1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

(12.1): $-u''(x) = f(x) = 1/x$, $0 < x < 1$.

(12.2): $u(0) = u(1) = 0$.

(12.3): $u(x) = \int_0^1 G(x,s) f(s)\, ds$, where, for any fixed x,

$$G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x, \\ x(1-s) & \text{if } x \le s \le 1. \end{cases}$$

$$\begin{aligned} u(x) &= \int_0^1 G(x,s)\frac{1}{s}\, ds \\ &= \int_0^x s(1-x)\frac{1}{s}\, ds + \int_x^1 x(1-s)\frac{1}{s}\, ds \\ &= (1-x)x + x\left[\int_x^1 \frac{1}{s} ds - \int_x^1 1\, ds\right] \\ &= (1-x)x + x\left[\ln s\Big|_x^1 - (1-x)\right] \\ &= -x\ln x \quad \text{for } 0 < x < 1. \end{aligned}$$

$\Rightarrow u'(x) = -\ln x - 1 \Rightarrow u''(x) = -\frac{1}{x}$.

So u', u'' do not exist at $x = 0$ and $u \in C^2((0,1))$. $\square$

Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_h)w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2.$$

(1)

$$\begin{aligned}
\Sigma_{j=0}^{n-1}(W_{j+1}-W_j)V_j &= \Sigma_{j=0}^{n-1} W_{j+1}V_j - \Sigma_{j=0}^{n-1} W_j V_j \\
&= \Sigma_{j=1}^{n} W_j V_{j-1} - \Sigma_{j=0}^{n-1} W_j V_j \\
&= W_n V_{n-1} - W_0 V_0 - \Sigma_{j=1}^{n-1}(V_j - V_{j-1})W_j \\
&= W_n V_{n-1} - W_0 V_0 - \Sigma_{j=0}^{n-2}(V_{j+1}-V_j)W_{j+1} \\
&= W_n V_{n-1} - W_0 V_0 + (V_n - V_{n-1})W_n - \Sigma_{j=0}^{n-1}(V_{j+1}-V_j)W_{j+1} \\
&= W_n V_n - W_0 V_0 - \Sigma_{j=0}^{n-1}(V_{j+1}-V_j)W_{j+1}
\end{aligned}$$

(2)

$$(L_h V_h)(X_j) = -\frac{V_h(X_{j+1}) - 2V_h(X_j) + V_h(X_{j-1})}{h^2}, \quad j = 1 \sim n-1.$$

$(W_h, V_h)_h = h\,\Sigma_{k=0}^{n} C_k W_k V_k$ for two discrete functions $W_h, V_h \in V_h$, where $C_0 = C_n = 1/2$ and $C_k = 1$ for $k = 1 \sim n-1$.

Since $V_h \in V_h^0$, $V_h(X_0) = V_h(X_n) = 0$.

$$\begin{aligned}
\text{So } (L_h V_h, V_h)_h &= h\,\Sigma_{j=1}^{n-1}(L_h V_h)(X_j)\,V_h(X_j) \\
&= h\,\Sigma_{j=1}^{n-1}\left(-\frac{V_h(X_{j+1}) - 2V_h(X_j) + V_h(X_{j-1})}{h^2}\right)V_h(X_j)
\end{aligned}$$

Note: $V_h \in V_h^0$

$(V_1 - V_0)^2 = V_1^2$

$(V_2 - V_1)^2$

$\vdots$

$(V_{n-1} - V_{n-2})^2$

$(V_n - V_{n-1})^2 = V_{n-1}^2$

$$\begin{aligned}
&= -\frac{1}{h}\Sigma_{j=1}^{n-1}\left[V_h(X_{j+1}) - 2V_h(X_j) + V_h(X_{j-1})\right]V_h(X_j) \\
&= -\frac{1}{h}\Sigma_{j=1}^{n-1}\left[V_h(X_{j+1}) - V_h(X_j)\right]V_h(X_j) - \left[V_h(X_j) - V_h(X_{j-1})\right]V_h(X_j) \\
&= -\frac{1}{h}\Big\{\left[V_h(X_2) - V_h(X_1)\right]V_h(X_1) - \left[V_h(X_1) - V_h(X_0)\right]V_h(X_1) \\
&\qquad + \left[V_h(X_3) - V_h(X_2)\right]V_h(X_2) - \left[V_h(X_2) - V_h(X_1)\right]V_h(X_2) \\
&\qquad + \cdots + \left[V_h(X_n) - V_h(X_{n-1})\right]V_h(X_{n-1}) - \left[V_h(X_{n-1}) - V_h(X_{n-2})\right]V_h(X_{n-1}) \\
&= \frac{1}{h}\Sigma_{j=0}^{n-1}\left[V_h(X_{j+1}) - V_h(X_j)\right]^2 \quad \square
\end{aligned}$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).
[*Solution*: we prove the result by verifying that $L_hG = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_hG = 0$ at every node x_l with $l = 0, \dots, k-1$ and $l = k+1, \dots, n+1$. Finally, a direct computation shows that $(L_hG)(x_k) = 1/h$ which concludes the proof.]

(12.4): $G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x, \\ x(1-s) & \text{if } x \le s \le 1 \end{cases}$ for any fixed x.

Define $G^k(x_j) = h\,G(x_j, x_k)$.

① Assume $j \neq k$.

(1-1) For $j < k$, $G(x_j, x_k) = x_j(1-x_k)$.

So $G(x_{j-1}, x_k) - G(x_j, x_k) = (x_{j-1} - x_j)(1-x_k) = h(1-x_k) = (x_j - x_{j+1})(1-x_k) = G(x_j, x_k) - G(x_{j+1}, x_k)$.

$\Rightarrow G(x_{j-1}, x_k) - 2G(x_j, x_k) + G(x_{j+1}, x_k) = 0$

(1-2) For $j > k$, $G(x_j, x_k) = x_k(1-x_j)$.

So $G(x_{j-1}, x_k) - G(x_j, x_k) = x_k(x_j - x_{j-1}) = h x_k = x_k(x_{j+1} - x_j) = G(x_j, x_k) - G(x_{j+1}, x_k)$.

$\Rightarrow G(x_{j-1}, x_k) - 2G(x_j, x_k) + G(x_{j+1}, x_k) = 0$

Thus $(L_hG^k)(x_j) = -\dfrac{G^k(x_{j+1}) - 2G^k(x_j) + G^k(x_{j-1})}{h^2} = -\dfrac{h[G(x_{j+1}, x_k) - 2G(x_j, x_k) + G(x_{j-1}, x_k)]}{h^2}$

$= 0$ for $j \neq k$.

② Assume $j = k$.

$G(x_{k-1}, x_k) - 2G(x_k, x_k) + G(x_{k+1}, x_k)$

$= x_{k-1}(1-x_k) - 2x_k(1-x_k) + x_k(1-x_{k+1})$

$= (k-1)h(1-kh) - 2kh(1-kh) + kh[1-(k+1)h]$

$= [(k-1)h - kh](1-kh) + \{-(1-kh) + [1-(k+1)h]\}kh$

$= -h(1-kh) - h(kh) = -h$

Thus $(L_hG^k)(x_j) = -\dfrac{h(-h)}{h^2} = 1$ for $j = k$.

By ①, ②, $(L_hG^k)(x_j) = \begin{cases} 0 & (j = k) \\ 1 & (j \neq k) \end{cases} = e_k$.

So $G^k(x_j) = h\,G(x_j, x_k)$. $\square$