On an octonionic construction of the groups of type E_6 and ${}^2E_6$

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Declaration

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Abstract

ABSTRACT HERE

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ACKNOWLEDGEMENTS HERE

dedication here

Contents

Chapter 1

Introduction

1.1 Motivation

1.2 Historical notes

1.3 Notation

Chapter 2

Octonions

2.1 Composition algebras

2.1.1 Quadratic and Bilinear Forms

Let V be a vector space over a field F. We define a *quadratic form* Q on V to be a map $Q : V \to F$ such that

(i) $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$ and $\lambda \in F$;

(ii) the form $\langle \cdot, \cdot \rangle : V \times V \to K$, defined by

$$\langle u, v \rangle = Q(u + v) - Q(u) - Q(v), \tag{2.1}$$

is bilinear. We usually refer to $\langle \cdot, \cdot \rangle$ as *the polar form of* Q.

From (2.1) we readily see that the form $\langle \cdot, \cdot \rangle$ is symmetric, i.e. $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$. We also observe that for all $v \in V$ we have

$$\langle v, v \rangle = 2Q(v), \tag{2.2}$$

It follows that in case $\mathrm{char}(F) = 2$ we get $\langle v, v \rangle = 0$ for all v, and the quadratic form carries strictly more information than the associated bilinear form. In all other characteristics, hovewer, we get $Q(v) = \frac{1}{2}\langle v, v \rangle$.

We say that two non-zero vectors $u, v \in V$ are *orthogonal*, if $\langle u, v \rangle = 0$. As already mentioned, this relation is symmetric. Now if U is any subspace of V (and even if it is just a subset), we define its *orthogonal complement* $U^\perp$ to be

$$U^\perp = \left\{ v \in V \,\middle|\, \langle u, v \rangle = 0 \text{ for all } u \in U \right\}. \tag{2.3}$$

A non-zero vector $v \in V$ is called *isotropic* if $Q(v) = 0$, otherwise v is *anisotropic*. Sometimes we also say that $Q(v)$ is the *norm* of v. Now, the quadratic form Q is isotropic if there exists an isotropic vector in V. The *radical* of $\langle \cdot, \cdot \rangle$ is $V^\perp$, and $\langle \cdot, \cdot \rangle$ is *non-degenerate* if the radical is trivial, or, otherwise speaking, if

$$\langle v, u \rangle = 0 \text{ for all } u \in V \text{ implies that } v = 0. \tag{2.4}$$

Similarly, the *radical* of Q is the subset of the radical of $\langle \cdot, \cdot \rangle$, consisting of isotropic vectors, i.e.

$$\mathrm{rad}_V(Q) = \{ v \in V \,|\, \langle v, u \rangle = 0 \text{ for all } u \in V,\ Q(v) = 0 \}. \tag{2.5}$$

If the radical of the form Q is trivial, then Q is said to be *non-singular*. Throughout this thesis we will be mostly interested in non-singular quadratic and non-degenerate bilinear forms. If U is a subspace of V and the restriction of $\langle \cdot, \cdot \rangle$ on U is non-degenerate, then $V = U \oplus U^\perp$, and the restriction of $\langle \cdot, \cdot \rangle$ on $U^\perp$ is also non-degenerate. A subspace U of V consisting entirely of isotropic vectors is called *totally isotropic*.

2.1.2 Isometries and Witt's Lemma

Let V_1, V_2 be vector spaces over fields F_1 and F_2 respectively, with non-singular quardatic forms Q_1 and Q_2. Denote by $\langle \cdot, \cdot \rangle_i$ the polar form of Q_i $(i = 1, 2)$. Suppose $\sigma : F_1 \to F_2$ is a field isomorphism. A map $s : V_1 \to V_2$, satisfying

$$Q_2(v^s) = \lambda_s Q_1(v)^\sigma \quad (v \in V_1), \tag{2.6}$$

where $\lambda_s \in F_2^\times$, is called a *σ-similarity*. The scalar λ_s is known as the *multiplier* of s. Using the definition of polar form, we obtain $\langle u^s, v^s \rangle_2 = \lambda_s \langle u, v \rangle_1^\sigma$, so s is a bijection. If $\lambda_s = 1$, then s is called a *σ-isometry*. In the case when a σ-similarity (or σ-isometry) between two spaces V_1 and V_2 exists, we say that V_1 and V_2 are σ-similar (or σ-isometric). If σ is the identity map, then σ-similarity (or σ-isometry) is simply called *similarity* (or *isometry*).

A key result about isometries, which also plays an important rôle in the study of the geometry of spaces with quadratic forms, is Witt's Lemma (also known as Witt's Theorem).

Theorem 2.1.1 (Witt's Lemma)**.** *If V_1, V_2 are two σ-isometric vector spaces of finite dimension with non-singular quadratic forms Q_1 on V_1 and Q_2 on V_2, then every σ-isometry between a subspace of V_1 and a subspace of V_2 extends to a σ-isometry between V_1 and V_2.*

If V is a vector space over F with a non-singular quadratic form Q, then an isometry from V onto itself is called an *orthogonal transformation* of V with respect to Q. These orthogonal transformations form the *(general) orthogonal group* $\mathrm{GO}(V,Q)$. Now suppose $s : V \to V$ is an invertible linear transformation such that $Q(v^s) = Q(v)$ for all $v \in V$ (and thus $\langle u^s, v^s\rangle = \langle u, v\rangle$ for all $u, v \in V$). Denote $n = \dim_F(V)$ and pick a basis $\mathcal{B} = \{ v_1, ..., v_n \}$. Then with respect to $\mathcal{B}$, s can be represented by an $n \times n$ matrix $[s]_{\mathcal{B}}$. The determinant of the resulting matrix is independent of the choice of basis, so there is a group homomorphism $\det : \mathrm{GO}(V,Q) \to F^\times$. Orthogonal transformations have determinant ± 1. In case of characteristic 2 we define the *quasideterminant* $\mathrm{qdet} : \mathrm{GO}(V,Q) \to \mathbb{F}_2$ to be the map

$$\mathrm{qdet} : g \mapsto \dim_F(\mathrm{Im}\,(\mathrm{id} - g)) \mod 2. \tag{2.7}$$

The subgroup $\mathrm{SO}(V,Q)$ of $\mathrm{GO}(V,Q)$ is the kernel of the (quasi-)determinant map. The group $\mathrm{SO}(V,Q)$ is referred to as *special orthogonal group* or *rotation* group of V with respect to Q.

Note that not every element of $\mathrm{GO}(V,Q)$ arises as a rotation. For an anisotropic vector $v \in V$ define r_v to be

$$r_v : u \mapsto u - \frac{\langle u, v\rangle}{Q(v)} v \quad (u \in V). \tag{2.8}$$

If the characteristic is not 2, then r_v is the *reflexion* in (the hyperplane orthogonal to) v. If $\mathrm{char}(K) = 2$, then r_v is the *orthogonal transvection* with centre v. For simplicity we use the word 'reflexion' in all cases.

We define the *spinor norm* to be a homomorphism $\mathrm{GO}(V, Q) \to F^\times/(F^\times)^2$, where $F^\times/(F^\times)^2$ is the *multiplicative group modulo squares* of F. The aforementioned homomorphism is defined as follows. Any element of $\mathrm{GO}(V, Q)$ arising as a reflexion in v is sent to the value $Q(v)$ modulo $(F^\times)^2$. This extends to a well-defined homomorphism. The subgroup $\Omega(V, Q)$ of $\mathrm{SO}(V, Q)$ is obtained as the kernel of spinor norm.

Witt's Lemma implies that all maximal totally isotropic subspaces of V (with respect to Q) have the same dimension, which is called the *Witt index* of Q. When Q is non-singular and V is finite-dimensional, Witt index of Q can be at most $\frac{1}{2}\dim_F(V)$. Moreover, the isometry group $\mathrm{GO}(V, Q)$ acts transitively on the set of maximal totally isotropic subspaces.

2.1.3 Definition of a composition algebra

Definition 2.1.2. *A composition algebra $C = C_F$ over a field F is a (not necessarily associative) unital algebra over F which admits a non-singular quadratic form $N : C \to F$ such that the polar form of N is non-degenerate and*

$$N(xy) = N(x)N(y) \quad \textit{for all } x, y \in C. \tag{2.9}$$

The quadratic form N on C is usually called the *norm* of C, and its polar form is referred to as the *inner product*. We also denote the identity element as 1_C.

Let D be a linear subspace of C such that the restriction of $\langle \cdot, \cdot \rangle$ on D is non-degenerate. If D is closed under multiplication and contains 1_C, then it is called the *subalgebra* of C.

Let C_1, C_2 be two composition algebras over fields F_1, F_2 respectively and suppose $\sigma : F_1 \to F_2$ is a field isomorphism. A bijective σ-linear transformation $s : C_1 \to C_2$ is called a *σ-isomorphism*, if

$$(xy)^s = x^s y^s \quad \text{for all } x, y \in C_1. \tag{2.10}$$

For simplicity, if $F_1 = F_2$ and $\sigma = \mathrm{id}$, then s is called an *isomorphism*.

Definition 2.1.2 allows us to derive a number of useful equations. First of all, we find that

$$N(x) = N(1_C \cdot x) = N(1_C)N(x)$$

for all $x \in C$, so it follows that

$$N(1_C) = 1. \tag{2.11}$$

Next, for any $x_1, x_2, y \in C$ we have

$$\begin{aligned} N(x_1 y + x_2 y) &= N((x_1 + x_2)y) = N(x_1 + x_2)N(y) \\ &= (N(x_1) + N(x_2) + \langle x_1, x_2 \rangle)N(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} N(x_1 y + x_2 y) &= N(x_1 y) + N(x_2 y) + \langle x_1 y, x_2 y \rangle \\ &= N(x_1)N(y) + N(x_2)N(y) + \langle x_1 y, x_2 y \rangle, \end{aligned}$$

and so

$$\langle x_1 y, x_2 y \rangle = \langle x_1, x_2 \rangle N(y) \tag{2.12}$$

for all $x_1, x_2, y \in C$. Similarly, we obtain

$$\langle x y_1, x y_2 \rangle = N(x) \langle y_1, y_2 \rangle \tag{2.13}$$

for all $x, y_1, y_2 \in C$. Replacing y by $y_1 + y_2$ in (2.12), we obtain

$$\langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \tag{2.14}$$

for all $x_1, x_2, y_1, y_2 \in C$.

Any composition algebra is quadratic, that is, every element satisfies a certain quadratic equation.

Proposition 2.1.3. *Every element x of a composition algebra C satisfies the following equation:*

$$x^2 - \langle x, 1_C \rangle x + N(x) \cdot 1_C = 0. \tag{2.15}$$

In the case when x is not a scalar multiple of 1_C, this is the minimal equation for

x. For all $x, y \in C$ we have

$$xy + yx - \langle x, 1_C \rangle y - \langle y, 1_C \rangle x + \langle x, y \rangle \cdot 1_C = 0. \tag{2.16}$$

For example, if x, y are orthogonal to 1_C and $\langle x, y \rangle = 0$, then $xy = -yx$, but most importantly we have the following corollary.

Corollary 2.1.4. *The norm N in a composition algebra C is uniquely determined by the algebra structure. Any σ-isomorphism of composition algebras is always a σ-isometry.*

Any composition algebra is *power associative*, i.e. for all $x \in C$ and $i, j \geqslant 1$, we have

$$x^i x^j = x^{i+j}. \tag{2.17}$$

2.2 Conjugation and inverses

We define *conjugation* in a composition algebra C to be the mapping $\bar{\ } : C \to C$ defined by

$$\bar{x} = \langle x, 1_C \rangle \cdot 1_C - x \quad (x \in C). \tag{2.18}$$

Note that geometrically speaking, the map $x \mapsto \bar{x}$ is $-r_{1_C}$, where r_{1_C} is the reflexion in 1_C. We call $\bar{x}$ the *conjugate* of x. The following lemma summarises the properties of $\mathbb{O}$ related to conjugation.

Lemma 2.2.1. *For all $x, y \in C$ the following identities hold:*

(i) $x\bar{x} = \bar{x}x = N(x) \cdot 1_C$,

(ii) $\overline{xy} = \bar{y}\bar{x}$,

(iii) $\bar{\bar{x}} = x$,

(iv) $\overline{x + y} = \bar{x} + \bar{y}$,

(v) $N(x) = N(\bar{x})$,

(vi) $\langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle$.

Furthermore, we have the following important properties.

Lemma 2.2.2. *For all $x, y, z \in C$ the following identities hold:*

(i) $x(\bar{x}y) = N(x)y$,

(ii) $(x\bar{y})y = N(y)x$,

(iii) $x(\bar{y}z) + y(\bar{x}z) = \langle x, y\rangle \cdot z$,

(iv) $(x\bar{y})z + (x\bar{z})y = x \cdot \langle y, z\rangle$.

If for an element $x \in C$ we have $N(x) \neq 0$, then x is said to be *invertible.* If this is the case, then the *inverse* of x is

$$x^{-1} = N(x)^{-1}\bar{x}. \tag{2.19}$$

Lemma 2.2.3. *If $x, y \in C$ are invertible, then*

$$(xy)^{-1} = y^{-1}x^{-1}. \tag{2.20}$$

2.3 Alternative laws and Moufang identities

Composition algebras are not necessarily associative, but there are certain results which can help us with the bracketing.

Lemma 2.3.1 (Moufang Identities)**.** *For all $x, y, z \in C$, the following identities hold:*

$$\begin{aligned} x(yz)x &= (xy)(zx), \\ x(yzy) &= ((xy)z)y, \\ (xyx)z &= x(y(xz)). \end{aligned} \tag{2.21}$$

This helps us to conclude that any composition algebra C is *alternative.* That is, for every element $x \in C$ the left-multiplication by x commutes with right-multiplication by x.

Lemma 2.3.2 (Alternative Laws)**.** *For all $x, y \in C$ the following are true:*

$$\begin{aligned}(xx)y &= x(xy),\\ (yx)x &= y(xx),\\ (xy)x &= x(yx).\end{aligned} \tag{2.22}$$

Theorem 2.3.3 (Artin)**.** *The subalgebra generated by any two elements of an alternative algebra is always associative.*

2.4 Octonion algebras

The most important structural result about composition algebras is the following theorem.

Theorem 2.4.1. *The possible dimensions of a composition algebra are* 1*,* 2*,* 4*, and* 8*. Composition algebras of dimension* 1 *only occur if the characreristic of the field is not* 2*. Composition algebras of dimension* 1 *and* 2 *are associative and commutative. Those of dimension* 4 *are associtaive but not commutative, and those of dimension* 8 *are neither associative nor commutative.*

In this thesis we will be mostly interested in the 8-dimensional composition algebras. To emphasise their importance in our work, we use a separate name for them.

Definition 2.4.2. *Let F be any field. An octonion algebra $\mathbb{O} = \mathbb{O}_F$ is an 8-dimensional composition algebra, i.e. it admits a norm defined as a quadratic form $\mathrm{N} : \mathbb{O} \to F$ such that the polar form of N is non-degenerate and $\mathrm{N}(xy) = \mathrm{N}(x)\mathrm{N}(y)$ for all $x, y \in \mathbb{O}$.*

The elements of $\mathbb{O}$ are called the *octonions*. The multiplicative identity in $\mathbb{O}$ is denoted $1_{\mathbb{O}}$, and for simplicity we sometimes omit the subscript. The polar form of N is denoted by $\langle \cdot, \cdot \rangle$ as usual. Define the *trace* of an octonion to be the inner product

$$\mathrm{T}(x) = \langle x, 1_{\mathbb{O}} \rangle. \tag{2.23}$$

It is easy to see that

$$\mathrm{T}(x) \cdot 1_{\mathbb{O}} = x + \bar{x}. \tag{2.24}$$

Although we define trace through the inner product, using Lemma 2.2.1 we can derive the following important relation.

Lemma 2.4.3. *For all $x, y \in \mathbb{O}$, the following identity holds:*

$$\langle x, y\rangle = \mathrm{T}(x\bar{y}). \tag{2.25}$$

Proof. Lemma 2.2.1 tells us that for all $x \in \mathbb{O}$, $\mathrm{N}(x) \cdot 1_{\mathbb{O}} = x\bar{x}$. Polarising N as usual, we obtain

$$\begin{aligned}\langle x, y\rangle \cdot 1_{\mathbb{O}} &= \mathrm{N}(x+y) \cdot 1_{\mathbb{O}} - \mathrm{N}(x) \cdot 1_{\mathbb{O}} - \mathrm{N}(y) \cdot 1_{\mathbb{O}} \\ &= (x+y)(\bar{x}+\bar{y}) - x\bar{x} - y\bar{y} = x\bar{y} + y\bar{x} = \mathrm{T}(x\bar{y}).\end{aligned}$$

□

Proposition 2.1.3 tells us that an arbitrary element $x \in \mathbb{O}$ satisfies the equation

$$x^2 - \mathrm{T}(x) \cdot x + \mathrm{N}(x) \cdot 1_{\mathbb{O}} = 0. \tag{2.26}$$

Finally, as we know, any octonion algebra $\mathbb{O}$ is neither associative nor commutative. However, we do have the following.

Lemma 2.4.4. *If $x, y, z \in \mathbb{O}$, then $\mathrm{T}(xy) = \mathrm{T}(yx)$ and $\mathrm{T}(x(yz)) = \mathrm{T}((xy)z)$.*

Note that although trace is 3-associative, it is not possible in this case to derive generalised associativity for the trace.

Lemma 2.4.5. *For all non-zero $C \in \mathbb{O}$ the map $\mathbb{O} \to F$, $x \mapsto \mathrm{T}(Cx)$ is onto.*

Proof. This is an F-linear map, so if it is not surjective, then it is a zero map. But if $\top(Cx) = \langle C, \bar{x}\rangle = 0$ for all $x \in \mathbb{O}$, then $C = 0$ (a contradiction), since the map $x \mapsto \bar{x}$ is surjective. □

Further in this thesis we will be interested in a certain class of subalgebras of $\mathbb{O}$. We say that a subalgebra $\mathbb{S}$ of $\mathbb{O}$ is *sociable*, if for any $x, y \in \mathbb{S}$ and any $z \in \mathbb{O}$, $x(zy) = (xz)y$.

2.5 Split octonion algebras

There is an important dichotomy with respect to the structure of an octonion algebra: either $\mathbb{O}$ is a division algebra or there exists an isotropic octonion. In the latter case $\mathbb{O}$ is called a *split octonion algebra*.

If $\mathbb{O}$ is split, then the Witt index of N is 4 (section 1.8 in [?]). Moreover, we have the following result.

Theorem 2.5.1. *Over any given field F there is a unique split octonion algebra, up to isomorphism.*

It turns out that any isotropic octonion left- and right-annihilates a 4-dimensional subspace of a split octonion algebra $\mathbb{O}$.

Proposition 2.5.2. *Let $\mathbb{O}$ be a split octonion algebra. Then for any isotropic $x \in \mathbb{O}$, the following is true:*

$$\dim_F(\mathbb{O}x) = \dim_F(x\mathbb{O}) = 4. \tag{2.27}$$

Moreover, $\mathbb{O}x$ is the set of octonions that are right-annihilated by $\bar{x}$, and $x\mathbb{O}$ is the set of octonions that are left-annihilated by $\bar{x}$.

Proof. We prove the statement for right multiplication by x. The proof for left multiplication is essentially the same. The map

$$\begin{aligned} R_x : \mathbb{O} &\to \mathbb{O} \\ y &\mapsto yx \end{aligned}$$

is an F-linear map with $\mathrm{Im}\,(R_x) = \mathbb{O}x$, which is a totally isotropic subspace of $\mathbb{O}$. Indeed, $(yx)(\bar{x}\bar{y}) = y(x\bar{x})\bar{y} = 0$ for any $y \in \mathbb{O}$. Since N is non-singular and its polar form is non-degenerate, we conclude that $\dim_F(\mathbb{O}x) \leqslant 4$.

If $x \neq 0$ and $yx = 0$, then y is isotropic for if that were not the case, we would get $x = y^{-1}(yx) = y^{-1} \cdot 0 = 0$, a contradiction. It follows that $\dim_F(\ker(R_x)) \leqslant 4$. The Rank–Nullity theorem implies that $\dim_F(\mathbb{O}x) = \dim_F(\ker(R_x)) = 4$. □

2.6 A basis for the split octonions

In this section we assume that $\mathbb{O}$ is a split octonion algebra. Theorem 2.5.1 allows us to choose a basis for $\mathbb{O}$ and to use it in our further constructuions. Otherwise speaking, we can 'redefine' split octonion algebras in the following way.

Definition 2.6.1. *If F is any field, then the split octonion algebra over F is defined as an 8-dimensional vector space $\mathbb{O} = \mathbb{O}_F$ with basis $\{ e_i \mid i \in \pm I \}$, where $I = \{ 0, 1, \omega, \overline{\omega} \}$, $\pm I = \{ \pm 0, \pm 1, \pm\omega, \pm\overline{\omega} \}$ and bilinear multiplication given by the following table.*

	e_{-1}	$e_{\overline{\omega}}$	e_{ω}	e_0	e_{-0}	$e_{-\omega}$	$e_{-\overline{\omega}}$	e_1
e_{-1}	0	0	0	0	e_{-1}	$e_{\overline{\omega}}$	$-e_{\omega}$	$-e_0$
$e_{\overline{\omega}}$	0	0	$-e_{-1}$	$e_{\overline{\omega}}$	0	0	$-e_{-0}$	$e_{-\omega}$
e_{ω}	0	e_{-1}	0	e_{ω}	0	$-e_{-0}$	0	$-e_{-\overline{\omega}}$
e_0	e_{-1}	0	0	e_0	0	$e_{-\omega}$	$e_{-\overline{\omega}}$	0
e_{-0}	0	$e_{\overline{\omega}}$	e_{ω}	0	e_{-0}	0	0	e_1
$e_{-\omega}$	$-e_{\overline{\omega}}$	0	$-e_0$	0	$e_{-\omega}$	0	e_1	0
$e_{-\overline{\omega}}$	e_{ω}	$-e_0$	0	0	$e_{-\overline{\omega}}$	$-e_1$	0	0
e_1	$-e_{-0}$	$-e_{-\omega}$	$e_{-\overline{\omega}}$	e_1	0	0	0	0

In other words, we get

(i) $e_1 e_\omega = -e_\omega e_1 = e_{-\omega}$;

(ii) $e_1 e_0 = -e_0 e_1 = e_1$;

(iii) $e_{-1} e_1 = -e_0$ and $e_0 e_0 = e_0$;

and images under negating all subscripts (including 0), and multiplying all subscripts by ω, where $\omega^2 = \overline{\omega}$ and $\omega\overline{\omega} = 1$. All other products of basis vectors are 0. Thus, e_0 and e_{-0} are orthogonal idempotents with $e_0 + e_{-0} = 1_{\mathbb{O}}$. Now, if $x = \sum_{i \in \pm I} \lambda_i e_i$, then the norm of x can be defined in the following way:

$$\mathrm{N}(x) = \lambda_{-1}\lambda_1 + \lambda_{\overline{\omega}}\lambda_{-\overline{\omega}} + \lambda_\omega\lambda_{-\omega} + \lambda_0\lambda_{-0}. \tag{2.28}$$

Lemma 2.6.2. *The norm N defined in (2.28) is multiplicative.*

Proof. Let $x = \sum_{i \in \pm I} \lambda_i e_i$ and $y = \sum_{i \in \pm I} \mu_i e_i$ be two arbitrary elements of $\mathbb{O}$. Their product is given by

$$\begin{aligned}
x \cdot y = {} & (\lambda_0\mu_{-0} - \lambda_{\overline{\omega}}\mu_\omega + \lambda_\omega\mu_{\overline{\omega}} + \lambda_0\mu_{-1}) \cdot e_{-1} \\
& + (\lambda_{-1}\mu_{-\omega} + \lambda_{\overline{\omega}}\mu_0 + \lambda_{-0}\mu_\omega - \lambda_{-\omega}\mu_{-1}) \cdot e_{\overline{\omega}} \\
& + (\lambda_{-\overline{\omega}}\mu_{-1} + \lambda_{-1}\mu_\omega - \lambda_{-1}\mu_{-\overline{\omega}} + \lambda_\omega\mu_0) \cdot e_\omega \\
& + (\lambda_0\mu_0 - \lambda_{-\omega}\mu_\omega - \lambda_{-\overline{\omega}}\mu_{\overline{\omega}} - \lambda_{-1}\mu_1) \cdot e_0 \\
& + (\lambda_{-0}\mu_{-0} - \lambda_1\mu_{-1} - \lambda_{\overline{\omega}}\mu_{-\overline{\omega}} - \lambda_\omega\mu_{-\omega}) \cdot e_{-0} \\
& + (\lambda_0\mu_{-\omega} - \lambda_1\mu_{\overline{\omega}} + \lambda_{-\omega}\mu_{-0} + \lambda_{\overline{\omega}}\mu_1) \cdot e_{-\omega} \\
& + (\lambda_{-\overline{\omega}}\mu_{-0} + \lambda_1\mu_\omega - \lambda_\omega\mu_1 + \lambda_0\mu_{-\overline{\omega}}) \cdot e_{-\overline{\omega}} \\
& + (\lambda_{-0}\mu_1 + \lambda_{-\omega}\mu_{-\overline{\omega}} - \lambda_{-\overline{\omega}}\mu_{-\omega} + \lambda_1\mu_0) \cdot e_1.
\end{aligned}$$

From this it is straightforward to derive

$$\begin{aligned}
\mathrm{N}(x \cdot y) = {} & (\lambda_0\mu_{-0} - \lambda_{\overline{\omega}}\mu_\omega + \lambda_\omega\mu_{\overline{\omega}} + \lambda_0\mu_{-1}) \cdot (\lambda_{-0}\mu_1 + \lambda_{-\omega}\mu_{-\overline{\omega}} - \lambda_{-\overline{\omega}}\mu_{-\omega} + \lambda_1\mu_0) \\
& + (\lambda_{-1}\mu_{-\omega} + \lambda_{\overline{\omega}}\mu_0 + \lambda_{-0}\mu_\omega - \lambda_{-\omega}\mu_{-1}) \cdot (\lambda_{-\overline{\omega}}\mu_{-0} + \lambda_1\mu_\omega - \lambda_\omega\mu_1 + \lambda_0\mu_{-\overline{\omega}}) \\
& + (\lambda_{-\overline{\omega}}\mu_{-1} + \lambda_{-1}\mu_\omega - \lambda_{-1}\mu_{-\overline{\omega}} + \lambda_\omega\mu_0) \cdot (\lambda_0\mu_{-\omega} - \lambda_1\mu_{\overline{\omega}} + \lambda_{-\omega}\mu_{-0} + \lambda_{\overline{\omega}}\mu_1) \\
& + (\lambda_0\mu_0 - \lambda_{-\omega}\mu_\omega - \lambda_{-\overline{\omega}}\mu_{\overline{\omega}} - \lambda_{-1}\mu_1) \cdot (\lambda_{-0}\mu_{-0} - \lambda_1\mu_{-1} - \lambda_{\overline{\omega}}\mu_{-\overline{\omega}} - \lambda_\omega\mu_{-\omega}) \\
= {} & \lambda_{-1}\lambda_1 \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
& + \lambda_{\overline{\omega}}\lambda_{\overline{\omega}} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
& + \lambda_\omega\lambda_{-\omega} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
& + \lambda_0\lambda_{-0} \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
= {} & (\lambda_{-0}\lambda_0 + \lambda_{\overline{\omega}}\lambda_{-\overline{\omega}} + \lambda_\omega\lambda_{-\omega} + \lambda_0\lambda_{-0}) \cdot (\mu_{-0}\mu_0 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
= {} & \mathrm{N}(x) \cdot \mathrm{N}(y).
\end{aligned}$$

□

It follows that $\mathbb{O}$ is indeed a composition algebra. Let x and y be the same as in Lemma 2.6.2. We find

$$
\begin{aligned}
\langle x, y\rangle &= \mathrm{N}(x+y) - \mathrm{N}(x) - \mathrm{N}(y) \\
&= (\lambda_{-1} + \mu_{-1}) \cdot (\lambda_1 + \mu_1) + (\lambda_{\overline{\omega}} + \mu_{\overline{\omega}}) \cdot (\lambda_{-\overline{\omega}} + \mu_{-\overline{\omega}}) \\
&+ (\lambda_\omega + \mu_\omega) \cdot (\lambda_{-\omega} + \mu_{-\omega}) + (\lambda_0 + \mu_0) \cdot (\lambda_{-0} + \mu_{-0}) \\
&- (\lambda_{-1}\lambda_1 + \lambda_{\overline{\omega}}\lambda_{-\overline{\omega}} + \lambda_\omega\lambda_{-\omega} + \lambda_0\lambda_{-0}) \\
&- (\mu_{-1}\mu_1 + \mu_{\overline{\omega}}\mu_{-\overline{\omega}} + \mu_\omega\mu_{-\omega} + \mu_0\mu_{-0}) \\
&= (\lambda_{-1}\mu_1 + \lambda_1\mu_{-1}) + (\lambda_{\overline{\omega}}\mu_{-\overline{\omega}} + \lambda_{-\overline{\omega}}\mu_{\overline{\omega}}) \\
&+ (\lambda_\omega\mu_{-\omega} + \lambda_{-\omega}\mu_\omega) + (\lambda_0\mu_{-0} + \lambda_{-0}\mu_0).
\end{aligned}
\tag{2.29}
$$

Thus, the trace of x becomes

$$
\mathrm{T}(x) = \langle x, 1_{\mathbb{O}}\rangle = \lambda_0 + \lambda_{-0}. \tag{2.30}
$$

Note that $\mathrm{N}(e_i) = 0$ for $i \neq \pm 0$, so $\mathbb{O}$ is indeed a split octonion algebra. Finally, the involution $x \mapsto \bar{x}$ is the extension by linearity of

$$
e_i \mapsto -e_i \ (i \neq \pm 0), \ \ e_0 \leftrightarrow e_{-0}. \tag{2.31}
$$

2.7 Centre of an octonion algebra

We define the centre of an octonion algebra $\mathbb{O}$ as

$$
\mathrm{Z}(\mathbb{O}) = \{\, c \in \mathbb{O} \,\big|\, cx = xc \text{ for all } x \in \mathbb{O} \,\}. \tag{2.32}
$$

In the literature, for example, in [**?**], it is sometimes required that central elements also "associate" with all other elements. We do not require this in our definition, however, it will be obvious that we have this property free of charge.

Proposition 2.7.1. *The centre of an octonion algebra $\mathbb{O} = \mathbb{O}_F$ is $F \cdot 1_{\mathbb{O}}$.*

This is essentially Proposition 1.9.1 in [**?**], however, we need to emphasise that in the proof of this proposition the following result is used without mentioning.

Lemma 2.7.2. *Let K be an extension field of F and let A be an F-algebra with centre $\mathrm{Z}(A)$. Then $\mathrm{Z}(A \otimes_F K) = \mathrm{Z}(A) \otimes_F K$.*

Proof. The proof is straightforward. Pick an arbitrary element $z = \sum_i (a_i \otimes e_i)$ in $\mathrm{Z}(A \otimes_F K)$. Here we may assume that the elements $e_i \in K$ are linearly independent, i.e. they form a (part of) basis for K. Since z is central, in particular it must commute with the elements of the form $a \otimes 1$. This means

$$\begin{aligned} 0 = z(a \otimes 1) - (a \otimes 1)z = \sum_i ((a_i a) \otimes e_i) - \sum_i ((a a_i) \otimes e_i) \\ = \sum_i ((a_i a - a a_i) \otimes e_i). \end{aligned}$$

This holds if and only if $a_i a = a a_i$, i.e. $a_i \in \mathrm{Z}(A)$. □

Therefore, any octonion algebra is central, and it follows from Proposition 2.7.1 that central elements "associate" with all other elements.

Proposition 2.7.3. *If an octonion $u \in \mathbb{O}$ satisfies*

$$(xy)u = x(yu) \tag{2.33}$$

for all $x, y \in \mathbb{O}$, then $u \in F \cdot 1_{\mathbb{O}}$. Condition (2.33) is equivalent to the condition $(xu)y = x(uy)$ for all $x, y \in \mathbb{O}$, and also to $(ux)y = u(xy)$ for all $x, y \in \mathbb{O}$.

Corollary 2.7.4. *Suppose that $u \in \mathbb{O}$ is an invertible octonion. Then*

$$(A\bar{u})(uB) = N(u)AB \tag{2.34}$$

holds for all $A, B \in \mathbb{O}$ if and only if $u \in F \cdot 1_{\mathbb{O}}$.

Proof. Proposition 2.7.3 tells us that if $(xu)y = x(uy)$ for all $x, y \in \mathbb{O}$, then $u \in \mathbb{O}$. Now put $x = A\bar{u}$ and $y = B$; using this together with the alternative laws, we get the result.

Conversely, if $u \in F \cdot 1_{\mathbb{O}}$, then obviously the statement holds. □

Chapter 3

Groups of type E_6

3.1 Albert vectors

3.1.1 Albert space $\mathbb{J}$

For the further discussion we consider $\mathbb{O} = \mathbb{O}_F$ to be an arbitrary octonion algebra over the field F. In the results which require $\mathbb{O}$ to be split, we specify this explicitly.

Define the *Albert space* $\mathbb{J} = \mathbb{J}_F$ to be the 27-dimensional vector space spanned by the elements of the form

$$(a, b, c \mid A, B, C) = \begin{bmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{bmatrix}, \tag{3.1}$$

where $a, b, c, A, B, C \in \mathbb{O}$ and furthermore $a, b, c \in \langle 1_{\mathbb{O}} \rangle$. Now, an *Albert vector* is an element of $\mathbb{J}$. To denote certain subspaces of $\mathbb{J}$ we use the following intuitive notation. The 10-dimensional subspace spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ is denoted $\mathbb{J}_{10}^{abC}$, while the 8-space spanned by the vectors $(0, 0, 0 \mid A, 0, 0)$ is denoted $\mathbb{J}_8^A$ and so on. That is, the subscript determines the dimension and the superscript shows which of the six 'coördinates' we use to span the corresponding subspace. Of course, this notation is by no means complete as it does not allow us to denote any possible subspace of $\mathbb{J}$. If this is the case, we specify the spanning vectors and denote the corresponding space in some other manner.

Suppose $X = (a, b, c \mid A, B, C) \in \mathbb{J}$ is an arbitrary Albert vector. We define the quadratic form Q on $\mathbb{J}$ via

$$\mathrm{Q}(X) = A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc. \tag{3.2}$$

As usual, this can be polarised to obtain the inner product

$$\begin{aligned} \mathrm{B}(X, Y) = \mathrm{T}(&A_1\overline{A}_2 + B_1\overline{B}_2 + C_1\overline{C}_2) \\ &- (a_1b_2 + a_2b_1) - (a_1c_2 + a_2c_1) - (b_1c_2 + b_2c_1), \end{aligned} \tag{3.3}$$

where $X = (a_1, b_1, c_1 \mid A_1, B_1, C_1)$ and $Y = (a_2, b_2, c_2 \mid A_2, B_2, C_2)$.

3.1.2 Dickson–Freudenthal determinant and $\mathrm{SE}_6(F)$

Lacking the associativity in $\mathbb{O}$ we also need to be slightly careful when we calculate the determinant of X. For these purposes we define the Dickson–Freudenthal determinant as

$$\Delta(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + \mathrm{T}(ABC). \tag{3.4}$$

This is a cubic form on $\mathbb{J}$ and it can be shown that it is equivalent to the original Dickson's cubic form [**?**] used to construct the group of type E_6.

We define the group $\mathrm{SE}_6(F)$ or $\mathrm{SE}_6(F, \mathbb{O})$ if we want to specify the octonion algebra, to be the group of all F-linear maps on $\mathbb{J}$ preserving the Dickson–Freudenthal determinant. If $F = \mathbb{F}_q$, then we denote this by $\mathrm{SE}_6(q)$. The group $\mathrm{E}_6(F)$ is defined as the quotient of $\mathrm{SE}_6(F)$ by its centre. Suppose M is a 3×3 matrix written over $\mathbb{O}$. If M is written over any sociable subalgebra of $\mathbb{O}$, then for an element $X \in \mathbb{J}$ the mapping $X \mapsto \overline{M}^{\top} X M$ makes sense. Indeed, every entry in the matrix $\overline{M}^{\top} X M$ is a sum of the terms of the form $m_1 x m_2$, where m_1 and m_2 belong to the same sociable subalgebra, and so $(m_1 x)m_2 = m_1(x m_2)$. Furthermore, the map $X \mapsto \overline{M}^{\top} X M$ is obviously F-linear:

$$\overline{M}^{\top}(\lambda X + \mu Y)M = \lambda(M^{\top} X M) + \mu(M^{\top} Y M).$$

3.2 Some elements of $\mathrm{SE}_6(F)$

Througout this section, let $X = (a, b, c \mid A, B, C)$ to be an arbitrary element of $\mathbb{J} = \mathbb{J}_F$. We encode some of the elements of $\mathrm{SE}_6(F)$ by the 3×3 matrices written over social subalgebras of $\mathbb{O} = \mathbb{O}_F$. As we mentioned before, if such a matrix M is written over any sociable subalgebra of $\mathbb{O}$, then the expression $\overline{M}^\top XM$ makes sense. If two matrices M and N are written over the same sociable subalgebra, then we have enough associativity to see that the action by the product MN is the same as the product of the actions, that is

$$(\overline{N}\,\overline{M})^\top X(MN) = \overline{N}^\top(\overline{M}^\top XM)N. \tag{3.5}$$

In general, the action by the product of two matrices is not defined whereas the product of the actions still is. Note that also $-\mathrm{I}_3$ acts trivially on $\mathbb{J}$.

We first notice that the elements

$$\delta = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.6}$$

preserve the Dickson–Freudenthal determinant. Their actions are given by

$$\begin{aligned} \delta &: (a, b, c \mid A, B, C) \mapsto (b, a, c \mid \overline{B}, \overline{A}, \overline{C}), \\ \tau &: (a, b, c \mid A, B, C) \mapsto (c, a, b \mid C, A, B). \end{aligned} \tag{3.7}$$

Now let x be any octonion and consider the matrices

$$M_x = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \quad M''_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}. \tag{3.8}$$

Note that the elements M'_x, M''_x can be obtained from M_x by applying the triality element τ, so to show that all three families described above preserve the Dickson–Freudenthal determinant, we only need to consider one of them.

Lemma 3.2.1. *The elements M_x, where $x \in \mathbb{O}$ is any octonion, preserve the*

Dickson–Freudenthal determinant, and hence they encode the elements of $\mathrm{SE}_6(F)$.

Proof. The action of M_x on $\mathbb{J}$ is given by

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), c \mid A + \bar{x}\overline{B}, B, C + ax).$$

The individual terms in the Dickson–Freudenthal determinant are being mapped in the following way:

$$\begin{aligned} abc &\mapsto abc + a^2c\mathrm{N}(x) + ac\,\mathrm{T}(\bar{x}C), \\ -aA\overline{A} &\mapsto -aA\overline{A} - a\,\mathrm{T}(ABx) - a\mathrm{N}(x)\mathrm{N}(B), \\ -bB\overline{B} &\mapsto -bB\overline{B} - a\mathrm{N}(x)\mathrm{N}(B) - \mathrm{T}(\bar{x}C)B\overline{B}, \\ -cC\overline{C} &\mapsto -cC\overline{C} - ac\,\mathrm{T}(\bar{x}C) - a^2c\mathrm{N}(x), \\ \mathrm{T}(ABC) &\mapsto \mathrm{T}(ABC) + B\overline{B}\,\mathrm{T}(\bar{x}C) + 2a\mathrm{N}(x)\mathrm{N}(B) + a\,\mathrm{T}(ABx). \end{aligned}$$

It is visibly obvious now that all the necessary terms on the right-hand side cancel out, so the result follows. □

It is obvious enough that we can also consider the transposes

$$L_x = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{bmatrix}, \quad L''_x = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

for an arbitrary $x \in \mathbb{O}$. A similar straightforward calculation as in Lemma 3.2.1 can be performed to show that these are also the elements of $\mathrm{SE}_6(F)$. Further in this thesis we will be able to show that the actions of the elements M_x, M'_x, M''_x, L_x, L'_x and L''_x generate the whole group $\mathrm{SE}_6(F)$.

Finally, we consider the elements of the form

$$P_u = \begin{bmatrix} u & 0 & 0 \\ 0 & \bar{u} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P'_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \bar{u} \end{bmatrix}, \quad P''_u = \begin{bmatrix} \bar{u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{bmatrix}, \tag{3.10}$$

where u is an octonion of norm one. The action of the element P_u on $\mathbb{J}$ is given by

$$P_u : (a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \bar{u}C\bar{u}). \tag{3.11}$$

It is a matter of straightforward computation to show that the elements P_u preserve the Dickson–Freudenthal determinant. Indeed, we have

$$\begin{aligned} abc &\mapsto abc, \\ aA\overline{A} &\mapsto a(uA)(\overline{A}\bar{u}) = aN(uA) = a\mathrm{N}(A).\mathrm{N}(u) = aA\overline{A}, \\ bB\overline{B} &\mapsto b(Bu)(\bar{u}\overline{B}) = b\mathrm{N}(Bu) = bB\overline{B}, \\ cC\overline{C} &\mapsto c(\bar{u}C\bar{u})(u\overline{C}u) = c\mathrm{N}(\bar{u}C\bar{u}) = c\mathrm{N}(C).\mathrm{N}(u)^2 = cC\overline{C}, \end{aligned} \tag{3.12}$$

and for the last term we get

$$\begin{aligned} \mathrm{T}((uA)(Bu)(\bar{u}C\bar{u})) &= \mathrm{T}((\bar{u}C\bar{u})(uA)(Bu)) = \mathrm{T}((\bar{u}(C(\bar{u}(uA))))(Bu)) \\ = \mathrm{T}((\bar{u}(CA))(Bu)) &= \mathrm{T}((Bu)(\bar{u}(CA))) = \mathrm{T}(B(u(\bar{u}(CA)))) = \mathrm{T}(B(CA)) \\ &= \mathrm{T}((BC)A) = \mathrm{T}(ABC). \end{aligned} \tag{3.13}$$

On the other hand, it is not difficult to see that $P_u = M_{u-1} \cdot L_1 \cdot M_{u^{-1}-1} \cdot L_{-u}$, so the fact that the matrices P_u preserve the determinant follows from the calculations already done for the elements M_x and L_x. We also notice that the elements P_u preserve the quadratic form Q_8^C defined on $\mathbb{J}_8^C$ via

$$\mathrm{Q}_8^C((0, 0, 0 \mid 0, 0, C)) = C\overline{C}. \tag{3.14}$$

We finish this section by showing that the action of the elements P_u on $\mathbb{J}_{10}^{abC}$, as u ranges through all the octonions of norm one, is that of $\Omega_8(F, \mathrm{Q}_8^C)$ when $\mathbb{O}$ is split.

Lemma 3.2.2. *If $\mathbb{O}$ is split, the actions of the elements P_u on $\mathbb{J}_8^C$, as u ranges through all the octonions of norm one, generate a group of type $\Omega_8^+(F)$. The action on $\mathbb{J}_{10}^{abC}$ is also that of $\Omega_8^+(F)$.*

Proof. Consider the action on the last octonionic 'coördinate', i.e. $C \mapsto \bar{u}C\bar{u}$. We will show now that this map can be represented as a product of two reflections. To avoid any predicaments in characteristic 2, we notice that since $\langle x, y\rangle = \mathrm{T}(x\bar{y})$, we get

$$\frac{2\langle x,y\rangle}{\langle y,y\rangle}=\frac{\langle x,y\rangle}{\mathrm{N}(y)}. \tag{3.15}$$

Now, the reflection in the hyperplane orthogonal to an arbitrary element $v\in\mathbb{O}$ is the map

$$r_v : x\mapsto x-\frac{\mathrm{T}(x\bar{u})}{\mathrm{N}(u)}\cdot u = x-\frac{x\bar{u}+u\bar{x}}{\mathrm{N}(u)}\cdot u = x-\frac{(x\bar{u})u-u\bar{x}u}{\mathrm{N}(u)} = -\frac{u\bar{x}u}{\mathrm{N}(u)}, \tag{3.16}$$

It is easy to see now that the given action of P_u on $\mathbb{J}_8^C$ is the composition $r_u\circ r_1$. As u ranges through all octonions of norm one, we get the action of $\Omega_8(F,\mathrm{Q}_8^C)$ on $\mathbb{J}_8^C$. Since we assume that $\mathbb{O}$ is split, the form Q_8 is of plus type, so we may denote this group as $\Omega_8^+(F)$. When acting on $\mathbb{J}_{10}^{abC}$, the form $ab-C\overline{C}$ is preserved, so we again get the action of $\Omega_8^+(F)$. □

3.3 The white points

3.3.1 The mixed form and the white vectors

Suppose $X=(a,b,c\mid A,B,C)$ and $Y=(d,e,f\mid D,E,F)$ are the arbitrary elements of $\mathbb{J}$. Define the mixed form $M(Y,X)$ as

$$\begin{aligned}M(Y,X)=bcd+ace+abf-dA\overline{A}-eB\overline{B}-fC\overline{C}\\ -a(D\overline{A}+A\overline{D})-b(E\overline{B}+B\overline{E})-c(F\overline{C}+C\overline{F})\\ +\mathrm{T}(DBC+ECA+FAB).\end{aligned} \tag{3.17}$$

Note that if $F\neq\mathbb{F}_2$, then $M(X,Y)$ can be obtained from the Dickson–Freudenthal determinant, for we have

$$\begin{aligned}M(X,Y)=\frac{1}{\alpha(\alpha-1)}\Delta(X+\alpha Y)\\ -\frac{1}{\alpha-1}\Delta(X+Y)+\frac{1}{\alpha}\Delta(X)-(\alpha+1)\Delta(Y),\end{aligned} \tag{3.18}$$

for any $\alpha\notin\{0,1\}$.

We colour the non-zero Albert vectors in $\mathbb{J}$ according to the following rules.

Definition 3.3.1. *A non-zero Albert vector $X \in \mathbb{J}$ is called*

(i) white if $M(Y, X) = 0$ for all $Y \in \mathbb{J}$;

(ii) grey if $\Delta(X) = 0$ and there exists $Y \in \mathbb{J}$ such that $M(Y, X) \neq 0$;

(iii) black if $\Delta(X) \neq 0$ and X is not white.

A white/grey/black point is a 1-dimensional subspace of $\mathbb{J}$ spanned by a white/grey/black vector.

For example, the vector $(0, 0, 1 \mid 0, 0, 0)$ is white, because if Y is an arbitrary Albert vector, then $M(Y, X) = 0$. Similarly, $(\lambda, 1, 1 \mid 0, 0, 0)$, where $\lambda \neq 0$, is black, since in this case $\Delta(X) = \lambda \neq 0$, and it is certainly not white as there exists $Y = (a, b, c \mid A, B, C)$ such that $M(Y, X) \neq 0$:

$$M(Y, X) = \lambda(bc - A\bar{A}) + (ac - B\bar{B}) + (ab - C\bar{C}). \tag{3.19}$$

Taking, for instance, $Y = (0, 1, 1 \mid 0, 0, 0)$, we get $M(Y, X) = \lambda \neq 0$. Finally, $(0, 1, 1 \mid 0, 0, 0)$ is grey as $\Delta(X) = 0$ and for $Y = (a, b, c \mid A, B, C)$ the value of M is given by

$$M(Y, X) = (ac - B\bar{B}) + (ab - C\bar{C}), \tag{3.20}$$

so we may take $Y = (1, 1, 0 \mid 0, 0, 0)$ to get $M(Y, X) = 1 \neq 0$. The terms white, grey and black were introduced by Cohen and Cooperstein [**?**]. In the paper by Aschbacher [**?**] they are called 'singular', 'brilliant non-singular' and 'dark' respectively. Jacobson [**?**] uses the terms 'rank 1', 'rank 2' and 'rank 3'.

It is clear that the action of $\mathrm{SE}_6(F)$ preserves the colour, except possibly in case $F = \mathbb{F}_2$, when white and grey vectors may be intermixed. Later we shall see that $\mathrm{SE}_6(\mathbb{F}_2)$ is also colour-preserving.

Let $X = (a, b, c \mid A, B, C)$ be an arbitrary white vector. A white vector W determines the quadratic form $\Delta(X + W) - \Delta(X) = M(W, X)$ on $\mathbb{J}$. Its radical is 17-dimensional and for any non-zero $\lambda \in F$ we have $\Delta(X + \lambda W) - \Delta(X) = \lambda(\Delta(X + W) - \Delta(X))$, so the form determined by λW has the same radical. Thus, the 17-dimensional space is determined by the white point $\langle W \rangle$.

For example, for the white vector $(0,0,1 \mid 0,0,0)$ the quadratic form is $ab-C\overline{C}$, whose radical is $\mathbb{J}_{17}^{cAB}$. For the vector $(0,0,0 \mid 0,0,D)$ with $D \neq 0 \neq D\overline{D}$ the form is $\widehat{Q}(X) = \mathrm{T}(D(AB - c\overline{C}))$ with $\widehat{B}(X,Y) = \mathrm{T}(D(AB' + A'B - c\overline{C}' - c'\overline{C}))$ being its polar form, where $Y = (a',b',c' \mid A',B',C')$. Now X is in the radical of $\widehat{Q}$ if and only if $\widehat{Q}(X) = 0$ and $\widehat{B}(X,Y) = 0$ for all Y. Taking $Y = (a',b',1 \mid 0,0,0)$ gives us $\mathrm{T}(D\overline{C}) = 0$ and taking $Y = (a',b',0 \mid 0,B',0)$ gives us $\mathrm{T}(DAB') = \mathrm{T}((DA)B') = 0$ for all B', so $DA = 0$. If $Y = (a',b',0 \mid A',0,0)$ then $\mathrm{T}(D(A'B)) = \mathrm{T}((BD)A') = 0$ for all A', so we get $BD = 0$. Finally, setting $Y = (a',b',0 \mid 0,0,C')$ gives us $\mathrm{T}(cD\overline{C}') = 0$ for all $\overline{C}'$, so $cD = 0$, and thus $c = 0$. Therefore the radical is

$$\left\{(a,b,0 \mid A,B,C) \;\middle|\; DA = BD = \mathrm{T}(D\overline{C}) = 0\right\}. \tag{3.21}$$

To obtain 17-spaces determined by other "coördinate" white vectors we apply a suitable power of τ to these two.

Next, we derive a system of conditions for an arbitrary vector $X \in \mathbb{J}$ to be white.

Lemma 3.3.2. *An Albert vector $X = (a,b,c \mid A,B,C)$ is white if and only if the following conditions hold:*

$$\left.\begin{aligned} A\overline{A} &= bc,\\ B\overline{B} &= ca,\\ C\overline{C} &= ab,\\ AB &= c\overline{C},\\ BC &= a\overline{A},\\ CA &= b\overline{B}. \end{aligned}\right\}. \tag{3.22}$$

If X is white, then $\Delta(X) = 0$.

Proof. Let $Y = (d,e,f \mid D,E,F)$. We rewrite $M(Y,X)$ in the form

$$\begin{aligned} M(Y,X) = (bc - A\overline{A})d + (ac - B\overline{B})e + (ab - C\overline{C})f \\ + \mathrm{T}(D(BC - a\overline{A}) + Q(CA - b\overline{B}) + R(AB - c\overline{C})). \end{aligned}$$

It is visibly clear now that if all the conditions in the statement are satisfied, then $M(Y,X) = 0$. Now, taking $Y = (1,0,0 \mid 0,0,0)$ forces $bc - A\overline{A} = 0$. Similarly, we may take $Y = (0,1,0 \mid 0,0,0)$ to get $ac - B\overline{B} = 0$ and, say, $Y = (0,0,0 \mid D,0,0)$ to obtain $\mathrm{T}(D(BC - a\overline{A})) = 0$ which forces $BC - a\overline{A} = 0$ as $D \in \mathbb{O}$ can be arbitrary. The other conditions are proved similarly.

Finally, if X is white, then we get $\mathrm{T}(ABC) = \mathrm{T}(aA\overline{A}) = \mathrm{T}(abc) = 2abc$. Also $bB\overline{B} = bca$, and so on. Overall we get

$$\Delta(X) = abc - abc - bca - cab + 2abc = 0$$

as required. This completes the proof. □

3.3.2 Action of $\mathrm{SE}_6(F)$ on white points

In this thesis we will be mostly interested in the action of $\mathrm{SE}_6(F)$ on the white points.

Consider $X = (a, b, c \mid A, B, C)$ and $Y = (0, 0, 1 \mid 0, 0, 0)$. Then we find $\Delta(X + Y) - \Delta(X) = ab - C\overline{C}$, which is a quadratic form with 17-dimensional radical in $\mathbb{J}$. In case when $Y = (0, 1, 1 \mid 0, 0, 0)$ we get $\Delta(X + Y) - \Delta(X) = a + ab + ac - B\overline{B} - C\overline{C}$. If $F = \mathbb{F}_2$, we have $a^2 = a$, so the latter form is quadratic with 9-dimensional radical. This shows that $(0, 0, 1 \mid 0, 0, 0)$ and $(0, 1, 1 \mid 0, 0, 0)$ are in different orbits of the isometry group for any field.

Finally, we investigate the orbits of $\mathrm{SE}_6(F)$ on Albert vectors. One of our main goals is to show that $\mathrm{SE}_6(F)$ acts transitively on white points.

Lemma 3.3.3. *Suppose X is an arbitrary Albert vector. Then X can be mapped under the action of $\mathrm{SE}_6(F)$ to a vector of the form $(a, b, c \mid 0, 0, 0)$ with $(a, b, c) \neq (0, 0, 0)$. In case when $\mathbb{O}$ is split, X can be mapped to precisely one of the following:*

(i) $(0, 0, 1 \mid 0, 0, 0)$*, a white vector;*

(ii) $(0, 1, 1 \mid 0, 0, 0)$*, a grey vector; or*

(iii) $(\lambda, 1, 1 \mid 0, 0, 0)$ *where $\lambda \neq 0$, a black vector.*

In the last case there is one orbit for each non-zero value of λ.

Proof. These vectors are indeed in the different orbits, except possibly for the white and grey vectors, since they have different values of Δ. We have already shown that these particular white and grey vectors are in different orbits in case of any field.

First, we show that each orbit of $\mathrm{SE}_6(F)$ contains an Albert vector of the form $(a, b, c \mid 0, 0, 0)$. Suppose that $X = (a, b, c \mid A, B, C)$ is non-zero. If $(a, b, c) = (0, 0, 0)$, then after applying the triality element τ a suitable number of times we may assume $C \neq 0$. Consider the action of the element L_x on the Albert vector $(0, 0, 0 \mid A, B, C)$:

$$L_x : (0, 0, 0 \mid A, B, C) \mapsto (\mathrm{T}(Cx), 0, 0 \mid A, B + \overline{A}x, C),$$

so we are allowed to choose orbit representatives with $(a, b, c) \neq (0, 0, 0)$.

As before, using a suitable power of τ, we may assume $c \neq 0$. Now we apply the element M_x with $x = -c^{-1}B$ to X, which gives us the vector of the form $(a, b, c \mid A, 0, C)$, where the 'coördinate' c stays the same, while a, b, A, C are possibly different. Next, the vector $(a, b, c \mid A, 0, C)$ is being mapped to the vector of the form $(a, b, c \mid 0, 0, C)$ under the action of L_x with $x = -c^{-1}A$, where the value of c stays the same while the values of a, b, C may be adjusted.

If $a = b = 0$, $C \neq 0$, then we apply the element L_x with x such that $\mathrm{T}(Cx) \neq 0$ to get the vector of the form $(\mathrm{T}(Cx), 0, c \mid 0, 0, C)$, i.e. we may assume that $a \neq 0$. With the latter assumption we apply the element M_x with $x = -a^{-1}C$ to $(a, b, c \mid 0, 0, C)$ to get the vector of the form $(a, b, c \mid 0, 0, 0)$ with the value of b being adjusted.

Finally, we use the elements τ, P_u and P''_v to standardise the vector of the form $(a, b, c \mid 0, 0, 0)$ to one the forms in the statement. $\square$

Note that the last part of the proof of this lemma used the fact that the map $\mathrm{N} : \mathbb{O} \to F$ is onto, which is the case when $\mathbb{O}$ is split. However, this is not true in any octonion algebra, which possibly leads to a bigger number of orbits. A vector of the form $(a, b, c \mid 0, 0, 0)$ is white if and only if precisely one of the a, b, c is non-zero, so we get the transitive action of $\mathrm{SE}_6(F)$ on white points regardless of the chosen octonion algebra.

Furthermore, we used the fact that N is a non-singular quadratic form on $\mathbb{O}$, i.e. provided $C \neq 0$, the map $x \mapsto \mathrm{T}(Cx)$ is surjective. This should be true for any octonion algebra.

Later we will use the transitivity on white points to calculate the group order in case $F = \mathbb{F}_q$ by finding the stabiliser of a white point and calculating the number of white points in case of a finite field.

Lemma 3.3.4. *Let $\mathbb{O}$ be an arbitrary octonion algebra over F. Let $X \in \mathbb{J}$ be white and let $\mathbb{J}_{17}$ be the 17-dimensional subspace of $\mathbb{J}$ determined by X. The stabiliser in $\mathrm{SE}_6(F)$ of $\langle X\rangle$, and even of X, is transitive on the white points spanned by the vectors in $\mathbb{J}_{17} \setminus \langle X\rangle$ (there are no such white points when $\mathbb{O}$ is non-split). It is also transitive on the white points spanned by the vectors in $\mathbb{J} \setminus \mathbb{J}_{17}$.*

Proof. Without loss of generality assume $X = (0, 0, 1 \mid 0, 0, 0)$. As we know, the white point $\langle X\rangle$ determines the 17-space $\mathbb{J}_{17}^{cAB}$. We also note that X is stabilised by the actions of the elements M_x, L_x, M'_x and L''_x. Those act on the elements in $\mathbb{J}_{17}^{cAB}$ in the following way:

$$\begin{aligned}
M_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c \mid A + \bar{x}\overline{B}, B, 0),\\
L_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c \mid A, B + \overline{A}x, 0),\\
M'_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c + \mathrm{T}(\bar{x}A) \mid A, B, 0),\\
L''_x &: (0, 0, c \mid A, B, 0) \mapsto (0, 0, c + \mathrm{T}(Bx) \mid A, B, 0).
\end{aligned}$$

It follows that a general white vector $(0, 0, c \mid A, B, 0) \in \mathbb{J}_{17}^{cAB} \setminus \langle X\rangle$ can easily be mapped to $(0, 0, 0 \mid A, B, 0)$ using the action of M'_x or L''_x for some suitable $x \in \mathbb{O}$. A vector $(0, 0, 0 \mid A, B, 0)$ is white if $(A, B) \neq (0, 0)$ and $A\overline{A} = B\overline{B} = AB = 0$. It is obvious enough that $\mathbb{J}_{17}^{cAB} \setminus \langle X\rangle$ is empty if $\mathbb{O}$ is not split, so we only need to show transitivity on the corresponding white points in case when $\mathbb{O}$ is split.

If $B = 0$ then evidently $A \neq 0$ and so we can apply the duality element δ to obtain a white vector of the form $(0, 0, 0 \mid A, B, 0)$ with $B \neq 0$. If now $A \neq 0$, we act by M_x to obtain $(0, 0, 0 \mid A + \bar{x}\overline{B}, B, 0)$. Our aim is to show that there exists such $x \in \mathbb{O}$ that $A + \bar{x}\overline{B} = 0$. Denote $U = \{y \in \mathbb{O} \mid \bar{y}B = 0\}$. Since for all $x \in \mathbb{O}$ we have $(\bar{x}\overline{B})B = \bar{x}(\overline{B}B) = 0$, we conclude that $\mathbb{O}\overline{B} \leqslant U$. Furthermore, we know that both subspaces are four-dimensional, so $\mathbb{O}\overline{B} = U$. As $AB = 0$, we have $A \in U$, and therefore there exists $y = \bar{x}\overline{B} \in U$ such that $A + y = 0$.

Now, the elements P''_u with $\mathrm{N}(u) = 1$ act on the Albert vectors of the form $(0, 0, 0 \mid 0, B, 0)$ as

$$(0, 0, 0 \mid 0, B, 0) \mapsto (0, 0, 0 \mid 0, \bar{u}B\bar{u}, 0),$$

and as u ranges through all the octonions of norm 1 the action generated is that of

$\Omega_8^+(F)$ which in case when $\mathbb{O}$ is split is transitive on isotropic vectors, i.e. those with $B\overline{B} = 0$. It follows that $\mathrm{SE}_6(F)$ is indeed transitive on the white points spanned by the vectors in $\mathbb{J}_{17}^{cAB} \setminus \langle X \rangle$.

To show the transitivity on white points spanned by the vectors in $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ we prove that every white point spanned by a white vector $(a, b, c \mid A, B, C) \in \mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ can be mapped to the white point spanned by $(1, 0, 0 \mid 0, 0, 0)$. Note that we require $(a, b, C) \neq (0, 0, 0)$.

In case $(a, b) = (0, 0)$ we choose $x \in \mathbb{O}$ such that $\mathrm{T}(Cx) \neq 0$ and apply the element L_x, which maps our vector $(0, 0, c \mid A, B, C)$ to $(\mathrm{T}(Cx), 0, c \mid A, B + \overline{A}x, C)$. If, on the other hand, $a = 0$ and $b \neq 0$, we apply δ. Hence, we may assume that we deal with a vector $(a, b, c \mid A, B, C)$ with $a \neq 0$. Take $x = -a^{-1}C$ and act by the element M_x:

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, b + aa^{-2}C\overline{C} - \mathrm{T}(a^{-1}\overline{C}C), c \mid A - a^{-1}\overline{C}\overline{B}, B, 0).$$

The whiteness conditions imply $C\overline{C} = ab$ and $BC = a\overline{A}$, so additionally we get $b + aa^{-2}C\overline{C} - \mathrm{T}(a^{-1}\overline{C}C) = b + b - \mathrm{T}(b) = 0$ and $A - a^{-1}\overline{C}\overline{B} = A - A = 0$. This means that the given M_x acts on the elements of $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ in the following way:

$$M_x : (a, b, c \mid A, B, C) \mapsto (a, 0, c \mid 0, B, 0),$$

where $a \neq 0$. It is still white, so $B\overline{B} = ca$. Finally, we act by L''_y with $y = -a^{-1}\overline{B}$:

$$L''_y : (a, 0, c \mid 0, B, 0) \mapsto (a, 0, 0 \mid 0, 0, 0),$$

where $a \neq 0$. In other words, any white point spanned by an element in $\mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ can be mapped by the action of the stabiliser of $\langle X \rangle$ to the white point spanned by $(1, 0, 0 \mid 0, 0, 0)$. $\square$

Lemma 3.3.5. *The action of* $\mathrm{SE}_6(F)$ *on white points is primitive.*

Proof. From the previous Lemma it follows that if $\mathbb{O}$ is non-split, then the action of $\mathrm{SE}_6(F)$ on white points in 2-transitive and hence primitive. It remains to prove the statement in case when $\mathbb{O}$ is split.

Suppose $X, Y \in \mathbb{J}$ are white vectors such that $\langle X \rangle \neq \langle Y \rangle$. Define $\sim$ to be an $\mathrm{SE}_6(F)$-congruence on white points and let $\langle X \rangle \sim \langle Y \rangle$. Our aim is to show that this generates the universal congruence. Since for $\mathbb{O}$ split the action on the white vectors is transitive, we may assume $X = (0, 0, 1 \mid 0, 0, 0)$. As mentioned in the beginning of this section, $\langle X \rangle$ determines the 17-dimensional space $\mathbb{J}_{17}^{cAB}$. We now distinguish two cases.

If $Y \in \mathbb{J}_{17}^{cAB}$, then acting by the stabiliser of $\langle X \rangle$ we get $\langle X \rangle \sim \langle \widehat{Y} \rangle$ for all white $\widehat{Y} \in \mathbb{J}_{17}^{cAB}$. Take $\widehat{Y} = (0, 0, 0 \mid e_0, 0, 0) \in \mathbb{J}_{17}^{cAB}$ and $\widehat{X} = (0, 1, 0 \mid 0, 0, 0) \notin \mathbb{J}_{17}^{cAB}$. As we see from the earlier calculations, both X and $\widehat{X}$ are in the 17-space determined by $\langle \widehat{Y} \rangle$. Acting by the stabiliser of $\langle \widehat{Y} \rangle$ we map $\langle X \rangle$ to $\langle \widehat{X} \rangle$, and so ensure $\langle \widehat{Y} \rangle \sim \langle \widehat{X} \rangle$, and so we have the chain $\langle X \rangle \sim \langle \widehat{Y} \rangle \sim \langle \widehat{X} \rangle$. To get $\langle X \rangle \sim \langle \widehat{X} \rangle$ for all white $\widehat{X}$ outside $\mathbb{J}_{17}^{cAB}$, we again act by the stabiliser of $\langle X \rangle$. It follows that $\langle X \rangle$ is congruent to any white point generated by a vector in $\mathbb{J}$, and so we get the universal congruence in this case.

On the other hand, if Y lies outside of $\mathbb{J}_{17}^{cAB}$, then we get $\langle X \rangle \sim \langle \widehat{Y} \rangle$ for all white $\widehat{Y} \in \mathbb{J} \setminus \mathbb{J}_{17}^{cAB}$ since the stabiliser of $\langle X \rangle$ is transitive on the white points spanned by those. In particular, we may take $\widehat{Y} = (1, 0, 0 \mid 0, 0, 0)$. Acting by the stabiliser of $\langle \widehat{Y} \rangle$ on both sides in $\langle X \rangle \sim \langle \widehat{Y} \rangle$, we map $\langle X \rangle$ to $\langle \widehat{X} \rangle$ with $\widehat{X} = (0, 0, 0 \mid e_0, 0, 0)$. Note that both X and $\widehat{X}$ are not in $\mathbb{J}_{17}^{aBC}$ which is the 17-space determined by $\widehat{Y}$, But $\widehat{X} \in \mathbb{J}_{17}^{cAB}$ and by transitivity we get $\langle X \rangle \sim \langle \widehat{X} \rangle$. Again, we act by the stabiliser of $\langle X \rangle$ to ensure $\langle X \rangle \sim \langle \widehat{X} \rangle$ for all white points $\langle \widehat{X} \rangle$ spanned by $\widehat{X} \in \mathbb{J}_{17}^{cAB}$, i.e. our $\mathrm{SE}_6(F)$-congruence is trivial in this case as well. $\square$

Suppose $\langle W \rangle$ and $\langle X \rangle$ are two white points and consider a *line* $\langle W, X \rangle$ as a 2-dimensional subspace of $\mathbb{J}$ spanned by white vectors W and X. Given a white point $\langle W \rangle$, we are interested in finding all the white points $\langle X \rangle$ such that $\langle W, X \rangle$ is totally white.

Since $\mathrm{SE}_6(F)$ acts transitively on white points, we may assume that $W = (0, 0, 1 \mid 0, 0, 0)$ and $X = (a, b, c \mid A, B, C)$. Consider an element $\lambda W + X \in \langle W, X \rangle$. First, we calculate the value of Dickson–Freudenthal determinant:

$$\begin{aligned}\Delta(\lambda W + X) &= ab(\lambda + c) - aA\overline{A} - bB\overline{B} - (\lambda + c)C\overline{C} + + \mathrm{T}(ABC) \\ &= \lambda(ab - C\overline{C}) + \Delta(X). \quad (3.23)\end{aligned}$$

As $X = (a, b, c \mid A, B, C)$ is white, we get $\Delta(X) = 0$ and $ab = C\overline{C}$, so we conclude $\Delta(\lambda W + X) = 0$.

Proposition 3.3.6. *If $\langle W \rangle$ and $\langle X \rangle$ are two white points, then any vector in (or any 1-subspace of) $\langle W, X \rangle$ is either white or grey.*

The conditions for $\lambda W + X$ to be white are

$$\left.\begin{aligned} A\overline{A} &= b(\lambda + c), \\ B\overline{B} &= (\lambda + c)a, \\ C\overline{C} &= ab, \\ AB &= (\lambda + c)\overline{C}, \\ BC &= a\overline{A}, \\ CA &= b\overline{B}. \end{aligned}\right\} \tag{3.24}$$

Since X is white by assumption, we get that $\lambda W + X$ is white if and only if $\lambda a = \lambda b = \lambda \overline{C} = 0$ for all $\lambda \in F$, which is equivalent to $a = b = C = 0$, i.e. $X \in \mathbb{J}_{17}^{cAB}$. Therefore, we conclude the following.

Proposition 3.3.7. *Given any white point $\langle W \rangle$, the line $\langle W, X \rangle$, where $\langle X \rangle$ is another white point, is totally white if and only if X belongs to the 17-space determined by W. Otherwise, $\langle W, X \rangle$ contains only two white points.*

3.3.3 The stabiliser of a white point

In this section we assume that $\mathbb{O}$ is a split octonion algebra. It is our aim now to obtain the stabiliser in $\mathrm{SE}_6(F)$ of a white point. In particular, we prove the following result.

Theorem 3.3.8. *If $\mathbb{O}$ is split, then the stabiliser of a white vector in $\mathrm{SE}_6(F)$ is isomorphic to the group generated by the actions of the elements M_x, L_x, M'_x and L''_x on $\mathbb{J}$ as x ranges over $\mathbb{O}$ and this is a group of shape*

$$F^{16}{:}\mathrm{Spin}_{10}^{+}(F). \tag{3.25}$$

The stabiliser of a white point is isomorphic to

$$F^{16}{:}\mathrm{Spin}_{10}^{+}(F).F^{\times}, \tag{3.26}$$

where $F^{\times}$ is the multiplicative group of the field F.

This whole section is devoted to proving this result. Some of this proof is in the running text, and some of it is contained in a series of technical lemmata. First, we prove that no invertible F-linear maps on $\mathbb{O}$ can change the order of the octonion product.

Lemma 3.3.9. *There are no invertible F-linear maps $\phi, \psi : \mathbb{O} \to \mathbb{O}$ such that for all $A, B \in \mathbb{O}$ it is true that $AB = (B\psi)(A\phi)$.*

Proof. For the sake of finding a contradiction, suppose that $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are invertible F-linear maps such that the identity $AB = (B\psi)(A\phi)$ holds for all $A, B \in \mathbb{O}$. In particilar, substituting $A = 1_{\mathbb{O}}$, we get $B = (B\psi)u$ for all $B \in \mathbb{O}$, where $u = 1\phi$, so $B\psi = Bu^{-1}$ for all $B \in \mathbb{O}$, which means that the map ψ is right multiplication by u^{-1}. Note that the existence of u^{-1} follows from the invertibility of the map ψ. Thus, our identity has the form $AB = (Bu^{-1})(A\phi)$ for all $A, B \in \mathbb{O}$. We can substitute $B = u$ which immediately gives us $A\phi = Au$ for all $A \in \mathbb{O}$, so the map ϕ is right multiplication by u. Finally, we get $AB = (Bu^{-1})(Au)$ for all $A, B \in \mathbb{O}$ and specifically for $B = 1_{\mathbb{O}}$ we get $A = u^{-1}(Au)$, or likewise $uA = Au$ for all $A \in \mathbb{O}$. Therefore u is a scalar multiple of $1_{\mathbb{O}}$, i.e. $u = \mu \cdot 1_{\mathbb{O}}$ for some $\mu \in F$. Since the linear maps ϕ and ψ are invertible, μ is non-zero, and we get $AB = (Bu^{-1})(Au) = (\mu^{-1}\mu \cdot 1_{\mathbb{O}})BA = BA$ for all $A, B \in \mathbb{O}$, which is definitely not true as $\mathbb{O}$ is not commutative. ◻

Second, we show that if two invertible linear maps commute with the octonion product, then these are mutually invertible scalar multiplication maps.

Lemma 3.3.10. *Suppose $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are two invertible F-linear maps such that $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$. Then $\psi : x \mapsto \mu x$ for some non-zero $\mu \in F$ and $\phi = \psi^{-1}$, i.e. $\phi : x \mapsto \mu^{-1}x$.*

Proof. Suppose $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are F-linear maps such that $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$. When $A = 1_{\mathbb{O}}$ we get $B\psi = uB$ for all $B \in \mathbb{O}$ where $u = (1_{\mathbb{O}}\phi)^{-1}$, so the map ψ is left multiplication by u. Substituting $B = 1_{\mathbb{O}}$ on the other hand gives us $A = (A\phi)(1_{\mathbb{O}}\psi)$ for all A and so $A\phi = Av$ where $v = (1_{\mathbb{O}}\psi)^{-1}$, so ϕ is the right multiplication by v. Therefore the condition in this case becomes $AB = (Av)(uB)$ for all $A, B \in \mathbb{O}$. Substituting $B = u^{-1}$, we get $Au^{-1} = Av$ for all $A \in \mathbb{O}$, and therefore $v = u^{-1}$, and our identity turns out to be $AB = (Au^{-1})(uB)$ for all $A, B \in \mathbb{O}$. Now since u is invertible, we can write $u^{-1} = N(u)^{-1}\bar{u}$. Finally, by Corollary 2.7.4, u must be a scalar multiple of $1_{\mathbb{O}}$, i.e. $u = \mu \cdot 1_{\mathbb{O}}$. □

The statements in Lemmas 3.3.9 and 3.3.10 are true even when $\mathbb{O}$ is not split. Everything is ready now for the investigation of the white vector stabiliser. Since it was shown that the group $\mathrm{SE}_6(F)$ acts transitively on the set of white points, it is sufficient to study the stabiliser of a specific white vector. For instance, it is convenient to take $v = (0, 0, 1 \mid 0, 0, 0)$. First thing to notice is that v is invariant under the action of the elements of the form

$$L''_x = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M'_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.27}$$

where $x, y \in \mathbb{O}$.

Lemma 3.3.11.

(a) Let Q be any of the $\{L, L', L'', M, M', M''\}$. Then the actions on $\mathbb{J}$ of the elements Q_x where x ranges over $\mathbb{O}$ generate an elementary abelian group isomorphic to F^8.

(b) Let (R, S) be any pair from the set $\{(L, M''), (L', M), (L'', M')\}$ or any of the $\{(L, M'), (L', M''), (L'', M)\}$. Then the actions of R_x and S_x, as x ranges through $\mathbb{O}$, generate an elementary abelian group isomorphic to F^{16}.

Proof. To show part (a) for the elements L_x, L'_x, L''_x it is enough to consider just, say, L''_x as to obtain the result for the rest of them we can apply the action of the triality element

$$\tau = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Similarly, out of M_x, M'_x, M''_x we only need to consider, for instance, M'_x. The actions of L''_x and M'_y on $\mathbb{J}$ are given by

$$\begin{aligned} L''_x : (a,b,c \mid A,B,C) &\mapsto (a,b,c+ax\bar{x}+\mathrm{T}(Bx) \mid A+\overline{C}x, B+A\bar{x}, C), \\ M'_y : (a,b,c \mid A,B,C) &\mapsto (a,b,c+by\bar{y}+\mathrm{T}(\bar{y}A) \mid A+By, B+\bar{y}\overline{C}, C). \end{aligned} \tag{3.28}$$

We notice that the action is nontrivial whenever x and y are non-zero. The element M'_y sends $(a,b,c+ax\bar{x}+\mathrm{T}(Bx) \mid A+\overline{C}x, B+a\bar{x}, C)$ to

$$(a,b,c+ax\bar{x}+\mathrm{T}(Bx)+by\bar{y}+\mathrm{T}(\bar{y}A) \mid A+\overline{C}x+by, B+a\bar{x}+\bar{y}\overline{C}, C),$$

and the element L''_x sends $(a,b,c+by\bar{y}+\mathrm{T}(\bar{y}A) \mid A+By, B+\bar{y}\overline{C}, C)$ to

$$(a,b,c+by\bar{y}+\mathrm{T}(\bar{y}A)+ax\bar{x}+\mathrm{T}(Bx) \mid A+by+\overline{C}x, B+\bar{y}\overline{C}+a\bar{x}, C).$$

Hence, the actions of these elements commute. Similarly, it is straightforward to verify that the actions of L''_x and L''_y commute as well as the actions of M'_x and M'_y. Moreover, the element L''_y sends $(a,b,c+ax\bar{x}+\mathrm{T}(Bx) \mid A+\overline{C}x, B+a\bar{x}, C)$ to

$$(a,b,c+ax\bar{x}+\mathrm{T}(Bx)+ay\bar{y}+\mathrm{T}(By)+a\,\mathrm{T}(\bar{x}y) \mid A+\overline{C}x+\overline{C}y, B+a\bar{x}+a\bar{y}, C),$$

and L''_{x+y} sends $(a,b,c \mid A,B,C)$ to

$$(a,b,c+ax\bar{x}+a\,\mathrm{T}(x\bar{y})+ay\bar{y}+\mathrm{T}(B(x+y)) \mid A+\overline{C}(x+y), B+a(\bar{x}+\bar{y}), C),$$

so the action of L''_{x+y} is the same as the product of the actions of L''_x and L''_y. A similar calculation shows that the action of M'_{x+y} is the same as the product of the actions of M'_x and M'_y. It follows that the action of L''_x on $\mathbb{J}$, $x \in \mathbb{O}$ generates an abelian group $(F^8,+)$ as well as the action of the element M'_y, $y \in \mathbb{O}$. We simply denote the abelian group $(F^n,+)$ as F^n in our further discussion.

To prove part (b) we need to verify that the intersection of the corresponding abelian groups, isomorphic to F^8 and generated by the actions of L''_x and M'_x is trivial. Suppose that the actions of L''_x and M'_y are equal. Then, according to (3.28), in the fourth "coördinate" we have

$$A + \overline{C}x = A + By$$

for arbitrary $A, B, C \in \mathbb{O}$. In other words, we get $\overline{C}x = By$ for arbitrary octonions B and C. In particular, if $B = 1_{\mathbb{O}}$ and $C = 0$, we get $y = 0$ and if $B = 0$ and $C = 1_{\mathbb{O}}$ we obtain $x = 0$. So, the intersection of two copies of F^8 consists of the identity element, as needed, and the result follows. Again, to get (b) for the rest of the pairs in the first set we apply the triality element. The calculations for the second set of pairs are essentially of the same nature. □

The next observation is that our white vector v is also invariant under the action of the elements

$$M_x = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_y = \begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.29}$$

First, we show that the actions of these on $\mathbb{J}_{10}^{abC}$ generate a group of type $\Omega_{10}^{+}(F)$. As we will see further, instead of arbitrary octonions it is enough for x to range through the scalar multiples of the basis elements e_i. Define the quadratic form Q_{10} on $\mathbb{J}$ via

$$\mathrm{Q}_{10}((a, b, c \mid A, B, C)) = ab - C\overline{C}. \tag{3.30}$$

We notice that Q_{10} is of *plus* type, so for convenience we denote the group $\Omega_{10}(F, \mathrm{Q}_{10})$ as $\Omega_{10}^{+}(F)$.

To construct $\Omega_{10}^{+}(F)$ we follow the series of steps. First, we consider the 4-space V_4 spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C_{-1}e_{-1} + C_1 e_1)$.

Lemma 3.3.12. *The actions of the elements $M_{\lambda e_{\pm 1}}$ and $L_{\lambda e_{\pm 1}}$ on V_4, where $\lambda \in F$, generate a group of type $\Omega_4^{+}(F)$.*

Proof. Consider the vectors v_1, v_2, v_3 and v_4 defined as

$$\begin{aligned} v_1 &= (1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_3 &= (0, 0, 0 \mid 0, 0, e_{-1}), \\ v_4 &= (0, 0, 0 \mid 0, 0, e_1). \end{aligned}$$

It is clear that these span V_4, so define $\mathcal{B} = \{v_1, v_4, v_3, v_2\}$ to be the basis of our 4-space. The element $M_{\lambda e_{-1}}$ acts on the basis elements in the following way:

$$\begin{aligned}
v_1 &\mapsto (1,0,0 \mid 0,0,\lambda e_{-1}) &= v_1 + \lambda v_3,\\
v_4 &\mapsto (0,\lambda,0 \mid 0,0,e_1) &= v_4 + \lambda v_2,\\
v_3 &\mapsto (0,0,0 \mid 0,0,e_{-1}) &= v_3,\\
v_2 &\mapsto (0,1,0 \mid 0,0,0) &= v_2.
\end{aligned}$$

As we can see, with respect to the basis $\mathcal{B}$ the action can be written as a 4×4 matrix

$$\begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\otimes$ is the Kronecker product. Similarly, the action of $M_{\lambda e_1}$ on $\mathcal{B}$ is given by

$$\begin{aligned}
v_1 &\mapsto (1,0,0 \mid 0,0,\lambda e_1) &= v_1 + \lambda v_4,\\
v_4 &\mapsto (0,0,0 \mid 0,0,e_1) &= v_4,\\
v_3 &\mapsto (0,\lambda,0 \mid 0,0,e_{-1}) &= v_3 + \lambda v_2,\\
v_2 &\mapsto (0,1,0 \mid 0,0,0) &= v_2,
\end{aligned}$$

so the corresponding 4×4 matrix has the form

$$\begin{bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

Now, for convenience, we do the same calculations for $L_{-\lambda e_{-1}}$: it acts on the elements of $\mathcal{B}$ as

$$\begin{aligned}
v_1 &\mapsto (1,0,0 \mid 0,0,0) &= v_1,\\
v_4 &\mapsto (\lambda,0,0 \mid 0,0,e_{-1}) &= \lambda v_1 + v_4,\\
v_3 &\mapsto (0,0,0 \mid 0,0,e_{-1}) &= v_3,\\
v_2 &\mapsto (0,1,0 \mid 0,0,\lambda e_{-1}) &= \lambda v_3 + v_2,
\end{aligned}$$

and it can be written in the matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}.$$

Finally, the action on $\mathcal{B}$ of $L_{-\lambda e_1}$ is given by

$$\begin{array}{rclcl} v_1 & \mapsto & (1,0,0 \mid 0,0,0) & = & v_1, \\ v_4 & \mapsto & (0,0,0 \mid 0,0,e_1) & = & v_4, \\ v_3 & \mapsto & (\lambda,0,0 \mid 0,0,e_{-1}) & = & \lambda v_1 + v_3, \\ v_2 & \mapsto & (0,1,0 \mid 0,0,\lambda e_1) & = & \lambda v_4 + v_2, \end{array}$$

and in the matrix form we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As we know,

$$\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \mathrm{SL}_2(F).$$

It follows that

$$\begin{aligned} &\left\langle \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \\ &\qquad \cong \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \;\middle|\; \lambda \in F \right\rangle \cong \mathrm{SL}_2(F), \end{aligned}$$

and since

$$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

we finally get

$$\left\langle \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&\lambda\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\\lambda&1\end{bmatrix}, \right.$$
$$\left. \begin{bmatrix}1&\lambda\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\\lambda&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \;\middle|\; \lambda\in F\right\rangle \cong \mathrm{SL}_2(F)\circ\mathrm{SL}_2(F).$$

Now, $\mathrm{SL}_2(F)\circ\mathrm{SL}_2(F)\cong\Omega_4^+(F)$ and this finishes the proof. □

In our construction we use the results of section **??**. Consider the 6-space V_6 spanned by the Albert vectors $(a,b,0\mid 0,0,C)$, where $C\in\langle e_{-1},e_{\overline{\omega}},e_{-\overline{\omega}},e_1\rangle$. Our copy of $\Omega_4^+(F)$ preserves two isotropic Albert vectors in V_6:

$$\begin{aligned} u_{\overline{\omega}} &= (0,0,0\mid 0,0,e_{\overline{\omega}}),\\ u_{-\overline{\omega}} &= (0,0,0\mid 0,0,e_{-\overline{\omega}}). \end{aligned} \tag{3.31}$$

The element $M_{e_{\overline{\omega}}}$ preserves $u_{\overline{\omega}}$ but not $u_{-\overline{\omega}}$. Therefore, adjoining $M_{e_{\overline{\omega}}}$ to $\Omega_4^+(F)$, we obtain a subgroup of $V_4{:}\Omega_4^+(F)$ (Lemma A.1), and since $\Omega_4^+(F)$ is maximal in the latter (Theorem **??**), we conclude that the action of $M_{\lambda e_{\pm1}}$, $L_{\lambda e_{\pm1}}$ and $M_{e_{\overline{\omega}}}$ on V_6 is that of $V_4{:}\Omega_4^+(F)$. That is, we have constructed the group $V_4{:}\Omega_4^+(F)$ as the stabiliser of $u_{\overline{\omega}}$ in $\Omega_6^+(F)$. Now we use the result of Theorem **??**. The element $M_{e_{-\overline{\omega}}}$ preserves V_6 but it does not preserve $u_{\overline{\omega}}$, and as a consequence it does not preserve the 1-space $\langle u_{\overline{\omega}}\rangle$. Therefore, if we adjoin $M_{e_{-\overline{\omega}}}$ to our copy of $V_4{:}\Omega_4^+(F)$, we get the action of the group $\Omega_6^+(F)$ on V_6.

Similarly, we consider the 8-space V_8 spanned by the vectors $(a,b,0\mid 0,0,C)$ with $C\in\langle e_{-1},e_{\overline{\omega}},e_{\omega},e_{-\omega},e_{-\overline{\omega}},e_1\rangle$. Consider two isotropic Albert vectors

$$\begin{aligned} u_{\omega} &= (0,0,0\mid 0,0,e_{\omega}),\\ u_{-\omega} &= (0,0,0\mid 0,0,e_{-\omega}), \end{aligned} \tag{3.32}$$

which are fixed by our copy of $\Omega_6^+(F)$. The action of the element M_{e_ω} on V_8 preserves u_ω but not $u_{-\omega}$ and therefore adjoining this element to $\Omega_6^+(F)$ we get the action of the group $V_6{:}\Omega_6^+(F)$. Next, the element $M_{e_{-\omega}}$ does not preserve the 1-space $\langle u_\omega \rangle$, so appending it to $V_6{:}\Omega_6^+(F)$ we get the action of the group $\Omega_8^+(F)$ on V_8.

Finally, we consider the 10-space $\mathbb{J}_{10}^{abC}$ with two isotropic Albert vectors

$$\begin{aligned} u_0 &= (0,0,0 \mid 0,0,e_0), \\ u_{-0} &= (0,0,0 \mid 0,0,e_{-0}). \end{aligned} \tag{3.33}$$

Following the same procedure, we adjoin the element M_{e_0} which fixes u_0 but not u_{-0} to get the action of the group of shape $V_8{:}\Omega_8^+(F)$. Appending the action of $M_{e_{-0}}$, which does not preserve $\langle u_0 \rangle$, to this yields the action of $\Omega_{10}^+(F)$ on $\mathbb{J}_{10}^{abC}$. Lemma 3.3.11 allows us to conclude that we have shown the following result.

Lemma 3.3.13. *The actions of M_x and L_x on $\mathbb{J}_{10}^{abC}$ generate the group $\Omega_{10}^+(F)$ as x ranges through $\mathbb{O}$.*

Now we need to understand the action of the elements M_x and L_x on the whole 27-space $\mathbb{J}$.

Lemma 3.3.14. *Suppose an element of the stabiliser in $\mathrm{SE}_6(F)$ of v preserves the decomposition of the Albert space into the direct sum of the form $\mathbb{J} = \mathbb{J}_1^c \oplus \mathbb{J}_{16}^{AB} \oplus \mathbb{J}_{10}^{abC}$.*

(a) If the action of this element on the 10-space $\mathbb{J}_{10}^{abC}$ is given by

$$\begin{aligned} (1,0,0 \mid 0,0,0) &\mapsto (\lambda,0,0 \mid 0,0,0), \\ (0,1,0 \mid 0,0,0) &\mapsto (0,\lambda^{-1},0 \mid 0,0,0), \\ (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C), \end{aligned}$$

then λ is a square in F.

(b) On the other hand, an action of the type

$$\begin{aligned} (1,0,0 \mid 0,0,0) &\mapsto (0,\lambda,0 \mid 0,0,0), \\ (0,1,0 \mid 0,0,0) &\mapsto (\lambda^{-1},0,0 \mid 0,0,0), \\ (0,0,0 \mid 0,0,C) &\mapsto (0,0,0 \mid 0,0,C) \end{aligned}$$

is impossible.

(c) Finally, if the action on the 10*-space is trivial, then the action on the corresponding* 16*-space is that of* $\pm\mathrm{I}_{16}$ *(hence, the action on* $\mathbb{J}$ *is that of* $P_{\pm 1}$*).*

Proof. We are considering the elements that fix $\mathbb{J}_8^C$ pointwise and either fix or swap the 1-dimensional spaces $\mathbb{J}_1^a$ and $\mathbb{J}_1^b$. So we may assume that these elements respectively fix or swap the corresponding 17-spaces $\mathbb{J}_{17}^{aBC}$ and $\mathbb{J}_{17}^{bAC}$. In particular, their intersection, i.e. the space $\mathbb{J}_8^C$ is fixed. If the action of the stabiliser swaps $\mathbb{J}_1^a$ and $\mathbb{J}_1^b$ while leaving the 1-space $\mathbb{J}_1^c$ in its place, then it also swaps the 8-spaces $\mathbb{J}_8^A$ and $\mathbb{J}_8^B$ as these subspaces are the intersections of the 17-space $\mathbb{J}_{17}^{cAB}$ with $\mathbb{J}_{17}^{bAC}$ and $\mathbb{J}_{17}^{aBC}$ respectively.

Suppose now that an element in the stabiliser acts in the following manner:

$$(a, b, c \mid A, B, C) \mapsto (\lambda a, \lambda^{-1} b, c \mid A\phi, B\psi, C),$$

where $\phi, \psi : \mathbb{O} \to \mathbb{O}$ are invertible F-linear maps. As this action is supposed to preserve the determinant, it has to preserve the cubic term $\mathrm{T}(ABC)$ in particular, i.e. we must have $\mathrm{T}(ABC) = \mathrm{T}((A\phi)(B\psi)C)$ for all $A, B, C \in \mathbb{O}$. This is equivalent to the condition $AB = (A\phi)(B\psi)$ for all $A, B \in \mathbb{O}$, since the original identity is equivalent to $\langle AB, \overline{C}\rangle = \langle (A\phi)(B\psi), \overline{C}\rangle$. By Lemma 3.3.10 we find that $A\phi = \mu^{-1}A$ and $B\psi = \mu B$ for all $A, B \in \mathbb{O}$ and some non-zero $\mu \in F$. The individual terms in the determinant are being changed in the following way:

$$\begin{aligned} abc &\mapsto abc, \\ aA\overline{A} &\mapsto \lambda\mu^{-2} aA\overline{A}, \\ bB\overline{B} &\mapsto \lambda^{-1}\mu^{2} bB\overline{B}, \\ cC\overline{C} &\mapsto cC\overline{C}, \\ \mathrm{T}(ABC) &\mapsto \mathrm{T}(ABC). \end{aligned}$$

It follows that in order to preserve the determinant we must have $\lambda^{-1}\mu^2 = 1$, i.e. $\lambda = \mu^2$.

In case when our element acts as

$$(a, b, c \mid A, B, C) \mapsto (\lambda^{-1} b, \lambda a, c \mid B\psi, A\phi, C),$$

we get $\mathrm{T}(ABC) = \mathrm{T}((B\psi)(A\phi)C)$ for all $A, B, C \in \mathbb{O}$. This holds if and only if

$AB = (B\psi)(A\phi)$ for all $A, B \in \mathbb{O}$. Lemma 3.3.9 asserts that there are no such maps ϕ and ψ, and so this rules out the latter case.

Finally, if we assume the trivial action on $\mathbb{J}_{10}^{abC}$, then we get $\lambda = 1$, i.e. $\mu^2 = 1$, so the action on $\mathbb{J}$ is indeed that of $P_{\pm 1}$. $\square$

Now let $X = (a, b, c \mid A, B, C)$ and let $Y = (a', b', c' \mid A', B', C')$. An isometry which maps X to Y and v to λv must send $\Delta(X + v) - \Delta(X) = ab - C\overline{C}$ to $\Delta(Y + \lambda v) - \Delta(Y) = \lambda(a'b' - C'\overline{C}')$. The 17-dimensional radical of both of these forms is fixed, and the quadratic form $ab - C\overline{C}$ is being scaled by a factor of λ. In particular, when $\lambda = 1$, the quadratic form is being preserved. So, the action of the vector stabiliser on the 10-dimensional quotient is that of a subgroup of $\mathrm{GO}_{10}^{+}(F)$.

Consider the white vectors of the form $(a, 0, c \mid A, B, 0)$ and $(0, b, c \mid A, B, 0)$ with $a, b \neq 0$. In the first case the conditions for being white are

$$\left.\begin{aligned} A\overline{A} &= 0, \\ B\overline{B} &= ac, \\ a\overline{A} &= 0, \\ AB &= 0. \end{aligned}\right\}$$

In other words, we have a white vector of the form $(a, 0, B\overline{B}/a \mid 0, B, 0)$. For the second vector we get

$$\left.\begin{aligned} bc &= A\overline{A}, \\ B\overline{B} &= 0, \\ b\overline{B} &= 0, \end{aligned}\right\}$$

so the vector has the form $(0, b, A\overline{A}/b \mid A, 0, 0)$. The elements M'_x and L''_x transform these in the following way:

$$\begin{aligned} M'_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a \mid 0, B, 0), \\ M'_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b + bx\bar{x} + \mathrm{T}(\bar{x}A) \mid A + bx, 0, 0), \\ L''_x : (a, 0, B\overline{B}/a \mid 0, B, 0) &\mapsto (a, 0, B\overline{B}/a + ax\bar{x} + \mathrm{T}(Bx) \mid 0, B + a\bar{x}, 0), \\ L''_x : (0, b, A\overline{A}/b \mid A, 0, 0) &\mapsto (0, b, A\overline{A}/b \mid A, 0, 0). \end{aligned}$$

Note that we already have an elementary abelian group F^{16} acting on the 17-space $\mathbb{J}_{17}^{cAB}$. We can now invoke Lemma 3.3.14 to conclude that the action of the stabiliser on the remaining 10-space $\mathbb{J}_{10}^{abC}$ is that of $\Omega_{10}^{+}(F)$ and the kernel of the action on $\mathbb{J}$ has order no more than two.

Theorem 3.3.15. *The actions of the elements M_x and L_x on $\mathbb{J}$ where x ranges through a split octonion algebra $\mathbb{O}$ generate a group of type $\mathrm{Spin}_{10}^{+}(F)$ understood as $\Omega_{10}^{+}(F)$ in case of characteristic 2.*

With the result of Lemma 3.3.11 we conclude that the stabiliser of a white vector is indeed a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$ as usual understood as $F^{16}{:}\Omega_{10}^{+}(F)$ in case of characteristic 2.

Now we have enough ingredients to produce the vector stabiliser. As before, we consider the stabiliser of the white vector $v = (0, 0, 1 \mid 0, 0, 0)$. As we know from Theorem 3.3.15 and Lemma 3.3.14, the actions of the elements M_x and L_x on $\mathbb{J}$ generate a group of type $\mathrm{Spin}_{10}^{+}(F)$. It is easy to check that this copy of $\mathrm{Spin}_{10}^{+}(F)$ normalises the elementary abelian group F^{16} from Lemma 3.3.11. A straighforward computation illustrates the following result:

$$\begin{aligned} (M'_x)^{L_y} &\text{ acts as } M'_x, \\ (M'_x)^{M_y} &\text{ acts as } L''_{-yx} \cdot M'_x, \\ (L''_x)^{L_y} &\text{ acts as } M'_{-yx} \cdot L''_x, \\ (L''_x)^{M_y} &\text{ acts as } L''_x, \end{aligned} \tag{3.34}$$

where the products in the right-hand side are understood as the products of the actions rather than as the matrix products. Furthermore, the intersection of the groups $\mathrm{Spin}_{10}^{+}(F)$ and F^{16} is trivial: the action of $\mathrm{Spin}_{10}^{+}(F)$ preserves the decomposition $\mathbb{J} = \mathbb{J}_1^c \oplus \mathbb{J}_{16}^{AB} \oplus \mathbb{J}_{10}^{abC}$, while any non-trivial action of the elementary abelian group F^{16} fails to do so. Indeed, a general element in F^{16} has the form $M'_x \cdot L''_y$ for some $x, y \in \mathbb{O}$ and it sends an Albert vector $(a, b, c \mid A, B, C)$ to

$$(a, b, c + a\mathrm{N}(y) + b\mathrm{N}(x) + \mathrm{T}(By) + \mathrm{T}(\bar{x}A) + \mathrm{T}(\bar{x}\overline{C}y) \mid A + \overline{C}y + bx, B + a\bar{y} + \bar{x}\overline{C}, C).$$

So, we have shown that the actions of the elements M'_x, L''_x, M_x, L_x on $\mathbb{J}$ generate a group of shape $F^{16}{:}\mathrm{Spin}_{10}^{+}(F)$, as x ranges through a split algebra $\mathbb{O}$.

Next, we consider the white point $\langle v \rangle$ spanned by our white vector. The stabiliser in $\mathrm{SE}_6(F)$ of $\langle v \rangle$, where $v = (0, 0, 1 \mid 0, 0, 0)$, maps v to λv for some non-zero $\lambda \in F$.

For instance, this can be achieved by the elements

$$P'_{u^{-1}} = \operatorname{diag}(1_{\mathbb{O}}, u^{-1}, u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u \end{bmatrix} \tag{3.35}$$

with u being an invertible octonion of arbitrary norm. Indeed, any such element $P'_{u^{-1}}$ sends $(0,0,1 \mid 0,0,0)$ to $(0,0,\mathrm{N}(u) \mid 0,0,0)$ and since $\mathrm{N}(u)$ can be any non-zero field element, we get an abelian group $F^{\times}$ on top of the vector stabiliser. This finishes the proof of the main theorem in this section.

Now, since the vector stabiliser is generated by the actions of M_x, L_x, M'_x, L''_x on $\mathbb{J}$, and the subgroup of $\mathrm{SE}_6(F)$ generated by $M_x, M'_x, M''_x, L_x, L'_x, L''_x$ acts transitively on the white points, we make the following conclusion.

Theorem 3.3.16. *The group $\mathrm{SE}_6(F)$ is generated by the actions of M_x, M'_x, M''_x and L_x, L'_x, L''_x on $\mathbb{J}$ as x ranges through $\mathbb{O}$.*

3.4 Simplicity of $\mathrm{E}_6(F)$

The construction we have obtained also allows us to show that the group $\mathrm{E}_6(F)$ is indeed simple without any references to Lie theory. The classical way of showing the simplicity of certain groups is the following lemma.

Lemma 3.4.1 (Iwasawa)**.** *If G is a perfect group acting faithfully and primitively on a set Ω, and the point stabilizer H has a normal abelian subgroup A whose conjugates generate G, then G is simple.*

First, we show that the subgroup of $\mathrm{SE}_6(F)$ stabilising all the white points simultaneously acts on $\mathbb{J}$ by scalar multiplications, and hence the action of $\mathrm{E}_6(F)$ on the set of white points is faithful.

Lemma 3.4.2. *The subgroup in $\mathrm{SE}_6(F)$ stabilising simultaneously all white points is the group of scalars.*

Proof. Consider the action of this stabiliser on $\mathbb{J}_{10}^{abC}$ and pick the basis

$$\begin{aligned} v_1 &= (1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_{i+2} &= (a_i, b_i, 0 \mid 0, 0, C_i), \end{aligned} \tag{3.36}$$

where $1 \leqslant i \leqslant 8$ and $C_i\overline{C}_i = a_i b_i$. Since in particular we stabilise $\langle v_1 \rangle, ..., \langle v_{10} \rangle$, the action on $\mathbb{J}_{10}^{cAB}$ is that of a 10×10 diagonal matrix $\mathrm{diag}(\lambda_1, ..., \lambda_{10})$ with respect to the basis $\{v_1, ..., v_{10}\}$. Consider an Albert vector $v = (a, b, 0 \mid 0, 0, C)$, where $C = C_1 + \cdots + C_8$ and a, b are such that v is white, i.e. $C\overline{C} = ab$. Now, if $F \neq \mathbb{F}_2$, we can choose $a, b \in F$ in such a way that v can be written as a linear combination $v = \alpha v_1 + \beta v_2 + v_3 + \cdots + v_{10}$ with $\alpha \neq 0$. The stabiliser of all white point maps v to λv for some non-zero $\lambda \in F$, so this ensures that $\lambda = \lambda_1 = \lambda_3 = \cdots = \lambda_{10}$. We now adjust the chosen values of a and b to obtain a linear combination with $\beta \neq 0$, and so $\lambda = \lambda_2 = \lambda_3 = \cdots \lambda_{10}$. It follows that the action on $\mathbb{J}_{10}^{abC}$ is just the multiplication by λ.

When $F = \mathbb{F}_2$, we take $\mathbb{O}$ to be the split octonion algebra with our favourite basis $\{e_i \mid i \in \pm\{0, 1, \omega, \overline{\omega}\}\}$. For the 10-space $\mathbb{J}_{10}^{abC}$ we choose the basis

$$\begin{aligned} v_1 &= (1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_{i+2} &= (0, 0, 0 \mid 0, 0, e_i), \end{aligned} \tag{3.37}$$

and then proceed in the same manner. The vector $v = v_1 + \cdots + v_{10}$ is white and since there is a single choice for a non-zero scalar in $\mathbb{F}_2$, it is being fixed and the action on the whole 10-space in this case is that of $\mathrm{diag}(1, ..., 1)$.

Now, by using the triality element, we map $\mathbb{J}_{10}^{abC}$ to $\mathbb{J}_{10}^{bcA}$ and further to $\mathbb{J}_{10}^{caB}$ and so we obtain that the stabiliser of all white points acts on $\mathbb{J}$ by scalar multiplications. That is, the stabiliser of all the white points is trivial in $\mathrm{E}_6(F)$.

□

From Lemma 3.3.5 we know that the action of $\mathrm{E}_6(F)$ on the white points is primitive. We need to show that the group is perfect.

Lemma 3.4.3. *The group* $\mathrm{SE}_6(F)$ *is perfect.*

Proof. This does not present great difficulties. A very straightforward computation shows that

$$\begin{aligned}
(L''_{-1})^{-1} \cdot L'_x \cdot L''_{-1} \cdot (L'_x)^{-1} &\text{ acts as } M_x, \\
(L_{-1})^{-1} \cdot L''_x \cdot L_{-1} \cdot (L''_x)^{-1} &\text{ acts as } M'_x, \\
(L'_{-1})^{-1} \cdot L_x \cdot L'_{-1} \cdot (L_x)^{-1} &\text{ acts as } M''_x, \\
(M'_{-1})^{-1} \cdot M''_x \cdot M'_{-1} \cdot (M''_x)^{-1} &\text{ acts as } L_x, \\
(M''_{-1})^{-1} \cdot M_x \cdot M''_{-1} \cdot (M_x)^{-1} &\text{ acts as } L'_x, \\
(M_{-1})^{-1} \cdot M'_x \cdot M_{-1} \cdot (M'_x)^{-1} &\text{ acts as } L''_x,
\end{aligned}$$

where as before $A \cdot B$ is understood as the product of the actions by the matrices A and B. Hence, every generator is in fact a commutator. □

Finally, using the Iwasawa's Lemma we obtain the following theorem.

Theorem 3.4.4. *The group* $\mathrm{E}_6(F)$ *is simple.*

3.5 Case of a finite field

In this section F is a finite field of q elements, that is, $F = \mathbb{F}_q$. Our aim is to count the white points in this case, and hence find the group order.

Theorem 3.5.1. *If* $F = \mathbb{F}_q$*, then there are precisely*

$$\frac{(q^{12}-1)(q^9-1)}{(q^4-1)} \tag{3.38}$$

white vectors in $\mathbb{J}$*.*

Proof. In the proof we study the series of subspaces

$$0 < \mathbb{J}_{10}^{abC} < \mathbb{J}_{26}^{abABC} < \mathbb{J}.$$

First, $(a, b, 0 \mid 0, 0, C) \in \mathbb{J}_{10}^{abC}$ is white if and only if $ab - C\overline{C} = 0$. We notice that $ab - C\overline{C}$ is a quadratic form of *plus* type defined on $\mathbb{J}_{10}^{abC}$, so there are $(q^5-1)(q^4+1)$ white vectors in this subspace.

Next, suppose $(a, b, c \mid A, B, C) \in \mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$ is white. Then $C = \overline{B}\overline{A}c^{-1}$, $b = A\overline{A}c^{-1}$ and $a = B\overline{B}c^{-1}$. We may choose A, B, c to be arbitrary (with $c \neq 0$), so there are $q^{16}(q-1)$ white vectors in $\mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$.

Finally, we investigate the white vectors $(a, b, 0 \mid A, B, C) \in \mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$. The conditions for such a vector to be white take the following form:

$$\left.\begin{aligned} A\overline{A} = B\overline{B} = AB = 0, \\ C\overline{C} = ab, \\ BC = a\overline{A}, \\ CA = b\overline{B}. \end{aligned}\right\}. \tag{3.39}$$

Note that we also require $(A, B) \neq (0, 0)$. In case $A = 0$, $B \neq 0$ we apply δ followed by τ to $(a, b, 0 \mid A, B, C)$ in order to obtain a vector of the form $(a, b, 0 \mid A, B, C)$ with $A \neq 0$. Note that the values of a, b, A, B, C are not the same as in the initial Albert vector. So, assuming $A \neq 0$, we have $AB = 0$ exactly when B is in a particular 4-dimensional subspace of $\mathbb{O}$ and any such B satisfies $B\overline{B} = 0$. For any octonion x, the action by the element L_x establishes a bijection between the white vectors of the form $(*, *, 0 \mid A, B, *)$ and those of the form $(*, *, 0 \mid A, B + \overline{A}x, *)$. Left-multiplication by $\overline{A}$ annihilates a 4-dimensional subspace of $\mathbb{O}$ (see Lemma **??**), so by the rank-nullity theorem we conclude that the image $\mathcal{A} = \{\overline{A}x \mid x \in \mathbb{O}\}$ is also 4-dimensional. Note that $A(\overline{A}x) = (A\overline{A})x = 0$, for any $x \in \mathbb{O}$, so $\mathcal{A}$ is the 4-space of all octonions left-annihilated by A, and therefore it contains $-B$. Now we pick an octonion x such that $\overline{A}x = -B$ to obtain the bijection between the white vectors of the form $(*, *, 0 \mid A, B, *)$ with $A \neq 0$ and those of the form $(*, *, 0 \mid A, 0, *)$. An Albert vector $(a, b, 0 \mid A, 0, C)$ is white if and only if $A\overline{A} = C\overline{C} = CA = 0$ and $a = 0$, with no dependence on b. As before, C lies in a particular 4-dimensional subspace of $\mathbb{O}$, hence $(0, b, 0 \mid 0, 0, C)$ lies in a particular 5-dimensional subspace of $\mathbb{J}_{10}^{abC}$, so for any choice of the pair (A, B) there are q^5 white vectors. If $A \neq 0$, then there are $(q^4 - 1)(q^3 + 1)$ choices for A, and for each of these q^4 choices for B. If $A = 0$, we have $(q^4 - 1)(q^3 + 1)$ choices for B. It follows that in total there are

$$q^5(q^4(q^4 - 1)(q^3 + 1) + (q^4 - 1)(q^3 + 1)) = q^5(q^8 - 1)(q^3 + 1)$$

white vectors in $\mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$.

The calculations above give the numbers of white vectors in certain subsets of $\mathbb{J}$ as shown in the following table.
Adding these numbers gives the result. □

subset	$\mathbb{J}_{10}^{abC}$	$\mathbb{J}_{26}^{abABC} \setminus \mathbb{J}_{10}^{abC}$	$\mathbb{J} \setminus \mathbb{J}_{26}^{abABC}$
number of white vectors	$(q^5-1)(q^4+1)$	$q^5(q^8-1)(q^3+1)$	$q^{16}(q-1)$

Corollary 3.5.2. *There are precisely*

$$\frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)} \tag{3.40}$$

white points in the case $F = \mathbb{F}_q$.

Theorem 3.3.8 allows us to find the stabiliser of a white point which in our case is a group of shape $q^{16}{:}\mathrm{Spin}_{10}^+(q)$. As a consequence, we now have:

$$|\mathrm{SE}_6(q)| = q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1). \tag{3.41}$$

We obtain $\mathrm{E}_6(q)$ as the quotient of $\mathrm{SE}_6(q)$ by any scalars it contains. Note that $\mathrm{SE}_6(q)$ contains non-trivial scalars if and only if $q \equiv 1 \pmod 3$, so

$$|\mathrm{E}_6(q)| = \frac{1}{\gcd(3, q-1)} q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1). \tag{3.42}$$

We can also invoke Proposition 3.3.7 to count totally white lines in the finite case.

Proposition 3.5.3. *Given any white point* $\langle W \rangle$,

(i) there are exactly $q(q^8-1)(q^3+1)/(q-1)$ *white points* $\langle X \rangle$ *such that all points in* $\langle W, X \rangle$ *are white;*

(ii) there are exactly $q^8(q^4+1)(q^5-1)/(q-1)$ *white points* $\langle Y \rangle$ *such that* $\langle W, Y \rangle$ *contains only two white points.*

Proof. Without loss of generality assume $W = (0,0,1 \mid 0,0,0)$. By Proposition 3.3.7 we need to count the white points spanned by the vectors in $\mathbb{J}_{17}^{cAB}$. Consider a general white vector $(0,0,c \mid A,B,0) \in \mathbb{J}_{17}^{cAB}$. The whiteness conditions are $A\overline{A} = B\overline{B} = 0 = AB$. We distinguish two cases. If $A \neq 0$, then there are $(q^4-1)(q^3+1)$ choices for A. For each of these there are q^4 choices for B and q choices for c. Therefore, there are $q^5(q^4-1)(q^3+1)$ white vectors in $\mathbb{J}_{17}^{cAB}$ with

$A \neq 0$. In case $A = 0$ we require $B \neq 0$, so there are $(q^4 - 1)(q^3 + 1)$ choices for B and q choices for c, giving $q(q^4 - 1)(q^3 + 1)$ white vectors. Adding these two values together, we get

$$q^5(q^4 - 1)(q^3 + 1) + q(q^4 - 1)(q^3 + 1) = q(q^8 - 1)(q^3 + 1)$$

white vectors in $\mathbb{J}_{17}^{cAB}$. To find the number of totally white lines, we divide this value by $q - 1$.

Obviously, all white points not already counted have the second property, so there are

$$\frac{(q^9 - 1)(q^{12} - 1)}{(q^4 - 1)(q - 1)} - \frac{q(q^8 - 1)(q^3 + 1)}{(q - 1)} - 1 = \frac{q^8(q^4 + 1)(q^5 - 1)}{(q - 1)}$$

choices for $\langle Y \rangle$. Note that we subtract 1 because we want $\langle W \rangle \neq \langle Y \rangle$. □

3.6 Arbitrary octonion algebras

Now that we have constructed the group of type E_6 over an arbitrary field, it is to be emphasised that our construction depends on the fact that $\mathbb{O}$ has to be split. Namely, it is a vital requirement in the proof of Theorem 3.3.8. This completely covers the possibilities in case $F = \mathbb{F}_q$, but while it was possible to prove many results independently of the choice of $\mathbb{O}$, there are still some questions to answer when $\mathbb{O}$ happens to be non-split. Note that a split octonion algebra exists over any field, so our construction is safe.

The main problem is to be able to tell whether the actions of the matrices M_x and L_x on $\mathbb{J}_{10}^{abC}$ generate $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$. At this stage it is possible to prove the following proposition.

Proposition 3.6.1. *The actions of the elements M_x and L_x on $\mathbb{J}_{10}^{abC}$ where x ranges through a non-split octonion algebra $\mathbb{O}$, generate at most a group of type $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$.*

Proof. To verify this we show that the elements encoded by M_x and L_x have the correct spinor norm. Recall that M_x acts on $\mathbb{J}_{10}^{abC}$ in the following way:

$$M_x : (a, b, 0 \mid 0, 0, C) \mapsto (a, b + a\mathrm{N}(x) + \mathrm{T}(\bar{x}C), 0 \mid 0, 0, C + ax).$$

Consider two Albert vectors $v = (0, 0, 0 \mid 0, 0, x)$ and $w = (0, x\bar{x}, 0 \mid 0, 0, x)$. Reflexion in v sends a vector $(a, b, 0 \mid 0, 0, C)$ to

$$(a, b, 0 \mid 0, 0, C) - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} \cdot (0, 0, 0 \mid 0, 0, x) = \left(a, b, 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x\right).$$

We then reflect the result in w to get

$$\begin{aligned}\left(a, b, 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x\right) &- \frac{-\mathrm{T}(C\bar{x}) - ax\bar{x}}{\mathrm{N}(x)} \cdot (0, x\bar{x}, 0 \mid 0, 0, x) \\ = \left(a, b + ax\bar{x} + \mathrm{T}(C\bar{x}), 0 \mid 0, 0, C - \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x + \frac{\mathrm{T}(C\bar{x})}{\mathrm{N}(x)} x + a\frac{x\bar{x}}{\mathrm{N}(x)} x\right) \\ &= (a, b + a\mathrm{N}(x) + \mathrm{T}(C\bar{x}), 0 \mid 0, 0, C + ax).\end{aligned}$$

Note that the action of M_x on $\mathbb{J}_{10}^{abC}$ is the same as the composition of reflexions in v and w. We find $\mathrm{Q}_{10}(v) = \mathrm{Q}_{10}(w) = \mathrm{N}(x)$ and conclude that M_x acts as an element of $\Omega(\mathbb{J}_{10}^{abC}, \mathrm{Q}_{10})$.

For the elements L_x we consider the reflexions in vectors $(0, 0, 0 \mid 0, 0, \bar{x})$ and $(x\bar{x}, 0, 0 \mid 0, 0, \bar{x})$ to obtain the same conclusion. □

Next, suppose that V is a vector space over F with a quadratic form Q, such that $V = \langle e, f\rangle \oplus W$, where (e, f) is a hyperbolic pair and $W = \langle e, f\rangle^{\perp}$. Consider an element g in $\mathrm{CGO}(V, Q)$ which scales of Q by some $\lambda \neq 0$. Then $V = \langle e^g, f^g\rangle \oplus W^g$ and $\langle e^g, f^g\rangle$ is isometric to $\langle e, f\rangle$. Therefore, W^g is isometric to W, and so there exists an isometry h in $\mathrm{GO}(V, Q)$ such that $\langle e^g, f^g\rangle^h = \langle e, f\rangle$. It follows that $(W^g)^h = W$, and gh is a λ-scaling of Q which fixes $\langle e, f\rangle$ and W. Hence, gh is a λ-scaling of Q_W.

Consider a λ-similarity on $\mathbb{O} = \mathbb{O}_F$ that sends $1_{\mathbb{O}}$ to some $u \in \mathbb{O}$. Then it gives rise to an element in the stabiliser of a white point which scales Q_8 by $\mathrm{N}(u)$. In other words, we have shown the following.

Proposition 3.6.2. *If $\mathbb{O}$ is an arbitrary octonion algebra over F, then the elements in the stabiliser of a white point can only scale a white vector by λ, where $\lambda \in F$ is such that there exists $u \in \mathbb{O}$ with $\mathrm{N}(u) = \lambda$.*

It is easy to check that all such scalings are possible. For example, the elements $P'_{u^{-1}}$, defined in (3.35), do the job.

Chapter 4

Groups of type $^2\mathrm{E}_6$

4.1 Quadratic field extensions

Let F and K be two fields such that F is a subfield of K. We say that K is an *extension field* of F. The *degree* of K over F, denoted by $[K : F]$, is the dimension of K as a vector space over F. We denote the extension of F by K as K/F.

A non-zero polynomial $f \in F[x]$ is called *separable*, if each root of f has multiplicity 1. If $\alpha \in F$ is algebraic, that is, α is a root of some polynomial $g \in F[x]$, then α is called *separable*, if its minimal polynomial is separable.

We have defined what it means for a field element and a polynomial to be separable. Suppose that $[K : F]$ is finite, then K/F is a *separable* extension, if every element of K is separable over F. We also say that K/F is *normal*, if every irreducible polynomial $f \in F[x]$ that has a root in K, splits into linear factors in $K[x]$. An extension K/F which is normal, separable, and of finite degree, is called a *Galois extension*. We are not going into too much detail here, for there are various well-known references on theory of field extensions, of very high quality, for instance, [?], [?], [?], or [?]. In this chapter we are interested in Galois extensions of degree 2. In case when $F = \mathbb{F}_q$, we have $|K| = q^2$, and so $K = \mathbb{F}_{q^2}$ (see, for example, [?]).

Given an extension K/F, we define the *Galois group* of K over F, denoted $\mathrm{Gal}(K/F)$, to be the group of automorphisms of K that fix F elementwise. In other words,

$$\mathrm{Gal}(K/F) = \{\ \varsigma \in \mathrm{Aut}(K)\ \big|\ \alpha^\varsigma = \alpha \text{ for all } \alpha \in F\ \}. \tag{4.1}$$

The most important result for us here is the following theorem.

Theorem 4.1.1. *Let K/F be a Galois extension. Then*

$$|\mathrm{Gal}(K/F)| = [K : F]. \tag{4.2}$$

It follows that in case $[K : F] = 2$ there is a unique non-trivial field automorphism $\varsigma : K \to K$, fixing F elementwise. If $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$, then ς takes the form $\varsigma : \lambda \mapsto \lambda^q$.

4.2 Spaces with two forms

Let K/F be a Galois extension of degree 2, and let s be a K-automorphism with F being its fixed field. Let also V be a $2m$-dimensional vector space over K with a quadratic form Q and a conjugate-symmetric sesquilinear form $B : V \times V \to K$, defined with respect to ς. Denote by f the polar form of Q, and suppose that $\mathcal{B} = \{e_1, f_1, e_2, f_2, ..., e_m, f_m\}$ is a hyperbolic basis of V with respect to f, i.e.

$$Q(e_i) = Q(f_i) = 0,\ f(e_i, f_i) = 1, \tag{4.3}$$

and $f(e_i, e_j) = f(e_i, f_j) = 0$ for $i \neq j$. The above said means that (V, Q) is a hyperbolic orthogonal space. Denote by G the maximal amongst all the subgroups of $\mathrm{GO}(V, Q)$ which preserve B. If U is a subspace of V, then we denote the restrictions of f and B on U as f_U and B_U respectively. We say that an element $v \in V$ is *singular isotropic*, if $Q(v) = B(v, v) = 0$.

Definition 4.2.1. *A (Q, B)-subspace of V is an F-subspace U of V such that $V = U \otimes_F K$, $f_U = B_U$ is an F-form on U, and Q_U is a non-degenerate quadratic form on U of Witt index at least* 2.

Proposition 4.2.2. *If U is a (Q, B)-subspace of V, then it is the unique (Q, B)-subspace of V, and $G = \mathrm{GO}(U, Q)$.*

Proposition 4.2.3. *Let U be a (Q, B)-subspace of V if Witt index at least 2. Fix $\lambda \in K \backslash F$. The group G has two orbits on doubly singular points with representatives $\langle u \rangle$ and $\langle u + \lambda v \rangle$, where $u, v \in U$ and $\langle u, v \rangle$ is a singular line.*

4.3 Hermitean form in $\mathbb{J}$ and the group ${}^2\mathrm{SE}_6(K/F)$

Suppose as before K/F is a quadratic Galois extension with F being a fixed field of a K-automorphism ς. In this chapter $\mathbb{O}_F$ will always be a split octonion algebra over F and $\mathbb{O}_K = \mathbb{O}_F \otimes_F K$. We also use the same basis $\{ e_i \mid i \in \pm I \}$ as in Section 2.6.

Denote by σ the automorphism of $\mathbb{O}_K$ induced by the field automorphism ς:

$$\left(\sum_{i \in \pm I} \lambda_i e_i \right)^{\sigma} = \sum_{i \in \pm I} \lambda_i^{\varsigma} e_i. \tag{4.4}$$

Consider the following Hermitean form defined on the elements of $\mathbb{O}_K$:

$$h(x) = A\overline{A}^{\sigma} + A^{\sigma}\overline{A} = \mathrm{T}(A\overline{A}^{\sigma}). \tag{4.5}$$

On the Albert space $\mathbb{J} = \mathbb{J}_K$ this induces the Hermitean form H, where

$$\mathrm{H}((a, b, c \mid A, B, C)) = aa^{\sigma} + bb^{\sigma} + cc^{\sigma} + \mathrm{T}(A\overline{A}^{\sigma} + B\overline{B}^{\sigma} + C\overline{C}^{\sigma}). \tag{4.6}$$

Using the construction from previous chapter, we obtain the group $\mathrm{SE}_6(K)$ in the usual way. Now define the group ${}^2\mathrm{SE}_6(K/F)$ as the subgroup of $\mathrm{SE}_6(K)$ which preserves H. In case $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$, we denote this by ${}^2\mathrm{SE}_6(q)$. As before, the group ${}^2\mathrm{E}_6(F/K)$ is defined as the quotient of ${}^2\mathrm{SE}_6(K/F)$ by its centre.

4.4 Some elements of ${}^2\mathrm{SE}_6(K/F)$

Let $X = (a, b, c \mid A, B, C)$ be an arbitrary element of $\mathbb{J} = \mathbb{J}_K$. First of all we notice that the matrices δ and τ preserve the Hermitean form, so they encode the elements of ${}^2\mathrm{SE}_6(K/F)$.

Lemma 4.4.1. *Let $x \in \mathbb{O}_K$ be such that $\bar{x}^\sigma x = x\bar{x}^\sigma = 0$. Then $x\bar{x} = 0$.*

Proof. If $x = 0$, then the result is trivial. Assume x is non-zero, $\bar{x}^\sigma x = x\bar{x}^\sigma = 0$. If $x\bar{x} \neq 0$, then x is invertible, and so $\bar{x}^s = 0$, which implies $x = 0$, a contradiction. □

Consider the matrices

$$N_x = \begin{bmatrix} 1 & x & 0 \\ -\bar{x}^\sigma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N'_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & -\bar{x}^\sigma & 1 \end{bmatrix}, \quad N''_x = \begin{bmatrix} 1 & 0 & -\bar{x}^\sigma \\ 0 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}, \tag{4.7}$$

where $x\bar{x}^\sigma = \bar{x}^\sigma x = 0$ and $x, \bar{x}^\sigma$ generate a sociable subalgebra. It is easy to see that these encode the elements of $\mathrm{SE}_6(K)$. Indeed,

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\bar{x}^\sigma & 1 \end{bmatrix} = \begin{bmatrix} 1 - x\bar{x}^\sigma & x \\ -\bar{x}^\sigma & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -\bar{x}^\sigma & 1 \end{bmatrix}. \tag{4.8}$$

So, the elements N_x, N'_x, and N''_x preserve the Dickson–Freudenthal determinant. To verify that they preserve H, we look at the action on $\mathbb{J}$:

$$\begin{aligned}
N_x : (a, b, c \mid A, B, C) &\mapsto (a - \mathrm{T}(C\bar{x}^\sigma), b + \mathrm{T}(\overline{C}x), c \mid \\
&\qquad \mid A + \bar{x}\overline{B}, B - \overline{A}\bar{x}^\sigma, C - x^\sigma \overline{C} x + ax - bx^\sigma), \\
N'_x : (a, b, c \mid A, B, C) &\mapsto (a, b - \mathrm{T}(a\bar{x}^\sigma), c + \mathrm{T}(\overline{A}x) \mid \\
&\qquad \mid A - x^\sigma \overline{A} x + bx - cx^\sigma, B + \bar{x}\overline{C}, C - \overline{B}\bar{x}^\sigma), \\
N''_x : (a, b, c \mid A, B, C) &\mapsto (a + \mathrm{T}(\overline{B}x), b, c - \mathrm{T}(B\bar{x}^\sigma) \mid \\
&\qquad \mid A - \overline{C}\bar{x}^\sigma, B - x^\sigma \overline{B} x + cx - ax^\sigma, C + \bar{x}\overline{A}).
\end{aligned} \tag{4.9}$$

We need to prove an auxiliary lemma.

Lemma 4.4.2. *Suppose that $x, y, z \in \mathbb{O}_F$ with $x\bar{x} = 0$. Then*

(i) $x\,\mathrm{T}(yx) = x(yx)$,

(ii) $\mathrm{T}((xy)(z\bar{x})) = 0$.

Proof.

(i) $x\,\mathrm{T}(yx) = x(yx + \bar{x}\bar{y}) = x(yx) + x(\bar{x}\bar{y}) = x(yx) + (x\bar{x})y = x(yx)$,

(ii) $\mathrm{T}((xy)(z\bar{x})) = \mathrm{T}((z\bar{x})(xy)) = \mathrm{T}(((z\bar{x})x)y) = \mathrm{T}(z(x\bar{x})y) = 0.$ $\square$

Obviously, it is enough to verify that the elements N_x preserve the Hermitean form H. The individual terms in $\mathrm{H}(X) = aa^\sigma + bb^\sigma + cc^\sigma + \mathrm{T}(A\overline{A}^\sigma + B\overline{B}^\sigma + C\overline{C}^\sigma)$ are being mapped in the following way:

$$\begin{aligned}
aa^\sigma &\mapsto aa^\sigma - a^\sigma\,\mathrm{T}(C\bar{x}^\sigma) - a\,\mathrm{T}(C^\sigma\bar{x}) + \mathrm{T}(C^\sigma\bar{x})\,\mathrm{T}(C\bar{x}^\sigma),\\
bb^\sigma &\mapsto bb^\sigma + b^\sigma\,\mathrm{T}(\overline{C}x) + b\,\mathrm{T}(\overline{C}^\sigma x^\sigma) + \mathrm{T}(\overline{C}x)\,\mathrm{T}(\overline{C}^\sigma x^\sigma),\\
cc^\sigma &\mapsto cc^\sigma,\\
\mathrm{T}(A\overline{A}^\sigma) &\mapsto \mathrm{T}(A\overline{A}^\sigma) + \mathrm{T}(AB^\sigma x^\sigma) + \mathrm{T}(\bar{x}\overline{B}\overline{A}^\sigma) + \mathrm{T}((\bar{x}\overline{B})(B^\sigma x^\sigma)),\\
\mathrm{T}(B\overline{B}^\sigma) &\mapsto \mathrm{T}(B\overline{B}^\sigma) - \mathrm{T}(\overline{A}\bar{x}^\sigma\overline{B}^\sigma) - \mathrm{T}(BxA^\sigma) + \mathrm{T}((\overline{A}\bar{x}^\sigma)(xA^\sigma)),\\
\mathrm{T}(C\overline{C}^\sigma) &\mapsto \mathrm{T}(C\overline{C}^\sigma) - \mathrm{T}(C(\bar{x}^\sigma C^\sigma\bar{x})) - \mathrm{T}(C^\sigma(\bar{x}C\bar{x}^\sigma)) + a\,\mathrm{T}(x\overline{C}^\sigma) +\\
&\qquad + a^\sigma\,\mathrm{T}(C\bar{x}^\sigma) - b^\sigma\,\mathrm{T}(C\bar{x}) - b\,\mathrm{T}(x^\sigma\overline{C}^\sigma).
\end{aligned} \tag{4.10}$$

Using Lemmas 4.4.1 and 4.4.2, we get $\mathrm{T}((\overline{A}\bar{x}^\sigma)(xA^\sigma)) = 0 = \mathrm{T}((\bar{x}\overline{B})(B^\sigma x^\sigma))$. Next, we also obtain $\mathrm{T}(C(\bar{x}^\sigma C^\sigma\bar{x})) = \mathrm{T}(C\bar{x}^\sigma\,\mathrm{T}(C^\sigma\bar{x})) = \mathrm{T}(C^\sigma\bar{x})\,\mathrm{T}(C\bar{x}^\sigma)$. Likewise, we get $\mathrm{T}(C^\sigma(\bar{x}C\bar{x}^\sigma)) = \mathrm{T}((x^\sigma\overline{C}x)\overline{C}^\sigma) = \mathrm{T}(\overline{C}^\sigma(x^\sigma\overline{C}x)) = \mathrm{T}(\overline{C}^\sigma x^\sigma)\,\mathrm{T}(\overline{C}x)$. We see that all the terms except $aa^\sigma, bb^\sigma, cc^\sigma$ and $\mathrm{T}(A\overline{A}^\sigma), \mathrm{T}(B\overline{B}^\sigma), \mathrm{T}(C\overline{C}^\sigma)$ cancel out, so it follows that the elements N_x preserve the Hermitean form. Hence, we have shown the following.

Proposition 4.4.3. *The matrices N_x, N'_x, and N''_x where $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$ and $x, \bar{x}^\sigma$ generate a sociable subalgebra, encode the elements of ${}^2\mathrm{SE}_6(K/F)$.*

Lemma 4.4.4. *If $x \in \mathbb{O}$ is such that $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$, with $x, \bar{x}^\sigma$ generating a sociable subalgebra, then $x^\sigma yx = xyx^\sigma$ for all $y \in \mathbb{O}_K$.*

Of great interest for us is the action of N_x on $\mathbb{J}_{10}^{abC}$. The rest of the section is devoted to proving the following result.

Theorem 4.4.5. *The actions of the elements N_x on $\mathbb{J}_{10}^{abC}$, where x is such that $x\bar{x}^\sigma = 0 = \bar{x}^\sigma x$, with $x, \bar{x}^\sigma$ generating a sociable subalgebra, generate a group of type $\Omega_{10}^{-,K}(F)$, as x ranges through all suitable octonions in $\mathbb{O}_K$.*

We prove this theorem in the series of steps. First, consider the 4-dimensional K-subspace V_4 of $\mathbb{J}_K$, spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_1\rangle$.

Lemma 4.4.6. *The actions of the elements $N_{\lambda e_{\pm 1}}$ on V_4, where $\lambda \in K$, generate a group of type $\Omega_4^{-,K}(F)$.*

Proof. Consider the basis $\mathcal{B} = \{ v_1, v_2, v_3, v_4 \}$ for V_4, where

$$\begin{aligned} v_1 &= (-1, 0, 0 \mid 0, 0, 0), \\ v_2 &= (0, 1, 0 \mid 0, 0, 0), \\ v_3 &= (0, 0, 0 \mid 0, 0, -e_{-1}), \\ v_4 &= (0, 0, 0 \mid 0, 0, e_1). \end{aligned}$$

Now we look at the action of $N_{\lambda e_{-1}}$ on these basis elements:

$$\begin{aligned} v_1 &\mapsto & (-1, 0, 0 \mid 0, 0, -\lambda e_{-1}) &= v_1 + \lambda v_3, \\ v_2 &\mapsto & (0, 1, 0 \mid 0, 0, -\lambda^\sigma e_{-1}) &= v_2 + \lambda^\sigma v_3, \\ v_3 &\mapsto & (0, 0, 0 \mid 0, 0, -e_{-1}) &= v_3, \\ v_4 &\mapsto & (-\lambda^\sigma, \lambda, 0 \mid 0, 0, -\lambda\lambda^\sigma e_{-1} + e1) &= -\lambda^\sigma v_1 + \lambda v_2 - \lambda\lambda^\sigma v_3 + v_4. \end{aligned}$$

It follows that the element $N_{\lambda e_{-1}}$ can be written as a 4×4 matrix over $\mathbb{F}_{q^2}$ with respect to $\mathcal{B}$:

$$[N_{\lambda e_{-1}}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & \lambda^\sigma & 0 \\ 0 & 0 & 1 & 0 \\ \lambda^\sigma & \lambda & \lambda\lambda^\sigma & 1 \end{bmatrix}.$$

For convenience, instead of the element $N_{\lambda e_1}$ we consider the element $N_{-\lambda^\sigma e_1}$ which acts on the elements of $\mathcal{B}$ as follows:

$$\begin{aligned} v_1 &\mapsto & (-1, 0, 0 \mid 0, 0, \lambda^\sigma e_1) &= v_1 + \lambda^\sigma v_4, \\ v_2 &\mapsto & (0, 1, 0 \mid 0, 0, \lambda e_1) &= v_2 + \lambda v_4, \\ v_3 &\mapsto & (-\lambda, \lambda^\sigma, 0 \mid 0, 0, -e_{-1} + \lambda\lambda^\sigma e_1) &= \lambda v_1 + \lambda^\sigma v_2 + v_3 + \lambda\lambda^\sigma v_4, \\ v_4 &\mapsto & (0, 0, 0 \mid 0, 0, e_1) &= v_4. \end{aligned}$$

The matrix $[N_{-\lambda^\sigma e_1}]_{\mathcal{B}}$ has the form

$$[N_{-\lambda^\sigma e_1}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \lambda^\sigma \\ 0 & 1 & 0 & \lambda \\ \lambda & \lambda^\sigma & 1 & \lambda\lambda^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the basis $\mathcal{B}' = \{\ v_3, v_4, v_1 v_4\ \}$ obtained as a permutation of the elements in $\mathcal{B}$. With respect to $\mathcal{B}'$ the 4×4 matrices $[N_{\lambda e_{-1}}]_{\mathcal{B}'}$ and $[N_{-\lambda^\sigma e_1}]_{\mathcal{B}'}$ take the form

$$[N_{\lambda e_{-1}}]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda^\sigma & 1 & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ \lambda\lambda^\sigma & \lambda & \lambda^\sigma & 1 \end{bmatrix}, \quad [N_{-\lambda^\sigma e_1}]_{\mathcal{B}'} = \begin{bmatrix} 1 & \lambda^\sigma & \lambda & \lambda\lambda^\sigma \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & \lambda^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We notice that

$$[N_{\lambda e_{-1}}]_{\mathcal{B}'} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda^\sigma & 1 \end{bmatrix}, \quad [N_{-\lambda^\sigma e_1}]_{\mathcal{B}'} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda^\sigma \\ 0 & 1 \end{bmatrix},$$

where $\otimes$ is the Kronecker product of two matrices. The mapping

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ \lambda^\sigma & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \lambda^\sigma \\ 0 & 1 \end{bmatrix}$$

can be extended to a homomorphism ϕ which is obviously surjective as λ ranges through the whole field K. Its kernel is a subgroup

$$\ker(\phi) = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

which has order 2, so we get the action of the group $\mathrm{PSL}_2(K)$ on V_4 since the matrices

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$$

generate a group $\mathrm{SL}_2(K)$ by a well-known result. Therefore, as $\mathrm{PSL}_2(K) \cong \Omega_4^-(q)$

(**??**), we have the action of $\Omega_4^{-,K}(F)$. $\square$

We again use the results of Appendix A. Consider the 6-dimensional K-subspace V_6 spanned by the Albert vectors of the form $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_{\overline{\omega}}, e_{-\overline{\omega}}, e_1 \rangle$. Our copy of $\Omega_4^{-,K}(F)$ preserves two isotropic Albert Vectors in V_6:

$$\begin{aligned} u_{\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{\overline{\omega}}), \\ u_{-\overline{\omega}} &= (0, 0, 0 \mid 0, 0, e_{-\overline{\omega}}). \end{aligned} \tag{4.11}$$

The element $N_{e_{\overline{\omega}}}$ preserves $u_{\overline{\omega}}$, but not $u_{-\overline{\omega}}$. Therefore, adjoining its action to our $\Omega_4^{-,K}(F)$, we obtain a subgroup of $V_4{:}\Omega_4^{-,K}(F)$. We know that $\Omega_4^{-,K}(F)$ is maximal in the latter, so we conclude that the actions of $N_{\lambda e_{\pm 1}}$ and $N_{e_{\overline{\omega}}}$ on V_6 is that of $V_4{:}\Omega_4^{-,K}(F)$. Next, the element $N_{e_{-\overline{\omega}}}$ preserves V_6 but it does not preserve $\langle u_{\overline{\omega}} \rangle$, so by Theorem **??**, adjoining $N_{e_{-\overline{\omega}}}$ to $V_4{:}\Omega_4^{-,K}(F)$, we get the action of $\Omega_6^{-,K}(F)$ on V_6.

Next, take the 8-space V_8 spanned by the Albert vectors $(a, b, 0 \mid 0, 0, C)$ with $C \in \langle e_{-1}, e_{\overline{\omega}}, e_{\omega}, e_{-\omega}, e_{-\overline{\omega}}, e_1 \rangle$, and consider two isotropic vectors

$$\begin{aligned} u_{\omega} &= (0, 0, 0 \mid 0, 0, e_{\omega}), \\ u_{-\omega} &= (0, 0, 0 \mid 0, 0, e_{-\omega}), \end{aligned} \tag{4.12}$$

which are fixed by our copy of $\Omega_6^{-,K}(F)$. The action of $N_{e_{\omega}}$ on V_8 preserves u_{ω} but not $u_{-\omega}$, and therefore adjoining this element to $\Omega_6^{-,K}(F)$ we get the action of the group $V_6{:}\Omega_6^{-,K}(F)$. The element $N_{e_{-\omega}}$ does not preserve the 1-space $\langle u_{\omega} \rangle$, so appending it to $V_6{:}\Omega_6^{-,K}(F)$, we get the action of $\Omega_8^{-,K}(F)$ on V_8.

Finally, we choose two isotropic Albert vectors

$$\begin{aligned} u_0 &= (0, 0, 0 \mid 0, 0, e_0), \\ u_{-0} &= (0, 0, 0 \mid 0, 0, e_{-0}). \end{aligned} \tag{4.13}$$

in $\mathbb{J}_{10}^{abC}$. We adjoin the element N_{e_0} which fixes u_0 but not u_{-0} to get the action of the group $V_8{:}\Omega_8^{-,K}(F)$. Appending to this the action of $N_{e_{-0}}$ which does not preserve $\langle u_0 \rangle$, we obtain the action of $\Omega_{10}^{-,K}(F)$.

4.5 Action of ${}^2\mathrm{SE}_6(K/F)$ on white points

As in the case of SE_6, we are interested in the action on white points. We will, however, see that although SE_6 acts transitively on white points, the action of ${}^2\mathrm{SE}_6(K/F)$ splits into several orbits. We say that a non-zero Albert vector X is *isotropic*, if $\mathrm{H}(X) = 0$.

We first consider some examples. Suppose $W_1 = (0,0,0 \mid 0,0,e_0)$. As we know (3.21), it determines a 17-space $\{\ (a,b,0 \mid A,B,C)\ \big|\ e_0A = Be_0 = \mathrm{T}(e_0\overline{C}) = 0\ \}$. A straightforward calculation shows that this 17-space U_1 is spanned by the Albert vectors of the form $(a,b,0 \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\rangle,\\ &B \in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\rangle,\\ &C \in \langle e_{-1}, e_{\overline{\omega}}, e_\omega, e_0, e_{-\omega}, e_{-\overline{\omega}}, e_1\rangle. \end{aligned} \tag{4.14}$$

We are also interested in the radical R_1 of H inside this 17-space. In our case it is spannned by the vectors of the form $(0,0,0 \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}}, e_\omega, e_{-0}, e_1\rangle,\\ &B \in \langle e_{-1}, e_{-0}, e_{-\omega}, e_{-\overline{\omega}}\rangle,\\ &C \in \langle e_0\rangle. \end{aligned} \tag{4.15}$$

In other words, our vector W_1 determines the 17-space U_1 and the 9-dimensional radical R_1 of H in U_1. Note that W_1 is isotropic with respect to H, and also $W_1 \in R_1$.

It turns out that there is another type of isotropic white vectors. Consider $W_2 = (0,0,0 \mid 0,0,e_0+\lambda e_1)$, where $\lambda \in K \setminus F$. It again determines a 17-space U_2, spanned by the Albert vectors of the form $(a,b,0 \mid A,B,C)$ with

$$\begin{aligned} &A \in \langle e_{\overline{\omega}} + \lambda e_{-\omega}, e_\omega - \lambda e_{-\overline{\omega}}, e_{-0}, e_1\rangle,\\ &B \in \langle e_{-1} + \lambda e_0, e_{-0} - \lambda e_1, e_{-\omega}, e_{-\overline{\omega}}\rangle,\\ &C \in \langle e_{-1} - \lambda^2 e_1 - \lambda, e_{\overline{\omega}}, e_\omega, e_0 + \lambda e_1, e_{-\omega}, e_{-\overline{\omega}}, e_1\rangle. \end{aligned} \tag{4.16}$$

We find that the radical R_2 of H in U_2 is spanned by the vectors of the form

$(0, 0, 0 \mid A, B, C)$ with

$$\begin{aligned} A &\in \langle e_{-0}, e_1 \rangle, \\ B &\in \langle e_{-\omega}, e_{-\overline{\omega}} \rangle, \\ C &\in \langle e_0 + \lambda^q e_1 \rangle, \end{aligned} \tag{4.17}$$

i.e. it is 5-dimensional. We notice that W_2 is isotropic, but in this case $W_2 \notin R_2$.

We conclude that the white points $\langle W_1 \rangle$ and $\langle W_2 \rangle$ belong to different orbuts under the action of ${}^2\mathrm{SE}_6(K/F)$. Of course, there is also at least one orbit on the non-isotropic white points.

4.5.1 Orbits of ${}^2\mathrm{SE}_6(K/F)$ on white points

4.5.2 The stabiliser of type 3 vector

We are now interested in the stabiliser in ${}^2\mathrm{SE}_6(K/F)$ of $W_3 = (0, 0, 1 \mid 0, 0, 0)$, which is non-isotropic. As we know, the elements N_x with $x\bar{x}^\sigma = \bar{x}^\sigma x = 0$ preserve W_3 (4.9). We now prove the following theorem.

Theorem 4.5.1. *The stabiliser in* ${}^2\mathrm{SE}_6(K/F)$ *of* $W_3 = (0, 0, 1 \mid 0, 0, 0)$ *is the subgroup of shape* $\mathrm{Spin}_{10}^{-,K}(F)$.

Proof. From Theorem 4.4.5 we know that the actions on $\mathbb{J}_{10}^{abC}$ of the elements N_x generate $\Omega_{10}^{-,K}(F)$. We use Lemma 3.3.14 to conclude that the action on $\mathbb{J}$ is that of $\mathrm{Spin}_{10}^{-,K}(F)$. Our aim is to show that this group is the whole stabiliser in ${}^2\mathrm{SE}_6(K/F)$ of W_3.

Let G be the stabiliser of W_3 in ${}^2\mathrm{SE}_6(K/F)$. In particular, G is a subgroup of $\mathrm{SE}_6(K)$, so it preserves the Dickson–Freudenthal determinant Δ, and hence it stabilises the 17-space $U_3 = \mathbb{J}_{17}^{cAB}$ determined by W_3. Next, G is a subgroup of ${}^2\mathrm{SE}_6(K/F)$, so it preserves the Hermitean form H, and so G stabilises $U_3^\perp$, where $\perp$ is taken with respect to H.

Consider the 10-dimensional F-subspace V_{10}^- spanned by the vectors of the form $(\lambda \cdot 1, -\lambda^\varsigma \cdot 1, 0 \mid 0, 0, C)$, where $\lambda \in K$ and $C \in \mathbb{O}_F$. First, we check that the action of our copy of $\Omega_{10}^{-,K}(F)$ preserves V_{10}^-. The action on this subspace is given by

$$\begin{aligned} N_x : (\lambda \cdot 1_{\mathbb{O}}, -\lambda^\varsigma \cdot 1_{\mathbb{O}}, 0 \mid 0, 0, C) \mapsto ((\lambda - \mathrm{T}(C\bar{x}^\sigma)) \cdot 1_{\mathbb{O}}, -(\lambda^\varsigma - \mathrm{T}(\overline{C}x)) \cdot 1_{\mathbb{O}}, 0 \mid \\ 0, 0, C - x^\sigma \overline{C} x + \lambda x + \lambda^\varsigma x^\sigma). \end{aligned}$$

Note that the product $x^\sigma \overline{C} x$ makes sense since x and x^σ generate a sociable subalgebra of $\mathbb{O}_K$. It is easy to see that $(\lambda - \mathrm{T}(C\bar{x}^\sigma))^\varsigma = \lambda^\varsigma - \mathrm{T}(\overline{C}x)$. Lemma 4.4.4 also implies that $x^\sigma \overline{C} x = x\overline{C} x^\sigma = (x^\sigma \overline{C} x)^\sigma$. That is, $x^\sigma \overline{C} x$ is an element of $\mathbb{O}_F$. Finally, $\lambda x + \lambda^\varsigma x^\sigma \in \mathbb{O}_F$, so we conclude that the elements N_x indeed preserve V_{10}^-.

Note that the stabiliser preserves restrictions of both Q and H on V_{10}^-. Proposition 4.2.2 asserts that such a subspace is unique, so we conclude that G is a subgroup of $\mathrm{GO}_{10}^{-,K}(F)$. Note that as a subgroup of $\mathrm{SE}_6(K)$, G also preserves the Dickson–Freudenthal determinant, so $G \leqslant \mathrm{SO}_{10}^{-,K}(F)$.

Now let us look at the action on V_{10}^- in more detail. The restriction of H on this 10-space is represented by the Gram matrix

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

which is block diagonal (here zeroes are replaced with dots). Consider the action by an element S such that it has the following matrix form when acting on V_{10}^-:

$$S_{10} = \left[\begin{array}{cc|c} \lambda & \cdot & \\ \cdot & \mu & \\ \hline & & \\ & & \mathrm{I}_8 \\ & & \end{array}\right],$$

where $\lambda\mu = 1$, i.e. $\mu = \lambda^{-1}$. Note that a generalised element P_λ acts on $\mathbb{J}$ in the following way:

$$P_\lambda : (a, b, c \mid A, B, C) \mapsto (\lambda^2 a, \lambda^{-2} b, c \mid \lambda^{-1} A, \lambda B, C).$$

It follows that the action on V_{10}^- of S^2 is the same as the action of P_λ, so the action of S on the 16-space $\mathbb{J}_{16}^{cAB}$ is determined up to sign.

Our element S also commutes with $\Omega_8^+(F)$, generated by the actions of P_u, as u ranges through all octonions of norm 1 in $\mathbb{O}_F$. Therefore, the action of S on $\mathbb{J}$ is given by

$$S_{27} = \left[\begin{array}{ccc|c|c|c}
\lambda & \cdot & \cdot & & & \\
\cdot & \lambda^{-1} & \cdot & & & \\
\cdot & \cdot & 1 & & & \\
\hline
 & & & \alpha \mathrm{I}_8 & & \\
\hline
 & & & & \beta \mathrm{I}_8 & \\
\hline
 & & & & & \mathrm{I}_8
\end{array}\right].$$

Since the action by S^2 coincides with the action by P_λ, we get

$$\left.\begin{array}{l} \alpha^2 = \lambda^{-1}, \\ \beta^2 = \lambda. \end{array}\right\}$$

Next, the action of S preserves the Dickson–Freudenthal determinant:

$$\left.\begin{array}{rcl}
abc &\mapsto& \lambda\lambda^{-1} abc, \\
aA\overline{A} &\mapsto& \lambda\alpha^2 aA\overline{A}, \\
bB\overline{B} &\mapsto& \lambda^{-1}\beta^2 bB\overline{B}, \\
cC\overline{C} &\mapsto& cC\overline{C}, \\
\mathrm{T}(ABC) &\mapsto& \alpha\beta\, \mathrm{T}(ABC).
\end{array}\right\}$$

It follows that $\alpha\beta = 1$. Similarly, by preserving the Hermitean form H we obtain the following conditions on α, β and λ:

$$\begin{aligned} \lambda\lambda^{\varsigma} &= 1, \\ \alpha\alpha^{\varsigma} &= 1, \\ \beta\beta^{\varsigma} &= 1. \end{aligned}$$

We see that on V_{10}^{-} S acts as

$$S_{10} = \left[\begin{array}{cc|c} \alpha & \cdot & \\ \cdot & \alpha^{-1} & \\ \hline & & \\ & & \mathrm{I}_8 \\ & & \end{array}\right]. \tag{4.18}$$

This is an element of $\Omega_{10}^{-,K}(F)$. With the help of Lemma 3.3.14 we find that the action of the stabiliser in ${}^2\mathrm{SE}_6(K/F)$ of W_3 is that of $\mathrm{Spin}_{10}^{-,K}(F)$. □

4.5.3 The stabiliser of type 1 vector

4.5.4 The stabiliser of type 2 vector

4.6 Case of a finite field

4.6.1 White vectors in $\mathbb{J}_8^C$

As a practical counting excersise, we count the isotropic white vectors in $\mathbb{J}_8^C$. As before, $K = \mathbb{F}_{q^2}$. First, we need the following auxiliary result.

Lemma 4.6.1. *Let V be a vector space over $\mathbb{F}_{q^2}$ of dimension $2m$. Define the map $Z_m : V \to \mathbb{F}_q$ in the following way:*

$$Z_m(x) = (x_1^q - x_1)(x_2^q - x_2) + (x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}),$$

where $x = (x_1, ..., x_{2m})$. Denote by z_m the number of $x \in V$ such that $Z_m(x) = 0$.

Then

$$z_n = q^{3m-1}(q^m + q - 1).$$

Proof. We proceed by induction on m. If $m = 1$, the equality $Z_m(x) = 0$ reduces to

$$(x_1^q - x_1)(x_2^q - x_2) = 0.$$

Note that this is possible when $x_1^q = x_1$ or $x_2^q = x_2$, i.e. when $x_1 \in \mathbb{F}_q$ or $x_2 \in \mathbb{F}_q$. Thus, when $m = 1$ there are precisely $2q^3 - q^2 = q^2(q + q - 1)$ solutions.

Now suppose that the statement holds for all integers k such that $1 \leqslant k \leqslant m-1$. In the case

$$\left.\begin{array}{l}(x_1^q - x_1)(x_2^q - x_2) = 0,\\(x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = 0,\end{array}\right\}$$

we get $z_1 z_{m-1}$ solutions. On the other hand, if

$$\left.\begin{array}{l}(x_1^q - x_1)(x_2^q - x_2) = \lambda,\\(x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = -\lambda\end{array}\right\}$$

for $0 \neq \lambda \in \mathbb{F}_q$, there are

$$(q^4 - z_1)\frac{(q^{4(m-1)} - z_{m-1})}{q-1}$$

solutions. We divide the second factor by $(q-1)$ since each pair (x_1, x_2) satisfying the first equation, fixes the value of λ for the second equation. Overall we have

$$z_m = z_1 z_{m-1} + (q^4 - z_1)\frac{(q^{4(m-1)} - z_{m-1})}{q-1}.$$

Thus, we have obtained a recurrence relation and by substituting z_1 and z_{m-1} in it, we finally obtain $z_m = q^{3m-1}(q^m + q - 1)$. □

The following theorem allows us to count the elements of V satisfying simultaneously a certain quadratic and a certain Hermitean form.

Theorem 4.6.2. *Let V be an vector space over $\mathbb{F}_{q^2}$ of dimension $2m$. Let the* *quadratic form* $Q_m : V \to \mathbb{F}_{q^2}$ *be defined as*

$$Q_m(x) = x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m}, \tag{4.19}$$

where $x = (x_1, ..., x_{2m})$, *and also define the Hermitean form* $H_m : V \to \mathbb{F}_q$ *by*

$$H_m(x) = x_1^q x_2 + x_1 x_2^q + x_3^q x_4 + x_3 x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1} x_{2m}^q. \tag{4.20}$$

Let w_m *be the number of* $x \in V$ *such that*

$$\left.\begin{aligned} Q_m(x) = 0, \\ H_m(x) = 0. \end{aligned}\right\} \tag{4.21}$$

Then

$$w_m = q^{2m} + q^{2m-1}(q^m - 1)(q^{m-2} + 1). \tag{4.22}$$

Proof. We again proceed by induction on m. When $m = 1$, the system (4.21) reduces to

$$\left.\begin{aligned} x_1x_2 = 0, \\ x_1^q x_2 + x_1 x_2^q = 0. \end{aligned}\right\}$$

Note that each pair (x_1, x_2) which satisfies the first equation also satisfies the second one, so in this case the number of solutions is $2q^2 - 1 = q^2 + q(q-1)(q^{-1} + 1)$.

Suppose now that the statement holds for all integers k such that $1 \leqslant k \leqslant m-1$ and consider the following system:

$$\left.\begin{aligned} &x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m} = 0, \\ &x_1^q x_2 + x_1 x_2^q + x_3^q x_4 + x_3 x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1} x_{2m}^q = 0. \end{aligned}\right\}$$

We distinguish two cases.

First, consider the case $x_1 = 0$. Then x_2 can take any of the q^2 possible values and the remaining system is equivalent to

$$\left.\begin{aligned} Q_{m-1}(x) = 0, \\ H_{m-1}(x) = 0, \end{aligned}\right\}$$

so there are $q^2 w_{m-1}$ solutions in this case.

Now suppose that $x_1 \neq 0$. Without loss of generality we may consider the case $x_1 = 1$. The system (4.21) takes the form

$$\left.\begin{array}{l} x_2 = -x_3x_4 - \cdots - x_{2m-1}x_{2m}, \\ x_2 + x_2^q + x_3^qx_4 + x_3x_4^q + \cdots + x_{2m-1}^q x_{2m} + x_{2m-1}x_{2m}^q = 0. \end{array}\right\}$$

We substitute x_2 from the first equation into the second one to obtain

$$(x_3^q - x_3)(x_4^q - x_4) + \cdots + (x_{2m-1}^q - x_{2m-1})(x_{2m}^q - x_{2m}) = 0.$$

Using the result of Lemma 4.6.1, we obtain that in this case there are $(q^2-1)z_{m-1}$ solutions. In total, we obtain the following recurrence relation:

$$w_n = q^2 w_{m-1} + (q^2-1)z_{m-1}.$$

By substituting the appropriate values for w_{m-1} and z_{m-1}, we obtain the result. $\square$

An Albert vector $(0,0,0 \mid 0,0,C) \in \mathbb{J}_8^C$ is white if and only if $C\overline{C} = 0$. Recall that $\mathbb{O}_K$ is split, so we can use our favourite basis $\{ e_i \bigm| i \in \pm I \}$. Note that with respect to this basis $C\overline{C} = 0$ is equivalent to

$$C_{-1}C_1 + C_{\varpi}C_{-\varpi} + C_{\omega}C_{-\omega} + C_{-0}C_0 = 0. \tag{4.23}$$

Next, $(0,0,0 \mid 0,0,C)$ is isotropic if and only if $\mathrm{T}(C\overline{C}^{\sigma}) = 0$, which is equivalent to

$$C_{-1}^qC_1 + C_{-1}C_1^q + C_{\varpi}^qC_{-\varpi} + C_{\varpi}C_{-\varpi}^q + C_{\omega}^qC_{-\omega} + C_{\omega}C_{-\omega}^q + C_{-0}^qC_0 + C_{-0}C_0^q = 0. \tag{4.24}$$

We know that there are exactly $(q^8-1)(q^6+1)$ white vectors in $\mathbb{J}_8^C$. Furthermore, there are

$$w_4 - 1 = (q^2+1)(q^3+1)(q^3(q^2+1)(q-1) + (q^5+1)) \tag{4.25}$$

isotropic white vectors of the form $(0,0,0 \mid 0,0,C)$ and

$$q^6(q^4-1)(q^3-1)(q-1) \tag{4.26}$$

non-isotropic white vectors of the same form.

Now, using Proposition 4.2.3, we find that the full subgroup of $\mathrm{SE}_6(K)$ which preserves Q and H on $\mathbb{J}_8^C$ has three orbits on white points, and the sizes of these orbits are given by

(i) $(q^4-1)(q^3+1)/(q-1)$;

(ii) $q(q^6-1)(q^4-1)(q^2+1)/(q^2-1)$;

(iii) $q^6(q^4-1)(q^3-1)/(q+1)$.

Of these, the first two are isotropic, while the last one is non-isotropic.

4.6.2 White vectors in $\mathbb{J}_{16}^{BC}$

We can also count the isotropic white vectors in $\mathbb{J}_{16}^{BC}$, $K = \mathbb{F}_{q^2}$. Suppose $X = (0,0,0 \mid 0,B,C)$ is white and note that the whiteness conditions take form $B\overline{B} = C\overline{C} = 0 = BC$.

First, we count the white vectors $(0,0,0 \mid 0,B,C)$ such that $B \neq 0$ and $C \neq 0$. We notice that given $B \neq 0$ and $B\overline{B} = 0 = BC$, we automatically have $C\overline{C} = 0$. Indeed, for if $C\overline{C} \neq 0$, C is invertible and $BC = 0$ implies $B = 0$, a contradiction. So, there are $(q^8-1)(q^6+1)$ choices for B and (q^8-1) choices for C (see Lemma 2.5.2). In total, there are $(q^8-1)^2(q^6+1)$ white vectors with $B \neq 0$ and $C \neq 0$.

To count the isotropic white vectors of the form $(0,0,0 \mid 0,B,C)$ we distiguish two cases:

$$\left.\begin{array}{l} B\overline{B} = BC = 0, \\ \mathrm{T}(B\overline{B}^\sigma) = 0, \\ \mathrm{T}(C\overline{C}^\sigma) = 0, \end{array}\right\}, \quad \left.\begin{array}{l} B\overline{B} = BC = 0, \\ \mathrm{T}(B\overline{B}^\sigma) = \lambda \neq 0, \\ \mathrm{T}(C\overline{C}^\sigma) = -\lambda. \end{array}\right\}. \tag{4.27}$$

In the previous section we learned that the subgroup of $\mathrm{SE}_6(K)$, preserving Q and H on $\mathbb{J}_8^C$ has two orbits on isotropic white points. Proposition 4.2.3 the first orbit consists of white points $\langle X \rangle$ where X is written over $\mathbb{F}_q$. That is, its representatives are one-dimensional $\mathbb{F}_{q^2}$-subspaces generated by the white vectors written over $\mathbb{F}_q$. Suppose $X_C = (0,0,0 \mid 0,0,C)$ belongs to the first orbit. By taking a particular candidate for C, it is easy to see that in this case there are q^8-1 choices for B. Now, if X_C belongs to the second orbit, there are $q^4(q^3+q^2-q)-1$ choices for B.

If, on the other hand, X_C is non-isotropic, then there are $q^3(q^4-1)$ choices for B, and overall we have

$$\begin{aligned}(q^4-1)(q^3+1)(q+1)(q^8-1)& \\ &+\ q(q^6-1)(q^4-1)(q^2+1)(q^4(q^3+q^2-q)-1) \\ &\quad+\ q^6(q^4-1)(q^3-1)(q-1)q^3(q^4-1) \\ &=\ (q^{13}+q^{11}-q^{10}+2q^9+q^8+2q^7-q^6+2q^4+1)(q^4-1)^2 \quad (4.28)\end{aligned}$$

isotropic white vectors of the form $(0,0,0 \mid 0,B,C)$ with $B,C\neq 0$.

Recall that the group preserving Q_4 and H_4 has two orbits on the isotropic white points in J_8 with $(q^4-1)(q^3+1)/(q-1)$ and $q(q^6-1)(q^4-1)(q^2+1)/(q^2-1)$ elements. Again, by PropositionThe first orbit consists of white points $\langle X\rangle$ where X is written over $\mathbb{F}_q$. That is, its representatives are one-dimensional $\mathbb{F}_{q^2}$-subspaces generated by the white vectors written over $\mathbb{F}_q$. Since we know the totality of white vectors of this form, we find that there are

$$q^6(q^4-1)^2(q^4+2)(q^3-1)(q-1) \quad (4.29)$$

non-isotropic white vectors in $\mathbb{J}_{16}^{BC}$ with $B,C\neq 0$. Overall there are

$$\begin{aligned}q^6(q^4-1)^2(q^4+2)(q^3-1)(q-1)& \\ &+\ 2q^6(q^4-1)(q^3-1)(q-1) \\ &=\ q^{10}(q^8-1)(q^3-1)(q-1) \quad (4.30)\end{aligned}$$

non-isotropic white vectors in $\mathbb{J}_{16}^{BC}$.

A: Some properties of $\Omega_{2m}(F,Q)$

Let V be a vector space over a field F of dimension n. We assume that there is a non-singular quadratic form Q defined on V. Denote by $\mathrm{GO}_n(F,Q)$ the group of non-singular linear transformations that preserve the form Q. In case of characteristic 2 we define the *quasideterminant* $\mathrm{qdet} : \mathrm{GO}_n(F,Q) \to \mathbb{F}_2$ to be the map

$$\mathrm{qdet} : g \mapsto \dim_F(\mathrm{Im}\,(\mathrm{I}-g)) \mod 2. \tag{31}$$

Further, the group $\mathrm{SO}_n(F,Q)$ is the kernel of the (quasi)determinant map. Define the *spinor norm* to be the homomorphism $\mathrm{sp} : \mathrm{SO}_n(F,Q) \to F^\times/(F^\times)^2$. This homomorphism is defined in the following way. Any element of $\mathrm{SO}_n(F,Q)$ arising as a reflection in v for some $v \in V$, is sent to the value $Q(v)$ modulo $(F^\times)^2$. This extends to a well-defined homomorphism. The subgroup $\Omega_n(F,Q)$ of $\mathrm{SO}_n(F,Q)$ is defined as the kernel of spinor norm. If the characteristic of the field is not 2, then there exists a double cover of $\Omega_n(F,Q)$, denoted as $\mathrm{Spin}_n(F,Q)$.

This section is devoted to some of the private life of the group $\Omega_{2m}(F,Q)$. Consider the vector space V of dimension $2m+2$ over F with a non-singular quadratic form Q defined on it. Let f be a polar form of Q. Assuming that the Witt index of Q is at least 1, we can pick the basis $\mathcal{B} = \{v_1, w_1, ..., w_{2m}, v_2\}$ in V such that (v_1, v_2) is a hyperbolic pair. Consider the decomposition $V = \langle v_1\rangle \oplus \langle w_1, ..., w_{2m}\rangle \oplus \langle v_2\rangle$ and denote $W = \langle w_1, ..., w_{2m}\rangle$. Further, denote by Q_W the restriction of Q on W.

Lemma A.1. *The stabiliser in $\Omega_{2m+2}(F,Q)$ of the vector v_1 is a subgroup of shape $W{:}\Omega_{2m}(F,Q_W)$, and the stabiliser of the pair (v_1,v_2) is a subgroup $\Omega_{2m}(F,Q_W)$.*

Proof. Any element in $\Omega_{2m+2}(F,Q)$ which fixes v_1 also stabilises $\langle v_1\rangle^\perp$, so it has the following form:

$$\widehat{A} = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline u_2^\top & A & 0 \\ \hline \mu & u_1 & \lambda \end{array}\right),$$

where the matrix A acts on the $2m$-dimensional subspace, spanned by $\{w_1, ..., w_{2m}\}$. Suppose $u_1 = (\nu_1, ..., \nu_{2m})$, so such an element acts on v_1 as

$$v_1 \quad \mapsto \quad \mu v_0 + \sum_{i=1}^{2m} \nu_i w_i + \lambda v_1,$$

but since the bilinear form f is preserved we get

$$1 = f(v_0, v_1) = f(v_0, \mu v_0) + \sum_{i=1}^{2m} f(v_0, \nu_i w_i) + \lambda f(v_0, v_1) = \lambda,$$

i.e. $\lambda = 1$. Consider the decomposition $F^{2m+2} = \langle v_0 \rangle \oplus \langle w_1, ..., w_{2m} \rangle \oplus \langle v_1 \rangle$ and denote by Q and β the quadratic and bilinear forms on the subspace $F^{2m} = \langle w_1, ..., w_{2m} \rangle$ obtained as the restriction of $\widehat{Q}$ and f respectively.

Since (v_0, v_1) is a hyperbolic pair, the form f on F^{2m+2} can be represented by the Gram matrix

$$[f]_{\mathcal{B}} = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right),$$

where B is the matrix of β with respect to the basis $\{w_1, ..., w_{2m}\}$. We explore the fact that an element in the stabiliser of v_0 preserves the form f:

$$\left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right) =$$

$$= \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline u_2^\top & A & 0 \\ \hline \mu & u_1 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & B & 0 \\ \hline 1 & 0 & 0 \end{array}\right) \left(\begin{array}{c|c|c} 1 & u_2 & \mu \\ \hline 0 & A^\top & u_1^\top \\ \hline 0 & 0 & 1 \end{array}\right) =$$

$$= \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & ABA^\top & ABu_1^\top + u_2^\top \\ \hline 1 & u_2 + u_1BA^\top & 2\mu + u_1Bu_1^\top \end{array}\right),$$

so we notice that $ABA^\top = B$. Furthermore, since $(0 \mid v \mid 0)\widehat{A} = (0 \mid vA \mid 0)$, where $v \in F^{2m}$, we obtain

$$Q(vA) = \widehat{Q}((0 \mid vA \mid 0)) = \widehat{Q}((0 \mid v \mid 0)) = Q(v),$$

so A is an element of $\mathrm{GO}_{2m}(F, Q)$. Additionally, $u_2 = -u_1BA^\top$ and we see that u_2 is uniquely determined by u_1. From the bottom right corner of the resulting matrix we obtain $\mu = -Q(u_1)$ in odd characteristic. In case of characteristic 2 we can explore the quadratic form again:

$$\begin{aligned} 0 = \widehat{Q}(v_1) = \widehat{Q}(v_1\widehat{A}) = \widehat{Q}((\mu \mid u_1 \mid 1)) = \\ = \widehat{Q}((\mu \mid u_1 \mid 0)) + \widehat{Q}(v_1) + f((\mu \mid u_1 \mid 0), v_1) = Q(u_1) + \mu. \end{aligned}$$

Consider the decomposition

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -ABu_1^\top & A & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right) = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu_1^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right),$$

The matrices of the form

$$C_{u_1} = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu_1^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u_1) & u_1 & 1 \end{array}\right)$$

generate an elementary abelian group F^{2m}. Indeed, since the product of two such matrices is given by

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bu^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u) & u & 1 \end{array}\right) \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -Bv^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(v) & v & 1 \end{array}\right) =$$

$$= \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -B(u+v)^\top & \mathrm{I}_{2m} & 0 \\ \hline -Q(u+v) & u+v & 1 \end{array}\right),$$

we see that the set of these matrices is closed under multiplication and moreover any two such matrices commute.

To show that the matrix A is an element of $\Omega_{2m}(F, Q)$, we use Proposition 1.6.11 from [?] to calculate the spinor norm and, in case of characteristic 2, the quasideterminant of the matrices C_{u_1}. Note that $\det(C_{u_1}) = \det(\widehat{A}) = 1$. Consider the matrix

$$\mathrm{I} - C_{u_1} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline Bu_1^\top & 0 & 0 \\ \hline Q(u_1) & -u_1 & 0 \end{array}\right).$$

For a vector v we denote by $[v]_i$ its i-th component. Now, if $u_1 = 0$, then $\mathrm{I} - C_{u_1}$ has rank 0, whereas if $u_1 \neq 0$, then there is an index i such that $[Bu_1^\top]_i \neq 0$ and it follows that the rank of $\mathrm{I} - C_{u_1}$ in this case is 2. Consequently, $k = \mathrm{rank}(\mathrm{I} - C_{u_1})$ is even, and so by the Proposition 1.6.11 in [?] the quasideterminant of C_{u_1} is 1. Further, if D is a $k \times (2m+2)$ matrix whose rows are the basis elements of a complement of the nullspace of $\mathrm{I} - C_{u_1}$, then the spinor norm of C_{u_1} is 1 if $\det(D(\mathrm{I} - C_{u_1})[f]_{\mathcal{B}} D^\top)$ is a square in F. If $u_1 \neq 0$, then the complement of the nullspace of $I - C_{u_1}$ has the basis $\{w_i, v_1\}$, where the index i is such that $[Bu_1^\top]_i \neq 0$. The matrix D has the following form:

$$D = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where 1 in the first row is in the $(i+1)$-st position. We calculate

$$D(\mathrm{I} - C_{u_1}) = \left(\begin{array}{c|c|c} \alpha & 0 & 0 \\ \hline Q(u_1) & -u_1 & 0 \end{array}\right), \quad [f]_{\mathcal{B}} D^\top = \begin{pmatrix} 0 & 1 \\ B_{1,i} & 0 \\ \vdots & \vdots \\ B_{2m,i} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\alpha = [Bu_1^\top]_i = [u_1 B]_i$. Finally,

$$D(\mathrm{I} - C_{u_1})[f]_{\mathcal{B}} D^\top = \begin{pmatrix} 0 & \alpha \\ -\alpha & Q(u_1) \end{pmatrix},$$

so $\det(D(\mathrm{I} - C_{u_1})FD^\top) = \alpha^2$ as needed. Since the quasideterminant and the spinor norm are multiplicative (Theorems 11.43 and 11.50 in [?]), and $\widehat{A} \in \Omega_{2m+2}(F, \widehat{Q})$, we conclude that A is an element of $\Omega_{2m}(F, Q)$ and it follows that the stabiliser of v_0 in $\Omega_{2m}(F, \widehat{Q})$ is indeed a subgroup of shape $W{:}\Omega_{2m}(F, Q)$.

Lastly, if we stabilise v_0 and v_1 simultaneously, a general element in the stabiliser takes the form

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline & & \\ 0 & A & 0 \\ & & \\ \hline 0 & 0 & 1 \end{array}\right),$$

so the stabiliser of (v_0, v_1) is $\Omega_{2m}(F, Q)$. □

B: $\Omega_4^+(F) \cong \mathrm{SL}_2(F) \circ \mathrm{SL}_2(F)$

C: $\Omega_4^{-,K}(F) \cong \mathrm{PSL}_2(K)$

D: Magma code