STAT400 Final Exam Review

1.

$$\begin{aligned}
P(|X-15|>3) &= 1-P(|X-15|\leq 3) \\
&= 1-P(-3\leq X-15\leq 3) \\
&= 1-P(12\leq X\leq 18) \\
&= 1-(P(X\leq 18)-P(X<12)) \\
&= 1-(F_X(18)-F_X(12)) \\
&= 1-F_X(18)+F_X(12)
\end{aligned}$$

2. todo

3. todo

4. todo

5.

DO THIS WHEN YOU GET HOME

5.a. todo

5.b. todo

6. todo

7. todo

8. todo

8.a. todo

$$E(XY) =$$

8.b. todo

9. todo

This one was in the homework so you can check

9.a. (done, not checked)

$$\begin{aligned}
P(X=4,Y=2) &= P(Y=2\mid X=4)P(X=4) \\
&= p_{\text{Binom}}(2;p=0.7,n=4)\cdot 0.2 \\
&= \binom{4}{2}0.7^2\cdot 0.3^2\cdot 0.2 \\
&= 0.05292
\end{aligned}$$

9.b. todo

$$P(X = Y) = \sum_{x=0}^{4} \binom{4}{x} 0.7^x \cdot 0.3^{4-x} \cdot P(X = x)$$

Expand manually I guess?

9.c. todo

Joint pmf is a table of $P(X = x, Y = y)$

Marginal pmf of Y is just $P(Y = y)$

10…11 todos

12. todo

$$\begin{aligned} P(X = 6 \mid Z = 20) &= \frac{P(X = 6 \cap Z = 20)}{P(Z = 20)} \\ &= \frac{P(X = 6 \cap Y = 14)}{P(X + Y = 20)} \end{aligned}$$

13. todo

13.a. (done, checked)

Helper:

$$\int x e^{bx} \, \mathrm{d}x = \frac{x e^{bx}}{b} + C - \int \frac{e^{bx}}{b} \, \mathrm{d}x = \frac{x e^{bx}}{b} - \frac{e^{bx}}{b^2} + C$$

$$
\begin{aligned}
P(3X+Y \leq 3) &= \int_0^1 \int_0^{3-3x} p_X(x) p_Y(y) \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_0^1 \int_0^{3-3x} 5e^{-5x} \cdot 3e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \int_0^{3-3x} e^{-3y} \,\mathrm{d}y \,\mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-3y}}{-3}\right)_0^{3-3x} \mathrm{d}x \\
&= 15 \int_0^1 e^{-5x} \left(\frac{e^{-9+9x}}{-3} - \frac{1}{-3}\right) \mathrm{d}x \\
&= 5 \int_0^1 e^{-5x} (-e^{9x-9} + 1) \,\mathrm{d}x \\
&= 5 \int_0^1 -e^{4x-9} \,\mathrm{d}x + 5 \int_0^1 e^{-5x} \,\mathrm{d}x \\
&= 5e^{-9} \int_0^1 -e^{4x} \,\mathrm{d}x + 5 \left(\frac{e^{-5x}}{-5}\right)_0^1 \\
&= -5e^{-9} \left(\frac{e^{4x}}{4}\right)_0^1 - \left(e^{-5x}\right)_0^1 \\
&= -5e^{-9} \left(\frac{e^4}{4} - \frac{1}{4}\right) - \left(e^{-5} - 1\right) \\
&= \frac{-5e^{-5}}{4} + \frac{5e^{-9}}{4} - e^{-5} + 1 \\
&= \frac{-9e^{-5}}{4} + \frac{5e^{-9}}{4} + 1
\end{aligned}
$$

Verified with Symbolab

13.b. todo

$$
P(X \geq Y) = \int_0^\infty \int_y^\infty p_X(x) \cdot
$$

todo incomplete

13.c. todo

14. todo

15.

15.a. (done, not checked)

Number of elements:

If has 1 green ball: $2 \cdot 2 \cdot 3$

If no green balls: $2 \cdot 2 \cdot 3$

Total: $2 \cdot (2 \cdot 2 \cdot 3) = 24$

15.b. (done)

Drawn on paper (page 146, outside notebook)

15.c. (done, checked)

Done on paper, more or less (page 147, outside notebook)

15.d. (done, checked)

This is the hypergeometric distribution with $N = 8$, $M = 5$, and $n = 3$

16. todo

17.

17.a. (done, not checked)

$P(F \cap H) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(U \cap H) = \frac{2}{3} \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid H) &= \frac{P(F \cap H)}{P(H)} \\ &= \frac{P(F \cap H)}{P(F \cap H) + P(U \cap H)} \\ &= \frac{\frac{1}{6}}{\frac{1}{6} + \frac{2}{3}} \\ &= \frac{1}{5} \end{aligned}$$

17.b. (done, not checked)

$P(F \cap \mathrm{HH}) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$

$P(U \cap \mathrm{HH}) = \frac{2}{3} \cdot 1 \cdot 1 = \frac{2}{3}$

$$\begin{aligned} P(F \mid \mathrm{HH}) &= \frac{P(F \cap \mathrm{HH})}{P(\mathrm{HH})} \\ &= \frac{P(F \cap \mathrm{HH})}{P(F \cap \mathrm{HH}) + P(U \cap \mathrm{HH})} \\ &= \frac{\frac{1}{12}}{\frac{1}{12} + \frac{2}{3}} \\ &= \frac{1}{9} \end{aligned}$$

17.c. (done)

0, because the unfair coins will never show tails.

18…21 todos

22. todo

$$
\begin{aligned}
E\left(\overline{X}^2\right) &= E\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right) \\
&= E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right)
\end{aligned}
$$

I have no idea how to solve this

23…25 todos

26. todo

26.a. (done, not checked)

$$
\begin{aligned}
E\left(\hat{\theta}_i\right) &= E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right) \\
&= \frac{1}{i} E\left(\sum_{j=1}^{i} X_j\right) \\
&= \frac{1}{i}(i\mu) \\
&= \mu
\end{aligned}
$$

26.b. todo

$$
\begin{aligned}
V\left(\hat{\theta}_i\right) &= E\left(\hat{\theta}_i^2\right) - E\left(\hat{\theta}_i\right)^2 \\
&= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2 \\
&= E\left(\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2\right) - E\left(\frac{1}{i}\sum_{j=1}^{i} X_j\right)^2
\end{aligned}
$$

todo i have no idea how to solve this

27. todo

28. (done, not checked)

$E(X) = 10, V(X) = 20$

We know that $\overline{X}\sim$ Normal$\left(\mu = 10, \sigma^2 = \frac{20}{49}\right)$ thanks to the Central Limit Theorem.

In terms of the standard normal distribution, this is $\overline{X} = \frac{20}{49}Z + 10$

So $\left(\overline{X} - 10\right) \cdot \frac{49}{20} = Z$

$$
\begin{aligned}
P\left(|\overline{X}-8|>3\right) &= 1-P\left(|\overline{X}-8|\le 3\right)\\
&= 1-P\left(-3\le \overline{X}-8\le 3\right)\\
&= 1-P\left(5\le \overline{X}\le 11\right)\\
&= 1-\left(P\left(\overline{X}\le 11\right)-P\left(\overline{X}<5\right)\right)\\
&= 1-P\left(\overline{X}\le 11\right)+P\left(\overline{X}<5\right)\\
&= 1-\Phi\left((11-10)\cdot\frac{49}{20}\right)+\Phi\left((5-10)\cdot\frac{49}{20}\right)\\
&= 1-\Phi\left(\frac{49}{20}\right)+\Phi\left(\frac{-49}{4}\right)
\end{aligned}
$$

29…32 todos

33. todo

The probabilities must sum to 1, so:

$$
\begin{aligned}
&1=\theta(1-\theta)+(1-\theta)+\theta^2\\
\Rightarrow\ &1=\theta-\theta^2+1-\theta+\theta^2\\
\Rightarrow\ &1=1
\end{aligned}
$$

θ is not constrained by that equation, so it can be any real number.

$$E(X)=2\theta(1-\theta)+4(1-\theta)+6\theta^2=4\theta^2-2\theta+4$$

33.a. todo

Method of moments:

$4\theta^2-2\theta+4=\frac{22}{7}$, use the quadratic formula to solve.

Maximum likelihood:

$$\ell(\theta)=\ln\left((\theta(1-\theta))^4\cdot(1-\theta)^2\cdot\theta^2\right)=4\ln(\theta(1-\theta))+2\ln(1-\theta)+\ln(\theta)$$

$$
\begin{aligned}
0 &= \ell'(\theta)\\
0 &= \frac{\mathrm{d}}{\mathrm{d}\theta}(4\ln(\theta(1-\theta))+2\ln(1-\theta)+\ln(\theta))\\
0 &= \frac{4(1-2\theta)}{\theta(1-\theta)}+\frac{-2}{1-\theta}+\frac{1}{\theta}\\
0 &= \frac{(4(1-2\theta))+(-2\theta)+(1-\theta)}{\theta(1-\theta)}\\
0 &= \frac{5-11\theta}{\theta(1-\theta)}\\
\theta-\theta^2 &= 5-11\theta
\end{aligned}
$$

Use quadratic formula to solve again.

33.b.

Method of moments: Same

Maximum likelihood: Same

33.c.

Method of moments: Same

Maximum likelihood: Same

33.d. todo

Method of moments: Same

Maximum likelihood: Same

Since $x = 3$ isn't included in the probability distribution table, our assumed distribution is probably not very good.

34…41 todos

42.

42.a. (checked)

$$\begin{aligned} E(X) &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ \frac{\hat{\alpha}}{\hat{\alpha}+1} &= \overline{X} \\ \hat{\alpha} &= \hat{\alpha}\overline{X} + \overline{X} \\ \hat{\alpha}\overline{X} - \hat{\alpha} &= -\overline{X} \\ \hat{\alpha} &= \frac{-\overline{X}}{\overline{X}-1} \end{aligned}$$

https://math.stackexchange.com/q/3185959/774737 has the same answer, so it's probably correct

42.b. (checked)

$$\begin{aligned} \ell(\alpha) &= \ln \prod_{i=1}^{n} f(x_i; \alpha) \\ &= \ln \prod_{i=1}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} \cdot x_i^{\alpha-1}(1-x_i)^{1-1} \\ &= \ln \prod_{i=1}^{n} \alpha \cdot x_i^{\alpha-1} \\ &= \sum_{i=1}^{n} \ln \alpha + \ln\left(x_i^{\alpha-1}\right) \\ &= \sum_{i=1}^{n} \ln \alpha + (\alpha-1)\ln x_i \end{aligned}$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\ell(\alpha) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\sum_{i=1}^{n}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\alpha}(\ln\alpha + (\alpha-1)\ln x_i) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i \frac{\mathrm{d}}{\mathrm{d}\alpha}(\alpha-1)\right) \\
0 &= \sum_{i=1}^{n}\left(\frac{1}{\alpha} + \ln x_i\right) \\
0 &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i \\
-\sum_{i=1}^{n}\ln x_i &= \frac{n}{\alpha} \\
-\alpha\sum_{i=1}^{n}\ln x_i &= n \\
\alpha &= \frac{-n}{\sum_{i=1}^{n}\ln x_i} \\
\hat{\alpha}_{\mathrm{MLE}} &= \frac{-n}{\sum_{i=1}^{n}\ln x_i}
\end{aligned}
$$

(this is the same MLE as https://math.stackexchange.com/q/4057637/774737, so it's probably correct)

Verify that this is a local maximum:

$$\ell''(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{n}{\alpha} + \sum_{i=1}^{n}\ln x_i\right) = \frac{-n}{a^2}$$

$\ell''(\alpha)$ must always be negative, because n is always positive and α^2 is always positive. Therefore, $\hat{\alpha}_{\mathrm{MLE}}$ is a maximum likelihood estimator for α.