STAT400 Final Exam Review

1.

$$\begin{aligned}
P(|X - 15| > 3) &= 1 - P(|X - 15| \leq 3) \\
&= 1 - P(-3 \leq X - 15 \leq 3) \\
&= 1 - P(12 \leq X \leq 18) \\
&= 1 - (P(X \leq 18) - P(X < 12)) \\
&= 1 - (F_X(18) - F_X(12)) \\
&= 1 - F_X(18) + F_X(12)
\end{aligned}$$

2. todo

3. todo

4. todo

5. todo

6. todo

7. todo

8. todo

8.a. todo

$$E(XY) =$$

8.b. todo

9…12 todos

13. todo

13.a. (done, checked)

Helper:

$$\int x e^{bx} \, \mathrm{d}x = \frac{x e^{bx}}{b} + C - \int \frac{e^{bx}}{b} \, \mathrm{d}x = \frac{x e^{bx}}{b} - \frac{e^{bx}}{b^2} + C$$

$$
\begin{aligned}
P(3X+Y\leq 3) &= \int_0^1 \int_0^{3-3x} p_X(x) p_Y(y) \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 \int_0^{3-3x} 5e^{-5x} \cdot 3e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \int_0^{3-3x} e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-3y}}{-3}\right)_0^{3-3x} \mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-9+9x}}{-3} - \frac{1}{-3}\right) \mathrm{d}x \\
&= 5 \int_0^1 e^{-5x} (-e^{9x-9} + 1) \,\mathrm{d}x \\
&= 5 \int_0^1 -e^{4x-9} \,\mathrm{d}x + 5 \int_0^1 e^{-5x} \,\mathrm{d}x \\
&= 5e^{-9} \int_0^1 -e^{4x} \,\mathrm{d}x + 5 \left(\frac{e^{-5x}}{-5}\right)_0^1 \\
&= -5e^{-9} \left(\frac{e^{4x}}{4}\right)_0^1 - \left(e^{-5x}\right)_0^1 \\
&= -5e^{-9} \left(\frac{e^4}{4} - \frac{1}{4}\right) - \left(e^{-5} - 1\right) \\
&= \frac{-5e^{-5}}{4} + \frac{5e^{-9}}{4} - e^{-5} + 1 \\
&= \frac{-9e^{-5}}{4} + \frac{5e^{-9}}{4} + 1
\end{aligned}
$$

Verified with Symbolab

13.b. todo

$$
P(X \geq Y) = \int_0^\infty \int_y^\infty p_X(x) \cdot
$$

todo incomplete

13.c. todo

14. todo

15.

15.a. (done, not checked)

Number of elements:

If has 1 green ball: $2 \cdot 2 \cdot 3$

If no green balls: $2 \cdot 2 \cdot 3$

Total: $2 \cdot (2 \cdot 2 \cdot 3) = 24$

15.b. (done)
Drawn on paper (page 146, outside notebook)

15.c. (done, checked)
Done on paper, more or less (page 147, outside notebook)

15.d. (done, checked)
This is the hypergeometric distribution with $N = 8$, $M = 5$, and $n = 3$

16. todo

17. todo

17.a. (done, not checked)
$P(F \cap H) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(U \cap H) = \frac{2}{3} \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid H) &= \frac{P(F \cap H)}{P(H)} \\ &= \frac{P(F \cap H)}{P(F \cap H) + P(U \cap H)} \\ &= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{2}{3}} \\ &= \frac{1}{5} \end{aligned}$$

17.b. (done, not checked)
$P(F \cap \mathrm{HH}) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$

$P(U \cap \mathrm{HH}) = \frac{2}{3} \cdot 1 \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid \mathrm{HH}) &= \frac{P(F \cap \mathrm{HH})}{P(\mathrm{HH})} \\ &= \frac{P(F \cap \mathrm{HH})}{P(F \cap \mathrm{HH}) + P(U \cap \mathrm{HH})} \\ &= \frac{\frac{1}{12}}{\frac{1}{12} + \frac{2}{3}} \\ &= \frac{1}{9} \end{aligned}$$

17.c. (done)
0, because the unfair coins will never show tails.

18…21 todos

22. todo

$$E\left(\overline{X}^2\right) = E\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right)$$
$$= E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right)$$

I have no idea how to solve this

23…25 todos

26. todo

26.a. (done, not checked)

$$E\left(\hat{\theta}_i\right) = E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)$$
$$= \frac{1}{i}E\left(\sum_{j=1}^{i} X_j\right)$$
$$= \frac{1}{i}(i\mu)$$
$$= \mu$$

26.b. todo

$$V\left(\hat{\theta}_i\right) = E\left(\hat{\theta}_i^2\right) - E\left(\hat{\theta}_i\right)^2$$
$$= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2$$
$$= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2$$

todo i have no idea how to solve this

27…32 todos

33. todo

The probabilities must sum to 1, so:

$$1 = \theta(1-\theta) + (1-\theta) + \theta^2$$
$$\Rightarrow 1 = \theta - \theta^2 + 1 - \theta + \theta^2$$
$$\Rightarrow 1 = 1$$

θ is not constrained by that equation, so it can be any real number.

$$E(X) = 2\theta(1-\theta) + 4(1-\theta) + 6\theta^2 = 4\theta^2 - 2\theta + 4$$

33.a. todo

Method of moments:

$4\theta^2 - 2\theta + 4 = \frac{22}{7}$, use the quadratic formula to solve.

Maximum likelihood:

$$\ell(\theta) = \ln\left((\theta(1-\theta))^4 \cdot (1-\theta)^2 \cdot \theta^2\right) = 4\ln(\theta(1-\theta)) + 2\ln(1-\theta) + \ln(\theta)$$

$$\begin{aligned}
0 &= \ell'(\theta) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\theta}(4\ln(\theta(1-\theta)) + 2\ln(1-\theta) + \ln(\theta)) \\
0 &= \frac{4(1-2\theta)}{\theta(1-\theta)} + \frac{-2}{1-\theta} + \frac{1}{\theta} \\
0 &= \frac{(4(1-2\theta)) + (-2\theta) + (1-\theta)}{\theta(1-\theta)} \\
0 &= \frac{5-11\theta}{\theta(1-\theta)} \\
\theta - \theta^2 &= 5 - 11\theta
\end{aligned}$$

Use quadratic formula to solve again.

33.b.

Method of moments: Same

Maximum likelihood: Same

33.c.

Method of moments: Same

Maximum likelihood: Same

33.d. todo

Method of moments: Same

Maximum likelihood: Same

Since $x = 3$ isn't included in the probability distribution table, our assumed distribution is probably not very good.

34…41 todos

42.

42.a. (checked)

$$
\begin{aligned}
E(X) &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\frac{\hat{\alpha}}{\hat{\alpha}+1} &= \overline{X} \\
\hat{\alpha} &= \hat{\alpha}\overline{X} + \overline{X} \\
\hat{\alpha}\overline{X} - \hat{\alpha} &= -\overline{X} \\
\hat{\alpha} &= \frac{-\overline{X}}{\overline{X}-1}
\end{aligned}
$$

https://math.stackexchange.com/q/3185959/774737 has the same answer, so it's probably correct

42.b. (checked)

$$
\begin{aligned}
\ell(\alpha) &= \ln \prod_{i=1}^{n} f(x_i;\alpha) \\
&= \ln \prod_{i=1}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} \cdot x_i^{\alpha-1}(1-x_i)^{1-1} \\
&= \ln \prod_{i=1}^{n} \alpha \cdot x_i^{\alpha-1} \\
&= \sum_{i=1}^{n} \ln\alpha + \ln\left(x_i^{\alpha-1}\right) \\
&= \sum_{i=1}^{n} \ln\alpha + (\alpha-1)\ln x_i
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\ell(\alpha) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\sum_{i=1}^{n}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\alpha}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i \frac{\mathrm{d}}{\mathrm{d}\alpha}(\alpha-1)\right) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i\right) \\
0 &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i \\
-\sum_{i=1}^{n}\ln x_i &= \frac{n}{\alpha} \\
-\alpha\sum_{i=1}^{n}\ln x_i &= n \\
\alpha &= \frac{-n}{\sum_{i=1}^{n}\ln x_i}
\end{aligned}
$$

$$\hat{\alpha}_{\text{MLE}} = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

(this is the same MLE as https://math.stackexchange.com/q/4057637/774737, so it's probably correct)

Verify that this is a local maximum:

$$\ell''(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i\right) = \frac{-n}{a^2}$$

$\ell''(\alpha)$ must always be negative, because n is always positive and α^2 is always positive. Therefore, $\hat{\alpha}_{\text{MLE}}$ is a maximum likelihood estimator for α.