Lecture Note
Quantum Mechanics of Light and Matters

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Contents

Chapter 1

Introduction

Quantum optics deals with quantum nature of light, where light is regarded as an ensemble of particles called photons. The quantum nature of light appears as 'noise' in various applications in optics and photonics such as optical measurement, optical manipulation, and optical communications, leading to the physical limit called quantum limit on the performance or precision achieved by these methods. To push the performance of various methods to the physical limit, it is crucial to understand the quantum limit. Furthermore, quantum nature of light is extensively utilized to develop various quantum technologies such as quantum cryptography, quantum teleportation, and quantum computing.

Optical measurement always involves the detection of light. It is categorized into direct detection and homodyne/heterodyne detection. Direct detection gives the intensity of light, and homodyne/heterodyne detection which gives the amplitude of light. Furthermore, optical amplification is often utilized before photodetection to mitigate the effect of detector noise. In every case, the signal-to-noise ratio is ultimately limited by quantum noise, and becomes the same order as the number of photons. This limit cannot be surpassed by classical (i.e., non-quantum) methods, while various methods to surpass the limit is developed by using quantum optics.

This lecture note aims at dealing with quantum noise of light. Chapter 2 summarizes the noise in optical measurements. Chapter 3 introduces quantum harmonic oscillators, which is an analogue of light in quantum optics. Chapter 4 describes the evolution of quantum states. Chapter 5 explains the quantization of light. Chapter 6 introduces representative quantum states. Chapter 7 describes two main interactions in quantum optics: mode mixing and parametric amplifications. Chapter 8 explains quantum optical treatment of optical measurements. Appendix contains basic quantum mechanics, Wigner function, and so on. Variables and operators are summarized in Table 1.1.

This lecture note was first prepared in Japanese in 2017, in which I referred to Prof. Kazuro Kikuchi's lecture note. Then I rearranged the contents in English in 2020. In particular, I tried to provide an intuitive picture of quantum optics by extensively using wavefunctions. I often hear that quantum optics

Table 1.1: List of variables and operators

Variable or operator	Explanation
$\hat{x}$	Normalized position (real part of complex amplitude)
$\hat{p}$	Normalized momentum (imaginary part of complex amplitude)
$\hat{a}$	Complex amplitude
$\hat{a}^\dagger\hat{a} = \hat{n}$	Number of photons
q	Elementary charge
$\hbar$	Planck constant divided by 2π
P	Optical power (energy per unit time)
I	Current (charge per unit time)
η	Quantum efficiency
τ	Time duration of a single time-frequency mode
B	Nyquist frequency
T	Temperature

is abstract; we can somehow calculate various properties of light using bra-ket and operators, while the physics behind them are quite unclear. Instead, typical quantum-optical calculation using operators can be understood as the rotation and distortion of wavefunctions. I hope that this lecture memo provide such intuitive pictures along with the calculation procedures. I appreciate many comments and feedback from students in my research group and in the lecture. Any feedback is appreciated at `https://github.com/ysozeki/quantum-optics`.

Chapter 2

Noise in optical measurement

This chapter introduces various detection methods of light and explains noise appearing in each method. Some explanations are phenomenological but they will be explained by quantum optics in later chapters.

2.1 Optical measurement

Figure 2.1(a) shows the **direct detection**. Photodetectors can convert photons to electrons to measure optical power, which is proportional to the number of photons per unit time.

Fig. 2.1(b) shows the **interferometric detection**, where a beamsplitter (BS) is used to mix the signal light wave to be measured and another light wave called local oscillator (LO) light, and the output light waves of the BS are detected with photodetectors to measure the amplitude of light. When the optical frequencies of signal and LO are the same, the method is called **homodyne**. When they are different, the method is called **heterodyne**.

Furthermore, an optical amplifier is often used before photodetection as shown in Fig. 2.1(c). This is called **preamplification**. Although not shown in the figure, it is also possible to conduct interferometric detection after preamplification.

In any case, the output signal of the photodetector contains noise due to various origins such as instability of light sources or optical systems, circuit noise of photodetector(s), and so on. We can somehow reduce these noises, but at last we will see 'quantum noise' that cannot be reduced by classical manner. Only quantum optics can control the quantum noise.

Here, before introducing various noise sources, we introduce direct detection, interferometric detection, and preamplification.

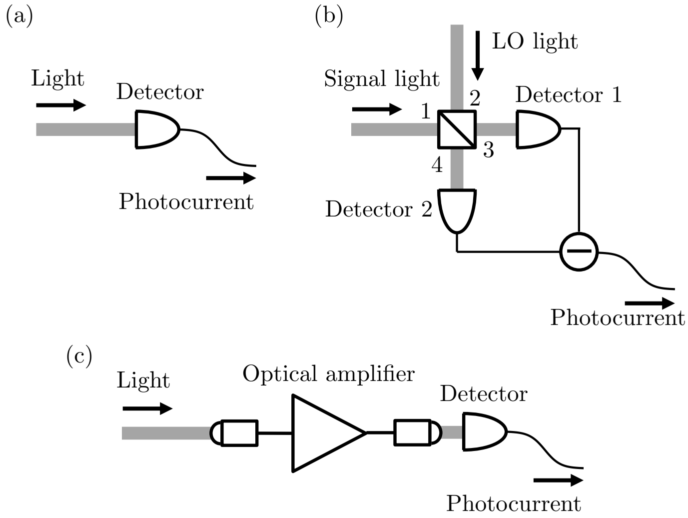

Figure 2.1: Various photodetection methods. (a) Direct detection. (b) Interferometric detection. (c) Optical preamplification with an optical amplifier.

2.1.1 Direct detection

In direct detection, a light wave is directly injected to a photodetector to measure the photocurrent I, which is proportional to the optical power P as follows:

$$I = \frac{\eta q P}{\hbar\omega},$$

where $\hbar\omega$ is the photon energy, and $q = 1.602\times10^{-19}$ C is the elementary charge. $P/\hbar\omega$ is the number of photons incident on the photodetector per unit time. η is the quantum efficiency, which is the ratio of the number of photoelectrons and the number of photons. [1]

2.1.2 Homodyne and heterodyne detection

Figure 2.1(b) shows a schematic of **balanced detector**, which is often used for homodyne and heterodyne. The light to be measured (signal) is combined with a local oscillator (LO) light by a beamsplitter (BS). Then, two light waves output from the BS are injected to photodetectors, and the difference of their photocurrents is taken. Denoting the optical frequencies of LO and signal light as ω and $\omega + \Delta\omega$, this method is called homodyne when $\Delta\omega = 0$ and heterodyne when $\Delta\omega \neq 0$.[2]

[1] For electrical engineers, it is worth remembering that the photon energy at the optical communication wavelength 1.55 µm is approximately 0.8 eV. Since $\hbar\omega/e$ is the photon energy in the unit of eV and typical photodiodes have a quantum efficiency of 90%, typical conversion efficiency is $I/P \sim 1.1$A/W (see some specsheets of InGaAs photodiodes).

[2] These terms originate from frequency mixing in electrical circuits.

The output signal of the balanced detector can be formulated as follows. We denote the complex amplitudes of signal and LO as α and β such that $|\alpha|^2$ and $|\beta|^2$ corresponds to the number of photons in the time duration of τ.[3] We assume that $\Delta\omega \ll \omega$ such that the photon energy difference between signal and LO is negligible.[4] Then the analytic signals of signal light and LO light are given by [5]

$$\begin{aligned} a(t) &= \alpha e^{-i(\omega+\Delta\omega)t}, \\ b(t) &= \beta e^{-i\omega t}. \end{aligned} \tag{2.1}$$

We assume that signal light and LO light are injected to the port 1 and the port 2 of BS, and that the coupling ratio of BS is 50%. The output light waves at the port 3 and the port 4 are given by

$$\begin{aligned} a' &= \frac{1}{\sqrt{2}}(a-b), \\ b' &= \frac{1}{\sqrt{2}}(a+b), \end{aligned} \tag{2.2}$$

respectively,[6] or equivalently,

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.3}$$

The photocurrents I_1, I_2 of the two photodiodes are respectively given by

$$\begin{aligned} I_1 &= \frac{q}{\tau}|a'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a-b)\right|^2, \\ I_2 &= \frac{q}{\tau}|b'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a+b)\right|^2. \end{aligned} \tag{2.4}$$

The output of the balanced detector is

$$\begin{aligned} I_2 - I_1 &= \frac{q}{\tau}(ab^* + a^*b) = \frac{2q}{\tau}\mathrm{Re}[ab^*] = 4qB\mathrm{Re}[\alpha\beta^* e^{-i\Delta\omega t}] \\ &= 4qB|\beta|\left\{\mathrm{Re}\left[\alpha e^{-i\phi}\right]\cos\Delta\omega t + \mathrm{Im}\left[\alpha e^{-i\phi}\right]\sin\Delta\omega t\right\}, \end{aligned} \tag{2.5}$$

where $B = 1/2\tau$ is the Nyquist frequency, and $\beta = |\beta|e^{i\phi}$. Equation 2.5 shows the following points:

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[3] That is, the optical power of signal light is $|\alpha|^2\hbar\omega/\tau$.

[4] Without this assumption, the linear combination of the electric field between signal and LO requires quantum mechanical treatment.

[5] When a complex sinusoidal wave $S(t) = \mathrm{Re}\ S_0 e^{-i\omega t}$ is given, S_0 is called complex amplitude, while $S_0 e^{-i\omega t}$ is called analytic signal, which is composed only of positive frequency components. By taking the real part of the analytic signal, we can obtain a real signal.

[6] In the right hand side of the first equation in Eq. (2.2), the sign of b is minus. This corresponds to the assumption of fixed end reflection from port 2 to port 4. The definition can be different: What's important is that Eq. (2.2) is a unitary transformation, which corresponding to the assumption that BS has no optical loss.

- When $\Delta\omega = 0$ (i.e., homodyne), $I_2 - I_1 = 4qB|\beta|\mathrm{Re}\ (\alpha e^{-i\phi})$. Therefore, homodyne gives the projection of α onto the axis at a phase of β.

- When $\Delta\omega \neq 0$ (i.e., heterodyne), the output signal is a sinusoidal wave at $\Delta\omega$, whose complex amplitude is proportional to α.

2.2 Noise sources

In this section, we summarize various noise sources such as shot noise , thermal noise , and amplifier noise . You will see that by avoiding the effect of thermal noise, the signal-to-noise ratio becomes on the order of the number of photons.

2.2.1 Shot noise

Shot noise refers to the noise due to the fluctuation in the number of photons. We assume that photons arrive in a stochastic and independent manner. The probability distribution $\mathrm{Pr}(X = k)$ of the number of photons X obeys the Poisson distribution given by

$$\mathrm{Pr}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{2.6}$$

where λ is the average value. We can easily show that

$$\sum_{k=0}^{\infty} \mathrm{Pr}(X = k) = 1, \tag{2.7}$$

and the expectation value and the variance are given by

$$E[X] = \sum_{k=0}^{\infty} kp(k) = \lambda, \tag{2.8}$$

$$V[X] = \sum_{k=0}^{\infty} (k-\lambda)^2 p(k) = \lambda, \tag{2.9}$$

and therefore the standard deviation of the number of photons is $\sqrt{\lambda}$. Consequently, the root-mean-square (RMS) noise due to the fluctuation of the number of photons is given by

$$I_{\mathrm{shot}} = q\sqrt{\lambda}/\tau = q\sqrt{\frac{I\tau}{q}}/\tau = \sqrt{\frac{qI}{\tau}}, \tag{2.10}$$

or equivalently,

$$I_{\mathrm{shot}} = \sqrt{2qIB}. \tag{2.11}$$

Shot-noise-limited SNR in direct detection

By defining the signal-to-noise ratio (SNR) as the energy ratio between signal and noise, we obtain the shot-noise-limited SNR as

$$\mathrm{SNR} = I^2/I_{\mathrm{shot}}^2 = I/2qB = 2qB|\alpha|^2/2qB = |\alpha|^2, \tag{2.12}$$

where $I = q|\alpha|^2/\tau = 2qB|\alpha|^2$. Since $|\alpha|^2$ corresponds to the number of photons, we can see that the shot-noise limited SNR is equal to the number of photons.

Shot-noise-limited SNR in homodyne detection

To calculate SNR of homodyne detection, let's assume that LO light is much stronger than signal light, and $\phi = 0$ for simplicity. Since the LO light is divided approximately by half by the BS in the balanced detector, the photocurrents are given by

$$I_1 \sim I_2 \sim \frac{1}{2}\frac{q}{\tau}|\beta|^2 = qB|\beta|^2, \tag{2.13}$$

and therefore the shot noise is given by

$$\sqrt{2qI_1B} \sim \sqrt{2qI_2B} \sim \sqrt{2q^2B^2|\beta|^2} = \sqrt{2}qB|\beta|. \tag{2.14}$$

The output of balanced detector is affected by two independent shot noise from I_1 and I_2. Therefore the shot noise of the balanced detector is given by

$$I_{\mathrm{shot}} = 2qB|\beta|. \tag{2.15}$$

By assuming $\Delta\omega = 0$ in Eq. (2.5), the homodyne output is described by

$$I_{\mathrm{homodyne}} = 4qB|\beta|\mathrm{Re}\ \alpha. \tag{2.16}$$

Therefore SNR in homodyne detection is given by

$$\mathrm{SNR}_{\mathrm{homodyne}} = I_{\mathrm{homodyne}}^2/I_{\mathrm{shot}}^2 = 4(\mathrm{Re}[\alpha])^2. \tag{2.17}$$

You can see that the SNR in homodyne is determined by the energy of signal light, and is independent of LO power.

Shot-noise-limited SNR in heterodyne detection

Let's consider SNR in heterodyne detection. We split Eq. (2.5) into two terms:

$$\begin{aligned} I_{\cos} &= 4qB|\beta|\mathrm{Re}[\alpha]\cos\Delta\omega t, \\ I_{\sin} &= 4qB|\beta|\mathrm{Im}[\alpha]\sin\Delta\omega t. \end{aligned} \tag{2.18}$$

Their mean square can be calculated as

$$\begin{aligned} \overline{I_{\cos}^2} &= (4qB|\beta|\mathrm{Re}[\alpha])^2\frac{1}{2}\overline{(1+\cos 2\Delta\omega t)} = 8(qB|\beta|\mathrm{Re}[\alpha])^2, \\ \overline{I_{\sin}^2} &= (4qB|\beta|\mathrm{Im}[\alpha])^2\frac{1}{2}\overline{(1-\cos 2\Delta\omega t)} = 8(qB|\beta|\mathrm{Im}[\alpha])^2. \end{aligned} \tag{2.19}$$

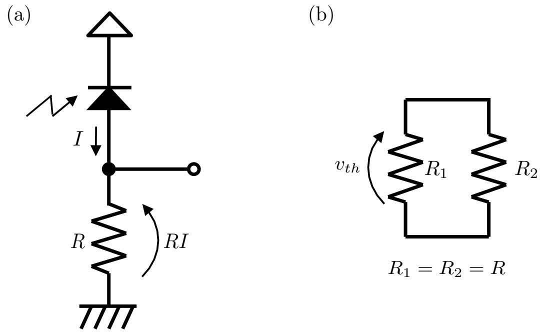

Figure 2.2: (a) Typical photodetection circuit, where a photodiode is inversely biased. The photocurrent I flows to a load resistor R and measure the voltage RI. (b) Two-resistor circuit, which is used to consider the Johnson noise.

Therefore, their SNRs are given by [7]

$$\begin{aligned} \overline{I_{\cos}^2}/I_{\text{shot}}^2 &= 2(\text{Re}[\alpha])^2, \\ \overline{I_{\sin}^2}/I_{\text{shot}}^2 &= 2(\text{Im}[\alpha])^2. \end{aligned} \tag{2.20}$$

From Eq. (2.17) and Eq. (2.20), we can see that the heterodyne gives both the real and imaginary parts of complex amplitude with 3-dB lower SNR than homodyne.

2.2.2 Thermal noise

Thermal noise appears as voltage noise or current noise of a resistor R, which is used for converting a photocurrent I to a voltage RI as shown Fig. 2.2(a). The RMS voltage of the thermal noise or Johnson noise is given by

$$v_{\text{th}} = \sqrt{4k_{\text{B}}TRB}, \tag{2.21}$$

where k_B is the Boltzmann constant, and T is the temperature of the resistor. It is important to suppress the thermal noise by optimizing the circuit design for achieving the shot-noise-limited SNR.

To derive Eq. (2.21), we consider a circuit shown in Fig. 2.2, where two resistors (R_1, R_2) are connected with each other, and they are impedance-matched.[2] Since v_{th} is the electromotive force in one of the resistors and the series resistance is $2R$, it leads to a current of $v_{\text{th}}/2R$. Therefore, each resistor generates power of $v_{\text{th}}^2/2R$ and is in the thermal equilibrium.[8] Here we consider

[7] Note that the discussion here omits various points on the quantification of shot noise; We should consider the shot noise in the frequency range from $\Delta\omega/2\pi - B$ to $\Delta\omega/2\pi + B$, whose bandwidth is $2B$. Nevertheless, here we have two shot noise components: sin and cos. Therefore, if we extract cos component of shot noise, its power is the same as Eq. (2.15).

[8] If the resistors have different temperature, their temperatures get closer to each other by transferring energy with thermal current.

the voltage noise in the frequency range from $-B = -1/2\tau$ to $B = 1/2\tau$. The sampling theorem tells us that the noise waveform can be captured by sampling it with a period of τ. The noise at each sampling point is independent of each other, and its average energy is Boltzmann energy $k_{\mathrm{B}}T$.[9] Therefore,

$$v_{\mathrm{th}}^2\tau/2R = v_{\mathrm{th}}^2/4RB = k_{\mathrm{B}}T, \tag{2.22}$$

which leads to Eq. (2.21).

Comparison between shot noise and thermal noise

To compare the amounts of shot noise and thermal noise, let's consider the case where they are the same, i.e., $RI_{\mathrm{shot}} = v_{\mathrm{th}}$ when the photocurrent is I'. This leads to

$$I' = \frac{2k_{\mathrm{B}}T}{qR}. \tag{2.23}$$

Since $k_{\mathrm{B}}T/q$ is the Boltzmann energy in the unit of eV, it is 26 meV at the room temperature. Assuming the load resistance of $R = 50\ \Omega$, $I' = 1$ mA. The photocurrent is larger than I', shot noise dominates, and *vice versa*. Since shot noise and thermal noise are proportional to R and $\sqrt{R}$, respectively, we can suppress the effect of thermal noise by increasing R, while the response time RC of the photodetector due to its capacitance C limits the bandwidth of the circuit. You can also see that homodyne/heterodyne detection is useful for suppressing the thermal noise because strong LO light is introduced to the photodetectors.

2.2.3 Optical amplifier noise

Optical amplifiers can literally amplify light, while amplified light is accompanied with optical noise called amplified spontaneous emission (ASE). There are various types of optical amplifiers such as fiber amplifiers using an optical fiber doped with various rare-earth ions ($\mathrm{Er}^{3+}, \mathrm{Yb}^{3+}, \mathrm{Tm}^{3+}$, etc.), semiconductor optical amplifiers using direct bandgap semiconductors, and optical parametric amplifiers using a nonlinear optical crystal or an optical fiber. Surprisingly, the SNR limit due to ASE applies to any type of optical amplifiers.

Without proof,[10] we introduce that the power of ASE in a single polarization state is given by

$$P_0 = n_{\mathrm{sp}}\hbar\omega(G-1)\Delta f, \tag{2.24}$$

where G is the gain, Δf is the optical bandwidth, and $n_{\mathrm{sp}} \geq 1$ is the spontaneous emission factor, which increases when the population inversion of laser material is imperfect and optical amplifier has optical loss. Since typical laser material

[9] The reason why each degree of freedom has an energy of $k_{\mathrm{B}}T$ instead of $k_{\mathrm{B}}T/2$ is that each degree of freedom corresponds to a thermally excited harmonic oscillator which has two degree of freedom of position and momentum. In the case of ideal gas, each atom has three momentum axis but not fixed at a point, leading to a kinetic energy of $3k_{\mathrm{B}}T/2$.

[10] The derivation will be given in the later chapter.

produces ASE in both vertical and horizontal polarizations, the total ASE power is given by

$$P_{\mathrm{ASE}} = 2n_{\mathrm{sp}}\hbar\omega(G-1)\Delta f. \tag{2.25}$$

To investigate how SNR is affected by optical amplification, let's calculate the SNR after amplification. We denote the optical power of a light wave before amplification as P_{in} and assume that its phase is zero. Also we consider the frequency range from $\omega/2\pi - B$ to $\omega/2\pi + B$, and therefore $\Delta f = 2B$. Since ASE has cos and sin components, ASE power only in the cos component and in a single polarization is given by

$$P_{\mathrm{ASE1}} = \frac{1}{2}n_{sp}\hbar\omega(G-1)2B = n_{sp}\hbar\omega(G-1)B. \tag{2.26}$$

When the ASE field interferes the amplified signal field, its power changes according to

$$\left(\sqrt{GP_{\mathrm{in}}} \pm \sqrt{P_{\mathrm{ASE1}}}\right)^2 = GP_{\mathrm{in}} \pm 2\sqrt{GP_{\mathrm{in}}P_{\mathrm{ASE1}}} + P_{\mathrm{ASE1}}. \tag{2.27}$$

The second term in the right hand side is the power change due to the interference between the amplified signal and ASE. The noise from this effect is called **signal-ASE beat noise**. The third term can also contribute to the noise and is significant when the ASE power is comparable with the signal. Assuming that the signal-ASE beat noise is dominant, we obtain the SNR as

$$\mathrm{SNR} = \left(\frac{GP_{\mathrm{in}}}{2\sqrt{GP_{\mathrm{in}}P_{\mathrm{ASE1}}}}\right)^2 = \frac{GP_{\mathrm{in}}}{4P_{\mathrm{ASE1}}} = \frac{GP_{\mathrm{in}}}{4n_{\mathrm{sp}}\hbar\omega(G-1)B} \to \frac{1}{2n_{\mathrm{sp}}}\frac{P_{\mathrm{in}}}{2\hbar\omega B}. \tag{2.28}$$

Here, $P_{in}/2\hbar\omega B$ is the number of photons of input light in a time duration of $\tau = 1/2B$, which equals to the shot-noise-limited SNR before amplification. Therefore, the optical amplification reduces SNR by $1/2n_{\mathrm{sp}}$ times, and hence the **noise figure** (NF) is $2n_{\mathrm{sp}}$. In particular, NF with $n_{\mathrm{sp}} = 1$ (i.e., 3 dB) is called quantum-limited noise figure. Typical NF in optical fiber amplifiers is 5-6 dB.

2.2.4 Other noise sources

So far, we have discussed shot noise, thermal noise, and amplifier noise. They are basic noise sources, while there are other noise sources as well. Here we introduce excess noise and $1/f$ noise.

Excess noise

When a light wave has a larger fluctuation that the shot noise, the other factors than the shot noise are called **excess noise**. Excess noise includes the instability of light sources and that of optical systems. ASE is also categorized as excess noise.

$1/f$ noise

It is known that active components and light sources exhibit a large fluctuation at low frequency region. Its power spectrum is often proportional to $1/f$. Such noise is called $1/f$ noise. This is in contrast to white noise such as shot noise and thermal noise whose power spectrum is independent on frequency. $1/f$ noise becomes significant when the signal is averaged over a long time.

2.3 Summary

We have reviewed the noise in optical measurement including direct detection, homodyne detection, and heterodyne detection. Direct detection is simple and useful but the detector noise can dominate when the optical power is low. Homodyne and heterodyne are useful for suppressing the effect of the thermal noise, and sensitive to the complex amplitude of signal light. Homodyne gives either real or imaginary part of the complex amplitude, while heterodyne gives both with 3-dB lower SNR. Optical amplification is useful for avoiding the detector noise, but the SNR is reduced by 3 dB. Some properties such as Poisson distribution and the amount of ASE were given without proof, but they will be supported by quantum optics. Furthermore, quantum optics allows for surpassing the shot-noise-limited SNR. These points will be discussed in the following chapters.

Chapter 3

Quantum-mechanical harmonic oscillator

The previous chapter introduced the phenomenological description of shot noise and ASE to calculate SNR in various optical measurements. Quantum optics tells you the physics behind these noise sources and provides ways to manipulate them.

In quantum optics, the physics of harmonic oscillators play a crucial role because electromagnetic field is decomposed into the collection of time-frequency modes, spatial modes, and polarizations, and each mode is assumed as a quantum-mechanical harmonic oscillator.

This chapter will introduce the quantum-mechanical harmonic oscillators. If you are familiar with the basics of quantum mechanics, you can skip this chapter.

3.1 Schrödinger equation for a harmonic oscillator

3.1.1 Classical harmonic oscillators

Let's discuss the motion of a one-dimensional mass-spring system without friction with a mass of m and a spring constant of k. The equation of motion of the position $X(t)$ of the mass as a function of time is given by

$$m\frac{d^2X}{dt^2} + kX = 0. \tag{3.1}$$

The solution is given by

$$X(t) = \frac{1}{2}A\exp(-i\omega t) + c.c. = \mathrm{Re}[A\exp(-i\omega t)] \tag{3.2}$$

where A is a complex number, $\omega = \sqrt{k/m}$, and *c.c.* stands for the complex conjugate. The momentum $P = m(dX/dt)$ is given by

$$P(t) = \frac{-im\omega}{2} A \exp(-i\omega t) + c.c. = m\omega \mathrm{Im}[A \exp(-i\omega t)]. \tag{3.3}$$

Eq. (3.2) and Eq. (3.3) show that $X(t)$ and $P(t)$ oscillate while keeping a phase difference of 90 degree.

The sum of potential energy and kinetic energy is given by

$$E = \frac{1}{2}kX^2 + \frac{m}{2}\left(\frac{dX}{dt}\right)^2 = \frac{1}{2}kX^2 + \frac{P^2}{2m}, \tag{3.4}$$

where we introduced the momentum $P = m(dX/dt)$. Substituting Eq. (3.2) to Eq. (3.4), we obtain

$$E = \frac{1}{2}k(\mathrm{Re}[A \exp(-i\omega t)])^2 + \frac{m\omega^2}{2}(\mathrm{Im}[A \exp(-i\omega t)])^2 = \frac{1}{2}k|A|^2. \tag{3.5}$$

In this way, we can see that the total energy is kept constant. The above results are general in harmonic oscillators: each oscillator has two degrees of freedom, and they oscillate with 90-degree phase difference. To further generalize the result, we introduce the normalized position x and the normalized momentum p such that the potential energy is given by $\hbar\omega x^2$ and the kinetic energy is given by $\hbar\omega p^2$, i.e., Eq. (3.4) becomes

$$E = \hbar\omega(x^2 + p^2). \tag{3.6}$$

This is made possible by defining x and p as

$$x = \sqrt{\frac{k}{2\hbar\omega}} X, \tag{3.7}$$

$$p = \frac{1}{\sqrt{2m\hbar\omega}} P, \tag{3.8}$$

respectively. Since $\hbar\omega$ is the energy unit of quantum harmonic oscillators, such normalization can simplify the notation and therefore they are often used. We also introduce the normalized complex amplitude

$$a = \sqrt{\frac{k}{2\hbar\omega}} A \tag{3.9}$$

so that the time evolution of x and p are given by

$$x(t) = \mathrm{Re}[a \exp(-i\omega t)], \tag{3.10}$$

$$p(t) = \mathrm{Im}[a \exp(-i\omega t)], \tag{3.11}$$

respectively. Also, we can define $a(t) = a \exp(-i\omega t)$, which satisfies

$$a(t) = x(t) + ip(t). \tag{3.12}$$

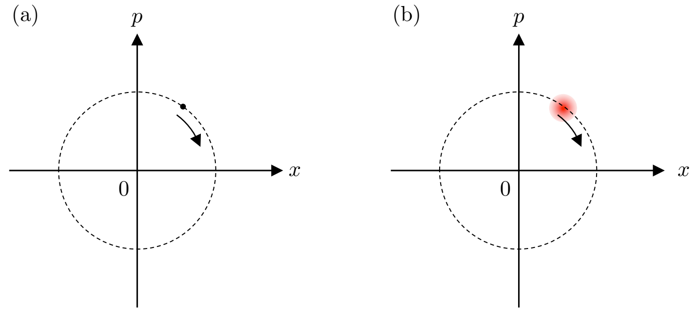

Figure 3.1: Evolution of position $x(t)$ and momentum $p(t)$ in the phase space. (a) Classical harmonic oscillator. (b) Quantum harmonic oscillator.

You can see that $x(t)$ and $p(t)$ in the x-p plane, which is called phase space, form a circular trajectory rotating at a frequency of ω, as shown in Fig. 3.2(a). This is a very general property of classical harmonic oscillator.

In the later chapters, you will see that in quantum harmonic oscillator $x(t)$ and $p(t)$ have certain fluctuation or uncertainty, as conceptually illustrated in Fig. 3.2(b). This results in the shot noise and optical amplifier noise.

3.1.2 Wavefunctions and Schrödinger equation

To introduce quantum-mechanical harmonic oscillators, we start from de Broglie's relation, which assumes a complex wavefunction with a temporal angular frequency Ω and spatial angular frequency K, which are related to the energy and the momentum by

$$E = \hbar\Omega, \tag{3.13}$$

$$P = \hbar K, \tag{3.14}$$

respectively. If we consider an exemplary wavefunction denoted as $\Phi(X,t) = \exp[i(KX - \Omega t)]$, we get

$$i\hbar\frac{\partial\Phi}{\partial t} = \hbar\Omega\Phi = E\Phi, \tag{3.15}$$

$$-i\hbar\frac{\partial\Phi}{\partial X} = \hbar K\Phi = P\Phi. \tag{3.16}$$

Suggested by the above equations, we define the following operators:

$$\hat{E} \equiv i\hbar\frac{\partial}{\partial t}, \tag{3.17}$$

$$\hat{P} \equiv -i\hbar\frac{\partial}{\partial X}, \tag{3.18}$$

which can extract the energy and the momentum, respectively, from the wavefunction.

Substituting (3.17) and (3.18) to Eq. (3.4), we obtain the Schrödinger equation for a one-dimensional harmonic oscillator given by

$$i\hbar\frac{\partial\Psi(X,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(X,t)}{\partial X^2} + \frac{1}{2}kX^2\Psi(X,t), \tag{3.19}$$

where $\Psi(X,t)$ is the **wavefunction** of the harmonic oscillator.

The meaning of wavefunction may be abstract at the moment, but we assume that $\int_{-\infty}^{\infty}|\Psi(X,t)|^2 dX = 1$ and that $|\Psi(X,t)|^2$ corresponds to the probability density of the position of the oscillator being at X. Once we assume $\Psi(X,t)$ at a certain time, we can calculate its time evolution by using Eq. (3.19) because the left-hand side is the time derivative of $\Psi(X,t)$. In the later sections, we will discuss that the wavefunction contains various information such as momentum and energy.

Before doing so, let's describe Eq. (3.19) with the normalized position x and the normalized momentum p given by Eqs. (3.7) and (3.8). From Eqs. (3.7), (3.8), and (3.18), the operator of normalized momentum $\hat{p}$ can be expressed by x as:

$$\hat{p} = \frac{\hat{P}}{\sqrt{2m\hbar\omega}} = \frac{-i\hbar\frac{\partial}{\partial X}}{\sqrt{2m\hbar\omega}} = -i\hbar\frac{\sqrt{\frac{K}{2\hbar\omega}}\frac{\partial}{\partial x}}{\sqrt{2m\hbar\omega}} = -\frac{i}{2}\frac{\partial}{\partial x}. \tag{3.20}$$

Therefore, Eq. (3.19) can be simplified as

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{k}{2\hbar\omega}\frac{\partial\psi}{\partial x} + \frac{1}{2}k\frac{2\hbar\omega}{k}x^2\psi = \hbar\omega\left(-\frac{1}{4}\frac{\partial^2\psi}{\partial x^2} + x^2\psi\right) \\ &= \hbar\omega(\hat{x}^2 + \hat{p}^2)\psi, \end{aligned} \tag{3.21}$$

where we introduced the normalized position operator $\hat{x} = x$. By using an operator $\hat{H} = \hbar\omega(\hat{x}^2 + \hat{p}^2)$ called Hamiltonian, we get

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t). \tag{3.22}$$

If $\psi(x,t)$ is known at a certain t, we can calculate the time evolution of probability distribution $|\psi(x,t)|^2$.[1]

3.1.3 Quantum state and bra-ket notation

Here we introduce **quantum state**, which is a generalized version of wavefunction. We also introduce the **bra-ket notation**, which is the most popular way of describing quantum state.

Before introducing quantum state, we discuss the motivation for using quantum state instead of wavefunction. As we discussed in the previous section, $\psi(x,t)$ contains the information not only on the position but also the momentum and the energy, and its time evolution is calculated by the Schrödinger

[1] Note that we implicitly assume that $\int_{-\infty}^{\infty}|\psi(x,t)|^2 dx = 1$. Therefore, $\Psi(X,t)$ in Eq. (3.19) and $\psi(x,t)$ in Eq. (3.21) should be normalized differently, and hence should have different values at corresponding X and x.

equation. We will see that, wavefunction can be described as a function of momentum, or that of energy. Nevertheless, if we change the expression of the wavefunction, we should change the expression of operators including position, momentum, Hamiltonian etc., which is quite inconvenient. To avoid such inconvenience, we utilize the idea of linear algebra, where a wavefunction is viewed as a vector, which can be expressed as linear combination of various set of orthonormal basis.

For example, a wavefunction[2] $\psi(x)$ can be viewed as a vector by expressing

$$\psi(x) = \int_{-\infty}^{\infty} \psi(x_0)\delta(x - x_0)dx_0, \tag{3.23}$$

where $\psi(x)$ is expanded as the linear combination of a set of basis that are consisted of the delta function[3] $\delta(x - x_0)$ for various x_0.

The basis can be transformed by using unitary transformation: If we have a set of orthonormal bases, we can expand some vector with the bases by taking inner product. In practice, by taking the inner product of $\psi(x)$ and a basis $\delta(x - x_0)$, we get $\psi(x_0)$ because

$$\int_{-\infty}^{\infty} \delta(x - x_0)\psi(x)dx = \psi(x_0), \tag{3.24}$$

and therefore we can recover the wavefunction at $x = x_0$. Another exemplary basis is

$$\phi_p(x) = \frac{1}{\sqrt{\pi}} e^{2ipx}, \tag{3.25}$$

which satisfies

$$\hat{p}\phi_p(x) = -\frac{i}{2}\frac{\partial}{\partial x}\frac{1}{\sqrt{\pi}} e^{2ipx} = p\phi_p(x), \tag{3.26}$$

indicating that $\phi_p(x)$ has a momentum p. Furthermore, since

$$\int_{-\infty}^{\infty} \phi_{p_1}^*(x)\phi_{p_2}(x)dx = \delta(p_1 - p_2), \tag{3.27}$$

$\phi_p(x)$ is a set of orthonormal basis.[4] Therefore, by taking the inner product with $\phi_p(x)$, we can express the wavefunction as the linear combination of $\phi_p(x)$ for various p. In this way, the basis of a wavefunction is interchangeable.

Quantum state is a complex vector expressed without apparently specifying any basis, and has the same information as wavefunction. Quantum state is

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[2] The dependence of $\psi(x, t)$ on t is not considered for a while to simplify the discussion.

[3] The delta function $\delta(x)$ is ∞ at $x = 0$ and 0 elsewhere, and $\int_{-\infty}^{\infty} \delta(x)dx = 1$. $\delta(x)$ can be defined in several ways but one of them is $\delta(x) = \lim_{\varepsilon\to+0} \mathrm{rect}(x/\varepsilon)/\varepsilon$, where $\mathrm{rect}(x) = 1$ for $-1/2 < x < 1/2$ and 0 for others.

[4] $\int \phi_{p_1}^*(x)\phi_{p_2}(x)dx = \frac{1}{\pi}\int e^{-2i(p_1-p_2)x}dx = \lim_{X\to\infty} \frac{1}{\pi}\int e^{-(x/X)^2} e^{-2i(p_1-p_2)x}dx$ $= \lim_{X\to\infty} \frac{1}{\pi}\int e^{-\frac{(x-iX^2(p_1-p_2))}{X^2}} e^{-X^2(p_1-p_2)^2}dx = \lim_{X\to\infty} \frac{X}{\sqrt{\pi}} e^{-X^2(p_1-p_2)^2}$. Because the last term is a Gaussian with a peak value of X and an area of 1, we get $\delta(p_1 - p_2)$.

described by ket $|\psi\rangle$. Inner product of two kets $|\phi\rangle$ and $|\psi\rangle$ is expressed as $\langle\phi|\psi\rangle$. If $|\phi\rangle$ and $|\psi\rangle$ can be expressed as wavefunctions of x (i.e., $\phi(x)$ and $\psi(x)$), respectively,

$$\langle\phi|\psi\rangle \equiv \int_{-\infty}^{\infty} \phi^*(x)\psi(x)dx, \tag{3.28}$$

which is analogous to the product of a row vector and a column vector. $\langle\phi|$ is called 'bra', and can be viewed as the transpose and complex conjugate of ket.

The inner product of the same ket and bra gives the squared norm:

$$\| \, |\phi\rangle \, \|^2 \equiv \langle\phi|\phi\rangle = \int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx \geq 0, \tag{3.29}$$

which quantum mechanics requests to be normalized to 1.

Not only wavefunctions but also bases can be expressed by ket. For example, $|x_0\rangle$ gives the delta function centered at $x = x_0$, i.e.,

$$\langle x|x_0\rangle = \delta(x - x_0). \tag{3.30}$$

Although we used x_0 in Eq. (3.23) and Eq. (3.24) to reserve x for integral calculation, it is no longer necessary in the bra-ket notation. Therefore we can denote $|x\rangle$ as the delta function centered at x.

$|p\rangle$ gives the engenfunction of momentum, i.e.,

$$\langle x|p\rangle = \phi_p(x) = \frac{1}{\sqrt{\pi}} e^{2ipx}. \tag{3.31}$$

Based on the above discussions, we can see that the wavefunction is an expression of a quantum state $|\psi\rangle$ with x basis, which can be obtained by taking the inner product with $|x\rangle$. Therefore,

$$\langle x|\psi\rangle = \psi(x). \tag{3.32}$$

In a similar manner, we can express $|\psi\rangle$ with p basis by taking the inner product as

$$\langle p|\psi\rangle \equiv \tilde{\psi}(p), \tag{3.33}$$

where $\langle p|\phi\rangle$ can be calculated in the position basis as

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} \langle p|x\rangle \langle x|\psi\rangle \, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-2ipx} dx, \tag{3.34}$$

which is the wavefunction expressed as a function of p. You can see that $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$.

3.1.4 Operators

Although we have already seen some operators, here we introduce some general idea of operators. By applying an operator $\hat{A}$, we can change the ket to $\hat{A}\,|\phi\rangle$.

In the position basis, the action of operator is expressed by a two-dimensional complex function $A(x, x')$ as

$$\hat{A}\phi(x) = \int_{-\infty}^{\infty} A(x, x')\phi(x')dx, \tag{3.35}$$

which is analogous to multiplication of a matrix and a vector.[5] For example, the position operator is given by

$$\hat{x}(x, x') = x\delta(x - x'). \tag{3.36}$$

The momentum operator is given by

$$\hat{p}(x, x') = -\frac{i}{2}\frac{\partial}{\partial x}\delta(x - x'). \tag{3.37}$$

so that

$$\begin{aligned} \langle x| \hat{p} |p\rangle &= \int \hat{p}(x, x')\phi_p(x')dx' \\ &= -\frac{i}{2}\int \frac{\partial}{\partial x}\delta(x - x')\frac{e^{2ipx'}}{\sqrt{\pi}}dx' \\ &= -\frac{i}{2}2ip\frac{e^{2ipx'}}{\sqrt{\pi}} = p\phi_p(x). \end{aligned} \tag{3.38}$$

more description is needed here.

Hermitian conjugate

We introduce the Hermitian conjugate $\hat{A}^\dagger$ of an operator $\hat{A}$, which is given by

$$A^\dagger(x, x') = A^*(x', x), \tag{3.39}$$

Hermitian conjugate appears in various situations. A representative case is the inner product between $\hat{A}|\phi\rangle$ and $|\psi\rangle$, which is given by

$$\begin{aligned} \iint \left[A(x, x')\phi(x')\right]^* \psi(x)dxdx' &= \iint \phi^*(x')A^*(x, x')\psi(x)dxdx' \\ &= \iint \phi^*(x')A^\dagger(x', x)\psi(x)dxdx' \\ &= \langle\phi| \hat{A}^\dagger |\psi\rangle . \end{aligned} \tag{3.40}$$

Therefore, the inner product of $\hat{A}|\phi\rangle$ and $|\psi\rangle$ is equal to the inner product of $|\phi\rangle$ and $\hat{A}^\dagger|\psi\rangle$.

Using Hermitian conjugate, we can define two important classes of operators: **unitary operator** $\hat{U}$ that satisfy $\hat{U}^\dagger\hat{U} = \hat{1}$, where $\hat{1}$ stands for the identity operator, and **Hermitian operator** $\hat{A}$ that satisfy $\hat{A}^\dagger = \hat{A}$. Their properties are discussed below.

[5] Defining $n \times n$ matrix $\mathbf{A} = (A_{ij})$ and a n-dimensional vector $\phi = (c_1, c_2, \cdots, n_n)^T$, the i-th component of $\mathbf{A}\phi$ is given by $\sum_{k=1}^{n} A_{ik}c_k$.

Unitary operators

A unitary operator $\hat{U}$ is composed of a set of orthonormal basis. As mentioned before, a unitary operator satisfies $\hat{U}^\dagger\hat{U} = \hat{1}$. This leads to

$$\hat{U}^\dagger\hat{U}\hat{U}^\dagger = \hat{U}^\dagger, \tag{3.41}$$

from which we obtain $\hat{U}\hat{U}^\dagger = \hat{1}$. In x basis, the $\hat{U}^\dagger\hat{U} = \hat{1}$ is expressed as

$$\int \hat{U}^\dagger(x, x'')U(x'', x')dx'' = \int \hat{U}^*(x'', x)U(x'', x')dx = \delta(x - x'), \tag{3.42}$$

where you can see that the norm of $U(x, x')$ at a certain x' is 1, while the inner product of $U(x, x')$ and $U(x, x'')$ becomes zero when $x' \neq x''$.

An important property of unitary operators is that it does not change the inner product or norm. This point is understood by calculating the inner product of $\hat{U}\left|\varphi_1\right\rangle$ and $\hat{U}\left|\varphi_2\right\rangle$ as

$$\left\langle\varphi_1\right|\hat{U}^\dagger\hat{U}\left|\varphi_2\right\rangle = \left\langle\varphi_1|\varphi_2\right\rangle. \tag{3.43}$$

Helmitian operators

Hermitian operators (or self-adjoint operators) are important in quantum mechanics because physical quantities called observables are expressed by Hermitian operators. It is easy to show that $\hat{x}$ and $\hat{p}$ are Hermitian operators. More explanation is needed.

Hermitian operators have the following important properties:

1. Eigenvalues are real.

2. Eigenvectors with different eigenvalues are orthogonal to each other.

These points are proved as follows. As for the first point, suppose that a Hermitian operator $\hat{A}$ has an eigenvalue λ and an eigenvector $|\varphi\rangle$, i.e.,

$$\hat{A}\left|\varphi\right\rangle = \lambda\left|\varphi\right\rangle. \tag{3.44}$$

By multiplying $\langle\phi|$, we get

$$\left\langle\varphi\right|\hat{A}\left|\varphi\right\rangle = \lambda\left\langle\varphi|\varphi\right\rangle. \tag{3.45}$$

By taking Hermitian conjugate of Eq. (3.44), we get

$$\left\langle\varphi\right|\hat{A}^\dagger = \lambda^*\left\langle\varphi\right|. \tag{3.46}$$

By multiplying $|\phi\rangle$ and using $\hat{A}^\dagger = \hat{A}$, we get

$$\left\langle\varphi\right|\hat{A}\left|\varphi\right\rangle = \lambda^*\left\langle\varphi|\varphi\right\rangle. \tag{3.47}$$

From Eq. (3.44) and Eq. (3.47), we can see that $\lambda^* = \lambda$, i.e., λ is real.

As for the second point, we assume

$$\hat{A}\left|\varphi_1\right\rangle = \lambda_1 \left|\varphi_1\right\rangle, \tag{3.48}$$

$$\hat{A}\left|\varphi_2\right\rangle = \lambda_2 \left|\varphi_2\right\rangle. \tag{3.49}$$

The Hermitian conjugate of Eq. (3.49) gives

$$\left\langle\varphi_2\right|\hat{A}^\dagger = \lambda_2^* \left\langle\varphi_2\right|. \tag{3.50}$$

Since $\hat{A}^\dagger = \hat{A}$ and λ_2 is real,

$$\left\langle\varphi_2\right|\hat{A} = \lambda_2 \left\langle\varphi_2\right|. \tag{3.51}$$

From Eq. (3.48) and (3.51), we obtain

$$\left\langle\varphi_2\right|\hat{A}\left|\varphi_1\right\rangle = \lambda_1 \left\langle\varphi_1|\varphi_2\right\rangle, \tag{3.52}$$

$$\left\langle\varphi_2\right|\hat{A}\left|\varphi_1\right\rangle = \lambda_2 \left\langle\varphi_1|\varphi_2\right\rangle, \tag{3.53}$$

Therefore,

$$(\lambda_1 - \lambda_2)\left\langle\varphi_1|\varphi_2\right\rangle = 0. \tag{3.54}$$

If $\lambda_1 \neq \lambda_2$, $\langle\varphi_1|\varphi_2\rangle = 0$, i.e., $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are orthogonal. When $\lambda_1 = \lambda_2$, $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are not always orthogonal, but they can be orthogonalized by Gramm-Schmidt orthogonalization. In this way, the eigenvectors of a Hermitian operator can form an orthonormal set of basis, which constitutes a unitary operator $\hat{U}$. Therefore, we can diagonalize a Hermitian operator $\hat{A}$ as

$$\hat{A} = \hat{U}\hat{D}\hat{U}^\dagger, \tag{3.55}$$

where $\hat{D}$ is a diagonal operator which gives

$$D(x, x') = d(x)\delta(x - x'). \tag{3.56}$$

This will be used for calculating the expectation value of observables.

Using this, we can calculate $e^{i\hat{A}}$ as

$$\begin{aligned} e^{i\hat{A}} &= \sum_{n=0}^{\infty} \frac{\left(i\hat{A}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(i\hat{U}\hat{D}\hat{U}^\dagger\right)^n}{n!} = \hat{U}\sum_{n=0}^{\infty}\frac{(i\hat{D})^n}{n!}\hat{U}^\dagger \\ &= \hat{U}e^{i\hat{D}}U^\dagger. \end{aligned} \tag{3.57}$$

Therefore, we can show that $e^{i\hat{A}}$ is unitary as follows:

$$\left(e^{i\hat{A}}\right)^\dagger e^{i\hat{A}} = \hat{U}e^{-i\hat{D}}\hat{U}^\dagger\hat{U}e^{i\hat{D}}\hat{U}^\dagger = \hat{U}\hat{U}^\dagger = \hat{1}. \tag{3.58}$$

Creation and annihilation operators

We introduce an operator defined as

$$\hat{a} = \hat{x} + i\hat{p}, \tag{3.59}$$

which is called annihilation operator. Its Hermitian conjugate is

$$\hat{a}^\dagger = \hat{x} - i\hat{p}, \tag{3.60}$$

which is called creation operator. Creation and annihilation operators play various important roles in quantum optics. They are not Hermitian nor unitary. As we will see, annilation operator $\hat{a}$ can be regarded as the quantum version of the complex amplitude a, which was introduced in Eq. (3.12).

Commutation relation

Quantum mechanics often considers commutation relation between arbitrary operators $\hat{A}$ and $\hat{B}$, which is defined as $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. For $\hat{x}$ and $\hat{p}$, the commutation relation is given by

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = -\frac{i}{2}\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\right) = -\frac{i}{2}\left(x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x} - 1\right) = \frac{i}{2}. \tag{3.61}$$

We can also see the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ as

$$[\hat{a}, \hat{a}^\dagger] = [\hat{x} + i\hat{p}, \hat{x} - i\hat{p}] = [\hat{x}, \hat{x}] + [\hat{p}, \hat{p}] + i[\hat{p}, \hat{x}] - i[\hat{x}, \hat{p}] = 1 \tag{3.62}$$

These relations will be used in various calculations.

3.1.5 Energy eigenstates

In Eq. (3.21), we saw the Schrödinger equation of a harmonic oscillator, which describes the time evolution of a wavefunction, which is a function of x. We can describe the same equation for a quantum state $|\psi\rangle$ as follows:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{3.63}$$

Here we aim at expressing quantum state as a function of energy. To this end, we look for the solution of Eq. (3.21) that corresponds to a specific energy, i.e., the eigenfunction of the Hamiltonian $\hat{H}$.

First, we denote $\hat{H} = \hbar\omega(\hat{x}^2 + \hat{p}^2)$ with $\hat{a}$ and $\hat{a}^\dagger$ as follows.

$$\begin{aligned} \hat{H} &= \hbar\omega\left[\left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)^2 + \left(\frac{\hat{a}-\hat{a}^\dagger}{2i}\right)^2\right] \\ &= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \equiv \hbar\omega\left(\hat{n} + \frac{1}{2}\right). \end{aligned} \tag{3.64}$$

Here we used the commutation relation in Eq. (3.61) and introduced $\hat{n} = \hat{a}^\dagger\hat{a}$. We denote an eigenstate of $\hat{n}$ with eigenvalue of n as $|n\rangle$. We assume that $|n\rangle$ is normalized, i.e., $\langle n|n\rangle = 1$. Since $\hat{n}$ is Hermitian,[6] we know that n is real. Then

[6] $\hat{n}^\dagger = (\hat{a}^\dagger\hat{a})^\dagger = \hat{a}^\dagger\hat{a} = \hat{n}$.

we show that $n \geq 0$.

$$n = n \langle n|n\rangle = \langle n|\hat{n}|n\rangle = \langle n|\hat{a}^\dagger \hat{a}|n\rangle = \|\hat{a}\,|n\rangle\,\|^2 \geq 0. \tag{3.65}$$

We can also show that n is an integer. To do so, let's show that $\hat{a}\,|n\rangle$ is also an eigenstate of $\hat{n}$.

$$\begin{aligned}\hat{n}\hat{a}\,|n\rangle &= \hat{a}^\dagger \hat{a}\hat{a}\,|n\rangle = (\hat{a}\hat{a}^\dagger - 1)\hat{a}\,|n\rangle = \hat{a}\hat{a}^\dagger \hat{a}\,|n\rangle - \hat{a}\,|n\rangle \\ &= \hat{a}\hat{n}\,|n\rangle - \hat{a}\,|n\rangle = (n-1)\hat{a}\,|n\rangle\,.\end{aligned} \tag{3.66}$$

This means that $\hat{a}\,|n\rangle$ is an eigenstate of $\hat{n}$ and its eigenvalue is $n-1$. Therefore we can assume $\hat{a}\,|n\rangle \propto |n-1\rangle$, or

$$\hat{a}\,|n\rangle = L_n\,|n-1\rangle\,. \tag{3.67}$$

The squared norm of LHS is given by

$$\|\hat{a}\,|n\rangle\,\|^2 = \langle n|\,\hat{a}^\dagger \hat{a}\,|n\rangle = n\,\langle n|n\rangle = n, \tag{3.68}$$

while that of RHS is $|L_n|^2$. Thus we obtain $|L_n| = \sqrt{n}e^{i\theta}$, where θ is a real arbitrary number, which defines the phase difference between $|n\rangle$ and $|n-1\rangle$. To simplify the notation, it is common to assume $\theta = 0$. Consequently, we obtain

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n\rangle\,. \tag{3.69}$$

Next, we show that $\hat{a}^\dagger\,|n\rangle$ is also an eigenstate of $\hat{n}$ as follows:

$$\hat{n}\hat{a}^\dagger\,|n\rangle = \hat{a}^\dagger \hat{a}\hat{a}^\dagger\,|n\rangle = \hat{a}^\dagger(\hat{a}^\dagger \hat{a} + 1)\,|n\rangle = \hat{a}^\dagger \hat{n}\,|n\rangle + \hat{a}^\dagger\,|n\rangle = (n+1)\hat{a}^\dagger\,|n\rangle\,. \tag{3.70}$$

This means that $\hat{a}^\dagger\,|n\rangle$ is an eigenstate of $\hat{n}$ and its eigenvalue is $n+1$. Therefore we can assume $\hat{a}^\dagger\,|n\rangle \propto |n+1\rangle$, or

$$\hat{a}^\dagger\,|n\rangle = K_n\,|n-1\rangle\,. \tag{3.71}$$

The squared norm of LHS is given by

$$\|\hat{a}^\dagger\,|n\rangle\,\|^2 = \langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle = \langle n|\,(\hat{a}^\dagger \hat{a} + 1)\,|n\rangle = (n+1)\,\langle n|n\rangle = n+1, \tag{3.72}$$

while that of RHS is $|K_n|^2$. Thus we obtain $|K_n| = \sqrt{n+1}e^{i\theta'}$, where θ' is a real arbitrary number. Nevertheless, it is common to assume $\theta' = 0$. Consequently, we obtain

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n\rangle\,. \tag{3.73}$$

Using Eq. (3.69) and Eq. (3.73), we can show that n is an integer; If n is not integer, we can obtain $|n-1\rangle$, $|n-2\rangle$, $\cdots$ by repetitively using Eq. (3.69), and n can be negative. This is contradictory to Eq. (3.65). When n is integer, we get $|n-1\rangle$, $|n-2\rangle$, $\cdots$, and we cannot go below 0 because $\hat{a}\,|0\rangle = 0$, which is consistent with Eq. (3.65).

Based on the above discussion, we can express $|n\rangle$ as

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle . \tag{3.74}$$

$|n\rangle$ is an eigenstate of the Hamiltonian of the harmonic oscillator $\hat{H}$ because

$$\hat{H} |n\rangle = \hbar\omega \left(\hat{n} + \frac{1}{2} \right) |n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle . \tag{3.75}$$

Therefore $E_n = \hbar\omega(n + 1/2)$ is the eigenvalue of $\hat{H}$ for $|n\rangle$.

Any quantum state can be expressed as a linear combination of $|n\rangle$ as

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \tag{3.76}$$

where c_n is complex. To normalize $|\psi\rangle$, we assume

$$\langle\psi|\psi\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m^* c_n \langle m|n\rangle = \sum_{n=0}^{\infty} |c_n|^2 = 1, \tag{3.77}$$

where we used $\langle m|n\rangle = 0$ for $m \neq n$.

This leads to

$$\hat{H} |\psi\rangle = \hbar\omega \left(\hat{n} + \frac{1}{2} \right) |\psi\rangle = \sum_{n=0}^{\infty} c_n \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle . \tag{3.78}$$

If $|\psi\rangle = |n\rangle$, the Schrödinger equation in Eq. (3.63) becomes

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hbar\omega(n + 1/2) |\psi(t)\rangle , \tag{3.79}$$

which leads to

$$|\psi(t)\rangle = e^{-i(n+1/2)\omega t} |n\rangle . \tag{3.80}$$

For a general quantum state given by Eq. (3.76), the solution of the Schrödinger equation is given by

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i(n+1/2)\omega t} |n\rangle . \tag{3.81}$$

In this way, the quantum state can be described as a superposition of $|n\rangle$ for various n with complex-valued weight c_n. Since $|n\rangle$ is an eigenstate of the Hamiltonian of a harmonic oscillator and its energy eigenvalue is E_n, we call $|n\rangle$ as **energy eigenstate**. We can see that any quantum state can be expressed by specifying the complex-valued weight c_n as a function of engenenergy E_n.

We will see that the probability for the quantum state to have an energy of E_n is given by $|c_n|^2$. This is analogous to wavefunction where complex value

$\psi(x)$ is specified as a function of x to express a quantum state, and the probability for the quantum state being at x is given by $|\psi(x)|^2$.

We can exchange the expression by taking inner product. Since $\langle n|\psi\rangle = c_n$, we get

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|\psi\rangle . \tag{3.82}$$

Similarly, we can express the quantum state by the linear combination of $|x\rangle$ or $|p\rangle$, and their weight is given by the inner product $\langle x|\psi\rangle$ or $\langle p|\psi\rangle$. Therefore,

$$|\psi\rangle = \int_{-\infty}^{\infty} |x\rangle \langle x|\psi\rangle \, dx = \int_{-\infty}^{\infty} |p\rangle \langle p|\psi\rangle \, dp. \tag{3.83}$$

Let's insert a figure to explain $\langle n|\psi\rangle$, $\langle x|\psi\rangle$, and $\langle p|\psi\rangle$

Position representation of energy eigenstates

Let's see how the energy eigenstates look like in the position representation. To this end, we use the x representation of $\hat{a}$ and $\hat{a}^\dagger$ given by

$$\hat{a} = \hat{x} + i\hat{p} = x + \frac{1}{2}\frac{\partial}{\partial x} \tag{3.84}$$

$$\hat{a}^\dagger = \hat{x} - i\hat{p} = x - \frac{1}{2}\frac{\partial}{\partial x} \tag{3.85}$$

By using them, we can show that

$$\langle x|0\rangle = (2/\pi)^{1/4} e^{-x^2}. \tag{3.86}$$

because it satisfies

$$\begin{aligned} \langle x|\hat{a}|0\rangle &= (\hat{x} + i\hat{p})(2/\pi)^{1/4} e^{-x^2} \\ &= (2/\pi)^{1/4} \left(x + \frac{1}{2}\frac{\partial}{\partial x} \right) e^{-x^2} = 0, \end{aligned} \tag{3.87}$$

and

$$\begin{aligned} \langle 0|0\rangle &= \int \langle 0|x\rangle \langle x|0\rangle \, dx = \int |\langle x|0\rangle|^2 dx \\ &= (2/\pi)^{1/2} \int e^{-2x^2} dx = 1. \end{aligned} \tag{3.88}$$

We can also find $\langle x|1\rangle$, $\langle x|2\rangle, \cdots$, using Eq. (3.74) as follows:

$$\begin{aligned} \langle x|1\rangle &= \langle x| \hat{a}^\dagger |0\rangle = (\hat{x} - i\hat{p})(2/\pi)^{1/4} e^{-x^2} \\ &= (2/\pi)^{1/4} \left(x - \frac{1}{2}\frac{\partial}{\partial x} \right) e^{-x^2} = (2/\pi)^{1/4} 2x e^{-x^2}, \end{aligned} \tag{3.89}$$

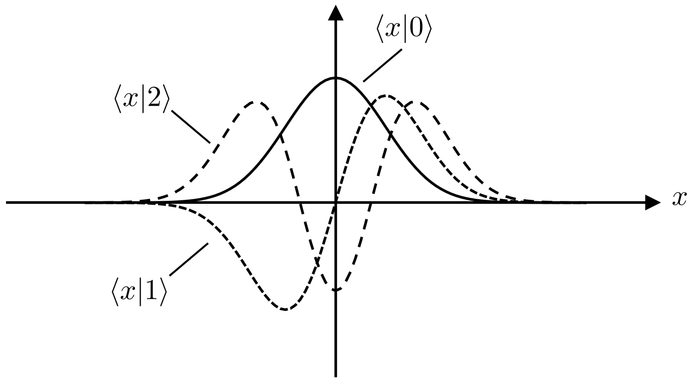

Figure 3.2: Position representation of the first three energy eigenstates: $\langle x|0\rangle \,, \langle x|1\rangle \,, \langle x|2\rangle$.

$$\begin{aligned}\langle x|2\rangle &= 2^{-1/2}\langle x|\, \hat{a}^\dagger \,|1\rangle = 2^{-1/2}(\hat{x} - i\hat{p})(2/\pi)^{1/4} 2xe^{-x^2} \\ &= 2^{-1/2}(2/\pi)^{1/4}\left(x - \frac{1}{2}\frac{\partial}{\partial x}\right) 2xe^{-x^2} \\ &= 2^{-1/2}(2/\pi)^{1/4}(4x^2-1)e^{-x^2},\end{aligned} \tag{3.90}$$

These are known as Hermite-Gaussian functions. Based on the above discussions, we can see that $\langle x|n\rangle$ constitutes a set of orthonormal basis; any ket ψ can be represented by the linear combination of $\langle x|n\rangle$.

Momentum representation of energy eigenstates

$$\langle p|n\rangle = \frac{1}{\sqrt{n!}}\langle p|(\hat{a}^\dagger)^n|0\rangle \tag{3.91}$$

In the p representation, $\hat{x}$ is given by

$$\hat{x} = \frac{i}{2}\frac{\partial}{\partial p}. \tag{3.92}$$

such that $[\hat{x}, \hat{p}] = i/2$ is satisfied. Therefore,

$$\hat{a} = \hat{x} + i\hat{p} = \frac{i}{2}\frac{\partial}{\partial p} + ip = i\left(p + \frac{1}{2}\frac{\partial}{\partial p}\right), \tag{3.93}$$

$$\hat{a}^\dagger = \hat{x} - i\hat{p} = \frac{i}{2}\frac{\partial}{\partial p} - ip = -i\left(p - \frac{1}{2}\frac{\partial}{\partial p}\right). \tag{3.94}$$

From Eq. (3.86) and Eq. (3.34),

$$\langle p|0\rangle = \int \langle p|x\rangle \langle x|0\rangle \, dx = \frac{(\pi/2)^{1/4}}{\sqrt{\pi}} \int e^{-x^2} e^{-2ipx} dx = (\pi/2)^{1/4} e^{-p^2}. \tag{3.95}$$

From Eq. (3.84), (3.93), (3.85), and (3.94), we can see the similarity of x representation and p representation; $\hat{a}$ $\hat{a}^\dagger$ have i and $-i$ terms, respectively. Therefore, we can find $\langle p|1\rangle, \langle p|2\rangle, \cdots$ as follows:

$$\langle p|1\rangle = -i(2/\pi)^{1/4} 2pe^{-p^2}, \tag{3.96}$$

$$\langle p|2\rangle = -2^{-1/2}(2/\pi)^{1/4}(4p^2-1)e^{-p^2}, \tag{3.97}$$

In this way, the energy eigenstates $|n\rangle$ in x and p representations are almost similar; the latter has $(-i)^n$ times difference. The reason behind this will be explained in later chapters.

3.2 Measurement of observables

Quantum mechanics uses Hermitian operators to express physical quantities called observables, such as number of photons, electric field, and so on. The expectation value of the observable can be calculated by specifying both the Hermitian operator of the observable and the quantum state. Here we explain how we calculate the observables.

3.2.1 Expectation value

Here we show that the expectation value of the observable $\hat{A}$ with a quantum state $|\psi\rangle$ is given by

$$\langle \hat{A}\rangle \equiv \langle\psi|\hat{A}|\psi\rangle \tag{3.98}$$

In Eq. (3.55), we saw that Hermitian operator can be diagonalized with a unitary operator $\hat{U}$. In the x representation, $\hat{U}$ can be represented by $U(x,x')$, which consists of the orthonormal eigenvector of $\hat{A}$ for various x' with a eigenvalue of $\lambda(x)$. Therefore

$$\int A(x,x'')U(x'',x')dx'' = \lambda(x')U(x,x'). \tag{3.99}$$

First, we express $|\psi\rangle$ by a linear combination of the eigenvector, i.e.,

$$|\psi\rangle = \hat{U}\,|\phi\rangle\,, \tag{3.100}$$

where $|\phi\rangle$ shows the weight of eigenvector. Then we multiply $\hat{A}$ to obtain

$$\hat{A}\,|\psi\rangle = \hat{A}\hat{U}\,|\phi\rangle \tag{3.101}$$

Since $\hat{A} = \hat{U}\hat{D}\hat{U}^\dagger$,

$$\hat{A}\,|\psi\rangle = \hat{U}\hat{D}\,|\phi\rangle \tag{3.102}$$

$$\langle\psi|\hat{A}|\psi\rangle = \langle\phi|\,\hat{U}^\dagger\hat{U}\hat{D}\,|\phi\rangle = \langle\phi|\hat{D}|\phi\rangle \tag{3.103}$$

The meaning of RHS can be clearer by describing in x-representation:

$$\langle\psi|\hat{A}|\psi\rangle = \int \phi^*(x)\lambda(x)\phi(x)dx = \int \lambda(x)|\phi(x)|^2 dx \tag{3.104}$$

The RHS is the weighted average of $\lambda(x)$ with a weight of $|\phi(x)|^2$.

In this way, when we evaluate some observable, we express ψ as a linear combination of eigenstates of the observable. Then we calculate the weighted sum of eigenvalue. Such calculation is embedded in $\langle\psi|\hat{A}|\psi\rangle$.

In the same manner, we can calculate the expectation of the variance as follows:

$$\begin{aligned}\langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle &= \langle\hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2\rangle \\ &= \langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle\langle\hat{A}\rangle + \langle\hat{A}\rangle^2 \\ &= \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 \\ &= \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2\end{aligned} \tag{3.105}$$

3.2.2 Commutation relation and simultaneous measurement

Here we show that, when two Hermitian operators $\hat{A}$ and $\hat{B}$ are commutative, i.e., $\hat{A}\hat{B} = \hat{B}\hat{A}$, there exists a quantum state that is an eigenstate $|\phi\rangle$ of $\hat{A}$ and $\hat{B}$. To prove this, we assume that $|\phi_A\rangle$ is an eigenstate of $\hat{A}$ with an eigenvalue of a_0, i.e.,

$$\hat{A}|\phi_0\rangle = a_0|\phi_0\rangle. \tag{3.106}$$

Then let's calculate $\hat{A}\hat{B}|\phi_0\rangle$ as

$$\hat{A}\hat{B}|\phi_0\rangle = \hat{B}\hat{A}|\phi_0\rangle = a_0\hat{B}|\phi_0\rangle. \tag{3.107}$$

This follows that $\hat{B}\phi_0$ is also an eigenvector of $\hat{A}$ with an eigenvalue of a_0. Therefore, $\hat{B}|\phi_0\rangle$ is proportional to $|\phi\rangle_A$, i.e.,

$$\hat{B}|\phi_0\rangle = b_0|\phi_0\rangle \tag{3.108}$$

Eq. (3.106) and Eq. (3.108) show that $|\phi_0\rangle$ is an eigenvector of $\hat{A}$ and $\hat{B}$. This means that we can measure the observables $\hat{A}$ and $\hat{B}$ simultaneously.

On the other hand, if $\hat{A}$ and $\hat{B}$ are not commutative, i.e., $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, the observables $\hat{A}$ and $\hat{B}$ cannot be measured simultaneously. $\hat{x}$ and $\hat{p}$ are representative examples of such observables. This leads to the uncertainty relation described below.

3.2.3 Uncertainty principle

Uncertainty principle states that, for observables $\hat{A}$, $\hat{B}$ with $[\hat{A},\hat{B}] \neq 0$,

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle \geq \left|\frac{1}{2}\langle[\hat{A},\hat{B}]\rangle\right|^2, \tag{3.109}$$

where

$$\langle \hat{A} \rangle \equiv \langle \phi | \hat{A} | \phi \rangle \, , \Delta\hat{A} = \hat{A} - \langle \hat{A} \rangle \, ,$$
$$\langle \hat{B} \rangle \equiv \langle \phi | \hat{B} | \phi \rangle \, , \Delta\hat{B} = \hat{B} - \langle \hat{B} \rangle \, . \tag{3.110}$$

This means that when $[\hat{A}, \hat{B}] \neq 0$, it is not possible to reduce the noise or fluctuation of the both observables to zero.

Eq. (3.109) can be derived as follows. First, we introduce $\hat{C} = i[\Delta\hat{A}, \Delta\hat{B}]$. We can show that $\hat{C}$ is Hermitian because

$$\begin{aligned} \hat{C}^\dagger &= -i(\Delta\hat{B}^\dagger \Delta\hat{A}^\dagger - \Delta\hat{A}^\dagger \Delta\hat{B}^\dagger) \\ &= -i(\Delta\hat{B}\Delta\hat{A} - \Delta\hat{A}\Delta\hat{B}) = i[\Delta\hat{A}, \Delta\hat{B}] = \hat{C}. \end{aligned} \tag{3.111}$$

Here we introduce $\hat{D} = \Delta A + i\lambda \Delta B$, where λ is a real number, and calculate $\langle \hat{D}^\dagger \hat{D} \rangle \equiv \langle \phi | \hat{D}^\dagger \hat{D} | \phi \rangle$ for an arbitrary quantum state $|\phi\rangle$.

$$\begin{aligned} \langle \hat{D}^\dagger \hat{D} \rangle &= \langle (\Delta\hat{A} - i\lambda\Delta\hat{B})(\Delta\hat{A} + i\lambda\Delta\hat{B}) \rangle \\ &= \langle \Delta\hat{A} \rangle^2 + i\lambda \langle [\Delta\hat{A}\Delta\hat{B} - \Delta\hat{B}\Delta\hat{A}] \rangle + \lambda^2 \langle \Delta\hat{B} \rangle^2 \\ &= \langle \Delta\hat{A} \rangle^2 + \lambda \langle \hat{C} \rangle + \lambda^2 \langle \Delta\hat{B} \rangle^2 \end{aligned} \tag{3.112}$$

Since $\langle \phi | \hat{D}^\dagger \hat{D} | \phi \rangle = \| \hat{D} \, |\phi\rangle \, \|^2 \geq 0$, the discriminant of the above equation should satisfy

$$\langle \hat{C} \rangle^2 - 4 \, \langle \Delta\hat{A}^2 \rangle \, \langle \Delta\hat{B}^2 \rangle \leq 0. \tag{3.113}$$

This leads to Eq. (3.109).

3.3 Multimode quantum states

So far, we have considered wavefunctions and quantum states of a one-dimensional harmonic oscillator at a single frequency. In later chapters, we consider light field as a collection of many harmonics oscillators that constitutes many spatial modes, time-frequency modes, and polarizations. Here we explain how to deal with quantum states of multi-mode harmonic oscillators.

The wavefunction of two-mode wavefunction can be denoted as $\psi(x_a, x_b)$, where x_a, x_b are the position of two modes denoted by a and b, respectively. $|\psi(x_a, x_b)|^2$ gives the probability density that the position of mode a and that of mode b are x_a and x_b, respectively. If $\psi(x_a, x_b)$ can be expressed by a product of two functions as

$$\psi(x_a, x_b) = \psi_a(x_a)\psi_b(x_b), \tag{3.114}$$

we regard that $\psi(x_a)$ is **separable**. If $\psi(x_a, x_b)$ cannot be expressed in such a way, we regards that the two modes are **entangled**.

In the bra-ket notation, separable quantum state is denoted as

$$|\psi\rangle = |\psi_a\rangle_a \otimes |\psi_b\rangle_b \, , \tag{3.115}$$

where $\otimes$ denotes tensor product. For example, if we assume that $|\psi_a\rangle_a$ and $|\psi_b\rangle_b$ are M- and N-dimensional vectors, respectively, i.e., $|\psi_a\rangle = (c_{a1}, c_{a2}, \cdots, c_{aM})^T$ and $|\psi_b\rangle = (c_{b1}, c_{b2}, \cdots, c_{bN})^T$,

$$|\psi_a\rangle_a \otimes |\psi_b\rangle_b = \begin{pmatrix} c_{a1}c_{b1} & c_{a1}c_{b2} & \cdots & c_{a1}c_{bN} \\ c_{a2}c_{b1} & c_{a2}c_{b2} & \cdots & c_{a2}c_{bN} \\ \vdots & \vdots & \ddots & \vdots \\ c_{aM}c_{b1} & c_{aM}c_{b2} & \cdots & c_{am}c_{bN} \end{pmatrix}, \tag{3.116}$$

which constitutes $M \times N$-dimensional vector space. We can see the similarity in Eq. (3.114) and Eq. (3.116).

Even if $|\psi\rangle$ is not separable, we can expand it by energy eigenstates $|n\rangle$ in the two modes as

$$|\psi\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m\rangle_a \otimes |n\rangle_b \left({}_a\langle m| \otimes {}_b\langle n| \right) |\psi\rangle . \tag{3.117}$$

That is, any quantum state can be expressed by the linear combination of $|m\rangle_a \otimes |n\rangle_b$, and the weight is given by the inner product between $|m\rangle_a \otimes |n\rangle_b$ and $|\psi\rangle$.

The above discussion applies to higher dimensions. For N-dimensional harmonic oscillator, the wavefunction is given by $\psi(x_1, x_2, \cdots, x_N)$. We can discuss the separability in the same way as the above.

3.4 Summary

We have reviewed the basics of quantum harmonic oscillators including the Schrödinger equation, wavefunction, energy eigenstates, operators and measurement. Important concepts are summarized below:

- Wavefunction $\psi(x)$ is the position representation of quantum state, and $|\psi(x)|^2$ gives the probability density of being at x.

- From the Schrödinger equation of a harmonic oscillator, we can derive energy eigenstate $|n\rangle$ with an energy $E_n = \hbar\omega(n + 1/2)$.

- The representation (position, momentum, and energy) of quantum state is interchangeable by taking the inner product with corresponding eigenstates.

- Expectation value of observable of a Hermitian operator $\hat{A}$ is given by $\langle\psi|A|\psi\rangle$.

Chapter 4

Evolution of quantum states

4.1 Schorödinger picture

4.2 Heisenberg picture

4.3 Unitary transformation of quantum states

4.3.1 Time evolution

4.3.2 Displacement

4.3.3 Mode mixing

4.3.4 Single-mode squeezing

4.3.5 Two-mode squeezing

4.4 Summary

Chapter 5

Quantization of light

5.1 Mode decomposition of electromagnetic waves

5.1.1 Time-frequency mode

5.1.2 Spatial mode

5.1.3 Polarization

5.2 Operator notation of electromagnetic waves

5.3 Summary

Chapter 6

Representative quantum states

6.1 Number states

6.2 Superposition states

6.3 Coherent states

6.4 Squeezed states

6.5 Two-mode squeezed states

6.5.1 EPR state

6.6 Summary

Chapter 7

Control of quantum states of light

7.1 Mode mixing

7.1.1 Beamsplitter

7.1.2 Waveplates

7.1.3 Optical loss

7.1.4 Fourier transform

7.2 Parametric amplification

7.2.1 Squeezing

7.2.2 Spontaneous parametric down conversion

7.2.3 Optical amplification

7.2.4 Raman scattering

7.3 Summary

Chapter 8

Quantum-optical measurement

8.1 Direct detection

8.2 Homodyne detection

8.3 Heterodyne detection

8.4 Preamplification

8.5 Quantum teleportation

8.6 Summary

Appendix A

Appendix

A.1 Bra-ket notation

A.2 Creation and annihilation operators

A.3 Pure states and mixed states

A.4 Wigner function

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1\end{aligned} \tag{A.1}$$

This is a simple calculation [1].

Index

Bibliography

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy.* San Val, 1995.

[2] H. Nyquist. Thermal agitation of electric charge in conductors. *Phys. Rev.*, 32:110–113, Jul 1928.