STAT400 Final Exam Review

1.

$$\begin{aligned} P(|X-15|>3) &= 1-P(|X-15|\le 3) \\ &= 1-P(-3\le X-15\le 3) \\ &= 1-P(12\le X\le 18) \\ &= 1-(P(X\le 18)-P(X<12)) \\ &= 1-(F_X(18)-F_X(12)) \\ &= 1-F_X(18)+F_X(12) \end{aligned}$$

2. todo

3. todo

4. todo

5. todo

6. todo

7. todo

8. todo

8.a. todo

$$E(XY) =$$

8.b. todo

9. todo

10. todo

11. todo

12. todo

… todos

42. todo

42.a. (checked)

$$
\begin{aligned}
E(X) &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\frac{\hat{\alpha}}{\hat{\alpha}+1} &= \overline{X} \\
\hat{\alpha} &= \hat{\alpha}\overline{X} + \overline{X} \\
\hat{\alpha}\overline{X} - \hat{\alpha} &= -\overline{X} \\
\hat{\alpha} &= \frac{-\overline{X}}{\overline{X}-1}
\end{aligned}
$$

https://math.stackexchange.com/q/3185959/774737 has the same answer, so it's probably correct

42.b. (checked)

$$
\begin{aligned}
\ell(\alpha) &= \ln \prod_{i=1}^{n} f(x_i;\alpha) \\
&= \ln \prod_{i=1}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} \cdot x_i^{\alpha-1}(1-x_i)^{1-1} \\
&= \ln \prod_{i=1}^{n} \alpha \cdot x_i^{\alpha-1} \\
&= \sum_{i=1}^{n} \ln\alpha + \ln\left(x_i^{\alpha-1}\right) \\
&= \sum_{i=1}^{n} \ln\alpha + (\alpha-1)\ln x_i
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\ell(\alpha) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\sum_{i=1}^{n}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\alpha}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i \frac{\mathrm{d}}{\mathrm{d}\alpha}(\alpha-1)\right) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i\right) \\
0 &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i \\
-\sum_{i=1}^{n}\ln x_i &= \frac{n}{\alpha} \\
-\alpha\sum_{i=1}^{n}\ln x_i &= n \\
\alpha &= \frac{-n}{\sum_{i=1}^{n}\ln x_i}
\end{aligned}
$$

$$\hat{\alpha}_{\text{MLE}} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

(this is the same MLE as https://math.stackexchange.com/q/4057637/774737, so it's probably correct)

Verify that this is a local maximum:

$$\ell''(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i\right) = \frac{-n}{a^2}$$

$\ell''(\alpha)$ must always be negative, because n is always positive and α^2 is always positive. Therefore, $\hat{\alpha}_{\text{MLE}}$ is a maximum likelihood estimator for α.