$fit, lty = 1, col = "red")
> lines(x, (base_field5$fit + 2*base_field5$se), lty = 2, col = "gray")
> lines(x, (base_field5$fit - 2*base_field5$se), lty = 2, col = "gray")
> lines(x, gop_field5$fit, lty = 1, col = "blue")
> lines(x, (gop_field5$fit + 2*gop_field5$se), lty = 2, col = "gray")
> lines(x, (gop_field5$fit - 2*gop_field5$se), lty = 2, col = "gray")
```

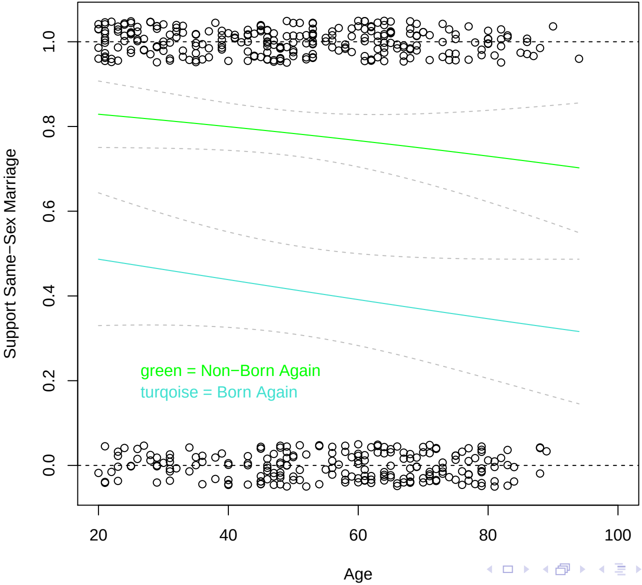
Quantities of Interest



Additional Quantities of Interest

- There is some overlap but clear separation for 40+ respondents.
- We can plot some other comparisons, e.g. born again Christians v. others; no overlap in these lines.

```
> plot(jitter(same_sex, .25)   age, mydata, xlab="Age", xlim=c(20,100),
+ ylab="Support Same-Sex Marriage")
> abline(h = 1, lty = 2); abline(h = 0, lty = 2)
> > lines(x, base_field5$fit, lty = 1, col = "green")
> lines(x, (base_field5$fit + 2*base_field5$se), lty = 2, col = "gray")
> lines(x, (base_field5$fit - 2*base_field5$se), lty = 2, col = "gray")
> > lines(x, born_field5$fit, lty = 1, col = "turquoise")
> lines(x, (born_field5$fit + 2*born_field5$se), lty = 2, col = "gray")
> lines(x, (born_field5$fit - 2*born_field5$se), lty = 2, col = "gray")
```



Support Same−Sex Marriage
Age
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
100
green = Non−Born Again
turqoise = Born Again

Additional Quantities of Interest

- One advantage of a logit model over a probit is the ability to substantively interpret the effects of predictors in terms of odds. Recall:

$$\log \frac{\pi_i}{1-\pi_i} = \frac{\dfrac{exp(\alpha+\beta X_i)}{1+exp(\alpha+\beta X_i)}}{1-\dfrac{exp(\alpha+\beta X_i)}{1+exp(\alpha+\beta X_i)}} = \alpha+\beta X_i$$

- The odds ratio is $exp(\alpha+\beta X_i)$. For a coefficient of an indicator variable, the odds ratio is $exp\beta$ holding other factors constant.
  - For example, the odds that a born again Christian supports same sex marriage is $exp(-1.631) = 0.196$ times that of others. Taking the inverse gives us the odds ratio for others vs. born again Christians, 5.109. The odd that someone who is <u>not</u> born again supports same sex marriage is <u>five times</u> that of a born again Christian.
- For an indicator variable, the interpretation is simple and intuitive.

Additional Quantities of Interest

- Since odds-ratios are proportional between adjacent categories, we can calculate the (approximate) percentage change in the odds given changes in X:

$$\%\Delta \frac{\pi_i}{1-\pi_i} = \frac{exp(\beta x) - exp(\beta x')}{exp \beta x'} * 100$$

- For born again Christians we have:

$$= \frac{exp(-1.631 * 1) - exp(-1.631 * 0)}{exp{-1.631 * 0}} * 100 = \left[\frac{0.961 - 1}{1}\right] * 100 = -80.42$$

- That means the odds are 80% lower.

Additional Quantities of Interest

- For DEMS, increasing age from 37 to 67 results in a 25.92% decrease in the odds.

$$\frac{exp(0.010 * 67) - exp(0.010 * 37)}{exp(0.010 * 37)} * 100 =$$
$$\left[\frac{0.512 - 0.691}{0.691}\right] * 100 =$$
$$-25.92$$

- For GOPs, increasing age from 37 to 67 results in a 48.31% decrease in the odds.

$$\frac{exp(0.010 * 67 - 0.012 * 67) - exp(0.010 * 37 - 0.012 * 37)}{exp(0.010 * 37 - 0.012 * 37)} * 100 =$$
$$\left[\frac{0.229 - 0.443}{0.443}\right] * 100 =$$
$$-48.31$$

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title

content...