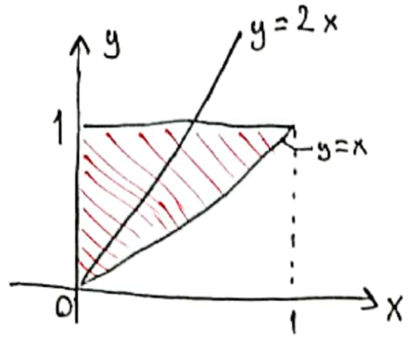
25 1)



4 a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy=1 \quad \text{so}$$

$$1=\int_0^1\int_0^y cxy\,dx\,dy=\int_0^1 cy\cdot\frac{x^2}{2}\Big|_0^y dy=\int_0^1 \frac{c}{2}y^3\,dy=\frac{c}{8}y^4\Big|_0^1=\frac{c}{8}\Rightarrow c=8$$

5 b)

$$f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy=\int_x^1 8xy\,dy=4x\cdot y^2\Big|_x^1=4x(1-x^2) \quad \text{for} \quad 0\leqslant x\leqslant 1$$

8 c)

$$P(A)=P(X\leqslant 0.5)=\int_{-\infty}^{0.5} f_X(x)\,dx=\int_0^{0.5}(4x-4x^3)\,dx=(2x^2-x^4)\Big|_0^{0.5}=\frac{1}{2}-\frac{1}{16}=\frac{7}{16}$$

$$f_{X|A}(x)=\frac{f(x)}{P(A)}=\frac{64}{7}x(1-x^2) \quad \text{for} \quad 0\leqslant x\leqslant 0.5$$

8 d)

$$P(Y\leqslant 2X)=\iint_{Y\leqslant 2X} f_{X,Y}(x,y)\,dx\,dy=\int_0^1\int_{y/2}^{y} 8xy\,dx\,dy=\int_0^1 4y\cdot x^2\Big|_{y/2}^{y} dy$$

$$=\int_0^1 \cancel{4}y\cdot\frac{3}{\cancel{4}}y^2\,dy=\frac{3}{4}y^4\Big|_0^1=3/4$$

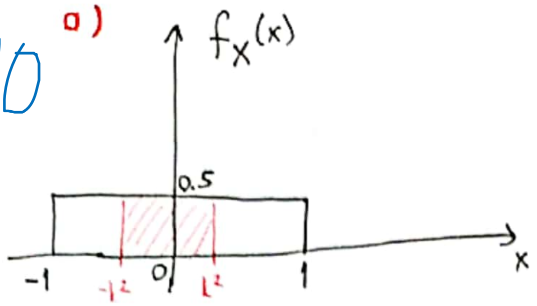
or

$$P(Y\leqslant 2X)=\int_0^1\int_x^{\min(2x,1)} 8xy\,dy\,dx=\int_0^{0.5}\int_x^{2x} 8xy\,dy\,dx+\int_{0.5}^1\int_x^1 8xy\,dy\,dx$$

$$=\int_0^{0.5} 4x\cdot 3x^2\,dx+\int_{0.5}^1 4x\cdot(1-x^2)\,dx=3x^4\Big|_0^{0.5}+2x^2\Big|_{0.5}^1-x^4\Big|_{0.5}^1=\frac{3}{16}+\frac{3}{2}-\frac{15}{16}=\frac{3}{4}$$

2) 20

a) 10



$$F_Y(k) = P(Y \leqslant k) = P(\sqrt{|X|} \leqslant k)$$

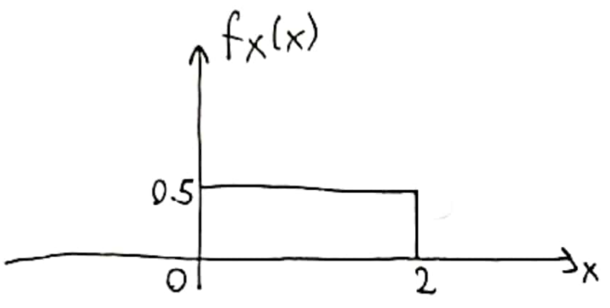
$$= P(|X| \leqslant k^2)$$

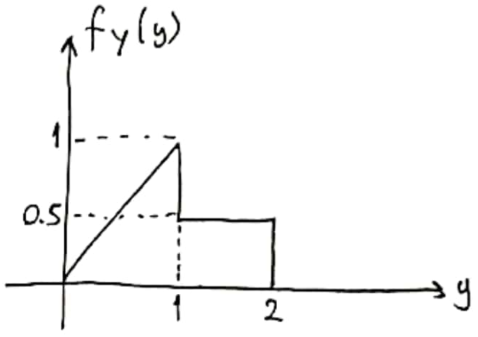
$$= P(-k^2 \leqslant X \leqslant k^2)$$

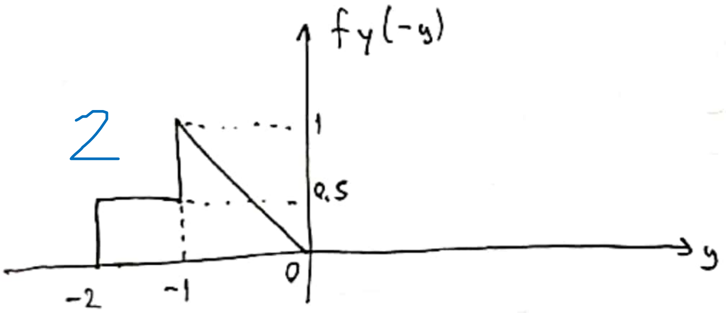
$$= k^2 \quad \text{for} \quad 0 \leqslant k \leqslant 1$$

$$f_Y(k) = \frac{d F_Y(k)}{dk} = 2k \quad \text{for} \quad 0 \leqslant k \leqslant 1$$

b) 10







$$f_Z(z) = f_X(t) * f_Y(t)$$

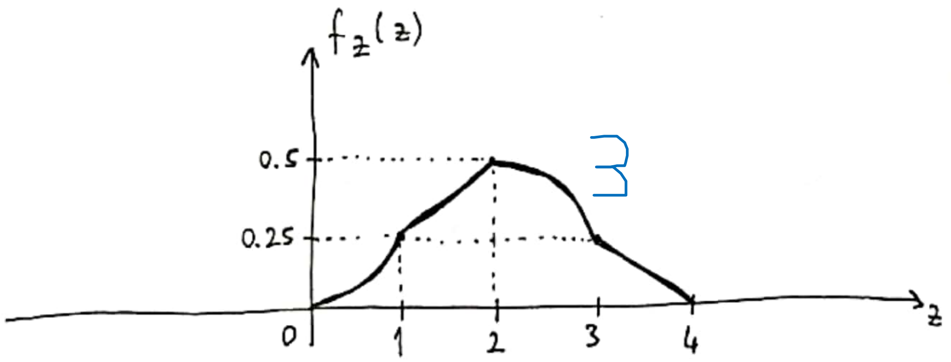
$f_Z(0) = 0$

$f_Z(1) = 0.25$

$f_Z(2) = 0.5$

$f_Z(3) = 0.25$

$f_Z(4) = 0$



3) X_i: the lifetime of i-th component

14

$$P\left(\sum_{i=1}^{n} X_i \geq 2000\right) \geq 0.95 \quad 4$$

Question asks to find n that makes above equation true.

$$\text{let } X = \sum_{i=1}^{n} X_i \qquad E[X] = E\left[\sum X_i\right] = \sum E[X_i] = \sum 100 = 100n$$

$$Var(X) = Var\left(\sum X_i\right) \underset{\text{i.i.d.}}{=} \sum Var(X_i) = \sum 900 = 900n$$

Then,

$$P\left(\frac{\sum X_i - 100n}{30\sqrt{n}} \geq \frac{2000-100n}{30\sqrt{n}}\right) \geq 0.95 \quad 4$$

$$P\left(Z \geq \frac{2000-100n}{30\sqrt{n}}\right) \geq 0.95 \quad 1 \qquad \text{where } Z \sim N(0,1) \text{ by CLT}$$

$$P\left(Z \leq \frac{2000-100n}{30\sqrt{n}}\right) \leq 0.05$$

$$P\left(Z \geq \frac{100n-2000}{30\sqrt{n}}\right) \leq 0.05 \quad 2$$

$$\frac{100n-2000}{30\sqrt{n}} \geq 1.645 \quad 2$$

$$n \geq 23 \quad 1$$

4)

14

$$\hat{\lambda}_{ML} = \arg\max_{\alpha} \prod_{i=1}^{n} \frac{1}{2\alpha} e^{-\frac{|x_i|}{\alpha}} \;3\; = \arg\max_{\alpha} \sum_{i=1}^{n} \log \frac{1}{2\alpha} e^{-\frac{|x_i|}{\alpha}} \quad 2$$

$$= \arg\max_{\alpha} \sum \left[-\log 2 - \log \alpha - \frac{|x_i|}{\alpha}\right] \quad 3$$

Then,

$$0 = \sum \left[-\frac{1}{\alpha} + \frac{|x_i|}{\alpha^2}\right] \;2\; = -\frac{n}{\alpha} + \frac{\sum |x_i|}{\alpha^2} \quad 2$$

Note that $\alpha \neq 0$. Then, $n\alpha = \sum |x_i| \Rightarrow \hat{\lambda} = \frac{\sum |x_i|}{n}$ 2

24

5)

17 **a)**

$$L(x)=\frac{f_{X|H_1}(x)}{f_{X|H_0}(x)}=\frac{\frac{1}{10}e^{-\frac{x}{10}}}{\frac{1}{20}e^{-\frac{x}{20}}}=2e^{-\frac{x}{20}}$$

1 1 2

$$R: \quad L(x) > \varepsilon$$

2

$$2e^{-\frac{x}{20}} > \varepsilon$$

$$e^{-\frac{x}{20}} > \frac{\varepsilon}{2}$$

$$-\frac{x}{20} > \ln\frac{\varepsilon}{2}$$

$$x < -20\ln\frac{\varepsilon}{2} = \gamma$$

2

So

$$R: \quad X < \gamma$$

$$\alpha = 0.1 = P(X<\gamma|H_0) = \int_0^{\gamma}\frac{1}{20}e^{-\frac{x}{20}}dx = -e^{-\frac{x}{20}}\Big|_0^{\gamma} = 1-e^{-\frac{\gamma}{20}}$$

2 3

$$\Rightarrow e^{-\frac{\gamma}{20}} = 0.9 \Rightarrow -\frac{\gamma}{20} = \ln 0.9 \Rightarrow \gamma = 2.1$$

2

So

$$R: \quad X < 2.1$$

2

7 **b)**

$$\beta = P(X>\gamma|H_1) = \int_{2.1}^{\infty}\frac{1}{10}e^{-\frac{x}{10}}dx = -e^{-\frac{x}{10}}\Big|_{2.1}^{\infty} = e^{-\frac{2.1}{10}} = 0.81$$

2 3 2

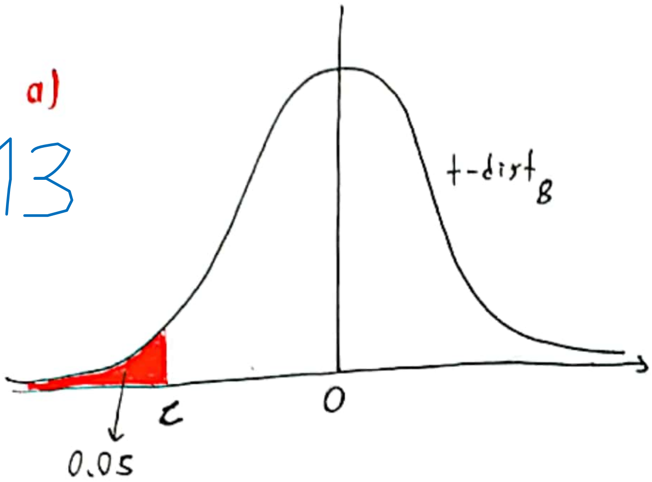
6)

23 let $X = \frac{1}{9}\sum X_i$ then $E[X] = \mu$ $M_n = 6.0798$

$Var(X) = \frac{\sigma^2}{9}$ $\frac{S_n^2}{n} = \frac{26.01}{9}$ $\frac{S_n}{\sqrt{n}} = \frac{5.1}{3} = 1.7$

a)

13



2 $\frac{X-\mu}{S_n/\sqrt{n}}$ has t-distribution with dof = 8

So rejection region becomes

R: $Z = \frac{X-10}{1.7} < c$ under H_0 2

we can find c easily since $P(Z < c) = 0.05$ 2

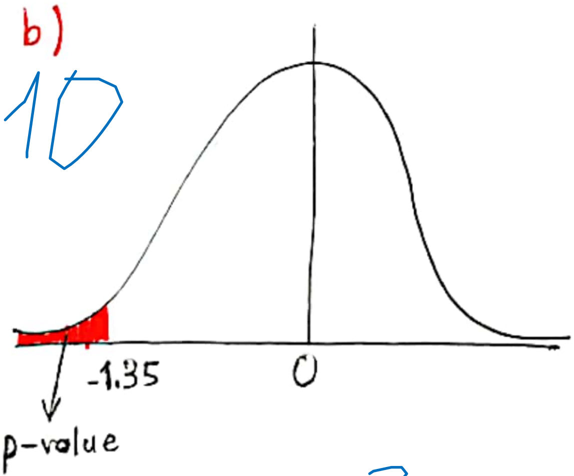
$\Rightarrow P(-c < Z) = 0.05 \Rightarrow -c = 1.86$ from table so $c = -1.86$ 2

R: $\frac{\frac{1}{n}\sum X_i - 10}{1.7} < -1.86$ 2 $\Rightarrow$ R: $\frac{1}{9}\sum X_i < 6.838$ 1

$M_n = 6.0798 < 6.838$ so we 2 reject H_0 for 0.05 significance level

b)

10



from observed data we get

$$z = \frac{M_n - \mu}{S_n/\sqrt{n}} = \frac{6.0798 - 10}{1.7} = -2.306$$ 2

2

we know it has t-distribution

p-value: false rejection probability for -2.306

$\Psi_8(-2.306) = $ 2 p-value $\Rightarrow \Psi_8(2.306) = 1 - $ p-value $\Rightarrow$ p-value $= 0.025$ 2

$0.01 < 0.025 = $ p-value so we accept H_0 for 0.01 significance level

2