Lecture Memo
Quantum Mechanics of Light and Matters

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Chapter 1

Introduction

Light is regarded as an ensemble of particles called photons. The particle nature of light appears as 'noise' in various applications of light waves such as optical measurement, optical manipulation, and optical communications, leading to the physical limit of the performance or precision achieved by these methods. To enhance

Chapter 2

Noise in optical measurements

This chapter introduces various detection methods of light and explains noise appearing in each method. Some explanations are phenomenological but they will be explained by quantum optics in later chapters.

2.1 Optical measurements

Figure 2.1(a) shows the **direct detection**. Photodetectors can convert photons to electrons to measure optical power, which is proportional to the number of photons per unit time.

Fig. 2.1(b) shows the **interferometric detection**, where a beamsplitter (BS) is used to mix the signal light wave to be measured and another light wave called local oscillator (LO) light, and the output light waves of the BS are detected with photodetectors to measure the amplitude of light. When the optical frequencies of signal and LO are the same, the method is called **homodyne**. When they are different, the method is called **heterodyne**.

Furthermore, an optical amplifier is often used before photodetection as shown in Fig. 2.1(c). This is called **preamplificatio**. Although not shown in the figure, it is also possible to conduct interferometric detection after preamplification.

In every case, the output signal of the photodetector contains noise due to various origins such as instability of light sources or optical systems, circuit noise of photodetector(s), and so on. We can somehow reduce these noises, but at last we will see 'quantum noise' that cannot be reduced by classical manner. Only quantum optics can control the quantum noise.

Here, before introducing various noise sources, we introduce direct detection, interferometric detection, and preamplification.

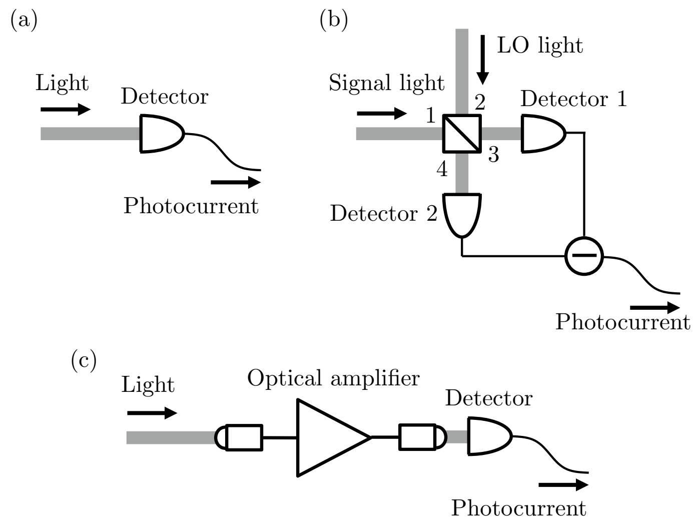

Figure 2.1: Various photodetection methods. (a) Direct detection. (b) Interferometric detection. (c) Optical preamplification with an optical amplifier.

2.1.1 Direct detection

Therefore

$$I = \frac{\eta q P}{\hbar \omega}$$

2.1.2 Homodyne and heterodyne detection

$$\begin{aligned} a(t) &= \alpha e^{-i(\omega+\Delta\omega)t} \\ b(t) &= \beta e^{-i\omega t} \end{aligned} \tag{2.1}$$

$$\begin{aligned} a' &= \frac{1}{\sqrt{2}}(a-b) \\ b' &= \frac{1}{\sqrt{2}}(a+b) \end{aligned} \tag{2.2}$$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.3}$$

$$\begin{aligned} I_1 &= \frac{q}{\tau}|a'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a-b)\right|^2 \\ I_2 &= \frac{q}{\tau}|b'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a+b)\right|^2 \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
I_2 - I_1 &= \frac{q}{\tau}(ab^* + a^*b) \\
&= 2qB(\alpha\beta^* e^{-i\Delta\omega t} + \alpha^*\beta e^{i\Delta\omega t}) \\
&= 4qB|\beta| \left\{ \text{Re } (\alpha e^{-i\phi}) \cos\Delta\omega t + \text{Im } (\alpha e^{-i\phi}) \sin\Delta\omega t \right\}
\end{aligned}
\tag{2.5}
$$

Here $B = 1/2\tau$ is the Nyquist frequency, and $\beta = |\beta| e^{i\phi}$.

2.2 Noise sources

2.2.1 Shot noise

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{2.6}$$

$$
\begin{aligned}
V[p(k)] &= \sum_k (k-\lambda)^2 p(k) \\
&= \sum_k k^2 p(k) - 2\lambda k p(k) + \lambda^2 p(k) = \sum_k k^2 p(k) - \lambda^2 \\
&= \sum_k k\lambda p(k-1) - \lambda^2 = \lambda \sum_k \{(k-1)p(k-1) + p(k-1)\} - \lambda^2 \\
&= \lambda(\lambda+1) - \lambda^2 = \lambda
\end{aligned}
\tag{2.7}
$$

$$I_{\text{shot}} = q\sqrt{\lambda}/\tau = q\sqrt{\frac{I\tau}{q}}/\tau = \sqrt{\frac{qI}{\tau}} \tag{2.8}$$

$$I_{\text{shot}} = \sqrt{2qIB} \tag{2.9}$$

$$\text{SNR} = I^2/I_{\text{shot}}^2 = I/2qB = 2qB|\alpha|^2/2qB = |\alpha|^2 \tag{2.10}$$

where $I = q|\alpha|^2/\tau = 2qB|\alpha|^2$. Since $|\alpha|^2$ corresponds to the number of photons, we can see that the shot-noise limited SNR is equal to the number of photons.

2.2.2 Thermal noise

2.2.3 Optical amplifier noise

2.3 Summary

Chapter 3

Quantum harmonic oscillators

3.1 Schrödinger equation

3.1.1 Wavefunction and energy eigenstates

3.1.2 Fock representation

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3.1.4 Momentum representation

3.2 Measurement of observables

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Chapter 4

Quantum states and their evolution

4.1 Evolution of quantum states

4.1.1 Schorödinger picture

4.1.2 Heisenberg picture

4.2 Unitary transformation of quantum states

4.2.1 Time evolution

4.2.2 Displacement

4.2.3 Mode mixing

4.2.4 Single-mode squeezing

4.2.5 Two-mode squeezing

Chapter 5

Quantization of light

5.1 Mode decomposition of electromagnetic waves

5.1.1 Time-frequency mode

5.1.2 Spatial mode

5.1.3 Polarization

5.2 Operator notation of electromagnetic waves

5.3 Summary

Chapter 6

Representative quantum states

6.1 Number states

6.2 Superposition states

6.3 Coherent states

6.4 Squeezed states

6.5 Two-mode squeezed states

6.5.1 EPR state

6.6 Summary

Chapter 7

Light-matter interaction

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7.1.1 Beamsplitter

7.1.2 Waveplates

7.1.3 Optical loss

7.1.4 Fourier transform

7.2 Parametric amplification

7.2.1 Squeezing

7.2.2 Spontaneous parametric down conversion

7.2.3 Optical amplification

7.2.4 Raman scattering

7.3 Summary

Chapter 8

Measurement of quantum states

8.1 Photodetection

8.2 Homodyne detection

8.3 Heterodyne detection

8.4 Quantum teleportation

Appendix A

Appendix

A.1 Bra-ket notation

A.2 Creation and annihilation operators

A.3 Pure states and mixed states

A.4 Wigner function

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{2^k} &= \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\dots \\ &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots \\ &= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1\end{aligned} \tag{A.1}$$

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

"I always thought something was fundamentally wrong with the universe" [1]

Figure A.1: The Universe

Bibliography

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy*. San Val, 1995.