7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

$$\begin{aligned}
\Gamma(z+1) &= \int_0^{\infty} e^{-t} t^{z} dt \\
&= -\int_0^{\infty} t^{z} de^{-t} \\
&= -t^{z} e^{-t} \Big|_0^{\infty} + \int_0^{\infty} e^{-t} dt^{z} \\
&= \int_0^{\infty} e^{-t} dt^{z} \\
&= \int_0^{\infty} e^{-t} z t^{z-1} dt \\
&= z \int_0^{\infty} e^{-t} t^{z-1} dt \\
&= z \Gamma(z). \qquad \square
\end{aligned}$$

Integration by parts.

① Since $\mathrm{Re} z > 0$, $\lim_{t \to 0^+} t^{z} e^{-t} = 0$.

② $\lim_{t \to \infty} t^{z} e^{-t} = 0$.

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1-\frac{\alpha}{2})f(x_n,u_n) + \frac{\alpha}{2}f(x_{n+1},u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

(1) Consistency:

Let $y(x)$ be the exact solution and denote $y(x_n)$ by y_n.

The local truncation error $\tau_{n+1} = \frac{y_{n+1}-y_n}{h} - [(1-\frac{\alpha}{2})f(x_n,y_n) + \frac{\alpha}{2}f(x_{n+1},y_{n+1})]$.

① By Taylor's expansion, $y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + O(h^3)$.

So $\frac{y_{n+1}-y_n}{h} = y'_n + \frac{h}{2}y''_n + O(h^2)$.

② By Taylor's expansion,

$f(x_{n+1},y_{n+1}) = f(x_n,y_n) + h\frac{\partial f}{\partial x}(x_n,y_n) + hy'_n\frac{\partial f}{\partial y}(x_n,y_n) + O(h^2)$.

Since $y'(x) = f(x,y(x))$,

$y''(x) = \frac{d}{dx}f(x,y(x)) = f_x(x,y(x)) + y'(x)f_y(x,y(x))$.

So $f(x_{n+1},y_{n+1}) = f(x_n,y_n) + hy''_n + O(h^2)$.

Then $(1-\frac{\alpha}{2})f(x_n,y_n) + \frac{\alpha}{2}f(x_{n+1},y_{n+1}) = f(x_n,y_n) + \frac{\alpha}{2}hy''_n + O(h^2)$

$= y'_n + \frac{\alpha}{2}hy''_n + O(h^2)$

Thus $\tau_{n+1} = (y'_n + \frac{h}{2}y''_n + O(h^2)) - (y'_n + \frac{\alpha}{2}hy''_n + O(h^2))$

$= \frac{h}{2}(1-\alpha)y''_n + O(h^2)$

If $\alpha = 1$, $\tau_{n+1} = O(h^2)$. $\Rightarrow$ The method has order 2 with respect to h.

If $\alpha \neq 1$, $\tau_{n+1} = O(h)$. $\Rightarrow$ The method has order 1 with respect to h.

Therefore the family of methods is consistent for any value of α.

(2) Take $\alpha = 1$ and solve $\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$

$u_{n+1} = u_n + \frac{h}{2}(f(x_n,u_n) + f(x_{n+1},u_{n+1}))$

$= u_n + \frac{h}{2}(-10u_n - 10u_{n+1})$

$\Rightarrow (1+5h)u_{n+1} = (1-5h)u_n \Rightarrow u_{n+1} = (\frac{1-5h}{1+5h})u_n$

Since $|\frac{1-5h}{1+5h}| < 1$ for all $h > 0$,

the method is absolutely stable for all $h > 0$. $\square$