STAT400 Final Exam Review

TODO get familiar with the different distributions

Which distributions can be used to approximate which distributions?
- Poisson can approximate binomial
- Binom can approximate hypergeometric

TODO why is this true? If $X\sim$ Pois$(\lambda = 7)$, then $P(-10 \leq X \leq 1) = e^{-7} + 7e^{-7}$

1.

$$\begin{aligned} P(|X-15| > 3) &= 1 - P(|X-15| \leq 3) \\ &= 1 - P(-3 \leq X - 15 \leq 3) \\ &= 1 - P(12 \leq X \leq 18) \\ &= 1 - (P(X \leq 18) - P(X < 12)) \\ &= 1 - (F_X(18) - F_X(12)) \\ &= 1 - F_X(18) + F_X(12) \end{aligned}$$

2. todo

3. todo

4. todo

5.

DO THIS WHEN YOU GET HOME

5.a. todo

5.b. todo

6. todo

7. todo

7.c. todo

8. todo

TODO FIGURE THIS ONE OUT!!!

8.a. todo

$$E(XY) =$$

8.b. todo

9. todo

This one was in the homework so you can check

9.a. (done, not checked)

$$\begin{aligned} P(X = 4, Y = 2) &= P(Y = 2 \mid X = 4)P(X = 4) \\ &= p_{\text{Binom}}(2; p = 0.7, n = 4) \cdot 0.2 \\ &= \binom{4}{2} 0.7^2 \cdot 0.3^2 \cdot 0.2 \\ &= 0.05292 \end{aligned}$$

9.b. todo

$$P(X = Y) = \sum_{x=0}^{4} \binom{4}{x} 0.7^x \cdot 0.3^{4-x} \cdot P(X = x)$$

Expand manually I guess?

9.c. todo

Joint pmf is a table of $P(X = x, Y = y)$

Marginal pmf of Y is just $P(Y = y)$

10…11 todos

12. todo

$$\begin{aligned} P(X = 6 \mid Z = 20) &= \frac{P(X = 6 \cap Z = 20)}{P(Z = 20)} \\ &= \frac{P(X = 6 \cap Y = 14)}{P(X + Y = 20)} \end{aligned}$$

13. todo

13.a. (done, checked)

Helper:

$$\int x e^{bx} \, \mathrm{d}x = \frac{x e^{bx}}{b} + C - \int \frac{e^{bx}}{b} \, \mathrm{d}x = \frac{x e^{bx}}{b} - \frac{e^{bx}}{b^2} + C$$

$$
\begin{aligned}
P(3X + Y \leq 3) &= \int_0^1 \int_0^{3-3x} p_X(x) p_Y(y) \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 \int_0^{3-3x} 5e^{-5x} \cdot 3e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \int_0^{3-3x} e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-3y}}{-3} \right)_0^{3-3x} \mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-9+9x}}{-3} - \frac{1}{-3} \right) \mathrm{d}x \\
&= 5 \int_0^1 e^{-5x} (-e^{9x-9} + 1) \,\mathrm{d}x \\
&= 5 \int_0^1 -e^{4x-9} \,\mathrm{d}x + 5 \int_0^1 e^{-5x} \,\mathrm{d}x \\
&= 5e^{-9} \int_0^1 -e^{4x} \,\mathrm{d}x + 5 \left(\frac{e^{-5x}}{-5} \right)_0^1 \\
&= -5e^{-9} \left(\frac{e^{4x}}{4} \right)_0^1 - \left(e^{-5x} \right)_0^1 \\
&= -5e^{-9} \left(\frac{e^4}{4} - \frac{1}{4} \right) - \left(e^{-5} - 1 \right) \\
&= \frac{-5e^{-5}}{4} + \frac{5e^{-9}}{4} - e^{-5} + 1 \\
&= \frac{-9e^{-5}}{4} + \frac{5e^{-9}}{4} + 1
\end{aligned}
$$

Verified with Symbolab

13.b. todo

$$
P(X \geq Y) = \int_0^\infty \int_y^\infty p_X(x) \cdot
$$

todo incomplete

13.c. todo

14. todo

15.

15.a. (done, not checked)

Number of elements:

If has 1 green ball: $2 \cdot 2 \cdot 3$

If no green balls: $2 \cdot 2 \cdot 3$

Total: $2 \cdot (2 \cdot 2 \cdot 3) = 24$

15.b. (done)

Drawn on paper (page 146, outside notebook)

15.c. (done, checked)

Done on paper, more or less (page 147, outside notebook)

15.d. (done, checked)

This is the hypergeometric distribution with $N = 8$, $M = 5$, and $n = 3$

16. todo

17.

17.a. (done, not checked)

$P(F \cap H) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(U \cap H) = \frac{2}{3} \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid H) &= \frac{P(F \cap H)}{P(H)} \\ &= \frac{P(F \cap H)}{P(F \cap H) + P(U \cap H)} \\ &= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{2}{3}} \\ &= \frac{1}{5} \end{aligned}$$

17.b. (done, not checked)

$P(F \cap \mathrm{HH}) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$

$P(U \cap \mathrm{HH}) = \frac{2}{3} \cdot 1 \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid \mathrm{HH}) &= \frac{P(F \cap \mathrm{HH})}{P(\mathrm{HH})} \\ &= \frac{P(F \cap \mathrm{HH})}{P(F \cap \mathrm{HH}) + P(U \cap \mathrm{HH})} \\ &= \frac{\frac{1}{12}}{\frac{1}{12} + \frac{2}{3}} \\ &= \frac{1}{9} \end{aligned}$$

17.c. (done)

0, because the unfair coins will never show tails.

18…21 todos

22. todo

$$
\begin{aligned}
E\left(\overline{X}^2\right) &= E\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right) \\
&= E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right)
\end{aligned}
$$

I have no idea how to solve this

23…25 todos

26. todo

26.a. (done, not checked)

$$
\begin{aligned}
E\left(\hat{\theta}_i\right) &= E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right) \\
&= \frac{1}{i}E\left(\sum_{j=1}^{i} X_j\right) \\
&= \frac{1}{i}(i\mu) \\
&= \mu
\end{aligned}
$$

26.b. todo

$$
\begin{aligned}
V\left(\hat{\theta}_i\right) &= E\left(\hat{\theta}_i^2\right) - E\left(\hat{\theta}_i\right)^2 \\
&= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2 \\
&= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2
\end{aligned}
$$

todo i have no idea how to solve this

27. todo

28. (done, not checked)

$E(X) = 10, V(X) = 20$

We know that $\overline{X} \sim$ Normal$\left(\mu = 10, \sigma^2 = \frac{\sqrt{20}}{7}\right)$ thanks to the Central Limit Theorem.

In terms of the standard normal distribution, this is $\overline{X} = \frac{\sqrt{20}}{7}Z + 10$

So $\left(\overline{X} - 10\right) \cdot \frac{49}{\sqrt{20}} = Z$

$$
\begin{aligned}
P\left(|\overline{X}-8|>3\right) &= 1-P\left(|\overline{X}-8|\le 3\right) \\
&= 1-P\left(-3\le \overline{X}-8\le 3\right) \\
&= 1-P\left(5\le \overline{X}\le 11\right) \\
&= 1-\left(P\left(\overline{X}\le 11\right)-P\left(\overline{X}<5\right)\right) \\
&= 1-P\left(\overline{X}\le 11\right)+P\left(\overline{X}<5\right) \\
&= 1-\Phi\left((11-10)\cdot\frac{7}{\sqrt{20}}\right)+\Phi\left((5-10)\cdot\frac{7}{\sqrt{20}}\right) \\
&= 1-\Phi\left(\frac{7}{\sqrt{20}}\right)+\Phi\left(\frac{-35}{\sqrt{20}}\right)
\end{aligned}
$$

29…30 todos

31. (simple but too much effort, not worth solving)

32. (done on paper, not checked)

Bias: $\frac{\sigma^2}{n}$

33. todo

The probabilities must sum to 1, so:

$$
\begin{aligned}
& 1=\theta(1-\theta)+(1-\theta)+\theta^2 \\
\Rightarrow\ & 1=\theta-\theta^2+1-\theta+\theta^2 \\
\Rightarrow\ & 1=1
\end{aligned}
$$

θ is not constrained by that equation, so it can be any real number.

$$E(X)=2\theta(1-\theta)+4(1-\theta)+6\theta^2=4\theta^2-2\theta+4$$

33.a. todo

Method of moments:

$4\theta^2-2\theta+4=\frac{22}{7}$, use the quadratic formula to solve.

Maximum likelihood:

$$\ell(\theta)=\ln\left((\theta(1-\theta))^4\cdot(1-\theta)^2\cdot\theta^2\right)=4\ln(\theta(1-\theta))+2\ln(1-\theta)+\ln(\theta)$$

$$
\begin{aligned}
0 &= \ell'(\theta) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\theta}(4\ln(\theta(1-\theta)) + 2\ln(1-\theta) + \ln(\theta)) \\
0 &= \frac{4(1-2\theta)}{\theta(1-\theta)} + \frac{-2}{1-\theta} + \frac{1}{\theta} \\
0 &= \frac{(4(1-2\theta)) + (-2\theta) + (1-\theta)}{\theta(1-\theta)} \\
0 &= \frac{5-11\theta}{\theta(1-\theta)} \\
\theta - \theta^2 &= 5 - 11\theta
\end{aligned}
$$

Use quadratic formula to solve again.

33.b.

Method of moments: Same

Maximum likelihood: Same

33.c.

Method of moments: Same

Maximum likelihood: Same

33.d. todo

Method of moments: Same

Maximum likelihood: Same

Since $x = 3$ isn't included in the probability distribution table, our assumed distribution is probably not very good.

34. (done on paper, checked)

35. (done on paper, checked)

36…37 todos

These are just MoM and MLE

38. (done on paper, checked)

39…41 todos

These are all just MoM and MLEs

42.

42.a. (checked)

$$
\begin{aligned}
E(X) &= \frac{1}{n} \sum_{i=1}^{n} X_i \\
\frac{\hat{\alpha}}{\hat{\alpha}+1} &= \overline{X} \\
\hat{\alpha} &= \hat{\alpha}\overline{X} + \overline{X} \\
\hat{\alpha}\overline{X} - \hat{\alpha} &= -\overline{X} \\
\hat{\alpha} &= \frac{-\overline{X}}{\overline{X}-1}
\end{aligned}
$$

https://math.stackexchange.com/q/3185959/774737 has the same answer, so it's probably correct

42.b. (checked)

$$
\begin{aligned}
\ell(\alpha) &= \ln \prod_{i=1}^{n} f(x_i;\alpha) \\
&= \ln \prod_{i=1}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} \cdot x_i^{\alpha-1} (1-x_i)^{1-1} \\
&= \ln \prod_{i=1}^{n} \alpha \cdot x_i^{\alpha-1} \\
&= \sum_{i=1}^{n} \ln \alpha + \ln\left(x_i^{\alpha-1}\right) \\
&= \sum_{i=1}^{n} \ln \alpha + (\alpha-1) \ln x_i
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \ell(\alpha) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \sum_{i=1}^{n} (\ln \alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}\alpha} (\ln \alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \ln x_i \frac{\mathrm{d}}{\mathrm{d}\alpha} (\alpha-1) \right) \\
0 &= \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \ln x_i \right) \\
0 &= \frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i \\
-\sum_{i=1}^{n} \ln x_i &= \frac{n}{\alpha} \\
-\alpha \sum_{i=1}^{n} \ln x_i &= n \\
\alpha &= \frac{-n}{\sum_{i=1}^{n} \ln x_i}
\end{aligned}
$$

$$\hat{\alpha}_{\text{MLE}} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

(this is the same MLE as https://math.stackexchange.com/q/4057637/774737, so it's probably correct)

Verify that this is a local maximum:

$$\ell''(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i\right) = \frac{-n}{a^2}$$

$\ell''(\alpha)$ must always be negative, because n is always positive and α^2 is always positive. Therefore, $\hat{\alpha}_{\text{MLE}}$ is a maximum likelihood estimator for α.