1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

Heun's method : $\bar{u}_{n+1} = u_n + hf(t_n, u_n)$ ← Forward Euler method

$$u_{n+1} = u_n + \underbrace{\frac{h}{2}(f(t_n, u_n) + f(t_{n+1}, \bar{u}_{n+1}))}_{\text{Trapezoidal method}}.$$

$E_1 = -\frac{h^3}{12} f''(\xi_1, y(\xi_1))$ for some $\xi_1 \in [t_n, t_{n+1}]$.

$\Rightarrow E_1 = O(h^3)$.

$E_2 = \frac{h}{2}\{ f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))\}$

在y方向上以y_{n+1}為展開點對$f(t_{n+1}, y)$做泰勒展開：

$f(t_{n+1}, y) = f(t_{n+1}, y_{n+1}) + f_y(t_{n+1}, y_{n+1})(y - y_{n+1}) + \frac{1}{2} f_{yy}(t_{n+1}, \xi)(y - y_{n+1})^2$

for some ξ between y and y_{n+1}.

Substitute $y = y_n + hf(t_n, y_n)$,

then $f(t_{n+1}, y_n + hf(t_n, y_n)) = f(t_{n+1}, y_{n+1}) + f_y(t_{n+1}, y_{n+1})[(y_n + hf(t_n, y_n)) - y_{n+1}]$
$\quad + \frac{1}{2} f_{yy}(t_{n+1}, \xi_2)[(y_n + hf(t_n, y_n)) - y_{n+1}]^2$

$\Rightarrow E_2 = \frac{h}{2}\{ f_y(t_{n+1}, y_{n+1})[y_{n+1} - (y_n + hf(t_n, y_n))]$
$\quad - \frac{1}{2} f_{yy}(t_{n+1}, \xi_2)[(y_n + hf(t_n, y_n)) - y_{n+1}]^2$

for some ξ_2 between $y_n + hf(t_n, y_n)$ and y_{n+1}.

$\Rightarrow E_2 = \frac{h}{2} O(h^2) = O(h^3)$.

$h\tau_{n+1} = E_1 + E_2 = O(h^3)$.

So $\tau_{n+1} = O(h^2)$.

Thus Heun's method has order 2 with respect to h. $\square$

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

Crank-Nicoloson method: $u_{n+1} = u_n + \frac{h}{2}[f(t_n, u_n) + f(t_{n+1}, u_{n+1})]$.

$\Rightarrow (u_{n+1} - u_n)/h = \frac{1}{2}[f(t_n, u_n) + f(t_{n+1}, u_{n+1})]$

By (9.12), $E_1(f) = -\frac{h^3}{12} f''(\xi_n)$ for some $\xi_n \in (t_n, t_{n+1})$

and $(y_{n+1} - y_n)/h = \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^2}{12} f''(\xi_n, y(\xi_n))$

So $\tau_{n+1} = -\frac{h^2}{12} f''(\xi_n, y(\xi_n)) = O(h^2)$

Thus Crank-Nicoloson method has order 2 with respect to h. $\square$