1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

(9.6) $E_0(f) = \frac{h^3}{3} f''(\xi)$, where $h = \frac{b-a}{2}$ and $\xi \in (a,b)$.

(9.12) $E_1(f) = -\frac{h^3}{12} f''(\xi)$, where $h = b-a$ and $\xi \in (a,b)$.

$E_0(f) = \left(\frac{b-a}{2}\right)^3 \frac{1}{3} f''(\xi_0) = \frac{(b-a)^3}{24} f''(\xi_0)$, where $\xi_0 \in (a,b)$.

$E_1(f) = -\frac{(b-a)^3}{12} f''(\xi_1)$, where $\xi_1 \in (a,b)$.

$$\Rightarrow E_1(f) = -2 E_0 \frac{f''(\xi_0)}{f''(\xi_1)}$$

$$\Rightarrow |E_1(f)| = 2|E_0| \left|\frac{f''(\xi_0)}{f''(\xi_1)}\right|$$

$$\Rightarrow |E_1(f)| \simeq 2|E_0| \qquad \square$$

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

(a) $f(x) = 1$: $\frac{2}{3}(2 - 1 + 2) = 2 = \int_{-1}^{1} 1\,dx$

$f(x) = x$: $\frac{2}{3}[2(-\frac{1}{2}) - 0 + 2(\frac{1}{2})] = 0 = \int_{-1}^{1} x\,dx$

$f(x) = x^2$: $\frac{2}{3}[2(\frac{1}{4}) - 0 + 2(\frac{1}{4})] = \frac{2}{3} = \int_{-1}^{1} x^2\,dx$

$f(x) = x^3$: $\frac{2}{3}[2(-\frac{1}{8}) - 0 + 2(\frac{1}{8})] = 0 = \int_{-1}^{1} x^3\,dx$

$f(x) = x^4$: $\frac{2}{3}[2(\frac{1}{16}) - 0 + 2(\frac{1}{16})] = \frac{1}{6} \neq \frac{2}{5} = \int_{-1}^{1} x^4\,dx$

$\Rightarrow r = 3$

(b) $f(x) = 1$: $\frac{1}{4}[1+3+3+1] = 2 = \int_{-1}^{1} 1\,dx$

$f(x) = x$: $\frac{1}{4}[-1 + 3(-\frac{1}{3}) + 3(\frac{1}{3}) + 1] = 0 = \int_{-1}^{1} x\,dx$

$f(x) = x^2$: $\frac{1}{4}[1 + 3(\frac{1}{9}) + 3(\frac{1}{9}) + 1] = \frac{2}{3} = \int_{-1}^{1} x^2\,dx$

$f(x) = x^3$: $\frac{1}{4}[-1 + 3(-\frac{1}{27}) + 3(\frac{1}{27}) + 1] = 0 = \int_{-1}^{1} x^3\,dx$

$f(x) = x^4$: $\frac{1}{4}[1 + 3(\frac{1}{81}) + 3(\frac{1}{81}) + 1] = \frac{14}{27} \neq \frac{2}{5} = \int_{-1}^{1} x^4\,dx$

$\Rightarrow r = 3$

Since I_2, I_4 are symmetric about 0 and r is odd, the first nonzero term in the reminder is of order $r+2 = 5$. So $p = 5$. $\square$

5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

$f(x)=1$: $\int_0^1 \sqrt{x}\, dx = \frac{2}{3} = af(x_1) = a \Rightarrow a = \frac{2}{3}$

$f(x)=x$: $\int_0^1 \sqrt{x}\, x\, dx = \frac{2}{5} = \frac{2}{3} f(x_1) = \frac{2}{3} x_1 \Rightarrow x_1 = \frac{3}{5}$

$f(x)=x^2$: $\int_0^1 \sqrt{x}\, x^2\, dx = \frac{2}{7} \neq \frac{6}{25} = \frac{2}{3} f(\frac{3}{5})$.

So $a = \frac{2}{3}$, $x_1 = \frac{3}{5}$, $r=1$. $\square$

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1,2,3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

$f(x)=1$: $\int_0^1 1\, dx = 1 = \alpha_1 \times 1 + \alpha_2 \times 1 + \alpha_3 \times 0 = \alpha_1 + \alpha_2$

$f(x)=x$: $\int_0^1 x\, dx = \frac{1}{2} = \alpha_1 \times 0 + \alpha_2 \times 1 + \alpha_3 \times 1 = \alpha_2 + \alpha_3$

$f(x)=x^2$: $\int_0^1 x^2\, dx = \frac{1}{3} = \alpha_1 \times 0 + \alpha_2 \times 1 + \alpha_3 \times 0 = \alpha_2$

So $\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2 + \alpha_3 = \frac{1}{2} \\ \alpha_2 = \frac{1}{3} \end{cases} \Rightarrow \alpha_1 = \frac{2}{3},\ \alpha_2 = \frac{1}{3},\ \alpha_3 = \frac{1}{6}$

Check that the $Q(f)$ fails at degree $r+1 = 2+1 = 3$:

$f(x)=x^3$: $\int_0^1 x^3\, dx = \frac{1}{4} \neq \frac{1}{3} = \frac{2}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{6} \times 0$ $\square$