Lecture Note
Quantum Mechanics of Light and Matters

Yasuyuki Ozeki

Department of Electrical Engineering and Information Systems
The University of Tokyo

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Contents

Chapter 1

Introduction

Quantum optics deals with quantum nature of light, where light is regarded as an ensemble of particles called photons. The quantum nature of light appears as 'noise' in various applications in optics and photonics such as optical measurement, optical manipulation, and optical communications, leading to the physical limit called quantum limit on the performance or precision achieved by these methods. To push the performance of various methods to the physical limit, it is crucial to understand the quantum limit. Furthermore, quantum nature of light is extensively utilized to develop various quantum technologies such as quantum cryptography, quantum teleportation, and quantum computing.

Optical measurement always involves the detection of light. It is categorized into direct detection and homodyne/heterodyne detection. Direct detection gives the intensity of light, and homodyne/heterodyne detection which gives the amplitude of light. Furthermore, optical amplification is often utilized before photodetection to mitigate the effect of detector noise. In every case, the signal-to-noise ratio is ultimately limited by quantum noise, and becomes the same order as the number of photons. This limit cannot be surpassed by classical (i.e., non-quantum) methods, while various methods to surpass the limit is developed by using quantum optics.

This lecture note aims at dealing with quantum noise of light. Chapter 2 summarizes the noise in optical measurements. Chapter 3 introduces quantum harmonic oscillators, which is an analogue of light in quantum optics. Chapter 4 describes the evolution of quantum states. Chapter 5 explains the quantization of light. Chapter 6 introduces representative quantum states. Chapter 7 describes two main interactions in quantum optics: mode mixing and parametric amplifications. Chapter 8 explains quantum optical treatment of optical measurements. Appendix contains basic quantum mechanics, Wigner function, and so on. Variables and operators are summarized in Table 1.1.

This lecture note was first prepared in Japanese in 2017, in which I referred to Prof. Kazuro Kikuchi's lecture note. Then I rearranged the contents in English in 2020. In particular, I tried to provide an intuitive picture of quantum optics by extensively using wavefunctions. I often hear that quantum optics

Table 1.1: List of variables and operators

Variable or operator	Explanation
$\hat{x}$	Normalized position (real part of complex amplitude)
$\hat{p}$	Normalized momentum (imaginary part of complex amplitude)
$\hat{a}$	Complex amplitude
$\hat{a}^\dagger\hat{a} = \hat{n}$	Number of photons
q	Elementary charge
$\hbar$	Planck constant divided by 2π
P	Optical power (energy per unit time)
I	Current (charge per unit time)
η	Quantum efficiency
τ	Time duration of a single time-frequency mode
B	Nyquist frequency
T	Temperature

is abstract; we can somehow calculate various properties of light using bra-ket and operators, while the physics behind them are quite unclear. Instead, typical quantum-optical calculation using operators can be understood as the rotation and distortion of wavefunctions. I hope that this lecture memo provide such intuitive pictures along with the calculation procedures. I appreciate many comments and feedback from students in my research group and in the lecture. Any feedback is appreciated at `https://github.com/ysozeki/quantum-optics`.

Chapter 2

Noise in optical measurement

This chapter introduces various detection methods of light and explains noise appearing in each method. Some explanations are phenomenological but they will be explained by quantum optics in later chapters.

2.1 Optical measurement

Figure 2.1(a) shows the **direct detection**. Photodetectors can convert photons to electrons to measure optical power, which is proportional to the number of photons per unit time.

Fig. 2.1(b) shows the **interferometric detection**, where a beamsplitter (BS) is used to mix the signal light wave to be measured and another light wave called local oscillator (LO) light, and the output light waves of the BS are detected with photodetectors to measure the amplitude of light. When the optical frequencies of signal and LO are the same, the method is called **homodyne**. When they are different, the method is called **heterodyne**.

Furthermore, an optical amplifier is often used before photodetection as shown in Fig. 2.1(c). This is called **preamplification**. Although not shown in the figure, it is also possible to conduct interferometric detection after preamplification.

In any case, the output signal of the photodetector contains noise due to various origins such as instability of light sources or optical systems, circuit noise of photodetector(s), and so on. We can somehow reduce these noises, but at last we will see 'quantum noise' that cannot be reduced by classical manner. Only quantum optics can control the quantum noise.

Here, before introducing various noise sources, we introduce direct detection, interferometric detection, and preamplification.

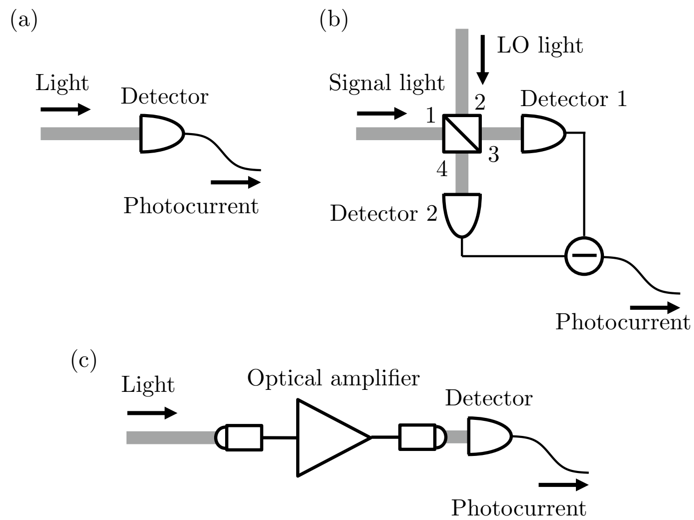

Figure 2.1: Various photodetection methods. (a) Direct detection. (b) Interferometric detection. (c) Optical preamplification with an optical amplifier.

2.1.1 Direct detection

In direct detection, a light wave is directly injected to a photodetector to measure the photocurrent I, which is proportional to the optical power P as follows:

$$I = \frac{\eta q P}{\hbar\omega},$$

where $\hbar\omega$ is the photon energy, and $q = 1.602\times10^{-19}$ C is the elementary charge. $P/\hbar\omega$ is the number of photons incident on the photodetector per unit time. η is the quantum efficiency, which is the ratio of the number of photoelectrons and the number of photons. [1]

2.1.2 Homodyne and heterodyne detection

Figure 2.1(b) shows a schematic of **balanced detector**, which is often used for homodyne and heterodyne. The light to be measured (signal) is combined with a local oscillator (LO) light by a beamsplitter (BS). Then, two light waves output from the BS are injected to photodetectors, and the difference of their photocurrents is taken. Denoting the optical frequencies of LO and signal light as ω and $\omega + \Delta\omega$, this method is called homodyne when $\Delta\omega = 0$ and heterodyne when $\Delta\omega \neq 0$.[2]

[1]For electrical engineers, it is worth remembering that the photon energy at the optical communication wavelength 1.55 µm is approximately 0.8 eV. Since $\hbar\omega/e$ is the photon energy in the unit of eV and typical photodiodes have a quantum efficiency of 90%, typical conversion efficiency is $I/P \sim 1.1$A/W (see some specsheets of InGaAs photodiodes).

[2]These terms originate from frequency mixing in electrical circuits.

The output signal of the balanced detector can be formulated as follows. We denote the complex amplitudes of signal and LO as α and β such that $|\alpha|^2$ and $|\beta|^2$ corresponds to the number of photons in the time duration of τ.[3] We assume that $\Delta\omega \ll \omega$ such that the photon energy difference between signal and LO is negligible.[4] Then the analytic signals of signal light and LO light are given by [5]

$$\begin{aligned} a(t) &= \alpha e^{-i(\omega+\Delta\omega)t}, \\ b(t) &= \beta e^{-i\omega t}. \end{aligned} \tag{2.1}$$

We assume that signal light and LO light are injected to the port 1 and the port 2 of BS, and that the coupling ratio of BS is 50%. The output light waves at the port 3 and the port 4 are given by

$$\begin{aligned} a' &= \frac{1}{\sqrt{2}}(a-b), \\ b' &= \frac{1}{\sqrt{2}}(a+b), \end{aligned} \tag{2.2}$$

respectively,[6] or equivalently,

$$\left(\begin{array}{c} a' \\ b' \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left(\begin{array}{c} a \\ b \end{array} \right). \tag{2.3}$$

The photocurrents I_1, I_2 of the two photodiodes are respectively given by

$$\begin{aligned} I_1 &= \frac{q}{\tau}|a'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a-b)\right|^2, \\ I_2 &= \frac{q}{\tau}|b'|^2 = \frac{q}{\tau}\left|\frac{1}{\sqrt{2}}(a+b)\right|^2. \end{aligned} \tag{2.4}$$

The output of the balanced detector is

$$\begin{aligned} I_2 - I_1 &= \frac{q}{\tau}(ab^* + a^*b) = \frac{2q}{\tau}\mathrm{Re}[ab^*] = 4qB\mathrm{Re}[\alpha\beta^* e^{-i\Delta\omega t}] \\ &= 4qB|\beta| \left\{ \mathrm{Re}\left[\alpha e^{-i\phi}\right] \cos\Delta\omega t + \mathrm{Im}\left[\alpha e^{-i\phi}\right] \sin\Delta\omega t \right\}, \end{aligned} \tag{2.5}$$

where $B = 1/2\tau$ is the Nyquist frequency, and $\beta = |\beta|e^{i\phi}$. Equation 2.5 shows the following points:

[3] That is, the optical power of signal light is $|\alpha|^2\hbar\omega/\tau$.

[4] Without this assumption, the linear combination of the electric field between signal and LO requires quantum mechanical treatment.

[5] When a complex sinusoidal wave $S(t) = \mathrm{Re}\ S_0 e^{-i\omega t}$ is given, S_0 is called complex amplitude, while $S_0 e^{-i\omega t}$ is called analytic signal, which is composed only of positive frequency components. By taking the real part of the analytic signal, we can obtain a real signal.

[6] In the right hand side of the first equation in Eq. (2.2), the sign of b is minus. This corresponds to the assumption of fixed end reflection from port 2 to port 4. The definition can be different: What's important is that Eq. (2.2) is a unitary transformation, which corresponding to the assumption that BS has no optical loss.

- When $\Delta\omega = 0$ (i.e., homodyne), $I_2 - I_1 = 4qB|\beta|\text{Re}\ (\alpha e^{-i\phi})$. Therefore, homodyne gives the projection of α onto the axis at a phase of β.
- When $\Delta\omega \neq 0$ (i.e., heterodyne), the output signal is a sinusoidal wave at $\Delta\omega$, whose complex amplitude is proportional to α.

2.2 Noise sources

In this section, we summarize various noise sources such as shot noise , thermal noise , and amplifier noise . You will see that by avoiding the effect of thermal noise, the signal-to-noise ratio becomes on the order of the number of photons.

2.2.1 Shot noise

Shot noise refers to the noise due to the fluctuation in the number of photons. We assume that photons arrive in a stochastic and independent manner. The probability distribution $\Pr(X = k)$ of the number of photons X obeys the Poisson distribution given by

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{2.6}$$

where λ is the average value. We can easily show that

$$\sum_{k=0}^{\infty} \Pr(X = k) = 1, \tag{2.7}$$

and the expectation value and the variance are given by

$$E[X] = \sum_{k=0}^{\infty} kp(k) = \lambda, \tag{2.8}$$

$$V[X] = \sum_{k=0}^{\infty} (k - \lambda)^2 p(k) = \lambda, \tag{2.9}$$

and therefore the standard deviation of the number of photons is $\sqrt{\lambda}$. Consequently, the root-mean-square (RMS) noise due to the fluctuation of the number of photons is given by

$$I_{\text{shot}} = q\sqrt{\lambda}/\tau = q\sqrt{\frac{I\tau}{q}}/\tau = \sqrt{\frac{qI}{\tau}}, \tag{2.10}$$

or equivalently,

$$I_{\text{shot}} = \sqrt{2qIB}. \tag{2.11}$$

Shot-noise-limited SNR in direct detection

By defining the signal-to-noise ratio (SNR) as the energy ratio between signal and noise, we obtain the shot-noise-limited SNR as

$$\mathrm{SNR} = I^2/I_{\mathrm{shot}}^2 = I/2qB = 2qB|\alpha|^2/2qB = |\alpha|^2, \tag{2.12}$$

where $I = q|\alpha|^2/\tau = 2qB|\alpha|^2$. Since $|\alpha|^2$ corresponds to the number of photons, we can see that the shot-noise limited SNR is equal to the number of photons.

Shot-noise-limited SNR in homodyne detection

To calculate SNR of homodyne detection, let's assume that LO light is much stronger than signal light, and $\phi = 0$ for simplicity. Since the LO light is divided approximately by half by the BS in the balanced detector, the photocurrents are given by

$$I_1 \sim I_2 \sim \frac{1}{2}\frac{q}{\tau}|\beta|^2 = qB|\beta|^2, \tag{2.13}$$

and therefore the shot noise is given by

$$\sqrt{2qI_1B} \sim \sqrt{2qI_2B} \sim \sqrt{2q^2B^2|\beta|^2} = \sqrt{2}qB|\beta|. \tag{2.14}$$

The output of balanced detector is affected by two independent shot noise from I_1 and I_2. Therefore the shot noise of the balanced detector is given by

$$I_{\mathrm{shot}} = 2qB|\beta|. \tag{2.15}$$

By assuming $\Delta\omega = 0$ in Eq. (2.5), the homodyne output is described by

$$I_{\mathrm{homodyne}} = 4qB|\beta|\mathrm{Re}\ \alpha. \tag{2.16}$$

Therefore SNR in homodyne detection is given by

$$\mathrm{SNR}_{\mathrm{homodyne}} = I_{\mathrm{homodyne}}^2/I_{\mathrm{shot}}^2 = 4(\mathrm{Re}[\alpha])^2. \tag{2.17}$$

You can see that the SNR in homodyne is determined by the energy of signal light, and is independent of LO power.

Shot-noise-limited SNR in heterodyne detection

Let's consider SNR in heterodyne detection. We split Eq. (2.5) into two terms:

$$\begin{aligned} I_{\cos} &= 4qB|\beta|\mathrm{Re}[\alpha]\cos\Delta\omega t, \\ I_{\sin} &= 4qB|\beta|\mathrm{Im}[\alpha]\sin\Delta\omega t. \end{aligned} \tag{2.18}$$

Their mean square can be calculated as

$$\begin{aligned} \overline{I_{\cos}^2} &= (4qB|\beta|\mathrm{Re}[\alpha])^2\frac{1}{2}\overline{(1+\cos 2\Delta\omega t)} = 8(qB|\beta|\mathrm{Re}[\alpha])^2, \\ \overline{I_{\sin}^2} &= (4qB|\beta|\mathrm{Im}[\alpha])^2\frac{1}{2}\overline{(1-\cos 2\Delta\omega t)} = 8(qB|\beta|\mathrm{Im}[\alpha])^2. \end{aligned} \tag{2.19}$$

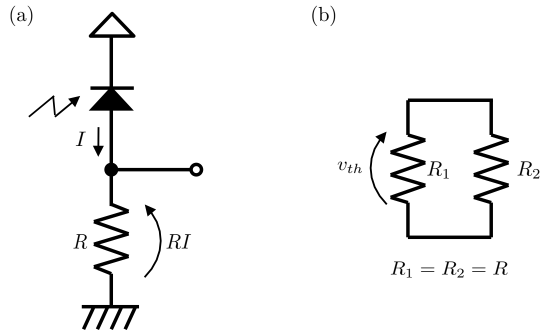

Figure 2.2: (a) Typical photodetection circuit, where a photodiode is inversely biased. The photocurrent I flows to a load resistor R and measure the voltage RI. (b) Two-resistor circuit, which is used to consider the Johnson noise.

Therefore, their SNRs are given by [7]

$$\begin{aligned} \overline{I_{\cos}^2}/I_{\text{shot}}^2 &= 2(\text{Re}[\alpha])^2, \\ \overline{I_{\sin}^2}/I_{\text{shot}}^2 &= 2(\text{Im}[\alpha])^2. \end{aligned} \tag{2.20}$$

From Eq. (2.17) and Eq. (2.20), we can see that the heterodyne gives both the real and imaginary parts of complex amplitude with 3-dB lower SNR than homodyne.

2.2.2 Thermal noise

Thermal noise appears as voltage noise or current noise of a resistor R, which is used for converting a photocurrent I to a voltage RI as shown Fig. 2.2(a). The RMS voltage of the thermal noise or Johnson noise is given by

$$v_{\text{th}} = \sqrt{4k_{\text{B}}TRB}, \tag{2.21}$$

where k_B is the Boltzmann constant, and T is the temperature of the resistor. It is important to suppress the thermal noise by optimizing the circuit design for achieving the shot-noise-limited SNR.

To derive Eq. (2.21), we consider a circuit shown in Fig. 2.2, where two resistors (R_1, R_2) are connected with each other, and they are impedance-matched.[2] Since v_{th} is the electromotive force in one of the resistors and the series resistance is $2R$, it leads to a current of $v_{\text{th}}/2R$. Therefore, each resistor generates power of $v_{\text{th}}^2/2R$ and is in the thermal equilibrium.[8] Here we consider

[7] Note that the discussion here omits various points on the quantification of shot noise; We should consider the shot noise in the frequency range from $\Delta\omega/2\pi - B$ to $\Delta\omega/2\pi + B$, whose bandwidth is $2B$. Nevertheless, here we have two shot noise components: sin and cos. Therefore, if we extract cos component of shot noise, its power is the same as Eq. (2.15).

[8] If the resistors have different temperature, their temperatures get closer to each other by transferring energy with thermal current.

the voltage noise in the frequency range from $-B = -1/2\tau$ to $B = 1/2\tau$. The sampling theorem tells us that the noise waveform can be captured by sampling it with a period of τ. The noise at each sampling point is independent of each other, and its average energy is Boltzmann energy $k_{\mathrm{B}}T$.[9] Therefore,

$$v_{\mathrm{th}}^2\tau/2R = v_{\mathrm{th}}^2/4RB = k_{\mathrm{B}}T, \tag{2.22}$$

which leads to Eq. (2.21).

Comparison between shot noise and thermal noise

To compare the amounts of shot noise and thermal noise, let's consider the case where they are the same, i.e., $RI_{\mathrm{shot}} = v_{\mathrm{th}}$ when the photocurrent is I'. This leads to

$$I' = \frac{2k_{\mathrm{B}}T}{qR}. \tag{2.23}$$

Since $k_{\mathrm{B}}T/q$ is the Boltzmann energy in the unit of eV, it is 26 meV at the room temperature. Assuming the load resistance of $R = 50\ \Omega$, $I' = 1$ mA. The photocurrent is larger than I', shot noise dominates, and *vice versa*. Since shot noise and thermal noise are proportional to R and $\sqrt{R}$, respectively, we can suppress the effect of thermal noise by increasing R, while the response time RC of the photodetector due to its capacitance C limits the bandwidth of the circuit. You can also see that homodyne/heterodyne detection is useful for suppressing the thermal noise because strong LO light is introduced to the photodetectors.

2.2.3 Optical amplifier noise

Optical amplifiers can literally amplify light, while amplified light is accompanied with optical noise called amplified spontaneous emission (ASE). There are various types of optical amplifiers such as fiber amplifiers using an optical fiber doped with various rare-earth ions ($\mathrm{Er}^{3+}, \mathrm{Yb}^{3+}, \mathrm{Tm}^{3+}$, etc.), semiconductor optical amplifiers using direct bandgap semiconductors, and optical parametric amplifiers using a nonlinear optical crystal or an optical fiber. Surprisingly, the SNR limit due to ASE applies to any type of optical amplifiers.

Without proof,[10] we introduce that the power of ASE in a single polarization state is given by

$$P_0 = n_{\mathrm{sp}}\hbar\omega(G-1)\Delta f, \tag{2.24}$$

where G is the gain, Δf is the optical bandwidth, and $n_{\mathrm{sp}} \geq 1$ is the spontaneous emission factor, which increases when the population inversion of laser material is imperfect and optical amplifier has optical loss. Since typical laser material

[9] The reason why each degree of freedom has an energy of $k_{\mathrm{B}}T$ instead of $k_{\mathrm{B}}T/2$ is that each degree of freedom corresponds to a thermally excited harmonic oscillator which has two degree of freedom of position and momentum. In the case of ideal gas, each atom has three momentum axis but not fixed at a point, leading to a kinetic energy of $3k_{\mathrm{B}}T/2$.

[10] The derivation will be given in the later chapter.

produces ASE in both vertical and horizontal polarizations, the total ASE power is given by

$$P_{\mathrm{ASE}} = 2n_{\mathrm{sp}}\hbar\omega(G-1)\Delta f. \tag{2.25}$$

To investigate how SNR is affected by optical amplification, let's calculate the SNR after amplification. We denote the optical power of a light wave before amplification as P_{in} and assume that its phase is zero. Also we consider the frequency range from $\omega/2\pi - B$ to $\omega/2\pi + B$, and therefore $\Delta f = 2B$. Since ASE has cos and sin components, ASE power only in the cos component and in a single polarization is given by

$$P_{\mathrm{ASE1}} = \frac{1}{2}n_{sp}\hbar\omega(G-1)2B = n_{sp}\hbar\omega(G-1)B. \tag{2.26}$$

When the ASE field interferes the amplified signal field, its power changes according to

$$\left(\sqrt{GP_{\mathrm{in}}} \pm \sqrt{P_{\mathrm{ASE1}}}\right)^2 = GP_{\mathrm{in}} \pm 2\sqrt{GP_{\mathrm{in}}P_{\mathrm{ASE1}}} + P_{\mathrm{ASE1}}. \tag{2.27}$$

The second term in the right hand side is the power change due to the interference between the amplified signal and ASE. The noise from this effect is called **signal-ASE beat noise**. The third term can also contribute to the noise and is significant when the ASE power is comparable with the signal. Assuming that the signal-ASE beat noise is dominant, we obtain the SNR as

$$\mathrm{SNR} = \left(\frac{GP_{\mathrm{in}}}{2\sqrt{GP_{\mathrm{in}}P_{\mathrm{ASE1}}}}\right)^2 = \frac{GP_{\mathrm{in}}}{4P_{\mathrm{ASE1}}} = \frac{GP_{\mathrm{in}}}{4n_{\mathrm{sp}}\hbar\omega(G-1)B} \to \frac{1}{2n_{\mathrm{sp}}}\frac{P_{\mathrm{in}}}{2\hbar\omega B}. \tag{2.28}$$

Here, $P_{in}/2\hbar\omega B$ is the number of photons of input light in a time duration of $\tau = 1/2B$, which equals to the shot-noise-limited SNR before amplification. Therefore, the optical amplification reduces SNR by $1/2n_{\mathrm{sp}}$ times, and hence the **noise figure** (NF) is $2n_{\mathrm{sp}}$. In particular, NF with $n_{\mathrm{sp}} = 1$ (i.e., 3 dB) is called quantum-limited noise figure. Typical NF in optical fiber amplifiers is 5-6 dB.

2.2.4 Other noise sources

So far, we have discussed shot noise, thermal noise, and amplifier noise. They are basic noise sources, while there are other noise sources as well. Here we introduce excess noise and $1/f$ noise.

Excess noise

When a light wave has a larger fluctuation that the shot noise, the other factors than the shot noise are called **excess noise**. Excess noise includes the instability of light sources and that of optical systems. ASE is also categorized as excess noise.

$1/f$ noise

It is known that active components and light sources exhibit a large fluctuation at low frequency region. Its power spectrum is often proportional to $1/f$. Such noise is called $1/f$ noise. This is in contrast to white noise such as shot noise and thermal noise whose power spectrum is independent on frequency. $1/f$ noise becomes significant when the signal is averaged over a long time.

2.3 Summary

We have reviewed the noise in optical measurement including direct detection, homodyne detection, and heterodyne detection. Direct detection is simple and useful but the detector noise can dominate when the optical power is low. Homodyne and heterodyne are useful for suppressing the effect of the thermal noise, and sensitive to the complex amplitude of signal light. Homodyne gives either real or imaginary part of the complex amplitude, while heterodyne gives both with 3-dB lower SNR. Optical amplification is useful for avoiding the detector noise, but the SNR is reduced by 3 dB. Some properties such as Poisson distribution and the amount of ASE were given without proof, but they will be supported by quantum optics. Furthermore, quantum optics allows for surpassing the shot-noise-limited SNR. These points will be discussed in the following chapters.

Chapter 3

Quantum harmonic oscillators

3.1 Schrödinger equation

3.1.1 Wavefunction and energy eigenstates

3.1.2 Fock representation

3.1.3 Position representation

3.1.4 Momentum representation

3.2 Measurement of observables

3.2.1 Expectation value

3.2.2 Expectation of variance

3.3 Multimode quantum states

3.4 Summary

Chapter 4

Evolution of quantum states

4.1 Schorödinger picture

4.2 Heisenberg picture

4.3 Unitary transformation of quantum states

4.3.1 Time evolution

4.3.2 Displacement

4.3.3 Mode mixing

4.3.4 Single-mode squeezing

4.3.5 Two-mode squeezing

4.4 Summary

Chapter 5

Quantization of light

5.1 Mode decomposition of electromagnetic waves

5.1.1 Time-frequency mode

5.1.2 Spatial mode

5.1.3 Polarization

5.2 Operator notation of electromagnetic waves

5.3 Summary

Chapter 6

Representative quantum states

6.1 Number states

6.2 Superposition states

6.3 Coherent states

6.4 Squeezed states

6.5 Two-mode squeezed states

6.5.1 EPR state

6.6 Summary

Chapter 7

Control of quantum states of light

7.1 Mode mixing

7.1.1 Beamsplitter

7.1.2 Waveplates

7.1.3 Optical loss

7.1.4 Fourier transform

7.2 Parametric amplification

7.2.1 Squeezing

7.2.2 Spontaneous parametric down conversion

7.2.3 Optical amplification

7.2.4 Raman scattering

7.3 Summary

Chapter 8

Quantum-optical measurement

8.1 Direct detection

8.2 Homodyne detection

8.3 Heterodyne detection

8.4 Preamplification

8.5 Quantum teleportation

8.6 Summary

Appendix A

Appendix

A.1 Bra-ket notation

A.2 Creation and annihilation operators

A.3 Pure states and mixed states

A.4 Wigner function

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\
&= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1
\end{aligned} \tag{A.1}$$

This is a simple calculation [1].

Index

Bibliography

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy.* San Val, 1995.

[2] H. Nyquist. Thermal agitation of electric charge in conductors. *Phys. Rev.*, 32:110–113, Jul 1928.