Assignment 1

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1.

2. Let $X_n =_{i.i.d} \mathbb{N}(\mu, \sigma^2)$ where $\mu = 1$, $\sigma^2 = 4$ and $n = 100$. The sample log likelihood function for a univariate normal distribution with mean μ and standard deviation σ is

$$
\begin{aligned}
l(x_i, \theta) &= \log p(x_i, \theta) \\
&= -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2\sigma^2}(x_i - \mu)^2
\end{aligned}
$$

where $p(X, \theta)$ is a density function of $X =_d \mathbb{N}(\mu, \sigma^2)$. The mean log likelihood is defined as

$$
\frac{1}{n}\sum_{i=1}^{n} l(x_i, \theta).
$$

For the numerical solution of the maximum likelihood estimate from X_n, see Figure 1.

3.

4.

5.

6. First, note that $y_t|\varepsilon_{t-1} =_d \mathbb{N}(\mu + \theta\varepsilon_{t-1}, \sigma^2)$ for all t and

$$
\begin{aligned}
p(y_t|\varepsilon_{t-1}) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}(y_t - \mu - \theta\varepsilon_{t-1})^2) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{\varepsilon_t^2}{2\sigma^2}).
\end{aligned}
$$

Second, $\sigma((y_s)_{s\leq t}) = \sigma((\varepsilon_s)_{s\leq t})$ for all t. Then

$$
\begin{aligned}
p(y_1, \cdots, y_t) &= p(y_t|y_1, \cdots, y_{t-1})p(y_1, \cdots, y_{t-1}) \\
&= p(y_t|y_1, \cdots, y_{t-1})p(y_{t-1}|y_1, \cdots, y_{t-2})p(y_1, \cdots, y_{t-2}) \\
&\cdots \\
&= p(y_t|y_1, \cdots, y_{t-1}) \times \cdots \times p(y_2|y_1) \times p(y_1)
\end{aligned}
$$

and

$$
l(y_1, \cdots, y_t, \mu, \sigma^2, \theta,) = -\frac{T}{2}\log 2\pi - \frac{T}{2}\log \sigma^2 - \sum_{t=1}^{T}\frac{\varepsilon_t^2}{2\sigma^2}.
$$

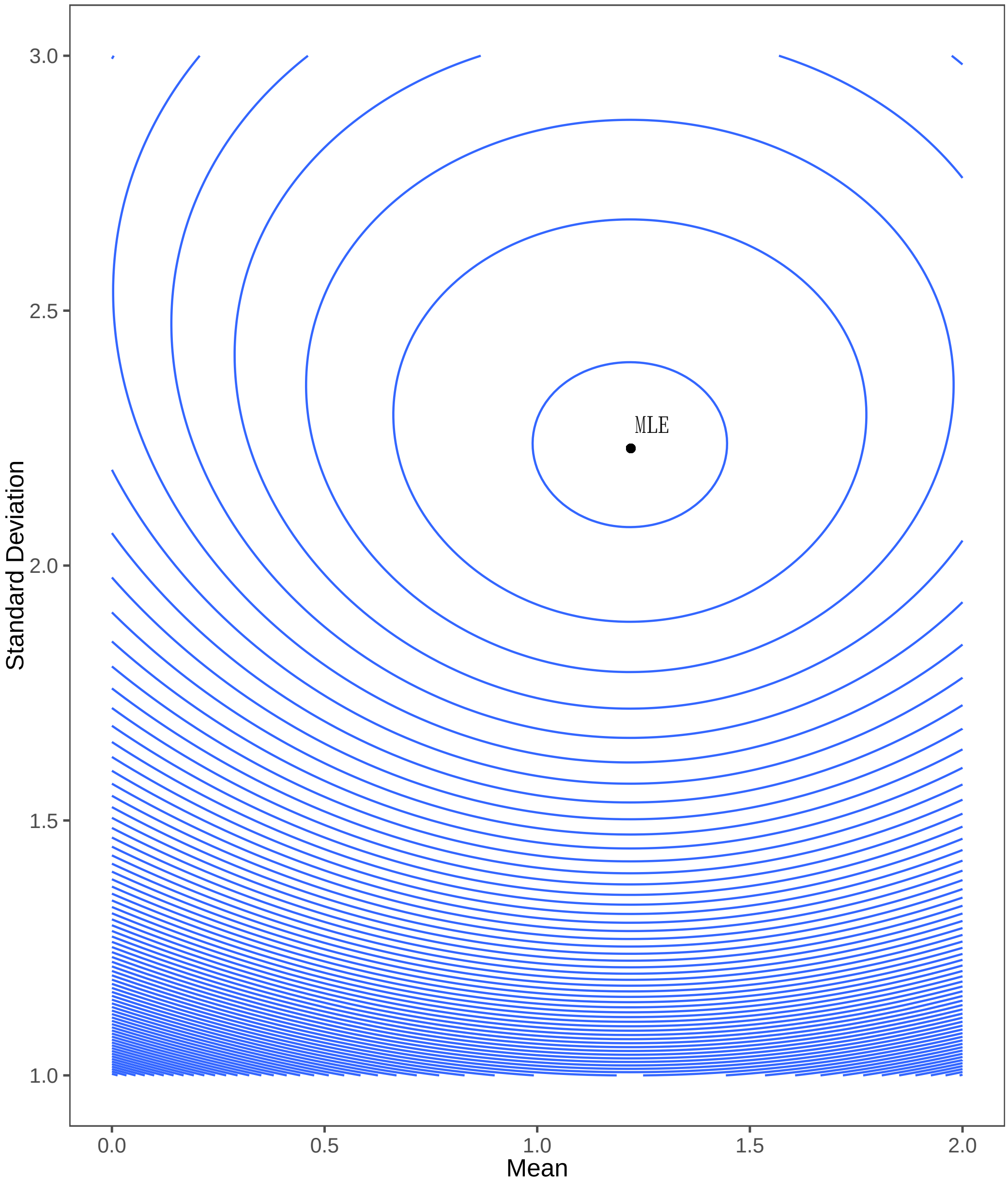

Figure 1: The contour of the mean log-likelihood. $\hat{\mu}_{ML} = 1.22$, $\hat{\sigma}_{ML} = 2.23$, and $l(x, \hat{\mu}_{ML}, \hat{\sigma}_{ML}) = -2.22$.

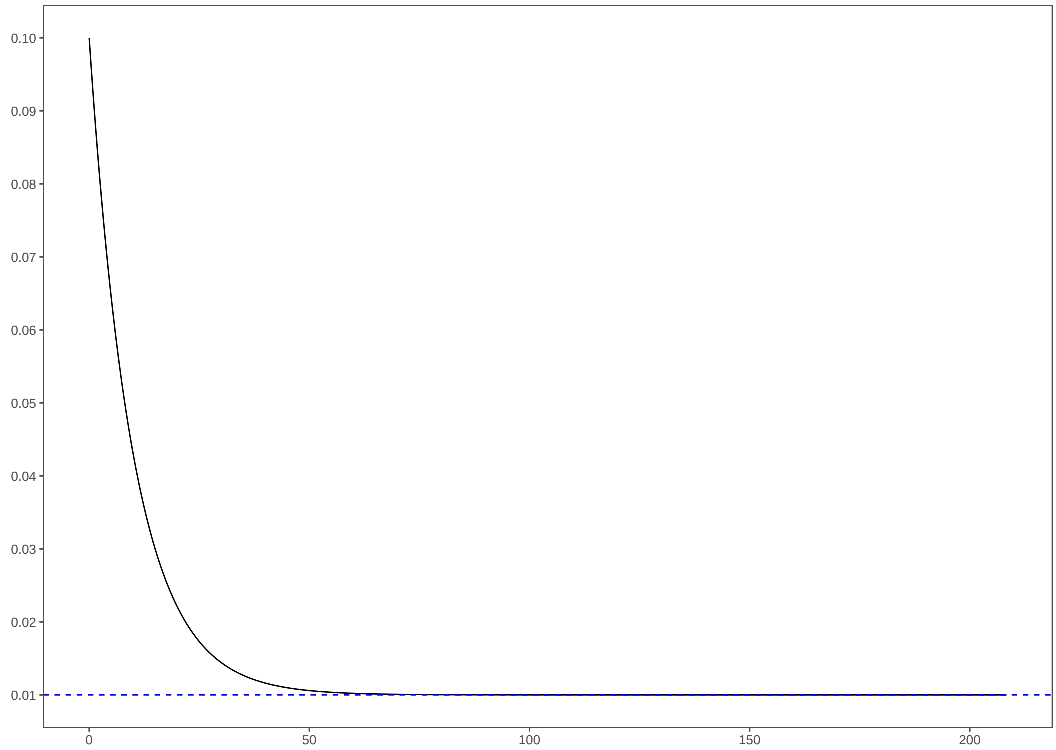

Figure 2: A discretized version of the ODE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$ and $t = 250$.

For the estimation, I use a simple grid search method. I randomly generate 10,000 parameters from $\hat{\mu} =_d U[0, 3]$, $\hat{\sigma}^2 =_d U[1, 3]$, and $\hat{\theta} =_d U[0, 0.99]$ and calculate the log likelihood of the sample given parameters. The result is

$$\hat{\mu}_{ML} = 2.9822,$$
$$\hat{\sigma}^2_{ML} = 1.9969,$$
$$\hat{\theta}_{ML} = 0.0897,$$

and

$$l(y, \hat{\mu}_{ML}, \hat{\sigma}^2_{ML}, \hat{\theta}_{ML}) = -1779.01$$

which is surprisingly bad.

7. (a)
   (b) See Figure 2.
   (c)
   (d) See Figure 3 and 4.

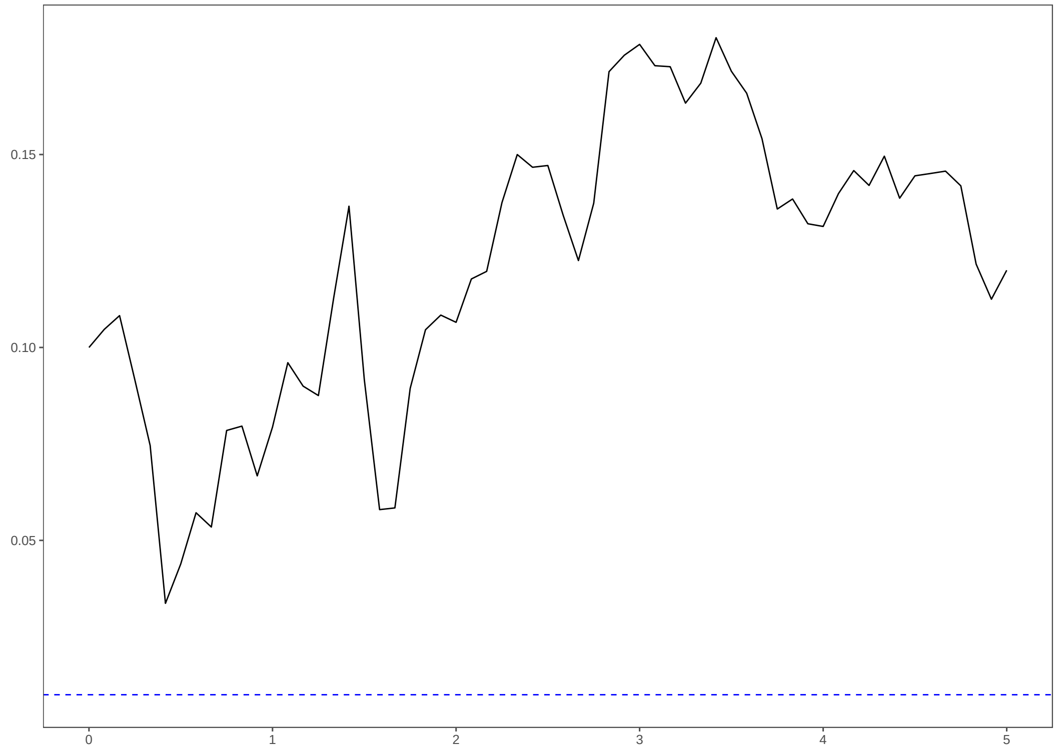

Figure 3: A discretized version of the SDE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t + \sigma(\sqrt{\Delta t})\varepsilon_t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$, $\sigma = 0.05$, $\varepsilon_t =_{i.i.d} \mathbb{N}(0, 1)$ and $t = 5$.

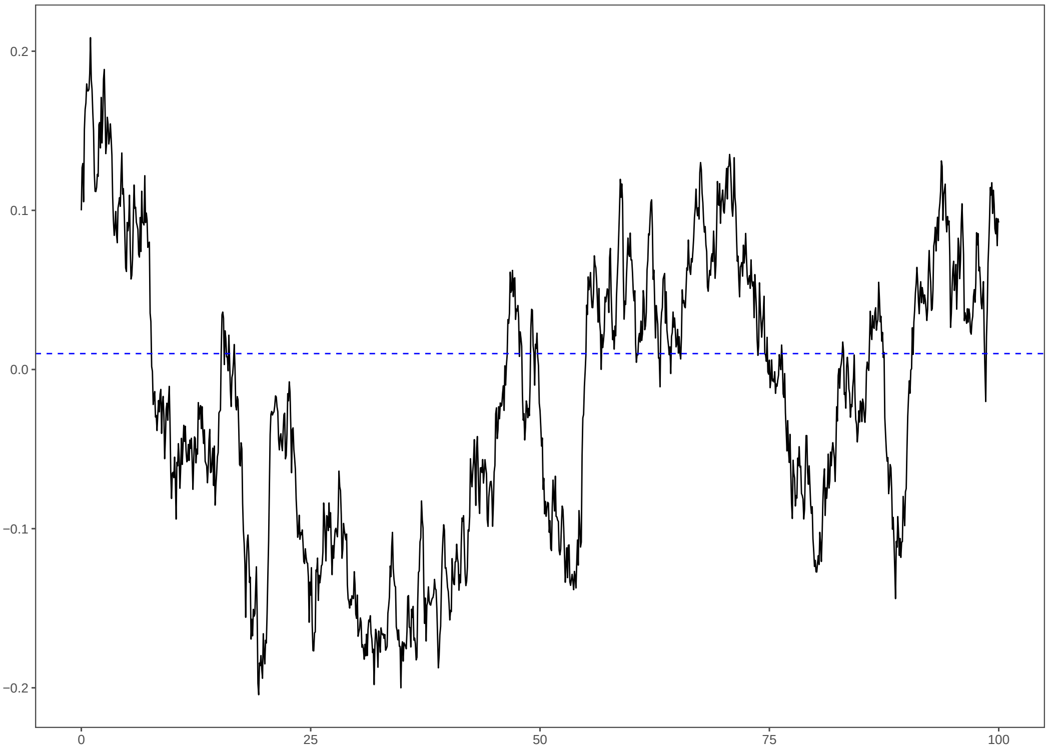

Figure 4: A discretized version of the SDE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t + \sigma(\sqrt{\Delta t})\varepsilon_t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$, $\sigma = 0.05$, $\varepsilon_t =_{i.i.d} \mathbb{N}(0,1)$ and $t = 100$.

Appendix

For R codes used to produce above results, see `https://github.com/ysugk/Course-FIN625/tree/master/HW/code`.