

# Constrained Geometric Approximation Approach for Robot Planning and Decentralized Formation Algorithm for Multi-Robot Systems

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# Outlines



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## CGA for Robot Planning

Overview

Problem Statement

## Range Space

Range Space Definition

Disk Range Space

Rectangle Range Space

Double-Rectangle Range Space

## Experiments

Conclusions

Introduction

## Lattice Graph

Algorithm

Robot Authority

# Constrained Geometric Approximation



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► **Goal:**

For an extremely simple robot with:

- ▶ computation limitations
- ▶ moving and sensing uncertainties

represent and reason about uncertainty  
in its own states efficiently.

► **Basic Idea:**

Explicitly represent what the robot  
knows as an information state (*I-state*).

► **Intuition:**

Accelerate time-consuming operations  
by maintaining only an  
**overapproximation** of the true *I-state*,  
and constraining this approximation to  
have a simple geometric form.



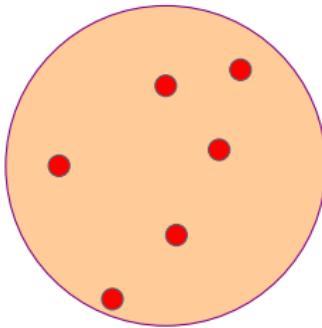
SRV-1 Surveyor Robot

# Robot Model



Assume that current real state of the robot could not be observed directly. The robot could maintain an *I-state*  $\eta_k$ , to make its decisions.

- ▶ Robot state at stage  $k$ :  $x_k \in X$ ,
- ▶ State transition function:  $F(x_k, u_k)$ .
- ▶ Robot action at stage  $k$ :  $u_k$ .
- ▶ Information state (*I-state*) at stage  $k$ :  $\eta_k$  is a set of possible states at stage  $k$



**Figure:** I-state  $\eta_k$  contains all possible states

# Prior Work



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- ▶ Prior research done by (B. Tovar and S. M. LaValle) and (J. van den Berg, P. Abbeel, and K. Goldberg) used probabilistic representations for planning
- ▶ Prior work by the J.O'Kane has used preliminary versions of the constrained geometric approximation method using specific, fixed range spaces.

## New contributions

1. A careful formulation of the operations in the range space  $\mathcal{R}$ .
2. Algorithms for double-rectangle range space  $\mathcal{R}_{direct}$ .
3. A series of experiments for effectiveness comparison of different range spaces.

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# Range Space



## Definition

A **range space**  $\mathcal{R} \subseteq \mathcal{I}$  is a set of I-states, contains approximation of I-states,  $A(\eta_k) \in \mathcal{R}$ , equipped with two operations:

1. An *approximate observation update function*

$O : \mathcal{R} \times Y \rightarrow \mathcal{R}$ , such that if  
 $\eta_k \subseteq A(\eta_k)$ , then

$$\eta_k \cap H(y_k) \subseteq O(A(\eta_k), u_k)$$

2. An *approximate action update function*  $T : \mathcal{R} \times U \rightarrow \mathcal{R}$ , such that if  $\eta_k \subseteq A(\eta_k)$ , then

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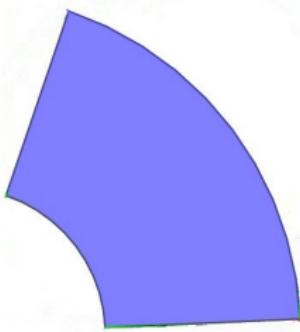


Figure: An I-state  $\eta_k$

# Range Space



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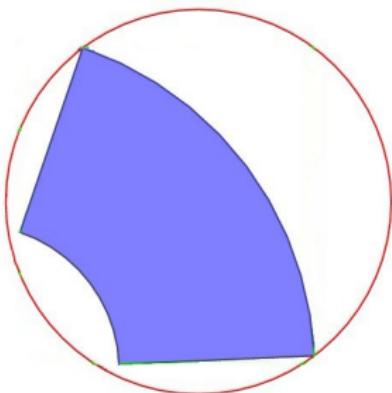
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**Figure:** Disk overapproximation  $A(\eta_k) \in \mathcal{R}_{disk}$  in red.

# Range Space



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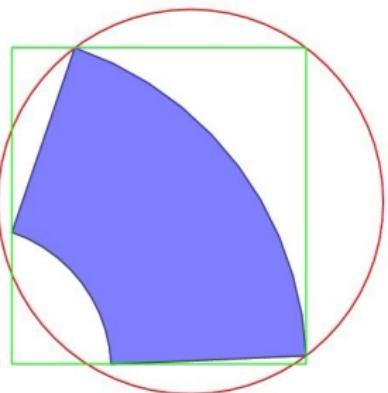
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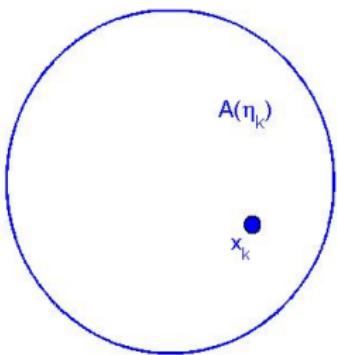
**Figure:** Rectangle overapproximation  $A(\eta_k) \in \mathcal{R}_{rect}$ .

# Observation Update in $\mathcal{R}_{disk}$



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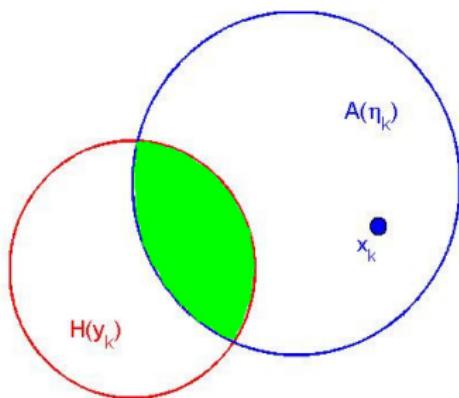
Approximated I-state  $A(\eta_k) \in \mathcal{R}_{disk}$ , where  $x_k$  denotes the real state but unknown to robot.



# Observation Update in $\mathcal{R}_{disk}$



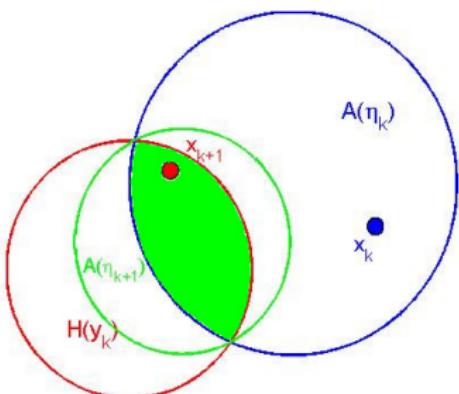
Approximated I-state  $A(\eta_k) \in \mathcal{R}_{disk}$  intersects with observation preimage  $H(y_k)$ .



# Observation Update in $\mathcal{R}_{disk}$



The green region of intersection is the updated I-state  $\eta_{k+1}$ , and the green disk is the approximation  $A(\eta_{k+1})$  of  $\eta_{k+1}$ .



# Observation Update in $\mathcal{R}_{rect}$

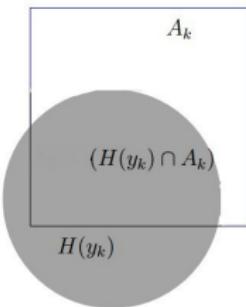


## Definition

$AABB(S)$  : For any compact set  $S \subset \mathbb{R}^2$ , let  $AABB(S)$  denote the its smallest “axis-aligned bounding box.”

In  $\mathcal{R}_{rect}$ , computing approximate observation update function  $O_{rect}$  takes  $O(1)$  time:

$$O_{rect}(A_k, y_k) = AABB(H(y_k) \cap A(\eta_k)), A_k = A(\eta_k) \in \mathcal{R}_{rect}$$



# Observation Update in $\mathcal{R}_{rect}$



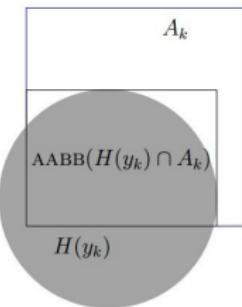
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# Action Update in $\mathcal{R}_{rect}$



In  $\mathcal{R}_{rect}$ , computing approximate action update function  $T_{rect}$ :

$$T_{rect}(A(\eta_k), u_k) = AABB(X_{free} \cap [A(\eta_k) \oplus \{u_k\} \oplus AABB(\Theta(u_k))])$$

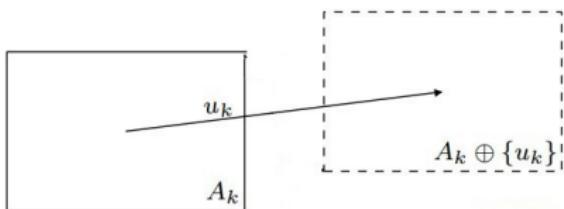


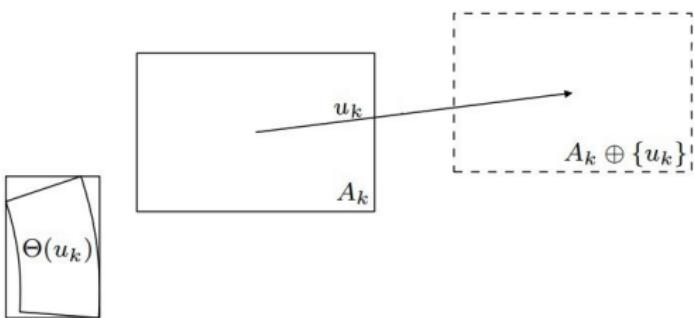
Figure: Transition of  $A(\eta_k)$  given action  $u_k$ ,  $\oplus$  is Minkowski addition of two sets.

# Action Update in $\mathcal{R}_{rect}$



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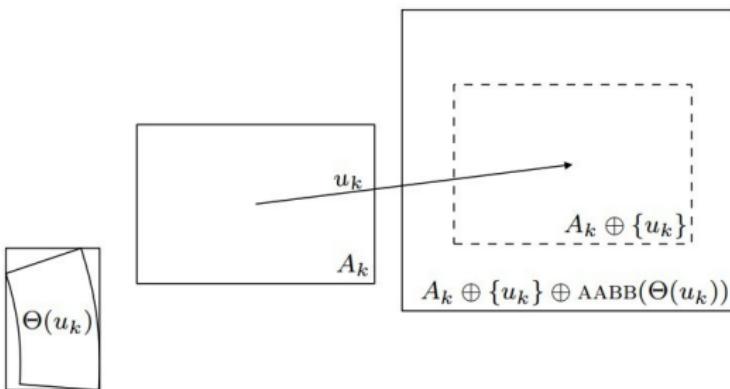
**Figure:** Consider approximation of the bounded motion uncertainty  $\Theta(u_k)$ .

# Action Update in $\mathcal{R}_{rect}$



In  $\mathcal{R}_{rect}$ , computing approximate action update function  $T_{rect}$ :

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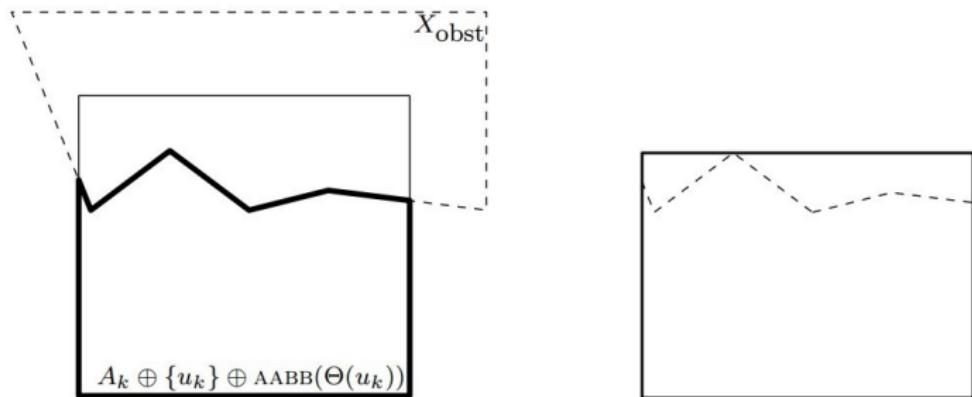
**Figure:** Find Minkowski sum of bounded noise and approximation transition

# Action Update in $\mathcal{R}_{rect}$



In  $\mathcal{R}_{rect}$ , computing approximate action update function  $T_{rect}$ :

$$T_{rect}(A(\eta_k), u_k) = AABB(X_{free} \cap [A(\eta_k) \oplus \{u_k\} \oplus AABB(\Theta(u_k))])$$



**Figure:** If there is obstacles, intersect with  $X_{free}$  first and then find the bounding box of the intersection.

# Double-Rectangle approximated I-state



- ▶ For better overapproximation quality for non-convex *I*-states, we proposed a more expressive range space of *double rectangles*:

$$\mathcal{R}_{direct} = \{R_1 \cup R_2 \mid R_1, R_2 \in \mathcal{R}_{rect}\} \quad (1)$$

- ▶ Aims to improve the approximation quality.

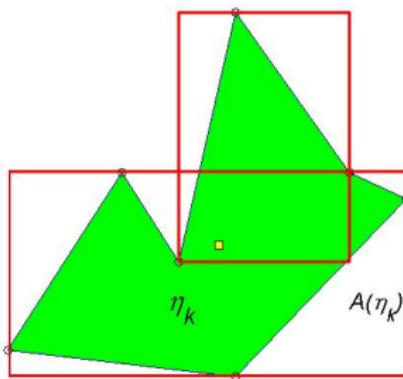


Figure: Non-convex *I*-state

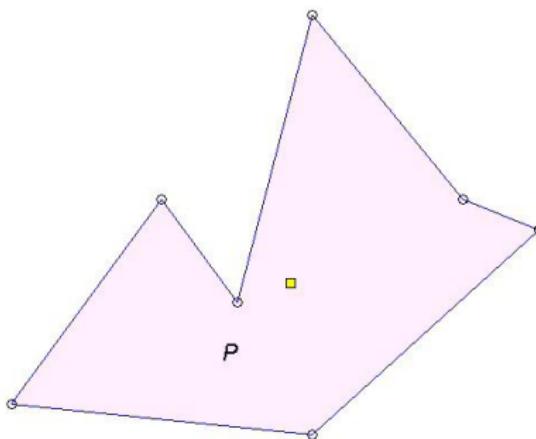
# DRAP Algorithm



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“Double Rectangle Around Polygon” (**DRAP**) algorithm:

- ▶ input is a  $n$ -edge polygonal region of the plane
- ▶ output is a small double rectangle containing that polygon
- ▶ run time:  $O(n^3)$



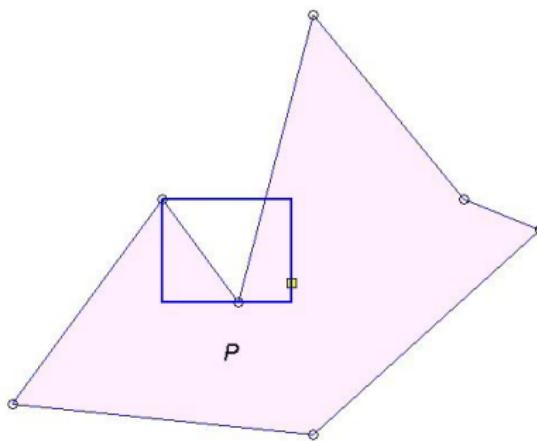
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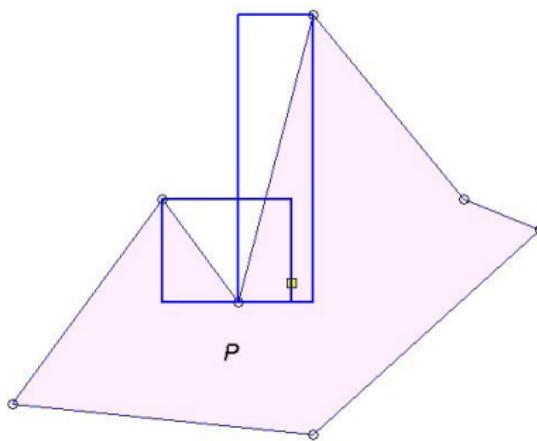
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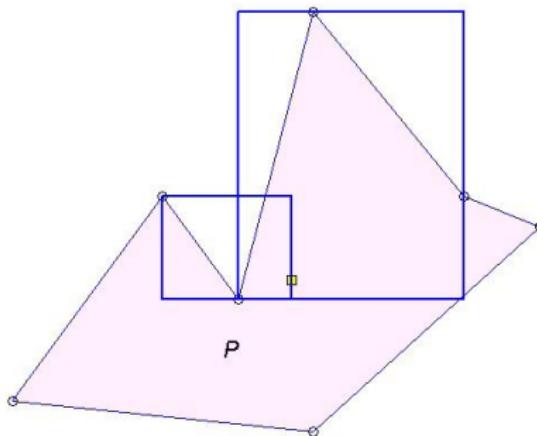
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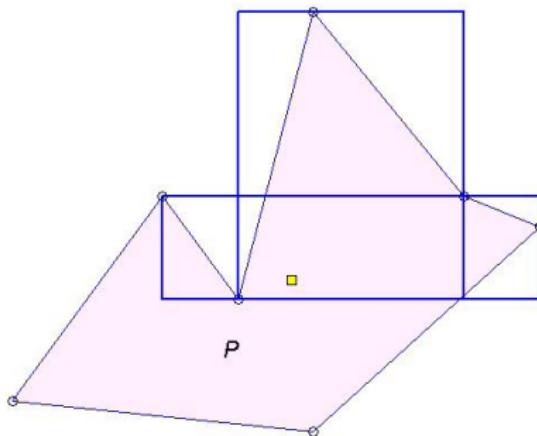
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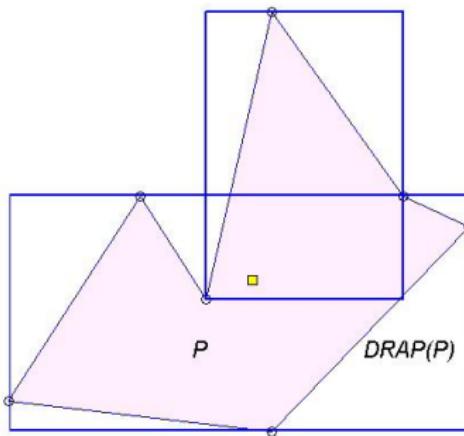


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# Experiments in Various Environments



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Comparison with using the true I-state, we conducted experiments using 3 environments, and 3 range spaces  $\mathcal{R}_{disk}$ ,  $\mathcal{R}_{rect}$ , and  $\mathcal{R}_{direct}$ .

## ASSUMPTIONS:

- ▶ Robot is guided by centroid point of the approximated *I-state*
- ▶ Robot can detect presence but not distance to the settled waypoints
- ▶ Landmarks are pseudo-randomly generated (Number Matters)
- ▶ Initial I-state  $\eta_0$  is given.

# Experiments

Experiments in Various Environments





# Experiments Results

For each environment, we measure:

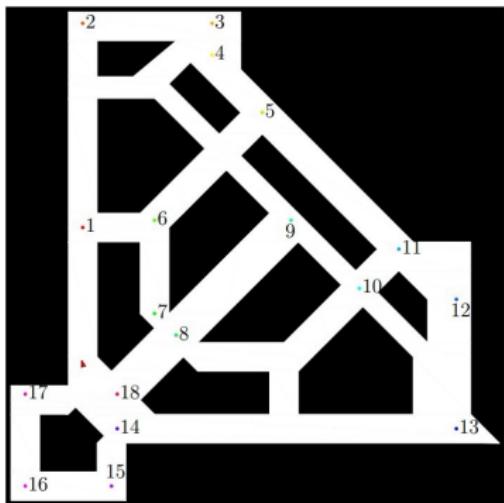
- ▶ the relationship between task completion and number of landmarks,
- ▶ the time required to compute approximated *I-state* compared to the high-quality polygonal representation of the exact *I-state*,
- ▶ the approximation ratio  $Q_k$

$$Q_k = \frac{1}{k} \sum_{i=1}^k \frac{\mathbb{A}(\eta_i)}{\mathbb{A}(A(\eta_i))} \quad (2)$$

in which  $\mathbb{A}(\diamond)$  denotes the area of set  $\diamond \subset \mathbb{R}^2$

# Comparison of Efficiency and Accuracy

## Office-like Environment



Range Space	Time (s)	Approximation Ratio
$\mathcal{R}_{disk}$	0.292	0.220
$\mathcal{R}_{rect}$	0.415	0.661
$\mathcal{R}_{dblrect}$	1.491	0.720
$\mathcal{I}$	26.895	1.000

**Figure:** Comparison of the computation time and approximation ratio for approximated I-states to the corresponding measurements of the exact I-states.

# Conclusions



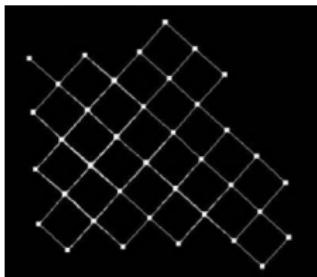
- ▶ CGA is effective for representing a robot's uncertain information about the current state.
- ▶ The form of double-rectangle is more accurate in approximating the non-convex *I-state*.
- ▶ The robot can complete the navigation task using approximated I-state with low approximation accuracy.

# Related Work

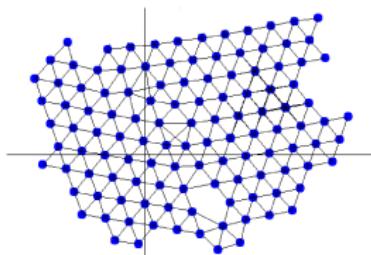
## Formation using virtual force



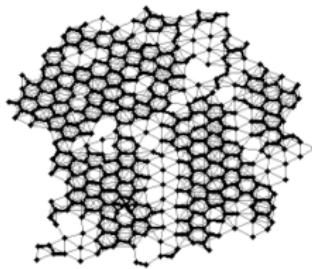
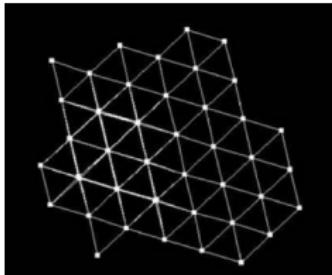
W. Spears, D. Spears, J. Hamann and R. Heil, 2004



I. Navarro, J. Pugh, A. Martinoli, and F. Matia, 2008



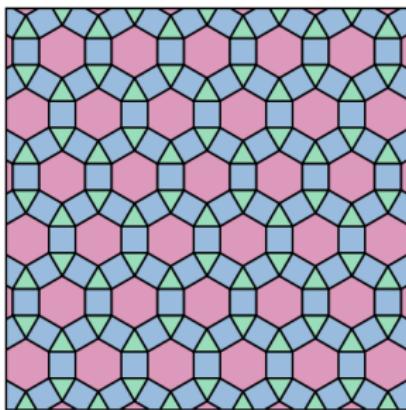
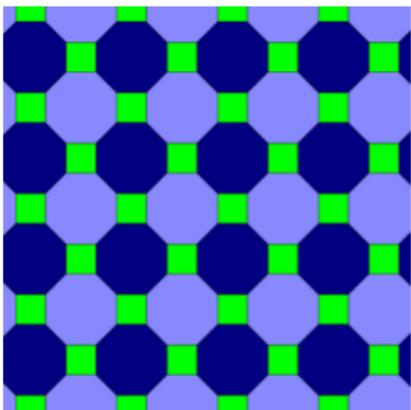
S. Prabhu, W. Li, J. McLurkin, 2012



# Motivation



Question: How to use one algorithm to generate various (repeating) lattice pattern formations?

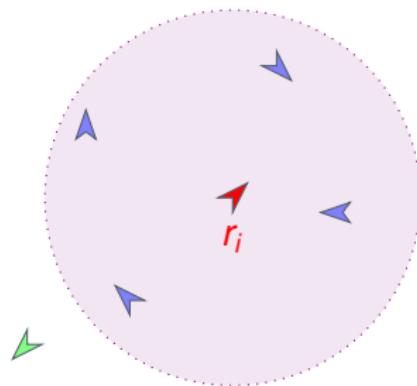


# Robot Model



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- ▶ Differential Drive robots.
- ▶ Each robot has an unique **ID**.
- ▶ Use a vector  $p = [x, y, \theta]^T$  to represent robot's **pose**.
- ▶ Each robot has a **range** within which it can sense and communicate with other robots.
- ▶ Each robot gets **observation** of its neighbors' IDs and relative poses in its body frame.

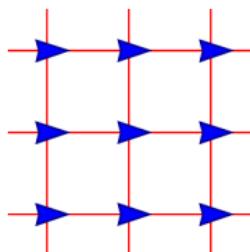
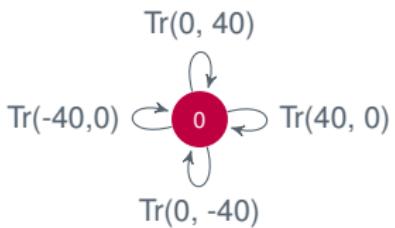


Robot  $r_i$  has 4 neighbors



## Definition

A **lattice graph** is a strongly connected directed multigraph in which each edge  $e$  is labeled with a rigid body transformation  $T(e)$  and each  $v \xrightarrow{T(e)} w$  has an inverse edge  $w \xrightarrow{T(e)^{-1}} v$ .

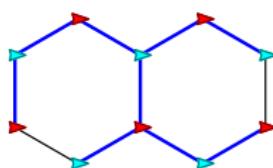
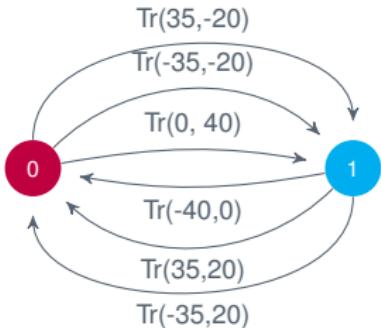




## Definition

Given a lattice graph  $G = (V, E)$  and a set of robots  $R = \{r_1, \dots, r_n\}$ ,  $R$  **satisfies**  $G$  if there exists a role function  $f : R \rightarrow V$  that preserves the neighborhood structure of  $G$ .

Specifically, for any  $i$  and  $j$ , if  $r_i$  and  $r_j$  are neighbors, there must exist an edge  $e_{ij} : f(r_i) \longrightarrow f(r_j)$  in  $E$ , such that  $T(r_j) = T(r_i)T(e_{ij})$ .



# Algorithm



## General Description

Robot broadcasts message containing its

- ▶ authority
  - ▶ matching.
1. Form tree structure.
  2. Use tree structure to computer local task assignment.
  3. Make movement decision.



## Define authority and comparison operator



An **authority** is an ordered list of robot IDs

$$(id_1, \dots, id_k)$$

The first ID in the list,  $id_1$  is called the **root** ID. The final ID in the list,  $id_k$  is called the **sender** ID.



Authority  $A_2$  is **higher than**  $A_1$  if:

- ▶ root ID of  $A_2 >$  root ID of  $A_1$ , or
- ▶ length of  $A_2 <$  length of  $A_1$  if they have the same root, or
- ▶ sender ID of  $A_2 >$  sender ID of  $A_1$  if they have the same root and length.

# 1. Construct Authority Tree

Decide to be root or descendant



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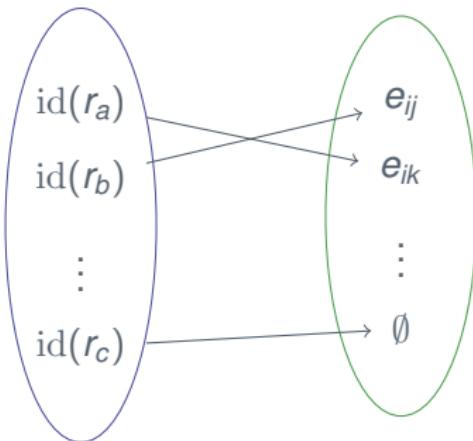
The robots use these authorities to establish a collection of authority trees

1. Discards any message in which the authority contains its own ID.
2. Forms an authority containing only its own ID, compares it with the authorities of remaining messages and selects the highest authority.
  - ▶ If its authority is the highest, then it is a **root**;
  - ▶ Otherwise, it selects the one who sends the highest authority as its parent. Append its own ID to the highest authority to create its own authority.

# Matching



A **matching** for a robot is a function  $\eta : \{\text{id}(r_a), \text{id}(r_b), \dots\} \rightarrow \{\emptyset, e_{ij}, e_{ik}, \dots\}$  that associates each neighbor ID with either a lattice graph edge from its role vertex or with the null value  $\emptyset$ .



## 2. Local Task Assignment

Hungarian Algorithm

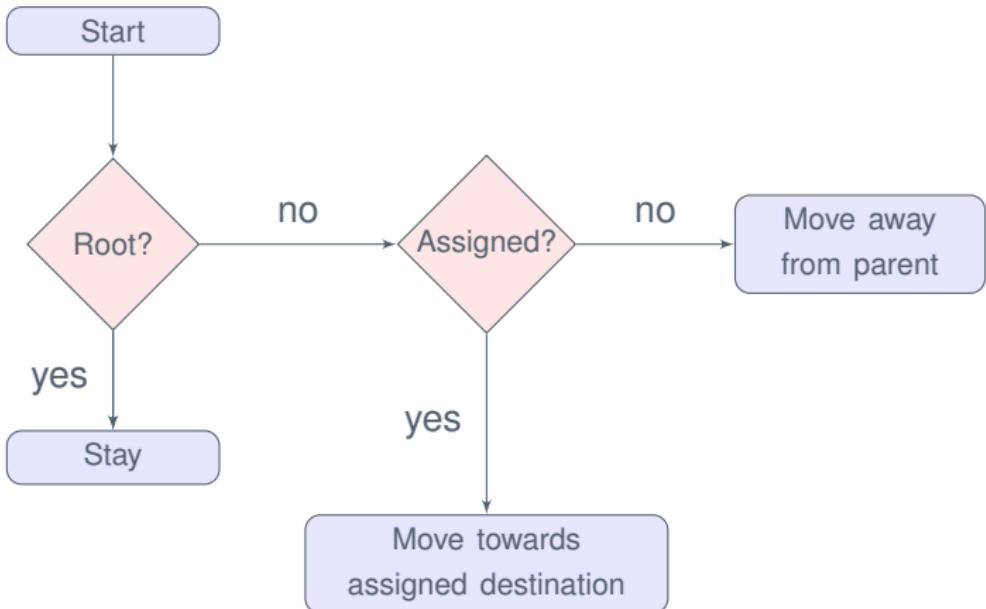


To compute an optimal matching of a robot with  $N$  neighbors and  $E$  out-going edges of its role in the lattice graph, define a weight matrix of size  $N \times \max(N, E)$  and apply **Hungarian Algorithm** (Harold W. Kuhn, 1955).

1. Each row corresponds to a neighbor;
2. Each column corresponds to an out-going edge of robot's role or a null value  $\emptyset$ .
3. The entries of the matrix are the Euclidean distance between current position of each neighbor and the desired position if matched with a lattice graph edge.

5 neighbors, 4 out-going edges.

### 3. Robot Movement Strategy



# Bounded Movement



Goal: let descendant stay in the range of its parent

- ▶ Within the set  $O$  (**Red circle**), the parent is guaranteed to get observation at next stage.
- ▶ Descendant can reach anywhere in set  $P$  (**blue circle**) at next stage.
- ▶ The real destination for descendant is the closest point on the boundary of the intersection ( $O \cap P$ ) to the assigned destination.

# Simulation

Octagon-square formation with 30 robots



# Simulation

Triangle-Hexagon-Square formation with 30 robots

