Assignment 1

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1. (a) Table 1 reports summary statistics of the data.
   (b) Let the excess return process $X_t =_{i.i.d} \mathbb{N}(\mu, \sigma^2)$ for $t = 1, \cdots, T$. Then, MLE estimators for (μ, σ^2) yield
   $$\hat{\mu} = \frac{1}{T}\sum_t x_t$$
   and
   $$\hat{\sigma}^2 = \frac{1}{T}\sum_t (X_t - \hat{\mu})^2$$
   (c) First, since $X = (X_t) =_d \mathbb{N}(\mu\iota, \sigma^2 I)$, we have
   $$\begin{aligned}\begin{bmatrix}\frac{1}{T}\iota' X \\ (I - P_\iota)X\end{bmatrix} &=_d \begin{bmatrix}\frac{1}{T}\iota' \\ (I - P_\iota)\end{bmatrix}\mathbb{N}(\mu\iota, \sigma^2 I) \\ &=_d \mathbb{N}(\begin{bmatrix}\mu \\ 0\end{bmatrix}, \begin{bmatrix}\sigma^2/T & 0 \\ 0 & \sigma^2(I - P_\iota)\end{bmatrix}\end{aligned}$$
   Consider a continuous function
   $$f(x, y) = \frac{x}{y}$$
   Since
   $$\nabla f(x, y) = \begin{bmatrix}1/y & -x/y^2\end{bmatrix}$$
   is also continuous, by the delta method, we obtain
   $$\frac{\mu}{\sigma}$$

Table 1: Summary Statistics
In this table, I report number of observations (N),

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
Mkt-RF	1,108	0.659	5.330	−29.130	−1.970	1.015	3.615	38.850
SMB	1,108	0.207	3.192	−16.870	−1.562	0.070	1.730	36.700
HML	1,108	0.369	3.484	−13.280	−1.320	0.140	1.740	35.460
RF	1,108	0.274	0.253	−0.060	0.030	0.230	0.430	1.350

Table 2: Summary Statistics

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Median	Pctl(75)	Max
Mkt-RF	1,108	0.659	5.330	−29.130	−1.970	1.015	3.615	38.850
SMB	1,108	0.207	3.192	−16.870	−1.562	0.070	1.730	36.700
HML	1,108	0.369	3.484	−13.280	−1.320	0.140	1.740	35.460

Table 3: OLS estimation results of Eq (1)

param	estimate	se	tstat	pvalue
alpha	0.082	0.092	0.892	0.372
beta	0.191	0.017		

2. Let $X_n =_{i.i.d} \mathbb{N}(\mu, \sigma^2)$ where $\mu = 1$, $\sigma^2 = 4$ and $n = 100$. The sample log likelihood function for a univariate normal distribution with mean μ and standard deviation σ is

$$\begin{aligned} l(x_i, \theta) &= \log p(x_i, \theta) \\ &= -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2\sigma^2}(x_i - \mu)^2 \end{aligned}$$

where $p(X, \theta)$ is a density function of $X =_d \mathbb{N}(\mu, \sigma^2)$. The mean log likelihood is defined as

$$\frac{1}{n}\sum_{i=1}^{n} l(x_i, \theta).$$

For the numerical solution of the maximum likelihood estimate from X_n, see Figure 1.

3. (a) Table 2 show summary statistics of the excess market returns, SMB, and HML. For OLS estimation results of Equation (1), see Table 3. Table 4 reports OLS estimation results of Equation (2).
   (b) Table and report the OLS estimation results of Equation (1) and (2), respectively.

4. (a) Define

$$\begin{aligned} h^1(\theta, r_{M,t}) &= r_{M,t} - \mu \\ h^2(\theta, r_{M,t}) &= (r_{M,t} - \mu)^2 - \sigma^2. \end{aligned}$$

Since the parameters are exactly identified, we set the sample moment conditions to zero

$$\tilde{h}^1_T(\theta) = \frac{1}{T}\sum r_{M,t} - \hat{\mu} = 0,$$

Table 4: OLS estimation results of Eq (2)

param	estimate	se	tstat	pvalue
alpha	0.268	0.102	2.612	0.009
beta	0.154	0.019		

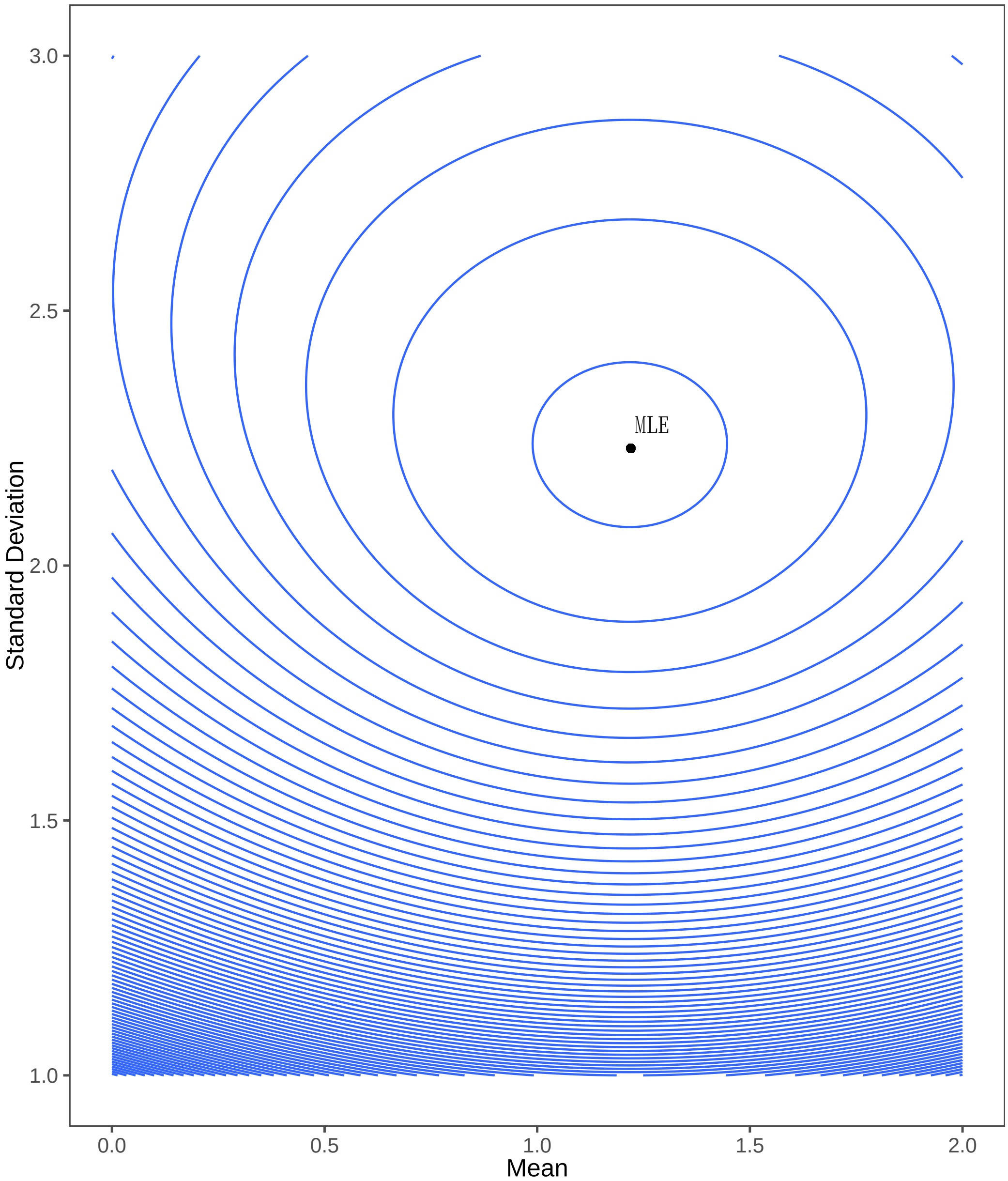

Figure 1: The contour of the mean log-likelihood. $\hat{\mu}_{ML} = 1.22$, $\hat{\sigma}_{ML} = 2.23$, and $l(x, \hat{\mu}_{ML}, \hat{\sigma}_{ML}) = -2.22$.

Table 5: OLS estimation results of Eq (1)

param	estimate	se	tstat	pvalue
alpha	0.082	0.085	0.960	0.337
beta	0.191	0.033		

Table 6: OLS estimation results of Eq (2)

param	estimate	se	tstat	pvalue
alpha	0.268	0.096	2.793	0.005
beta	0.154	0.054		

Table 7: Summary Statistics when $T = 840$

param	n	mean	sd	skewness	kurtosis
beta	$10,000$	0.284	0.156	0.739	4.065
rho	$10,000$	0.967	0.009	-0.805	4.206

$$\tilde{h}_T^2(\theta) = \frac{1}{T}\sum(r_{M,t} - \hat{\mu})^2 - \hat{\sigma}^2 = 0.$$

Then, we obtain

$$\begin{aligned}\hat{\mu} &= \frac{1}{T}\sum r_{M,t} \\ &= 0.6590,\end{aligned}$$

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{T}\sum(r_{M,t} - \hat{\mu})^2 \\ &= 28.3814\end{aligned}$$

(b) Suppose

$$r_{M,t} =_{i.i.d} \mathbb{N}(0, \sigma^2).$$

Define

$$h(\mu, \sigma^2, r_{M,t}) = \begin{bmatrix} r_{M,t} - \mu \\ (r_{M,t} - \mu)^2 - \sigma^2 \end{bmatrix}$$

5. Table 7 and 8 summarise sample distributions of $\hat{\beta}$ and $\hat{\rho}$ when $T = 840$ and $T = 240$, repectively. As we expect from the finite sample properties, $\hat{\beta}$ is underestimated ($\hat{\beta} < \beta$) on average and skewed to the right (positive skewness), while $\hat{\rho}$ is overestimated ($\hat{\rho} > \rho$) on average and skewed to the left (negative skewness). However, since both estimators are consistent, those estimation errors are quite small when sample size (T) is large.

Table 8: Summary Statistics when $T = 240$

param	n	mean	sd	skewness	kurtosis
beta	$10,000$	0.498	0.376	1.106	5.099
rho	$10,000$	0.953	0.023	-1.163	5.224

6. First, note that $y_t|\varepsilon_{t-1} =_d \mathbb{N}(\mu + \theta\varepsilon_{t-1}, \sigma^2)$ for all t and

$$\begin{aligned} p_{\mu,\sigma^2,\theta}(y_t|\varepsilon_{t-1}) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(y_t - \mu - \theta\varepsilon_{t-1})^2) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\varepsilon_t^2}{2\sigma^2}) \end{aligned}$$

Second, $\sigma((y_s)_{s\le t}, \mu, \theta) = \sigma((\varepsilon_s)_{s\le t})$ for all t. Then

$$\begin{aligned} p_{\mu,\sigma^2,\theta}(y_1, \cdots, y_t) &= p_{\mu,\sigma^2,\theta}(y_t|y_1, \cdots, y_{t-1}) p_{\mu,\sigma^2,\theta}(y_1, \cdots, y_{t-1}) \\ &= p_{\mu,\sigma^2,\theta}(y_t|y_1, \cdots, y_{t-1}) p_{\mu,\sigma^2,\theta}(y_{t-1}|y_1, \cdots, y_{t-2}) p_{\mu,\sigma^2,\theta}(y_1, \cdots, y_{t-2}) \\ &\cdots \\ &= p_{\mu,\sigma^2,\theta}(y_t|y_1, \cdots, y_{t-1}) \times \cdots \times p_{\mu,\sigma^2,\theta}(y_2|y_1) \times p_{\mu,\sigma^2,\theta}(y_1) \end{aligned}$$

and

$$l(y_1, \cdots, y_t, \mu, \sigma^2, \theta,) = -\frac{T}{2}\log 2\pi - \frac{T}{2}\log\sigma^2 - \sum_{t=1}^{T} \frac{\varepsilon_t^2}{2\sigma^2}$$

where l is the log likelihood function. Based on the above discussion, I conduct a Monte Carlo simulation and numerically estimate MLE. For a numerical estimation, I use a simple gradient method. Using initial parameter values $\mu = 1.9$, $\sigma^2 = 1.7$ and $\theta = 0.2$, the results from the estimation is

$$\begin{aligned} \hat{\mu}_{ML} &= 1.460, \\ \hat{\sigma}^2_{ML} &= 2.040, \\ \hat{\theta}_{ML} &= 0.080, \end{aligned}$$

and

$$l(y, \hat{\mu}_{ML}, \hat{\sigma}^2_{ML}, \hat{\theta}_{ML}) = -1775.47$$

which is quite reasonable.

7. (a) Due to the Ito formula, we have

$$\begin{aligned} d(e^{\kappa t}(x_t - \bar{x})) &= \kappa e^{\kappa t}(x_t - \bar{x})dt + e^{\kappa t}dx_t \\ &= \kappa e^{\kappa t}(x_t - \bar{x})dt + e^{\kappa t}\kappa(\bar{x} - x_t)dt \\ &= 0 \end{aligned}$$

It follows that

$$\begin{aligned} e^{\kappa t}(x_t - \bar{x}) - (x - \bar{x}) &= 0 \\ x_t &= \bar{x} + e^{-\kappa t}(x - \bar{x}) \end{aligned}$$

Note that

$$\begin{aligned} \lim_{t\to\infty} x_t &= \bar{x} + (x - \bar{x}) \lim_{t\to\infty} e^{-\kappa t} \\ &= \bar{x} \end{aligned}$$

(b) See Figure 2.

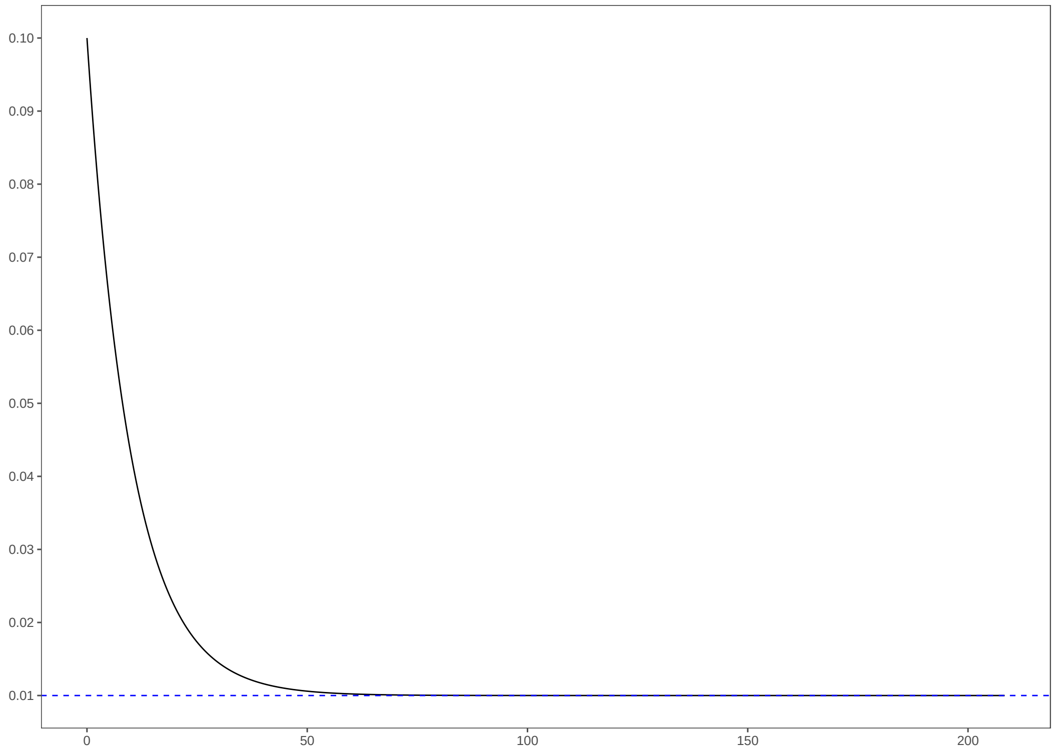

Figure 2: A discretized version of the ODE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$ and $t = 250$.

(c) Due to the Ito formula, we have

$$
\begin{aligned}
d(e^{\kappa t}(x_t - \bar{x})) &= \kappa e^{\kappa t}(x_t - \bar{x})dt + e^{\kappa t}dx_t \\
&= \kappa e^{\kappa t}(x_t - \bar{x})dt + e^{\kappa t}\kappa(\bar{x} - x_t)dt + e^{\kappa t}\sigma dW_t \\
&= e^{\kappa t}\sigma dW_t
\end{aligned}
$$

It follows that

$$
e^{\kappa t}(x_t - \bar{x}) - (x - \bar{x}) = \sigma \int_0^t e^{\kappa s} dW_s
$$

$$
\Rightarrow x_t = \bar{x} + e^{-\kappa t}(x - \bar{x}) + \sigma \int_0^t e^{\kappa(s-t)} dW_s.
$$

Then

$$
\begin{aligned}
\mathbb{E}_t(x_s) &= \bar{x} + e^{-\kappa s}(x - \bar{x}) + \mathbb{E}_t(\sigma \int_0^s e^{\kappa(u-s)} dW_u) \\
&= \bar{x} + e^{-\kappa t}(x - \bar{x}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u + e^{-\kappa s}(x - \bar{x}) - e^{-\kappa t}(x - \bar{x}) \\
&= x_t + (e^{-\kappa s} - e^{-\kappa t})(x - \bar{x}).
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{var}_t(x_s) &= \mathbb{E}_t(\sigma \int_t^s e^{\kappa(u-s)} dW_u)^2 \\
&= \sigma^2 \int_t^s e^{2\kappa(u-s)} du \\
&= \sigma^2 [\frac{1}{2\kappa} e^{2\kappa(u-s)}]_t^s \\
&= \frac{\sigma^2(1 - e^{2\kappa(t-s)})}{2\kappa}.
\end{aligned}
$$

Note that for any given t,

$$
\lim_{s \to \infty} \mathbb{E}_t(x_s) = x_t + (1 - e^{-\kappa t})(x - \bar{x})
$$

$$
\begin{aligned}
\lim_{s \to \infty} \mathrm{var}_t(x_s) &= \frac{\sigma^2}{2\kappa} - \frac{1}{2\kappa} \lim_{s \to \infty} e^{2\kappa(t-s)} \\
&= \frac{\sigma^2}{2\kappa}
\end{aligned}
$$

(d) See Figure 3 and 4.

Appendix

For R codes used to produce above results, see `https://github.com/ysugk/Course-FIN625/tree/master/HW/code`.

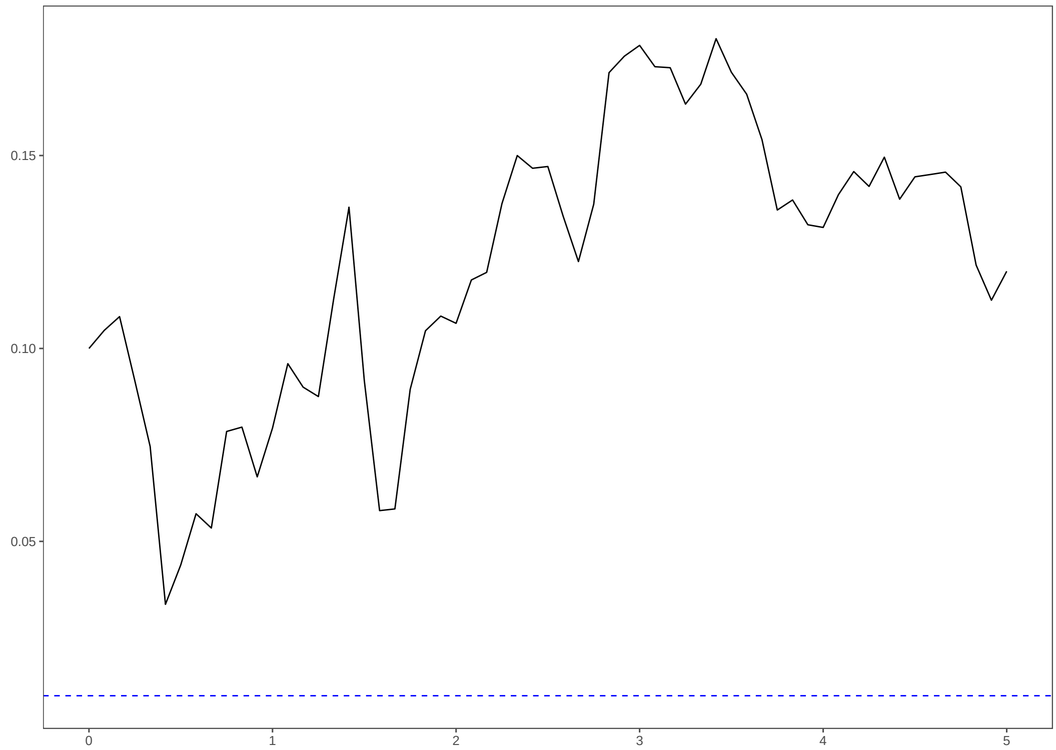

Figure 3: A discretized version of the SDE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t + \sigma(\sqrt{\Delta t})\varepsilon_t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$, $\sigma = 0.05$, $\varepsilon_t =_{i.i.d} \mathbb{N}(0, 1)$ and $t = 5$.

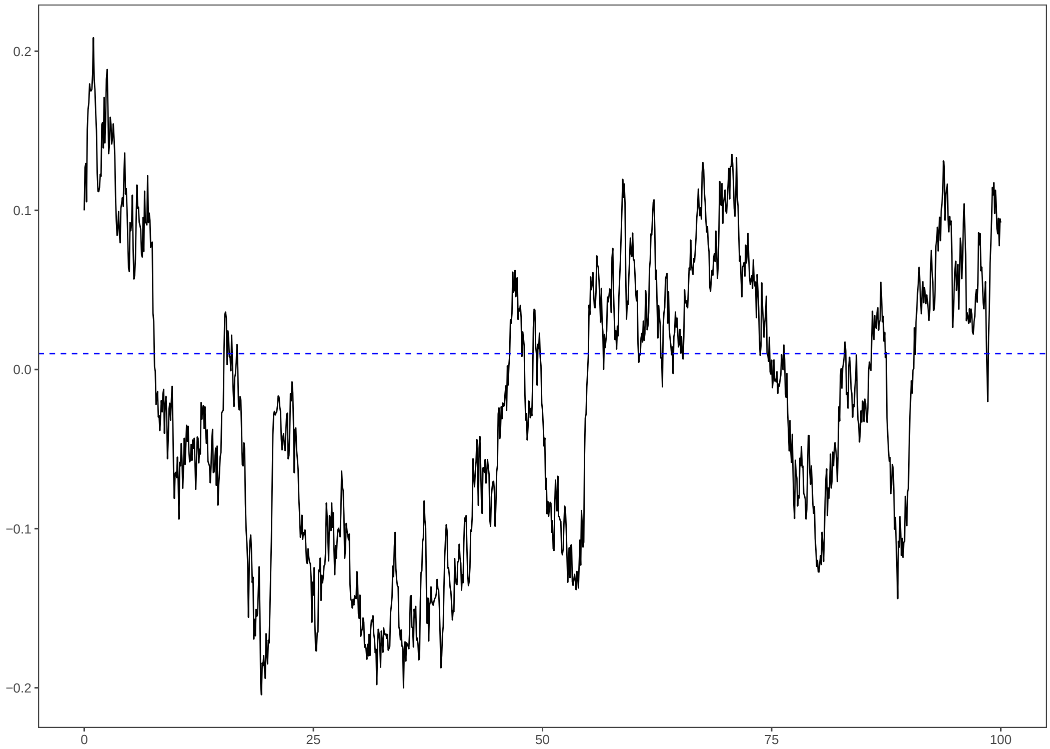

Figure 4: A discretized version of the SDE $\Delta x_t = \kappa(\bar{x} - x_t)\Delta t + \sigma(\sqrt{\Delta t})\varepsilon_t$. $\Delta t = 1/12$, $\kappa = 0.1$, $\bar{x} = 0.01$, $x_0 = 0.1$, $\sigma = 0.05$, $\varepsilon_t =_{i.i.d} \mathbb{N}(0,1)$ and $t = 100$.