#1. Show that $\frac{1}{w_i} = W'_{n+1}(x_i)$

Sol.

We have $w_i = \frac{1}{\prod_{j \neq i} (x_i - x_j)}$ and $W_{n+1}(x) = \prod_{k=0}^{n} (x - x_k)$.

Then $W_{n+1}(x) = \prod_{k=0}^{n} (x - x_k) = (x - x_i) \prod_{k \neq i} (x - x_k)$

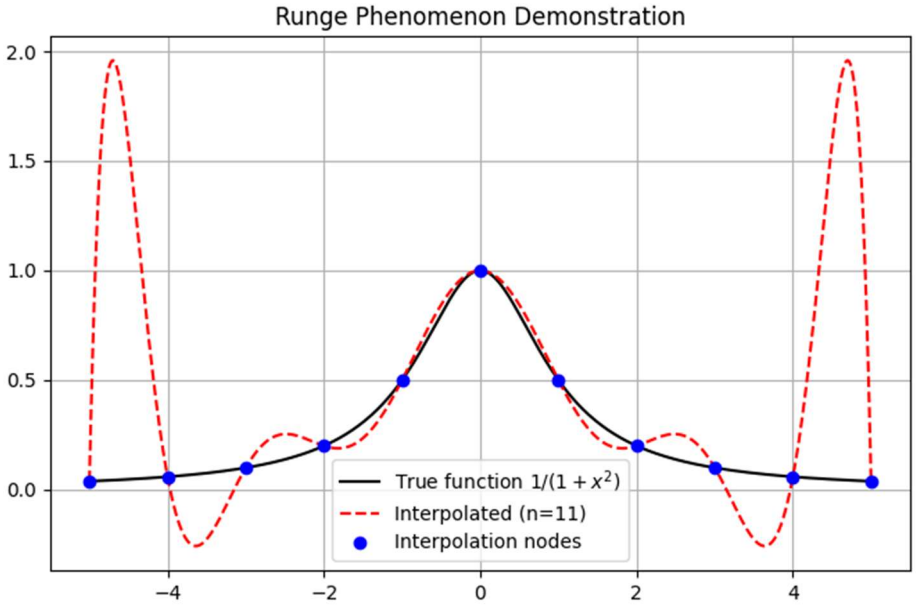
$\Rightarrow W'_{n+1}(x) = \prod_{k \neq i} (x - x_k) \frac{d}{dx}(x - x_i) + (x - x_i) \frac{d}{dx} \prod_{k \neq i} (x - x_k)$

$= \prod_{k \neq i} (x - x_k) + (x - x_i) \frac{d}{dx} \prod_{k \neq i} (x - x_k)$

$\Rightarrow W'_{n+1}(x_i) = \prod_{k \neq i} (x_i - x_k)$

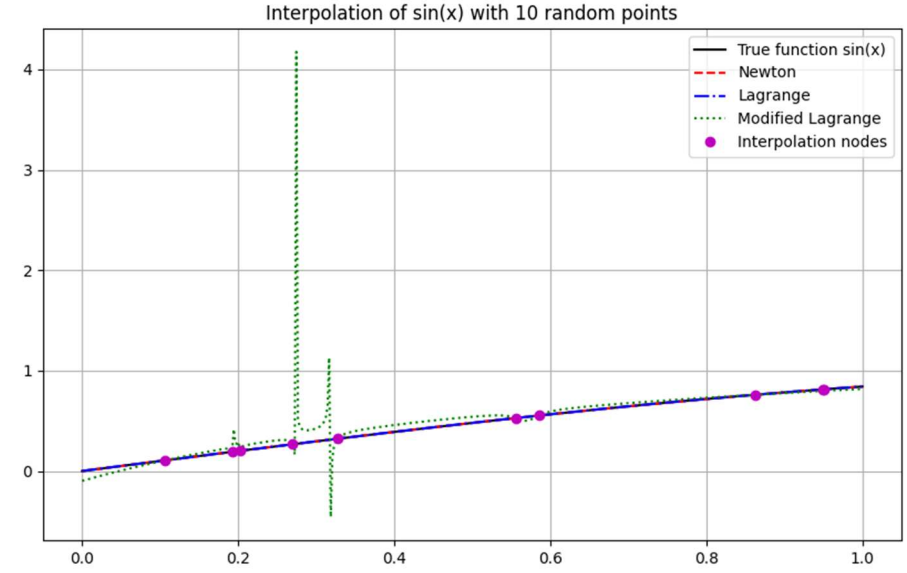
$= \frac{1}{w_i}$ □

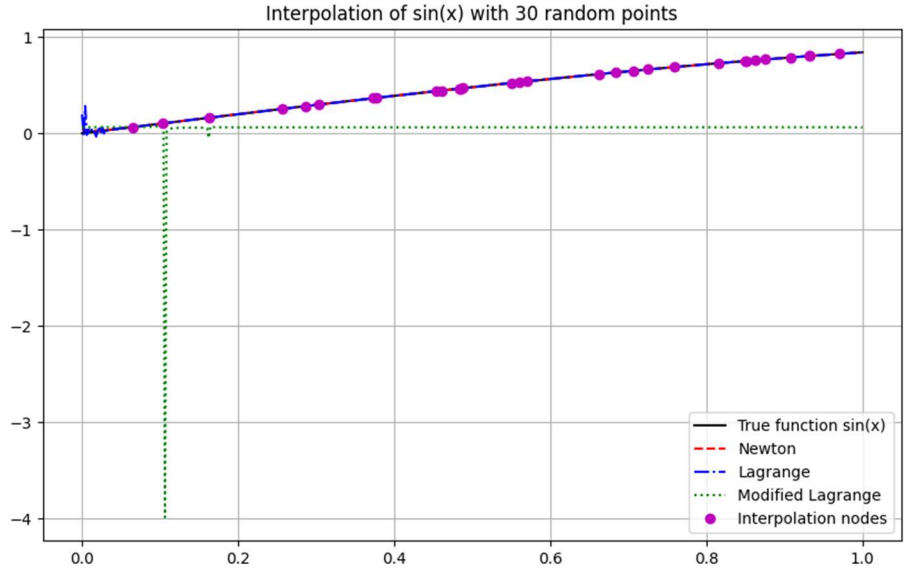
1. Runge Phenonmenon

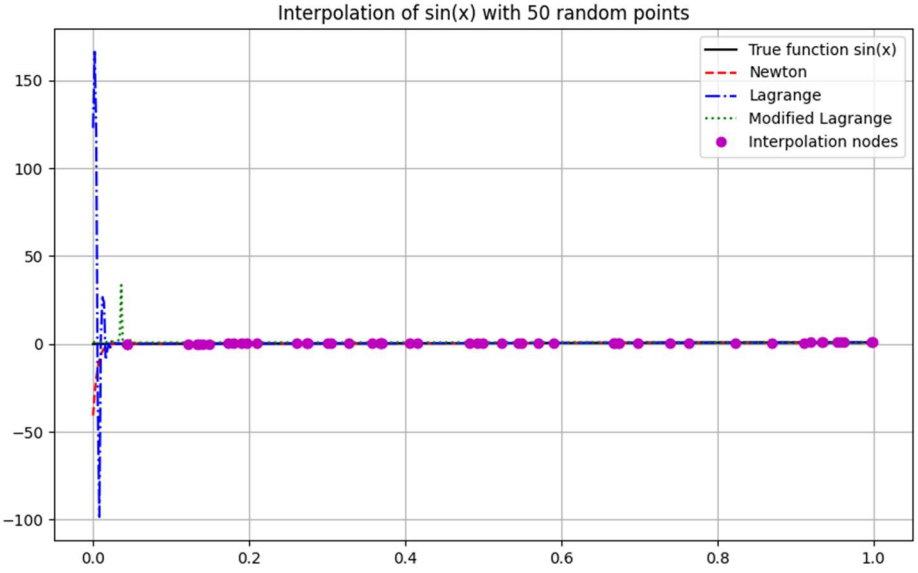


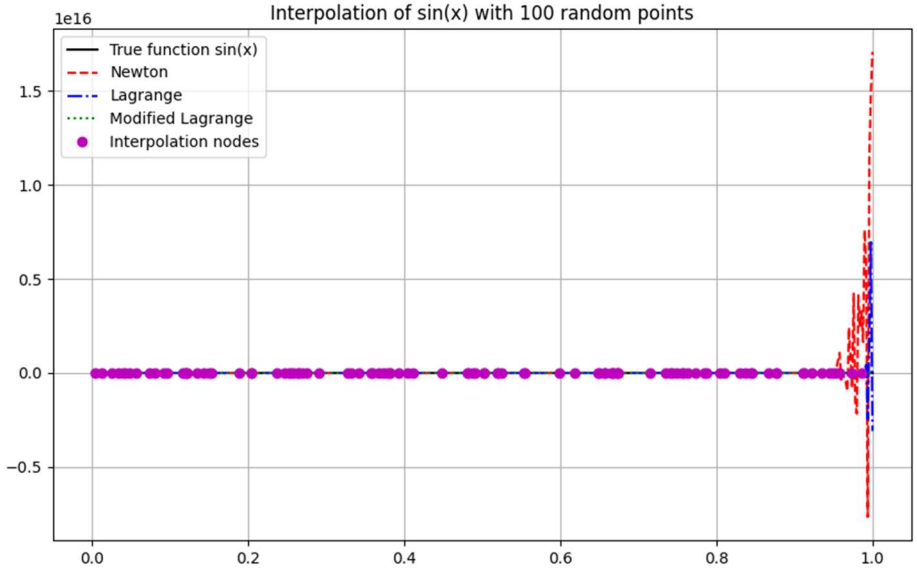
2. Interpolate

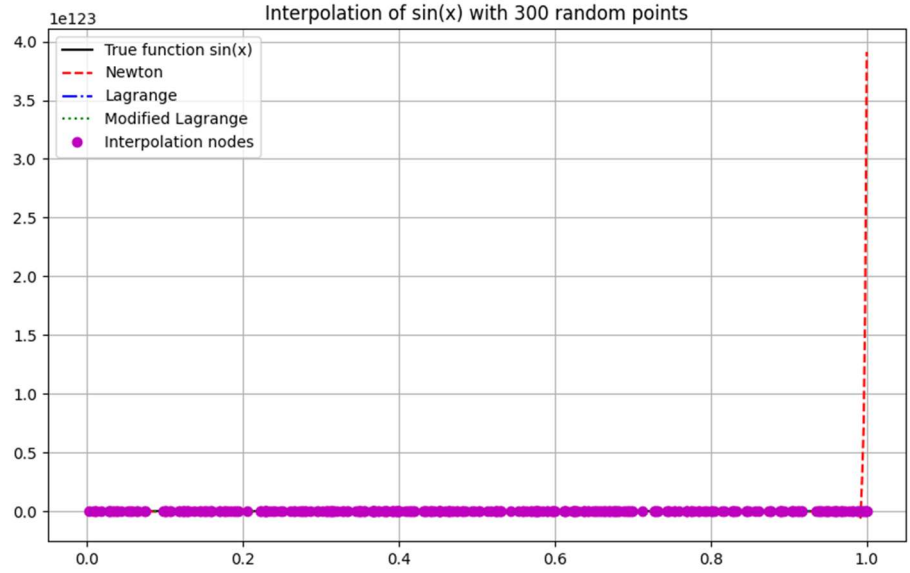
在插值實驗中，我們觀察到不同方法在節點數量變化下的表現有所差異。當節點數少於 30 個時，Newton 與 Lagrange 比 Modified Lagrange 表現的更好；然而，當節點數超過 30 個時，Modified Lagrange 的振盪情形則相對較輕微。另一方面，數值穩定性也隨節點數增加而顯現差異：在大約 500 個節點時，Newton 插值法因計算過程中分母過小而導致結果爆掉；而當節點數進一步增加至約 1000 個時，Lagrange 與 Modified Lagrange 也因運算過程中數值過大或過小而出現無法正常運算的情況。

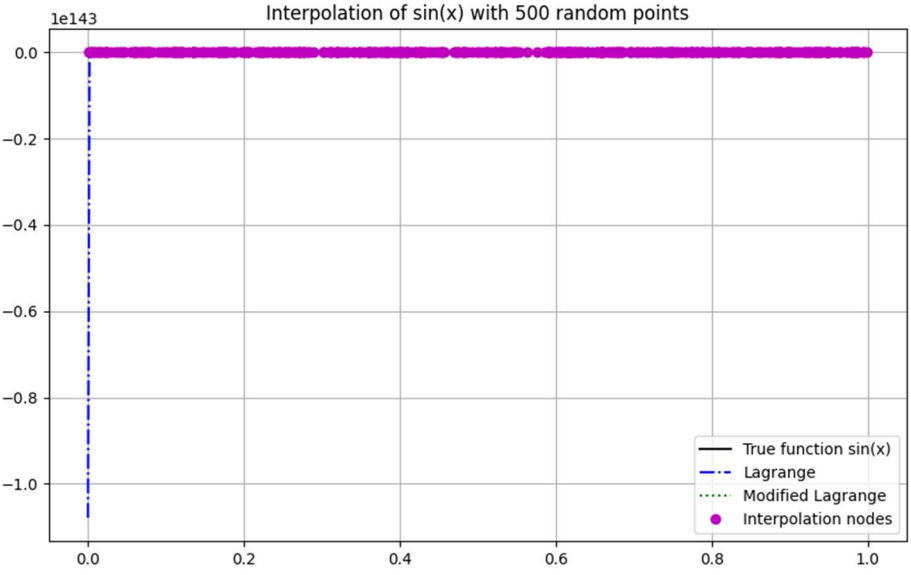

Interpolation of sin(x) with 10 random points
True function sin(x)
Newton
Lagrange
Modified Lagrange
Interpolation nodes
4
3
2
1
0
0.0
0.2
0.4
0.6
0.8
1.0


Interpolation of sin(x) with 30 random points
True function sin(x)
Newton
Lagrange
Modified Lagrange
Interpolation nodes
1
0
−1
−2
−3
−4
0.0
0.2
0.4
0.6
0.8
1.0


Interpolation of sin(x) with 50 random points
True function sin(x)
Newton
Lagrange
Modified Lagrange
Interpolation nodes
150
100
50
0
−50
−100
0.0
0.2
0.4
0.6
0.8
1.0


Interpolation of sin(x) with 100 random points
1e16
True function sin(x)
Newton
Lagrange
Modified Lagrange
Interpolation nodes
1.5
1.0
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0


Interpolation of sin(x) with 300 random points
1e123
True function sin(x)
Newton
Lagrange
Modified Lagrange
Interpolation nodes
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0


Interpolation of sin(x) with 500 random points
1e143
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
True function sin(x)
Lagrange
Modified Lagrange
Interpolation nodes
0.0
0.2
0.4
0.6
0.8
1.0