

Lecture Note
Quantum Mechanics of Light and Matters

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Chapter 1

Quantum harmonic oscillators

The previous chapter introduced the phenomenological description of shot noise and ASE to calculate SNR in various optical measurements. Quantum optics tells you the physics behind these noise sources and provides ways to manipulate them.

In quantum optics, the physics of harmonic oscillators play a crucial role because electromagnetic field is decomposed into the collection of time-frequency modes, spatial modes, and polarizations, and each mode is assumed as a harmonic oscillator.

This chapter will introduce the quantum harmonic oscillators. If you are familiar with the basics of quantum mechanics, you can skip this chapter.

1.1 Schrödinger equation for a harmonic oscillator

1.1.1 Classical harmonic oscillators

Let's discuss the motion of a one-dimensional mass-spring system without friction with a mass of m and a spring constant of k . The equation of motion is given by

$$m \frac{d^2 X}{dt^2} + kX = 0. \quad (1.1)$$

The solution is

$$X(t) = \frac{1}{2}A \exp(-i\omega t) + c.c. = \text{Re}[A \exp(-i\omega t)] \quad (1.2)$$

where A is a complex number, $\omega = \sqrt{k/m}$, and *c.c.* stands for the complex conjugate. The momentum $P = m(dX/dt)$ is given by

$$P(t) = \frac{-im\omega}{2} A \exp(-i\omega t) + c.c. = m\omega \text{Im}[A \exp(-i\omega t)]. \quad (1.3)$$

Therefore $X(t)$ and $P(t)$ oscillate with 90deg. phase shift.

The sum of potential energy and kinetic energy is given by

$$E = \frac{1}{2}kX^2 + \frac{m}{2} \left(\frac{dX}{dt} \right)^2 = \frac{1}{2}kX^2 + \frac{P^2}{2m}, \quad (1.4)$$

where we introduced the momentum $P = m(dX/dt)$. Substituting Eq. (1.2) to Eq. (1.4), we can see that the total energy is kept constant as

$$E = \frac{1}{2}k(\text{Re}[A \exp(-i\omega t)])^2 + \frac{m\omega^2}{2}(\text{Im}[A \exp(-i\omega t)])^2 = \frac{1}{2}k|A|^2. \quad (1.5)$$

The above results are quite common in harmonic oscillators: each oscillator has two degrees of freedom, and they oscillate with 90 degree phase shift with each other. To further generalize the result, we introduce normalized position x and normalized momentum p such that the potential energy is given by $\hbar\omega x^2$ and the kinetic energy is given by $\hbar\omega p^2$, i.e., Eq. (1.4) becomes

$$E = \hbar\omega(x^2 + p^2). \quad (1.6)$$

Specifically, x and p are given by

$$x = \sqrt{\frac{k}{2\hbar\omega}} X, \quad (1.7)$$

$$p = \frac{1}{\sqrt{2m\hbar\omega}} P, \quad (1.8)$$

respectively. Since $\hbar\omega$ is the energy unit of quantum harmonic oscillators, such normalization drastically simplifies the notation and therefore they are often used. We also introduce normalized complex amplitude

$$a = \sqrt{\frac{k}{2\hbar\omega}} A \quad (1.9)$$

so that the time evolution of x and p are given by

$$x(t) = \text{Re}[a \exp(-i\omega t)], \quad (1.10)$$

$$p(t) = \text{Im}[a \exp(-i\omega t)], \quad (1.11)$$

respectively. Also, we can define $a(t) = a \exp(-i\omega t)$, which satisfies

$$a(t) = x(t) + ip(t). \quad (1.12)$$

You can see that $x(t)$ and $p(t)$ in the x - p plane (phase space) form a circular trajectory at a frequency of ω , as shown in Fig. 1.1(a). This is a very general property of classical harmonic oscillator.

In the later chapters, you will see that in quantum harmonic oscillator $x(t)$ and $p(t)$ have certain fluctuation or uncertainty, as conceptually shown in Fig. 1.1(b). This results in the shot noise and optical amplifier noise.

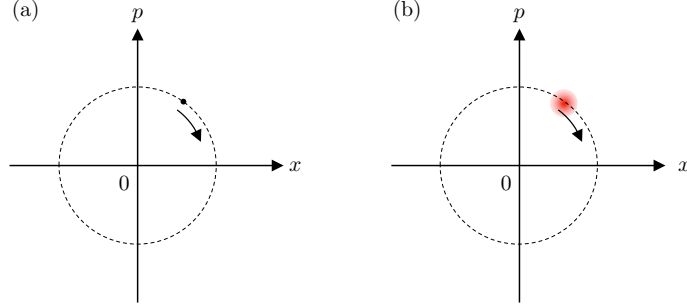


Figure 1.1: Evolution of position $x(t)$ and momentum $p(t)$ in the phase space. (a) Classical harmonic oscillator. (b) Quantum harmonic oscillator.

1.1.2 Wavefunction and operators

To introduce quantum mechanical harmonic oscillators, we start from de Broglie's relation, which assumes that a complex wavefunction with a temporal angular frequency Ω and spatial angular frequency K , which are related to the energy and the momentum by

$$E = \hbar\Omega, \quad (1.13)$$

$$P = \hbar K, \quad (1.14)$$

respectively. If we consider an exemplary wavefunction denoted as $\phi(X, t) = \exp[i(KX - \Omega t)]$, we get

$$i\hbar \frac{\partial \phi}{\partial t} = \hbar\Omega \phi = E\phi, \quad (1.15)$$

$$-i\hbar \frac{\partial \phi}{\partial X} = \hbar K \phi = P\phi. \quad (1.16)$$

Suggested by the above equations, we define the following operators:

$$\hat{E} \equiv i\hbar \frac{\partial}{\partial t}, \quad (1.17)$$

$$\hat{P} \equiv -i\hbar \frac{\partial}{\partial X}, \quad (1.18)$$

which extract the energy and the momentum, respectively, from the wavefunction.

Substituting (1.17) and (1.18) to Eq. (1.4), we obtain the Schrödinger equation for a one-dimensional harmonic oscillator given by

$$i\hbar \frac{\partial \psi(X, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(X, t)}{\partial X^2} + \frac{1}{2}kX^2 \psi(X, t), \quad (1.19)$$

where $\psi(X, t)$ is the **wavefunction** of the harmonic oscillator.

The meaning of wavefunction may be abstract at the moment, but we assume that $\int_{-\infty}^{\infty} |\psi(X, t)|^2 dX = 1$ and that $|\psi(X, t)|^2$ corresponds to the probability

density of the position of the oscillator being at X . Once we assume $\psi(X, t)$ at a certain time, we can calculate its time evolution by using Eq. (1.19) because the left-hand side is the time derivative of $\psi(X, t)$. In the later sections, we will discuss that the wavefunction contains various information such as momentum and energy.

Before doing so, let's describe Eq. (1.19) with normalized position x and normalized momentum p given by Eqs. (1.7) and (1.8). From Eqs. (1.7), (1.8), and (1.18), the operator of normalized momentum \hat{p} can be expressed by x as:

$$\hat{p} = \frac{\hat{P}}{\sqrt{2m\hbar\omega}} = \frac{-i\hbar \frac{\partial}{\partial X}}{\sqrt{2m\hbar\omega}} = -i\hbar \frac{\sqrt{\frac{K}{2\hbar\omega}} \frac{\partial}{\partial x}}{\sqrt{2m\hbar\omega}} = -\frac{i}{2} \frac{\partial}{\partial x}. \quad (1.20)$$

Therefore, Eq. (1.19) can be simplified as

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} \frac{k}{2\hbar\omega} \frac{\partial \psi}{\partial x} + \frac{1}{2} k \frac{2\hbar\omega}{k} x^2 \psi = \hbar\omega \left(-\frac{1}{4} \frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) \\ &= \hbar\omega (\hat{x}^2 + \hat{p}^2) \psi \equiv \hat{H} \psi, \end{aligned} \quad (1.21)$$

where we introduced the normalized position operator $\hat{x} = x$, and an operator $\hat{H} = \hbar\omega(\hat{x}^2 + \hat{p}^2)$ called Hamiltonian. If $\psi(x, t)$ is known at a certain t , we can calculate the time evolution of probability distribution $|\psi(x, t)|^2$.¹

1.1.3 Quantum state and bra-ket notation

So far, we assumed that ψ is a wavefunction as a function of x and t . As mentioned before, the wavefunction contains information not only on the position but also the momentum and the energy. We will see that, the wavefunction can be described as a function of momentum, or that of energy. The Schrödinger equation of a normalized harmonic oscillator (Eq. (1.21)) tells us that the time derivative of a wavefunction is given by applying the Hamiltonian to the wavefunction. This relationship holds even when the wavefunction is expressed not by position but by momentum or energy because we can express the Hamiltonian in corresponding forms. Here we introduce **quantum state**, which is a generalized version of wavefunction.

To describe the idea of quantum state, let's consider a wavefunction as a vector. This is done by

$$\psi(x) = \int_{-\infty}^{\infty} \psi(x_0) \delta(x - x_0) dx_0, \quad (1.22)$$

where $\psi(x)$ is expanded as the linear combination of a set of basis that are consisted of the delta function² $\delta(x - x_0)$ for various x_0 . The basis can be

¹Note that we implicitly assume that $\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1$. Therefore, $\psi(X, t)$ in Eq. (1.19) and $\psi(x, t)$ in Eq. (1.21) should be normalized differently, and hence should have different values at corresponding X and x .

²The delta function $\delta(x)$ gives ∞ at $x = 0$ and 0 at $x \neq 0$, and $\int_{-\infty}^{\infty} \delta(x) dx = 1$. $\delta(x)$ can be defined in several ways but one of them is $\delta(x) = \lim_{\varepsilon \rightarrow +0} \text{rect}(x/\varepsilon)/\varepsilon$, where $\text{rect}(x) = 1$ for $-1/2 < x < 1/2$ and 0 for others.

transformed by using unitary transformation: If we have a set of orthonormal bases, we can expand some vector with the bases by taking the inner product.

For example, if we take the inner product of $\psi(x)$ and a basis $\delta(x - x_0)$, we get

$$\int_{-\infty}^{\infty} \delta(x - x_0) \psi(x) dx = \psi(x_0), \quad (1.23)$$

and therefore we can recover the wavefunction at $x = x_0$. Another exemplary basis is

$$\phi_p(x) = \frac{1}{\sqrt{\pi}} e^{2ipx}, \quad (1.24)$$

which satisfies

$$\hat{p}\phi_p(x) = -\frac{i}{2} \frac{\partial}{\partial x} \frac{1}{\sqrt{\pi}} e^{2ipx} = p\phi_p(x), \quad (1.25)$$

indicating that $\phi_p(x)$ has a momentum p . Furthermore, since

$$\int_{-\infty}^{\infty} \phi_{p_1}^*(x) \phi_{p_2}(x) dx = \delta(p_1 - p_2), \quad (1.26)$$

$\phi_p(x)$ is a set of orthonormal basis.³ Therefore, by taking the inner product with $\phi_p(x)$, we can express the wavefunction as the linear combination of $\phi_p(x)$ for various p . In this way, the basis of a wavefunction can be changed by taking inner product.

Quantum state is a generalized version of wavefunction, and allows us to describe the inner product and the effect of operators regardless of the choice of a set of basis.

Quantum state is described by a ket $|\psi\rangle$. Inner product of two kets $|\phi\rangle$ and $|\psi\rangle$ is expressed as $\langle\phi|\psi\rangle$.

Quantum mechanics often considers commutation relation between \hat{x} and \hat{p} , which is given by

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = -\frac{i}{2} \left(x \frac{\partial}{\partial x} - \frac{\partial}{\partial x} x \right) = -\frac{i}{2} \left(x \frac{\partial}{\partial x} - x \frac{\partial}{\partial x} - 1 \right) = \frac{i}{2}. \quad (1.27)$$

$$aaa \quad (1.28)$$

$$\hat{H} = \hbar\omega(\hat{x}^2 + \hat{p}^2) \quad (1.29)$$

$$\frac{k}{2}x^2 \equiv \frac{1}{2}(\omega q)^2 \quad (1.30)$$

$$\frac{m}{2} \left(\frac{dx}{dt} \right)^2 \equiv \frac{1}{2}p^2 \quad (1.31)$$

³ $\int_{-\infty}^{\infty} \phi_{p_1}^*(x) \phi_{p_2}(x) dx = \int_{-\infty}^{\infty} \pi^{-1} e^{-2i(p_1 - p_2)x} dx =$
 $\lim_{X \rightarrow \infty} \pi^{-1} \int_{-\infty}^{\infty} e^{-(x/X)^2} e^{-2i(p_1 - p_2)x} dx$ to be revised later.

1.1.4 Wavefunction and energy eigenstates**1.1.5 Fock representation****1.1.6 Position representation****1.1.7 Momentum representation****1.2 Measurement of observables****1.2.1 Expectation value****1.2.2 Expectation of variance****1.3 Multimode quantum states****1.4 Summary**

Chapter 2

Evolution of quantum states

2.1 Schorödinger picture

2.2 Heisenberg picture

2.3 Unitary transformation of quantum states

2.3.1 Time evolution

2.3.2 Displacement

2.3.3 Mode mixing

2.3.4 Single-mode squeezing

2.3.5 Two-mode squeezing

2.4 Summary

Chapter 3

Quantization of light

3.1 Mode decomposition of electromagnetic waves

3.1.1 Time-frequency mode

3.1.2 Spatial mode

3.1.3 Polarization

3.2 Operator notation of electromagnetic waves

3.3 Summary

Chapter 4

Representative quantum states

- 4.1 Number states
- 4.2 Superposition states
- 4.3 Coherent states
- 4.4 Squeezed states
- 4.5 Two-mode squeezed states
 - 4.5.1 EPR state
- 4.6 Summary

Chapter 5

Control of quantum states of light

5.1 Mode mixing

5.1.1 Beamsplitter

5.1.2 Waveplates

5.1.3 Optical loss

5.1.4 Fourier transform

5.2 Parametric amplification

5.2.1 Squeezing

5.2.2 Spontaneous parametric down conversion

5.2.3 Optical amplification

5.2.4 Raman scattering

5.3 Summary

Chapter 6

Quantum-optical measurement

- 6.1 Direct detection
- 6.2 Homodyne detection
- 6.3 Heterodyne detection
- 6.4 Preamplification
- 6.5 Quantum teleportation
- 6.6 Summary

Appendix A

Appendix

A.1 Bra-ket notation

A.2 Creation and annihilation operators

A.3 Pure states and mixed states

A.4 Wigner function

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1\end{aligned}\tag{A.1}$$

This is a simple calculation [1].

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Bibliography

- [1] D. Adams. *The Hitchhiker's Guide to the Galaxy*. San Val, 1995.