On an octonionic construction of the groups of type E_6 and ${}^2\mathrm{E}_6$

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Abstract

ABSTRACT HERE

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Contents

Contents **1**

1 Introduction **3**

1.1 Historical notes . . . 3
1.2 Motivation . . . 5
1.3 Notation . . . 6

2 Octonions **7**

2.1 Composition algebras . . . 7
2.1.1 Quadratic and Bilinear Forms . . . 7
2.1.2 Isometries and Witt's Lemma . . . 8
2.1.3 Definition of a composition algebra . . . 10
2.2 Conjugation and inverses . . . 12
2.3 Alternative laws and Moufang identities . . . 13
2.4 Octonion algebras . . . 14
2.5 Split octonion algebras . . . 16
2.6 A basis for the split octonions . . . 17
2.7 Centre of an octonion algebra . . . 19

3 Groups of type E_6 **21**

3.1 Albert vectors . . . 21
3.1.1 Albert space $\mathbb{J}$ . . . 21
3.1.2 Dickson–Freudenthal determinant and $\mathrm{SE}_6(F)$ . . . 22
3.2 Some elements of $\mathrm{SE}_6(F)$ . . . 23
3.3 The white points . . . 26

3.3.1 The mixed form and the white vectors 26
3.3.2 Action of $\mathrm{SE}_6(F)$ on white points 29
3.3.3 The stabiliser of a white point 34
3.4 Some related geometry 50
3.4.1 The stabiliser in $\mathrm{SE}_6(F)$ of $\mathbb{J}_{10}^{abC}$ 51
3.4.2 Joint stabilisers of $\mathbb{J}_{10}^{abC}$ and a white point 54
3.5 Simplicity of $\mathrm{E}_6(F)$ 55
3.6 Case of a finite field 57
3.7 Arbitrary octonion algebras 60

4 Groups of type $^2\mathrm{E}_6$ 63

4.1 Quadratic field extensions 63
4.2 Spaces with two forms 64
4.3 Hermitean form in $\mathbb{J}$ and the group $^2\mathrm{SE}_6^K(F)$ 65
4.4 Some elements of $^2\mathrm{SE}_6^K(F)$ 66
4.5 Action of $^2\mathrm{SE}_6^K(F)$ on white points 72
4.5.1 Orbits of $^2\mathrm{SE}_6^K(F)$ on white points 73
4.5.2 The stabiliser of type 3 vector 73
4.5.3 The stabiliser of type 1 vector 76
4.5.4 The stabiliser of type 2 vector 78
4.6 Case of a finite field 82
4.6.1 White vectors in $\mathbb{J}_8^C$ 82
4.6.2 White vectors in $\mathbb{J}_{16}^{BC}$ 86

A: Some properties of $\Omega_{2m}(F,Q)$ 88

B: $\Omega_4^+(F) \cong \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$ 98

C: $\Omega_4^{-,K}(F) \cong \mathrm{PSL}_2(K)$ 99

D: Magma code 100

Chapter 1

Introduction

1.1 Historical notes

One may think of the exceptional groups of Lie type as of the wonders of our world which are still very far from our understanding. Yet these groups can be found almost everywhere in contemporary mathematics and physics. For instance, physicists have some high hopes that exceptional groups will somehow reveal themselves in the theory of quantum mechanics.

The history of the exceptional groups began roughly at the turn of the 19th and 20th centuries. Elie Cartan (1869–1951) first completely classified simple Lie algebras, and then by determining the real forms of complex algebras he classified the simple real Lie algebras. A rigorous and highly readable historical account of this classification can be found in [**?**]. Apart from the four infinite families of "classical" Lie algebras and Lie groups corresponding to them, there are five isolated ones, known to the world as G_2, F_4, E_6, E_7, and E_8.

In this thesis we devote our interest to the construction of the groups of type E_6. This goes back more than a hundred years to the works of Leonard Eugene Dickson (1874–1954), who characterised E_6 as a 27-dimensional group with an invariant cubic form [**?**, **?**]. Dickson also managed to write down a large number of group generators. He uses 27 coordinates labelled x_i, y_i, and $z_{ij} = -z_{ji}$ where $i, j \in \{1, 2, 3, 4, 5, 6\}$

and $i \neq j$. His group is defined as the stabiliser of a cubic form with 45 terms

$$\sum_{i \neq j} x_i y_j z_{ij} + \sum z_{ij} z_{kl} z_{mn}, \tag{1.1}$$

where the second sum is taken over all partitions $\{\{i,j\},\{k,l\},\{m,n\}\}$ of $\{1,2,3,4,5,6\}$, ordered so that $(i\ j\ k\ l\ m\ n)$ is an even permutation.

In 1955 Claude Chevalley (1909–1984) obtained a uniform construction of what are now known as Chevalley groups [?]. Chevalley's construction included E_6 albeit he constructs the 78-dimensional representation whereas Dickson obtained the smallest 27-dimensional representation.

There was another major breakthrough following Dickson's construction of E_6: in 1932 a new subatomic particle called neutron was discovered, which indicated the need for a new algebraic underpinning of quantum mechanics.

In the same 1932 a German physicist Pascual Jordan[1] (1902–1980) introduced Jordan algebras as a tool which was supposed to illuminate the behaviour of observable particles in quantum mechanics.

In 1934, Pascual Jordan, John von Neumann (1903–1957), and Eugene Wigner (1902–1995) introduced Jordan algebras and octonions into physics [?]. Although their attempt to formulate a new quantum mechanics was unsuccessful, Jordan algebras turned out to have astonishing connections with many areas of mathematics. The by-product of physical investigation was the discovery of 27-dimensional exceptional Jordan algebra also known as Albert algebra, named after Abraham Adrian Albert (1905–1972).

Although being abandoned by physicists, Albert algebra turned out to be of high interest to mathematicians. This 27-dimensional algebra consists of 3×3 Hermitian matrices written over octonions, with the multiplication given by

$$X \circ Y = \frac{1}{2}(XY + YX). \tag{1.2}$$

Hans Freudenthal (1905–1990) showed that the stabiliser of a certain cubic form on

[1] Naming the first-born son Pascual was a family tradition. Furthermore, while in English 'Jordan' is pronounced with the first sound being the same as in the word 'judge', in German the first sound is the same as in the word 'yolk'.

this 27-dimensional space is the group of type E_6 [?]. Next, George Seligman (born 1927) proved that the automorphism group of a split Albert algebra over any field F is isomorphic to the Chevalley group $F_4(F)$ [?]. Nathan Jacobson (1910–1999), inspired by the works of Dickson, Chevalley, and Seligman, studied the automorphism group of an Albert algebra, and the stabiliser of the determinant over the fields of characteristic not two or three in a series of papers [?, ?, ?]. For instance, he proved that if an Albert algebra contains nilpotent elements, then the group is simple. It must have been implicit that the determinant of the elements in an Albert algebra is essentially the same as Dickson's cubic form, although Jacobson does not refer to Dickson. Moreover, although cases of characteristic 2 and 3 were of no problem to Dickson, they were still problematic in Jacobson's construction.

Michael Aschbacher (born 1944) also addresses the construction of groups of type E_6 without mentioning Albert algebra or octonions at all. Though of fundamental importance, Aschbacher's construction is of rather abstract nature and some of the aspects are not burdened with things like proofs. Thus, some of the structural questions require further research.

In his famous book, 'The Finite Simple Groups' [?], and also in subsequent preprint [?], Robert Arnott Wilson (born 1958) sketches the construction of finite simple groups $F_4(q)$, $E_6(q)$, and ${}^2E_6(q)$. The purpose of this thesis is, having this sketch as a basis, to fill in the major gaps and obtain a complete and self-contained construction of the groups $E_6(F)$ and ${}^2E_6(F)$.

1.2 Motivation

In the late 1980s the problem of classifying maximal subgroups came into prominence. One of the most notable examples is Kay Magaard's unpublished thesis [?], which deals with the maximal subgroups of finite simple groups $F_4(q)$ where the characteristic is not 2 or 3.

– Why is this important? – What new ideas does it uncover? – Practical advantages

1.3 Notation

Chapter 2

Octonions

2.1 Composition algebras

2.1.1 Quadratic and Bilinear Forms

Let V be a vector space over a field F. We define a *quadratic form* Q on V to be a map $Q : V \to F$ such that

(i) $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$ and $\lambda \in F$;

(ii) the form $\langle \cdot, \cdot \rangle : V \times V \to K$, defined by

$$\langle u, v \rangle = Q(u + v) - Q(u) - Q(v), \tag{2.1}$$

is bilinear. We usually refer to $\langle \cdot, \cdot \rangle$ as *the polar form of* Q.

From (2.1) we readily see that the form $\langle \cdot, \cdot \rangle$ is symmetric, i.e. $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$. We also observe that for all $v \in V$ we have

$$\langle v, v \rangle = 2Q(v), \tag{2.2}$$

It follows that in case $\mathrm{char}(F) = 2$ we get $\langle v, v \rangle = 0$ for all v, and the quadratic form carries strictly more information than the associated bilinear form. In all other characteristics, hovewer, we get $Q(v) = \frac{1}{2}\langle v, v \rangle$.

We say that two non-zero vectors $u, v \in V$ are *orthogonal*, if $\langle u, v\rangle = 0$. As already mentioned, this relation is symmetric. Now if U is any subspace of V (and even if it is just a subset), we define its *orthogonal complement* $U^\perp$ to be

$$U^\perp = \left\{ v \in V \,\middle|\, \langle u, v\rangle = 0 \text{ for all } u \in U \right\}. \tag{2.3}$$

A non-zero vector $v \in V$ is called *isotropic* if $Q(v) = 0$, otherwise v is *anisotropic*. Sometimes we also say that $Q(v)$ is the *norm* of v. Now, the quadratic form Q is isotropic if there exists an isotropic vector in V. The *radical* of $\langle\cdot,\cdot\rangle$ is $V^\perp$, and $\langle\cdot,\cdot\rangle$ is *non-degenerate* if the radical is trivial, or, otherwise speaking, if

$$\langle v, u\rangle = 0 \text{ for all } u \in V \text{ implies that } v = 0. \tag{2.4}$$

Similarly, the *radical* of Q is the subset of the radical of $\langle\cdot,\cdot\rangle$, consisting of isotropic vectors, i.e.

$$\mathrm{rad}_V(Q) = \{\, v \in V \,|\, \langle v, u\rangle = 0 \text{ for all } u \in V,\ Q(v) = 0 \,\}. \tag{2.5}$$

If the radical of the form Q is trivial, then Q is said to be *non-singular*. Throughout this thesis we will be mostly interested in non-singular quadratic and non-degenerate bilinear forms. If U is a subspace of V and the restriction of $\langle\cdot,\cdot\rangle$ on U is non-degenerate, then $V = U \oplus U^\perp$, and the restriction of $\langle\cdot,\cdot\rangle$ on $U^\perp$ is also non-degenerate. A subspace U of V consisting entirely of isotropic vectors is called *totally isotropic*.

2.1.2 Isometries and Witt's Lemma

Let V_1, V_2 be vector spaces over fields F_1 and F_2 respectively, with non-singular quardatic forms Q_1 and Q_2. Denote by $\langle\cdot,\cdot\rangle_i$ the polar form of Q_i $(i = 1, 2)$. Suppose $\sigma : F_1 \to F_2$ is a field isomorphism. A map $s : V_1 \to V_2$, satisfying

$$Q_2(v^s) = \lambda_s Q_1(v)^\sigma \quad (v \in V_1), \tag{2.6}$$

where $\lambda_s \in F_2^\times$, is called a *σ-similarity.* The scalar λ_s is known as the *multiplier* of s. Using the definition of polar form, we obtain $\langle u^s, v^s\rangle_2 = \lambda_s \langle u, v\rangle_1^\sigma$, so s is a bijection. If $\lambda_s = 1$, then s is called a *σ-isometry*. In the case when a σ-similarity (or σ-isometry) between two spaces V_1 and V_2 exists, we say that V_1 and V_2 are σ-similar (or σ-isometric). If σ is the identity map, then σ-similarity (or σ-isometry) is simply called *similarity* (or *isometry*).

A key result about isometries, which also plays an important rôle in the study of the geometry of spaces with quadratic forms, is Witt's Lemma (also known as Witt's Theorem).

Theorem 2.1.1 (Witt's Lemma)**.** *If V_1, V_2 are two σ-isometric vector spaces of finite dimension with non-singular quadratic forms Q_1 on V_1 and Q_2 on V_2, then every σ-isometry between a subspace of V_1 and a subspace of V_2 extends to a σ-isometry between V_1 and V_2.*

If V is a vector space over F with a non-singular quadratic form Q, then an isometry from V onto itself is called an *orthogonal transformation* of V with respect to Q. These orthogonal transformations form the *(general) orthogonal group* $\mathrm{GO}(V, Q)$. Now suppose $s : V \to V$ is an invertible linear transformation such that $Q(v^s) = Q(v)$ for all $v \in V$ (and thus $\langle u^s, v^s \rangle = \langle u, v \rangle$ for all $u, v \in V$). Denote $n = \dim_F(V)$ and pick a basis $\mathcal{B} = \{ v_1, ..., v_n \}$. Then with respect to $\mathcal{B}$, s can be represented by an $n \times n$ matrix $[s]_{\mathcal{B}}$. The determinant of the resulting matrix is independent of the choice of basis, so there is a group homomorphism $\det : \mathrm{GO}(V, Q) \to F^\times$. Orthogonal transformations have determinant ± 1. In case of characteristic 2 we define the *quasideterminant* $\mathrm{qdet} : \mathrm{GO}(V, Q) \to \mathbb{F}_2$ to be the map

$$\mathrm{qdet} : g \mapsto \dim_F(\mathrm{Im}\,(\mathrm{id} - g)) \mod 2. \tag{2.7}$$

The subgroup $\mathrm{SO}(V, Q)$ of $\mathrm{GO}(V, Q)$ is the kernel of the (quasi-)determinant map. The group $\mathrm{SO}(V, Q)$ is referred to as *special orthogonal group* or *rotation* group of V with respect to Q.

Note that not every element of $\mathrm{GO}(V, Q)$ arises as a rotation. For an anisotropic vector $v \in V$ define r_v to be

$$r_v : u \mapsto u - \frac{\langle u, v \rangle}{Q(v)} v \quad (u \in V). \tag{2.8}$$

If the characteristic is not 2, then r_v is the *reflexion* in (the hyperplane orthogonal to) v. If $\mathrm{char}(K) = 2$, then r_v is the *orthogonal transvection* with centre v. For simplicity we use the word ‘reflexion’ in all cases.

We define the *spinor norm* to be a homomorphism $\mathrm{GO}(V, Q) \to F^\times/(F^\times)^2$, where $F^\times/(F^\times)^2$ is the *multiplicative group modulo squares* of F. The aforementioned homomorphism is defined as follows. Any element of $\mathrm{GO}(V, Q)$ arising as a reflexion in v is sent to the value $Q(v)$ modulo $(F^\times)^2$. This extends to a well-defined homomorphism. The subgroup $\Omega(V, Q)$ of $\mathrm{SO}(V, Q)$ is obtained as the kernel of spinor norm.

Witt’s Lemma implies that all maximal totally isotropic subspaces of V (with respect to Q) have the same dimension, which is called the *Witt index* of Q. When Q is non-singular and V is finite-dimensional, Witt index of Q can be at most $\frac{1}{2}\dim_F(V)$. Moreover, the isometry group $\mathrm{GO}(V, Q)$ acts transitively on the set of maximal totally isotropic subspaces.

2.1.3 Definition of a composition algebra

Definition 2.1.2. *A composition algebra $C = C_F$ over a field F is a (not necessarily associative) unital algebra over F which admits a non-singular quadratic form $N : C \to F$ such that the polar form of N is non-degenerate and*

$$N(xy) = N(x)N(y) \quad \textit{for all } x, y \in C. \tag{2.9}$$

The quadratic form N on C is usually called the *norm* of C, and its polar form is referred to as the *inner product*. We also denote the identity element as 1_C.

Let D be a linear subspace of C such that the restriction of $\langle \cdot, \cdot \rangle$ on D is non-degenerate. If D is closed under multiplication and contains 1_C, then it is called the *subalgebra* of C.

Let C_1, C_2 be two composition algebras over fields F_1, F_2 respectively and suppose $\sigma : F_1 \to F_2$ is a field isomorphism. A bijective σ-linear transformation $s : C_1 \to C_2$ is called a *σ-isomorphism*, if

$$(xy)^s = x^s y^s \quad \text{for all } x, y \in C_1. \tag{2.10}$$

For simplicity, if $F_1 = F_2$ and $\sigma = \mathrm{id}$, then s is called an *isomorphism*.

Definition 2.1.2 allows us to derive a number of useful equations. First of all, we find that

$$N(x) = N(1_C \cdot x) = N(1_C)N(x)$$

for all $x \in C$, so it follows that

$$N(1_C) = 1. \tag{2.11}$$

Next, for any $x_1, x_2, y \in C$ we have

$$\begin{aligned} N(x_1y + x_2y) &= N((x_1 + x_2)y) = N(x_1 + x_2)N(y) \\ &= (N(x_1) + N(x_2) + \langle x_1, x_2 \rangle)N(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} N(x_1y + x_2y) &= N(x_1y) + N(x_2y) + \langle x_1y, x_2y \rangle \\ &= N(x_1)N(y) + N(x_2)N(y) + \langle x_1y, x_2y \rangle, \end{aligned}$$

and so

$$\langle x_1y, x_2y \rangle = \langle x_1, x_2 \rangle N(y) \tag{2.12}$$

for all $x_1, x_2, y \in C$. Similarly, we obtain

$$\langle xy_1, xy_2 \rangle = N(x)\langle y_1, y_2 \rangle \tag{2.13}$$

for all $x, y_1, y_2 \in C$. Replacing y by $y_1 + y_2$ in (2.12), we obtain

$$\langle x_1y_1, x_2y_2 \rangle + \langle x_1y_2, x_2y_1 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \tag{2.14}$$

for all $x_1, x_2, y_1, y_2 \in C$.

Any composition algebra is quadratic, that is, every element satisfies a certain quadratic equation.

Proposition 2.1.3. *Every element x of a composition algebra C satisfies the following equation:*

$$x^2 - \langle x, 1_C \rangle x + N(x) \cdot 1_C = 0. \tag{2.15}$$

In the case when x is not a scalar multiple of 1_C, this is the minimal equation for

x. For all $x, y \in C$ we have

$$xy + yx - \langle x, 1_C \rangle y - \langle y, 1_C \rangle x + \langle x, y \rangle \cdot 1_C = 0. \tag{2.16}$$

For example, if x, y are orthogonal to 1_C and $\langle x, y \rangle = 0$, then $xy = -yx$, but most importantly we have the following corollary.

Corollary 2.1.4. *The norm N in a composition algebra C is uniquely determined by the algebra structure. Any σ-isomorphism of composition algebras is always a σ-isometry.*

Any composition algebra is *power associative*, i.e. for all $x \in C$ and $i, j \geqslant 1$, we have

$$x^i x^j = x^{i+j}. \tag{2.17}$$

2.2 Conjugation and inverses

We define *conjugation* in a composition algebra C to be the mapping $\bar{} : C \to C$ defined by

$$\bar{x} = \langle x, 1_C \rangle \cdot 1_C - x \quad (x \in C). \tag{2.18}$$

Note that geometrically speaking, the map $x \mapsto \bar{x}$ is $-r_{1_C}$, where r_{1_C} is the reflexion in 1_C. We call $\bar{x}$ the *conjugate* of x. The following lemma summarises the properties of $\mathbb{O}$ related to conjugation.

Lemma 2.2.1. *For all $x, y \in C$ the following identities hold:*

(i) $x\bar{x} = \bar{x}x = N(x) \cdot 1_C$,

(ii) $\overline{xy} = \bar{y}\bar{x}$,

(iii) $\bar{\bar{x}} = x$,

(iv) $\overline{x + y} = \bar{x} + \bar{y}$,

(v) $N(x) = N(\bar{x})$,

(vi) $\langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle$.

Furthermore, we have the following important properties.

Lemma 2.2.2. *For all $x, y, z \in C$ the following identities hold:*

(i) $x(\bar{x}y) = N(x)y$,

(ii) $(x\bar{y})y = N(y)x$,

(iii) $x(\bar{y}z) + y(\bar{x}z) = \langle x, y\rangle \cdot z$,

(iv) $(x\bar{y})z + (x\bar{z})y = x \cdot \langle y, z\rangle$.

If for an element $x \in C$ we have $N(x) \neq 0$, then x is said to be *invertible*. If this is the case, then the *inverse* of x is

$$x^{-1} = N(x)^{-1}\bar{x}. \tag{2.19}$$

Lemma 2.2.3. *If $x, y \in C$ are invertible, then*

$$(xy)^{-1} = y^{-1}x^{-1}. \tag{2.20}$$

2.3 Alternative laws and Moufang identities

Composition algebras are not necessarily associative, but there are certain results which can help us with the bracketing.

Lemma 2.3.1 (Moufang Identities)**.** *For all $x, y, z \in C$, the following identities hold:*

$$\begin{aligned} x(yz)x &= (xy)(zx), \\ x(yzy) &= ((xy)z)y, \\ (xyx)z &= x(y(xz)). \end{aligned} \tag{2.21}$$

This helps us to conclude that any composition algebra C is *alternative*. That is, for every element $x \in C$ the left-multiplication by x commutes with right-multiplication by x.

Lemma 2.3.2 (Alternative Laws)**.** *For all $x, y \in C$ the following are true:*

$$\begin{aligned}(xx)y &= x(xy),\\ (yx)x &= y(xx),\\ (xy)x &= x(yx).\end{aligned} \tag{2.22}$$

Theorem 2.3.3 (Artin)**.** *The subalgebra generated by any two elements of an alternative algebra is always associative.*

2.4 Octonion algebras

The most important structural result about composition algebras is the following theorem.

Theorem 2.4.1. *The possible dimensions of a composition algebra are* 1*,* 2*,* 4*, and* 8*. Composition algebras of dimension* 1 *only occur if the characreristic of the field is not* 2*. Composition algebras of dimension* 1 *and* 2 *are associative and commutative. Those of dimension* 4 *are associtaive but not commutative, and those of dimension* 8 *are neither associative nor commutative.*

In this thesis we will be mostly interested in the 8-dimensional composition algebras. To emphasise their importance in our work, we use a separate name for them.

Definition 2.4.2. *Let F be any field. An octonion algebra $\mathbb{O} = \mathbb{O}_F$ is an 8-dimensional composition algebra, i.e. it admits a norm defined as a quadratic form $\mathrm{N} : \mathbb{O} \to F$ such that the polar form of N is non-degenerate and $\mathrm{N}(xy) = \mathrm{N}(x)\mathrm{N}(y)$ for all $x, y \in \mathbb{O}$.*

The elements of $\mathbb{O}$ are called the *octonions*. The multiplicative identity in $\mathbb{O}$ is denoted $1_{\mathbb{O}}$, and for simplicity we sometimes omit the subscript. The polar form of N is denoted by $\langle \cdot, \cdot \rangle$ as usual. Define the *trace* of an octonion to be the inner product

$$\mathrm{T}(x) = \langle x, 1_{\mathbb{O}} \rangle. \tag{2.23}$$

It is easy to see that

$$\mathrm{T}(x) \cdot 1_{\mathbb{O}} = x + \bar{x}. \tag{2.24}$$

Although we define trace through the inner product, using Lemma 2.2.1 we can derive the following important relation.

Lemma 2.4.3. *For all $x, y \in \mathbb{O}$, the following identity holds:*

$$\langle x, y\rangle = \mathrm{T}(x\bar{y}). \tag{2.25}$$

Proof. Lemma 2.2.1 tells us that for all $x \in \mathbb{O}$, $\mathrm{N}(x) \cdot 1_{\mathbb{O}} = x\bar{x}$. Polarising N as usual, we obtain

$$\begin{aligned}\langle x, y\rangle \cdot 1_{\mathbb{O}} &= \mathrm{N}(x+y) \cdot 1_{\mathbb{O}} - \mathrm{N}(x) \cdot 1_{\mathbb{O}} - \mathrm{N}(y) \cdot 1_{\mathbb{O}} \\ &= (x+y)(\bar{x}+\bar{y}) - x\bar{x} - y\bar{y} = x\bar{y} + y\bar{x} = \mathrm{T}(x\bar{y}).\end{aligned}$$

□

Proposition 2.1.3 tells us that an arbitrary element $x \in \mathbb{O}$ satisfies the equation

$$x^2 - \mathrm{T}(x) \cdot x + \mathrm{N}(x) \cdot 1_{\mathbb{O}} = 0. \tag{2.26}$$

Finally, as we know, any octonion algebra $\mathbb{O}$ is neither associative nor commutative. However, we do have the following.

Lemma 2.4.4. *If $x, y, z \in \mathbb{O}$, then $\mathrm{T}(xy) = \mathrm{T}(yx)$ and $\mathrm{T}(x(yz)) = \mathrm{T}((xy)z)$.*

Note that although trace is 3-associative, it is not possible in this case to derive generalised associativity for the trace.

Lemma 2.4.5. *For all non-zero $C \in \mathbb{O}$ the map $\mathbb{O} \to F$, $x \mapsto \mathrm{T}(Cx)$ is onto.*

Proof. This is an F-linear map, so if it is not surjective, then it is a zero map. But if $\top(Cx) = \langle C, \bar{x}\rangle = 0$ for all $x \in \mathbb{O}$, then $C = 0$ (a contradiction), since the map $x \mapsto \bar{x}$ is surjective. □

Further in this thesis we will be interested in a certain class of subalgebras of $\mathbb{O}$. We say that a subalgebra $\mathbb{S}$ of $\mathbb{O}$ is *sociable*, if for any $x, y \in \mathbb{S}$ and any $z \in \mathbb{O}$, $x(zy) = (xz)y$.

2.5 Split octonion algebras

There is an important dichotomy with respect to the structure of an octonion algebra: either $\mathbb{O}$ is a division algebra or there exists an isotropic octonion. In the latter case $\mathbb{O}$ is called a *split octonion algebra*.

If $\mathbb{O}$ is split, then the Witt index of N is 4 (section 1.8 in [**?**]). Moreover, we have the following result.

Theorem 2.5.1. *Over any given field F there is a unique split octonion algebra, up to isomorphism.*

It turns out that any isotropic octonion left- and right-annihilates a 4-dimensional subspace of a split octonion algebra $\mathbb{O}$.

Proposition 2.5.2. *Let $\mathbb{O}$ be a split octonion algebra. Then for any isotropic $x \in \mathbb{O}$, the following is true:*

$$\dim_F(\mathbb{O}x) = \dim_F(x\mathbb{O}) = 4. \tag{2.27}$$

Moreover, $\mathbb{O}x$ is the set of octonions that are right-annihilated by $\bar{x}$, and $x\mathbb{O}$ is the set of octonions that are left-annihilated by $\bar{x}$.

Proof. We prove the statement for right multiplication by x. The proof for left multiplication is essentially the same. The map

$$\begin{aligned} R_x : \mathbb{O} &\to \mathbb{O} \\ y &\mapsto yx \end{aligned}$$

is an F-linear map with $\mathrm{Im}\,(R_x) = \mathbb{O}x$, which is a totally isotropic subspace of $\mathbb{O}$. Indeed, $(yx)(\bar{x}\bar{y}) = y(x\bar{x})\bar{y} = 0$ for any $y \in \mathbb{O}$. Since N is non-singular and its polar form is non-degenerate, we conclude that $\dim_F(\mathbb{O}x) \leqslant 4$.

If $x \neq 0$ and $yx = 0$, then y is isotropic for if that were not the case, we would get $x = y^{-1}(yx) = y^{-1} \cdot 0 = 0$, a contradiction. It follows that $\dim_F(\ker(R_x)) \leqslant 4$. The Rank–Nullity theorem implies that $\dim_F(\mathbb{O}x) = \dim_F(\ker(R_x)) = 4$. $\square$

2.6 A basis for the split octonions

In this section we assume that $\mathbb{O}$ is a split octonion algebra. Theorem 2.5.1 allows us to choose a basis for $\mathbb{O}$ and to use it in our further constructuions. Otherwise speaking, we can 'redefine' split octonion algebras in the following way.

Definition 2.6.1. *If F is any field, then the split octonion algebra over F is defined as an 8-dimensional vector space $\mathbb{O} = \mathbb{O}_F$ with basis $\{ e_i \mid i \in \pm I \}$, where $I = \{ 0, 1, \omega, \overline{\omega} \}$, $\pm I = \{ \pm 0, \pm 1, \pm\omega, \pm\overline{\omega} \}$ and bilinear multiplication given by the following table.*

	e_{-1}	$e_{\overline{\omega}}$	e_{ω}	e_0	e_{-0}	$e_{-\omega}$	$e_{-\overline{\omega}}$	e_1
e_{-1}	0	0	0	0	e_{-1}	$e_{\overline{\omega}}$	$-e_{\omega}$	$-e_0$
$e_{\overline{\omega}}$	0	0	$-e_{-1}$	$e_{\overline{\omega}}$	0	0	$-e_{-0}$	$e_{-\omega}$
e_{ω}	0	e_{-1}	0	e_{ω}	0	$-e_{-0}$	0	$-e_{-\overline{\omega}}$
e_0	e_{-1}	0	0	e_0	0	$e_{-\omega}$	$e_{-\overline{\omega}}$	0
e_{-0}	0	$e_{\overline{\omega}}$	e_{ω}	0	e_{-0}	0	0	e_1
$e_{-\omega}$	$-e_{\overline{\omega}}$	0	$-e_0$	0	$e_{-\omega}$	0	e_1	0
$e_{-\overline{\omega}}$	e_{ω}	$-e_0$	0	0	$e_{-\overline{\omega}}$	$-e_1$	0	0
e_1	$-e_{-0}$	$-e_{-\omega}$	$e_{-\overline{\omega}}$	e_1	0	0	0	0

In other words, we get

(i) $e_1 e_\omega = -e_\omega e_1 = e_{-\omega}$;

(ii) $e_1 e_0 = -e_0 e_1 = e_1$;

(iii) $e_{-1} e_1 = -e_0$ and $e_0 e_0 = e_0$;

and images under negating all subscripts (including 0), and multiplying all subscripts by ω, where $\omega^2 = \overline{\omega}$ and $\omega\overline{\omega} = 1$. All other products of basis vectors are 0. Thus, e_0 and e_{-0} are orthogonal idempotents with $e_0 + e_{-0} = 1_{\mathbb{O}}$. Now, if $x = \sum_{i\in\pm I} \lambda_i e_i$, then the norm of x can be defined in the following way:

$$\mathrm{N}(x) = \lambda_{-1}\lambda_1 + \lambda_{\overline{\omega}}\lambda_{-\overline{\omega}} + \lambda_\omega\lambda_{-\omega} + \lambda_0\lambda_{-0}. \tag{2.28}$$

Lemma 2.6.2. *The norm N defined in (2.28) is multiplicative.*

Proof. Let $x = \sum_{i\in\pm I} \lambda_i e_i$ and $y = \sum_{i\in\pm I} \mu_i e_i$ be two arbitrary elements of $\mathbb{O}$. Their product is given by

$$\begin{aligned} x \cdot y = {} & (\lambda_{-1}\mu_{-0} - \lambda_{\overline{\omega}}\mu_{\omega} + \lambda_{\omega}\mu_{\overline{\omega}} + \lambda_0\mu_{-1}) \cdot e_{-1} \\ & + (\lambda_{-1}\mu_{-\omega} + \lambda_{\overline{\omega}}\mu_0 + \lambda_{-0}\mu_{\omega} - \lambda_{-\omega}\mu_{-1}) \cdot e_{\overline{\omega}} \\ & + (\lambda_{-\overline{\omega}}\mu_{-1} + \lambda_{-1}\mu_{\omega} - \lambda_{-1}\mu_{-\overline{\omega}} + \lambda_{\omega}\mu_0) \cdot e_{\omega} \\ & + (\lambda_0\mu_0 - \lambda_{-\omega}\mu_{\omega} - \lambda_{-\overline{\omega}}\mu_{\overline{\omega}} - \lambda_{-1}\mu_1) \cdot e_0 \\ & + (\lambda_{-0}\mu_{-0} - \lambda_1\mu_{-1} - \lambda_{\overline{\omega}}\mu_{-\overline{\omega}} - \lambda_{\omega}\mu_{-\omega}) \cdot e_{-0} \\ & + (\lambda_0\mu_{-\omega} - \lambda_1\mu_{\overline{\omega}} + \lambda_{-\omega}\mu_{-0} + \lambda_{\overline{\omega}}\mu_1) \cdot e_{-\omega} \\ & + (\lambda_{-\overline{\omega}}\mu_{-0} + \lambda_1\mu_{\omega} - \lambda_{\omega}\mu_1 + \lambda_0\mu_{-\overline{\omega}}) \cdot e_{-\overline{\omega}} \\ & + (\lambda_{-0}\mu_1 + \lambda_{-\omega}\mu_{-\overline{\omega}} - \lambda_{-\overline{\omega}}\mu_{-\omega} + \lambda_1\mu_0) \cdot e_1. \end{aligned}$$

From this it is straightforward to derive

$$\begin{aligned} \mathrm{N}(x \cdot y) = {} & (\lambda_{-1}\mu_{-0} - \lambda_{\overline{\omega}}\mu_{\omega} + \lambda_{\omega}\mu_{\overline{\omega}} + \lambda_0\mu_{-1}) \cdot (\lambda_{-0}\mu_1 + \lambda_{-\omega}\mu_{-\overline{\omega}} - \lambda_{-\overline{\omega}}\mu_{-\omega} + \lambda_1\mu_0) \\ & + \ (\lambda_{-1}\mu_{-\omega} + \lambda_{\overline{\omega}}\mu_0 + \lambda_{-0}\mu_{\omega} - \lambda_{-\omega}\mu_{-1}) \cdot (\lambda_{-\overline{\omega}}\mu_{-0} + \lambda_1\mu_{\omega} - \lambda_{\omega}\mu_1 + \lambda_0\mu_{-\overline{\omega}}) \\ & + (\lambda_{-\overline{\omega}}\mu_{-1} + \lambda_{-1}\mu_{\omega} - \lambda_{-1}\mu_{-\overline{\omega}} + \lambda_{\omega}\mu_0) \cdot (\lambda_0\mu_{-\omega} - \lambda_1\mu_{\overline{\omega}} + \lambda_{-\omega}\mu_{-0} + \lambda_{\overline{\omega}}\mu_1) \\ & + (\lambda_0\mu_0 - \lambda_{-\omega}\mu_{\omega} - \lambda_{-\overline{\omega}}\mu_{\overline{\omega}} - \lambda_{-1}\mu_1) \cdot (\lambda_{-0}\mu_{-0} - \lambda_1\mu_{-1} - \lambda_{\overline{\omega}}\mu_{-\overline{\omega}} - \lambda_{\omega}\mu_{-\omega}) \\ = {} & \lambda_{-1}\lambda_1 \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_{\omega}\mu_{-\omega} + \mu_0\mu_{-0}) \\ & + \lambda_{\overline{\omega}}\lambda_{\overline{\omega}} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_{\omega}\mu_{-\omega} + \mu_0\mu_{-0}) \\ & + \lambda_{\omega}\lambda_{-\omega} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_{\omega}\mu_{-\omega} + \mu_0\mu_{-0}) \\ & + \lambda_0\lambda_{-0} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_{\omega}\mu_{-\omega} + \mu_0\mu_{-0}) \\ = {} & (\lambda_{-0}\lambda_0 + \lambda_{\overline{\omega}}\lambda_{-\overline{\omega}} + \lambda_{\omega}\lambda_{-\omega} + \lambda_0\lambda_{-0}) \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_{\omega}\mu_{-\omega} + \mu_0\mu_{-0}) \\ = {} & \mathrm{N}(x) \cdot \mathrm{N}(y). \end{aligned}$$

□

It follows that $\mathbb{O}$ is indeed a composition algebra. Let x and y be the same as in Lemma 2.6.2. We find

$$\begin{aligned}\langle x,y\rangle &= \mathrm{N}(x+y)-\mathrm{N}(x)-\mathrm{N}(y)\\ &= (\lambda_{-1}+\mu_{-1})\cdot(\lambda_1+\mu_1)+(\lambda_{\overline{\omega}}+\mu_{\overline{\omega}})\cdot(\lambda_{-\overline{\omega}}+\mu_{-\overline{\omega}})\\ &\quad+(\lambda_\omega+\mu_\omega)\cdot(\lambda_{-\omega}+\mu_{-\omega})+(\lambda_0+\mu_0)\cdot(\lambda_{-0}+\mu_{-0})\\ &\quad-(\lambda_{-1}\lambda_1+\lambda_{\overline{\omega}}\lambda_{-\overline{\omega}}+\lambda_\omega\lambda_{-\omega}+\lambda_0\lambda_{-0})\\ &\quad-(\mu_{-1}\mu_1+\mu_{\overline{\omega}}\mu_{-\overline{\omega}}+\mu_\omega\mu_{-\omega}+\mu_0\mu_{-0})\\ &= (\lambda_{-1}\mu_1+\lambda_1\mu_{-1})+(\lambda_{\overline{\omega}}\mu_{-\overline{\omega}}+\lambda_{-\overline{\omega}}\mu_{\overline{\omega}})\\ &\quad+(\lambda_\omega\mu_{-\omega}+\lambda_{-\omega}\mu_\omega)+(\lambda_0\mu_{-0}+\lambda_{-0}\mu_0).\end{aligned} \tag{2.29}$$

Thus, the trace of x becomes

$$\mathrm{T}(x)=\langle x,1_{\mathbb{O}}\rangle=\lambda_0+\lambda_{-0}. \tag{2.30}$$

Note that $\mathrm{N}(e_i)=0$ for $i\neq\pm0$, so $\mathbb{O}$ is indeed a split octonion algebra. Finally, the involution $x\mapsto\bar{x}$ is the extension by linearity of

$$e_i\mapsto -e_i\ (i\neq\pm0),\ \ e_0\leftrightarrow e_{-0}. \tag{2.31}$$

2.7 Centre of an octonion algebra

We define the centre of an octonion algebra $\mathbb{O}$ as

$$\mathrm{Z}(\mathbb{O})=\{\,c\in\mathbb{O}\;\big|\;cx=xc \text{ for all } x\in\mathbb{O}\,\}. \tag{2.32}$$

In the literature, for example, in [?], it is sometimes required that central elements also "associate" with all other elements. We do not require this in our definition, however, it will be obvious that we have this property free of charge.

Proposition 2.7.1. *The centre of an octonion algebra $\mathbb{O}=\mathbb{O}_F$ is $F\cdot 1_{\mathbb{O}}$.*

This is essentially Proposition 1.9.1 in [?], however, we need to emphasise that in the proof of this proposition the following result is used without mentioning.

Lemma 2.7.2. *Let K be an extension field of F and let A be an F-algebra with centre $\mathrm{Z}(A)$. Then $\mathrm{Z}(A\otimes_F K)=\mathrm{Z}(A)\otimes_F K$.*

Proof. The proof is straightforward. Pick an arbitrary element $z = \sum_i (a_i \otimes e_i)$ in $\mathrm{Z}(A \otimes_F K)$. Here we may assume that the elements $e_i \in K$ are linearly independent, i.e. they form a (part of) basis for K. Since z is central, in particular it must commute with the elements of the form $a \otimes 1$. This means

$$\begin{aligned} 0 = z(a \otimes 1) - (a \otimes 1)z &= \sum_i ((a_i a) \otimes e_i) - \sum_i ((a a_i) \otimes e_i) \\ &= \sum_i ((a_i a - a a_i) \otimes e_i). \end{aligned}$$

This holds if and only if $a_i a = a a_i$, i.e. $a_i \in \mathrm{Z}(A)$. ☐

Therefore, any octonion algebra is central, and it follows from Proposition 2.7.1 that central elements "associate" with all other elements.

Proposition 2.7.3. *If an octonion $u \in \mathbb{O}$ satisfies*

$$(xy)u = x(yu) \tag{2.33}$$

for all $x, y \in \mathbb{O}$, then $u \in F \cdot 1_{\mathbb{O}}$. Condition (2.33) is equivalent to the condition $(xu)y = x(uy)$ for all $x, y \in \mathbb{O}$, and also to $(ux)y = u(xy)$ for all $x, y \in \mathbb{O}$.

Corollary 2.7.4. *Suppose that $u \in \mathbb{O}$ is an invertible octonion. Then*

$$(A\bar{u})(uB) = N(u)AB \tag{2.34}$$

holds for all $A, B \in \mathbb{O}$ if and only if $u \in F \cdot 1_{\mathbb{O}}$.

Proof. Proposition 2.7.3 tells us that if $(xu)y = x(uy)$ for all $x, y \in \mathbb{O}$, then $u \in \mathbb{O}$. Now put $x = A\bar{u}$ and $y = B$; using this together with the alternative laws, we get the result.

Conversely, if $u \in F \cdot 1_{\mathbb{O}}$, then obviously the statement holds. ☐

Chapter 3

Groups of type E_6

In this chapter we derive a complete self-contained construction of the group $\mathrm{E}_6(F)$ acting on its 27-dimensional module. As already mentioned in the Introduction, we do not make any assumptions on whether the underlying field F is finite or infinite. The approach presented in this chapter is also characteristic-free.

Of great interest to us is the action of $\mathrm{E}_6(F)$ on a certain set of elements in the 27-dimensional module. It turns out, that careful and detailed study of this action reveals many hidden wonders of the group structure.

3.1 Albert vectors

3.1.1 Albert space $\mathbb{J}$

For the further discussion we consider $\mathbb{O} = \mathbb{O}_F$ to be an arbitrary octonion algebra over the field F. In the results which require $\mathbb{O}$ to be split, we specify this explicitly.

Define the *Albert space* $\mathbb{J} = \mathbb{J}_F$ to be the 27-dimensional vector space spanned by the elements of the form

$$(a, b, c \mid A, B, C) = \begin{bmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{bmatrix}, \tag{3.1}$$

where $a, b, c, A, B, C \in \mathbb{O}$ and furthermore $a, b, c \in \langle 1_{\mathbb{O}} \rangle$. Now, an *Albert vector* is an element of $\mathbb{J}$. To denote certain subspaces of $\mathbb{J}$ we use the following intuitive notation. The 10-dimensional subspace spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ is denoted $\mathbb{J}_{10}^{abC}$, while the 8-space spanned by the vectors $(0, 0, 0 \mid A, 0, 0)$ is denoted $\mathbb{J}_8^A$ and so on. That is, the subscript determines the dimension and the superscript shows which of the six 'coördinates' we use to span the corresponding subspace. Of course, this notation is by no means complete as it does not allow us to denote any possible subspace of $\mathbb{J}$. If this is the case, we specify the spanning vectors and denote the corresponding space in some other manner.

Suppose $X = (a, b, c \mid A, B, C) \in \mathbb{J}$ is an arbitrary Albert vector. We define the quadratic form Q on $\mathbb{J}$ via

$$\mathrm{Q}(X) = A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc. \tag{3.2}$$

As usual, this can be polarised to obtain the inner product

$$\begin{aligned} \mathrm{B}(X, Y) = \mathrm{T}(A_1\overline{A}_2 + B_1\overline{B}_2 + C_1\overline{C}_2) \\ - (a_1b_2 + a_2b_1) - (a_1c_2 + a_2c_1) - (b_1c_2 + b_2c_1), \end{aligned} \tag{3.3}$$

where $X = (a_1, b_1, c_1 \mid A_1, B_1, C_1)$ and $Y = (a_2, b_2, c_2 \mid A_2, B_2, C_2)$.

3.1.2 Dickson–Freudenthal determinant and $\mathrm{SE}_6(F)$

Lacking the associativity in $\mathbb{O}$ we also need to be slightly careful when we calculate the determinant of X. For these purposes we define the Dickson–Freudenthal determinant as

$$\Delta(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + \mathrm{T}(ABC). \tag{3.4}$$

This is a cubic form on $\mathbb{J}$ and it can be shown that it is equivalent to the original Dickson's cubic form [?] used to construct the group of type E_6.

We define the group $\mathrm{SE}_6(F)$ or $\mathrm{SE}_6(F, \mathbb{O})$ if we want to specify the octonion algebra, to be the group of all F-linear maps on $\mathbb{J}$ preserving the Dickson–Freudenthal determinant. If $F = \mathbb{F}_q$, then we denote this by $\mathrm{SE}_6(q)$. The group $\mathrm{E}_6(F)$ is defined as the quotient of $\mathrm{SE}_6(F)$ by its centre. Suppose M is a 3×3 matrix written over $\mathbb{O}$. If M is written over any sociable subalgebra of $\mathbb{O}$, then for an element $X \in \mathbb{J}$ the mapping $X \mapsto \overline{M}^\top XM$ makes sense. Indeed, every entry in the matrix $\overline{M}^\top XM$ is a sum of the terms of the form m_1xm_2, where m_1 and m_2 belong to the same sociable subalgebra, and so $(m_1x)m_2 = m_1(xm_2)$. Furthermore, the map $X \mapsto \overline{M}^\top XM$ is obviously F-linear:

$$\overline{M}^\top(\lambda X + \mu Y)M = \lambda(M^\top XM) + \mu(M^\top YM).$$

3.2 Some elements of $\mathrm{SE}_6(F)$

Througout this section, let $X = (a, b, c \mid A, B, C)$ to be an arbitrary element of $\mathbb{J} = \mathbb{J}_F$. We encode some of the elements of $\mathrm{SE}_6(F)$ by the 3×3 matrices written over social subalgebras of $\mathbb{O} = \mathbb{O}_F$. As we mentioned before, if such a matrix M is written over any sociable subalgebra of $\mathbb{O}$, then the expression $\overline{M}^\top XM$ makes sense. If two matrices M and N are written over the same sociable subalgebra, then we have enough associativity to see that the action by the product MN is the same as the product of the actions, that is

$$(\overline{N}\,\overline{M})^\top X(MN) = \overline{N}^\top(\overline{M}^\top XM)N. \tag{3.5}$$

In general, the action by the product of two matrices is not defined whereas the product of the actions still is. Note that also $-\mathrm{I}_3$ acts trivially on $\mathbb{J}$.

We first notice that the elements

$$\delta = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.6}$$

preserve the Dickson–Freudenthal determinant. Their actions are given by

$$\begin{aligned} \delta &: (a, b, c \mid A, B, C) \mapsto (b, a, c \mid \overline{B}, \overline{A}, \overline{C}), \\ \tau &: (a, b, c \mid A, B, C) \mapsto (c, a, b \mid C, A, B). \end{aligned} \tag{3.7}$$

Now let x be any octonion and consider the matrices

$$M_x = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \quad M''_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}. \tag{3.8}$$

Note that the elements M'_x, M''_x can be obtained from M_x by applying the triality element τ, so to show that all three families described above preserve the Dickson–Freudenthal determinant, we only need to consider one of them.

Lemma 3.2.1. *The elements M_x, where $x \in \mathbb{O}$ is any octonion, preserve the Dickson–Freudenthal determinant, and hence they encode the elements of* $\mathrm{SE}_6(F)$.

Proof. The action of M_x on $\mathbb{J}$ is given by

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), c \mid A + \bar{x}\overline{B}, B, C + ax).$$

The individual terms in the Dickson–Freudenthal determinant are being mapped in the following way:

$$\begin{aligned} abc &\mapsto abc + a^2c\mathrm{N}(x) + ac\,\mathrm{T}(\bar{x}C), \\ -aA\overline{A} &\mapsto -aA\overline{A} - a\,\mathrm{T}(ABx) - a\mathrm{N}(x)\mathrm{N}(B), \\ -bB\overline{B} &\mapsto -bB\overline{B} - a\mathrm{N}(x)\mathrm{N}(B) - \mathrm{T}(\bar{x}C)B\overline{B}, \\ -cC\overline{C} &\mapsto -cC\overline{C} - ac\,\mathrm{T}(\bar{x}C) - a^2c\mathrm{N}(x), \\ \mathrm{T}(ABC) &\mapsto \mathrm{T}(ABC) + B\overline{B}\,\mathrm{T}(\bar{x}C) + 2a\mathrm{N}(x)\mathrm{N}(B) + a\,\mathrm{T}(ABx). \end{aligned}$$

It is visibly obvious now that all the necessary terms on the right-hand side cancel out, so the result follows. □

It is obvious enough that we can also consider the transposes

$$L_x = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix}, \quad L''_x = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

for an arbitrary $x \in \mathbb{O}$. A similar straightforward calculation as in Lemma 3.2.1 can be performed to show that these are also the elements of $\mathrm{SE}_6(F)$. Further in this thesis we will be able to show that the actions of the elements M_x, M'_x, M''_x, L_x, L'_x and L''_x generate the whole group $\mathrm{SE}_6(F)$.

Finally, we consider the elements of the form

$$P_u = \begin{bmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P'_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u^{-1} \end{bmatrix}, \quad P''_u = \begin{bmatrix} u^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{bmatrix}, \tag{3.10}$$

where u is an octonion of norm one. Note that in this case $u^{-1} = \bar{u}$. The action of the element P_u on $\mathbb{J}$ is given by

$$P_u : (a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \bar{u}C\bar{u}). \tag{3.11}$$

It is a matter of straightforward computation to show that the elements P_u preserve the Dickson–Freudenthal determinant. Indeed, we have

$$\begin{aligned} abc &\mapsto abc, \\ aA\overline{A} &\mapsto a(uA)(\overline{A}\bar{u}) = aN(uA) = a\mathrm{N}(A).\mathrm{N}(u) = aA\overline{A}, \\ bB\overline{B} &\mapsto b(Bu)(\bar{u}\overline{B}) = b\mathrm{N}(Bu) = bB\overline{B}, \\ cC\overline{C} &\mapsto c(\bar{u}C\bar{u})(u\overline{C}u) = c\mathrm{N}(\bar{u}C\bar{u}) = c\mathrm{N}(C).\mathrm{N}(u)^2 = cC\overline{C}, \end{aligned} \tag{3.12}$$

and for the last term we get

$$\begin{aligned} \mathrm{T}((uA)(Bu)(\bar{u}C\bar{u})) &= \mathrm{T}((\bar{u}C\bar{u})(uA)(Bu)) = \mathrm{T}((\bar{u}(C(\bar{u}(uA))))(Bu)) \\ &= \mathrm{T}((\bar{u}(CA))(Bu)) = \mathrm{T}((Bu)(\bar{u}(CA))) = \mathrm{T}(B(u(\bar{u}(CA)))) = \mathrm{T}(B(CA)) \\ &= \mathrm{T}((BC)A) = \mathrm{T}(ABC). \end{aligned} \tag{3.13}$$

On the other hand, it is not difficult to see that $P_u = M_{u-1} \cdot L_1 \cdot M_{u^{-1}-1} \cdot L_{-u}$, so the fact that the matrices P_u preserve the determinant follows from the calculations already done for the elements M_x and L_x. We also notice that the elements P_u preserve the quadratic form Q_8^C defined on $\mathbb{J}_8^C$ via

$$\mathrm{Q}_8^C((0, 0, 0 \mid 0, 0, C)) = C\overline{C}. \tag{3.14}$$

We finish this section by showing that the action of the elements P_u on $\mathbb{J}_{10}^{abC}$, as u ranges through all the octonions of norm one, is that of $\Omega_8(F, \mathrm{Q}_8^C)$ when $\mathbb{O}$ is split.

Lemma 3.2.2. *If $\mathbb{O}$ is split, the actions of the elements P_u on $\mathbb{J}_8^C$, as u ranges through all the octonions of norm one, generate a group of type $\Omega_8^+(F)$. The action on $\mathbb{J}_{10}^{abC}$ is also that of $\Omega_8^+(F)$.*

Proof. Consider the action on the last octonionic 'coördinate', i.e. $C \mapsto \bar{u}C\bar{u}$. We will show now that this map can be represented as a product of two reflections. To avoid any predicaments in characteristic 2, we notice that since $\langle x, y\rangle = \mathrm{T}(x\bar{y})$, we get

$$\frac{2\langle x, y\rangle}{\langle y, y\rangle} = \frac{\langle x, y\rangle}{\mathrm{N}(y)}. \tag{3.15}$$

Now, the reflection in the hyperplane orthogonal to an arbitrary element $v \in \mathbb{O}$ is the map

$$r_v : x \mapsto x - \frac{\mathrm{T}(x\bar{u})}{\mathrm{N}(u)} \cdot u = x - \frac{x\bar{u} + u\bar{x}}{\mathrm{N}(u)} \cdot u = x - \frac{(x\bar{u})u - u\bar{x}u}{\mathrm{N}(u)} = -\frac{u\bar{x}u}{\mathrm{N}(u)}, \tag{3.16}$$

It is easy to see now that the given action of P_u on $\mathbb{J}_8^C$ is the composition $r_u \circ r_1$. As u ranges through all octonions of norm one, we get the action of $\Omega_8(F, \mathrm{Q}_8^C)$ on $\mathbb{J}_8^C$. Since we assume that $\mathbb{O}$ is split, the form Q_8 is of plus type, so we may denote this group as $\Omega_8^+(F)$. When acting on $\mathbb{J}_{10}^{abC}$, the form $ab - C\overline{C}$ is preserved, so we again get the action of $\Omega_8^+(F)$. $\square$

3.3 The white points

3.3.1 The mixed form and the white vectors

Suppose $X = (a, b, c \mid A, B, C)$ and $Y = (d, e, f \mid D, E, F)$ are the arbitrary elements of $\mathbb{J}$. Define the mixed form $M(Y, X)$ as

$$\begin{aligned} M(Y, X) = bcd + ace + abf - dA\overline{A} - eB\overline{B} - fC\overline{C} \\ - a(D\overline{A} + A\overline{D}) - b(E\overline{B} + B\overline{E}) - c(F\overline{C} + C\overline{F}) \\ + \mathrm{T}(DBC + ECA + FAB). \end{aligned} \tag{3.17}$$

Note that if $F \neq \mathbb{F}_2$, then $M(X,Y)$ can be obtained from the Dickson–Freudenthal determinant, for we have

$$\begin{aligned} M(X,Y) = {} & \frac{1}{\alpha(\alpha-1)}\Delta(X+\alpha Y) \\ & - \frac{1}{\alpha-1}\Delta(X+Y) + \frac{1}{\alpha}\Delta(X) - (\alpha+1)\Delta(Y), \end{aligned} \tag{3.18}$$

for any $\alpha \notin \{0,1\}$.

We colour the non-zero Albert vectors in $\mathbb{J}$ according to the following rules.

Definition 3.3.1. *A non-zero Albert vector $X \in \mathbb{J}$ is called*

(i) white if $M(Y,X) = 0$ for all $Y \in \mathbb{J}$;

(ii) grey if $\Delta(X) = 0$ and there exists $Y \in \mathbb{J}$ such that $M(Y,X) \neq 0$;

(iii) black if $\Delta(X) \neq 0$ and X is not white.

A white/grey/black point is a 1-dimensional subspace of $\mathbb{J}$ spanned by a white/grey/black vector.

For example, the vector $(0,0,1 \mid 0,0,0)$ is white, because if Y is an arbitrary Albert vector, then $M(Y,X) = 0$. Similarly, $(\lambda,1,1 \mid 0,0,0)$, where $\lambda \neq 0$, is black, since in this case $\Delta(X) = \lambda \neq 0$, and it is certainly not white as there exists $Y = (a,b,c \mid A,B,C)$ such that $M(Y,X) \neq 0$:

$$M(Y,X) = \lambda(bc - A\overline{A}) + (ac - B\overline{B}) + (ab - C\overline{C}). \tag{3.19}$$

Taking, for instance, $Y = (0,1,1 \mid 0,0,0)$, we get $M(Y,X) = \lambda \neq 0$. Finally, $(0,1,1 \mid 0,0,0)$ is grey as $\Delta(X) = 0$ and for $Y = (a,b,c \mid A,B,C)$ the value of M is given by

$$M(Y,X) = (ac - B\overline{B}) + (ab - C\overline{C}), \tag{3.20}$$

so we may take $Y = (1,1,0 \mid 0,0,0)$ to get $M(Y,X) = 1 \neq 0$. The terms white, grey and black were introduced by Cohen and Cooperstein [**?**]. In the paper by Aschbacher [**?**] they are called 'singular', 'brilliant non-singular' and 'dark' respectively. Jacobson [**?**] uses the terms 'rank 1', 'rank 2' and 'rank 3'.

It is clear that the action of $\mathrm{SE}_6(F)$ preserves the colour, except possibly in case $F = \mathbb{F}_2$, when white and grey vectors may be intermixed. Later we shall see that $\mathrm{SE}_6(\mathbb{F}_2)$ is also colour-preserving.

Let $X = (a, b, c \mid A, B, C)$ be an arbitrary white vector. A white vector W determines the quadratic form $\Delta(X + W) - \Delta(X) = M(W, X)$ on $\mathbb{J}$. Its radical is 17-dimensional and for any non-zero $\lambda \in F$ we have $\Delta(X + \lambda W) - \Delta(X) = \lambda(\Delta(X + W) - \Delta(X))$, so the form determined by λW has the same radical. Thus, the 17-dimensional space is determined by the white point $\langle W \rangle$.

For example, for the white vector $(0, 0, 1 \mid 0, 0, 0)$ the quadratic form is $ab - C\overline{C}$, whose radical is $\mathbb{J}_{17}^{cAB}$. For the vector $(0, 0, 0 \mid 0, 0, D)$ with $D \neq 0 = D\overline{D}$ the form is $\widehat{Q}(X) = \mathrm{T}(D(AB - c\overline{C}))$ with $\widehat{B}(X, Y) = \mathrm{T}(D(AB' + A'B - c\overline{C}' - c'\overline{C}))$ being its polar form, where $Y = (a', b', c' \mid A', B', C')$. Now X is in the radical of $\widehat{Q}$ if and only if $\widehat{Q}(X) = 0$ and $\widehat{B}(X, Y) = 0$ for all Y. Taking $Y = (a', b', 1 \mid 0, 0, 0)$ gives us $\mathrm{T}(D\overline{C}) = 0$ and taking $Y = (a', b', 0 \mid 0, B', 0)$ gives us $\mathrm{T}(DAB') = \mathrm{T}((DA)B') = 0$ for all B', so $DA = 0$. If $Y = (a', b', 0 \mid A', 0, 0)$ then $\mathrm{T}(D(A'B)) = \mathrm{T}((BD)A') = 0$ for all A', so we get $BD = 0$. Finally, setting $Y = (a', b', 0 \mid 0, 0, C')$ gives us $\mathrm{T}(cD\overline{C}') = 0$ for all $\overline{C}'$, so $cD = 0$, and thus $c = 0$. Therefore the radical is

$$\left\{(a, b, 0 \mid A, B, C) \;\middle|\; DA = BD = \mathrm{T}(D\overline{C}) = 0\right\}. \tag{3.21}$$

To obtain 17-spaces determined by other "coördinate" white vectors we apply a suitable power of τ to these two.

Next, we derive a system of conditions for an arbitrary vector $X \in \mathbb{J}$ to be white.

Lemma 3.3.2. *An Albert vector $X = (a, b, c \mid A, B, C)$ is white if and only if the following conditions hold:*

$$\left.\begin{aligned} A\overline{A} &= bc, \\ B\overline{B} &= ca, \\ C\overline{C} &= ab, \\ AB &= c\overline{C}, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned}\right\}. \tag{3.22}$$

If X is white, then $\Delta(X) = 0$.

Proof. Let $Y = (d, e, f \mid D, E, F)$. We rewrite $M(Y, X)$ in the form

$$M(Y, X) = (bc - A\overline{A})d + (ac - B\overline{B})e + (ab - C\overline{C})f \\ + \mathrm{T}(D(BC - a\overline{A}) + Q(CA - b\overline{B}) + R(AB - c\overline{C})).$$

It is visibly clear now that if all the conditions in the statement are satisfied, then $M(Y, X) = 0$. Now, taking $Y = (1, 0, 0 \mid 0, 0, 0)$ forces $bc - A\overline{A} = 0$. Similarly, we may take $Y = (0, 1, 0 \mid 0, 0, 0)$ to get $ac - B\overline{B} = 0$ and, say, $Y = (0, 0, 0 \mid D, 0, 0)$ to obtain $\mathrm{T}(D(BC - a\overline{A})) = 0$ which forces $BC - a\overline{A} = 0$ as $D \in \mathbb{O}$ can be arbitrary. The other conditions are proved similarly.

Finally, if X is white, then we get $\mathrm{T}(ABC) = \mathrm{T}(aA\overline{A}) = \mathrm{T}(abc) = 2abc$. Also $bB\overline{B} = bca$, and so on. Overall we get

$$\Delta(X) = abc - abc - bca - cab + 2abc = 0$$

as required. This completes the proof. ☐

3.3.2 Action of $\mathrm{SE}_6(F)$ on white points

In this thesis we will be mostly interested in the action of $\mathrm{SE}_6(F)$ on the white points.

Consider $X = (a, b, c \mid A, B, C)$ and $Y = (0, 0, 1 \mid 0, 0, 0)$. Then we find $\Delta(X + Y) - \Delta(X) = ab - C\overline{C}$, which is a quadratic form with 17-dimensional radical in $\mathbb{J}$. In case when $Y = (0, 1, 1 \mid 0, 0, 0)$ we get $\Delta(X + Y) - \Delta(X) = a + ab + ac - B\overline{B} - C\overline{C}$. If $F = \mathbb{F}_2$, we have $a^2 = a$, so the latter form is quadratic with 9-dimensional radical. This shows that $(0, 0, 1 \mid 0, 0, 0)$ and $(0, 1, 1 \mid 0, 0, 0)$ are in different orbits of the isometry group for any field.

Finally, we investigate the orbits of $\mathrm{SE}_6(F)$ on Albert vectors. One of our main goals is to show that $\mathrm{SE}_6(F)$ acts transitively on white points.

Lemma 3.3.3. *Suppose X is an arbitrary Albert vector. Then X can be mapped under the action of $\mathrm{SE}_6(F)$ to a vector of the form $(a, b, c \mid 0, 0, 0)$ with $(a, b, c) \neq (0, 0, 0)$. In case when $\mathbb{O}$ is split, X can be mapped to precisely one of the following:*

(i) $(0, 0, 1 \mid 0, 0, 0)$*, a white vector;*

(ii) $(0,1,1 \mid 0,0,0)$*, a grey vector; or*

(iii) $(\lambda,1,1 \mid 0,0,0)$ *where* $\lambda \neq 0$*, a black vector.*

In the last case there is one orbit for each non-zero value of λ*.*

Proof. These vectors are indeed in the different orbits, except possibly for the white and grey vectors, since they have different values of Δ. We have already shown that these particular white and grey vectors are in different orbits in case of any field.

First, we show that each orbit of $\mathrm{SE}_6(F)$ contains an Albert vector of the form $(a,b,c \mid 0,0,0)$. Suppose that $X=(a,b,c \mid A,B,C)$ is non-zero. If $(a,b,c)=(0,0,0)$, then after applying the triality element τ a suitable number of times we may assume $C \neq 0$. Consider the action of the element L_x on the Albert vector $(0,0,0 \mid A,B,C)$:

$$L_x : (0,0,0 \mid A,B,C) \mapsto (\mathrm{T}(Cx),0,0 \mid A, B+\overline{A}x, C),$$

so we are allowed to choose orbit representatives with $(a,b,c) \neq (0,0,0)$.

As before, using a suitable power of τ, we may assume $c \neq 0$. Now we apply the element M_x with $x=-c^{-1}B$ to X, which gives us the vector of the form $(a,b,c \mid A,0,C)$, where the 'coördinate' c stays the same, while a,b,A,C are possibly different. Next, the vector $(a,b,c \mid A,0,C)$ is being mapped to the vector of the form $(a,b,c \mid 0,0,C)$ under the action of L_x with $x=-c^{-1}A$, where the value of c stays the same while the values of a,b,C may be adjusted.

If $a=b=0$, $C \neq 0$, then we apply the element L_x with x such that $\mathrm{T}(Cx) \neq 0$ to get the vector of the form $(\mathrm{T}(Cx),0,c \mid 0,0,C)$, i.e. we may assume that $a \neq 0$. With the latter assumption we apply the element M_x with $x=-a^{-1}C$ to $(a,b,c \mid 0,0,C)$ to get the vector of the form $(a,b,c \mid 0,0,0)$ with the value of b being adjusted.

Finally, we use the elements τ, P_u and P''_v to standardise the vector of the form $(a,b,c \mid 0,0,0)$ to one the forms in the statement. □

Note that the last part of the proof of this lemma used the fact that the map $\mathrm{N} : \mathbb{O} \to F$ is onto, which is the case when $\mathbb{O}$ is split. However, this is not true in any octonion algebra, which possibly leads to a bigger number of orbits. A vector of the form $(a,b,c \mid 0,0,0)$ is white if and only if precisely one of the a,b,c is non-zero, so we get the transitive action of $\mathrm{SE}_6(F)$ on white points regardless of the chosen octonion algebra.

Furthermore, we used the fact that N is a non-singular quadratic form on $\mathbb{O}$, i.e. provided $C \neq 0$, the map $x \mapsto \mathrm{T}(Cx)$ is surjective. This should be true for any octonion algebra.

Later we will use the transitivity on white points to calculate the group order in case $F = \mathbb{F}_q$ by finding the stabiliser of a white point and calculating the number of white points in case of a finite field.

Lemma 3.3.4. *Let $\mathbb{O}$ be an arbitrary octonion algebra over F. Let $X \in \mathbb{J}$ be white and let $\mathbb{J}_{17}$ be the 17-dimensional subspace of $\mathbb{J}$ determined by X. The stabiliser in $\mathrm{SE}_6(F)$ of $\langle X \rangle$, and even of X, is transitive on the white points spanned by the vectors in $\mathbb{J}_{17} \setminus \langle X \rangle$ (there are no such white points when $\mathbb{O}$ is non-split). It is also transitive on the white points spanned by the vectors in $\mathbb{J} \setminus \mathbb{J}_{17}$.*

Proof. Without loss of generality assume $X = (0, 0, 1 \mid 0, 0, 0)$. As we know, the white point $\langle X \rangle$ determines the 17-space $\mathbb{J}_{17}^{cAB}$. We also note that X is stabilised by the actions of the elements M_x, L_x, M'_x and L''_x. Those act on the elements in $\mathbb{J}_{17}^{cAB}$ in the following way:

$$\begin{aligned}
M_x &: (0,0,c \mid A,B,0) \mapsto (0,0,c \mid A + \bar{x}\overline{B}, B, 0),\\
L_x &: (0,0,c \mid A,B,0) \mapsto (0,0,c \mid A, B + \overline{A}x, 0),\\
M'_x &: (0,0,c \mid A,B,0) \mapsto (0,0,c + \mathrm{T}(\bar{x}A) \mid A, B, 0),\\
L''_x &: (0,0,c \mid A,B,0) \mapsto (0,0,c + \mathrm{T}(Bx) \mid A, B, 0).
\end{aligned}$$

It follows that a general white vector $(0, 0, c \mid A, B, 0) \in \mathbb{J}_{17}^{cAB} \setminus \langle X \rangle$ can easily be mapped to $(0, 0, 0 \mid A, B, 0)$ using the action of M'_x or L''_x for some suitable $x \in \mathbb{O}$. A vector $(0, 0, 0 \mid A, B, 0)$ is white if $(A, B) \neq (0, 0)$ and $A\overline{A} = B\overline{B} = AB = 0$. It is obvious enough that $\mathbb{J}_{17}^{cAB} \setminus \langle X \rangle$ is empty if $\mathbb{O}$ is not split, so we only need to show transitivity on the corresponding white points in case when $\mathbb{O}$ is split.

If $B = 0$ then evidently $A \neq 0$ and so we can apply the duality element δ to obtain a white vector of the form $(0, 0, 0 \mid A, B, 0)$ with $B \neq 0$. If now $A \neq 0$, we act by M_x to obtain $(0, 0, 0 \mid A + \bar{x}\overline{B}, B, 0)$. Our aim is to show that there exists such $x \in \mathbb{O}$ that $A + \bar{x}\overline{B} = 0$. Denote $U = \{y \in \mathbb{O} \mid \bar{y}B = 0\}$. Since for all $x \in \mathbb{O}$ we have $(\bar{x}\overline{B})B = \bar{x}(\overline{B}B) = 0$, we conclude that $\mathbb{O}\overline{B} \leqslant U$. Furthermore, we know that both subspaces are four-dimensional, so $\mathbb{O}\overline{B} = U$. As $AB = 0$, we have $A \in U$, and therefore there exists $y = \bar{x}\overline{B} \in U$ such that $A + y = 0$.

Now, the elements P''_u with $\mathrm{N}(u) = 1$ act on the Albert vectors of the form $(0, 0, 0 \mid 0, B, 0)$ as

$$(0, 0, 0 \mid 0, B, 0) \mapsto (0, 0, 0 \mid 0, \bar{u}B\bar{u}, 0),$$

and as u ranges through all the octonions of norm 1 the action generated is that of $\Omega_8^+(F)$ which in case when $\mathbb{O}$ is split is transitive on isotropic vectors, i.e. those with $B\overline{B} = 0$. It follows that $\mathrm{SE}_6(F)$ is indeed transitive on the white points spanned by the vectors in $\mathbb{J}_{17}^{cAB} \setminus \langle X \rangle$.

To show the transitivity on white points spanned by the vectors in $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ we prove that every white point spanned by a white vector $(a, b, c \mid A, B, C) \in \mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ can be mapped to the white point spanned by $(1, 0, 0 \mid 0, 0, 0)$. Note that we require $(a, b, C) \neq (0, 0, 0)$.

In case $(a, b) = (0, 0)$ we choose $x \in \mathbb{O}$ such that $\mathrm{T}(Cx) \neq 0$ and apply the element L_x, which maps our vector $(0, 0, c \mid A, B, C)$ to $(\mathrm{T}(Cx), 0, c \mid A, B + \overline{A}x, C)$. If, on the other hand, $a = 0$ and $b \neq 0$, we apply δ. Hence, we may assume that we deal with a vector $(a, b, c \mid A, B, C)$ with $a \neq 0$. Take $x = -a^{-1}C$ and act by the element M_x:

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, b + aa^{-2}C\overline{C} - \mathrm{T}(a^{-1}\overline{C}C), c \mid A - a^{-1}\overline{C}\overline{B}, B, 0).$$

The whiteness conditions imply $C\overline{C} = ab$ and $BC = a\overline{A}$, so additionally we get $b + aa^{-2}C\overline{C} - \mathrm{T}(a^{-1}\overline{C}C) = b + b - \mathrm{T}(b) = 0$ and $A - a^{-1}\overline{C}\overline{B} = A - A = 0$. This means that the given M_x acts on the elements of $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ in the following way:

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, 0, c \mid 0, B, 0),$$

where $a \neq 0$. It is still white, so $B\overline{B} = ca$. Finally, we act by L''_y with $y = -a^{-1}\overline{B}$:

$$L''_y : (a, 0, c \mid 0, B, 0) \mapsto (a, 0, 0 \mid 0, 0, 0),$$

where $a \neq 0$. In other words, any white point spanned by an element in $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$

can be mapped by the action of the stabiliser of $\langle X\rangle$ to the white point spanned by $(1,0,0 \mid 0,0,0)$. $\square$

Lemma 3.3.5. *The action of* $\mathrm{SE}_6(F)$ *on white points is primitive.*

Proof. From the previous Lemma it follows that if $\mathbb{O}$ is non-split, then the action of $\mathrm{SE}_6(F)$ on white points in 2-transitive and hence primitive. It remains to prove the statement in case when $\mathbb{O}$ is split.

Suppose $X, Y \in \mathbb{J}$ are white vectors such that $\langle X\rangle \neq \langle Y\rangle$. Define $\sim$ to be an $\mathrm{SE}_6(F)$-congruence on white points and let $\langle X\rangle \sim \langle Y\rangle$. Our aim is to show that this generates the universal congruence. Since for $\mathbb{O}$ split the action on the white vectors is transitive, we may assume $X = (0,0,1 \mid 0,0,0)$. As mentioned in the beginning of this section, $\langle X\rangle$ determines the 17-dimensional space $\mathbb{J}_{17}^{cAB}$. We now distinguish two cases.

If $Y \in \mathbb{J}_{17}^{cAB}$, then acting by the stabiliser of $\langle X\rangle$ we get $\langle X\rangle \sim \langle \widehat{Y}\rangle$ for all white $\widehat{Y} \in \mathbb{J}_{17}^{cAB}$. Take $\widehat{Y} = (0,0,0 \mid e_0,0,0) \in \mathbb{J}_{17}^{cAB}$ and $\widehat{X} = (0,1,0 \mid 0,0,0) \notin \mathbb{J}_{17}^{cAB}$. As we see from the earlier calculations, both X and $\widehat{X}$ are in the 17-space determined by $\langle \widehat{Y}\rangle$. Acting by the stabiliser of $\langle \widehat{Y}\rangle$ we map $\langle X\rangle$ to $\langle \widehat{X}\rangle$, and so ensure $\langle \widehat{Y}\rangle \sim \langle \widehat{X}\rangle$, and so we have the chain $\langle X\rangle \sim \langle \widehat{Y}\rangle \sim \langle \widehat{X}\rangle$. To get $\langle X\rangle \sim \langle \widehat{X}\rangle$ for all white $\widehat{X}$ outside $\mathbb{J}_{17}^{cAB}$, we again act by the stabiliser of $\langle X\rangle$. It follows that $\langle X\rangle$ is congruent to any white point generated by a vector in $\mathbb{J}$, and so we get the universal congruence in this case.

On the other hand, if Y lies outside of $\mathbb{J}_{17}^{cAB}$, then we get $\langle X\rangle \sim \langle \widehat{Y}\rangle$ for all white $\widehat{Y} \in \mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ since the stabiliser of $\langle X\rangle$ is transitive on the white points spanned by those. In particular, we may take $\widehat{Y} = (1,0,0 \mid 0,0,0)$. Acting by the stabiliser of $\langle \widehat{Y}\rangle$ on both sides in $\langle X\rangle \sim \langle \widehat{Y}\rangle$, we map $\langle X\rangle$ to $\langle \widehat{X}\rangle$ with $\widehat{X} = (0,0,0 \mid e_0,0,0)$. Note that both X and $\widehat{X}$ are not in $\mathbb{J}_{17}^{aBC}$ which is the 17-space determined by $\widehat{Y}$, But $\widehat{X} \in \mathbb{J}_{17}^{cAB}$ and by transitivity we get $\langle X\rangle \sim \langle \widehat{X}\rangle$. Again, we act by the stabiliser of $\langle X\rangle$ to ensure $\langle X\rangle \sim \langle \widehat{X}\rangle$ for all white points $\langle \widehat{X}\rangle$ spanned by $\widehat{X} \in \mathbb{J}_{17}^{cAB}$, i.e. our $\mathrm{SE}_6(F)$-congruence is trivial in this case as well. $\square$

Suppose $\langle W\rangle$ and $\langle X\rangle$ are two white points and consider a *line* $\langle W, X\rangle$ as a 2-dimensional subspace of $\mathbb{J}$ spanned by white vectors W and X. Given a white point $\langle W \rangle$, we are interested in finding all the white points $\langle X \rangle$ such that $\langle W, X \rangle$ is totally white.

Since $\mathrm{SE}_6(F)$ acts transitively on white points, we may assume that $W = (0, 0, 1 \mid 0, 0, 0)$ and $X = (a, b, c \mid A, B, C)$. Consider an element $\lambda W + X \in \langle W, X \rangle$. First, we calculate the value of Dickson–Freudenthal determinant:

$$\begin{aligned} \Delta(\lambda W + X) = ab(\lambda + c) - aA\overline{A} - bB\overline{B} - (\lambda + c)C\overline{C} + + \mathrm{T}(ABC) \\ = \lambda(ab - C\overline{C}) + \Delta(X). \end{aligned} \tag{3.23}$$

As $X = (a, b, c \mid A, B, C)$ is white, we get $\Delta(X) = 0$ and $ab = C\overline{C}$, so we conclude $\Delta(\lambda W + X) = 0$.

Proposition 3.3.6. *If $\langle W \rangle$ and $\langle X \rangle$ are two white points, then any vector in (or any 1-subspace of) $\langle W, X \rangle$ is either white or grey.*

The conditions for $\lambda W + X$ to be white are

$$\left. \begin{aligned} A\overline{A} &= b(\lambda + c), \\ B\overline{B} &= (\lambda + c)a, \\ C\overline{C} &= ab, \\ AB &= (\lambda + c)\overline{C}, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned} \right\} \tag{3.24}$$

Since X is white by assumption, we get that $\lambda W + X$ is white if and only if $\lambda a = \lambda b = \lambda \overline{C} = 0$ for all $\lambda \in F$, which is equivalent to $a = b = C = 0$, i.e. $X \in \mathbb{J}_{17}^{cAB}$. Therefore, we conclude the following.

Proposition 3.3.7. *Given any white point $\langle W \rangle$, the line $\langle W, X \rangle$, where $\langle X \rangle$ is another white point, is totally white if and only if X belongs to the 17-space determined by W. Otherwise, $\langle W, X \rangle$ contains only two white points.*

3.3.3 The stabiliser of a white point

In this section we assume that $\mathbb{O}$ is a split octonion algebra. It is our aim now to obtain the stabiliser in $\mathrm{SE}_6(F)$ of a white point. In particular, we prove the following result.

Theorem 3.3.8. *If $\mathbb{O}$ is split, then the stabiliser of a white vector in* $\mathrm{SE}_6(F)$ *is isomorphic to the group generated by the actions of the elements M_x, L_x, M'_x and L''_x on $\mathbb{J}$ as x ranges over $\mathbb{O}$ and this is a group of shape*

$$F^{16}{:}\mathrm{Spin}^+_{10}(F). \tag{3.25}$$

The stabiliser of a white point is isomorphic to

$$F^{16}{:}\mathrm{Spin}^+_{10}(F).F^{\times}, \tag{3.26}$$

where $F^{\times}$ is the multiplicative group of the field F.

This whole section is devoted to proving this result. Some of this proof is in the running text, and some of it is contained in a series of technical lemmata. First, we prove that no invertible F-linear maps on $\mathbb{O}$ can change the order of the octonion product.

Lemma 3.3.9. *There are no invertible F-linear maps $\phi, \psi : \mathbb{O} \to \mathbb{O}$ such that for all $A, B \in \mathbb{O}$ it is true that $AB = (B\psi)(A\phi)$.*

Proof. For the sake of finding a contradiction, suppose that $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are invertible F-linear maps such that the identity $AB = (B\psi)(A\phi)$ holds for all $A, B \in \mathbb{O}$. In particilar, substituting $A = 1_{\mathbb{O}}$, we get $B = (B\psi)u$ for all $B \in \mathbb{O}$, where $u = 1\phi$, so $B\psi = Bu^{-1}$ for all $B \in \mathbb{O}$, which means that the map ψ is right multiplication by u^{-1}. Note that the existence of u^{-1} follows from the invertibility of the map ψ. Thus, our identity has the form $AB = (Bu^{-1})(A\phi)$ for all $A, B \in \mathbb{O}$. We can substitute $B = u$ which immediately gives us $A\phi = Au$ for all $A \in \mathbb{O}$, so the map ϕ is right multiplication by u. Finally, we get $AB = (Bu^{-1})(Au)$ for all $A, B \in \mathbb{O}$ and specifically for $B = 1_{\mathbb{O}}$ we get $A = u^{-1}(Au)$, or likewise $uA = Au$ for all $A \in \mathbb{O}$. Therefore u is a scalar multiple of $1_{\mathbb{O}}$, i.e. $u = \mu \cdot 1_{\mathbb{O}}$ for some $\mu \in F$. Since the linear maps ϕ and ψ are invertible, μ is non-zero, and we get $AB = (Bu^{-1})(Au) = (\mu^{-1}\mu \cdot 1_{\mathbb{O}})BA = BA$ for all $A, B \in \mathbb{O}$, which is definitely not true as $\mathbb{O}$ is not commutative. ∎

Second, we show that if two invertible linear maps commute with the octonion product, then these are mutually invertible scalar multiplication maps.

Lemma 3.3.10. *Suppose $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are two invertible F-linear maps such that $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$. Then $\psi : x \mapsto \mu x$ for some non-zero $\mu \in F$ and $\phi = \psi^{-1}$, i.e. $\phi : x \mapsto \mu^{-1}x$.*

Proof. Suppose $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are F-linear maps such that $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$. When $A = 1_{\mathbb{O}}$ we get $B\psi = uB$ for all $B \in \mathbb{O}$ where $u = (1_{\mathbb{O}}\phi)^{-1}$, so the map ψ is left multiplication by u. Substituting $B = 1_{\mathbb{O}}$ on the other hand gives us $A = (A\phi)(1_{\mathbb{O}}\psi)$ for all A and so $A\phi = Av$ where $v = (1_{\mathbb{O}}\psi)^{-1}$, so ϕ is the right multiplication by v. Therefore the condition in this case becomes $AB = (Av)(uB)$ for all $A, B \in \mathbb{O}$. Substituting $B = u^{-1}$, we get $Au^{-1} = Av$ for all $A \in \mathbb{O}$, and therefore $v = u^{-1}$, and our identity turns out to be $AB = (Au^{-1})(uB)$ for all $A, B \in \mathbb{O}$. Now since u is invertible, we can write $u^{-1} = N(u)^{-1}\bar{u}$. Finally, by Corollary 2.7.4, u must be a scalar multiple of $1_{\mathbb{O}}$, i.e. $u = \mu \cdot 1_{\mathbb{O}}$. □

The statements in Lemmas 3.3.9 and 3.3.10 are true even when $\mathbb{O}$ is not split. Everything is ready now for the investigation of the white vector stabiliser. Since it was shown that the group $\mathrm{SE}_6(F)$ acts transitively on the set of white points, it is sufficient to study the stabiliser of a specific white vector. For instance, it is convenient to take $v = (0, 0, 1 \mid 0, 0, 0)$. First thing to notice is that v is invariant under the action of the elements of the form

$$L''_x = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M'_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.27}$$

where $x, y \in \mathbb{O}$.

Lemma 3.3.11.

(a) Let Q be any of the $\{L, L', L'', M, M', M''\}$. Then the actions on $\mathbb{J}$ of the elements Q_x where x ranges over $\mathbb{O}$ generate an elementary abelian group isomorphic to F^8.

(b) Let (R, S) be any pair from the set $\{(L, M''), (L', M), (L'', M')\}$ or any of the $\{(L, M'), (L', M''), (L'', M)\}$. Then the actions of R_x and S_x, as x ranges through $\mathbb{O}$, generate an elementary abelian group isomorphic to F^{16}.

Proof. To show part (a) for the elements L_x, L'_x, L''_x it is enough to consider just, say, L''_x as to obtain the result for the rest of them we can apply the action of the triality element

$$\tau = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Similarly, out of M_x, M'_x, M''_x we only need to consider, for instance, M'_x. The actions of L''_x and M'_y on $\mathbb{J}$ are given by

$$\begin{aligned} L''_x : (a, b, c \mid A, B, C) &\mapsto (a, b, c + ax\bar{x} + \mathrm{T}(Bx) \mid A + \overline{C}x, B + A\bar{x}, C), \\ M'_y : (a, b, c \mid A, B, C) &\mapsto (a, b, c + by\bar{y} + \mathrm{T}(\bar{y}A) \mid A + By, B + \bar{y}\overline{C}, C). \end{aligned} \tag{3.28}$$

We notice that the action is nontrivial whenever x and y are non-zero. The element M'_y sends $(a, b, c + ax\bar{x} + \mathrm{T}(Bx) \mid A + \overline{C}x, B + a\bar{x}, C)$ to

$$(a, b, c + ax\bar{x} + \mathrm{T}(Bx) + by\bar{y} + \mathrm{T}(\bar{y}A) \mid A + \overline{C}x + by, B + a\bar{x} + \bar{y}\overline{C}, C),$$

and the element L''_x sends $(a, b, c + by\bar{y} + \mathrm{T}(\bar{y}A) \mid A + By, B + \bar{y}\overline{C}, C)$ to

$$(a, b, c + by\bar{y} + \mathrm{T}(\bar{y}A) + ax\bar{x} + \mathrm{T}(Bx) \mid A + by + \overline{C}x, B + \bar{y}\overline{C} + a\bar{x}, C).$$

Hence, the actions of these elements commute. Similarly, it is straightforward to verify that the actions of L''_x and L''_y commute as well as the actions of M'_x and M'_y. Moreover, the element L''_y sends $(a, b, c + ax\bar{x} + \mathrm{T}(Bx) \mid A + \overline{C}x, B + a\bar{x}, C)$ to

$$(a, b, c + ax\bar{x} + \mathrm{T}(Bx) + ay\bar{y} + \mathrm{T}(By) + a\,\mathrm{T}(\bar{x}y) \mid A + \overline{C}x + \overline{C}y, B + a\bar{x} + a\bar{y}, C),$$

and L''_{x+y} sends $(a, b, c \mid A, B, C)$ to

$$(a, b, c + ax\bar{x} + a\,\mathrm{T}(x\bar{y}) + ay\bar{y} + \mathrm{T}(B(x + y)) \mid A + \overline{C}(x + y), B + a(\bar{x} + \bar{y}), C),$$

so the action of L''_{x+y} is the same as the product of the actions of L''_x and L''_y. A similar calculation shows that the action of M'_{x+y} is the same as the product of the actions of M'_x and M'_y. It follows that the action of L''_x on $\mathbb{J}$, $x \in \mathbb{O}$ generates an abelian group $(F^8, +)$ as well as the action of the element M'_y, $y \in \mathbb{O}$. We simply denote the abelian group $(F^n, +)$ as F^n in our further discussion.

To prove part (b) we need to verify that the intersection of the corresponding abelian groups, isomorphic to F^8 and generated by the actions of L''_x and M'_x is trivial. Suppose that the actions of L''_x and M'_y are equal. Then, according to (3.28), in the fourth "coördinate" we have

$$A + \overline{C}x = A + By$$

for arbitrary $A, B, C \in \mathbb{O}$. In other words, we get $\overline{C}x = By$ for arbitrary octonions B and C. In particular, if $B = 1_{\mathbb{O}}$ and $C = 0$, we get $y = 0$ and if $B = 0$ and $C = 1_{\mathbb{O}}$ we obtain $x = 0$. So, the intersection of two copies of F^8 consists of the identity element, as needed, and the result follows. Again, to get (b) for the rest of the pairs in the first set we apply the triality element. The calculations for the second set of pairs are essentially of the same nature. □

Lemma 3.3.12. *The actions on $\mathbb{J}$ of the elements P_u, P'_u, and P''_u with u being any invertible octonion, normalise the abelian groups isomorphic to F^8, generated by L_x (x ranges through $\mathbb{O}$), L'_x, L''_x, M_x, M'_x, M''_x respectively*

Proof. This is a straightforward calculation. Since P_u, P'_u, and P''_u are all mutual conjugates by a suitable power of the triality element τ, it is enough to consider one of them. We have the following relations:

$$\begin{aligned}
(L_x)^{P'_u} &\text{ acts as } L_{u^{-1}x}, \\
(L'_x)^{P'_u} &\text{ acts as } L'_{uxu}, \\
(L''_x)^{P'_u} &\text{ acts as } L''_{xu^{-1}}, \\
(M_x)^{P'_u} &\text{ acts as } M_{xu}, \\
(M'_x)^{P'_u} &\text{ acts as } M'_{u^{-1}xu^{-1}}, \\
(M''_x)^{P'_u} &\text{ acts as } M''_{ux}.
\end{aligned} \tag{3.29}$$

□

The next observation is that our white vector v is also invariant under the action of the elements

$$M_x = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_y = \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.30}$$

First, we show that the actions of these on $\mathbb{J}_{10}^{abC}$ generate a group of type $\Omega_{10}^{+}(F)$. As we will see further, instead of arbitrary octonions it is enough for x to range through the scalar multiples of the basis elements e_i. Define the quadratic form Q_{10} on $\mathbb{J}$ via

$$\mathrm{Q}_{10}((a, b, c \mid A, B, C)) = ab - C\overline{C}. \tag{3.31}$$

We notice that Q_{10} is of *plus* type, so for convenience we denote the group $\Omega_{10}(F, \mathrm{Q}_{10})$ as $\Omega_{10}^{+}(F)$.

To construct $\Omega_{10}^{+}(F)$ we follow the series of steps. First, we consider the 4-space V_4 spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C_{-1}e_{-1} + C_1e_1)$.

Lemma 3.3.13. *The actions of the elements $M_{\lambda e_{\pm 1}}$ and $L_{\lambda e_{\pm 1}}$ on V_4, where $\lambda \in F$, generate a group of type $\Omega_4^{+}(F)$.*

Proof. Consider the vectors v_1, v_2, v_3 and v_4 defined as

$$\begin{aligned} v_1 &= (1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_3 &= (0, 0, 0 \mid 0, 0, e_{-1}), \\ v_4 &= (0, 0, 0 \mid 0, 0, e_1). \end{aligned}$$

It is clear that these span V_4, and let $\mathcal{B}$ be the [ordered] basis of V_4, whose elements are v_1, v_4, v_3, and v_2, in that order.

The element $M_{\lambda e_{-1}}$ acts on the basis elements in the following way:

$$\begin{array}{lclcl} v_1 & \mapsto & (1, 0, 0 \mid 0, 0, \lambda e_{-1}) & = & v_1 + \lambda v_3, \\ v_4 & \mapsto & (0, \lambda, 0 \mid 0, 0, e_1) & = & v_4 + \lambda v_2, \\ v_3 & \mapsto & (0, 0, 0 \mid 0, 0, e_{-1}) & = & v_3, \\ v_2 & \mapsto & (0, 1, 0 \mid 0, 0, 0) & = & v_2. \end{array}$$

As we can see, with respect to the basis $\mathcal{B}$ the action can be written as a 4×4

matrix

$$\begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\otimes$ is the Kronecker product. Similarly, the action of $M_{\lambda e_1}$ on $\mathcal{B}$ is given by

$$\begin{array}{rclcl} v_1 & \mapsto & (1,0,0 \mid 0,0,\lambda e_1) & = & v_1 + \lambda v_4, \\ v_4 & \mapsto & (0,0,0 \mid 0,0,e_1) & = & v_4, \\ v_3 & \mapsto & (0,\lambda,0 \mid 0,0,e_{-1}) & = & v_3 + \lambda v_2, \\ v_2 & \mapsto & (0,1,0 \mid 0,0,0) & = & v_2, \end{array}$$

so the corresponding 4×4 matrix has the form

$$\begin{bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

Now, for convenience, we do the same calculations for $L_{-\lambda e_{-1}}$: it acts on the elements of $\mathcal{B}$ as

$$\begin{array}{rclcl} v_1 & \mapsto & (1,0,0 \mid 0,0,0) & = & v_1, \\ v_4 & \mapsto & (\lambda,0,0 \mid 0,0,e_{-1}) & = & \lambda v_1 + v_4, \\ v_3 & \mapsto & (0,0,0 \mid 0,0,e_{-1}) & = & v_3, \\ v_2 & \mapsto & (0,1,0 \mid 0,0,\lambda e_{-1}) & = & \lambda v_3 + v_2, \end{array}$$

and it can be written in the matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}.$$

Finally, the action on $\mathcal{B}$ of $L_{-\lambda e_1}$ is given by

$$\begin{array}{rcrcl} v_1 & \mapsto & (1,0,0 \mid 0,0,0) & = & v_1, \\ v_4 & \mapsto & (0,0,0 \mid 0,0,e_1) & = & v_4, \\ v_3 & \mapsto & (\lambda,0,0 \mid 0,0,e_{-1}) & = & \lambda v_1 + v_3, \\ v_2 & \mapsto & (0,1,0 \mid 0,0,\lambda e_1) & = & \lambda v_4 + v_2, \end{array}$$

and in the matrix form we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As we know,

$$\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \mathrm{SL}_2(F).$$

It follows that

$$\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong$$
$$\cong \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \mathrm{SL}_2(F).$$

These two copies of $\mathrm{SL}_2(F)$ clearly commute, thus their intersection is contained in the centre of each copy, which means that it can only contain $\pm\mathrm{I}_2$. Since

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we finally get

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F).$$

Now, $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F) \cong \Omega_4^+(F)$, and what left is to show that those matrices which generate $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$ also generate the whole of $\Omega_4^+(F)$. Indeed, if F is infinite, having just an isomorphism is not sufficient, since in this case the group can easily be isomorphic to a proper subgroup of itself. First, we show that the action of $G = \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F) \leqslant \Omega_4^+(F)$ on the isotropic vectors in V_4 is transitive.

Let $v = (a, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1)$ be an arbitrary isotropic vector in V_4 with $\alpha, \beta \in F$. We find $Q(v) = ab - \alpha\beta$, so v is isotropic if and only if $ab = \alpha\beta$ with $(a, b, \alpha, \beta) \neq (0, 0, 0, 0)$. The actions of the generators on v are given by

$$\begin{aligned} M_{\lambda e_{-1}} &: (a, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1) \mapsto (a, b - \lambda\beta, 0 \mid 0, 0, (\alpha + \lambda a)e_{-1} + \beta e_1), \\ M_{\lambda e_1} &: (a, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1) \mapsto (a, b - \lambda\alpha, 0 \mid 0, 0, \alpha e_{-1} + (\beta + \lambda a)e_1), \\ L_{-\lambda e_{-1}} &: (a, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1) \mapsto (a + \lambda\beta, b, 0 \mid 0, 0, (\alpha - \lambda b)e_{-1} + \beta e_1), \\ L_{-\lambda e_1} &: (a, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1) \mapsto (a + \lambda\alpha, b, 0 \mid 0, 0, \alpha e_{-1} + (\beta - \lambda b)e_1). \end{aligned}$$

Consider the case $(a, b) = (0, 0)$. Then for v to be isotropic, it has to be either $\alpha = 0, \beta \neq 0$ or $\alpha \neq 0, \beta = 0$. Acting on $(0, 0, 0 \mid 0, 0, \beta e_1)$ by $L_{-\beta^{-1}e_{-1}}$, we obtain the vector $(1, 0, 0 \mid 0, 0, \beta e_1)$. Next, we act by $M_{-\beta e_1}$ to obtain $(1, 0, 0 \mid 0, 0, 0)$. Similarly, acting on $(0, 0, 0 \mid 0, 0, \alpha e_{-1})$ with $\alpha \neq 0$ by $L_{-\alpha^{-1}e_1}$, and then by $M_{-\alpha e_{-1}}$, we get $(1, 0, 0 \mid 0, 0, 0)$ as well.

If $a \neq 0$ and $b \neq 0$, we are forced to have $\alpha \neq 0$ and $\beta \neq 0$. Acting by $M_{b\alpha^{-1}e_1}$, we obtain the vector $(a, 0, 0 \mid 0, 0, \alpha e_{-1} + (\beta + b\alpha^{-1}a)e_1)$. Next, we consider $(a, 0, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1)$ with $a \neq 0$. If $\alpha \neq 0$, then we may act by $L_{a\alpha^{-1}e_1}$ to obtain $(0, 0, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1)$; on the other hand, if $\beta \neq 0$, we may act by $L_{a\beta^{-1}e_{-1}}$ to obtain the same vector. If $(\alpha, \beta) = (0, 0)$, then we map $(a, 0, 0, \mid 0, 0, 0)$ to $(a, 0, 0 \mid 0, 0, e_1)$ by the action of $M_{a^{-1}e_1}$, and further to $(0, 0, 0 \mid 0, 0, e_1)$ by the action of $L_{ae_{-1}}$. In other words, we have reduced to the case $(a, b) = (0, 0)$.

Similarly, if $a = 0$ and $b \neq 0$, we consider $(0, b, 0 \mid 0, 0, \alpha e_{-1} + \beta e_1)$, and act on it by $M_{b\alpha^{-1}e_1}$ if $\alpha \neq 0$ or by $M_{b\beta^{-1}e_{-1}}$ if $\beta \neq 0$ to obtain $(0, 0, 0, \mid 0, 0, \alpha e_{-1} + \beta e_1)$. Finally, if $(\alpha, \beta) = (0, 0)$, then we act on $(0, b, 0 \mid 0, 0, 0)$ by $L_{-b^{-1}e_1}$, followed by the action of $M_{be_{-1}}$ to obtain $(0, 0, 0 \mid 0, 0, e_1)$ which again brings us back to the case $(a, b) = (0, 0)$, and we are done proving transitivity of G on the isotropic vectors in V_4.

The rest is to show that the matrices $[M_{\lambda \pm 1}]_{\mathcal{B}}$ and $[L_{-\lambda \pm 1}]_{\mathcal{B}}$ indeed generate the whole $\Omega_4^+(F)$. Let g be an element of $\Omega_4^+(F)$. Since now we know that $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$ generated by these matrices is transitive on isotropic vectors in V_4, we may choose an element $h_1 \in \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$ such that gh_1^{-1} stabilises v_1. We let $w = v_4^{gh_1^{-1}}$ and write down the conditions on w:

$$\left.\begin{aligned} \langle v_1, w\rangle &= \quad \langle v_1, v_4\rangle &&= 0,\\ \mathrm{Q}_4(w) &= \quad \mathrm{Q}_4(v_4) &&= 0,\\ \dim_F\langle v_1, w\rangle_F &= \dim_F\langle v_1, v_4\rangle_F &&= 2. \end{aligned}\right\}$$

Note that we use Q_4 to denote the restriction of Q_{10} on V_4. With respect to $\mathcal{B}$, w has the form $(\alpha, \beta, \gamma, \delta)$ for some $\alpha, \beta, \gamma, \delta \in F$. By polarising the quadratic form, we find that in this basis the inner product is represented by the matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix},$$

so using the conditions above we find $\langle v_1, w\rangle = v_1 B w^\top = \delta$, so $\delta = 0$. Next, $\mathrm{Q}_4(w) = -\beta\gamma$, so either $\beta = 0,\ \gamma \neq 0$ or $\beta \neq 0,\ \gamma = 0$. The case $(\beta, \gamma) = (0, 0)$ does not satisfy the condition on dimension, so we do not consider it.

It follows that either $w = \alpha v_1 + \beta v_4$, $\beta \neq 0$ or $w = \alpha v_1 + \gamma v_3$, $\gamma \neq 0$. Consider an element $h_2 \in \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$, which has the following matrix form:

$$[h_2]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 \end{bmatrix}.$$

We have $(v_4^{h_2})^{gh_1^{-1}} = (\lambda v_1 + v_4)^{gh_1^{-1}}$, which is either $(\alpha+\lambda)v_1 + \beta v_4$, or $(\alpha+\lambda)v_1 + \gamma v_3$. Take $\lambda = -\alpha$ to get $v_4^{h_2gh_1^{-1}}$ to be either βv_4 for some $\beta \neq 0$, or γv_3 for some $\gamma \neq 0$.

In $\Omega_4^+(F)$ our element $h_2gh_1^{-1}$ has the following matrix form with respect to the basis $\mathcal{B}$:

$$[h_2gh_1^{-1}]_{\mathcal{B}} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline * & A & 0 \\ \hline * & * & 1 \end{array}\right],$$

where A represents an element of $\Omega_2^+(F)$. Now, $h_2gh_1^{-1}$ has spinor norm 1, as well as the element represented by A. In the case when $h_2gh_1^{-1}$ maps v_4 to βv_4, A takes the form

$$A = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}.$$

On the other hand, if v_4 is mapped to γv_3, then

$$A = \begin{bmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{bmatrix}.$$

The latter is of no interest to us, because its determinant is -1, and in characteristic 2 it has the wrong quasideterminant. Since also the spinor norm of the element represented by A is 1, β is a square in F. Take $\lambda \in F$ such that $\lambda^2 = \beta$, and consider the 4×4 matrix C:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that C represents an element of $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$. Indeed,

$$C = \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \right)$$

Denote by h_3 the element of $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$, represented by C. Now, the matrix representing $h_2 g h_1^{-1} h_3^{-1}$ with respect to the basis $\mathcal{B}$ has ones on the diagonal:

$$[h_2 g h_1^{-1} h_3^{-1}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \\ \zeta & 0 & 1 & 0 \\ \eta & \iota & \kappa & 1 \end{bmatrix},$$

where $\varepsilon, \zeta, \eta, \iota, \kappa \in F$. Denote by w_1, w_2, w_4 the images of v_1, v_2, and v_4 respectively, under the action of $h_2 g h_1^{-1} h_3^{-1}$. Since $\mathrm{Q}_4(v_4) = 0$ and $\mathrm{Q}_4(w_4) = \eta - \iota\kappa$, we get $\eta = \iota\kappa$. Next, $\langle v_2, v_4 \rangle = 0$ and $\langle v_2, w_4 \rangle = \varepsilon - \kappa$, so $\varepsilon = \kappa$. Finally, $\langle v_1, v_4 \rangle = 0$ and $\langle v_1, w_4 \rangle = \zeta - \iota$, so $\zeta = \iota$. It turns out that

$$[h_2 g h_1^{-1} h_3^{-1}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \varepsilon & 1 & 0 & 0 \\ \zeta & 0 & 1 & 0 \\ \varepsilon\zeta & \zeta & \varepsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \zeta & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}.$$

Thus, $h_2 g h_1^{-1} h_3^{-1}$ is an element of $\mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$, and therefore so is g. $\square$

In our construction we use the results of the Appendix A. Consider the 6-space V_6 spanned by the Albert vectors $(a, b, 0 \mid 0, 0, C)$, where $C \in \langle e_{-1}, e_{\overline{\omega}}, e_{-\overline{\omega}}, e_1 \rangle$. Our copy of $\Omega_4^+(F)$ preserves two isotropic Albert vectors in V_6:

$$\begin{aligned} u_{\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{\overline{\omega}}), \\ u_{-\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{-\overline{\omega}}). \end{aligned} \tag{3.32}$$

The element $M_{e_{\overline{\omega}}}$ preserves $u_{\overline{\omega}}$ but not $u_{-\overline{\omega}}$. Therefore, adjoining $M_{e_{\overline{\omega}}}$ to $\Omega_4^+(F)$, we obtain a subgroup of $V_4{:}\Omega_4^+(F)$ (Lemma A.1), and since $\Omega_4^+(F)$ is maximal in the latter (Theorem A.3), we conclude that the action of $M_{\lambda e_{\pm 1}}$, $L_{\lambda e_{\pm 1}}$ and $M_{e_{\overline{\omega}}}$ on

V_6 is that of $V_4{:}\Omega_4^+(F)$. That is, we have constructed the group $V_4{:}\Omega_4^+(F)$ as the stabiliser of $u_{\overline{\omega}}$ in $\Omega_6^+(F)$. Now we use the result of Theorem A.4. The element $M_{e_{-\overline{\omega}}}$ preserves V_6 but it does not preserve $u_{\overline{\omega}}$, and as a consequence it does not preserve the 1-space $\langle u_{\overline{\omega}} \rangle$. Therefore, if we adjoin $M_{e_{-\overline{\omega}}}$ to our copy of $V_4{:}\Omega_4^+(F)$, we get the action of the group $\Omega_6^+(F)$ on V_6.

Similarly, we consider the 8-space V_8 spanned by the vectors $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_{\overline{\omega}}, e_\omega, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle$. Consider two isotropic Albert vectors

$$\begin{aligned} u_\omega &= (0, 0, 0 \mid 0, 0, e_\omega), \\ u_{-\omega} &= (0, 0, 0 \mid 0, 0, e_{-\omega}), \end{aligned} \tag{3.33}$$

which are fixed by our copy of $\Omega_6^+(F)$. The action of the element M_{e_ω} on V_8 preserves u_ω but not $u_{-\omega}$ and therefore adjoining this element to $\Omega_6^+(F)$ we get the action of the group $V_6{:}\Omega_6^+(F)$. Next, the element $M_{e_{-\omega}}$ does not preserve the 1-space $\langle u_\omega \rangle$, so appending it to $V_6{:}\Omega_6^+(F)$ we get the action of the group $\Omega_8^+(F)$ on V_8.

Finally, we consider the 10-space $\mathbb{J}_{10}^{abC}$ with two isotropic Albert vectors

$$\begin{aligned} u_0 &= (0, 0, 0 \mid 0, 0, e_0), \\ u_{-0} &= (0, 0, 0 \mid 0, 0, e_{-0}). \end{aligned} \tag{3.34}$$

Following the same procedure, we adjoin the element M_{e_0} which fixes u_0 but not u_{-0} to get the action of the group of shape $V_8{:}\Omega_8^+(F)$. Appending the action of $M_{e_{-0}}$, which does not preserve $\langle u_0 \rangle$, to this yields the action of $\Omega_{10}^+(F)$ on $\mathbb{J}_{10}^{abC}$. Lemma 3.3.11 allows us to conclude that we have shown the following result.

Lemma 3.3.14. *The actions of M_x and L_x on $\mathbb{J}_{10}^{abC}$ generate the group $\Omega_{10}^+(F)$ as x ranges through $\mathbb{O}$.*

Now we need to understand the action of the elements M_x and L_x on the whole 27-space $\mathbb{J}$.

Lemma 3.3.15. *Suppose an element of the stabiliser in $\mathrm{SE}_6(F)$ of v preserves the decomposition of the Albert space into the direct sum of the form $\mathbb{J} = \mathbb{J}_1^c \oplus \mathbb{J}_{16}^{AB} \oplus \mathbb{J}_{10}^{abC}$.*

(a) If the action of this element on the 10*-space* $\mathbb{J}_{10}^{abC}$ *is given by*

$$\begin{aligned}(1,0,0 \mid 0,0,0) &\mapsto (\lambda,0,0 \mid 0,0,0),\\ (0,1,0 \mid 0,0,0) &\mapsto (0,\lambda^{-1},0 \mid 0,0,0),\\ (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C),\end{aligned}$$

then λ *is a square in* F*.*

(b) On the other hand, an action of the type

$$\begin{aligned}(1,0,0 \mid 0,0,0) &\mapsto (0,\lambda,0 \mid 0,0,0),\\ (0,1,0 \mid 0,0,0) &\mapsto (\lambda^{-1},0,0 \mid 0,0,0),\\ (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C)\end{aligned}$$

is impossible.

(c) Finally, if the action on the 10*-space is trivial, then the action on the corresponding* 16*-space is that of* $\pm \mathrm{I}_{16}$ *(hence, the action on* $\mathbb{J}$ *is that of* $P_{\pm 1}$*).*

Proof. We are considering the elements that fix $\mathbb{J}_8^C$ pointwise and either fix or swap the 1-dimensional spaces $\mathbb{J}_1^a$ and $\mathbb{J}_1^b$. So we may assume that these elements respectively fix or swap the corresponding 17-spaces $\mathbb{J}_{17}^{aBC}$ and $\mathbb{J}_{17}^{bAC}$. In particular, their intersection, i.e. the space $\mathbb{J}_8^C$ is fixed. If the action of the stabiliser swaps $\mathbb{J}_1^a$ and $\mathbb{J}_1^b$ while leaving the 1-space $\mathbb{J}_1^c$ in its place, then it also swaps the 8-spaces $\mathbb{J}_8^A$ and $\mathbb{J}_8^B$ as these subspaces are the intersections of the 17-space $\mathbb{J}_{17}^{cAB}$ with $\mathbb{J}_{17}^{bAC}$ and $\mathbb{J}_{17}^{aBC}$ respectively.

Suppose now that an element in the stabiliser acts in the following manner:

$$(a,b,c \mid A,B,C) \mapsto (\lambda a, \lambda^{-1} b, c \mid A\phi, B\psi, C),$$

where $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are invertible F-linear maps. As this action is supposed to preserve the determinant, it has to preserve the cubic term $\mathrm{T}(ABC)$ in particular, i.e. we must have $\mathrm{T}(ABC) = \mathrm{T}((A\phi)(B\psi)C)$ for all $A, B, C \in \mathbb{O}$. This is equivalent to the condition $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$, since the original identity is equivalent to $\langle AB, \overline{C}\rangle = \langle (A\phi)(B\psi), \overline{C}\rangle$. By Lemma 3.3.10 we find that $A\phi = \mu^{-1}A$ and $B\psi = \mu B$ for all $A, B \in \mathbb{O}$ and some non-zero $\mu \in F$. The individual terms in the determinant are being changed in the following way:

$$\begin{aligned}
abc &\mapsto abc,\\
aA\overline{A} &\mapsto \lambda\mu^{-2}aA\overline{A},\\
bB\overline{B} &\mapsto \lambda^{-1}\mu^{2}bB\overline{B},\\
cC\overline{C} &\mapsto cC\overline{C},\\
\mathrm{T}(ABC) &\mapsto \mathrm{T}(ABC).
\end{aligned}$$

It follows that in order to preserve the determinant we must have $\lambda^{-1}\mu^2 = 1$, i.e. $\lambda = \mu^2$.

In case when our element acts as

$$(a, b, c \mid A, B, C) \mapsto (\lambda^{-1}b, \lambda a, c \mid B\psi, A\phi, C),$$

we get $\mathrm{T}(ABC) = \mathrm{T}((B\psi)(A\phi)C)$ for all $A, B, C \in \mathbb{O}$. This holds if and only if $AB = (B\psi)(A\phi)$ for all $A, B \in \mathbb{O}$. Lemma 3.3.9 asserts that there are no such maps ϕ and ψ, and so this rules out the latter case.

Finally, if we assume the trivial action on $\mathbb{J}_{10}^{abC}$, then we get $\lambda = 1$, i.e. $\mu^2 = 1$, so the action on $\mathbb{J}$ is indeed that of $P_{\pm 1}$. $\square$

Now let $X = (a, b, c \mid A, B, C)$ and let $Y = (a', b', c' \mid A', B', C')$. An isometry which maps X to Y and v to λv must send $\Delta(X + v) - \Delta(X) = ab - C\overline{C}$ to $\Delta(Y + \lambda v) - \Delta(Y) = \lambda(a'b' - C'\overline{C}')$. The 17-dimensional radical of both of these forms is fixed, and the quadratic form $ab - C\overline{C}$ is being scaled by a factor of λ. In particular, when $\lambda = 1$, the quadratic form is being preserved. So, the action of the vector stabiliser on the 10-dimensional quotient is that of a subgroup of $\mathrm{GO}_{10}^{+}(F)$.

Consider the white vectors of the form $(a, 0, c \mid A, B, 0)$ and $(0, b, c \mid A, B, 0)$ with $a, b \neq 0$. In the first case the conditions for being white are

$$\left.\begin{aligned}
A\overline{A} &= 0,\\
B\overline{B} &= ac,\\
a\overline{A} &= 0,\\
AB &= 0.
\end{aligned}\right\}$$

In other words, we have a white vector of the form $(a, 0, B\overline{B}/a \mid 0, B, 0)$. For the second vector we get

$$\left.\begin{aligned} bc &= A\overline{A}, \\ B\overline{B} &= 0, \\ b\overline{B} &= 0, \end{aligned}\right\}$$

so the vector has the form $(0, b, A\overline{A}/b \mid A, 0, 0)$. The elements M'_x and L''_x transform these in the following way:

$$\begin{aligned} M'_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a \mid 0, B, 0), \\ M'_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b + bx\bar{x} + \mathrm{T}(\bar{x}A) \mid A + bx, 0, 0), \\ L''_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a + ax\bar{x} + \mathrm{T}(Bx) \mid 0, B + a\bar{x}, 0), \\ L''_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b \mid A, 0, 0). \end{aligned}$$

Note that we already have an elementary abelian group F^{16} acting on the 17-space $\mathbb{J}_{17}^{cAB}$. We can now invoke Lemma 3.3.15 to conclude that the action of the stabiliser on the remaining 10-space $\mathbb{J}_{10}^{abC}$ is that of $\Omega_{10}^{+}(F)$ and the kernel of the action on $\mathbb{J}$ has order no more than two.

Theorem 3.3.16. *The actions of the elements M_x and L_x on $\mathbb{J}$ where x ranges through a split octonion algebra $\mathbb{O}$ generate a group of type $\mathrm{Spin}_{10}^{+}(F)$ understood as $\Omega_{10}^{+}(F)$ in case of characteristic 2.*

With the result of Lemma 3.3.11 we conclude that the stabiliser of a white vector is indeed a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$ as usual understood as $F^{16}{:}\Omega_{10}^{+}(F)$ in case of characteristic 2.

Now we have enough ingredients to produce the vector stabiliser. As before, we consider the stabiliser of the white vector $v = (0, 0, 1 \mid 0, 0, 0)$. As we know from Theorem 3.3.16 and Lemma 3.3.15, the actions of the elements M_x and L_x on $\mathbb{J}$ generate a group of type $\mathrm{Spin}_{10}^{+}(F)$. It is easy to check that this copy of $\mathrm{Spin}_{10}^{+}(F)$ normalises the elementary abelian group F^{16} from Lemma 3.3.11. A straightforward computation illustrates the following result:

$$\begin{aligned} (M'_x)^{L_y} &\text{ acts as } M'_x, \\ (M'_x)^{M_y} &\text{ acts as } L''_{-yx} \cdot M'_x, \\ (L''_x)^{L_y} &\text{ acts as } M'_{-yx} \cdot L''_x, \\ (L''_x)^{M_y} &\text{ acts as } L''_x, \end{aligned} \tag{3.35}$$

where the products in the right-hand side are understood as the products of the actions rather than as the matrix products. Furthermore, the intersection of the groups $\mathrm{Spin}_{10}^{+}(F)$ and F^{16} is trivial: the action of $\mathrm{Spin}_{10}^{+}(F)$ preserves the decomposition $\mathbb{J} = \mathbb{J}_1^c \oplus \mathbb{J}_{16}^{AB} \oplus \mathbb{J}_{10}^{abC}$, while any non-trivial action of the elementary abelian group F^{16} fails to do so. Indeed, a general element in F^{16} has the form $M'_x \cdot L''_y$ for some $x, y \in \mathbb{O}$ and it sends an Albert vector $(a, b, c \mid A, B, C)$ to

$$(a, b, c + a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(By) + \mathrm{T}(\bar{x}A) + \mathrm{T}(\bar{x}\overline{C}y) \mid A + \overline{C}y + bx, B + a\bar{y} + \bar{x}\overline{C}, C).$$

So, we have shown that the actions of the elements M'_x, L''_x, M_x, L_x on $\mathbb{J}$ generate a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$, as x ranges through a split algebra $\mathbb{O}$.

Next, we consider the white point $\langle v \rangle$ spanned by our white vector. The stabiliser in $\mathrm{SE}_6(F)$ of $\langle v \rangle$, where $v = (0, 0, 1 \mid 0, 0, 0)$, maps v to λv for some non-zero $\lambda \in F$. For instance, this can be achieved by the elements

$$P'_{u^{-1}} = \mathrm{diag}(1_{\mathbb{O}}, u^{-1}, u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u \end{bmatrix} \tag{3.36}$$

with u being an invertible octonion of arbitrary norm. Indeed, any such element $P'_{u^{-1}}$ sends $(0, 0, 1 \mid 0, 0, 0)$ to $(0, 0, \mathrm{N}(u) \mid 0, 0, 0)$ and since $\mathrm{N}(u)$ can be any non-zero field element, we get an abelian group $F^{\times}$ on top of the vector stabiliser. This finishes the proof of the main theorem in this section.

Now, since the vector stabiliser is generated by the actions of M_x, L_x, M'_x, L''_x on $\mathbb{J}$, and the subgroup of $\mathrm{SE}_6(F)$ generated by $M_x, M'_x, M''_x, L_x, L'_x, L''_x$ acts transitively on the white points, we make the following conclusion.

Theorem 3.3.17. *The group $\mathrm{SE}_6(F)$ is generated by the actions of M_x, M'_x, M''_x and L_x, L'_x, L''_x on $\mathbb{J}$ as x ranges through $\mathbb{O}$.*

3.4 Some related geometry

In this section we are interested in some of the underlying geometry related to white points. Consider first a 10-dimensional space $\mathbb{J}_{10}^{abC}$ and note that it contains only white and grey vectors. In this section we are interested in finding the stabiliser of $\mathbb{J}_{10}^{abC}$, discovering some of its properties, and also finding the joint stabiliser of such a 10-space and a white point. Note that throughout the whole section $\mathbb{O}$ is a split octonion algebra.

3.4.1 The stabiliser in $\mathrm{SE}_6(F)$ of $\mathbb{J}_{10}^{abC}$

The following lemma helps to get an idea what the stabiliser we are looking for can be.

Lemma 3.4.1. *The stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *contains a subgroup of shape*

$$F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}. \tag{3.37}$$

Proof. We take an arbitrary vector $(a, b, 0 \mid 0, 0, C)$ in $\mathbb{J}_{10}^{abC}$ and look how the elements M_x, M'_x, M''_x, L_x, L'_x, and L''_x act on it:

$$\begin{aligned}
M_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), 0 \mid 0, 0, C + ax),\\
M'_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, a\mathrm{N}(x) \mid bx, \bar{x}\overline{C}, C),\\
M''_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, 0 \mid 0, 0, C),\\
L_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a + b\mathrm{N}(x) + \mathrm{T}(Cx), b, 0 \mid 0, 0, C + b\bar{x}),\\
L'_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, 0 \mid 0, 0, C),\\
L''_x &: (a, b, 0 \mid 0, 0, C) \mapsto (a, b, a\mathrm{N}(x) + \mathrm{T}(Bx) \mid \overline{C}x, a\bar{x}, C).
\end{aligned}$$

It is visibly clear now that the elements M_x, L_x, M''_x, L'_x preserve $\mathbb{J}_{10}^{abC}$. We have been in a similar situation before (Theorem 3.3.8), so we just note here that the abelian group F^{16} generated by the actions of M''_x and L'_x is different from the one generated by M'_x and L''_x in the theorem we refer to. Thus, the stabiliser of $\mathbb{J}_{10}^{abC}$ is at least a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$.

Finally, we notice that each of the elements P_u, P'_u, P''_u preserve $\mathbb{J}_{10}^{abC}$. It is straightforward to see that for any invertible octonion u, $P_u \cdot P'_u \cdot P''_u$ acts on $\mathbb{J}$ as the identity matrix. Therefore we only consider the action on $\mathbb{J}_{10}^{abC}$ of two of them, say P_u and P'_u, and since P_u acts on $\mathbb{J}$ as $M_{u-1} \cdot L_1 \cdot M_{u^{-1}-1} \cdot L_{-u}$, we conclude that they represent elements of $\mathrm{Spin}_{10}^{+}(F)$, which we already have as a part of the stabiliser, so it is enough to consider the elements P'_u.

Note that the elements P'_u, where u is an arbitrary invertible octonion, preserve $\mathbb{J}_{10}^{abC}$:

$$P'_u : (a, b, 0 \mid 0, 0, C) \mapsto (a, b\mathrm{N}(u), c\mathrm{N}(u)^{-1} \mid \bar{u}A\bar{u}\mathrm{N}(u)^{-1}, uB\mathrm{N}(u)^{-1}, Cu),$$

so we get $F^{\times}$ on top of the stabiliser, and the result follows. $\square$

The following theorem strengthens this result: we prove that the stabiliser of $\mathbb{J}_{10}^{abC}$ is precisely a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}$.

Theorem 3.4.2. *The stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *is a subgroup of shape*

$$F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}, \tag{3.38}$$

generated by the actions on $\mathbb{J}$ *of the elements* M_x, M''_x, L_x, L'_x *as* x *ranges through* $\mathbb{O}$, *and* P'_u *as* u *ranges through invertible octonions in* $\mathbb{O}$.

Proof. Consider the white point W spanned by $(0, 0, 1 \mid 0, 0, 0)$. We are interested in the joint stabiliser of $\mathbb{J}_{10}^{abC}$ and W. Theorem 3.3.8 tell us that the stabiliser in $\mathrm{SE}_6(F)$ of W has the shape $G_W = F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}$, and $H \cong \mathrm{Spin}_{10}^{+}(F).F^{\times}$, generated by the actions of M_x, L_x, P'_u, stabilises $\mathbb{J}_{10}^{abC}$. The normal subgroup $T_1 \cong F^{16}$ of G_W is a left (or right) transversal of H in G_W. It is easy to see that no non-trivial element of T_1 stabilises $\mathbb{J}_{10}^{abC}$. Indeed, a general element in T_1 has the form $M'_x \cdot L''_y$ for some $x, y \in \mathbb{O}$ and it sends $(a, b, 0 \mid 0, 0, C) \in \mathbb{J}_{10}^{abC}$ to

$$(a, b, a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(\bar{x}\overline{C}y) \mid \overline{C}y + bx, a\bar{y} + \bar{x}\overline{C}, C).$$

Therefore, such an element preserves $\mathbb{J}_{10}^{abC}$ if and only if the following conditions hold for arbitrary a, b, and C:

$$\left.\begin{aligned} a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(\bar{x}\overline{C}y) &= 0, \\ \overline{C}y + bx &= 0, \\ a\bar{y} + \bar{x}\overline{C} &= 0. \end{aligned}\right\}$$

In particular, if we take $C = 0$, $b = 1$ then we obtain $x = 0$, and when $C = 0$, $a = 1$, we get $y = 0$.

Let T_2 be the subgroup of $\mathrm{SE}_6(F)$ generated by the actions on $\mathbb{J}$ of M''_x and L'_y as x, y range through $\mathbb{O}$. It can be shown that T_2 is isomorphic to F^{16}; this proof is rather similar to the proof $T_1 \cong F^{16}$. Consider the 26-dimensional space $\mathbb{J}_{26}^{abABC}$ spanned by the 17-spaces corresponding to the white vectors in $\mathbb{J}_{10}^{abC}$. Let $(a, b, c \mid A, B, C)$ be a white vector outside $\mathbb{J}_{26}^{abABC}$, that is, with $c \neq 0$. The whiteness conditions imply that such a vector has the form $(B\overline{B}/c, A\overline{A}/c, c \mid A, B, \overline{B}\overline{A}/c)$. T_2 acts sharply transitively on white points spanned by these vectors. Therefore, the full stabiliser of $\mathbb{J}_{10}^{abC}$ is indeed $F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}$. $\square$

Next, we investigate the orbits of the stabiliser of $\mathbb{J}_{10}^{abC}$ on white vectors and white points. First, we consider white vectors in $\mathbb{J}_{10}^{abC}$. An arbitrary non-zero vector $(a, b, 0 \mid 0, 0, C)$ is white if and only if $ab - C\overline{C} = 0$. Note that the stabiliser of $\mathbb{J}_{10}^{abC}$ acts on the [quotient] 10-space as $\Omega_{10}^{+}(F)$, and is transitive on such vectors. Therefore, the stabiliser of $\mathbb{J}_{10}^{abC}$ is transitive on white vectors in $\mathbb{J}_{10}^{abC}$, and on the white points spanned by them.

Suppose now that $(a, b, c \mid A, B, C)$ is a white vector such that $(c, A, B) \neq (0, 0, 0)$. We consider two cases. First, assume $c \neq 0$. The element $M''_{-c^{-1}B}$ maps our vector to $(a - c^{-1}\mathrm{N}(B), b, c \mid A, 0, C - c^{-1}\overline{B}\overline{A})$, and since the latter is white, we have $a - c^{-1}\mathrm{N}(B) = 0 = C - c^{-1}\overline{B}\overline{A}$, so our new vector is of the form $(0, b, c \mid A, 0, 0)$. Similarly, we act on it by $L_{-c^{-1}\overline{A}}$ to obtain $(0, b - c^{-1}\mathrm{N}(A), c \mid 0, 0, 0)$, and since this vector is also white, we have $b - c^{-1}\mathrm{N}(A) = 0$, so the resulting vector is of the form $(0, 0, c \mid 0, 0, 0)$, $c \neq 0$.

Second, we consider the case $c = 0$, so we start with $(a, b, 0 \mid A, B, C)$ where $(A, B) \neq (0, 0)$. The whiteness conditions are

$$\left.\begin{aligned} A\overline{A} &= 0, \\ B\overline{B} &= 0, \\ C\overline{C} &= ab, \\ AB &= 0, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned}\right\} \tag{3.39}$$

If $(a, b) = 0$, then we choose a suitable M_x, M''_x, L_x, or L'_x to map our vector to a vector of the form $(a', b', 0 \mid A', B', C')$ with $(a', b') \neq (0, 0)$. Thus, we may assume

$(a, b) \neq (0, 0)$.

We again distinguish two cases. If $a \neq 0$, then we act on $(a, b, 0 \mid A, B, C)$ by $M_{-a^{-1}C}$ to obtain $(a, 0, 0 \mid 0, B, 0)$. Next, we act by M''_y to get $(a + \mathrm{T}(\bar{y}B), 0, 0 \mid 0, B, 0)$, and choosing a suitable y we obtain $(0, 0, 0 \mid 0, B, 0)$.

If on the other hand $b \neq 0$, the action by $L_{-b^{-1}\overline{C}}$ maps $(a, b, 0 \mid A, B, C)$ to $(0, b, 0 \mid A, 0, 0)$ and similarly acting by a suitable L'_y, we obtain $(0, 0, 0 \mid A, 0, 0)$. Recall that the duality element δ preserves $\mathbb{J}_{10}^{abC}$, so it is enough to consider vectors $(0, 0, 0 \mid 0, 0, 0)$ with $A \in \mathbb{O}$. By choosing a copy of $\Omega_8^+(F)$ generated by the elements P'_u with u being an octonion of norm one, we can map $(0, 0, 0 \mid A, 0, 0)$ to $(0, 0, 0 \mid e_0, 0, 0)$.

Finally, it is impossible to map $(a, b, c \mid A, B, C)$ with $c \neq 0$ to $(a, b, 0 \mid A, B, C)$ using any of the elements M_x, M''_x, L_x, and L'_x, so these two belong to different orbits. In other words, we have shown the following result.

Theorem 3.4.3. *The stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *has three orbits on white points:*

(i) images of $\langle(1, 0, 0 \mid 0, 0, 0)\rangle$ *under the action of the stabiliser,*

(ii) images of $\langle(0, 0, 0 \mid e_0, 0, 0)\rangle$ *under the action of the stabiliser,*

(iii) images of $\langle(0, 0, 1 \mid 0, 0, 0)\rangle$ *under the action of the stabiliser.*

3.4.2 Joint stabilisers of $\mathbb{J}_{10}^{abC}$ and a white point

Now that we know the structure of the stabiliser of $\mathbb{J}_{10}^{abC}$ in $\mathrm{SE}_6(F)$, and moreover, we know the orbits of its action on white points, it is possible to figure out the joint stabilisers. The results are presented in a series of lemmas.

Lemma 3.4.4. *The joint stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *and* $\langle(1, 0, 0 \mid 0, 0, 0)\rangle$ *is the subgroup of shape*

$$F^{16}{:}F^8{:}\mathrm{Spin}_8^+(F).F^\times.F^\times. \tag{3.40}$$

Proof. The orbit of $\langle(1, 0, 0 \mid 0, 0, 0)\rangle$ under the action of the stabiliser of $\mathbb{J}_{10}^{abC}$ is the set of all white points spanned by the vectors in this 10-space. Any white point $\langle(a, b, 0 \mid 0, 0, C)\rangle$ satisfies $ab - C\overline{C} = 0$. The normal subgroup $F^{16} \triangleleft F^{16}{:}\mathrm{Spin}_{10}^+(F).F^\times$ acts trivially on $\mathbb{J}_{10}^{abC}$. Preserving a white point is equivalent to preserving an isotropic point in $\mathrm{Spin}_{10}^+(F)$ and thus we obtain $F^{16}{:}F^8{:}\mathrm{Spin}_8^+(F).F^\times.F^\times$. $\square$

Lemma 3.4.5. *The joint stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *and* $\langle(0,0,0 \mid e_0,0,0)\rangle$ *is the subgroup of shape*

$$F^{11}{:}F^{10}{:}\mathrm{SL}_5(F).F^\times.F^\times. \tag{3.41}$$

Proof. The 17-space of $\langle(0,0,0 \mid e_0,0,0)\rangle$ is $\langle(0,b,c \mid e_0,0,0)\rangle$ where

$$\begin{aligned}
A &\in \langle e_{-1}, e_{\overline{\omega}}, e_\omega, e_0, e_{-\omega}, e_{-\overline{\omega}}, e_1\rangle,\\
B &\in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\rangle,\\
C &\in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\rangle.
\end{aligned}$$

The joint stabiliser preserves this space and hence also its intersection with $\mathbb{J}_{10}^{abC}$, which is a 5-dimensional space $\langle(0,b,0 \mid 0,0,C)\rangle$ where $C \in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\rangle$. Consider the actions of M_x, L_x, M''_x, and L'_x on this intersection:

$$\begin{aligned}
M_x &: (0,b,0 \mid 0,0,C) \mapsto (0, b + \mathrm{T}(\bar{x}C), 0 \mid 0,0,C),\\
L_x &: (0,b,0 \mid 0,0,C) \mapsto (b\mathrm{N}(x) + \mathrm{T}(Cx), b, 0 \mid 0,0,C),\\
M''_x &: (0,b,0 \mid 0,0,C) \mapsto (0,b,0 \mid 0,0,C),\\
L'_x &: (0,b,0 \mid 0,0,C) \mapsto (0,b,0 \mid 0,0,C).
\end{aligned}$$

It follows that M_x preserves the 5-space for any $x \in \mathbb{O}$, and L_x does so if and only if $b\mathrm{N}(x) + \mathrm{T}(Cx) = 0$, from which we find $x \in \langle e_{-1}, e_0, e_{-\omega}, e_{-\overline{\omega}}\rangle$. Finally, M''_x and L'_x act trivially on the intersection. $\square$

We have already seen the proof of the following lemma (Theorem 3.4.2), so we just state the result here.

Lemma 3.4.6. *The joint stabiliser in* $\mathrm{SE}_6(F)$ *of* $\mathbb{J}_{10}^{abC}$ *and* $\langle(0,0,1 \mid 0,0,0)\rangle$ *is the subgroup of shape*

$$\mathrm{Spin}^+_{10}(F).F^\times. \tag{3.42}$$

3.5 Simplicity of $\mathrm{E}_6(F)$

The construction we have obtained also allows us to show that the group $\mathrm{E}_6(F)$ is indeed simple without any references to Lie theory. The classical way of showing the simplicity of certain groups is the following lemma.

Lemma 3.5.1 (Iwasawa)**.** *If G is a perfect group acting faithfully and primitively on a set Ω, and the point stabilizer H has a normal abelian subgroup A whose conjugates generate G, then G is simple.*

First, we show that the subgroup of $\mathrm{SE}_6(F)$ stabilising all the white points simultaneously acts on $\mathbb{J}$ by scalar multiplications, and hence the action of $\mathrm{E}_6(F)$ on the set of white points is faithful.

Lemma 3.5.2. *The subgroup in* $\mathrm{SE}_6(F)$ *stabilising simultaneously all white points is the group of scalars.*

Proof. Consider the action of this stabiliser on $\mathbb{J}_{10}^{abC}$ and pick the basis

$$\begin{aligned} v_1 &= (1,0,0 \mid 0,0,0), \\ v_2 &= (0,1,0 \mid 0,0,0), \\ v_{i+2} &= (a_i, b_i, 0 \mid 0,0,C_i), \end{aligned} \tag{3.43}$$

where $1 \leqslant i \leqslant 8$ and $C_i\overline{C}_i = a_i b_i$. Since in particular we stabilise $\langle v_1\rangle, ..., \langle v_{10}\rangle$, the action on $\mathbb{J}_{10}^{cAB}$ is that of a 10×10 diagonal matrix $\mathrm{diag}(\lambda_1, ..., \lambda_{10})$ with respect to the basis $\{v_1, ..., v_{10}\}$. Consider an Albert vector $v = (a, b, 0 \mid 0, 0, C)$, where $C = C_1 + \cdots + C_8$ and a, b are such that v is white, i.e. $C\overline{C} = ab$. Now, if $F \neq \mathbb{F}_2$, we can choose $a, b \in F$ in such a way that v can be written as a linear combination $v = \alpha v_1 + \beta v_2 + v_3 + \cdots + v_{10}$ with $\alpha \neq 0$. The stabiliser of all white point maps v to λv for some non-zero $\lambda \in F$, so this ensures that $\lambda = \lambda_1 = \lambda_3 = \cdots = \lambda_{10}$. We now adjust the chosen values of a and b to obtain a linear combination with $\beta \neq 0$, and so $\lambda = \lambda_2 = \lambda_3 = \cdots \lambda_{10}$. It follows that the action on $\mathbb{J}_{10}^{abC}$ is just the multiplication by λ.

When $F = \mathbb{F}_2$, we take $\mathbb{O}$ to be the split octonion algebra with our favourite basis $\{e_i \mid i \in \pm\{0, 1, \omega, \overline{\omega}\}\}$. For the 10-space $\mathbb{J}_{10}^{abC}$ we choose the basis

$$\begin{aligned} v_1 &= (1,0,0 \mid 0,0,0), \\ v_2 &= (0,1,0 \mid 0,0,0), \\ v_{i+2} &= (0,0,0 \mid 0,0,e_i), \end{aligned} \tag{3.44}$$

and then proceed in the same manner. The vector $v = v_1 + \cdots + v_{10}$ is white and since there is a single choice for a non-zero scalar in $\mathbb{F}_2$, it is being fixed and the action on the whole 10-space in this case is that of $\operatorname{diag}(1, ..., 1)$.

Now, by using the triality element, we map $\mathbb{J}_{10}^{abC}$ to $\mathbb{J}_{10}^{bcA}$ and further to $\mathbb{J}_{10}^{caB}$ and so we obtain that the stabiliser of all white points acts on $\mathbb{J}$ by scalar multiplications. That is, the stabiliser of all the white points is trivial in $\mathrm{E}_6(F)$.

□

From Lemma 3.3.5 we know that the action of $\mathrm{E}_6(F)$ on the white points is primitive. We need to show that the group is perfect.

Lemma 3.5.3. *The group* $\mathrm{SE}_6(F)$ *is perfect.*

Proof. This does not present great difficulties. A very straightforward computation shows that

$$\begin{aligned}
(L''_{-1})^{-1} \cdot L'_x \cdot L''_{-1} \cdot (L'_x)^{-1} &\text{ acts as } M_x, \\
(L_{-1})^{-1} \cdot L''_x \cdot L_{-1} \cdot (L''_x)^{-1} &\text{ acts as } M'_x, \\
(L'_{-1})^{-1} \cdot L_x \cdot L'_{-1} \cdot (L_x)^{-1} &\text{ acts as } M''_x, \\
(M'_{-1})^{-1} \cdot M''_x \cdot M'_{-1} \cdot (M''_x)^{-1} &\text{ acts as } L_x, \\
(M''_{-1})^{-1} \cdot M_x \cdot M''_{-1} \cdot (M_x)^{-1} &\text{ acts as } L'_x, \\
(M_{-1})^{-1} \cdot M'_x \cdot M_{-1} \cdot (M'_x)^{-1} &\text{ acts as } L''_x,
\end{aligned}$$

where as before $A \cdot B$ is understood as the product of the actions by the matrices A and B. Hence, every generator is in fact a commutator. □

Finally, using the Iwasawa's Lemma we obtain the following theorem.

Theorem 3.5.4. *The group* $\mathrm{E}_6(F)$ *is simple.*

3.6 Case of a finite field

In this section F is a finite field of q elements, that is, $F = \mathbb{F}_q$. Our aim is to count the white points in this case, and hence find the group order.

Theorem 3.6.1. *If $F = \mathbb{F}_q$, then there are precisely*

$$\frac{(q^{12}-1)(q^9-1)}{(q^4-1)} \tag{3.45}$$

white vectors in $\mathbb{J}$.

Proof. In the proof we study the series of subspaces

$$0 < \mathbb{J}_{10}^{abC} < \mathbb{J}_{26}^{abABC} < \mathbb{J}.$$

First, $(a, b, 0 \mid 0, 0, C) \in \mathbb{J}_{10}^{abC}$ is white if and only if $ab - C\overline{C} = 0$. We notice that $ab - C\overline{C}$ is a quadratic form of *plus* type defined on $\mathbb{J}_{10}^{abC}$, so there are $(q^5 - 1)(q^4 + 1)$ white vectors in this subspace.

Next, suppose $(a, b, c \mid A, B, C) \in \mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$ is white. Then $C = \overline{B}\,\overline{A}c^{-1}$, $b = A\overline{A}c^{-1}$ and $a = B\overline{B}c^{-1}$. We may choose A, B, c to be arbitrary (with $c \neq 0$), so there are $q^{16}(q - 1)$ white vectors in $\mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$.

Finally, we investigate the white vectors $(a, b, 0 \mid A, B, C) \in \mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$. The conditions for such a vector to be white take the following form:

$$\left.\begin{aligned} A\overline{A} = B\overline{B} = AB &= 0, \\ C\overline{C} &= ab, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned}\right\}. \tag{3.46}$$

Note that we also require $(A, B) \neq (0, 0)$. In case $A = 0$, $B \neq 0$ we apply δ followed by τ to $(a, b, 0 \mid A, B, C)$ in order to obtain a vector of the form $(a, b, 0 \mid A, B, C)$ with $A \neq 0$. Note that the values of a, b, A, B, C are not the same as in the initial Albert vector. So, assuming $A \neq 0$, we have $AB = 0$ exactly when B is in a particular 4-dimensional subspace of $\mathbb{O}$ and any such B satisfies $B\overline{B} = 0$. For any octonion x, the action by the element L_x establishes a bijection between the white vectors of the form $(*, *, 0 \mid A, B, *)$ and those of the form $(*, *, 0 \mid A, B + \overline{A}x, *)$. Left-multiplication by $\overline{A}$ annihilates a 4-dimensional subspace of $\mathbb{O}$ (see Lemma **??**), so by the rank-nullity theorem we conclude that the image $\mathcal{A} = \{\overline{A}x \mid x \in \mathbb{O}\}$ is also 4-dimensional. Note that $A(\overline{A}x) = (A\overline{A})x = 0$, for any $x \in \mathbb{O}$, so $\mathcal{A}$ is the 4-space of all octonions left-annihilated by A, and therefore it contains $-B$. Now we pick an octonion x such that $\overline{A}x = -B$ to obtain the bijection between the white vectors of the form $(*, *, 0 \mid A, B, *)$ with $A \neq 0$ and those of the form $(*, *, 0 \mid A, 0, *)$. An Albert vector $(a, b, 0 \mid A, 0, C)$ is white if and only if $A\overline{A} = C\overline{C} = CA = 0$ and $a = 0$, with no dependence on b. As before, C lies in a particular 4-dimensional subspace of $\mathbb{O}$, hence $(0, b, 0 \mid 0, 0, C)$ lies in a particular 5-dimensional subspace of

$\mathbb{J}_{10}^{abC}$, so for any choice of the pair (A, B) there are q^5 white vectors. If $A \neq 0$, then there are $(q^4-1)(q^3+1)$ choices for A, and for each of these q^4 choices for B. If $A = 0$, we have $(q^4-1)(q^3+1)$ choices for B. It follows that in total there are

$$q^5(q^4(q^4-1)(q^3+1)+(q^4-1)(q^3+1)) = q^5(q^8-1)(q^3+1)$$

white vectors in $\mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$.

The calculations above give the numbers of white vectors in certain subsets of $\mathbb{J}$ as shown in the following table.

subset	$\mathbb{J}_{10}^{abC}$	$\mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$	$\mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$
number of white vectors	$(q^5-1)(q^4+1)$	$q^5(q^8-1)(q^3+1)$	$q^{16}(q-1)$

Adding these numbers gives the result. □

Corollary 3.6.2. *There are precisely*

$$\frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)} \tag{3.47}$$

white points in the case $F = \mathbb{F}_q$.

Theorem 3.3.8 allows us to find the stabiliser of a white point which in our case is a group of shape $q^{16}{:}\mathrm{Spin}_{10}^{+}(q)$. As a consequence, we now have:

$$|\mathrm{SE}_6(q)| = q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1). \tag{3.48}$$

We obtain $\mathrm{E}_6(q)$ as the quotient of $\mathrm{SE}_6(q)$ by any scalars it contains. Note that $\mathrm{SE}_6(q)$ contains non-trivial scalars if and only if $q \equiv 1 \pmod 3$, so

$$|\mathrm{E}_6(q)| = \frac{1}{\gcd(3, q-1)} q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1). \tag{3.49}$$

We can also invoke Proposition 3.3.7 to count totally white lines in the finite case.

Proposition 3.6.3. *Given any white point $\langle W \rangle$,* *(i) there are exactly $q(q^8-1)(q^3+1)/(q-1)$ white points $\langle X\rangle$ such that all points in $\langle W, X\rangle$ are white;*

(ii) there are exactly $q^8(q^4+1)(q^5-1)/(q-1)$ white points $\langle Y\rangle$ such that $\langle W, Y\rangle$ contains only two white points.

Proof. Without loss of generality assume $W = (0, 0, 1 \mid 0, 0, 0)$. By Proposition 3.3.7 we need to count the white points spanned by the vectors in $\mathbb{J}_{17}^{cAB}$. Consider a general white vector $(0, 0, c \mid A, B, 0) \in \mathbb{J}_{17}^{cAB}$. The whiteness conditions are $A\overline{A} = B\overline{B} = 0 = AB$. We distinguish two cases. If $A \neq 0$, then there are $(q^4-1)(q^3+1)$ choices for A. For each of these there are q^4 choices for B and q choices for c. Therefore, there are $q^5(q^4-1)(q^3+1)$ white vectors in $\mathbb{J}_{17}^{cAB}$ with $A \neq 0$. In case $A = 0$ we require $B \neq 0$, so there are $(q^4-1)(q^3+1)$ choices for B and q choices for c, giving $q(q^4-1)(q^3+1)$ white vectors. Adding these two values together, we get

$$q^5(q^4-1)(q^3+1) + q(q^4-1)(q^3+1) = q(q^8-1)(q^3+1)$$

white vectors in $\mathbb{J}_{17}^{cAB}$. To find the number of totally white lines, we divide this value by $q-1$.

Obviously, all white points not already counted have the second property, so there are

$$\frac{(q^9-1)(q^{12}-1)}{(q^4-1)(q-1)} - \frac{q(q^8-1)(q^3+1)}{(q-1)} - 1 = \frac{q^8(q^4+1)(q^5-1)}{(q-1)}$$

choices for $\langle Y\rangle$. Note that we subtract 1 because we want $\langle W\rangle \neq \langle Y\rangle$. □

3.7 Arbitrary octonion algebras

Now that we have constructed the group of type E_6 over an arbitrary field, it is to be emphasised that our construction depends on the fact that $\mathbb{O}$ has to be split. Namely, it is a vital requirement in the proof of Theorem 3.3.8. This completely covers the possibilities in case $F = \mathbb{F}_q$, but while it was possible to prove many results independently of the choice of $\mathbb{O}$, there are still some questions to answer when $\mathbb{O}$ happens to be non-split. Note that a split octonion algebra exists over any field, so our construction is safe.

The main problem is to be able to tell whether the actions of the matrices M_x and L_x on $\mathbb{J}_{10}^{abC}$ generate $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$. At this stage it is possible to prove the following proposition.

Proposition 3.7.1. *The actions of the elements M_x and L_x on $\mathbb{J}_{10}^{abC}$ where x ranges through a non-split octonion algebra $\mathbb{O}$, generate at most a group of type $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$.*

Proof. To verify this we show that the elements encoded by M_x and L_x have the correct spinor norm. Recall that M_x acts on $\mathbb{J}_{10}^{abC}$ in the following way:

$$M_x : (a, b, 0 \mid 0, 0, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), 0 \mid 0, 0, C + ax).$$

Consider two Albert vectors $v = (0, 0, 0 \mid 0, 0, x)$ and $w = (0, x\bar{x}, 0 \mid 0, 0, x)$. Reflexion in v sends a vector $(a, b, 0 \mid 0, 0, C)$ to

$$(a, b, 0 \mid 0, 0, C) - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} \cdot (0, 0, 0 \mid 0, 0, x) = \left(a, b, 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x\right).$$

We then reflect the result in w to get

$$\begin{aligned}
&\left(a, b, 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x\right) - \frac{-\mathrm{T}(C\bar{x}) - ax\bar{x}}{\mathrm{N}(x)} \cdot (0, x\bar{x}, 0 \mid 0, 0, x) \\
&\quad = \left(a, b + ax\bar{x} + \mathrm{T}(C\bar{x}), 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x + \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x + a\frac{x\bar{x}}{\mathrm{N}(x)} x\right) \\
&\quad = (a, b + a\mathrm{N}(x) + \mathrm{T}(C\bar{x}), 0 \mid 0, 0, C + ax).
\end{aligned}$$

Note that the action of M_x on $\mathbb{J}_{10}^{abC}$ is the same as the composition of reflexions in v and w. We find $\mathrm{Q}_{10}(v) = \mathrm{Q}_{10}(w) = \mathrm{N}(x)$ and conclude that M_x acts as an element of $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$.

For the elements L_x we consider the reflexions in vectors $(0, 0, 0 \mid 0, 0, \bar{x})$ and $(x\bar{x}, 0, 0 \mid 0, 0, \bar{x})$ to obtain the same conclusion. $\square$

Next, suppose that V is a vector space over F with a quadratic form Q, such that $V = \langle e, f\rangle \oplus W$, where (e, f) is a hyperbolic pair and $W = \langle e, f\rangle^{\perp}$. Consider an element g in $\mathrm{CGO}(V, Q)$ which scales of Q by some $\lambda \neq 0$. Then $V = \langle e^g, f^g \rangle \oplus W^g$ and $\langle e^g, f^g \rangle$ is isometric to $\langle e, f \rangle$. Therefore, W^g is isometric to W, and so there exists an isometry h in $\mathrm{GO}(V, Q)$ such that $\langle e^g, f^g \rangle^h = \langle e, f \rangle$. It follows that $(W^g)^h = W$, and gh is a λ-scaling of Q which fixes $\langle e, f \rangle$ and W. Hence, gh is a λ-scaling of Q_W.

Consider a λ-similarity on $\mathbb{O} = \mathbb{O}_F$ that sends $1_{\mathbb{O}}$ to some $u \in \mathbb{O}$. Then it gives rise to an element in the stabiliser of a white point which scales Q_8 by $\mathrm{N}(u)$. In other words, we have shown the following.

Proposition 3.7.2. *If $\mathbb{O}$ is an arbitrary octonion algebra over F, then the elements in the stabiliser of a white point can only scale a white vector by λ, where $\lambda \in F$ is such that there exists $u \in \mathbb{O}$ with $\mathrm{N}(u) = \lambda$.*

It is easy to check that all such scalings are possible. For example, the elements $P'_{u^{-1}}$, defined in (3.36), do the job.

Chapter 4

Groups of type ${}^2\mathrm{E}_6$

4.1 Quadratic field extensions

Let F and K be two fields such that F is a subfield of K. We say that K is an *extension field* of F. The *degree* of K over F, denoted by $[K : F]$, is the dimension of K as a vector space over F. We denote the extension of F by K as $K : F$.

A non-zero polynomial $f \in F[x]$ is called *separable*, if each root of f has multiplicity 1. If $\alpha \in F$ is algebraic, that is, α is a root of some polynomial $g \in F[x]$, then α is called *separable*, if its minimal polynomial is separable.

We have defined what it means for a field element and a polynomial to be separable. Suppose that $[K : F]$ is finite, then $K : F$ is a *separable* extension, if every element of K is separable over F. We also say that $K : F$ is *normal*, if every irreducible polynomial $f \in F[x]$ that has a root in K, splits into linear factors in $K[x]$. An extension $K : F$ which is normal, separable, and of finite degree, is called a *Galois extension*. We are not going into too much detail here, for there are various well-known references on theory of field extensions, of very high quality, for instance, [?], [?], [?], or [?]. In this chapter we are interested in Galois extensions of degree 2. In case when $F = \mathbb{F}_q$, we have $|K| = q^2$, and so $K = \mathbb{F}_{q^2}$ (see, for example, [?]).

Given an extension $K : F$, we define the *Galois group* of K over F, denoted $\mathrm{Gal}(K : F)$, to be the group of automorphisms of K that fix F elementwise. In other words,

$$\mathrm{Gal}(K:F) = \{\ \sigma \in \mathrm{Aut}(K)\ \big|\ \alpha^\sigma = \alpha \text{ for all } \alpha \in F\ \}. \tag{4.1}$$

The most important result for us here is the following theorem.

Theorem 4.1.1. *Let $K:F$ be a Galois extension. Then*

$$|\mathrm{Gal}(K:F)| = [K:F]. \tag{4.2}$$

It follows that in case $[K:F] = 2$ there is a unique non-trivial field automorphism $\sigma : K \to K$, fixing F elementwise. If $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$, then σ takes the form $\sigma : \lambda \mapsto \lambda^q$.

4.2 Spaces with two forms

Let $K:F$ be a Galois extension of degree 2, and let s be a K-automorphism with F being its fixed field. Let also V be a $2m$-dimensional vector space over K with a quadratic form Q and a conjugate-symmetric sesquilinear form $B : V \times V \to K$, defined with respect to σ. Denote by f the polar form of Q, and suppose that $\mathcal{B} = \{e_1, f_1, e_2, f_2, ..., e_m, f_m\}$ is a hyperbolic basis of V with respect to f, i.e.

$$Q(e_i) = Q(f_i) = 0,\ f(e_i, f_i) = 1, \tag{4.3}$$

and $f(e_i, e_j) = f(e_i, f_j) = 0$ for $i \neq j$. The above said means that (V, Q) is a hyperbolic orthogonal space. Denote by G the maximal amongst all the subgroups of $\mathrm{GO}(V, Q)$ which preserve B. If U is a subspace of V, then we denote the restrictions of f and B on U as f_U and B_U respectively. We say that an element $v \in V$ is *singular isotropic*, if $Q(v) = B(v, v) = 0$.

Definition 4.2.1. *A (Q, B)-subspace of V is an F-subspace U of V such that $V = U \otimes_F K$, $f_U = B_U$ is an F-form on U, and Q_U is a non-degenerate quadratic form on U of Witt index at least* 2.

Proposition 4.2.2. *If U is a (Q, B)-subspace of V, then it is the unique (Q, B)-subspace of V, and $G = \mathrm{GO}(U, Q)$.*

Proposition 4.2.3. *Let U be a (Q, B)-subspace of V if Witt index at least 2. Fix $\lambda \in K \backslash F$. The group G has two orbits on doubly singular points with representatives $\langle u \rangle$ and $\langle u + \lambda v \rangle$, where $u, v \in U$ and $\langle u, v \rangle$ is a singular line.*

4.3 Hermitean form in $\mathbb{J}$ and the group ${}^2\mathrm{SE}_6^K(F)$

Suppose as before $K : F$ is a quadratic Galois extension with F being a fixed field of a K-automorphism σ. In this chapter $\mathbb{O}_F$ will always be a split octonion algebra over F and $\mathbb{O}_K = \mathbb{O}_F \otimes_F K$. We also use the same basis $\{ e_i \mid i \in \pm I \}$ as in Section 2.6.

We slightly abuse the notation here denoting by σ also the automorphism of $\mathbb{O}_K$ induced by the field automorphism σ:

$$\left(\sum_{i \in \pm I} \lambda_i e_i \right)^{\sigma} = \sum_{i \in \pm I} \lambda_i^{\sigma} e_i. \tag{4.4}$$

This, however, should not create any difficulties for the reader. Consider the following Hermitean form defined on the elements of $\mathbb{O}_K$:

$$h(x) = A\overline{A}^{\sigma} + A^{\sigma}\overline{A} = \mathrm{T}(A\overline{A}^{\sigma}). \tag{4.5}$$

On the Albert space $\mathbb{J} = \mathbb{J}_K$ this induces the Hermitean form H, where

$$\mathrm{H}((a, b, c \mid A, B, C)) = aa^{\sigma} + bb^{\sigma} + cc^{\sigma} + \mathrm{T}(A\overline{A}^{\sigma} + B\overline{B}^{\sigma} + C\overline{C}^{\sigma}). \tag{4.6}$$

As usual, the sesquilinear form associated with H is obtained by polarising the Hermitean form:

$$\langle X, Y \rangle_{\mathrm{H}} = H(X + Y) - H(X) - H(Y). \tag{4.7}$$

Using the construction from previous chapter, we obtain the group $\mathrm{SE}_6(K)$ in the usual way. Now define the group ${}^2\mathrm{SE}_6^K(F)$ as the subgroup of $\mathrm{SE}_6(K)$ which preserves H. In case $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$, we denote this by ${}^2\mathrm{SE}_6(q)$. As before, the group ${}^2\mathrm{E}_6(F/K)$ is defined as the quotient of ${}^2\mathrm{SE}_6^K(F)$ by its centre.

4.4 Some elements of ${}^2\mathrm{SE}_6^K(F)$

Let $X = (a, b, c \mid A, B, C)$ be an arbitrary element of $\mathbb{J} = \mathbb{J}_K$. First of all we notice that the matrices δ and τ preserve the Hermitean form, so they encode the elements of ${}^2\mathrm{SE}_6^K(F)$.

The elements P_u with u written over the small field and such that $\mathrm{N}(u) = 1$ are also of interest.

Lemma 4.4.1. *The actions on* $\mathbb{J}$ *of the elements* $P_u = \mathrm{diag}(u, \bar{u}, 1)$ *such that* $u \in \mathbb{O}_F$ *and* $\mathrm{N}(u) = 1$*, preserve* H.

Proof. Recall that the action on $\mathbb{J}$ is given by

$$P_u : (a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \bar{u}C\bar{u}),$$

so the individual terms are being mapped in the following way:

$$\begin{aligned} aa^\sigma &\mapsto aa^\sigma, \\ bb^\sigma &\mapsto bb^\sigma, \\ cc^\sigma &\mapsto cc^\sigma, \\ \mathrm{T}(A\overline{A}^\sigma) &\mapsto \mathrm{T}((uA)(\overline{A}^\sigma\bar{u})), \\ \mathrm{T}(B\overline{B}^\sigma) &\mapsto \mathrm{T}((Bu)(\bar{u}\overline{B}^\sigma), \\ \mathrm{T}(\bar{u}C\bar{u}) &\mapsto \mathrm{T}((\bar{u}C\bar{u})(u\overline{C}^\sigma u)). \end{aligned}$$

Note that $u^\sigma = u$ since $u \in \mathbb{O}_F$. We find

$$\begin{aligned} \mathrm{T}((uA)(\overline{A}^\sigma\bar{u})) = \mathrm{T}(((uA)\overline{A}^\sigma)\bar{u}) &= \mathrm{T}(\bar{u}((uA)\overline{A}^\sigma)) \\ &= \mathrm{T}((\bar{u}(uA))\overline{A}^\sigma) = \mathrm{T}(((\bar{u}u)A)\overline{A}^\sigma) = \mathrm{T}(A\overline{A}^\sigma). \end{aligned}$$

Similarly,

$$\mathrm{T}((Bu)(\bar{u}\overline{B}^\sigma) = \mathrm{T}(((Bu)\bar{u})\overline{B}^\sigma) = \mathrm{T}((B(u\bar{u}))\overline{B}^\sigma) = \mathrm{T}(B\overline{B}^\sigma).$$

For the last term we have

$$\mathrm{T}((\bar{u}C\bar{u})(u\overline{C}^\sigma u)) = \langle \bar{u}C\bar{u}, \bar{u}C^\sigma\bar{u} \rangle.$$

Now, as we know from Lemma 3.2.2, the map $x \mapsto \bar{u}x\bar{u}$ is a product of two reflexions, hence,

$$\langle \bar{u}C\bar{u}, \bar{u}C^\sigma\bar{u}\rangle = \langle C, C^\sigma\rangle = \mathrm{T}(C\overline{C}^\sigma).$$

□

Lemma 4.4.2. *Let $x \in \mathbb{O}_K$ be such that $\bar{x}^\sigma x = x\bar{x}^\sigma = 0$. Then $x\bar{x} = 0$.*

Proof. If $x = 0$, then the result is trivial. Assume x is non-zero, $\bar{x}^\sigma x = x\bar{x}^\sigma = 0$. If $x\bar{x} \neq 0$, then x is invertible, and so $\bar{x}^s = 0$, which implies $x = 0$, a contradiction. □

Consider the matrices

$$N_x = \begin{bmatrix} 1 & x & 0 \\ -\bar{x}^\sigma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & -\bar{x}^\sigma & 1 \end{bmatrix}, \quad N''_x = \begin{bmatrix} 1 & 0 & -\bar{x}^\sigma \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}, \tag{4.8}$$

where $x\bar{x}^\sigma = \bar{x}^\sigma x = 0$ and $x, \bar{x}^\sigma$ generate a sociable subalgebra. It is easy to see that these encode the elements of $\mathrm{SE}_6(K)$. Indeed,

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\bar{x}^\sigma & 1 \end{bmatrix} = \begin{bmatrix} 1 - x\bar{x}^\sigma & x \\ -\bar{x}^\sigma & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -\bar{x}^\sigma & 1 \end{bmatrix}. \tag{4.9}$$

So, the elements N_x, N'_x, and N''_x preserve the Dickson–Freudenthal determinant. To verify that they preserve H, we look at the action on $\mathbb{J}$:

$$\begin{aligned} N_x : (a, b, c \mid A, B, C) \mapsto (&a - \mathrm{T}(C\bar{x}^\sigma), b + \mathrm{T}(\overline{C}x), c \mid \\ &\mid A + \bar{x}\overline{B}, B - \overline{A}\bar{x}^\sigma, C - x^\sigma\overline{C}x + ax - bx^\sigma), \\ N'_x : (a, b, c \mid A, B, C) \mapsto (&a, b - \mathrm{T}(A\bar{x}^\sigma), c + \mathrm{T}(\overline{A}x) \mid \\ &\mid A - x^\sigma\overline{A}x + bx - cx^\sigma, B + \bar{x}\overline{C}, C - \overline{B}\bar{x}^\sigma), \\ N''_x : (a, b, c \mid A, B, C) \mapsto (&a + \mathrm{T}(\overline{B}x), b, c - \mathrm{T}(B\bar{x}^\sigma) \mid \\ &\mid A - \overline{C}\bar{x}^\sigma, B - x^\sigma\overline{B}x + cx - ax^\sigma, C + \bar{x}\overline{A}). \end{aligned} \tag{4.10}$$

We need to prove an auxiliary lemma.

Lemma 4.4.3. *Suppose that $x, y, z \in \mathbb{O}_F$ with $x\bar{x} = 0$. Then*

(i) $x\,\mathrm{T}(yx) = x(yx)$,

(ii) $\mathrm{T}((xy)(z\bar{x})) = 0$.

Proof.

(i) $x\,\mathrm{T}(yx) = x(yx + \bar{x}\bar{y}) = x(yx) + x(\bar{x}\bar{y}) = x(yx) + (x\bar{x})y = x(yx)$,

(ii) $\mathrm{T}((xy)(z\bar{x})) = \mathrm{T}((z\bar{x})(xy)) = \mathrm{T}(((z\bar{x})x)y) = \mathrm{T}(z(x\bar{x})y) = 0$. $\square$

Obviously, it is enough to verify that the elements N_x preserve the Hermitean form H. The individual terms in $\mathrm{H}(X) = aa^\sigma + bb^\sigma + cc^\sigma + \mathrm{T}(A\overline{A}^\sigma + B\overline{B}^\sigma + C\overline{C}^\sigma)$ are being mapped in the following way:

$$\begin{aligned}
aa^\sigma &\mapsto aa^\sigma - a^\sigma\,\mathrm{T}(C\bar{x}^\sigma) - a\,\mathrm{T}(C^\sigma\bar{x}) + \mathrm{T}(C^\sigma\bar{x})\,\mathrm{T}(C\bar{x}^\sigma),\\
bb^\sigma &\mapsto bb^\sigma + b^\sigma\,\mathrm{T}(\overline{C}x) + b\,\mathrm{T}(\overline{C}^\sigma x^\sigma) + \mathrm{T}(\overline{C}x)\,\mathrm{T}(\overline{C}^\sigma x^\sigma),\\
cc^\sigma &\mapsto cc^\sigma,\\
\mathrm{T}(A\overline{A}^\sigma) &\mapsto \mathrm{T}(A\overline{A}^\sigma) + \mathrm{T}(AB^\sigma x^\sigma) + \mathrm{T}(\bar{x}\overline{B}\overline{A}^\sigma) + \mathrm{T}((\bar{x}\overline{B})(B^\sigma x^\sigma)),\\
\mathrm{T}(B\overline{B}^\sigma) &\mapsto \mathrm{T}(B\overline{B}^\sigma) - \mathrm{T}(\overline{A}\bar{x}^\sigma\overline{B}^\sigma) - \mathrm{T}(BxA^\sigma) + \mathrm{T}((\overline{A}\bar{x}^\sigma)(xA^\sigma)),\\
\mathrm{T}(C\overline{C}^\sigma) &\mapsto \mathrm{T}(C\overline{C}^\sigma) - \mathrm{T}(C(\bar{x}^\sigma C^\sigma\bar{x})) - \mathrm{T}(C^\sigma(\bar{x}C\bar{x}^\sigma)) + a\,\mathrm{T}(x\overline{C}^\sigma) +\\
&\qquad +a^\sigma\,\mathrm{T}(C\bar{x}^\sigma) - b^\sigma\,\mathrm{T}(C\bar{x}) - b\,\mathrm{T}(x^\sigma\overline{C}^\sigma).
\end{aligned} \tag{4.11}$$

Using Lemmas 4.4.2 and 4.4.3, we get $\mathrm{T}((\overline{A}\bar{x}^\sigma)(xA^\sigma)) = 0 = \mathrm{T}((\bar{x}\overline{B})(B^\sigma x^\sigma))$. Next, we also obtain $\mathrm{T}(C(\bar{x}^\sigma C^\sigma\bar{x})) = \mathrm{T}(C\bar{x}^\sigma\,\mathrm{T}(C^\sigma\bar{x})) = \mathrm{T}(C^\sigma\bar{x})\,\mathrm{T}(C\bar{x}^\sigma)$. Likewise, we get $\mathrm{T}(C^\sigma(\bar{x}C\bar{x}^\sigma)) = \mathrm{T}((x^\sigma\overline{C}x)\overline{C}^\sigma) = \mathrm{T}(\overline{C}^\sigma(x^\sigma\overline{C}x)) = \mathrm{T}(\overline{C}^\sigma x^\sigma)\,\mathrm{T}(\overline{C}x)$. We see that all the terms except $aa^\sigma, bb^\sigma, cc^\sigma$ and $\mathrm{T}(A\overline{A}^\sigma), \mathrm{T}(B\overline{B}^\sigma), \mathrm{T}(C\overline{C}^\sigma)$ cancel out, so it follows that the elements N_x preserve the Hermitean form. Hence, we have shown the following.

Proposition 4.4.4. *The matrices N_x, N'_x, and N''_x where $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$ and $x, \bar{x}^\sigma$ generate a sociable subalgebra, encode the elements of ${}^2\mathrm{SE}_6^K(F)$.*

Lemma 4.4.5. *If $x \in \mathbb{O}$ is such that $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$, with $x, \bar{x}^\sigma$ generating a sociable subalgebra, then $x^\sigma yx = xyx^\sigma$ for all $y \in \mathbb{O}_K$.*

Of great interest for us is the action of N_x on $\mathbb{J}_{10}^{abC}$. The rest of the section is devoted to proving the following result.

Theorem 4.4.6. *The actions of the elements N_x on $\mathbb{J}_{10}^{abC}$, where x is such that $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$, with $x, \bar{x}^\sigma$ generating a sociable subalgebra, generate a group of type $\Omega_{10}^{-,K}(F)$, as x ranges through all suitable octonions in $\mathbb{O}_K$.*

We prove this theorem in the series of steps. First, consider the 4-dimensional K-subspace V_4 of $\mathbb{J}_K$, spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_1 \rangle$.

Lemma 4.4.7. *The actions of the elements $N_{\lambda e_{\pm 1}}$ on V_4, where $\lambda \in K$, generate a group of type $\Omega_4^{-,K}(F)$.*

Proof. Consider the basis $\mathcal{B} = \{ v_1, v_2, v_3, v_4 \}$ for V_4, where

$$\begin{aligned} v_1 &= (-1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_3 &= (0, 0, 0 \mid 0, 0, -e_{-1}), \\ v_4 &= (0, 0, 0 \mid 0, 0, e_1). \end{aligned}$$

Now we look at the action of $N_{\lambda e_{-1}}$ on these basis elements:

$$\begin{aligned} v_1 &\mapsto (-1, 0, 0 \mid 0, 0, -\lambda e_{-1}) &&= v_1 + \lambda v_3, \\ v_2 &\mapsto (0, 1, 0 \mid 0, 0, -\lambda^\sigma e_{-1}) &&= v_2 + \lambda^\sigma v_3, \\ v_3 &\mapsto (0, 0, 0 \mid 0, 0, -e_{-1}) &&= v_3, \\ v_4 &\mapsto (-\lambda^\sigma, \lambda, 0 \mid 0, 0, -\lambda\lambda^\sigma e_{-1} + e1) &&= -\lambda^\sigma v_1 + \lambda v_2 - \lambda\lambda^\sigma v_3 + v_4. \end{aligned}$$

It follows that the element $N_{\lambda e_{-1}}$ can be written as a 4×4 matrix over $\mathbb{F}_{q^2}$ with respect to $\mathcal{B}$:

$$[N_{\lambda e_{-1}}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & \lambda^\sigma & 0 \\ 0 & 0 & 1 & 0 \\ \lambda^\sigma & \lambda & \lambda\lambda^\sigma & 1 \end{bmatrix}.$$

For convenience, instead of the element $N_{\lambda e_1}$ we consider the element $N_{-\lambda^\sigma e_1}$ which acts on the elements of $\mathcal{B}$ as follows:

$$\begin{aligned}
v_1 &\mapsto & (-1,0,0 \mid 0,0,\lambda^\sigma e_1) &= v_1 + \lambda^\sigma v_4,\\
v_2 &\mapsto & (0,1,0 \mid 0,0,\lambda e_1) &= v_2 + \lambda v_4,\\
v_3 &\mapsto & (-\lambda,\lambda^\sigma,0 \mid 0,0,-e_{-1}+\lambda\lambda^\sigma e_1) &= \lambda v_1 + \lambda^\sigma v_2 + v_3 + \lambda\lambda^\sigma v_4,\\
v_4 &\mapsto & (0,0,0 \mid 0,0,e_1) &= v_4.
\end{aligned}$$

The matrix $[N_{-\lambda^\sigma e_1}]_\mathcal{B}$ has the form

$$[N_{-\lambda^\sigma e_1}]_\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & \lambda^\sigma \\ 0 & 1 & 0 & \lambda \\ \lambda & \lambda^\sigma & 1 & \lambda\lambda^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the basis $\mathcal{B}' = \{\ v_3, v_4, v_1 v_4\ \}$ obtained as a permutation of the elements in $\mathcal{B}$. With respect to $\mathcal{B}'$ the 4×4 matrices $[N_{\lambda e_{-1}}]_{\mathcal{B}'}$ and $[N_{-\lambda^\sigma e_1}]_{\mathcal{B}'}$ take the form

$$[N_{\lambda e_{-1}}]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda^\sigma & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ \lambda\lambda^\sigma & \lambda & \lambda^\sigma & 1 \end{bmatrix}, \quad [N_{-\lambda^\sigma e_1}]_{\mathcal{B}'} = \begin{bmatrix} 1 & \lambda^\sigma & \lambda & \lambda\lambda^\sigma \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & \lambda^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We notice that

$$[N_{\lambda e_{-1}}]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda^\sigma & 1 \end{bmatrix}, \quad [N_{-\lambda^\sigma e_1}]_{\mathcal{B}'} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda^\sigma \\ 0 & 1 \end{bmatrix},$$

where $\otimes$ is the Kronecker product of two matrices. The mapping

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda^\sigma & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda^\sigma \\ 0 & 1 \end{bmatrix}$$

can be extended to a homomorphism ϕ which is obviously surjective as λ ranges through the whole field K. Its kernel is a subgroup

$$\ker(\phi) = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

which has order 2, so we get the action of the group $\mathrm{PSL}_2(K)$ on V_4 since the matrices

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

generate a group $\mathrm{SL}_2(K)$ by a well-known result. Therefore, as $\mathrm{PSL}_2(K) \cong \Omega_4^{-,K}(F)$ (**??**), we have the action of $\Omega_4^{-,K}(F)$. $\square$

We again use the results of Appendix A. Consider the 6-dimensional K-subspace V_6 spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_{\overline{\omega}}, e_{-\overline{\omega}}, e_1 \rangle$. Our copy of $\Omega_4^{-,K}(F)$ preserves two isotropic Albert Vectors in V_6:

$$\begin{aligned} u_{\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{\overline{\omega}}), \\ u_{-\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{-\overline{\omega}}). \end{aligned} \tag{4.12}$$

The element $N_{e_{\overline{\omega}}}$ preserves $u_{\overline{\omega}}$, but not $u_{-\overline{\omega}}$. Therefore, adjoining its action to our $\Omega_4^{-,K}(F)$, we obtain a subgroup of $V_4{:}\Omega_4^{-,K}(F)$. We know that $\Omega_4^{-,K}(F)$ is maximal in the latter, so we conclude that the actions of $N_{\lambda e_{\pm 1}}$ and $N_{e_{\overline{\omega}}}$ on V_6 is that of $V_4{:}\Omega_4^{-,K}(F)$. Next, the element $N_{e_{-\overline{\omega}}}$ preserves V_6 but it does not preserve $\langle u_{\overline{\omega}} \rangle$, so by Theorem **??**, adjoining $N_{e_{-\overline{\omega}}}$ to $V_4{:}\Omega_4^{-,K}(F)$, we get the action of $\Omega_6^{-,K}(F)$ on V_6.

Next, take the 8-space V_8 spanned by the Albert vectors $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_{\overline{\omega}}, e_{\omega}, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle$, and consider two isotropic vectors

$$\begin{aligned} u_{\omega} &= (0, 0, 0 \mid 0, 0, e_{\omega}), \\ u_{-\omega} &= (0, 0, 0 \mid 0, 0, e_{-\omega}), \end{aligned} \tag{4.13}$$

which are fixed by our copy of $\Omega_6^{-,K}(F)$. The action of $N_{e_{\omega}}$ on V_8 preserves u_{ω} but not $u_{-\omega}$, and therefore adjoining this element to $\Omega_6^{-,K}(F)$ we get the action of the group $V_6{:}\Omega_6^{-,K}(F)$. The element $N_{e_{-\omega}}$ does not preserve the 1-space $\langle u_{\omega} \rangle$, so appending it to $V_6{:}\Omega_6^{-,K}(F)$, we get the action of $\Omega_8^{-,K}(F)$ on V_8.

Finally, we choose two isotropic Albert vectors

$$
\begin{aligned}
u_0 &= (0, 0, 0 \mid 0, 0, e_0), \\
u_{-0} &= (0, 0, 0 \mid 0, 0, e_{-0}).
\end{aligned}
\tag{4.14}
$$

in $\mathbb{J}_{10}^{abC}$. We adjoin the element N_{e_0} which fixes u_0 but not u_{-0} to get the action of the group $V_8{:}\Omega_8^{-,K}(F)$. Appending to this the action of $N_{e_{-0}}$ which does not preserve $\langle u_0 \rangle$, we obtain the action of $\Omega_{10}^{-,K}(F)$.

4.5 Action of ${}^2\mathrm{SE}_6^K(F)$ on white points

As in the case of SE_6, we are interested in the action on white points. We will, however, see that although SE_6 acts transitively on white points, the action of ${}^2\mathrm{SE}_6^K(F)$ splits into several orbits. We say that a non-zero Albert vector X is *isotropic*, if $\mathrm{H}(X) = 0$.

We first consider some examples. Suppose $X_1 = (0, 0, 0 \mid 0, 0, e_0)$. As we know (3.21), it determines a 17-space $\{\ (a, b, 0 \mid A, B, C)\ \big|\ e_0 A = B e_0 = \mathrm{T}(e_0 \overline{C}) = 0\ \}$. A straightforward calculation shows that this 17-space U_1 is spanned by the Albert vectors of the form $(a, b, 0 \mid A, B, C)$ with

$$
\begin{aligned}
&A \in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1 \rangle, \\
&B \in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\
&C \in \langle e_{-1}, e_{\overline{\omega}}, e_\omega, e_0, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle.
\end{aligned}
\tag{4.15}
$$

We are also interested in the radical R_1 of H inside this 17-space. In our case it is spanned by the vectors of the form $(0, 0, 0 \mid A, B, C)$ with

$$
\begin{aligned}
&A \in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1 \rangle, \\
&B \in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\
&C \in \langle e_0 \rangle.
\end{aligned}
\tag{4.16}
$$

In other words, our vector X_1 determines the 17-space U_1 and the 9-dimensional radical R_1 of H in U_1. Note that X_1 is isotropic with respect to H, and also $X_1 \in R_1$.

It turns out that there is another type of isotropic white vectors. Consider

$X_2 = (0,0,0 \mid 0,0,e_0+\lambda e_1)$, where $\lambda \in K \setminus F$. It again determines a 17-space U_2, spanned by the Albert vectors of the form $(a,b,0 \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}} + \lambda e_{-\omega}, e_{\omega} - \lambda e_{-\overline{\omega}}, e_{-0}, e_1 \rangle, \\ &B \in \langle e_{-1} + \lambda e_0, e_{-0} - \lambda e_1, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &C \in \langle e_{-1} - \lambda^2 e_1 - \lambda \cdot 1_{\mathbb{O}}, e_{\overline{\omega}}, e_{\omega}, e_0 + \lambda e_1, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle. \end{aligned} \tag{4.17}$$

We find that the radical R_2 of H in U_2 is spanned by the vectors of the form $(0,0,0 \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{-0}, e_1 \rangle, \\ &B \in \langle e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &C \in \langle e_0 + \lambda^{\sigma} e_1 \rangle, \end{aligned} \tag{4.18}$$

i.e. it is 5-dimensional. We notice that X_2 is isotropic, but in this case $X_2 \notin R_2$.

We conclude that the white points $\langle X_1 \rangle$ and $\langle X_2 \rangle$ belong to different orbuts under the action of ${}^2\mathrm{SE}_6^K(F)$. Of course, there is also at least one orbit on the non-isotropic white points.

4.5.1 Orbits of ${}^2\mathrm{SE}_6^K(F)$ on white points

4.5.2 The stabiliser of type 3 vector

We are now interested in the stabiliser in ${}^2\mathrm{SE}_6^K(F)$ of $X_3 = (0,0,1 \mid 0,0,0)$, which is non-isotropic. As we know, the elements N_x with $x\bar{x}^{\sigma} = \bar{x}^{\sigma}x = 0$ preserve X_3 (4.10). We now prove the following theorem.

Theorem 4.5.1. *The stabiliser in* ${}^2\mathrm{SE}_6^K(F)$ *of* $X_3 = (0,0,1 \mid 0,0,0)$ *is the subgroup of shape* $\mathrm{Spin}_{10}^{-,K}(F)$.

Proof. From Theorem 4.4.6 we know that the actions on $\mathbb{J}_{10}^{abC}$ of the elements N_x generate $\Omega_{10}^{-,K}(F)$. We use Lemma 3.3.15 to conclude that the action on $\mathbb{J}$ is that of $\mathrm{Spin}_{10}^{-,K}(F)$. Our aim is to show that this group is the whole stabiliser in ${}^2\mathrm{SE}_6^K(F)$ of X_3.

Let G be the stabiliser of X_3 in ${}^2\mathrm{SE}_6^K(F)$. In particular, G is a subgroup of $\mathrm{SE}_6(K)$, so it preserves the Dickson–Freudenthal determinant Δ, and hence it stabilises the 17-space $U_3 = \mathbb{J}_{17}^{cAB}$ determined by X_3. Next, G is a subgroup of ${}^2\mathrm{SE}_6^K(F)$, so it preserves the Hermitean form H, and so G stabilises $U_3^\perp$, where $\perp$ is taken with respect to the sesquilinear form. Now, $U_3^\perp$ has two forms defined on it, so we use Aschbacher's result (Proposition 4.2.2) to conclude that there exists a (Q, H)-subspace.

Consider the 10-dimensional F-subspace V_{10}^- spanned by the vectors of the form $(\lambda \cdot 1, -\lambda^\sigma \cdot 1, 0 \mid 0, 0, C)$, where $\lambda \in K$ and $C \in \mathbb{O}_F$. First, we check that the action of our copy of $\Omega_{10}^{-,K}(F)$ preserves V_{10}^-. The action on this subspace is given by

$$N_x : (\lambda \cdot 1_{\mathbb{O}}, -\lambda^\sigma \cdot 1_{\mathbb{O}}, 0 \mid 0, 0, C) \mapsto ((\lambda - \mathrm{T}(C\bar{x}^\sigma)) \cdot 1_{\mathbb{O}}, -(\lambda^\sigma - \mathrm{T}(\overline{C}x)) \cdot 1_{\mathbb{O}}, 0 \mid \\ 0, 0, C - x^\sigma \overline{C} x + \lambda x + \lambda^\sigma x^\sigma).$$

Note that the product $x^\sigma \overline{C} x$ makes sense since x and x^σ generate a sociable subalgebra of $\mathbb{O}_K$. It is easy to see that $(\lambda - \mathrm{T}(C\bar{x}^\sigma))^\sigma = \lambda^\sigma - \mathrm{T}(\overline{C}x)$. Lemma 4.4.5 also implies that $x^\sigma \overline{C} x = x \overline{C} x^\sigma = (x^\sigma \overline{C} x)^\sigma$. That is, $x^\sigma \overline{C} x$ is an element of $\mathbb{O}_F$. Finally, $\lambda x + \lambda^\sigma x^\sigma \in \mathbb{O}_F$, so we conclude that the elements N_x indeed preserve V_{10}^-.

Note that the stabiliser preserves restrictions of both Q and H on V_{10}^-. Proposition 4.2.2 asserts that such a subspace is unique, so we conclude that G is a subgroup of $\mathrm{GO}_{10}^{-,K}(F)$ in its action on V_{10}^-. In fact, since G is a subgroup of white vector stabiliser, namely $K^{16}{:}\mathrm{Spin}_{10}^+(K)$, we conclude that $G \leqslant \mathrm{SO}_{10}^{-,K}(F)$ in its action on V_{10}^-.

Now let us look at the action on V_{10}^- in more detail. The restriction of $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ on this 10-space is represented by the Gram matrix

$$\begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}$$

which is block diagonal (here zeroes are replaced with dots). Consider the action by an element S such that it has the following matrix form when acting on V_{10}^{-}:

$$S_{10} = \left[\begin{array}{cc|c} \lambda & \cdot & \\ \cdot & \mu & \\ \hline & & \mathrm{I}_8 \end{array}\right],$$

where $\lambda\mu = 1$, i.e. $\mu = \lambda^{-1}$. Note that a generalised element P_λ acts on $\mathbb{J}$ in the following way:

$$P_\lambda : (a, b, c \mid A, B, C) \mapsto (\lambda^2 a, \lambda^{-2} b, c \mid \lambda^{-1} A, \lambda B, C).$$

It follows that the action on V_{10}^{-} of S^2 is the same as the action of P_λ, so the action of S on the 16-space $\mathbb{J}_{16}^{cAB}$ is determined up to sign.

Our element S also commutes with $\Omega_8^{+}(F)$, generated by the actions of P_u, as u ranges through all octonions of norm 1 in $\mathbb{O}_F$. Therefore, the action of S on $\mathbb{J}$ is given by

$$S_{27} = \left[\begin{array}{ccc|c|c|c} \lambda & \cdot & \cdot & & & \\ \cdot & \lambda^{-1} & \cdot & & & \\ \cdot & \cdot & 1 & & & \\ \hline & & & \alpha\mathrm{I}_8 & & \\ \hline & & & & \beta\mathrm{I}_8 & \\ \hline & & & & & \mathrm{I}_8 \end{array}\right].$$

Since the action by S^2 coincides with the action by P_λ, we get

$$\left.\begin{aligned}\alpha^2 &= \lambda^{-1},\\ \beta^2 &= \lambda.\end{aligned}\right\}$$

Next, the action of S preserves the Dickson–Freudenthal determinant:

$$\left.\begin{aligned}abc &\mapsto \lambda\lambda^{-1}abc,\\ aA\overline{A} &\mapsto \lambda\alpha^2 aA\overline{A},\\ bB\overline{B} &\mapsto \lambda^{-1}\beta^2 bB\overline{B},\\ cC\overline{C} &\mapsto cC\overline{C},\\ \mathrm{T}(ABC) &\mapsto \alpha\beta\,\mathrm{T}(ABC).\end{aligned}\right\}$$

Setting, for example $a = b = c = 0$ and $A = B = 1$, $C = e_0$, we obtain $\alpha\beta = 1$. Similarly, by preserving the Hermitean form H we obtain the following conditions on α, β and λ:

$$\begin{aligned}\lambda\lambda^\sigma &= 1,\\ \alpha\alpha^\sigma &= 1,\\ \beta\beta^\sigma &= 1.\end{aligned}$$

We see that S acts on V_{10}^- as

$$S_{10} = \left[\begin{array}{cc|c} \lambda & \cdot & \\ \cdot & \lambda^{-1} & \\ \hline & & \\ & & \mathrm{I}_8 \\ & & \end{array}\right]. \tag{4.19}$$

This is an element of $\Omega_{10}^{-,K}(F)$. With the help of Lemma 3.3.15 we find that the action of the stabiliser in ${}^2\mathrm{SE}_6^K(F)$ of X_3 is that of $\mathrm{Spin}_{10}^{-,K}(F)$. □

4.5.3 The stabiliser of type 1 vector

We are ready to investigate the stabiliser in ${}^2\mathrm{SE}_6^K(F)$ of $X_1 = (0, 0, 0 \mid 0, 0, e_0)$. From the previous section we know that the stabiliser of X_3 is the subgroup of shape

$\mathrm{Spin}_{10}^{-,K}(F)$. By stabilising X_3 and X_1 simultaneously, with the help of Lemma A.1, we find that the stabiliser of X_1 is at least a subgroup $F^8{:}\mathrm{Spin}_8^{-,K}(F)$.

To find the rest of the stabiliser, we consider the elements $N_{\mu x}$, $N'_{\mu x}$, and $N''_{\mu x}$ which preserve X_1 but do not preserve X_3. It turns out that only the elements $N'_{\mu x}$ with $x \in \{e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\}$ and $N''_{\nu y}$ with $y \in \{e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\}$ for any $\mu, \nu \in K$, do what we want, except when $(\mu, \nu) = (0, 0)$. These elements move X_3 in the following way:

$$\begin{aligned} N'_{\mu x} &: (0, 0, 1 \mid 0, 0, 0) \mapsto (0, 0, 1 \mid -\mu^\sigma x, 0, 0), \\ N''_{\nu y} &: (0, 0, 1 \mid 0, 0, 0) \mapsto (0, 0, 1 \mid 0, \nu y, 0). \end{aligned} \tag{4.20}$$

Consider the subspace $U_1^\perp$ (with respect to $\langle \cdot, \cdot \rangle_{\mathrm{H}}$). We find that $U_1^\perp$ is 10-dimensional and spanned by the vectors $(0, 0, c \mid A, B, C)$ with

$$\begin{aligned} A &\in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1 \rangle, \\ B &\in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ C &\in \langle e_0 \rangle. \end{aligned} \tag{4.21}$$

Note that $X_3 \in U_1^\perp$, so the stabiliser in ${}^2\mathrm{SE}_6^K(F)$ sends X_3 to some vector in $U_1^\perp$. The actions on $U_1^\perp$ of the elements $N'_{\mu x}$ and $N''_{\nu x}$ from above are given by

$$\begin{aligned} N'_{\mu x} : (0, 0, c \mid A, B, C) \mapsto\ & (0, -\mu^\sigma\, \mathrm{T}(A\bar{x}^\sigma), c + \mu\, \mathrm{T}(\overline{A}x) \mid \\ & \mid A - \mu\mu^\sigma x^\sigma \overline{A}x - \mu^\sigma c x^\sigma, \\ & B + \mu\bar{x}\overline{C}, C - \mu^\sigma \overline{B}\bar{x}^s), \\ N''_{\nu y} : (0, 0, c \mid A, B, C) \mapsto\ & (\nu\, \mathrm{T}(\overline{B}y), 0, c - \nu^\sigma\, \mathrm{T}(B\bar{y}^s) \mid \\ & \mid A - \nu^\sigma \overline{C}\bar{y}^\sigma, \\ & B - \nu\nu^\sigma y^\sigma \overline{B}y + \nu c y, C + \nu\bar{y}\overline{A}). \end{aligned} \tag{4.22}$$

Since $x \in \{e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\}$, we find $\mathrm{T}(A\bar{x}^\sigma) = \mathrm{T}(\overline{A}x) = 0 = x^\sigma \overline{A}x = \bar{x}\overline{C}$, and also $x^\sigma = x$, $\overline{B}\bar{x}^\sigma \in \langle e_0 \rangle$. Similarly, $\mathrm{T}(\overline{B}y) = \mathrm{T}(B\bar{y}^\sigma) = 0 = y^\sigma \overline{B}y = \overline{C}\bar{y}^\sigma$, and $y^\sigma = y$, $\bar{y}\overline{A} \in \langle e_0 \rangle$ as $y \in \{e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\}$. After these simplifications the action takes the form

$$\begin{aligned} N'_{\mu x} &: (0, 0, c \mid A, B, C) \mapsto (0, 0, c \mid A - \mu^\sigma c x, B, C - \mu^\sigma \overline{B}\bar{x}), \\ N''_{\nu y} &: (0, 0, c \mid A, B, C) \mapsto (0, 0, c \mid A, B + \nu c y, C + \nu\bar{y}\overline{A}). \end{aligned} \tag{4.23}$$

The value of H on $U_1^\perp$ is given by

$$\mathrm{H}((0,0,c \mid A,B,C)) = cc^\sigma, \tag{4.24}$$

so we conclude that the subgroup of the stabiliser of X_1, which does not preserve X_3, sends X_3 to a vector of the form $(0,0,1 \mid A, B, \overline{B}\,\overline{A})$. These vectors span a 9-subspace of $U_1^\perp$. We are interested in the action on its subspaces S_A and S_B, spanned by the vectors of the form $(0,0,1 \mid A,0,0)$ and $(0,0,1 \mid 0,B,0)$ respectively. The action of the elements $N'_{\mu x}$ and $N''_{\nu y}$ on these subspaces is given by

$$\begin{aligned}
N'_{\mu x} &: (0,0,1 \mid A,0,0) \mapsto (0,0,1 \mid A - \mu^\sigma x, 0, 0),\\
N'_{\mu x} &: (0,0,1 \mid 0,B,0) \mapsto (0,0,1 \mid 0,B,0),\\
N''_{\nu y} &: (0,0,1 \mid A,0,0) \mapsto (0,0,1 \mid A,0,0),\\
N''_{\nu y} &: (0,0,1 \mid 0,B,0) \mapsto (0,0,1 \mid 0,B+\nu y,0).
\end{aligned} \tag{4.25}$$

We see that the action of $N'_{\mu x}$ on S_A is that of K^4, as well as the action of $N''_{\nu y}$ on S_B. We also see that the intersection of these two copies of K^4 is trivial, hence we get the action of $K^4 \times K^4 \cong K^8$ on $U_1^\perp$. Finally, we have shown the following.

Theorem 4.5.2. *The stabiliser in* ${}^2\mathrm{SE}_6^K(F)$ *of* $X_1 = (0,0,0 \mid 0,0,e_0)$ *is the subgroup of shape* $F^8.K^8{:}\mathrm{Spin}_8^{-,K}(F)$.

4.5.4 The stabiliser of type 2 vector

Finally, we investigate the stabiliser in ${}^2\mathrm{SE}_6^K(F)$ of $X_2 = (0,0,0 \mid 0,0,e_0+\lambda e_1)$, where $\lambda \in K \setminus F$. For this we need to prepare certain ingredients. Recall that the 17-space U_1 determined by X_1 is spanned by the vectors of the form $(a,b,0 \mid A,B,C)$ with

$$\begin{aligned}
A &\in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\rangle,\\
B &\in \langle e_{-1}, e_{-0}, e_{-\omega} e_{-\overline{\omega}}\rangle,\\
C &\in \langle e_{-1}, e_{\overline{\omega}}, e_\omega e_0, e_{-\omega}, e_{-\overline{\omega}}, e_1\rangle.
\end{aligned} \tag{4.26}$$

Recall that the stabiliser of a white vector in $\mathrm{SE}_6(K)$ is a subgroup of shape $K^{16}{:}\mathrm{Spin}_{10}^+(K)$ (Theorem 3.3.8). As we know, an element in the stabiliser, preserving the $1 \oplus 16 \oplus 10$ decomposition, belongs to $\mathrm{Spin}_{10}^+(K)$. The 10-space of X_1, denoted W_1, preserved by such an element, is spanned by the Albert vectors

$(0,0,c \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{-1}, e_0, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &B \in \langle e_{\overline{\omega}}, e_\omega, e_0, e_1 \rangle, \\ &C \in \langle e_{-0} \rangle. \end{aligned} \tag{4.27}$$

To obtain the corresponding 10-space for X_2, we find an element in $\mathrm{SE}_6(K)$ which maps X_1 to X_2. For example, the element P_u with $u = 1 - \lambda e_1$ does exactly what we want. Now, W_2 is spanned by the vectors of the form $(0,0,c \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{-1} + \lambda e_{-0}, e_0 - \lambda e_1, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &B \in \langle e_{\overline{\omega}} - \lambda e_{-\omega}, e_\omega + \lambda e_{-\overline{\omega}}, e_0, e_1 \rangle, \\ &C \in \langle e_{-0} + \lambda e_1 \rangle. \end{aligned} \tag{4.28}$$

Note that since $P_{1-\lambda e_1}$ is an element of $\mathrm{SE}_6(K)$, it preserves colour, so all the white vectors in W_1 are mapped to the white vectors in W_2. An Albert vector $(0,0,c \mid A,B,C) \in W_1$ is white if and only if $A\overline{A} = B\overline{B} = C\overline{C} = 0$, $AB = c\overline{C}$, and $BC = CA = 0$. Most of these conditions, except possibly $AB = c\overline{C}$, are satisfied automatically. Now, we find that $AB \in \langle e_0 \rangle$ by an explicit calculation, so if $c \neq 0$, there is a unique value of C such that an element $(0,0,c \mid A,B,C)$ in W_2 is white. The action of $P_{1-\lambda e_1}$ on the third 'coördinate' is trivial, so we may conclude that there is a unique value of C such that an element $(0,0,c \mid A,B,C)$ in W_2 is white given $c \neq 0$.

Next, the subgroup of $\mathrm{Spin}^+_{10}(K)$, stabilising X_2, not only preserves the colour, but it also preserves the value of H on W_2. For $X = (0,0,c \mid A,B,C) \in W_2$, the value of H is given by

$$\mathrm{H}(X) = cc^\sigma + \mathrm{T}(A\overline{A}^\sigma) + \mathrm{T}(B\overline{B}^\sigma) + \mathrm{T}(C\overline{C}^\sigma). \tag{4.29}$$

As we know,

$$\begin{aligned} A &= A_{-1}(e_{-1} + \lambda e_{-0}) + A_0(e_0 - \lambda e_1) + A_{-\omega} e_{-\omega} + A_{-\overline{\omega}} e_{-\overline{\omega}}, \\ B &= B_{\overline{\omega}}(e_{\overline{\omega}} - \lambda e_{-\omega}) + B_\omega(e_\omega + \lambda e_{-\overline{\omega}}) + B_0 e_0 + B_1 e_1, \\ C &= C_{-0}(e_{-0} + \lambda e_1). \end{aligned} \tag{4.30}$$

We then find

$$\begin{aligned}A\overline{A}^\sigma = A_{-1}A_0^\sigma(e_{-1}+\lambda e_{-0}-\lambda^\sigma e_0+\lambda\lambda^\sigma e_1) - A_{-1}A_{-\omega}^\sigma e_{\overline{\omega}} + A_{-1}A_{-\overline{\omega}}^\sigma e_\omega \\ + A_0A_{-1}^\sigma(\lambda^\sigma e_0 - \lambda\lambda^\sigma e_1 - e_{-1} - \lambda e_{-0}) - A_0A_{-\omega}^\sigma e_{-\omega} - A_0 A_{-\overline{\omega}}^\sigma e_{-\overline{\omega}} \\ + A_{-\omega}A_{-1}^\sigma e_{\overline{\omega}} + A_{-\omega}A_0^\sigma e_{-\omega} - A_{-\omega}A_{-\overline{\omega}}^\sigma e_1 \\ + A_{-\overline{\omega}}A_{-\omega}^\sigma e_{-\overline{\omega}} + A_{-\overline{\omega}}A_{-1}^\sigma e_\omega + A_{-\overline{\omega}}A_{-\omega}^\sigma e_1,\end{aligned} \quad (4.31)$$

so

$$\begin{aligned}\mathrm{T}(A\overline{A}^\sigma) = -\lambda^\sigma A_{-1}A_0^\sigma + \lambda^\sigma A_0A_{-1}^\sigma + \lambda A_{-1}A_0^\sigma - \lambda A_0 A_{-1}^\sigma \\ = (\lambda^\sigma - \lambda)(A_0A_{-1}^\sigma - A_{-1}A_0^\sigma).\end{aligned} \quad (4.32)$$

Next,

$$\begin{aligned}B\overline{B}^\sigma = -B_{\overline{\omega}}B_\omega^\sigma(-e_{-1}+\lambda e_0 - \lambda^\sigma e_{-0} - \lambda\lambda^\sigma e_1) - B_{\overline{\omega}}B_0^\sigma \lambda e_{-\omega} + B_{\overline{\omega}}B_1^\sigma e_{-\omega} \\ + B_\omega B_{\overline{\omega}}^\sigma(-\lambda^\sigma e_{-0} - \lambda\lambda^\sigma e_1 - e_{-1} + \lambda e_0) + B_\omega B_0^\sigma \lambda e_{-\overline{\omega}} - B_\omega B_1^\sigma e_{-\overline{\omega}} \\ + B_0B_{\overline{\omega}}^\sigma \lambda^\sigma e_{-\omega} - B_0 B_\omega^\sigma \lambda^\sigma e_{-\overline{\omega}} + B_1 B_{\overline{\omega}}^\sigma e_{-\omega} - B_1 B_\omega^\sigma e_{-\overline{\omega}},\end{aligned} \quad (4.33)$$

and so similarly,

$$\mathrm{T}(B\overline{B}^\sigma) = (\lambda^\sigma - \lambda)(B_{\overline{\omega}}B_\omega^\sigma - B_\omega B_{\overline{\omega}}^\sigma). \quad (4.34)$$

Finally,

$$C\overline{C}^\sigma = C_{-0}C_{-0}^\sigma(\lambda e_1 - \lambda^\sigma e_1), \quad (4.35)$$

so

$$\mathrm{T}(C\overline{C}^\sigma) = 0. \quad (4.36)$$

It follows that on the elements of W_2, Hermitian form H becomes

$$\mathrm{H}(X) = cc^\sigma + (\lambda^\sigma - \lambda)(A_0A_{-1}^\sigma - A_{-1}A_0^\sigma + B_{\overline{\omega}}B_\omega^\sigma - B_\omega B_{\overline{\omega}}^\sigma). \quad (4.37)$$

This is a unitary form in 5 variables, which is preserved by the subgroup of the stabiliser of X_2 inside $\mathrm{Spin}_{10}^+(K)$. Note that $\lambda^\sigma - \lambda \neq 0$ since $\lambda \in K \setminus F$. Given $X = (a, b, c \mid A, B, C) \in \mathbb{J}$, the value of the quadratic form $Q(X) = \Delta(X + X_2) - \Delta(X)$, determined by X_2, is

$$\begin{aligned} Q(X) = \mathrm{T}((e_0 + \lambda e_1)(AB - c\overline{C})) \\ = A_0(B_0 - \lambda B_{-1}) - (A_{-\omega} - \lambda A_{\overline{\omega}})B_\omega \\ - (A_{-\overline{\omega}} + \lambda A_\omega)B_{\overline{\omega}} - A_{-1}(B_1 + \lambda B_{-0}) \\ - c(C_{-0} + \lambda C_{-1}). \end{aligned} \tag{4.38}$$

On W_2 this becomes

$$Q(X) = A_0B_0 - A_{-\omega}B_\omega - A_{-\overline{\omega}}B_{\overline{\omega}} - A_{-1}B_1 - cC_{-0}. \tag{4.39}$$

Consider the 5-dimensional subspace of W_2 spanned by the vectors $(0, 0, c \mid A, B, C)$ with

$$\begin{aligned} &A \in \langle e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &B \in \langle e_0, e_1 \rangle, \\ &C \in \langle e_{-0} \rangle. \end{aligned} \tag{4.40}$$

This is the radical of H in W_2. We also notice that this subspace coincides with a maximal totally isotropic subspace of Q on W_2. Therefore, the action of the subgroup of $\mathrm{Spin}^+_{10}(K)$, preserving W_2, also preserves a $5 \oplus 5$ decomposition of W_2. As a result, we get the action of the group of type $\mathrm{SU}^K_5(F, \mathrm{H})$.

Using the element $P_{1-\lambda e_1}$ we can also easily find the 17-space U_2, determined by X_2. We conjugate U_1 by $P_{1-\lambda e_1}$, and find that U_2 is spanned by the vectors of the form $(a, b, 0 \mid A, B, C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}} + \lambda e_{-\omega}, e_\omega - \lambda e_{-\overline{\omega}}, e_{-0}, e_1 \rangle, \\ &B \in \langle e_{-1} + \lambda e_0, e_{-0} - \lambda e_1, e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ &C \in \langle e_{-1} - \lambda^2 e_1 - \lambda \cdot 1_{\mathbb{O}}, e_{\overline{\omega}}, e_\omega, e_0 + \lambda e_1, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle. \end{aligned} \tag{4.41}$$

The subspace U_2 is preserved by the stabiliser of W_2, and therefore so is the orthogonal complement $U_2^\perp$ in $\mathbb{J}$, which is 10-dimensional and spanned by the Albert vectors of the form $(0,0,c \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}} + \lambda^\sigma e_{-\omega}, e_\omega - \lambda^\sigma e_{-\overline{\omega}}, e_{-0}, e_1\rangle, \\ &B \in \langle e_{-1} + \lambda^\sigma e_0, e_{-0} - \lambda^\sigma e_1, e_{-\omega}, e_{-\overline{\omega}}\rangle, \\ &C \in \langle e_0 + \lambda^\sigma e_1\rangle. \end{aligned} \tag{4.42}$$

As a result, we get the action of $K^{10}{:}\mathrm{SU}_5^K(F,\mathrm{H})$. The $\mathrm{SU}_5^K(F,\mathrm{H})$ acts on the corresponding image of W_2, which is being moved around.

Now assume the trivial action on $U_2^\perp$. Then the action is also trivial on $U_2^\perp \cap U_2$, and so it is trivial on $(U_2^\perp \cap U_2)^\perp$, and finally, the same holds for $W_2 \cap (U_2^\perp \cap U_2)^\perp$. The latter is a 5-dimensional subspace of W_2, spanned by the vectors $(0,0,c \mid A,B,0)$ with

$$\begin{aligned} &A \in \langle e_{-\omega}, e_{-\overline{\omega}}\rangle, \\ &B \in \langle e_0, e_1\rangle. \end{aligned} \tag{4.43}$$

4.6 Case of a finite field

4.6.1 White vectors in $\mathbb{J}_8^C$

As a practical counting excersise, we count the isotropic white vectors in $\mathbb{J}_8^C$. As before, $K = \mathbb{F}_{q^2}$. First, we need the following auxiliary result.

Lemma 4.6.1. *Let V be a vector space over $\mathbb{F}_{q^2}$ of dimension $2m$. Define the map $Z_m : V \to \mathbb{F}_q$ in the following way:*

$$Z_m(x) = (x_1^q - x_1)(x_2^q - x_2) + (x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}),$$

where $x = (x_1, ..., x_{2m})$. Denote by z_m the number of $x \in V$ such that $Z_m(x) = 0$. Then

$$z_n = q^{3m-1}(q^m + q - 1).$$

Proof. We proceed by induction on m. If $m = 1$, the equality $Z_m(x) = 0$ reduces to

$$(x_1^q - x_1)(x_2^q - x_2) = 0.$$

Note that this is possible when $x_1^q = x_1$ or $x_2^q = x_2$, i.e. when $x_1 \in \mathbb{F}_q$ or $x_2 \in \mathbb{F}_q$.

Thus, when $m = 1$ there are precisely $2q^3 - q^2 = q^2(q + q - 1)$ solutions.

Now suppose that the statement holds for all integers k such that $1 \leqslant k \leqslant m-1$. In the case

$$\left.\begin{array}{l}(x_1^q - x_1)(x_2^q - x_2) = 0,\\ (x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = 0,\end{array}\right\}$$

we get $z_1 z_{m-1}$ solutions. On the other hand, if

$$\left.\begin{array}{l}(x_1^q - x_1)(x_2^q - x_2) = \lambda,\\ (x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = -\lambda\end{array}\right\}$$

for $0 \neq \lambda \in \mathbb{F}_q$, there are

$$(q^4 - z_1)\frac{(q^{4(m-1)} - z_{m-1})}{q-1}$$

solutions. We divide the second factor by $(q-1)$ since each pair (x_1, x_2) satisfying the first equation, fixes the value of λ for the second equation. Overall we have

$$z_m = z_1 z_{m-1} + (q^4 - z_1)\frac{(q^{4(m-1)} - z_{m-1})}{q-1}.$$

Thus, we have obtained a recurrence relation and by substituting z_1 and z_{m-1} in it, we finally obtain $z_m = q^{3m-1}(q^m + q - 1)$. □

The following theorem allows us to count the elements of V satisfying simultaneously a certain quadratic and a certain Hermitean form.

Theorem 4.6.2. *Let V be an vector space over $\mathbb{F}_{q^2}$ of dimension $2m$. Let the quadratic form $Q_m : V \to \mathbb{F}_{q^2}$ be defined as*

$$Q_m(x) = x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m}, \tag{4.44}$$

where $x = (x_1, ..., x_{2m})$, and also define the Hermitean form $H_m : V \to \mathbb{F}_q$ by

$$H_m(x) = x_1^q x_2 + x_1 x_2^q + x_3^q x_4 + x_3 x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1} x_{2m}^q. \tag{4.45}$$

Let w_m be the number of $x \in V$ such that

$$\left.\begin{aligned} Q_m(x) = 0, \\ H_m(x) = 0. \end{aligned}\right\} \tag{4.46}$$

Then

$$w_m = q^{2m} + q^{2m-1}(q^m - 1)(q^{m-2} + 1). \tag{4.47}$$

Proof. We again proceed by induction on m. When $m = 1$, the system (4.46) reduces to

$$\left.\begin{aligned} &x_1 x_2 = 0, \\ &x_1^q x_2 + x_1 x_2^q = 0. \end{aligned}\right\}$$

Note that each pair (x_1, x_2) which satisfies the first equation also satisfies the second one, so in this case the number of solutions is $2q^2 - 1 = q^2 + q(q-1)(q^{-1} + 1)$.

Suppose now that the statement holds for all integers k such that $1 \leqslant k \leqslant m-1$ and consider the following system:

$$\left.\begin{aligned} &x_1 x_2 + x_3 x_4 + \cdots + x_{2m-1} x_{2m} = 0, \\ &x_1^q x_2 + x_1 x_2^q + x_3^q x_4 + x_3 x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1} x_{2m}^q = 0. \end{aligned}\right\}$$

We distinguish two cases.

First, consider the case $x_1 = 0$. Then x_2 can take any of the q^2 possible values and the remaining system is equivalent to

$$\left.\begin{aligned} Q_{m-1}(x) = 0, \\ H_{m-1}(x) = 0, \end{aligned}\right\}$$

so there are $q^2 w_{m-1}$ solutions in this case.

Now suppose that $x_1 \neq 0$. Without loss of generality we may consider the case $x_1 = 1$. The system (4.46) takes the form

$$\left.\begin{aligned} &x_2 = -x_3 x_4 - \cdots - x_{2m-1} x_{2m}, \\ &x_2 + x_2^q + x_3^q x_4 + x_3 x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1} x_{2m}^q = 0. \end{aligned}\right\}$$

We substitute x_2 from the first equation into the second one to obtain

$$(x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = 0.$$

Using the result of Lemma 4.6.1, we obtain that in this case there are $(q^2 - 1)z_{m-1}$ solutions. In total, we obtain the following recurrence relation:

$$w_n = q^2 w_{m-1} + (q^2 - 1)z_{m-1}.$$

By substituting the appropriate values for w_{m-1} and z_{m-1}, we obtain the result. $\square$

An Albert vector $(0, 0, 0 \mid 0, 0, C) \in \mathbb{J}_8^C$ is white if and only if $C\overline{C} = 0$. Recall that $\mathbb{O}_K$ is split, so we can use our favourite basis $\{ e_i \bigm| i \in \pm I \}$. Note that with respect to this basis $C\overline{C} = 0$ is equivalent to

$$C_{-1}C_1 + C_{\overline{\omega}}C_{-\overline{\omega}} + C_{\omega}C_{-\omega} + C_{-0}C_0 = 0. \tag{4.48}$$

Next, $(0, 0, 0 \mid 0, 0, C)$ is isotropic if and only if $\mathrm{T}(C\overline{C}^{\sigma}) = 0$, which is equivalent to

$$C_{-1}^q C_1 + C_{-1}C_1^q + C_{\overline{\omega}}^q C_{-\overline{\omega}} + C_{\overline{\omega}}C_{-\overline{\omega}}^q + C_{\omega}^q C_{-\omega} + C_{\omega}C_{-\omega}^q + C_{-0}^q C_0 + C_{-0}C_0^q = 0. \tag{4.49}$$

We know that there are exactly $(q^8 - 1)(q^6 + 1)$ white vectors in $\mathbb{J}_8^C$. Furthermore, there are

$$w_4 - 1 = (q^2 + 1)(q^3 + 1)(q^3(q^2 + 1)(q - 1) + (q^5 + 1)) \tag{4.50}$$

isotropic white vectors of the form $(0, 0, 0 \mid 0, 0, C)$ and

$$q^6(q^4 - 1)(q^3 - 1)(q - 1) \tag{4.51}$$

non-isotropic white vectors of the same form.

Now, using Proposition 4.2.3, we find that the full subgroup of $\mathrm{SE}_6(K)$ which preserves Q and H on $\mathbb{J}_8^C$ has three orbits on white points, and the sizes of these orbits are given by

(i) $(q^4 - 1)(q^3 + 1)/(q - 1)$;

(ii) $q(q^6 - 1)(q^4 - 1)(q^2 + 1)/(q^2 - 1)$;

(iii) $q^6(q^4-1)(q^3-1)/(q+1)$.

Of these, the first two are isotropic, while the last one is non-isotropic.

4.6.2 White vectors in $\mathbb{J}_{16}^{BC}$

We can also count the isotropic white vectors in $\mathbb{J}_{16}^{BC}$, $K = \mathbb{F}_{q^2}$. Suppose $X = (0,0,0 \mid 0,B,C)$ is white and note that the whiteness conditions take form $B\overline{B} = C\overline{C} = 0 = BC$.

First, we count the white vectors $(0,0,0 \mid 0,B,C)$ such that $B \neq 0$ and $C \neq 0$. We notice that given $B \neq 0$ and $B\overline{B} = 0 = BC$, we automatically have $C\overline{C} = 0$. Indeed, for if $C\overline{C} \neq 0$, C is invertible and $BC = 0$ implies $B = 0$, a contradiction. So, there are $(q^8-1)(q^6+1)$ choices for B and (q^8-1) choices for C (see Lemma 2.5.2). In total, there are $(q^8-1)^2(q^6+1)$ white vectors with $B \neq 0$ and $C \neq 0$.

To count the isotropic white vectors of the form $(0,0,0 \mid 0,B,C)$ we distiguish two cases:

$$\left.\begin{array}{l} B\overline{B} = BC = 0, \\ \mathrm{T}(B\overline{B}^{\sigma}) = 0, \\ \mathrm{T}(C\overline{C}^{\sigma}) = 0, \end{array}\right\}, \quad \left.\begin{array}{l} B\overline{B} = BC = 0, \\ \mathrm{T}(B\overline{B}^{\sigma}) = \lambda \neq 0, \\ \mathrm{T}(C\overline{C}^{\sigma}) = -\lambda. \end{array}\right\}. \tag{4.52}$$

In the previous section we learned that the subgroup of $\mathrm{SE}_6(K)$, preserving Q and H on $\mathbb{J}_8^C$ has two orbits on isotropic white points. Proposition 4.2.3 the first orbit consists of white points $\langle X \rangle$ where X is written over $\mathbb{F}_q$. That is, its representatives are one-dimensional $\mathbb{F}_{q^2}$-subspaces generated by the white vectors written over $\mathbb{F}_q$. Suppose $X_C = (0,0,0 \mid 0,0,C)$ belongs to the first orbit. By taking a particular candidate for C, it is easy to see that in this case there are q^8-1 choices for B. Now, if X_C belongs to the second orbit, there are $q^4(q^3+q^2-q)-1$ choices for B. If, on the other hand, X_C is non-isotropic, then there are $q^3(q^4-1)$ choices for B, and overall we have

$$\begin{aligned} &(q^4-1)(q^3+1)(q+1)(q^8-1) \\ &\qquad + q(q^6-1)(q^4-1)(q^2+1)(q^4(q^3+q^2-q)-1) \\ &\qquad\quad + q^6(q^4-1)(q^3-1)(q-1)q^3(q^4-1) \\ &= (q^{13}+q^{11}-q^{10}+2q^9+q^8+2q^7-q^6+2q^4+1)(q^4-1)^2 \end{aligned} \tag{4.53}$$

isotropic white vectors of the form $(0,0,0 \mid 0,B,C)$ with $B, C \neq 0$.

Recall that the group preserving Q_4 and H_4 has two orbits on the isotropic white points in J_8 with $(q^4-1)(q^3+1)/(q-1)$ and $q(q^6-1)(q^4-1)(q^2+1)/(q^2-1)$ elements. Again, by PropositionThe first orbit consists of white points $\langle X \rangle$ where X is written over $\mathbb{F}_q$. That is, its representatives are one-dimensional $\mathbb{F}_{q^2}$-subspaces generated by the white vectors written over $\mathbb{F}_q$. Since we know the totality of white vectors of this form, we find that there are

$$q^6(q^4-1)^2(q^4+2)(q^3-1)(q-1) \tag{4.54}$$

non-isotropic white vectors in $\mathbb{J}_{16}^{BC}$ with $B, C \neq 0$. Overall there are

$$\begin{aligned} q^6(q^4-1)^2(q^4+2)(q^3-1)(q-1) & \\ + \ 2q^6(q^4-1)(q^3-1)(q-1) & \\ = \ q^{10}(q^8-1)(q^3-1)(q-1) & \end{aligned} \tag{4.55}$$

non-isotropic white vectors in $\mathbb{J}_{16}^{BC}$.

A: Some properties of $\Omega_{2m}(F, Q)$

Let V be a vector space over a field F of dimension n. We assume that there is a non-singular quadratic form Q defined on V. Denote by $\mathrm{GO}_n(F, Q)$ the group of non-singular linear transformations that preserve the form Q. In case of characteristic 2 we define the *quasideterminant* qdeat : $\mathrm{GO}_n(F, Q) \to \mathbb{F}_2$ to be the map

$$\mathrm{qdet} : g \mapsto \dim_F(\mathrm{Im}\,(\mathrm{I} - g)) \mod 2. \tag{56}$$

Further, the group $\mathrm{SO}_n(F, Q)$ is the kernel of the (quasi)determinant map. Define the *spinor norm* to be the homomorphism $\mathrm{sp} : \mathrm{SO}_n(F, Q) \to F^\times/(F^\times)^2$. This homomorphism is defined in the following way. Any element of $\mathrm{SO}_n(F, Q)$ arising as a reflection in v for some $v \in V$, is sent to the value $Q(v)$ modulo $(F^\times)^2$. This extends to a well-defined homomorphism. The subgroup $\Omega_n(F, Q)$ of $\mathrm{SO}_n(F, Q)$ is defined as the kernel of spinor norm. If the characteristic of the field is not 2, then there exists a double cover of $\Omega_n(F, Q)$, denoted as $\mathrm{Spin}_n(F, Q)$.

This section is devoted to some of the private life of the group $\Omega_{2m}(F, Q)$. Consider the vector space V of dimension $2m + 2$ over F with a non-singular quadratic form Q defined on it. Let f be a polar form of Q. Assuming that the Witt index of Q is at least 1, we can pick the basis $\mathcal{B} = \{v_1, w_1, ..., w_{2m}, v_2\}$ in V such that (v_1, v_2) is a hyperbolic pair. Consider the decomposition $V = \langle v_1\rangle \oplus \langle w_1, ..., w_{2m}\rangle \oplus \langle v_2\rangle$ and denote $W = \langle w_1, ..., w_{2m}\rangle$. Further, denote by Q_W and f_W the restrictions of Q and f on W.

Lemma A.1. *The stabiliser in $\Omega_{2m+2}(F, Q)$ of the vector v_1 is a subgroup of shape $W{:}\Omega_{2m}(F, Q_W)$, and the stabiliser of the pair (v_1, v_2) is a subgroup $\Omega_{2m}(F, Q_W)$.*

Proof. Any element g in $\Omega_{2m+2}(F, Q)$ which fixes v_1 also stabilises $\langle v_1\rangle^\perp$, so it has the following form with respect to $\mathcal{B}$:

$$[g]_{\mathcal{B}} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline u_2^\top & A & 0 \\ \hline \mu & u_1 & \lambda \end{array}\right],$$

where the matrix A acts on the $2m$-dimensional subspace, spanned by $\{w_1, ..., w_{2m}\}$. Such an element g acts on v_2 as

$$v_2 \mapsto [\mu \mid u_1 \mid \ \lambda],$$

and since the bilinear form f is preserved we get

$$1 = f(v_1, v_2) = f(v_1, \mu v_2) + f(v_1, [0 \mid u_1 \mid \ 0]) + \lambda f(v_1, v_2) = \lambda,$$

Since (v_1, v_2) is a hyperbolic pair, f can be represented by the Gram matrix

$$[f]_{\mathcal{B}} = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right],$$

where B is the matrix of f_W with respect to the basis $\{w_1, ..., w_{2m}\}$. We explore the fact that an element in the stabiliser of v_1 preserves the form f:

$$\left[\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline u_2^\top & A & 0 \\ \hline \mu & u_1 & 1 \end{array}\right] \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c|c|c} 1 & u_2 & \mu \\ \hline 0 & A^\top & u_1^\top \\ \hline 0 & 0 & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & ABA^\top & ABu_1^\top + u_2^\top \\ \hline 1 & u_2 + u_1BA^\top & 2\mu + u_1Bu_1^\top \end{array}\right],$$

so we notice that $ABA^\top = B$. Furthermore, since $[0 \mid v \mid 0][g]_{\mathcal{B}} = [0 \mid vA \mid 0]$, where $v \in W$, we obtain

$$Q_W(vA) = Q([0 \mid vA \mid 0]) = Q([0 \mid v \mid 0]) = Q_W(v),$$

so A is an element of $\mathrm{GO}_{2m}(F, Q)$. Additionally, $u_2 = -u_1BA^\top$ and we see that u_2 is uniquely determined by u_1. From the bottom right corner of the resulting matrix we obtain $\mu = -Q(u_1)$ in odd characteristic. In case of characteristic 2 we can explore the quadratic form again:

$$\begin{aligned} 0 = Q(v_2) = Q(v_2[g]_{\mathcal{B}}) = Q([\mu \mid u_1 \mid 1]) = \\ = Q([\mu \mid u_1 \mid 0]) + Q(v_2) + f([\mu \mid u_1 \mid 0], v_2) = Q(u_1) + \mu. \end{aligned}$$

Consider the decomposition

$$\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -ABu_1^\top & A & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu_1^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right].$$

The matrices of the form

$$C_{u_1} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu_1^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right]$$

generate an elementary abelian group isomorphic to W (as abelian groups). Indeed, since the product of two such matrices is given by

$$\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u) & u & 1 \end{array}\right] \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bv^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(v) & v & 1 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -B(u+v)^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u+v) & u+v & 1 \end{array}\right],$$

we see that the set of these matrices is closed under multiplication and moreover any two such matrices commute.

To show that the matrix A is an element of $\Omega_{2m}(F, Q)$, we use Proposition 1.6.11 from [?] to calculate the spinor norm and, in case of characteristic 2, the quasideterminant of the matrices C_{u_1}. Note that $\det(C_{u_1}) = \det([g]_{\mathcal{B}}) = 1$. Consider the matrix

$$\mathrm{I} - C_{u_1} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline Bu_1^\top & 0 & 0 \\ \hline Q(u_1) & -u_1 & 0 \end{array}\right].$$

For a vector v we denote by $[v]_i$ its i-th component. Now, if $u_1 = 0$, then $\mathrm{I} - C_{u_1}$ has rank 0, whereas if $u_1 \neq 0$, then there is an index i such that $[Bu_1^\top]_i \neq 0$ and it follows that the rank of $\mathrm{I} - C_{u_1}$ in this case is 2. Consequently, $k = \operatorname{rank}(\mathrm{I} - C_{u_1})$ is even, and so by the Proposition 1.6.11 in [?] the quasideterminant of C_{u_1} is 1. Further, if D is a $k \times (2m+2)$ matrix whose rows are the basis elements of a complement of the nullspace of $\mathrm{I} - C_{u_1}$, then the spinor norm of C_{u_1} is 1 if $\det(D(\mathrm{I} - C_{u_1})[f]_{\mathcal{B}} D^\top)$ is a square in F. If $u_1 \neq 0$, then the complement of the nullspace of $I - C_{u_1}$ has the basis $\{w_i, v_1\}$, where the index i is such that $[Bu_1^\top]_i \neq 0$. The matrix D has the following form:

$$D = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where 1 in the first row is in the $(i+1)$-st position. We calculate

$$D(\mathrm{I} - C_{u_1}) = \left[\begin{array}{c|c|c} \alpha & 0 & 0 \\ \hline Q(u_1) & -u_1 & 0 \end{array}\right], \quad [f]_{\mathcal{B}} D^\top = \begin{bmatrix} 0 & 1 \\ B_{1,i} & 0 \\ \vdots & \vdots \\ B_{2m,i} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha = [Bu_1^\top]_i = [u_1 B]_i$. Finally,

$$D(\mathrm{I} - C_{u_1})[f]_{\mathcal{B}} D^\top = \begin{bmatrix} 0 & \alpha \\ -\alpha & Q(u_1) \end{bmatrix},$$

so $\det(D(\mathrm{I} - C_{u_1}) F D^\top) = \alpha^2$ as needed. Since the quasideterminant and the spinor norm are multiplicative (Theorems 11.43 and 11.50 in [?]), and $g \in \Omega_{2m+2}(F, \widehat{Q})$, we conclude that A acts on W as an element of $\Omega_{2m}(F, Q)$ and it follows that the stabiliser of v_1 in $\Omega_{2m}(F, \widehat{Q})$ is indeed a subgroup of shape $W{:}\Omega_{2m}(F, Q)$.

Lastly, if we stabilise v_1 and v_2 simultaneously, a general element in the stabiliser takes the form

$$\left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

so the stabiliser of the pair (v_1, v_2) is $\Omega_{2m}(F, Q)$. $\square$

Witt's lemma tells us that the group $\mathrm{GO}_{2m}(F, Q)$ acts transitively on the non-zero vectors of each norm. The following result allows us to use the fact that the same is true for $\Omega_{2m}(F, Q)$ in case when Q is of Witt index at least 1.

Lemma A.2. *The group $\Omega_{2m+2}(F, Q)$, where Q is of Witt index at least* 1*, acts transitively on*

$$O_\lambda = \left\{ \ v \in V \ \middle| \ Q(v) = \lambda, \ v \neq 0 \ \right\}$$

for each value of $\lambda \in F$.

Proof. Suppose $u, v \in O_\lambda$. Since by Witt's lemma $\mathrm{GO}_{2m+2}(F, Q)$ acts transitively on O_λ, there is an element $g \in \mathrm{GO}_{2m+2}(F, Q)$ which sends u to v. Consider the basis for V, $\mathcal{B} = \{v_1, w_1, ..., w_{2m-2}, v_1\}$ such that as before (v_1, v_2) is a hyperbolic pair and $u \in \langle v_1, v_2\rangle$, and so $V = \langle v_1\rangle \oplus \langle w_1, ..., w_{2m-2}\rangle \oplus \langle v_2\rangle$. Suppose h is an element in the stabiliser of (v_1, v_2). As a consequence, h stabilises u and with respect to $\mathcal{B}$ it has the form

$$[h]_{\mathcal{B}} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

where the matrix A acts on $W = \langle w_1, ..., w_{2m-2}\rangle$ as an element of $\mathrm{GO}_{2m}(W, Q_W)$. If the determinant of g is 1, then we may take

$$A = \begin{bmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $\mu \in F$. On the other hand, if $\det(g) = -1$, then we take

$$A = \begin{bmatrix} 0 & \mu^{-1} & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

instead, so we can always get $\det(hg) = 1$. Note that the latter choice of A also adjusts the quasideterminant in characteristic 2 as the rank of $\mathrm{I} - [h]_{\mathcal{B}}$ in this case is odd. Finally, choosing μ accordingly we can ensure that the spinor norm of hg is 1, i.e. $hg \in \Omega_{2m+2}(F, Q)$. $\square$

The following theorems are explicitly used in the constructions of certain orthogonal subgroups of $\mathrm{E}_6(F)$ and ${}^2\mathrm{E}_6^K(F)$.

Theorem A.3. *Let Q_W be of Witt index at least 1. The subgroup $\Omega_{2m}(F, Q_W)$ is maximal in $W{:}\Omega_{2m}(F, Q_W)$.*

Proof. Recall that $v_2 \in V$ is mapped under the action of $\mathcal{G} = W{:}\Omega_{2m}(F, Q_W)$ to a vector of the form $[-Q_W(u) \mid u \mid 1]$, where u is an element of W. Since the stabiliser of v_2 in $\mathcal{G}$ is $\Omega_{2m}(F, Q_W)$, we conclude that the orbit of v_2 under the action of $\mathcal{G}$ is the following set:

$$\mathcal{O}_{\mathcal{G}}(v_2) = \left\{ [-Q_W(u) \mid u \mid 1] \;\middle|\; u \in W \right\}.$$

Since the elements of this orbit are in one-to-one correspondence with the cosets of

$\Omega_{2m}(F, Q_W)$ in $\mathcal{G}$, it is enough to show the primitive action on $\mathcal{O}_{\mathcal{G}}(v_2)$.

Consider the action of $\mathcal{G}$ on $\mathcal{O}_{\mathcal{G}}(v_2)$. A general element in $\mathcal{G}$ acts on the elements of $\mathcal{O}_{\mathcal{G}}(v_2)$ in the following way:

$$\begin{aligned}[-Q_W(u) \mid u \mid 1] \;\mapsto\; &[-Q_W(u) - uABv^\top - Q(v) \mid uA + v \mid 1] = \\ &= [-Q_W(uA + v) \mid uA + v \mid 1].\end{aligned}$$

Note that $uABv^\top = f_W(uA, v)$. We see that this action is isomorphic to the action on W defined by $u \mapsto uA + v$, where $u, v \in W$. In case when A is the identity matrix, this map is a translation. On the other hand, taking $v = 0$, we obtain the action of $\Omega_{2m}(F, Q_W)$. Denote the group generated by the described action on W as $\mathrm{A}\Omega_{2m}(F, Q_W)$.

Since Q_W is of Witt index at least 1, we may choose a hyperbolic pair (u_1, u_2) in W such that $W = \langle u_1, u_2 \rangle \oplus U$, where $U = \langle u_1, u_2 \rangle^\perp$. We aim to show that any $\mathrm{A}\Omega_{2m}(F, Q_W)$-congruence on W is trivial (and hence, the action is primitive). Suppose $w_1 \sim w_2$, where $w_1, w_2 \in W$ and $\sim$ is some congruence relation preserved by $\mathrm{A}\Omega_{2m}(F, Q_W)$. It follows that $w_1 - w_2 \sim 0$, and so we may start with $v \sim 0$ for some $v \in W$. We distinguish two cases.

First, if $Q_W(v) = 0$, then since $\Omega_{2m}(F, Q_W)$ acts transitively on isotropic vectors in W, we get $u_1 \sim 0$, $u_2 \sim 0$, and also $-\lambda u_2 \sim 0$ for any $\lambda \in F$. Since $\sim$ is a congruence relation, it is transitive and so $u_1 \sim -\lambda u_2$, from which it follows $u_1 + \lambda u_2 \sim 0$. Now, $Q_W(u_1 + \lambda u_2) = \lambda$, and λ is an arbitrary field element, so $\sim$ is trivial.

Next, if $Q_W(v) = \lambda$ for some non-zero $\lambda \in F$, we consider two vectors $w_1 = u_1 + \lambda u_2$ and $w_2 = u_1 + (\lambda - Q_W(u))u_2 + u$ for some $u \in U$. Note that $Q_W(w_1) = Q_W(w_2) = \lambda$, so since $\Omega_{2m}(F, Q_W)$ acts transitively on the vectors of norm λ, we obtain $w_1 \sim 0$ and $w_2 \sim 0$, from which it immediately follows that $w_1 \sim w_2$, and further $w_1 - w_2 \sim 0$. We find $Q_W(w_1 - w_2) = Q_W(u - Q(u)u_2) = Q_W(u)$, so in fact we have $u \sim 0$ for some $u \in U$. From $Q(u_1 + u) = Q(u)$ it follows that $u_1 + u \sim 0$, and so by transitivity $u_1 + u \sim u$. We subtract u from both sides to obtain $u_1 \sim 0$, which is covered by the previous case. $\square$

As we already know from Lemma A.1, the stabiliser in $\Omega_{2m+2}(F, Q)$ of an isotropic vector $v_1 \in V$ is a subgroup of shape $W:\Omega_{2m}(F, Q_W)$. We also find that every proper subgroup of $\Omega_{2m+2}(F, Q)$, containing $W:\Omega_{2m}(F, Q_W)$ as a subgroup, stabilises the 1-space spanned by v_1.

Theorem A.4. *Let Q_W be of Witt index at least 1. Any subgroup H such that*

$$W:\Omega_{2m}(F, Q_W) \leqslant H < \Omega_{2m+2}(F, Q), \tag{57}$$

stabilises the 1-space $\langle v_1 \rangle$.

Proof. Let $G = \Omega_{2m+2}(F, Q)$. We aim to prove that if $v_1 \sim v$ for some $v \in V$, where $\sim$ is any non-trivial G-congruence, then $v = \lambda v_1$ for some $\lambda \in F$. For the sake of finding a contradiction, suppose that $v = \lambda v_1 + u + \mu v_2$, where $u + \mu v_2 \neq 0$, i.e. either $u \neq 0$ or $\mu \neq 0$. We distinguish two cases.

First, if $\mu = 0$, then $v_1 \sim v$, where $v = \lambda v_1 + u$ with $0 \neq u \in W$. We have $0 = Q(v) = Q_W(u)$. Recall that $W:\Omega_{2m}(F, Q_W)$ is the stabiliser in $\Omega_{2m+2}(F, Q)$ of v_1, so with respect to the familiar basis $\mathcal{B} = \{v_1, w_1, ..., w_{2m}, v_2\}$, its general element has the form

$$[g]_{\mathcal{B}} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline u_2^\top & A & 0 \\ \hline \mu & u_1 & 1 \end{array}\right],$$

where $u_2 = -u_1 B A^\top$. Now, g maps v to a vector of the form $v^g = (\lambda + u u_2^\top) v_1 + uA$. Since $u u_2^\top = -uABu_1^\top = f_W(u, u_1)$, we get $v^g = (\lambda - f_W(u, u_1)) v_1 + uA$. Setting $A = \mathrm{I}_{2m}$, we see that it is possible to send v to a vector of the form $\alpha v_1 + u$ for any $\alpha \in F$. On the other hand, taking $u_1 = 0$, we obtain the action on $W = \langle w_1, ..., w_{2m} \rangle$ of $\Omega_{2m}(F, Q_W)$, which is transitive on the isotropic vectors in W. We have shown that $v_1 \sim v$ for any v of the form $\alpha v_1 + u$ for arbitrary $\alpha \in F$ and $0 \neq u \in W$. The group $\Omega_{2m+2}(F, Q)$ in its turn is transitive on the isotropic vectors in V, so there exists an element $h \in \Omega_{2m+2}(F, Q)$ such that $v_1^h = v_2$ and $v^h \in \langle v_1 \rangle \oplus W$. It follows that $v_2 \sim v^h$ and so, by transitivity of $\sim$ we find that $v_1 \sim v_2$. Finally, it is easy to derive the congruences of the form $v_1 \sim \alpha v_1$ and $v_2 \sim \beta v_2$ for any non-zero $\alpha, \beta \in F$.

For instance, v_2 can be scaled using an element of the form

$$[s(\beta)]_{\mathcal{B}} = \left[\begin{array}{c|c|c} \beta^{-1} & 0 & 0 \\ \hline & & \\ 0 & \mathrm{I}_{2m} & 0 \\ & & \\ \hline 0 & 0 & \beta \end{array}\right].$$

It follows that if $\mu = 0$, the congruence relation $\sim$ is universal.

$\square$

B: $\Omega_4^+(F) \cong \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$

C: $\Omega_4^{-,K}(F) \cong \mathrm{PSL}_2(K)$

D: Magma code