Lecture Note
Quantum Mechanics of Light and Matters

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Contents

Chapter 1

Introduction

Quantum optics deals with quantum nature of light, where light is regarded as an ensemble of particles called photons. The quantum nature of light appears as 'noise' in various applications in optics and photonics such as optical measurement, optical manipulation, and optical communications, leading to the physical limit called quantum limit on the performance or precision achieved by these methods. To push the performance of various methods to the physical limit, it is crucial to understand the quantum limit. Furthermore, quantum nature of light is extensively utilized to develop various quantum technologies such as quantum cryptography, quantum teleportation, and quantum computing.

Optical measurement always involves the detection of light. It is categorized into direct detection and homodyne/heterodyne detection. Direct detection gives the intensity of light, and homodyne/heterodyne detection which gives the amplitude of light. Furthermore, optical amplification is often utilized before photodetection to mitigate the effect of detector noise. In every case, the signal-to-noise ratio is ultimately limited by quantum noise, and becomes the same order as the number of photons. This limit cannot be surpassed by classical (i.e., non-quantum) methods, while various methods to surpass the limit is developed by using quantum optics.

This lecture note aims at dealing with quantum noise of light. Chapter 2 summarizes the noise in optical measurements. Chapter 3 introduces quantum harmonic oscillators, which is an analogue of light in quantum optics. Chapter 4 describes the evolution of quantum states. Chapter 5 explains the quantization of light. Chapter 6 introduces representative quantum states. Chapter 7 describes two main interactions in quantum optics: mode mixing and parametric amplifications. Chapter 8 explains quantum optical treatment of optical measurements. Appendix contains basic quantum mechanics, Wigner function, and so on. Variables and operators are summarized in Table 1.1.

This lecture note was first prepared in Japanese in 2017, in which I referred to Prof. Kazuro Kikuchi's lecture note. Then I rearranged the contents in English in 2020. In particular, I tried to provide an intuitive picture of quantum optics by extensively using wavefunctions. I often hear that quantum optics

Table 1.1: List of variables and operators

Variable or operator	Explanation
$\hat{x}$	Normalized position (real part of complex amplitude)
$\hat{p}$	Normalized momentum (imaginary part of complex amplitude)
$\hat{a}$	Complex amplitude
$\hat{a}^\dagger \hat{a} = \hat{n}$	Number of photons
q	Elementary charge
$\hbar$	Planck constant divided by 2π
P	Optical power (energy per unit time)
I	Current (charge per unit time)
η	Quantum efficiency
τ	Time duration of a single time-frequency mode
B	Nyquist frequency
T	Temperature

is abstract; we can somehow calculate various properties of light using bra-ket and operators, while the physics behind them are quite unclear. Instead, typical quantum-optical calculation using operators can be understood as the rotation and distortion of wavefunctions. I hope that this lecture memo provide such intuitive pictures along with the calculation procedures. I appreciate many comments and feedback from students in my research group and in the lecture. Any feedback is appreciated at `https://github.com/ysozeki/quantum-optics`.

Chapter 2

Noise in optical measurement

This chapter introduces various detection methods of light and explains noise appearing in each method. Some explanations are phenomenological but they will be explained by quantum optics in later chapters.

2.1 Optical measurement

Figure 2.1(a) shows the **direct detection**. Photodetectors can convert photons to electrons to measure optical power, which is proportional to the number of photons per unit time.

Fig. 2.1(b) shows the **interferometric detection**, where a beamsplitter (BS) is used to mix the signal light wave to be measured and another light wave called local oscillator (LO) light, and the output light waves of the BS are detected with photodetectors to measure the amplitude of light. When the optical frequencies of signal and LO are the same, the method is called **homodyne**. When they are different, the method is called **heterodyne**.

Furthermore, an optical amplifier is often used before photodetection as shown in Fig. 2.1(c). This is called **preamplification**. Although not shown in the figure, it is also possible to conduct interferometric detection after preamplification.

In any case, the output signal of the photodetector contains noise due to various origins such as instability of light sources or optical systems, circuit noise of photodetector(s), and so on. We can somehow reduce these noises, but at last we will see 'quantum noise' that cannot be reduced by classical manner. Only quantum optics can control the quantum noise.

Here, before introducing various noise sources, we introduce direct detection, interferometric detection, and preamplification.

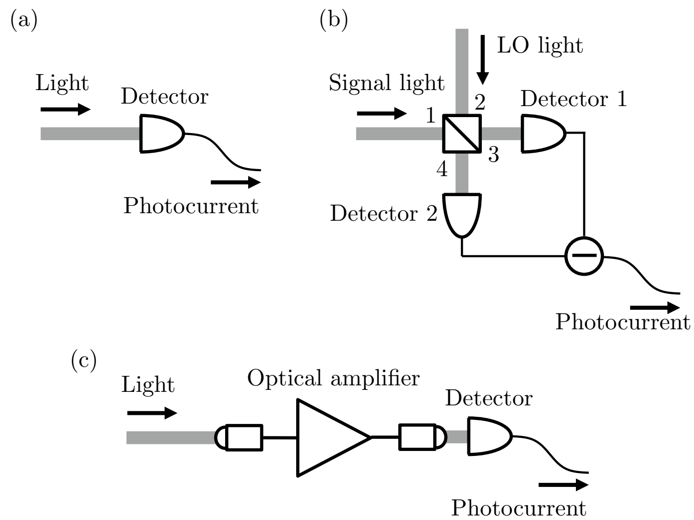

Figure 2.1: Various photodetection methods. (a) Direct detection. (b) Interferometric detection. (c) Optical preamplification with an optical amplifier.

2.1.1 Direct detection

In direct detection, a light wave is directly injected to a photodetector to measure the photocurrent I, which is proportional to the optical power P as follows:

$$I = \frac{\eta q P}{\hbar\omega},$$

where $\hbar\omega$ is the photon energy, and $q = 1.602\times10^{-19}$ C is the elementary charge. $P/\hbar\omega$ is the number of photons incident on the photodetector per unit time. η is the quantum efficiency, which is the ratio of the number of photoelectrons and the number of photons. [1]

2.1.2 Homodyne and heterodyne detection

Figure 2.1(b) shows a schematic of **balanced detector**, which is often used for homodyne and heterodyne. The light to be measured (signal) is combined with a local oscillator (LO) light by a beamsplitter (BS). Then, two light waves output from the BS are injected to photodetectors, and the difference of their photocurrents is taken. Denoting the optical frequencies of LO and signal light as ω and $\omega+\Delta\omega$, this method is called homodyne when $\Delta\omega = 0$ and heterodyne when $\Delta\omega \neq 0$.[2]

[1] For electrical engineers, it is worth remembering that the photon energy at the optical communication wavelength 1.55 μm is approximately 0.8 eV. Since $\hbar\omega/e$ is the photon energy in the unit of eV and typical photodiodes have a quantum efficiency of 90%, typical conversion efficiency is $I/P \sim 1.1$A/W (see catalogues of InGaAs photodiodes).

[2] These terms originate from frequency mixing in electrical circuits.

The output signal of the balanced detector can be formulated as follows. We denote the complex amplitudes of signal and LO as α and β such that $|\alpha|^2$ and $|\beta|^2$ corresponds to the number of photons in the time duration of τ.[3] We assume that $\Delta\omega \ll \omega$ such that the photon energy difference between signal and LO is negligible.[4] Then the analytic signals of signal light and LO light are given by [5]

$$\begin{aligned} a(t) &= \alpha e^{-i(\omega+\Delta\omega)t}, \\ b(t) &= \beta e^{-i\omega t}. \end{aligned} \tag{2.1}$$

We assume that signal light and LO light are injected to the port 1 and the port 2 of BS, and that the coupling ratio of BS is 50%. The output light waves at the port 3 and the port 4 are given by

$$\begin{aligned} a' &= \frac{1}{\sqrt{2}}(a-b), \\ b' &= \frac{1}{\sqrt{2}}(a+b), \end{aligned} \tag{2.2}$$

respectively,[6] or equivalently,

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.3}$$

The photocurrents I_1, I_2 of the two photodiodes are respectively given by

$$\begin{aligned} I_1 &= \frac{q}{\tau}|a'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a-b)\right|^2, \\ I_2 &= \frac{q}{\tau}|b'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a+b)\right|^2. \end{aligned} \tag{2.4}$$

The output of the balanced detector is

$$\begin{aligned} I_2 - I_1 &= \frac{q}{\tau}(ab^* + a^*b) = \frac{2q}{\tau}\mathrm{Re}[ab^*] = 4qB\mathrm{Re}[\alpha\beta^* e^{-i\Delta\omega t}] \\ &= 4qB|\beta|\left\{\mathrm{Re}\left[\alpha e^{-i\phi}\right]\cos\Delta\omega t + \mathrm{Im}\left[\alpha e^{-i\phi}\right]\sin\Delta\omega t\right\}, \end{aligned} \tag{2.5}$$

where $B = 1/2\tau$ is the Nyquist frequency, and $\beta = |\beta|e^{i\phi}$. Equation 2.5 shows the following points:

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[3]That is, the optical power of signal light is $|\alpha|^2\hbar\omega/\tau$.

[4]Without this assumption, the linear combination of the electric field between signal and LO requires quantum mechanical treatment.

[5]When a complex sinusoidal wave $S(t) = \mathrm{Re}\ S_0 e^{-i\omega t}$ is given, S_0 is called complex amplitude, while $S_0 e^{-i\omega t}$ is called analytic signal, which is composed only of positive frequency components. By taking the real part of the analytic signal, we can obtain a real signal.

[6]In the right hand side of the first equation in Eq. (2.2), the sign of b is minus. This corresponds to the assumption of fixed end reflection from port 2 to port 4. The definition can be different: What's important is that Eq. (2.2) is a unitary transformation, which corresponding to the assumption that BS has no optical loss.

- When $\Delta\omega = 0$ (i.e., homodyne), $I_2 - I_1 = 4qB|\beta|\mathrm{Re}\ (\alpha e^{-i\phi})$. Therefore, homodyne gives the projection of α onto the axis at a phase of β.
- When $\Delta\omega \neq 0$ (i.e., heterodyne), the output signal is a sinusoidal wave at $\Delta\omega$, whose complex amplitude is proportional to α.

2.2 Noise sources

In this section, we summarize various noise sources such as shot noise , thermal noise , and amplifier noise . You will see that by avoiding the effect of thermal noise, the signal-to-noise ratio becomes on the order of the number of photons.

2.2.1 Shot noise

Shot noise refers to the noise due to the fluctuation in the number of photons. We assume that photons arrive in a stochastic and independent manner. The probability distribution $\Pr(X = k)$ of the number of photons X obeys the Poisson distribution given by

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{2.6}$$

where λ is the average value. We can easily show that

$$\sum_{k=0}^{\infty} \Pr(X = k) = 1, \tag{2.7}$$

and the expectation value and the variance are given by

$$E[X] = \sum_{k=0}^{\infty} kp(k) = \lambda, \tag{2.8}$$

$$V[X] = \sum_{k=0}^{\infty} (k - \lambda)^2 p(k) = \lambda, \tag{2.9}$$

and therefore the standard deviation of the number of photons is $\sqrt{\lambda}$. Consequently, the root-mean-square (RMS) noise due to the fluctuation of the number of photons is given by

$$I_{\mathrm{shot}} = q\sqrt{\lambda}/\tau = q\sqrt{\frac{I\tau}{q}}/\tau = \sqrt{\frac{qI}{\tau}}, \tag{2.10}$$

or equivalently,

$$I_{\mathrm{shot}} = \sqrt{2qIB}. \tag{2.11}$$

By defining the signal-to-noise ratio (SNR) as the energy ratio between signal and noise, we obtain the shot-noise-limited SNR as

$$\mathrm{SNR} = I^2/I_{\mathrm{shot}}^2 = I/2qB = 2qB|\alpha|^2/2qB = |\alpha|^2, \tag{2.12}$$

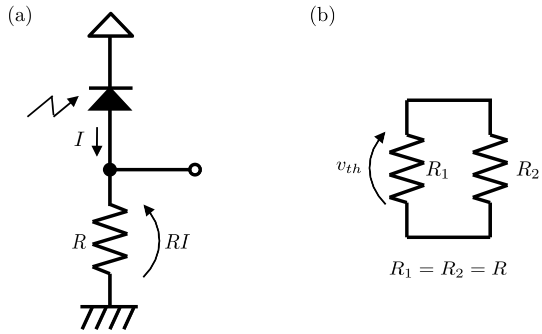

Figure 2.2: (a) Typical photodetection circuit, where a photodiode is inversely biased. The photocurrent I flows to a load resistor R and measure the voltage RI. (b) Two-resistor circuit, which is used to consider the Johnson noise.

where $I = q|\alpha|^2/\tau = 2qB|\alpha|^2$. Since $|\alpha|^2$ corresponds to the number of photons, we can see that the shot-noise limited SNR is equal to the number of photons.

2.2.2 Thermal noise

Thermal noise appears as voltage noise or current noise of a resistor R, which is used for converting a photocurrent I to a voltage RI as shown Fig. 2.2(a). The RMS voltage of the thermal noise or Johnson noise is given by

$$v_{\mathrm{th}} = \sqrt{4k_{\mathrm{B}}TRB}, \tag{2.13}$$

where k_B is the Boltzmann constant, and T is the temperature of the resistor. It is important to suppress the thermal noise by optimizing the circuit design for achieving the shot-noise-limited SNR.

To derive Eq. (2.13), we consider a circuit shown in Fig. 2.2, where two resistors (R_1, R_2) are connected with each other, and they are impedance-matched.[2] Since v_{th} is the electromotive force in one of the resistors and the series resistance is $2R$, it leads to a current of $v_{\mathrm{th}}/2R$. Therefore, each resistor generates power of $v_{\mathrm{th}}^2/2R$ and is in the thermal equilibrium.[7] Here we consider the voltage noise in the frequency range from $-B = -1/2\tau$ to $B = 1/2\tau$. The sampling theorem tells us that the noise waveform can be captured by sampling it with a period of τ. The noise at each sampling point is independent of each other, and its average energy is Boltzmann energy $k_{\mathrm{B}}T$.[8] Therefore,

$$v_{\mathrm{th}}^2\tau/2R = v_{\mathrm{th}}^2/4RB = k_{\mathrm{B}}T, \tag{2.14}$$

which leads to Eq. (2.13).

[7] If the resistors have different temperature, their temperatures get closer to each other by transferring energy with thermal current.

[8] The reason why each degree of freedom has an energy of $k_{\mathrm{B}}T$ instead of $k_{\mathrm{B}}T/2$ is that each degree of freedom corresponds to a thermally excited harmonic oscillator which has two degree of freedom of position and momentum. In the case of ideal gas, each atom has three momentum axis but not fixed at a point, leading to a kinetic energy of $3k_{\mathrm{B}}T/2$.

2.2.3 Optical amplifier noise

2.3 Summary

Chapter 3

Quantum harmonic oscillators

3.1 Schrödinger equation

3.1.1 Wavefunction and energy eigenstates

3.1.2 Fock representation

3.1.3 Position representation

3.1.4 Momentum representation

3.2 Measurement of observables

3.2.1 Expectation value

3.2.2 Expectation of variance

3.3 Multimode quantum states

3.4 Summary

Chapter 4

Evolution of quantum states

4.1 Schorödinger picture

4.2 Heisenberg picture

4.3 Unitary transformation of quantum states

4.3.1 Time evolution

4.3.2 Displacement

4.3.3 Mode mixing

4.3.4 Single-mode squeezing

4.3.5 Two-mode squeezing

4.4 Summary

Chapter 5

Quantization of light

5.1 Mode decomposition of electromagnetic waves

5.1.1 Time-frequency mode

5.1.2 Spatial mode

5.1.3 Polarization

5.2 Operator notation of electromagnetic waves

5.3 Summary

Chapter 6

Representative quantum states

6.1 Number states

6.2 Superposition states

6.3 Coherent states

6.4 Squeezed states

6.5 Two-mode squeezed states

6.5.1 EPR state

6.6 Summary

Chapter 7

Control of quantum states of light

7.1 Mode mixing

7.1.1 Beamsplitter

7.1.2 Waveplates

7.1.3 Optical loss

7.1.4 Fourier transform

7.2 Parametric amplification

7.2.1 Squeezing

7.2.2 Spontaneous parametric down conversion

7.2.3 Optical amplification

7.2.4 Raman scattering

7.3 Summary

Chapter 8

Quantum-optical measurement

8.1 Direct detection

8.2 Homodyne detection

8.3 Heterodyne detection

8.4 Preamplification

8.5 Quantum teleportation

8.6 Summary

Appendix A

Appendix

A.1 Bra-ket notation

A.2 Creation and annihilation operators

A.3 Pure states and mixed states

A.4 Wigner function

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1
\end{aligned}
\tag{A.1}
$$

This is a simple calculation [1].

Index

Bibliography

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy.* San Val, 1995.

[2] H. Nyquist. Thermal agitation of electric charge in conductors. *Phys. Rev.*, 32:110–113, Jul 1928.