9. Given the following set of data

$$\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \dots, 3$ is singular.]

$$H_3(X) = a_3X^3 + a_2X^2 + a_1X + a_0$$

$$\Rightarrow H_3'(X) = 3a_3X^2 + 2a_2X + a_1$$

$$\Rightarrow \begin{cases} 1 = H_3(-1) = -a_3 + a_2 - a_1 + a_0 \\ 1 = H_3'(-1) = 3a_3 - 2a_2 + a_1 \\ 2 = H_3'(1) = 3a_3 + 2a_2 + a_1 \\ 1 = H_3(2) = 8a_3 + 4a_2 + 2a_1 + a_0 \end{cases} \quad \rightarrow a_2 = \frac{1}{4}$$

$$\Rightarrow \begin{cases} \frac{3}{4} = -a_3 - a_1 + a_0 \\ \frac{3}{2} = 3a_3 + a_1 \\ 0 = 8a_3 + 2a_1 + a_0 \end{cases} \rightarrow \begin{array}{l} \frac{9}{4} = 2a_3 + a_0 \\ \frac{3}{2} = 6a_3 + 3a_0 \Rightarrow \frac{1}{2} = 2a_3 + a_0 \end{array} \rightarrow (\rightarrow\leftarrow)$$

So the Hermite-Birkoff interpolating polynomial H_3 does not exist. $\square$

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \tag{8.75}$$

called the *Padé approximation*. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10}x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

$$(1+b_2x^2)^{-1} = 1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + \cdots$$

$$\Rightarrow r(x) = (a_0 + a_2x^2 + a_4x^4)(1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + \cdots)$$

$$= a_0 + (a_2 - a_0b_2)x^2 + (a_4 - a_2b_2 + a_0b_2^2)x^4 + (-a_4b_2 + a_2b_2^2 - a_0b_2^3)x^6 + \cdots$$

$\Rightarrow$ x^0 : $a_0 = 1$,

x^2 : $a_2 - a_0b_2 = -\frac{1}{2!} = -\frac{1}{2} \Rightarrow a_2 - b_2 = -\frac{1}{2}$,

x^4 : $a_4 - a_2b_2 + a_0b_2^2 = \frac{1}{4!} = \frac{1}{24} \Rightarrow a_4 - a_2b_2 + b_2^2 = \frac{1}{24}$,

x^6 : $-a_4b_2 + a_2b_2^2 - a_0b_2^3 = -\frac{1}{6!} = -\frac{1}{720} \Rightarrow -a_4b_2 + a_2b_2^2 - b_2^3 = -\frac{1}{720}$.

$\Rightarrow a_0 = 1$, $a_2 = -\frac{7}{15}$, $a_4 = \frac{1}{40}$, $b_2 = \frac{1}{30}$. $\square$