where the products in the right-hand side are understood as the products of the actions rather than as the matrix products. Furthermore, the intersection of the groups $\mathrm{Spin}_{10}^{+}(F)$ and F^{16} is trivial: the action of $\mathrm{Spin}_{10}^{+}(F)$ preserves the decomposition $\mathbb{J} = \mathbb{J}_1^c \oplus \mathbb{J}_{16}^{AB} \oplus \mathbb{J}_{10}^{abC}$, while any non-trivial action of the elementary abelian group F^{16} fails to do so. Indeed, a general element in F^{16} has the form $M'_x \cdot L''_y$ for some $x, y \in \mathbb{O}$ and it sends an Albert vector $(a, b, c \mid A, B, C)$ to

$$(a, b, c + a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(By) + \mathrm{T}(\bar{x}A) + \mathrm{T}(\bar{x}\overline{C}y) \mid A + \overline{C}y + bx, B + a\bar{y} + \bar{x}\overline{C}, C).$$

So, we have shown that the actions of the elements M'_x, L''_x, M_x, L_x on $\mathbb{J}$ generate a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$, as x ranges through a split algebra $\mathbb{O}$.

Next, we consider the white point $\langle v \rangle$ spanned by our white vector. The stabiliser in $\mathrm{SE}_6(F)$ of $\langle v \rangle$, where $v = (0, 0, 1 \mid 0, 0, 0)$, maps v to λv for some non-zero $\lambda \in F$. For instance, this can be achieved by the elements

$$P'_{u^{-1}} = \mathrm{diag}(1_{\mathbb{O}}, u^{-1}, u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u \end{bmatrix} \tag{3.36}$$

with u being an invertible octonion of arbitrary norm. Indeed, any such element $P'_{u^{-1}}$ sends $(0, 0, 1 \mid 0, 0, 0)$ to $(0, 0, \mathrm{N}(u) \mid 0, 0, 0)$ and since $\mathrm{N}(u)$ can be any non-zero field element, we get an abelian group $F^{\times}$ on top of the vector stabiliser. This finishes the proof of the main theorem in this section.

Now, since the vector stabiliser is generated by the actions of M_x, L_x, M'_x, L''_x on $\mathbb{J}$, and the subgroup of $\mathrm{SE}_6(F)$ generated by $M_x, M'_x, M''_x, L_x, L'_x, L''_x$ acts transitively on the white points, we make the following conclusion.

Theorem 3.3.17. *The group $\mathrm{SE}_6(F)$ is generated by the actions of M_x, M'_x, M''_x and L_x, L'_x, L''_x on $\mathbb{J}$ as x ranges through $\mathbb{O}$.*

3.4 Some related geometry

In this section we are interested in some of the underlying geometry related to white points. Consider first a 10-dimensional space $\mathbb{J}_{10}^{abC}$ and note that it contains only white and grey vectors. In this section we are interested in finding the stabiliser of $\mathbb{J}_{10}^{abC}$, discovering some of its properties, and also finding the joint stabiliser of such a 10-space and a white point. Note that throughout the whole section $\mathbb{O}$ is a split octonion algebra.

3.4.1 The stabiliser in $\mathrm{SE}_6(F)$ of $\mathbb{J}_{10}^{abC}$

The following lemma helps to get an idea what the stabiliser we are looking for can be.

Lemma 3.4.1. *The stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *contains a subgroup of shape*

$$F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}. \tag{3.37}$$

Proof. We take an arbitrary vector $(a, b, 0 \mid 0, 0, C)$ in $\mathbb{J}_{10}^{abC}$ and look how the elements M_x, M'_x, M''_x, L_x, L'_x, and L''_x act on it:

$$\begin{aligned}
M_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), 0 \mid 0, 0, C + ax),\\
M'_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, a\mathrm{N}(x) \mid bx, \bar{x}\overline{C}, C),\\
M''_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, 0 \mid 0, 0, C),\\
L_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a + b\mathrm{N}(x) + \mathrm{T}(Cx), b, 0 \mid 0, 0, C + b\bar{x}),\\
L'_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, 0 \mid 0, 0, C),\\
L''_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, a\mathrm{N}(x) + \mathrm{T}(Bx) \mid \overline{C}x, a\bar{x}, C).
\end{aligned}$$

It is visibly clear now that the elements M_x, L_x, M''_x, L'_x preserve $\mathbb{J}_{10}^{abC}$. We have been in a similar situation before (Theorem 3.3.8), so we just note here that the abelian group F^{16} generated by the actions of M''_x and L'_x is different from the one generated by M'_x and L''_x in the theorem we refer to. Thus, the stabiliser of $\mathbb{J}_{10}^{abC}$ is at least a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$.

Finally, we notice that each of the elements P_u, P'_u, P''_u preserve $\mathbb{J}_{10}^{abC}$. It is straightforward to see that for any invertible octonion u, $P_u \cdot P'_u \cdot P''_u$ acts on $\mathbb{J}$ as the identity matrix. Therefore we only consider the action on $\mathbb{J}_{10}^{abC}$ of two of them, say P_u and P'_u, and since P_u acts on $\mathbb{J}$ as $M_{u-1} \cdot L_1 \cdot M_{u^{-1}-1} \cdot L_{-u}$, we conclude that they represent the elements of $\mathrm{Spin}_{10}^{+}(F)$, which we already have as a part of the stabiliser, so it is enough to consider the elements P'_u.

Note that the elements P'_u, where u is an arbitrary invertible octonion, preserve $\mathbb{J}_{10}^{abC}$:

$$P'_u : (a, b, 0 \mid 0, 0, C) \mapsto (a, b\mathrm{N}(u), c\mathrm{N}(u)^{-1} \mid \bar{u}A\bar{u}\mathrm{N}(u)^{1}, uBN(u)^{-1}, Cu),$$

so we get $F^\times$ on top of the stabiliser, and the result follows. $\square$

The following theorem strengthens this result: we prove that the stabiliser of $\mathbb{J}_{10}^{abC}$ is precisely a group of shape $F^{16}{:}\mathrm{Spin}_{10}^+(F).F^\times$.

Theorem 3.4.2. *The stabiliser in $\mathrm{SE}_6(F)$ of $\mathbb{J}_{10}^{abC}$ is a subgroup of shape*

$$F^{16}{:}\mathrm{Spin}_{10}^+(F).F^\times, \tag{3.38}$$

generated by the actions on $\mathbb{J}$ of the elements M_x, M''_x, L_x, L'_x as x ranges through $\mathbb{O}$, and P'_u as u ranges through invertible octonions in $\mathbb{O}$.

Proof. Consider the white point W spanned by $(0, 0, 1 \mid 0, 0, 0)$. We are interested in the joint stabiliser of $\mathbb{J}_{10}^{abC}$ and W. Theorem 3.3.8 tell us that the stabiliser in $\mathrm{SE}_6(F)$ of W has the shape $G_W = F^{16}{:}\mathrm{Spin}_{10}^+(F).F^\times$, and by stabilising $\mathbb{J}_{10}^{abC}$ we go down to $H = \mathrm{Spin}_{10}^+(F).F^\times$. The normal subgroup $T_1 \cong F^{16}$ of G_W is a left (or right) transversal of H in G_W. It is easy to see that no non-trivial element of T_1 stabilises $\mathbb{J}_{10}^{abC}$. Indeed, a general element in T_1 has the form $M'_x \cdot L''_y$ for some $x, y \in \mathbb{O}$ and it sends $(a, b, 0 \mid 0, 0, C) \in \mathbb{J}_{10}^{abC}$ to

$$(a, b, a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(\bar{x}\overline{C}y) \mid \overline{C}y + bx, a\bar{y} + \bar{x}\overline{C}, C).$$

Therefore, such an element preserves $\mathbb{J}_{10}^{abC}$ if and only if the following conditions hold for arbitrary a, b, and C:

$$\left.\begin{aligned} a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(\bar{x}\overline{C}y) &= 0,\\ \overline{C}y + bx &= 0,\\ a\bar{y} + \bar{x}\overline{C} &= 0. \end{aligned}\right\}$$

In particular, if we take $C = 0$, $b = 1$ then we obtain $x = 0$, and when $C = 0$, $a = 1$, we get $y = 0$. Next, every element of H stabilises $\mathbb{J}_{10}^{abC}$, and it confirms that

$H = \mathrm{Spin}_{10}^{+}(F).F^{\times}$.

Let T_2 be the subgroup of $\mathrm{SE}_6(F)$ generated by the actions on $\mathbb{J}$ of M_x'' and L_y' as x, y range through $\mathbb{O}$. Consider the 26-dimensional space $\mathbb{J}_{26}^{abABC}$ spanned by the 17-spaces corresponding to the white vectors in $\mathbb{J}_{10}^{abC}$. Let $(a, b, c \mid A, B, C)$ be a white vector outside $\mathbb{J}_{26}^{abABC}$, i.e. with $c \neq 0$. The whiteness conditions imply that such a vector has the form $(B\overline{B}/c, A\overline{A}/c, c \mid A, B, \overline{B}\,\overline{A}/c)$. These span a 16-space, and so T_2 acts sharply transitively on such vectors and points spanned by them. Therefore, the full stabiliser of $\mathbb{J}_{10}^{abC}$ is indeed $F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}$, understood as $F^{16}{:}\Omega_{10}^{+}(F).F^{\times}$ in characteristic 2. $\square$

Next, we investigate the orbits of the stabiliser of $\mathbb{J}_{10}^{abC}$ on white vectors and white points. First, consider white vectors in $\mathbb{J}_{10}^{abC}$. An arbitrary non-zero vector $(a, b, 0 \mid 0, 0, C)$ is white if and only if $ab - C\overline{C} = 0$, and the action of $\Omega_{10}^{+}(F)$ is transitive on such vectors. Therefore, the stabiliser of $\mathbb{J}_{10}^{abC}$ is transitive on white vectors in $\mathbb{J}_{10}^{abC}$, and on the white points spanned by them.

Suppose now that $(a, b, c \mid A, B, C)$ is a white vector such that $(c, A, B) \neq (0, 0, 0)$. We consider two cases. First, assume $c \neq 0$. The element $M''_{-c^{-1}B}$ maps our vector to $(a - c^{-1}\mathrm{N}(B), b, c \mid A, 0, C - c^{-1}\overline{B}\,\overline{A})$, and since the latter is white, we have $a - c^{-1}\mathrm{N}(B) = 0 = C - c^{-1}\overline{B}\,\overline{A}$, so our new vector is of the form $(0, b, c \mid A, 0, 0)$. Similarly, we act on it by $L_{-c^{-1}\overline{A}}$ to obtain $(0, b - c^{-1}\mathrm{N}(A), c \mid 0, 0, 0)$, and since this vector is also white, we have $b - c^{-1}\mathrm{N}(A) = 0$, so the resulting vector is of the form $(0, 0, c \mid 0, 0, 0)$, $c \neq 0$.

Second, we consider the case $c = 0$, so we start with $(a, b, 0 \mid A, B, C)$ with $(A, B) \neq (0, 0)$. The whiteness conditions are

$$\left.\begin{aligned} A\overline{A} &= 0, \\ B\overline{B} &= 0, \\ C\overline{C} &= ab, \\ AB &= 0, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned}\right\} \tag{3.39}$$

If $(a, b) = 0$, then we choose a suitable M_x, M_x'', L_x, or L_x' to map our vector to the vector of the form $(a', b', 0 \mid A', B', C')$ with $(a', b') \neq (0, 0)$. Thus, we may assume

$(a, b) \neq (0, 0)$.

We again distinguish two cases. If $a \neq 0$, then we act on $(a, b, 0 \mid A, B, C)$ by $M_{-a^{-1}C}$ to obtain $(a, 0, 0 \mid 0, B, 0)$. Next, we act by M''_y to get $(a + \mathrm{T}(\bar{y}B), 0, 0 \mid 0, B, 0)$, and choosing a suitable y we obtain $(0, 0, 0 \mid 0, B, 0)$.

If on the other hand $b \neq 0$, the action by $L_{-b^{-1}\bar{C}}$ maps $(a, b, 0 \mid A, B, C)$ to $(0, b, 0 \mid A, 0, 0)$ and similarly acting by a suitable L'_y, we obtain $(0, 0, 0 \mid A, 0, 0)$. Recall that the duality element δ preserves $\mathbb{J}_{10}^{abC}$, so it is enough to consider vectors $(0, 0, 0 \mid 0, 0, 0)$ with $A \in \mathbb{O}$. By choosing a copy of $\Omega_8^+(F)$ generated by the elements P'_u with u being an octonion of norm one, we can map $(0, 0, 0 \mid A, 0, 0)$ to $(0, 0, 0 \mid e_0, 0, 0)$.

Finally, it is impossible to map $(a, b, c \mid A, B, C)$ with $c \neq 0$ to $(a, b, 0 \mid A, B, C)$ using any of the elements M_x, M''_x, L_x, and L'_x, so these two belong to different orbits. In other words, we have shown the following result.

Theorem 3.4.3. *The stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *has three orbits on white vectors:*

(i) vectors in $\mathbb{J}_{10}^{abC}$*,*

(ii) images of $(0, 0, 1 \mid 0, 0, 0)$ *under the action of stabiliser,*

(iii) images of $(0, 0, 0 \mid e_0, 0, 0)$ *under the action of stabiliser.*

3.5 Simplicity of $\mathrm{E}_6(F)$

The construction we have obtained also allows us to show that the group $\mathrm{E}_6(F)$ is indeed simple without any references to Lie theory. The classical way of showing the simplicity of certain groups is the following lemma.

Lemma 3.5.1 (Iwasawa)**.** *If G is a perfect group acting faithfully and primitively on a set Ω, and the point stabilizer H has a normal abelian subgroup A whose conjugates generate G, then G is simple.*

First, we show that the subgroup of $\mathrm{SE}_6(F)$ stabilising all the white points simultaneously acts on $\mathbb{J}$ by scalar multiplications, and hence the action of $\mathrm{E}_6(F)$ on the set of white points is faithful.