RO(G)-GRADED NORMS FOR PRISMATIC AND DE RHAM-WITT THEORY

Abstract. We make two case studies of RO(G)-graded homotopy in connection with p-adic cohomology. First, we identify the RO($\mathbf{T}$)-graded TC^- and TP of quasiregular semiperfectoid rings as multi-Rees algebras for the Nygaard and $\phi^\bullet(I)$-adic filtrations. Next, we compute the (contiguous) polyrepresentation-graded TR of smooth algebras over a perfectoid base, leading to a structure we call the *poly-de Rham-Witt complex* $\mathrm{W}\Omega^\bigstar$. In each situation we study the Norm map in RO(G)-graded degrees: this extends the author's prismatic norm maps to Breuil-Kisin twists, and Angeltveit's Norm map of Witt vectors to the de Rham-Witt complex. We find a close relationship with the prismatic Gamma function as well as a q-analogue of Wilson's theorem.

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1. Introduction

This paper establishes some foundational computations for studying p-adic cohomology through the lens of genuine equivariant homotopy theory. Recall that the interface between these is topological Hochschild homology (here viewed as a genuine $\mathbf{T}$-spectrum), and its various types of fixed point spectra. For nice input rings, the homotopy groups of these are given by important arithmetic invariants, often with additional *filtrations* and *twists*.

Example 1.1. If R is a quasiregular semiperfectoid ring, the (p-completed) TC^- and TP of R are related to (Nygaard-filtered, Nygaard-completed, Breuil-Kisin twisted, absolute) prismatic cohomology via

$$\mathrm{TC}^-_{2i}(R;\mathbf{Z}_p) = \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}_R\{i\}$$

$$\mathrm{TP}_{2i}(R;\mathbf{Z}_p) = \hat{\mathbb{\Delta}}_R\{i\}$$

(and the odd groups vanish) by work of Bhatt-Morrow-Scholze [BMS19]. In the q-de Rham case, the twist can be understood as tensoring with a free module on the symbol $(p^\infty)_q = \frac{q^{p^\infty}-1}{q-1} = \frac{\log_{\Delta}(q^p)}{q-1}$ (see Remark 1.8 and [BL22, Proposition 2.6.1]).

Example 1.2. When S is formally smooth over any perfectoid ring R (i.e. the p-completion of a smooth R-algebra), Hesselholt [Hes96] showed that $\mathrm{TR}^n(S; \mathbf{Z}_p)$ is given by the p-complete relative de Rham-Witt complex [LZ04] adjoined with a polynomial generator in degree 2:

$$\mathrm{TR}^n_*(S; \mathbf{Z}_p) \cong \mathrm{W}_n\Omega^*_{\operatorname{Spf} S/\operatorname{Spf} R}[\sigma_n] \qquad |\sigma_n| = 2 \tag{1}$$

Here the p-complete de Rham-Witt complex is

$$\mathrm{W}_n\Omega^*_{\operatorname{Spf} S/\operatorname{Spf} R} := \varprojlim_s \mathrm{W}_n\Omega^*_{(S/p^s)/(R/p^s)}$$

as in [BMS18, Definition 10.11]; we will subsequently abuse notation and simply write this as $\mathrm{W}_n\Omega^*_{S/R}$. The identification (1) can be made canonical by interpreting σ_n as tensoring with the symbol ξ_{n+1} ($= (p^{n+1})_{q^{1/p^{n+1}}}$ in the q-de Rham case), see Remark 1.8.

In particular, the Restriction maps on the twists get multiplied by powers of $\xi_{n+1}/\xi_n = \phi^{-n}(\xi)$, so there are potentially $\lim^1$ terms when we take the limit over this result to understand $\mathrm{TR}(S; \mathbf{Z}_p)$. Fortunately, these vanish in the main cases of interest: if $R = k$ is a perfect $\mathbf{F}_p$-algebra, then

$$\mathrm{TR}_*(S) = \mathrm{W}\Omega^*_S,$$

and if $R = \mathcal{O}_C$ is the ring of integers of a spherically complete perfectoid field C, then we have

$$\mathrm{TR}_*(S; \mathbf{Z}_p) = \mathrm{W}\Omega^*_{S/\mathcal{O}_C}[\beta], \qquad |\beta| = 2$$

by [Mat21, Lemma 7.22]. This identification can be made canonical by interpreting β as tensoring with the ideal $(1-q)$ of $\mathrm{A}_{\inf}(\mathcal{O}_C)$.

By Kan extending the Postnikov filtration (or its double-speed acceleration), one gets filtrations on these spectra for arbitrary animated rings, whose associated gradeds are given by derived versions of the same arithmetic invariants.

The genuine equivariant[1] perspective provides a significant enlargement of the homotopy groups of a spectrum: a G-spectrum E naturally has homotopy groups $\pi^H_V E$ for each subgroup $H \leq G$ and each real virtual representation $V \in \mathrm{RO}(H)$ of H. This has a great deal of functoriality in H, leading to $\mathrm{RO}(G)$-*graded homotopy Mackey functors* $\underline{\pi}_\bigstar E$. When E is a G-$\mathbf{E}_\infty$-ring, we even get an $\mathrm{RO}(G)$-*graded Tambara functor* in the sense of Angeltveit-Bohmann [AB18].

Remark 1.3. For all notation and background, the reader can consult

- [Sul20, Part 1] and [Hil20] for equivariant homotopy theory in general;
- [Sul23a, §4.2] for poly-representations;
- [Sul23b, §2] for Tambara functors;
- §2 for all the notation used in this paper.

In this paper, we will generalize the two computations above to $\mathrm{RO}(G)$-grading. Here are some reasons for doing so.

Motivation 1.4. In general, the $\mathrm{RO}(G)$-graded homotopy groups of a G-spectrum tend to give a simpler or more holistic picture, in which the $\mathbf{Z}$-graded groups sit as a funny diagonal. For example, while the integer-graded TC^- mixes the Nygaard filtration with Breuil-Kisin twists, the $\mathrm{RO}(\mathbf{T})$-graded groups will give us the Nygaard filtration on the nose:

$$\mathrm{TC}^-_{2i}(R; \mathbf{Z}_p) = \mathcal{N}^{\geq i}\hat{\Delta}_R\{i\} \quad \text{but} \quad \mathrm{TC}^-_{i\lambda}(R; \mathbf{Z}_p) = \mathcal{N}^{\geq i}\hat{\Delta}_R.$$

This example showcases another important point: the $\mathrm{RO}(G)$-graded groups considered here do not yield "new" arithmetic invariants. Rather, they vastly increase the supply of additional *filtrations* and *twists* that can be passed through the topological dictionary. However, one can obtain genuinely new invariants

[1] $\mathrm{RO}(G)$-graded homotopy already exists for Borel G-spectra, but the notion is emphasized less there.

by taking the spirit of this paper farther; we give two examples (iterated Nygaard filtrations and Legendre filtrations) in Remarks 1.35 and 1.36.

Motivation 1.5. The original motivation for studying $\mathrm{RO}(G)$-graded TR specifically comes from algebraic K-theory: for any commutative ring A, we may consider the reduced p-adic K-theory of truncated polynomial algebras over A,

$$\widetilde{\mathrm{K}}(A[x]/x^e; \mathbf{Z}_p) := \mathrm{fib}\left(\mathrm{K}(A[x]; \mathbf{Z}_p) \to \mathrm{K}(A; \mathbf{Z}_p)\right)$$

By [Hes05, Proposition 8], this can be understood in terms of "$\mathrm{RO}(G)$-graded big TR": there is a fiber sequence

$$\widetilde{\mathrm{K}}(A[x]/x^e; \mathbf{Z}_p) \to \mathbf{TR}(\Sigma^{[\![1]\!]_\lambda}\mathrm{THH}(A; \mathbf{Z}_p)) \to \mathbf{TR}(\Sigma^{[\![1/e]\!]_\lambda}\mathrm{THH}(A; \mathbf{Z}_p))$$

and the latter two terms are extremely computable. This example is explained further in Example 1.18.

In this range, the answer has in some sense been known since the author was two and a half years old:

- the base case was proved in [HM97b, Proposition 9.1], and re-proven in [AG11, Theorem 8.3] from the perspective of the homotopy orbits to TR spectral sequence (our main tool in this paper);
- the base case essentially implies the result for smooth algebras over it by [Hes05, Theorem 11];
- the perfectoid base case was done in [Sul23a]—but in this range it behaves exactly like the characteristic p case.

However, it took the author over thirty years to understand this.

Even in the negative range, we get two new things. Our canonical identifications make it possible to compute the functoriality of $\widetilde{\mathrm{K}}(A[x]/x^e; \mathbf{Z}_p)$ (or more general monoid algebras) in e. We also obtain an interesting filtration on these groups by computing with cell structures instead of the homotopy orbits to TR spectral sequence.

While the $\mathrm{RO}(G)$-graded groups themselves are not "new", they provide a home for something which is.

Motivation 1.6 (Tambara structure)**.** Given $K \leq H \leq G$, there are $\mathrm{RO}(G)$-graded Frobenius (restriction) and Verschiebung (transfer) maps that go

$$\pi_V^H E \underset{V}{\overset{F}{\rightleftarrows}} \pi^K_{\mathrm{res}^H_K(V)} E$$

Since the restriction of a trivial representation is trivial, these restrict to F and V maps on the $\mathbf{Z}$-graded homotopy groups.

Genuine equivariant homotopy theory emphasizes another map, the *Norm*[2] *map*, which goes

$$N_K^H : \pi_V^K E \longrightarrow \pi^H_{\mathrm{ind}^H_K(V)} E$$

when E is a G-$\mathbf{E}_\infty$-ring spectrum. Since the induction of a trivial representation is never trivial, except for the zero representation, studying the norm in nonzero $\mathbf{Z}$-graded degrees forces us to contemplate $\mathrm{RO}(G)$-graded degrees. Understanding these norm maps was our primary motivation for these calculations.

As for the zero representation, the π_0 Norm was introduced in [Ang15] for Witt vectors (and therefore perfect prisms), and for transversal prisms in [Sul23b]. Two applications of the latter are a conceptual construction of de Rham-Witt comparison maps for prismatic cohomology [Mol20], [Sul23b, §4.1], and (at odd primes) a refinement of the prismatic logarithm [Mao24]. It would be very interesting to try to generalize these applications to $\mathrm{RO}(G)$-graded norms.

Motivation 1.7. $\mathrm{RO}(G)$-graded homotopy groups also serve as calculational input for studying the equivariant slice filtration. Understanding the role of norms is also very important for understanding the slice filtration, since it is tailormade to interact well with them. This was our primary motivation for the calculations in §3; we discuss it some more in Remark 1.36.

Remark 1.8 (Canonicity)**.** Since it is one of the main focuses of this paper, and the author was confused about it for a very long time, let us clarify what "canonical" really means. Usually we are told it means "without making choices", but this gets a bit philosophical. More concretely, we are going to compute

[2] While the Frobenius map is named for German mathematician Ferdinand Frobenius, the Norm map is named for Canadian mathematician Norm Macdonald.

a bunch of groups $\mathrm{TR}^{\bullet}_{\bigstar}$, and there are several naturally defined maps between these groups. By making "canonical identifications", we mean that we are going to identify these maps as well. (Note for example that every single group $\mathrm{TP}_\alpha(S;\mathbf{Z}_p)$ is non-canonically isomorphic to $\hat{\mathbb{\Delta}}_S$.)

We do this not by writing down a bunch of matrices, but by expressing the answer as tensored with various line bundles—a sort of dimensional analysis—so that all the maps are simply "the obvious ones". For example, in the TR case there are F and R maps, which act on the Bökstedt generator by

$$F(\sigma_n) \approx \sigma_{n-1}$$
$$R(\sigma_n) \approx \phi^{-n}(\xi)\sigma_{n-1}$$

where $\approx$ means up to units. These can be explained by the natural maps

$$\xi_{n+1}\mathrm{A}_{\mathrm{inf}} \underset{R}{\overset{F}{\rightrightarrows}} \xi_n \mathrm{A}_{\mathrm{inf}}$$

where R is the inclusion, and $F(f) = \phi(f)/\widetilde{\xi}$ is a divided Frobenius.

The module $\xi_n\mathrm{A}_{\mathrm{inf}}/\xi_n^2\mathrm{A}_{\mathrm{inf}}$ is often denoted by $(\mathrm{A}_{\mathrm{inf}}/\xi_n)\{1\}$, but we will need a much larger range of twists than this, so we will instead write this as $(\mathrm{A}_{\mathrm{inf}} \otimes \{\xi_n\})/\xi_n$. In general, if M is an $A = \mathrm{A}_{\mathrm{inf}}(R)$-module, a notation like $M \otimes \{\xi^2\phi^{-1}(\xi)^{-3}\}$ means

$$M \otimes_A \xi^2 A \otimes_A \mathrm{Hom}_A\left(\phi^{-1}(\xi)^3 A, A\right)$$

While slightly abusive, this notation is very compact, and the braces are similar to the usual twist notation. A principal ideal notation (ξ^2) could also work, but one needs to be clear that the principal ideal is happening in $\mathrm{A}_{\mathrm{inf}}$ (where ξ is a non-zerodivisor) and not in W_n. We will sometimes write $M \otimes \{\xi\}^2 \otimes \{\phi^{-1}(\xi)\}^{-3}$ to emphasize that these are "formal" products.

We will sometimes need to make sense of this notation for certain *infinite* products. In the prismatic context, the infinite product $I\phi(I)\cdots$ can be made sense of as the Breuil-Kisin twist, as in [BL22, §2.2]. In the de Rham-Witt context, the symbol $\{\xi\phi^{-1}(\xi)\cdots\}$ means the ideal

$$\bigcap_{n=0}^{\infty} \xi_n \mathrm{A}_{\mathrm{inf}}.$$

We only consider this intersection when the perfectoid base is spherically complete, in which case this intersection is equal to $(1-q)\mathrm{A}_{\mathrm{inf}}$ by [BMS18, Lemma 3.23]. We do not know what the intersection looks like in any other cases (besides $\phi^{-r}(\xi)\phi^{-(r+1)}(\xi)\cdots = \phi^{-r}(1-q)$).

Example 1.9. Let A be a ring, $f \in A$ a non-zerodivisor, $I = (f)$ the principal ideal generated by A, and $\bar{A} = A/I$. It is well-known that for any A-module M,

$$\mathrm{Tor}_1^A(\bar{A}, M) \cong {}_I M := \{m \in M \mid Im = 0\}$$
$$\mathrm{Ext}^1_A(\bar{A}, M) \cong M/I$$

as one sees by resolving A/I as $A \xrightarrow{f} A$.

It is less well-known that these isomorphisms are non-canonical: the chosen resolution depends on f, while $\bar{A}$ only depends on I. Let $\bar{A}\{1\} = I/I^2$, and for any $\bar{A}$-module N let $N\{n\} = N \otimes_{\bar{A}} (\bar{A}\{1\})^{\otimes n}$. In general, we have

$$\mathrm{gr}^i_I A = \begin{cases} \bar{A}\{i\} & i \geq 0 \\ 0 & i < 0 \end{cases} \qquad \mathrm{gr}^i_I(A[1/I]) = \bar{A}\{i\}$$

Now a *canonical* free resolution of $\bar{A}$ is $I \to A$, which gives *canonical* identifications

$$\mathrm{Tor}_1^A(\bar{A}, M) = {}_I M\{1\}$$
$$\mathrm{Ext}^1_A(\bar{A}, M) = M/I\{-1\}$$

In particular, if there is additional structure like a Galois action or a Frobenius floating around, these twists will give the correct actions on Tor and Ext.

Please mention this point the next time you teach homological algebra.

1.1. **Borel spectra and prismatic cohomology.** We first compute the $\mathrm{RO}(\mathbf{T})$-graded TC^- and TP of quasiregular semiperfectoid rings (§3). Let us quickly review the representation theory of $\mathbf{T}$ and set some notation. The complex representation ring of $\mathbf{T}$ is $\mathrm{RU}(\mathbf{T}) = \mathbf{Z}[\lambda^{\pm}]$, with λ the natural representation of $\mathbf{T}$ on $\mathbf{C}$; explicitly, $\lambda^i = \mathbf{C}$ with $z \in \mathbf{T}$ acting as multiplication by z^i. Complex conjugation induces a real isomorphism $\lambda^i \cong \lambda^{-i}$, so the real representation ring is $\mathrm{RO}(\mathbf{T}) = \mathbf{Z}[\lambda]$. Since we are only concerned with the homotopy type of representation spheres, we will work with $\mathbf{Z}[\lambda]$; but λ^0 will still mean $\mathbf{C}^{\mathrm{triv}}$ rather than $\mathbf{R}^{\mathrm{triv}}$. In the p-typical context, we have $S^{\lambda^i} \simeq S^{\lambda^j}$ iff $v_p(i) = v_p(j)$, so we set $\lambda_i = \lambda^{p^i}$ and $\lambda_\infty = \lambda^0$ (so we should really say $\mathrm{RO}(C_{p^\infty})$-graded). In addition to the irreducible decomposition

$$\alpha = k_0\lambda_0 + k_1\lambda_1 + \cdots + k_n\lambda_n + k_\infty\lambda_\infty,$$

it will be convenient to encode representations via the eventually-constant sequence

$$d_n(\alpha) = \dim_{\mathbf{C}} \alpha^{C_{p^n}}$$

which we write as $(d_0, \ldots, d_n; d_\infty)$. This encoding was introduced by Gerhardt and Angeltveit-Gerhardt [Ger08, AG11], and as we will see, it is suited to working in "ghost coordinates".

Recall the ideals

$$I_n := I\phi(I)\cdots\phi^{n-1}(I)$$

from [BL22, Notation 2.2.2] (warning: our notation is off-by-one from that used in [Sul23b]).

Theorem 1.10. *Let R be a quasiregular semiperfectoid ring with associated prism $(\mathbb{\Delta}_R, I)$, and let*

$$\begin{aligned}\alpha &= k_0\lambda_0 + \cdots + k_n\lambda_n + k_\infty\lambda_\infty \\ &= (d_0, \ldots, d_n; d_\infty)\end{aligned}$$

be a virtual C_{p^∞}-representation. Then there are canonical identifications

$$\begin{aligned}\mathrm{TC}^-_\alpha(R; \mathbf{Z}_p) &= \left(\prod_{i=1}^n I_i^{k_i}\right)\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R\{k_\infty\} & \mathrm{TP}_\alpha(R; \mathbf{Z}_p) &= \left(\prod_{i=1}^n I_i^{k_i}\right)\hat{\mathbb{\Delta}}_R\{k_\infty\} \\ &= \left(\prod_{i=0}^\infty \phi^{i-1}(I^{d_i})\right)\hat{\mathbb{\Delta}}_R & &= \left(\prod_{i=1}^\infty \phi^{i-1}(I)^{d_i}\right)\hat{\mathbb{\Delta}}_R\end{aligned}$$

To interpret the second line: we have absorbed the Breuil-Kisin twist into the product via $\mathbb{\Delta}\{1\} = \prod_{i=1}^\infty \phi^{i-1}(I)$, and note that $\phi^{i-1}(I^{d_i}) = \phi^{i-1}(I)^{d_i}$ for $i > 0$.

Warning 1.11. Given a G-spectrum E, there are two notations for the integer-graded homotopy groups of E^G: either $\pi_n E^G$ or E^G_n. We can imagine these as

$$\begin{aligned}\pi_n E^G &:= [S^n, E^G]_{\mathrm{Sp}} \\ E^G_n &:= [S^n, E]_{\mathrm{Sp}^G}\end{aligned}$$

In other words, for the first version we take G-fixed points of E and then take π_n of the resulting ordinary spectrum, while in the second we compute the mapping space in Sp^G directly.

In the $\mathrm{RO}(G)$-graded context, the first version *does not make sense*: for $\alpha \in \mathrm{RO}(G)$, we define

$$E^G_\alpha := [S^\alpha, E]_{\mathrm{Sp}^G}$$

(where S^α is the one-point compactification of α when α is an actual representation), and this cannot be recovered from the plain spectrum E^G. We will often abuse notation by writing $\pi_\alpha E^G$, but it should be remembered that this is technically incorrect.

Thus, when we write TC^-_α and TP_α, these are defined as

$$\begin{aligned}\mathrm{TC}^-_\alpha &:= [S^\alpha, \mathrm{THH}^{E\mathbf{T}_+}]_{\mathrm{Sp}^\xi} & \mathrm{TP}_\alpha &:= [S^\alpha, \widetilde{E\mathbf{T}} \otimes \mathrm{THH}^{E\mathbf{T}_+}]_{\mathrm{Sp}^\xi} \\ &= \pi_0(\Sigma^{-\alpha}\mathrm{THH})^{h\mathbf{T}} & &= \pi_0(\Sigma^{-\alpha}\mathrm{THH})^{t\mathbf{T}}\end{aligned}$$

with Sp^ξ the category of cyclonic[3] spectra (i.e. proper-genuine $\mathbf{T}$-spectra).

[3]We use the notation Sp^ξ because ξ looks like a tornado.

We can Kan extend this computation to get a statement for more general rings. Repeating the sheafiness arguments of [Rig22a, §4], our calculations imply

Corollary 1.12. *Let*

$$\alpha = (d_0, \ldots, d_n; d_\infty)$$

be a virtual $\mathbf{T}$*-representation. Then there are equivalences of quasisyntomic sheaves*

$$\begin{aligned}
\mathrm{gr}^i(\Sigma^{-\alpha}\mathrm{THH}(-;\mathbf{Z}_p))^{h\mathbf{T}} &= \left(\bigotimes_{r=1}^{n}(\varphi^{r-1})^*(\mathcal{I})^{\otimes(d_r-d_\infty)}\right)\otimes \mathcal{N}^{\geq d_0+i}\mathcal{O}_{\hat{\mathbb{A}}}\{d_\infty+i\}[2i] \\
\mathrm{gr}^i(\Sigma^{-\alpha}\mathrm{THH}(-;\mathbf{Z}_p))^{t\mathbf{T}} &= \left(\bigotimes_{r=1}^{n}(\varphi^{r-1})^*(\mathcal{I})^{\otimes(d_r-d_\infty)}\right)\otimes \mathcal{O}_{\hat{\mathbb{A}}}\{d_\infty+i\}[2i] \\
\mathrm{gr}^i(\Sigma^{-\alpha}\mathrm{THH}(-;\mathbf{Z}_p))^{hC_{p^n}} &= \left(\bigotimes_{r=1}^{n}(\varphi^{r-1})^*(\mathcal{I})^{\otimes(d_r-d_\infty)}\right)\otimes \frac{\mathcal{N}^{\geq d_0+i}\mathcal{O}_{\hat{\mathbb{A}}}}{\mathcal{N}^{\geq d_0+i+1}\mathcal{O}_{\hat{\mathbb{A}}}\otimes\mathcal{I}_n}\{d_\infty+i\}[2i] \\
\mathrm{gr}^i(\Sigma^{-\alpha}\mathrm{THH}(-;\mathbf{Z}_p))^{tC_{p^n}} &= \left(\bigotimes_{r=1}^{n}(\varphi^{r-1})^*(\mathcal{I})^{\otimes(d_r-d_\infty)}\right)\otimes \frac{\mathcal{O}_{\hat{\mathbb{A}}}}{\mathcal{I}_n}\{d_\infty+i\}[2i]
\end{aligned}$$

The proof of Theorem 1.10 is very easy, and the argument works much more generally; all we need to do is formulate it. Recall the following generalization of cyclotomic polynomials.

Definition 1.13 ([DM23a, Remark 2.4.3])**.** Let $\mathbf{G}$ be a formal group law with n-series $[n]_{\mathbf{G}}(t)$. Define

$$\Phi_{n,\mathbf{G}}(t) = \prod_{d|n}[d]_{\mathbf{G}}(t)^{\mu(n/d)}$$

where μ is the Möbius function. Note that by Möbius inversion we have

$$[n]_{\mathbf{G}}(t) = \prod_{d|n}\Phi_{d,\mathbf{G}}(t)$$

Theorem 1.14 (Remark 3.6)**.** *Let E be a $\mathbf{T}$-equivariant even $\mathbf{E}_\infty$-ring with associated formal group $\mathbf{G}$. Let $A\{i\} = \pi_{2i}E^{t\mathbf{T}}$, and let $\mathrm{Fil}^\bullet A\{i\}$ be the abutment filtration of the Tate spectral sequence. Fix a choice of complex orientation $t \in A\{-1\}$, define $\langle n\rangle_{\mathbf{G}} = \frac{[n]_{\mathbf{G}}(t)}{t} \in A$ and $\Phi_{n,\mathbf{G}} = \Phi_{n,\mathbf{G}}(t) \in A$ $(n > 1)$ as above, and assume that $\langle n\rangle_{\mathbf{G}}$ is a non-zerodivisor in A for all n.*

Let α be a virtual complex $\mathbf{T}$-representation with irreducible decomposition

$$\alpha = \sum_{i=0}^{\infty} k_i\lambda^i$$

and dimension sequence $d_n = \dim_{\mathbf{C}}\alpha^{C_n}$, $d_\infty = \dim_{\mathbf{C}}\alpha^{\mathbf{T}}$.

Then the $\mathrm{RO}(\mathbf{T})$-graded homotopy groups of $E^{h\mathbf{T}}$ and $E^{t\mathbf{T}}$ are given canonically by

$$\begin{aligned}
E^{h\mathbf{T}}_\alpha &= \mathrm{Fil}^{d_1} A\{k_0\}\otimes_A \bigotimes_{n>1}(\langle n\rangle_{\mathbf{G}} A)^{\otimes k_n} \\
&= \mathrm{Fil}^{d_1} A\otimes_A \bigotimes_{n>1}(\Phi_{n,\mathbf{G}}A)^{\otimes d_n} \\
&:= \mathrm{Fil}^{d_1} A\{d_\infty\}\otimes_A \bigotimes_{n>1}(\Phi_{n,\mathbf{G}}A)^{\otimes(d_n-d_\infty)}
\end{aligned}$$

$$\begin{aligned}
E^{t\mathbf{T}}_\alpha &= A\{k_0\}\otimes_A \bigotimes_{n>1}(\langle n\rangle_{\mathbf{G}} A)^{\otimes k_n} \\
&= \bigotimes_{n>1}(\Phi_{n,\mathbf{G}}A)^{\otimes d_n} \\
&:= A\{d_\infty\}\otimes_A \bigotimes_{n>1}(\Phi_{n,\mathbf{G}}A)^{\otimes(d_n-d_\infty)}
\end{aligned}$$

1.2. **Polygonic spectra and de Rham-Witt complexes.** We next shift our attention to TR. Since the Restriction maps involve geometric fixed points, TR is not quite graded by representations, but rather *poly-representations* (introduced in [Sul23a, §4.2]). A poly-representation $\underline{\alpha}$ consists of representations $\underline{\alpha}(C_n) \in \mathrm{RO}(C_n)$ for each n, compatible under the maps

$$\begin{aligned} \mathrm{RO}(C_{mn}) &\to \mathrm{RO}(C_{mn}/C_n) \\ &\cong \mathrm{RO}(C_m) \\ \alpha &\mapsto \alpha^{C_n} \end{aligned}$$

The name *poly*-representation is chosen partly because it is a family of many representations, but more importantly to emphasize the connection with polygonic spectra [KMN23]: the representation spheres $S^{\underline{\alpha}(C_n)}$ assemble into a polygonic spectrum $S^{\underline{\alpha}}$ (which in particular is $\otimes$-invertible). Equivalently, a poly-representation is a continuous $\widehat{\mathbf{Z}}$-representation $\underline{\alpha}$ such that $\underline{\alpha}(C_n) = \underline{\alpha}^{n\widehat{\mathbf{Z}}}$ is finite-dimensional for all $n > 0$. We will mostly work with p-typical poly-representations, in which case $\widehat{\mathbf{Z}}$ is replaced by $\mathbf{Z}_p$. We will study the polyrepresentation-graded TR of smooth algebras S over a perfectoid base R.

Flipping and reversing the dimensional encoding, we write a p-typical poly-representation as

$$\underline{\alpha} = (\underline{d}_0, \underline{d}_1, \dots),$$

where $\underline{d}_i = \dim_{\mathbf{C}} \underline{\alpha}(C_{p^n})$ should be thought of $\dim_{\mathbf{C}}(\text{“}\alpha^{1/C_{p^i}}\text{”})$, and indeed we sometimes write $d_{-i}(\underline{\alpha})$ instead of $\underline{d}_i$. We highlight some important families of poly-representations.

Definition 1.15. The *irreducible* poly-representations are

$$\lambda^{1/k} \colon C_n \mapsto \begin{cases} \lambda^{n/k} & k \mid n \\ 0 & k \nmid n \end{cases}$$

In the p-typical case, we set $\underline{\lambda}_i = \lambda^{1/p^i}$. We have $\underline{d}_j(\underline{\lambda}_i) = [j \geq i]$, where $[P]$ is the Iverson bracket.

Definition 1.16. The *unit* poly-representations are

$$\underline{\lambda}_i - \underline{\lambda}_{(i+1)}$$

These are unit vectors for the dimensional encoding: $\underline{d}_j(\underline{\lambda}_i - \underline{\lambda}_{(i+1)}) = [i = j]$. While we rarely refer to these poly-representations explicitly, they are the degrees of our classes $\underline{a}_i$ and $\underline{u}_i$.

Definition 1.17. For a positive rational number $\nu \in \mathbf{Q}_{>0}$, the *regular poly-representation of slope* ν is

$$[\![\nu]\!]_\lambda \colon C_n \mapsto [[\lceil n\nu \rceil]]_\lambda := \lambda^0 + \dots + \lambda^{\lceil n\nu \rceil - 1}.$$

These are so-named because $[\![1]\!]_\lambda$ specializes to the complex regular representation of each C_n. Beware that the irreducible decomposition of $[\![\nu]\!]_\lambda$ is infinite. This has dimension sequence $\underline{d}_i([\![\nu]\!]_\lambda) = \lceil p^i\nu \rceil$, or integrally $d_{1/n}([\![\nu]\!]_\lambda) = \lceil n\nu \rceil$.

Example 1.18. Let k be a perfectoid ring, then there is a fiber sequence

$$\widetilde{\mathbf{Z}_p(i)}(k[x]/x^e) \to \mathbf{TR}_{2i-[\![1]\!]_\lambda}(k) \xrightarrow{\mathbf{V}^e} \mathbf{TR}_{2i-[\![1/e]\!]_\lambda}(k) \tag{2}$$

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A basic computation is that

$$\mathbf{TR}_{2i-[\![\nu]\!]_\lambda}(k) = \mathbf{W}_{\lfloor i/\nu \rfloor}(k);$$

this is essentially [HM97b, Proposition 9.1] + [HM97a, Theorem 4.2.10], but see [Sul23a, Corollary 4.6] for the present formulation (which is mildly more general). This immediately implies the syntomic cohomology

$$\widetilde{\mathbf{Z}_p(i)}(k[x]/x^e) = \frac{\mathbf{W}_{ei}(k)}{\mathbf{V}^e\mathbf{W}_i(k)}[-1]$$

which in turn gives the (reduced, p-adic) K-theory $\widetilde{\mathrm{K}}_*(k[x]/x^e; \mathbf{Z}_p)$ of $k[x]/x^e$.

In [Sul23a], this syntomic cohomology is computed by first computing $\mathcal{N}^{\geq *}\mathbb{\Delta}_{k[x]/x^e}$ and $\mathbb{\Delta}_{k[x]/x^e}$, but the big Witt vectors appear somewhat miraculously. This method makes the appearance of big Witt vectors clearer, and also generalizes to perfectoid rings. The calculation of $\widetilde{\mathbf{Z}}_p(i)(k[x]/x^e)$ for k perfectoid was done independently by Riggenbach using different methods [Rig22a, Theorem 1.1], and we will build on some of his tools in §3.

We therefore view this method of computing syntomic cohomology of monoid algebras as superior to the one used in [Sul23a, §3]. The main disadvantage is that it is extremely difficult to establish the fiber sequence (2) [HM97a]. However, this should be much easier to do in the modern prismatic approach, and to extend to more general monoid algebras. We will not pursue this here, but our work is intended to be a repository of computations to accompany such an endeavor.

Remark 1.19. The truncated polynomial algebra $S[x]/x^e$ can also be written as $S[y^{1/e}]/y$, where $y = x^e$. This perspective is fruitful as the the limit $R[y^{1/p^\infty}]_p^\wedge$ is an important example of a quasi-regular semiperfectoid ring. The poly-representation perspective suggests that we should expand "truncated polynomial algebras" to include $S[y^\nu]/y$ for general $\nu \in \mathbf{Q}_{>0}$. We expect (but have not checked) that our results hold for such rational-slope truncated polynomial algebras.

In the base cases of a perfect $\mathbf{F}_p$-algebra k or a p-torsionfree perfectoid ring R^+, the full rings $\mathrm{TF}_\bigstar(k)$ and $\mathrm{TF}_\bigstar(R^+;\mathbf{Z}_p)$ were computed in [Sul22]. This in principle implies the computation of $\mathrm{TR}_\bigstar$ via the long exact sequences

$$\mathrm{TF}_{\alpha+\lambda_i}(R;\mathbf{Z}_p) \xrightarrow{a_{\lambda_i}} \mathrm{TF}_\alpha(R;\mathbf{Z}_p) \to \boxed{\mathrm{TR}^i_\alpha(R;\mathbf{Z}_p)} \xrightarrow{\partial} \mathrm{TF}_{\alpha+\lambda_i-1}(R;\mathbf{Z}_p),$$

but it is nontrivial to unpack this; we will instead proceed directly by adapting the homotopy orbits to TR spectral sequence. The results of [Sul22] may be informally summarized as follows:

- in even degrees, $\mathrm{TF}_\bigstar(R;\mathbf{Z}_p)$ is torsionfree and satisfies an $\mathrm{RU}(\mathbf{T})$-graded form of Bökstedt periodicity;
- in the characteristic p case, there are additional classes witnessing the "collision" of the distinguished elements $d = \phi(d) = \phi^2(d) = \cdots$;
- the odd degrees contain deep caverns with dangerous monsters.

In light of the extraordinary complexity of even the base case, we will not attempt to compute the full $\mathrm{TR}_\bigstar(S)$, but rather focus our attention on certain safe regions:

Definition 1.20. Consider the following swathes of poly-representations.

- the *negative range* consists of $\bigstar = *-V$ with $* \in \mathbf{Z}$ and V an actual poly-representation; equivalently, $\underline{\alpha} = (\underline{d}_0, \underline{d}_1, \dots)$ is in the negative range if $\underline{d}_i \geq \underline{d}_{(i+1)}$ for all i. This is the range that arises in algebraic K-theory calculations.
- the *positive range* consists of $\bigstar = V-*$ with $* \in \mathbf{Z}$ and V an actual poly-representation; equivalently, $\underline{\alpha} = (\underline{d}_0, \underline{d}_1, \dots)$ is in the positive range if $\underline{d}_i \leq \underline{d}_{(i+1)}$ for all i. This range is of interest because the Norm from the $\mathbf{Z}$-graded de Rham-Witt complex lands in here, allowing us to extend the Witt vector Norm [Ang15, Sul23b] to the de Rham-Witt complex. (Also, a tiny portion of it is used in computing the slice filtration [Sul20, Lemma 4.11].)
- the *contiguous range* consisting of those $\underline{\alpha} = (\underline{d}_0, \underline{d}_1, \dots)$ for which $\{i \mid d_i(\underline{\alpha}) \geq 0\}$ is a contiguous interval $[-\ell, -k]$ or $(-\infty, k]$. We say that $\underline{\alpha}$ is *non-negatively concentrated in* $[-\ell, -k]$ or $(-\infty, k]$.

The negative and positive ranges are exactly the ranges that can be handled by the cellular method; however, this is a very difficult way to proceed. We instead skip straight to the contiguous range and employ the *homotopy orbits to TR spectral sequence* (HOTRSS) introduced in [AG11] and upgraded in [Sul22]. The homotopy orbits to TR spectral sequence can handle all of $\mathrm{TR}_\bigstar$, but the answer becomes unwieldy outside of the contiguous range.

We need two more ingredients to state the result. Recall that the *canonical filtration* of the de Rham-Witt complex is given by

$$\begin{aligned}\mathrm{Fil}^{k+1}\,\mathrm{W}_n\Omega^j_{S/R} &= \ker(\mathrm{W}_n\Omega^j_{S/R} \xrightarrow{R^{n-k}} \mathrm{W}_k\Omega^j_{S/R}) \\ &= \mathrm{im}(V^{k+1} + dV^{k+1})\end{aligned}$$

for $n \in \mathbf{N} \cup \{\infty\}$. (Remember that in our notation Witt vectors start at W_0.) Next, in the $\mathbf{Z}$-graded case, Hesselholt's theorem [Hes96] says that TR_* splits into contributions from individual $\mathrm{W}_\bullet\Omega^j$ terms. This is also true in the fancy-graded case (either from the proof of Hesselholt's theorem, or by the homotopy orbits to TR spectral sequence), so we let $\mathrm{TR}_{\bigstar,j}(S)$ denote the $\mathrm{W}\Omega^j$ contribution to $\mathrm{TR}_{\bigstar+j}(S)$. Pulling out j this way, the computation becomes essentially the same for the base case $S = R$, although we have to be slightly more careful with Fil.

Theorem 1.21 (Contiguous range of $\mathrm{TR}_\bigstar$, Theorem 4.12)**.** *Let $\underline{\alpha} = (\underline{d}_0, \dots, \underline{d}_n)$ be a $\mathbf{Z}/p^n$-polyrepresentation which is non-negatively concentrated in $[-\ell, -k]$.*

If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then there are canonical identifications

$$\mathrm{TR}^n_{\underline{\alpha},j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) = \mathrm{Fil}^{k-(\underline{d}_k+\cdots+\underline{d}_\ell)}\,\mathrm{W}_\ell\Omega^j_S \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_\ell}\}$$
$$\mathrm{TR}^n_{\underline{\alpha}-1,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) = \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\underline{d}_k+\cdots+\underline{d}_\ell}} \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_{(k-1)}}\}$$

This also holds for $n = \infty$.

If S is formally smooth over a p-torsionfree perfectoid ring R^+, then there are canonical identifications

$$\mathrm{TR}^n_{\underline{\alpha},j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)}) = \mathrm{Fil}^k\,\mathrm{W}_\ell\Omega^j_{S/R^+} \otimes \bigotimes_{i=0}^{\ell}\{\phi^{-i}(\xi)\}^{\underline{d}_i}$$
$$\mathrm{TR}^n_{\underline{\alpha}-1,j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)}) = \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k}\cdots\phi^{-\ell}(\xi)^{\underline{d}_\ell}} \otimes \bigotimes_{i=0}^{k-1}\{\phi^{-i}(\xi)\}^{\underline{d}_i}$$

This also holds for $n = \infty$ if $\ell < \infty$ or if $R^+ = \mathcal{O}_C$.

While the restriction to the contiguous range is disappointing, note that the contiguous range is already massively larger than the range of degrees that have been previously studied; besides, the point of this paper is to open the doors to more $\mathrm{RO}(G)$-graded ideas in the field, not to serve as the final word on it. Restricting to the contiguous range is most distressing for the following reason:

$$\{\underline{\alpha} \mid \underline{\alpha} + i \text{ is contiguous for all } i \in \mathbf{Z}\}$$

is easily seen to be the union of the positive and negative ranges, so only in these ranges can we fully mix our result with the Postnikov filtration.

Returning to the negative and positive ranges, we obtain the following special cases.

Corollary 1.22 (Negative range)**.** *Let $\underline{\alpha} = \sum_{r=0}^{\infty} \underline{k}_r \underline{\lambda}_r$ be an actual poly-representation. Given $i \geq 0$, let ℓ be such that*

$$i - \underline{d}_{\ell+1}(\underline{\alpha}) < 0 \leq i - \underline{d}_\ell(\underline{\alpha}).$$

If S is formally smooth over an arbitrary perfectoid ring R, there are canonical identifications

$$\begin{aligned}\mathrm{TR}^n_{2i-\underline{\alpha},j}(S;\mathbf{Z}_p) &= \mathrm{W}_\ell\Omega^j_{S/R} \otimes \bigotimes_{r=0}^{\ell}\{\phi^{-r}(\xi)\}^{i-\underline{d}_r}\\ &= \mathrm{W}_\ell\Omega^j_{S/R} \otimes ???\\ \mathrm{TR}^n_{(2i-\underline{\alpha})-1,j}(S;\mathbf{Z}_p) &= 0\end{aligned}$$

positive range

application to $\widetilde{\mathrm{K}}(S[x]/x^e;\mathbf{Z}_p)$, compare with [Rig22a]

Now that we know the answer in advance, we can work out how the cellular method actually plays out. This leads to an interesting filtration on the groups.

Theorem 1.23 (Cellular filtration - negative)**.** *todo*

Theorem 1.24 (Cellular filtration - positive)**.** *todo*

Remark 1.25. The notations $\mathbf{Z}/n$ and C_n are often used interchangeably for a cyclic group of order n. In this paper, we will think of $\mathbf{Z}/n$ as a quotient of $\widehat{\mathbf{Z}}$, while we think of C_n as embedded in $\mathbf{T}$ (so perhaps a better notation would be $\mathbf{Z}/n\{-1\}$ as in Example 1.9, the Pontryagin dual $(\mathbf{Z}/n)^\wedge$, or μ_n, although the latter can be confused for a group scheme). As we will see in §4, the passage between these two perspectives involves the Cartier isomorphism (Example 4.1); it would be interesting to know if there is a deeper connection between Pontryagin duality and the Cartier isomorphism.

1.3. **Norms.** While the $\mathrm{RO}(G)$-graded groups are interesting in their own right, our main reason for computing them is to be able to understand the Norm even in integer degrees $\neq 0$. For a general C_{p^∞}-E_∞-ring, the norm maps are determined in ghost coordinates by

$$FNx = x^p \tag{3}$$
$$Nx = \Phi^p(x) \bmod V \tag{4}$$

for some ring map $\Phi^p \colon E^{\Phi C_{p^{n-1}}} \to E^{\Phi C_{p^n}}$. We summarize this in the following diagram:

$$\begin{tikzcd} E_\alpha^{C_{p^n}} \arrow[r] & E_{\mathrm{ind}\,\alpha}^{\Phi C_{p^n}} \\ E_\alpha^{C_{p^{n-1}}} \arrow[r] \arrow[u, "N"] & E_{\mathrm{ind}\,\alpha}^{\Phi C_{p^{n-1}}} \arrow[u, "\Phi^p"'] \end{tikzcd}$$

In the de Rham-Witt situation, Φ^p is the pth Adams operation ψ^p, while in the prismatic setting it is ψ^p followed by the Nikolaus-Scholze Frobenius φ:

$$\begin{tikzcd} \mathrm{THH}^{hC_{p^n}} \arrow[r] & \mathrm{THH}^{\tau C_{p^n}} \\ \mathrm{THH}^{hC_{p^{n-1}}} \arrow[r] \arrow[u, "N"] & \mathrm{THH}^{\tau C_{p^{n-1}}} \arrow[u, "\varphi\psi^p"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{TR}^n \arrow[r] & \mathrm{THH}_{\otimes C_{p^n}} \\ \mathrm{TR}^{n-1} \arrow[r] \arrow[u, "N"] & \mathrm{THH}_{\otimes C_{p^{n-1}}} \arrow[u, "\psi^p"'] \end{tikzcd}$$

In both settings, the identities (3), (4) uniquely determine N for p-torsionfree input rings, and we can reduce to that case by functoriality. So there are three steps to understanding the norm:

(a) describe the $\mathrm{RO}(G)$-graded groups that serve as the domain and codomain;
(b) understand the Adams operation ψ^p;
(c) find a unified formula which satisfies (3), (4), but makes sense for possibly p-torsionful rings.

We have basically already done (a), but we need to specialize the result to $\pi_{\mathrm{ind}\,\alpha}$. To do so, we make the following definition.

Definition 1.26. Let (A, I) be a prism. The (ideals generated by the) *prismatic factorial* and *prismatic Gamma function* (relative to (A, I)) are defined as

$$\{(n)_{\Delta}!\} = \bigotimes_{r=0}^{\infty} \phi^r(I)^{\left\lfloor \frac{n}{p^{r+1}} \right\rfloor}$$
$$\{(n)_{\Delta}?\} = \{(n-1)_{\Delta}!\}$$

When $(A, I) = (\mathbf{Z}_p[\![q-1]\!], (p)_q)$ is the q-de Rham prism, we have $\{(n)_{\Delta}!\} = (n)_q! A$ and $\{(n)_{\Delta}?\} = \Gamma_q(n) A$. We do not know how to define actual elements $(n)_{\Delta}!, (n)_{\Delta}? \in A$ for other prisms, only the ideals they generate. Readers are free to use $\Gamma_{\Delta}(n)$ instead of $(n)_{\Delta}?$ if they prefer, but the latter notation fits better with the philosophy of [Sul23a, §6].

The norms out of the $\mathbf{Z}$-graded degrees will then go

$$N^k = N_n^{n+k} \colon \frac{\mathcal{N}^{\geq i} \hat{\Delta}_R \otimes I_n^i}{\mathcal{N}^{\geq i+1} \hat{\Delta}_R \otimes I_n^{i+1}} \longrightarrow \frac{\mathcal{N}^{\geq p^k i} \hat{\Delta}_R \otimes I_{n-1}^{p^k i} \otimes \left\{ \dfrac{\phi^{n-1}(p^{k+1} i)_{\Delta}!}{\phi^{n+k} \lfloor i/p \rfloor_{\Delta}!} \right\}}{\mathcal{N}^{\geq p^k i+1} \hat{\Delta}_R \otimes I_{n-1}^{p^k i} \otimes \left\{ \dfrac{\phi^{n-1}(p^{k+1} i)_{\Delta}!}{\phi^{n+k} \lfloor i/p \rfloor_{\Delta}!} \right\} \otimes I_{n+k}}$$

$$N^k = N_n^{n+k} \colon \mathrm{W}_n\Omega^j \otimes \{\xi_{n+1}^i\} \longrightarrow \mathrm{W}_{n+1}\Omega^j \otimes \{\xi_{n+1}^i\}$$

By multiplicativity, it suffices to identify the following special cases: *prismatic norms*

$$N_n^{n+k}\colon \frac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}_R}{\mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}_R\otimes I_n} \longrightarrow \frac{\mathcal{N}^{\geq p^k i}\hat{\mathbb{\Delta}}_R}{\mathcal{N}^{\geq p^k i+1}\hat{\mathbb{\Delta}}_R\otimes I_{n+k}} \qquad n>0$$

$$N_0^{k}\colon \frac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}_R}{\mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}_R} \longrightarrow \frac{\mathcal{N}^{\geq p^k i}\hat{\mathbb{\Delta}}_R\otimes\left\{\dfrac{(p^k i)_{\mathbb{\Delta}}!}{\phi^k(i)_{\mathbb{\Delta}}!}\right\}}{\mathcal{N}^{\geq p^k i+1}\hat{\mathbb{\Delta}}_R\otimes\left\{\dfrac{(p^k i)_{\mathbb{\Delta}}!}{\phi^k(i)_{\mathbb{\Delta}}!}\right\}\otimes I_{n+k}},$$

Breuil-Kisin norms (in both prismatic and de Rham-Witt notation)

$$N_n^{n+k}\colon \frac{\mathrm{A_{inf}}\otimes\phi^i(I)}{\mathrm{A_{inf}}\otimes\phi^i(I)\otimes I_n} \longrightarrow \frac{\mathrm{A_{inf}}\otimes\phi^i(I)^{p^k}}{\mathrm{A_{inf}}\otimes\phi^i(I)^{p^k}\otimes I_{n+k}} \qquad 0\leq i<n-1$$

$$N_n^{n+k}\colon \frac{\mathrm{A_{inf}}\otimes\phi^{n-1}(I)}{\mathrm{A_{inf}}\otimes\phi^{n-1}(I)\otimes I_n} \longrightarrow \frac{\mathrm{A_{inf}}\otimes\left\{\dfrac{\phi^{n-1}(p^k)_{\mathbb{\Delta}}!}{}\right\}}{\mathrm{A_{inf}}\otimes\left\{\dfrac{\phi^{n-1}(p^k)_{\mathbb{\Delta}}!}{}\right\}\otimes I_{n+k}}$$

and *de Rham-Witt norms* (p odd, $j\in\frac{1}{2}\mathbf{N}$)

$$N_n^{n+k}\colon \mathrm{W}_n\Omega^{2j}_{S/R}\longrightarrow\bigoplus_{r=0}^{k}\mathrm{Fil}^r\,\mathrm{W}_{n+k}\Omega^{2jp^r}_{S/R}\otimes\left\{\frac{\phi^{-(k+1)}(\xi_n^{p^k}(p^{k+1})_{\mathbb{\Delta}}!)}{\xi^{p^r}_{n+k+1}}\right\}^{\otimes j}\subset\mathrm{TR}^{n+k}_{\mathrm{ind}_n^{n+k}(2j)}(S;\mathbf{Z}_p)$$

Tensoring by the half-integer j makes sense since $(p^k-p^r)j\in\mathbf{N}$ for all k,j,p. It would of course suffice to determine the case $k=1$, but the general perspective is clarifying, especially for integral generalizations.

As for (b), it is well-known when $p\nmid\ell$ that the Adams operation ψ^ℓ acts on all invariants in sight as multiplication by ℓ^i in weight i [BMS19, Proposition 9.14], [Rak20, Proposition 6.4.12]. However, we need to understand the Adams operation ψ^p, or more generally ψ^{p^k}, which are mostly ignored in the literature. While ψ^p does not make sense on invariants like prismatic cohomology or the de Rham-Witt complex, it still acts on invariants like Hodge-Tate cohomology and topological Hochschild homology. We show in Appendix A that ψ^p still acts as multiplication by p^i in weight i for these invariants.

As for (c), the formula used in [Ang15] and [Sul23b] is

$$N(f)=\Phi^p(f)-V\delta(f) \tag{5}$$

making use of the δ-structures available on Witt vectors and on prisms. It is easy to see that this satisfies the congruences (3) and (4). This formula is not suitable for our generalizations, since it is not *a priori* clear how to apply δ to a Breuil-Kisin twist or a differential form.

Theorem 1.27 (Prismatic norms)**.**

todo

Now, to fix ideas, we examine the Norm map

$$N_1^2\colon\frac{\hat{\mathbb{\Delta}}_R\otimes I}{\hat{\mathbb{\Delta}}_R\otimes I^2}=\pi_2\mathrm{THH}(R;\mathbf{Z}_p)^{tC_p}\to\pi_{[p]_{\lambda^p}}\mathrm{THH}(R;\mathbf{Z}_p)^{tC_{p^2}}=\frac{\hat{\mathbb{\Delta}}_R\otimes\{(p^2)_{\mathbb{\Delta}}!\}}{\hat{\mathbb{\Delta}}_R\otimes\{(p^2)_{\mathbb{\Delta}}!\}\otimes I_1}$$

Let $\widetilde{\xi}$ be a generator of I. The F map from the target lands in

$$\pi_{2p}\mathrm{THH}(R;\mathbf{Z}_p)^{tC_p}=\frac{\hat{\mathbb{\Delta}}_R\otimes I^p}{\hat{\mathbb{\Delta}}_R\otimes I^{p+1}}$$

and is given by dividing the natural inclusion by $p\equiv\phi(\widetilde{\xi})/\delta(\widetilde{\xi})$. Over an arbitrary prism, the only obvious generator of $\{(p^2)_{\mathbb{\Delta}}!\}$ is $\widetilde{\xi}^p\phi(\widetilde{\xi})$, and we find that

$$N_1^2(f\otimes\widetilde{\xi})=N_1^2(f)\otimes\frac{\widetilde{\xi}^p\phi(\widetilde{\xi})}{\delta(\widetilde{\xi})}$$

In general, it is fairly easy to show that Breuil-Kisin norms take a generator to a generator (which does not imply that the norms are isomorphisms, since they are not ring maps). However, pinning down the unit is an interesting challenge.

Theorem 1.28 (Breuil-Kisin norms, general prisms)**.**

Breuil-Kisin norms

However, over the q-de Rham prism, there is a more natural choice of generator for $\{(p^2)_{\mathbb{A}}!\}$, namely $(p^2)_q!$. Therefore, we would like to understand the unit $\dfrac{(p^2)_q!}{(p)_q^p(p)_{q^p}}\delta((p)_q)$. The prototype for such congruences is *Wilson's theorem*

$$(p-1)! \equiv -1 \bmod p$$

The paper cite meclaims that no good q-analogue of Wilson's theorem exists for $(n)_q!$. However, it depends on what one wants out of such a generalization. We find in cite methe following generalization of Wilson's theorem:

$$\frac{(np)!}{n!p^n} \equiv -1 \bmod p$$

and this is clearly the kind of thing we want. In fact, one easily sees that

$$\frac{(p^2)_q!}{(p)_q^p(p)_{q^p}} \equiv (p-1)_q!^p(p-1)! \bmod (p)_q$$

Theorem 1.29 (q-de Rham norms, q-Wilson's theorem)**.**

Wilson's theorem thing

norms of Breuil-Kisin twist given by

$$N_n^{n+k}(f\otimes(p)_{q^{p^i}}) \approx N_n^{n+k}(f)\otimes(p)_{q^{p^i}}^{p^k} \quad 0\le i<n$$
$$N_n^{n+k}(f\otimes(p)_{q^{p^n}}) \approx N_n^{n+k}(f)\otimes(p^{k+1})_{q^{p^n}}!$$

where $\approx$ means up to units; what is the unit?

Another issue arises in the de Rham-Witt situation. For a simple example, take $p=3$ and look at the norm map

$$\begin{aligned}\Omega_S^0\{1\}\oplus\Omega_S^2 &= \mathrm{TR}_2^0(S;\mathbf{Z}_p)\\ &\xrightarrow{N} \mathrm{TR}^1_{[p]_\lambda}(S;\mathbf{Z}_p)\\ &= \bigoplus\begin{pmatrix} \mathrm{W}_1\Omega_S^0\otimes\{\phi^{-2}(p^2)_q!\} & \mathrm{W}_1\Omega_S^2\otimes\left\{\phi^{-2}\left(\dfrac{(p^2)_q!}{(p^2)_q}\right)\right\}\\ \mathrm{Fil}^1\,\mathrm{W}_1\Omega_S^4\otimes\{\xi^{-1}\phi^{-1}(\xi)\} & \mathrm{Fil}^1\,\mathrm{W}_1\Omega_S^6\otimes\{\xi^{-2}\}\end{pmatrix}\end{aligned}$$

hmmm double check the negative twists

This restricts to a map

$$\Omega_S^2 \xrightarrow{N} \left(\mathrm{W}_1\Omega_S^2\otimes\{\phi^{-1}(\xi)^2\}\right)\oplus\mathrm{Fil}^1\,\mathrm{W}_1\Omega_S^6$$

so we need a formula that expresses N as the sum of something in $\ker F$ and something in Fil^1, which is not the shape of (5).

The missing formula is

$$N(\omega) = (p-V(1))\frac{\psi^p(\omega)}{p} + V\frac{\omega^p}{p} \tag{6}$$

It is easy to check that, as a formal expression, this satisfies the necessary congruences. This formula does not make sense in the degree 0 case, but it does in positive degree. Indeed, if ω is a differential form of positive degree, then ω^p and $\psi^p(\omega)$ are divisible by p, in fact canonically so in the p-torsionfree case, which we can reduce to by functoriality. Another subtlety is that the factor $p-V(1)$ gets partly absorbed into the ideal twist (Example 1.32), although this can largely be ignored in practice.

Theorem 1.30 (Norms for the de Rham-Witt complex)**.** *Let R be any perfectoid ring. For a formally smooth R-algebra S, there are Norm maps*

$$N^k = N_n^{n+k} \colon \mathrm{W}_n\Omega^{2j}_{S/R} \longrightarrow \bigoplus_{r=0}^{k} \mathrm{Fil}^r\, \mathrm{W}_{n+k}\Omega^{2jp^r}_{S/R} \otimes \left\{ \frac{\phi^{-(k+1)}(\xi_n^{p^k}(p^{k+1})_{\Delta}!)}{\xi_{n+k+1}^{p^r}} \right\}^{\otimes j}$$

When $R = k$ is a perfect $\mathbf{F}_p$-algebra, these assemble to an infinite Norm map

$$\begin{aligned} N^k \colon \mathrm{W}\Omega^{2j}_S &\to \mathrm{Fil}^k\, \mathrm{W}_{n+k}\Omega^{2jp^k}_S \otimes \left\{ \frac{(p^{k+1})!}{p^{(k+1)p^k}} \right\}^{\otimes j} \\ &= \mathrm{Fil}^k\, \mathrm{W}_{n+k}\Omega^{2jp^k}_S \otimes \left\{ p^{[k+1]_p-(k+1)p^k} \right\}^{\otimes j} \end{aligned}$$

When $R = \mathcal{O}_C$ is spherically complete, they assemble to an infinite Norm map

$$N^k \colon \mathrm{W}\Omega^{2j}_{S/\mathcal{O}_C} \longrightarrow \bigoplus_{r=0}^{k} \mathrm{Fil}^r\, \mathrm{W}\Omega^{2jp^r}_{S/\mathcal{O}_C} \otimes \left\{ \frac{\phi^{-(k+1)}(\mu^{p^k}(p^{k+1})_q!)}{\mu^{p^r}} \right\}^{\otimes j}$$

These maps satisfy the following properties:

- *when $j = 0$, $N(f) = f - V\delta(f)$ is the Norm map of Witt vectors* [Ang15]*;*
- *when $j > 0$, N is given by*

$$\begin{aligned} N(\omega) &= (p - V(1))\frac{\psi^p(\omega)}{p} + V\frac{\omega^p}{p} \\ &= (p - V(1))p^{2j-1}\omega + V\frac{\omega^p}{p} \end{aligned}$$

  or more generally

$$\begin{aligned} N^k(\omega) &= \sum_{r=0}^{k} V^r(p^{k-r} - V\lfloor p^{k-r-1} \rfloor)\frac{\psi^{p^{k-r}}(\omega)^{p^r}}{p^k} \\ &= \sum_{r=0}^{k} V^r(p^{k-r} - V\lfloor p^{k-r-1} \rfloor)p^{2(k-r)jp^r-k}\omega^{p^r} \end{aligned}$$

- *when $R = k$ is a perfect $\mathbf{F}_p$-algebra, the preceding formula simplifies to*

$$N^k(\omega) = V^k\frac{\omega^{p^k}}{p^k},$$

  and in particular $N(\omega) = 0$ whenever ω is of odd degree;
- *N satisfies the following identities for all ω, τ:*

$$\begin{aligned} FN\omega &= \omega^p \\ NV\omega &= p^{p-2}V^2\omega^p \\ NR\omega &= RN\omega \\ N(\omega\tau) &= N(\omega)N(\tau) \\ N(\omega+\tau) &= N(\omega) + N(\tau) + Vs_1(\omega,\tau) \\ &\qquad \textit{where } (a+b)^p = a^p + b^p + ps_1(a,b) \end{aligned}$$

Warning 1.31. It is not obvious that (6) simplifies to $N(\omega) = V\frac{\omega^p}{p}$ in characteristic p, since some of the factor $(p - V(1))$ gets absorbed into the ideal twist. However, a small calculation shows that this factor still vanishes in characteristic p.

Example 1.32. Let p be odd, let R be any perfectoid ring, and consider the p-complete affine line $X = (\mathbf{A}^1)^\wedge_p = \operatorname{Spf} R[x]^\wedge_p$. We have

$$N^1_0(dx) = \left(-\frac{\phi^{-1}(\xi)^{\frac{p+1}{2}}}{\phi^{-1}\delta(\xi)}dx\right) \otimes \phi^{-1}(\xi)^{\frac{p-1}{2}}$$
$$\in \mathrm{W}_1\Omega^1_X \otimes \{\phi^{-1}(\xi)^{\frac{p-1}{2}}\}$$

In characteristic p, we have $\frac{p+1}{2} \geq \frac{3+1}{2} = 2$, so the term $\phi^{-1}(\xi)^{\frac{p+1}{2}} = p^{\frac{p+1}{2}}$ vanishes in $\mathrm{W}_1(k)$.

describe $N^{n+1}_n(dx)$

Example 1.33. Consider (p-complete) affine $2p$-space $X = (\mathbf{A}^{2p}_R)^\wedge_p = \operatorname{Spf} R[x_1, y_1, \ldots, x_p, y_p]^\wedge_p$ over an arbitrary perfectoid ring R. Let

$$\tau = \sum_{i=1}^{p} dx_i\, dy_i \in \mathrm{W}\Omega^2_X$$
$$\omega = \prod_{i=1}^{p} dx_i\, dy_i \in \mathrm{W}\Omega^{2p}_X$$

Then

$$N(\tau) = (p - V(1))p\tau + V(p-1)!\omega$$
$$\in \mathrm{W}\Omega^2_X \otimes \left\{\frac{\phi^{-2}(\mu)^p\xi\phi^{-1}(\xi)^p}{\mu}\right\} \oplus \mathrm{Fil}^1\, \mathrm{W}\Omega^{2p}_X \otimes \left\{\frac{\phi^{-2}(\mu)^p\xi\phi^{-1}(\xi)^p}{\mu^p}\right\}$$

what is going on with the twists here

When R is a perfect $\mathbf{F}_p$-algebra k, this simplifies to

$$N(\tau) = V(p-1)!\omega$$

Remark 1.34 (Notation)**.** We warn the reader that this paper is very dense with notation. In fact, the notation system we introduce should be regarded with the same status as a theorem—the "fundamental technical lemma" from which all else in the paper follows. (This was also largely the case in [Sul23a].) All the notation is reviewed in §2. Here we will comment on the evolution of the notation:

(1) the first major maneuver is to index a representation α by its dimension sequence $d_\bullet(\alpha)$, rather than the coefficients of its irreducible decomposition. This insight is due to Angeltveit-Gerhardt [AG11], and would be a good lesson for the rest of equivariant homotopy theory to learn—for example, it changes the "double cone" pattern seen in computations into a simple quadrant pattern.

To appreciate the difference, note that the dimensional encoding is suited for computing with the isotropy separation square, while the irreducible encoding is suited for computing with cell structures. The isotropy separation method is multiplicative, and this is a major advantage over the cellular method, which is not. See [Zen17] for a detailed comparison of these methods for computing $\underline{\pi}_\bigstar(\mathbf{Z})$ over C_{p^2}.

(2) the next powerful move is to unravel the induction computing $\mathrm{TR}^n_\bigstar$ into a spectral sequence, the homotopy orbits to TR spectral sequence. This again is due to Angeltveit-Gerhardt [AG11, §3]. We gave a more conceptual construction of this spectral sequence (that makes it easy to see the multiplicative and Mackey structure) in [Sul22, §3], and it is this formulation that we will use.

(3) to make the multiplicative and Mackey structures transparent, we shift to indexing using the gold elements. We learned this from Dylan Wilson and [Zen18].

(4) to study $\mathrm{TR}_\bigstar$ rather than just $\mathrm{TR}^n_\bigstar$, one switches to using poly-representations, introduced in [Sul23a, §4.2] in order to conceptualize [HM97a, Proposition 4.2.3].

(5) one major advance in this paper is switching from gold elements to canonical identifications. When R_0 is a perfectoid ring, the canonical identification $\mathrm{TF}_\lambda(R_0; \mathbf{Z}_p) = \xi \mathrm{A}_{\inf}(R_0)$ was explained to the author by Mike Hill at MSRI in 2019. Unreadable canonical identifications of $\mathrm{TF}_\bigstar(R_0; \mathbf{Z}_p)$

and $\mathrm{TR}^{\bullet}_{\bigstar}(R_0;\mathbf{Z}_p)$ in the positive and negative ranges appeared in the author's PhD thesis. We discovered the q-gold relation [Sul20, Lemma 4.9] a few months later; this essentially implies the canonical identifications, but it took years for the author to understand this. The first canonical identifications in the generality of quasiregular semiperfectoid rings are due to Riggenbach [Rig22a].

(6) to extend the homotopy orbits to TR spectral sequence from perfectoid rings to smooth algebras over such, the main simplification is to pull out $\mathrm{TR}_{\bigstar,j}(S;\mathbf{Z}_p)$, the contribution of $\mathrm{W}\Omega^j_S$ to $\mathrm{TR}_{\bigstar+j}(S;\mathbf{Z}_p)$. This is effectively Hesselholt's theorem [Hes05, Theorem 11] that

$$\mathrm{TR}_{\bigstar}(S;\mathbf{Z}_p) = \mathrm{W}\Omega^*_S \otimes_{\mathrm{W}(R)} \mathrm{TR}_{\bigstar}(R;\mathbf{Z}_p).$$

check that this is literally true beyond the negative range

(7) (the power of negative thinking) in both [Sul22] and [Sul23a], there is a lot of fussing over whether a given expression is negative or not. Usually this is because we have some filtration like $\xi^i A$, and when i is negative we want to interpret this as $\xi^{\max(i,0)}A$ (i.e. make the filtration constant in negative degrees), rather than the ξ-adic filtration on the localization $A[\xi^{\pm}]$. By interpreting filtrations this way, and similarly defining $\mathrm{W}_i(R) = 0$ for $i < 0$, we can avoid handling $2^\infty - 1$ cases in exchange for complicated expressions like $\mathrm{W}_{-(\underline{d}_1+\underline{d}_2)}\Omega^j$ that will simplify in all practical situations.

(8) (accepting what we cannot know) in §4, we first compute $\mathrm{TR}^{\bullet}_{\bigstar}$ explicitly as modules, then show how to make canonical identifications. This is partly responsible for our restriction to the contiguous range: in characteristic p, we could compute $\mathrm{TR}^{\bullet}_{\bigstar}$ with no restrictions, but the formulas are overwhelming; in mixed characteristic, determining the kernels of the differentials in the homotopy orbits to TR spectral sequence is a non-trivial commutative algebra problem once one leaves the contiguous range. However, it should still be possible to give these more complicated groups "arithmetically meaningful names", and leave it to to the arithmetic geometers to determine the precise module structure of these invariants. For obtaining these arithmetically meaningful names, we believe the best route is the philosophy of Remark 1.35(2).

1.4. **Further directions.**

We briefly speculate on some natural follow-ups to this paper.

Remark 1.35 (Iterated Nygaard filtrations)**.** Define *higher topological negative cyclic homology* by

$$\mathrm{TC}^{-r} = (\mathrm{TR}^r)^{h\mathbf{T}/C_{p^r}}$$

This is equivalently the pullback

$$\mathrm{TC}^- \times_{\mathrm{TP}} \cdots \times_{\mathrm{TP}} \mathrm{TC}^-,$$

where the structure maps are can: $\mathrm{TC}^- \to \mathrm{TP}$ on the right and $\varphi\colon \mathrm{TC}^- \to \mathrm{TP}$ on the left. In particular we have $\mathrm{TC}^- = \mathrm{TC}^{-0}$, while morally $\mathrm{TP} = \mathrm{TC}^{-(-1)}$. Andriopoulos studied these spectra in his thesis [And24].

There are several motivations for considering these spectra:

(1) Nygaard's original paper [Nyg81] considered not just $\mathcal{N}^{\geq i}\mathrm{W}\Omega = \varphi^{-1}(p^i\mathrm{W}\Omega)$, but also

$$\mathcal{N}^{\geq i}_r\mathrm{W}\Omega := \varphi^{-r}(p^{ir}\mathrm{W}\Omega).$$

The modern prismatic version would be

$$\mathcal{N}^{\geq i}_r {\Delta}^{(r)}_{R/A} = \varphi^{-r}(I^i_r {\Delta}_{R/A})$$

where $I_r = I\phi(I)\cdots\phi^{r-1}(I)$; in particular, the elements $\xi_r = \xi\phi^{-1}(\xi)\cdots\phi^{-(r-1)}(\xi)$ from [BMS18, Lemma 3.12] appear naturally in this context. When R is a perfectoid ring, we have

$$\mathrm{TC}^{-r}_*(R;\mathbf{Z}_p) = \frac{\mathrm{A}_{\mathrm{inf}}(R)[\sigma,t_r]}{\sigma t_r - \xi_{r+1}} \quad \text{with } |\sigma| = 2,\ |t_r| = -2$$

by [And24, Proposition 3.4].

(2) notably, Andriopoulos introduces the idea of viewing TR^r as the quotient TC^{-r}/t_r, generalizing the identification $\mathrm{TC}^-/t = \mathrm{THH}$ [And24, Proposition 3.6]. Just as TC^- is easier to understand than THH (since it has less torsion), TC^{-r} and $\mathrm{TR}^{h\mathbf{T}}$ are easier to understand than TR^r and TR, and this should clarify the passage between the cyclotomic spectra perspective and the topological Cartier module/polygonic spectra perspective [AN21, KMN23].

(3) it is hard to understand what the norm

$$N\colon \mathrm{THH}^{hC_{p^n}} \to \mathrm{THH}^{hC_{p^{n+1}}}$$

is really doing arithmetically, since the finite-level homotopy groups have a lot of torsion. If we could take $(-)^{h\mathbf{T}}$ of the norm $N\colon \mathrm{THH} \to \mathrm{TR}^1$, we might be able to get a norm $N\colon \mathrm{TC}^- \to \mathrm{TC}^{-1}$, which should be easier to understand. However, it is unclear if this works.

(4) it is natural to ask if $\mathrm{RO}(\mathbf{T})$-grading makes sense for TC. The answer turns out to be no, but you'd stumble upon TC^{-r} while trying. (However, [DM23b, Corollary 1.2] suggests that it might be possible to define *polyrepresentation*-graded TC; we do not explore this in this paper.)

(5) (integral versions of) these spectra arise when constructing Habiro cohomology via refined topological Hochschild homology: see [MW24, 1.23] and [SG25, Theorem 6.5]. Relatedly, they arise in Devalapurkar-Hahn-Raksit-Yuan's work on prismatization of ring spectra [DHRY25] for the purpose of decompleting the Nygaard filtration, see the discussion in [Dev25, Chapter 7].

A thorough treatment of these spectra should touch on all of these threads, so we do not really study them in this paper. However, we show how to compute $\mathrm{TC}^{-1}_{\bigstar}$ of quasiregular semiperfectoid rings in Example 3.5. The answer involves *iterated* Nygaard filtrations such as

$$\mathcal{N}^{\geq d_0}\mathcal{N}^{\geq d_1}\Delta^{(2)}_{R/A} = \{f \in \Delta^{(2)}_{R/A} \mid \phi(f) \in I^{d_1}\Delta^{(1)}_{R/A},\ \phi^2(f) \in I^{d_0}\phi(I)^{d_1}\Delta_{R/A}\}.$$

Remark 1.36 (Legendre filtrations)**.** Our research program got started with the following question, posed to the author by Mike Hill in 2018: what happens if we take the BMS filtration on THH and replace the Postnikov filtration with the regular equivariant slice filtration? This requires us to calculate the slice filtration of (the genuine spectra whose fixed points are) $\mathrm{TC}^-(R;\mathbf{Z}_p)$ and $\mathrm{TP}(R;\mathbf{Z}_p)$ when R is quasiregular semiperfectoid.

We computed the case where R is perfectoid in [Sul20]; the result says that

$$\mathrm{gr}^n_{\mathrm{slice}}\,\mathrm{THH}(R;\mathbf{Z}_p) = \Sigma^{[n]_\lambda}\,\mathrm{inj}_{C_n}\,\underline{\pi}_{[n]_\lambda}\mathrm{THH}(R;\mathbf{Z}_p)$$

which implies cite methe same is true for TC^- and TP, at least for $i \geq 0$. With the results of this paper, we can make canonical identifications

compute slice filtration

where (A, I) is the perfect prism corresponding to R, and $\underline{(A, I)}$ is the Mackey (in fact Tambara) functor constructed in [Sul23b].

We conjecture that this identification also holds for R quasiregular semiperfectoid, but we don't know how to prove it: the identification in [Sul20] relied heavily on the fact that $\tau^{\mathrm{Post}}_{\geq 2i}\mathrm{THH}(R;\mathbf{Z}_p) = \Sigma^{2i}\mathrm{THH}(R;\mathbf{Z}_p)$, which is not true for general quasiregular semiperfectoid rings.

Remark 1.37 (Poly-de Rham-Witt complex)**.** Our computations imply a new expansion of the de Rham-Witt complex. In general, there is a map

$$\mathrm{W}_\bullet\Omega^*_{S/R} \to \mathrm{TR}^\bullet_*(S \,|\, \mathbf{S}_{\mathrm{A_{inf}}(R)})$$

because $\mathrm{W}_\bullet\Omega^*$ is defined exactly as the universal example of "the kind of structure $\mathrm{TR}^\bullet_*$ has". This is quite vague, but in fact different authors get away with axiomatizing different fragments of the d, F, V, R structure; and when TR becomes more complicated than the classical algebraic case, we can always generalize the definition of this structure, as with the p-typical Cartier complexes of [AN21, Definition 3.34] or the η-deformed (p-typical) Cartier complexes of [AR24, Definition 4.17].

The polyrepresentation-graded groups $\mathrm{TR}^\bullet_\bigstar$ possess d, F, V, and R maps just as the $\mathbf{Z}$-graded groups do, but now also have N maps. We can define the four variants (p-typical vs big, truncated vs full) of the *poly-de Rham Witt complex*

$$\begin{array}{cc} \mathrm{W}\Omega^\bigstar_{S/R}, & \mathrm{W}_\bullet\Omega^\bigstar_{S/R}, \\ \mathbf{W}\Omega^\bigstar_{S/R}, & \mathbf{W}_\bullet\Omega^\bigstar_{S/R} \end{array}$$

as the universal examples of this structure. We will not study the poly-de Rham-Witt complex in this paper, but our computations provide an upper bound for it. Again, the poly-de Rham-Witt complex is a filtered and twisted version of the usual de Rham-Witt complex, rather than a totally new object, with the norms being the most interesting new piece of structure.

Remark 1.38 (Picard-de Rham-Witt complex)**.** Angeltveit computes the Picard group $\mathrm{Pic}(\mathrm{Sp}^{C_n})$ of genuine C_n-equivariant homotopy theory in [Ang21], and finds that it splits as the Picard group of the Burnside ring $\mathbf{W}_{C_n}(\mathbf{Z})$ and a free factor corresponding to equivariant representation spheres:

$$\begin{aligned}\mathrm{Pic}(\mathrm{Sp}^{C_n}) &\cong \mathrm{Pic}(\mathbf{W}_{C_n(\mathbf{Z})}) \times \prod_{d|n} \mathbf{Z} \\ &= \prod_{\substack{d|n \\ d\neq 1,2}} (\mathbf{Z}/d)^\times/\{\pm 1\} \times \prod_{d|n} \mathbf{Z}\end{aligned}$$

We only study the second factor in this paper. It would be interesting to extend our computations to all of $\mathrm{Pic}(\mathrm{Sp}^{C_{p^n}})$ and $\mathrm{Pic}(\mathrm{PgcSp}_p)$, and define the *Picard-de Rham-Witt* complex as the universal example of the structure enjoyed by Pic-graded TR.

Remark 1.39 (Reality)**.** There is a great deal of work on Real topological Hochschild homology, especially recently, e.g. [AKGH21, AKKQ25, DMPR17, HW22, HP23, Yan25]. It would be very fun to extend our results to the Real case.

1.5. **Overview.** We review all our notation in §2. We study the $\mathrm{RO}(G)$-graded homotopy of even Borel spectra in §3: primarily the $\mathrm{RO}(C_{p^\infty})$-graded TC^- and TP of quasiregular semiperfectoid rings, and some parallel results for even $\mathbf{E}_\infty$-rings with $\mathbf{T}$-action. We study the de Rham-Witt setting in §4. In Appendix A, we study the p-primary Adams operations.

1.6. **Acknowledgments.** We thank Benjamin Antieau, Andrew Blumberg, Elden Elmanto, Lars Hesselholt, Jacob Lurie, Zhouhang Mao, Shubhodip Mondal, and Noah Riggenbach for helpful conversations related to this work. We are especially grateful to Antieau for discussions about the Adams operation ψ^p; Mao for explaining Example 1.9 to us; and to Riggenbach for clarifying the difference between absolute and relative Ngyaard filtrations for us.

2. Notation

Here we collect all the notation used in the paper. In many ways, this is the most difficult and important part of the paper.

In §3, R denotes a quasiregular semiperfectoid ring; in §4, R instead denotes an arbitrary perfectoid ring. We reserve k for a perfect $\mathbf{F}_p$-algebra, R^+ for a p-torsionfree perfectoid ring, and $\mathcal{O}_C$ for the ring of integers of a spherically complete perfectoid field C. Actually, $\mathcal{O}_C$ could be any perfectoid ring such that $\varprojlim^1 I_r \mathrm{A}_{\mathrm{inf}}(\mathcal{O}_C) = 0$, if you happen to know of other sources of such rings. Finally, we use S for a formally smooth algebra (i.e. the p-completion of a smooth algebra) over any of these bases.

If (A, I) is a prism, then $I_n = I\phi(I)\cdots\phi^{n-1}(I)$ [BL22, Notation 2.2.2]. We recall the following elements from [BMS18, Lemma 3.12]: if R is a perfectoid ring with corresponding perfect prism $(\Delta_R = \mathrm{A}_{\mathrm{inf}}(R), I)$, then

$$\begin{aligned}
\widetilde{\xi} &\text{ is a generator of } I \\
\xi = \phi^{-1}(\widetilde{\xi}) &\text{ is a generator of } \mathcal{N}^{\geq 1}\Delta_R = \phi^{-1}(I) \\
\widetilde{\xi}_n = \widetilde{\xi}\phi(\widetilde{\xi})\cdots\phi^{n-1}(\widetilde{\xi}) &\text{ is a generator of } I_n \\
\xi_n = \xi\phi^{-1}(\xi)\cdots\phi^{-(n-1)}(\xi) &\text{ is a generator of } \phi^{-n}(I_n)
\end{aligned}$$

In the q-de Rham case, we have the element $q = [\epsilon] \in \mathrm{A}_{\mathrm{inf}}(R)$ from from [BMS18, Proposition 3.17], allowing us to make these more concrete as q-analogues:

$$\begin{aligned}
\mu &= 1 - q \\
\widetilde{\xi}_n &= (p^n)_q = \frac{\phi^n(\mu)}{\mu} \\
\xi_n &= (p^n)_{q^{1/p^n}} = \frac{\mu}{\phi^{-n}(\mu)}
\end{aligned}$$

We will sometimes use the *relative* theory

$$\mathrm{TR}(S \mid \mathbf{S}_{\mathrm{W}(k)}) \quad \text{or} \quad \mathrm{TR}(S \mid \mathbf{S}_{\mathrm{A}_{\mathrm{inf}}(R)})$$

where $\mathbf{S}_{\mathrm{A}_{\mathrm{inf}}(R)} := \mathbf{S}_{\mathrm{W}(R^\flat)}$, and $\mathbf{S}_{\mathrm{W}(k)}$ is the functor of spherical Witt vectors [Lur18, Example 5.2.7], [Ant23], [BSY22, §2], [NY25]. This is equivalent to $\mathrm{TR}(S; \mathbf{Z}_p)$ by [Zho20, Lemma 5.20], an avatar of the identification between absolute and relative prismatic cohomology under a perfect prism [BS22, Lemma 4.8], [BL22, Proposition 4.4.12], [BMS19, Remark 11.9]. However, the relative notation is helpful for emphasizing the base in results that depend on it, and for reducing the context needed when reading our formulas on the go.

Our notation for the homotopy groups of the spectra in the isotropy separation square is

$$\pi_* \left(\begin{tikzcd} \mathrm{TF}(R; \mathbf{Z}_p) \arrow[r, "R"] \arrow[d] & (\mathrm{THH}(R; \mathbf{Z}_p)^{\Phi C_p})^{\mathbf{T}/C_p} \arrow[d, "\varphi"] \\ \mathrm{TC}^-(R; \mathbf{Z}_p) \arrow[r, "\mathrm{can}"'] & \mathrm{TP}(R; \mathbf{Z}_p) \end{tikzcd} \right) = \left(\begin{tikzcd} A[\sigma] \arrow[r] \arrow[d] & A[t^{-1}] \arrow[d] \\ \dfrac{A[\sigma, t]}{\sigma t - \xi} \arrow[r] & A[t^{\pm}] \end{tikzcd} \right)$$

with R a perfectoid ring, $A = \mathrm{A}_{\mathrm{inf}}(R)$, $|\sigma| = 2$ and $|t| = -2$. In the de Rham-Witt context, we use σ_n for the generator of $\mathrm{TR}^n_2(R; \mathbf{Z}_p) = \pi_2 \mathrm{THH}(R; \mathbf{Z}_p)^{hC_{p^n}}$; since these are preserved by the F maps (but not the R maps), in §3 we just use σ for all of them.

The circle group is denoted by $\mathbf{T}$. We write λ^n for the 1-dimensional complex $\mathbf{T}$-representation in which $z \cdot w = z^n w$ for $z \in \mathbf{T}$ and $w \in \mathbf{C}$; this is the n^{th} power of $\lambda = \lambda^1$ in the complex representation ring $\mathrm{RU}(\mathbf{T})$. We set $\lambda_i = \lambda^{p^i}$, with $\lambda_\infty = \lambda^0 = \mathbf{C}^{\mathrm{triv}}$.

Usually, we will write a virtual representation α as $\alpha = (d_0, \ldots, d_n; d_\infty)$, where $d_i = \dim_{\mathbf{C}}(\alpha^{C_{p^i}})$. Note that $(d_0, \ldots, d_n; d_\infty) = (d_0, \ldots, d_n, d_\infty; d_\infty)$. To convert between this encoding and the irreducible decomposition $\alpha = k_0\lambda_0 + \cdots + k_n\lambda_n + k_\infty\lambda_\infty$, the formulas are

$$\begin{aligned} d_i &= \sum_{i \le j \le \infty} k_j \\ k_i &= d_i - d_{i+1} \qquad i < \infty \\ k_\infty &= d_\infty \end{aligned}$$

Given an actual representation V, we write $a_V \in \pi^G_{-V}\mathbf{S}_G$ for the *Euler class* of V, obtained by suspending the inclusion $\{0\} \hookrightarrow V$ (see [HHR16, Definition 3.11]). In particular we can map a_V to any G-spectrum X. In favorable cases there are also *Thom classes* $u_V \in \pi^G_{V-\dim_{\mathbf{R}} V}(X)$ giving an isomorphism

$$u_V \colon \Sigma^V X \xrightarrow{\sim} \Sigma^{\dim_{\mathbf{R}} V} X.$$

Such an isomorphism typically only exists when X is Borel-complete; however, if X is a genuine G-spectrum and $X^h = X^{EG_+}$ is its Borel completion, the classes u_V (but not u_V^{-1}) often lift along $X \to X^h$. For the case we need, the u_V are constructed in [Rig22b, §4]. Other good sources to learn about these classes are [Zen18, §5.2] and [HZ18].

There are divisibility relations between these classes coming from the n^{th} power maps $S^{\lambda^k} \to S^{\lambda^{nk}}$, and we set

$$\begin{aligned} a_i &:= a_{\lambda_{i-1}}^{-1} a_{\lambda_i} \\ u_i &:= u_{\lambda_{i-1}} u_{\lambda_i}^{-1} \end{aligned}$$

When $i = 0$, we interpret these formulas by setting $a_{\lambda_{-1}} := 1$ and $u_{\lambda_{-1}} = \sigma$. (There is some interesting philosophy here). Setting $T = \mathrm{THH}(R; \mathbf{Z}_p)$, these classes are summarized in the following diagram:

$$\begin{tikzcd}
& T \arrow[dl, bend right, "a_{\lambda_{-1}}:=\mathrm{id}"'] \arrow[d, "a_{\lambda_0}"] \arrow[dr, "a_{\lambda_1}"] \arrow[drr, bend left, "a_{\lambda_2}"] & & \\
T \arrow[r, shift left, "a_0"] & \Sigma^{\lambda_0}T \arrow[l, shift left, "u_0"] \arrow[r, shift left, "a_1"] & \Sigma^{\lambda_1}T \arrow[l, shift left, "u_1"] \arrow[r, shift left, "a_2"] & \cdots \arrow[l, shift left, "u_2"] \\
& \Sigma^2 T \arrow[ul, bend left, "u_{\lambda_{-1}}:=\sigma"] \arrow[u, "u_{\lambda_0}"'] \arrow[ur, "u_{\lambda_1}"'] \arrow[urr, bend right, "u_{\lambda_2}"'] & &
\end{tikzcd}$$

Note that $t = a_{\lambda_0} u_{\lambda_0}^{-1}$.

The gold classes are related by the "q-gold relations"

$$\begin{aligned} \phi^{-1}(\widetilde{\xi}_{i+1}) a_{\lambda_i} &= 0 \\ a_{\lambda_j} u_{\lambda_i} &= \phi^i(\widetilde{\xi}_{j-i}) a_{\lambda_i} u_{\lambda_j} \quad (-1 \le i < j) \\ a_i u_i &= \phi^i(\xi) \end{aligned}$$

from [Sul20, Lemma 4.9], as well as the "infinite products"

$$u_{\lambda_{i-1}} = u_i u_{i+1} u_{i+2} \cdots$$

These infinite products can be made precise by viewing $\mathrm{TF}(R; \mathbf{Z}_p) = \varprojlim_F \mathrm{TR}^n(R; \mathbf{Z}_p)$ as a condensed spectrum with the limit topology, or by using Breuil-Kisin twists as explained in Remark 1.8.

explain ϑ and $\underline{\vartheta}$

In the TR setting, we denote a poly-representation $\underline{\alpha}$ by $(\underline{d}_0, \underline{d}_1, \underline{d}_2, \dots)$, with $\underline{d}_n = \dim_C \underline{\alpha}(C_{p^n})$. We have poly-graded gold classes

$$\underline{a}_{\lambda_i} : C_{p^n} \mapsto a_{\lambda_{n-i}}$$
$$\underline{u}_{\lambda_i} : C_{p^n} \mapsto u_{\lambda_{n-i}}$$
$$\underline{a}_i : C_{p^n} \mapsto a_{n-i}$$
$$\underline{u}_i : C_{p^n} \mapsto u_{n-i}$$

check that this works for σ_n

We also have the class

$$\vartheta^{\underline{\alpha}}_{r} = \underline{u}_0^{\underline{d}_0} \underline{u}_1^{\underline{d}_1} \cdots \underline{u}_r^{\underline{d}_r} \underline{a}_{r+1}^{-\underline{d}_{r+1}} \cdots$$

which generates which equivariant thing?

Warning 2.1. We should really write $d_{-n}(\underline{\alpha})$, but this notation gets too bulky; the underline notation emphasizes the connection to Mackey functors, and $\underline{d}_n$ still looks pretty close to d_{-n}. However, notational macros like this can obscure the bigger picture: for integral poly-representations, you want to write $d_{1/n}(\underline{\alpha})$.

In particular, interpreting $\underline{a}_i = a_{-i}$ and $\underline{u}_i = u_{-i}$, the polygonic gold relation is still just $a_i u_i = \phi^i(\xi)$, but for negative i. does this work for the annihilator of a? and the uv one? or is our indexing off? (Again, there is some interesting philosophy here.) In $\mathrm{TR}_{\bigstar}(\mathcal{O}_C; \mathbf{Z}_p)$, we have the additional relation

$$\beta a_{\underline{\lambda}_i} = \phi^{-i}(\mu) u_{\underline{\lambda}_i}$$

as well as the "infinite product"

$$??? = \underline{u}_i \underline{u}_{(i+1)} \cdots$$

Warning 2.2. The symbol $u_{\lambda_{-1}}$ is used for two different things: the generator of $\pi_2 \mathrm{TC}^-(R_0; \mathbf{Z}_p)$, and the poly-graded class $C_{p^n} \mapsto u_{\lambda_{n-1}}$. say more

Notation

Symbol	Meaning
k	perfect $\mathbf{F}_p$-algebra
R	qrsp ring (§3) or arbitrary perfectoid ring (§4)
R^+	p-torsionfree perfectoid ring
C	spherically complete perfectoid field
α	virtual $\mathbf{T}$-representation
$\underline{\alpha}$	virtual polyrepresentation
λ^n	$\mathbf{C}$ with $z \in \mathbf{T}$ acting as z^n
λ_i	$:= \lambda^{p^i}$
$\lambda^{1/k}$	polyrepresentation $C_k \mapsto \lambda^{n/k}$
$\underline{\lambda}_i$	$:= \lambda^{1/p^i}$
$[\![\nu]\!]_\lambda$	polyrepresentation $C_n \mapsto \lambda^0 + \cdots + \lambda^{\lceil n\nu \rceil - 1}$
σ	generator of $\mathrm{TC}^-_2(R'; \mathbf{Z}_p)$
t	generator of $\mathrm{TC}^-_{-2}(R'; \mathbf{Z}_p)$
σ_n	generator of $\mathrm{TR}^n(R'; \mathbf{Z}_p)$
a_V	Euler class of V
u_V	Thom class of V
a_i	$:= a^{-1}_{\lambda_{i-1}} a_{\lambda_i}$
u_i	$:= u_{\lambda_{i-1}} u^{-1}_{\lambda_i}$
$\underline{a}_i$	poly-class $C_{p^n} \mapsto a_{n-i}$
$\underline{u}_i$	poly-class $C_{p^n} \mapsto u_{n-i}$

Identities

$$\begin{aligned}
a_i u_i &= \phi^i(\xi) \\
\sigma a_{\lambda_i} &= \phi^{-1}(\widetilde{\xi}_{i+1}) u_{\lambda_i} \\
\beta \underline{a}_{\lambda_i} &= \phi^{-i}(\mu) \underline{u}_{\lambda_i} \\
\mathrm{ind}_n^{n+1}(d_0, \ldots, d_{n-1}; d_\infty) &= (pd_0, \ldots, pd_{n-1}, pd_\infty; d_\infty) \\
\mathrm{ind}_n^{n+1}(\underline{d}_0, \ldots, \underline{d}_n) &= (\underline{d}_0, p\underline{d}_0, \ldots, p\underline{d}_n) \\
N a_V &= a_{\mathrm{ind}\, V} \\
N u_V &= \frac{u_{\mathrm{ind}\, V}}{u_{\mathrm{ind}\, |V|}} \\
N(f) &= \phi(f) - V\delta(f) && \text{(prismatic)} \\
N(f) &= f - V\delta(f) && \text{(Witt vectors)} \\
N(\omega) &= V\frac{\omega^p}{p} + (p - V(1))\frac{\psi^p(\omega)}{p}
\end{aligned}$$

3. RO($\mathbf{T}$)-graded TC^- and TP of quasiregular semiperfectoid rings

In this section we study the RO($\mathbf{T}$) (really RO(C_{p^∞}))-graded TC^- and TP of quasiregular semiperfectoid rings R. We compute the relevant groups in §3.1; the result is essentially the multi-Rees algebra of the Nygaard and $\phi^\bullet(I)$-adic filtrations on $\hat{\mathbb{\Delta}}_R[1/\phi^\bullet(I)]$, so that we encounter *prismatic fractional ideals* like $I^2 \otimes \phi(I)^{-3}$. We study the norm maps between these groups in §3.2. Finally, in §3.3 we comment on non-orientable representations at the prime 2, which will be needed in §4.5. Along the way, we give integral generalizations of some of our results to even $\mathbf{E}_\infty$-rings with $\mathbf{T}$-action.

3.1. Prismatic fractional ideals.

In this section we explain how to see Theorem 1.10. We fix a quasiregular semiperfectoid ring R and abbreviate $\mathrm{TC}^- := \mathrm{TC}^-(R; \mathbf{Z}_p)$, $\mathrm{TP} := \mathrm{TP}(R; \mathbf{Z}_p)$, $\hat{\mathbb{\Delta}} := \hat{\mathbb{\Delta}}_R$, etc.

Our main tools will be the cell structure

$$\mathbf{T}/C_{n+} \to S^0 \to S^{\lambda^n} \tag{7}$$

and the motivic filtrations

$$\mathrm{TC}^-_{2i} = \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\} \qquad \mathrm{TP}_{2i} = \hat{\mathbb{\Delta}}\{i\}$$

$$\mathrm{THH}^{hC_{p^n}}_{2i} = \frac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}}{I_n \otimes \mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}\{i\}} \qquad \mathrm{THH}^{tC_{p^n}}_{2i} = \frac{\hat{\mathbb{\Delta}}\{i\}}{I_n\hat{\mathbb{\Delta}}\{i\}}$$

from [BMS19, Theorem 1.12(4)] and [Rig22a, Corollary 3.3]. We will just explain the case of TC^-, leaving the (easier) case of TP to the reader.

Remark 3.1. Since THH of perfectoid rings is even, there are invertible classes $u_{\lambda_k} \in \mathrm{TC}^-_{2-\lambda_k}$ giving non-canonical isomorphisms

$$\mathrm{TC}^-_\alpha \cong \mathrm{TC}^-_{2d_0(\alpha)}$$

for every $\alpha \in \mathrm{RU}(\mathbf{T})$ [Rig22b, Proposition 4.2]. Of course the whole point of this section is to avoid these classes and instead give canonical identifications. However, this observation is useful for solving extensions and identifying images.

Example 3.2. The cell structure (7) gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{2i+\lambda_k} \arrow[r, "a_{\lambda_k}"] \arrow[d, equal] & \mathrm{TC}^-_{2i} \arrow[r] \arrow[d, equal] & \mathrm{THH}^{hC_{p^k}}_{2i} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & ? \arrow[r] & \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & \dfrac{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}}{I_k \otimes \mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}\{i\}} \arrow[r] & 0
\end{tikzcd}$$

which shows that $\mathrm{TC}^-_{2i+\lambda_k} = I_k \otimes \mathcal{N}^{\geq i+1}\hat{\mathbb{\Delta}}\{i\}$, as claimed by Theorem 1.10.

Example 3.3. The cell structure (7) gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{2i} \arrow[r, "a_{\lambda_k}"] \arrow[d, equal] & \mathrm{TC}^-_{2i-\lambda_k} \arrow[r] \arrow[d, equal] & \mathrm{THH}^{hC_{p^k}}_{2(i-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & ? \arrow[r] & \dfrac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}}{I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}} \arrow[r] & 0
\end{tikzcd}$$

Rewriting the final term as

$$\frac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}}{I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}} = \frac{I_k^{-1} \otimes \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\}}{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}}$$

and using Remark 3.1, we see that $\mathrm{TC}^-_{2i-\lambda_k} = I_k^{-1} \otimes \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\}$, as claimed by Theorem 1.10.

Alternatively, we can use the multiplication map

$$\begin{tikzcd}
\mathrm{TC}^-_{2i-\lambda_k} \otimes \mathrm{TC}^-_{\lambda_k-2} \arrow[r] \arrow[d, equal] & \mathrm{TC}^-_{2(i-1)} \arrow[d, equal] \\
\mathrm{TC}^-_{2i-\lambda_k} \otimes I_k \otimes \hat{\mathbb{\Delta}}\{-1\} \arrow[r] & \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}
\end{tikzcd}$$

where the left vertical identification is Example 3.2. Using Remark 3.1, we see that the rows are isomorphisms. We thus get

$$\mathrm{TC}^-_{2i-\lambda_k} = \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\} \otimes \Big(I_k \otimes \hat{\mathbb{\Delta}}\{-1\}\Big)^{-1} = I_k^{-1}\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\}.$$

Example 3.4. We now explain how to mix representations. If $k \leq n$, then $\mathrm{res}^{\mathbf{T}}_{C_{p^k}} \lambda_n$ is trivial, but $\mathrm{res}^{\mathbf{T}}_{C_{p^n}} \lambda_k$ is not unless $k = n$. We will thus induct by using λ_n before we use λ_k. Now the cell structure (7) gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{TC}^-_{2i+\lambda_k-\lambda_n} \arrow[r, "a_{\lambda_k}"] \arrow[d, equal] & \mathrm{TC}^-_{2i-\lambda_n} \arrow[r] \arrow[d, equal] & \mathrm{THH}^{hC_{p^k}}_{2(i-1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & ? \arrow[r] & I_n^{-1}\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\} \arrow[r] & \dfrac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}}{I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}} \arrow[r] & 0.
\end{tikzcd}$$

Rewriting the final term as

$$\frac{\mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i-1\}}{I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}} = \frac{I_k^{-1} \otimes \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\}}{\mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}} = \frac{I_n^{-1} \otimes \mathcal{N}^{\geq i-1}\hat{\mathbb{\Delta}}\{i\}}{I_k \otimes I_n^{-1} \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}},$$

we see that $\mathrm{TC}^-_{2i+\lambda_k-\lambda_n} = I_k \otimes I_n^{-1} \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}$, as claimed by Theorem 1.10.

Alternatively, we can use the multiplication map

$$\begin{tikzcd}
\mathrm{TC}^-_{2i+\lambda_k-\lambda_n} \otimes \mathrm{TC}^-_{\lambda_n-2} \arrow[r] \arrow[d, equal] & \mathrm{TC}^-_{2(i-1)+\lambda_k} \arrow[d, equal] \\
\mathrm{TC}^-_{2i+\lambda_k-\lambda_n} \otimes I_n \otimes \hat{\mathbb{\Delta}}\{-1\} \arrow[r] & I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\}
\end{tikzcd}$$

where the left vertical identification is Example 3.2. Using Remark 3.1, we see that the rows are isomorphisms. We thus get

$$\mathrm{TC}^-_{2i+\lambda_k-\lambda_n} = I_k \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i-1\} \otimes \Big(I_n \otimes \hat{\mathbb{\Delta}}\{-1\}\Big)^{-1} = I_k \otimes I_n^{-1} \otimes \mathcal{N}^{\geq i}\hat{\mathbb{\Delta}}\{i\}.$$

Combining the techniques in the above examples, the general case of Theorem 1.10 follows immediately.

Example 3.5 (Higher TC^-)**.** We explain how to compute $\pi_\bigstar$ of $\mathrm{TC}^{-1} = (\mathrm{THH}^{C_p})^{h\mathbf{T}/C_p}$. If $\alpha = (d_0, d_1, \dots)$, recall that $\alpha' = \alpha^{C_p} = (d_1, d_2, \dots)$ and $\alpha^{(r)} = \alpha^{C_{p^r}} = (d_r, d_{r+1}, \dots)$. To ease notation, set

$$\begin{aligned}
\varphi^{-n}(I^\alpha\hat{\mathbb{\Delta}}_R) &= \mathcal{N}^{\geq d_0} \cdots \mathcal{N}^{\geq d_{n-1}} I^{d_n}\phi(I)^{d_{n+1}} \cdots \hat{\mathbb{\Delta}}_R \\
&= \left\{ f \in \hat{\mathbb{\Delta}}_R\{d_\infty\} \;\middle|\; \varphi^k(f) \in \left(\prod_{i=0}^{\infty} \phi^i(I)^{d_{n-k+i}-d_\infty}\right)\hat{\mathbb{\Delta}}_R\{d_\infty\} \text{ for all } 0 \leq k \leq n \right\} \\
&= \{ f \in \hat{\mathbb{\Delta}}_R\{d_\infty\} \mid \varphi^k(f) \in I^{\alpha^{(n-k)}}\hat{\mathbb{\Delta}}_R \text{ for all } 0 \leq k \leq n \}
\end{aligned}$$

Write $\mathrm{TC}^{-,\mathrm{can}} = \mathrm{THH}^{h\mathbf{T}}$ for the copy of TC^- that maps to TP via the canonical map, and $\mathrm{TC}^{-,\varphi} = (\mathrm{THH}^{\Phi C_p})^{h\mathbf{T}/C_p}$ for the copy of TC^- that maps to TP via Frobenius. Then

$$\begin{aligned}
\mathrm{TC}^{-,\mathrm{can}}_\alpha(R;\mathbf{Z}_p) &= \varphi^{-1}(I^\alpha \hat{\mathbb{\Delta}}_R) = \mathcal{N}^{\geq d_0} I^{\alpha'} \hat{\mathbb{\Delta}}_R \\
\mathrm{TP}_\alpha(R;\mathbf{Z}_p) &= I^{\alpha'} \hat{\mathbb{\Delta}}_R \\
\mathrm{TC}^{-,\varphi}_\alpha(R;\mathbf{Z}_p) &= \varphi^{-1}(I^{\alpha'} \hat{\mathbb{\Delta}}_R) = \mathcal{N}^{\geq d_1} I^{\alpha''} \hat{\mathbb{\Delta}}_R
\end{aligned}$$

The spectral sequence computing $\mathrm{TC}^{-1}_{\alpha-*}(R;\mathbf{Z}_p)$, where $* \in \{0,1\}$, is

$$E_0 = \begin{tikzcd} \mathrm{TP}_\alpha(R;\mathbf{Z}_p) & \mathrm{TC}^{-,\mathrm{can}}_\alpha(R;\mathbf{Z}_p) \arrow[l] \\ & \mathrm{TC}^{-,\varphi}_\alpha(R;\mathbf{Z}_p) \end{tikzcd}$$

$$= \begin{tikzcd} I^{\alpha'} \hat{\mathbb{\Delta}}_R & \varphi^{-1}(I^\alpha \hat{\mathbb{\Delta}}_R) \arrow[l, "\mathrm{can}"'] \\ & \varphi^{-1}(I^{\alpha'} \hat{\mathbb{\Delta}}_R) \end{tikzcd}$$

$$E_1 = \begin{tikzcd} \dfrac{I^{\alpha'} \hat{\mathbb{\Delta}}_R}{\varphi^{-1}(I^\alpha)} & \\ & \varphi^{-1}(I^{\alpha'} \hat{\mathbb{\Delta}}_R) \arrow[ul, "\varphi"'] \end{tikzcd}$$

$$E_\infty = \begin{array}{ll} \dfrac{I^{\alpha'} \hat{\mathbb{\Delta}}_R}{\varphi^{-1}(I^\alpha) + \varphi(\varphi^{-1}(I^{\alpha'}))} & \\ & \varphi^{-2}(I^\alpha \hat{\mathbb{\Delta}}_R) \end{array}$$

$$= \begin{array}{ll} \dfrac{I^{\alpha'} \hat{\mathbb{\Delta}}_R}{\mathcal{N}^{\geq d_0} I^{\alpha'} + \varphi(\mathcal{N}^{\geq d_1} I^{\alpha''})} & \\ & \mathcal{N}^{\geq d_0} \mathcal{N}^{\geq d_1} I^{\alpha''} \hat{\mathbb{\Delta}}_R \end{array}$$

We conclude that

$$\begin{aligned}
\mathrm{TC}^{-1}_\alpha(R;\mathbf{Z}_p) &= \mathcal{N}^{\geq d_0} \mathcal{N}^{\geq d_1} I^{\alpha''} \hat{\mathbb{\Delta}}_R \\
\mathrm{TC}^{-1}_{\alpha-1}(R;\mathbf{Z}_p) &= \frac{I^{\alpha'} \hat{\mathbb{\Delta}}_R}{\mathcal{N}^{\geq d_0} I^{\alpha'} + \varphi(\mathcal{N}^{\geq d_1} I^{\alpha''})}
\end{aligned}$$

Remark 3.6. The only inputs needed for Theorem 1.10 were:

- $\mathrm{THH}(R;\mathbf{Z}_p)$ is orientable in the Borel category [Rig22b, Proposition 4.2];
- a description of the Mackey functors $\underline{\pi}_\bigstar \mathrm{THH}(R;\mathbf{Z}_p)^h$ and $\underline{\pi}_\bigstar \mathrm{THH}(R;\mathbf{Z}_p)^t$ [Rig22a, Corollary 3.3].

Riggenbach in fact shows the first point for all $\mathbf{T}$-equivariant even $\mathbf{E}_\infty$-rings, while he and Antieau provide a version of the second in this generality in [AR24, Corollary 3.20]. We now explain this integral generalization (Theorem 1.14).

Let E be an even $\mathbf{E}_\infty$-ring with $\mathbf{T}$-action. Set $A = \pi_0 E^{t\mathbf{T}}$, $A\{1\} = \pi_2 E^{t\mathbf{T}}$, and $A\{n\} = A\{1\}^{\otimes n}$; by the Tate spectral sequence, we have

$$\pi_* E^{t\mathbf{T}} = \bigoplus_{i \in \mathbf{Z}} A\{i\}[2i]$$

Let $\mathrm{Fil}^\bullet A\{i\}$ be the abutment filtration of the Tate spectral sequence. By the homotopy fixed points spectral sequence, we have

$$\pi_* E^{h\mathbf{T}} = \bigoplus_{i \in \mathbf{Z}} \mathrm{Fil}^i A\{i\}[2i]$$

The natural map

$$\mathrm{Fil}^i A\{i\} \otimes_A A\{-i\} \to \mathrm{Fil}^i A$$

induces an isomorphism

$$\pi_{2i}E = \mathrm{gr}^i A\{i\} \xrightarrow{\sim} \mathrm{gr}^i A$$

and therefore the short exact sequences

$$0 \to E^{h\mathbf{T}}_{2i+(j+1)\lambda} \xrightarrow{a_\lambda} E^{h\mathbf{T}}_{2i+j\lambda} \to \pi_{2(i+j)}E \to 0$$

give identifications

$$E^{h\mathbf{T}}_{2i+j\lambda} = \mathrm{Fil}^{i+j} A\{i\}$$

for all $i, j \in \mathbf{Z}$, as before.

To go further, let $\mathbf{G}$ be the formal group law of E, and let $t \in A\{-1\}$ be a choice of complex orientation. Write $[n]_{\mathbf{G}}(t) \in A\{-1\}$ for the n-series of $\mathbf{G}$, $\langle n\rangle_{\mathbf{G}}(t) = \frac{[n]_{\mathbf{G}}(t)}{t} \in A$ for the divided n-series; we sometimes suppress the dependence on t and just write $\langle n\rangle_{\mathbf{G}}$ (since we only care about the ideal $\langle n\rangle_{\mathbf{G}}(t)A$). Assume furthermore that $\langle n\rangle_{\mathbf{G}}(t)$ is a non-zerodivisor for all n. By [AR24, Corollary 3.20], we have

$$\pi_{2i}E^{hC_n} = \frac{\mathrm{Fil}^i A\{i\}}{\langle n\rangle_{\mathbf{G}} \mathrm{Fil}^{i+1} A\{i\}} \qquad\qquad \pi_{2i}E^{tC_n} = \frac{A\{i\}}{\langle n\rangle_{\mathbf{G}}}$$

It follows that

$$\begin{aligned} E^{h\mathbf{T}}_{2i+j\lambda^n} &= \mathrm{Fil}^{i+j} A\{i\} \otimes_A (\langle n\rangle_{\mathbf{G}} A)^{\otimes j} \\ E^{t\mathbf{T}}_{2i+j\lambda^n} &= A\{i\} \otimes_A (\langle n\rangle_{\mathbf{G}} A)^{\otimes j} \end{aligned}$$

To go fully general, we need a way to describe the geometric fixed points. Recall $\Phi_{d,\mathbf{G}}(t)$ from Definition 1.13. In our topological situation, we have $\Phi_{d,\mathbf{G}}(t) \in A$ for $d > 1$, and we suppress the dependence on t and simply write $\Phi_{d,\mathbf{G}} \in A$. Note that for $n > 1$, we have

$$\pi_{2i}E^{\tau C_n} = \frac{A\{i\}}{\Phi_{n,\mathbf{G}}}$$

and heuristically $A\{1\} = \bigotimes_{n>1} \Phi_{n,\mathbf{G}} A$. With these notations, Theorem 1.14 follows immediately.

Example 3.7. Let E be an even $\mathbf{E}_\infty$-ring with $\mathbf{T}$-action, and assume that $\langle n\rangle_{\mathbf{G}} \in A = \pi_0 E^{t\mathbf{T}}$ is a non-zerodivisor for all n. Then for all $i \in \mathbf{Z}$,

$$\begin{aligned} E^{h\mathbf{T}}_{(n)_{\lambda^m}+2i} &= \mathrm{Fil}^{i+n} A\{i+1\} \otimes \langle n\rangle_{\mathbf{G}^{(m)}}?\} & \qquad E^{t\mathbf{T}}_{(n)_{\lambda^m}+2i} &= A\{1+i\} \otimes \{\langle n\rangle_{\mathbf{G}^{(m)}}?\} \\ E^{h\mathbf{T}}_{\lambda^m(n)_{\lambda^m}+2i} &= \mathrm{Fil}^{i+n} A\{i\} \otimes \{\langle n\rangle_{\mathbf{G}^{(m)}}!\} & \qquad E^{t\mathbf{T}}_{\lambda^m(n)_{\lambda^m}+2i} &= A\{i\} \otimes \{\langle n\rangle_{\mathbf{G}^{(m)}}!\} \end{aligned}$$

From here we can get more complicated examples like

$$E^{h\mathbf{T}}_{\psi^m(\lambda((n)_\lambda-(k)_\lambda-(n-k)_\lambda))+2i} = \mathrm{Fil}^i A\{i\} \otimes \left\{ \binom{n}{k}_{\mathbf{G}^{(m)}} \right\}$$

3.2. **Norm maps.** In this section we identify the Tambara structure of $\underline{\pi}_\bigstar \mathrm{TC}^-(R;\mathbf{Z}_p)$ and $\underline{\pi}_\bigstar \mathrm{TP}(R;\mathbf{Z}_p)$. This means studying the F, V, and N maps between $\pi_\alpha \mathrm{THH}(R;\mathbf{Z}_p)^{\{h,t\}C_{p^n}}$ as n varies. In the $\mathrm{RO}(G)$-graded context, these maps go

$$\pi^H_\alpha E \underset{V^H_K}{\overset{F^H_K}{\rightleftarrows}} \pi^K_{\mathrm{res}^H_K(\alpha)} E \qquad\qquad \pi^K_\alpha E \xrightarrow{N^H_K} \pi^H_{\mathrm{ind}^H_K(\alpha)} E$$

When E is cyclotomic (as opposed to merely cyclonic), there are additional maps

$$R^H_{H/K} \colon \pi^H_\alpha E \longrightarrow \pi^{H/K}_{\alpha^K} E$$

which will be studied in §4.

To identify these maps, we will temporarily switch to using the gold elements, then deduce canonical descriptions from these. By construction, we have

$$\begin{aligned}\operatorname{res}^G_H(u_V) &= u_{\operatorname{res}^G_H V}\\ \operatorname{res}^G_H(a_V) &= a_{\operatorname{res}^G_H V},\end{aligned}$$

and thus also

$$\begin{aligned}\operatorname{tr}^H_K(u_{\operatorname{res} V}) &= \operatorname{tr}(1)u_V\\ \operatorname{tr}^H_K(a_{\operatorname{res} V}) &= \operatorname{tr}(1)a_V.\end{aligned}$$

It follows that the F maps are given by the natural quotient maps, while V from C_{p^n} to $C_{p^{n+1}}$ is given by multiplying by a generator of $\phi^n(I)$ which is congruent to p mod $\mathcal{N}^{\geq 1}I\cdots\phi^{n-1}(I)$. It remains to identify the N maps.

Let us begin by recording how induction works.

Proposition 3.8. *In dimensional notation, induction of representations is given by*

$$\operatorname{ind}_n^{n+1}(d_0,\ldots,d_{n-1};d_\infty) = (pd_0,\ldots,pd_{n-1},pd_\infty;d_\infty).$$

Proof. This follows from $\operatorname{res}^G_H\operatorname{ind}^G_H V = |G/H|V$ and $(\operatorname{ind}^G_H V)^G = V^H$. ☐

Corollary 3.9. *We have*

$$\begin{aligned}\operatorname{ind}_n^{n+1}(\lambda_\infty) &= \lambda_\infty + (p-1)\lambda_n\\ \operatorname{ind}_n^{n+1}(\lambda_i) &= p\lambda_i\end{aligned}$$

Lemma 3.10. *The norms of the gold elements are given by*

$$\begin{aligned}N_n^{n+1}(a_{\lambda_i}) &= a_{\lambda_i}^p & \qquad N_n^{n+1}(a_j) &= a_j^p\\ N_n^{n+1}(u_{\lambda_i}) &= u_{\lambda_i}^p u_{\lambda_n}^{-(p-1)} & \qquad N_n^{n+1}(u_j) &= u_j^p\end{aligned}$$

for $i \geq -1$ and $j \geq 0$.

Proof. By [HHR16, Lemma 3.13], we have

$$\begin{aligned}Na_V &= a_{\operatorname{ind} V}\\ Nu_V &= \frac{u_{\operatorname{ind} V}}{u_{\operatorname{ind}|V|}}\end{aligned}$$

where $|V|$ means the trivial representation with the same dimension as V. By Corollary 3.9, this implies all of the claims except for $N(u_{\lambda_{-1}})$ and $N(u_0)$. To identify the norm of the Bökstedt generator $u_{\lambda_{-1}}$, we borrow a trick from §4. Since $u_{\lambda_{-1}}$ lifts to TR_2^n, we can identify its norm there to deduce the result in $\mathrm{THH}^{hC_{p^n}}$. In this context the R map makes sense, and we have $R(u_{\lambda_i}) = u_{\lambda_{i-1}}$ for all $i \geq 0$ cite me. Since R commutes with N, we have $N(u_{\lambda_{-1}}) = RN(u_{\lambda_0})$ which completes the proof. ☐

Remark 3.11 (Lurie, c.f. the paragraph preceding [NS18, §IV.3])**.** To understand the norm more conceptually, we relate it to Adams operations; under some additional assumptions this gives another way to identify the norm of σ.

The composite RN is a map of $\mathbf{T}$-equivariant $\mathbf{E}_\infty$-rings, so it factors through the C_p orbits of THH in the category of $\mathbf{E}_\infty$-rings, which we denote by $\mathrm{THH}_{\otimes hC_p}$. The induced map $\mathrm{THH}_{\otimes hC_p} \to \mathrm{THH}^{\Phi C_p}$ is an equivalence. If we identify the domain with THH and further map the codomain to THH^{tC_p}, we obtain the Nikolaus-Scholze Frobenius.

$$\begin{tikzcd}
\mathrm{THH} \arrow[r, "N"] \arrow[d, "\psi^p"'] & \mathrm{TR}^1 \arrow[r, "R"] & \mathrm{THH}^{\Phi C_p} \arrow[d, "\varphi_{\mathrm{NS}}"] \\
\mathrm{THH}_{\otimes hC_p} \arrow[urr, dotted, "\simeq"'] \arrow[rr, "\varphi_{\mathrm{NS}}"'] & & \mathrm{THH}^{tC_p}
\end{tikzcd}$$

It follows that $RN\colon \mathrm{THH} \to \mathrm{THH}^{\Phi C_p} \simeq \mathrm{THH}_{\otimes hC_p}$ is given by the pth Adams operation ψ^p.

To identify $N(\sigma)$ over quasiregular semiperfectoid rings, it suffices to do so over perfectoid rings. By [Bha18, Proposition IV.3.2], every perfectoid ring can be written as the pullback of a p-torsionfree perfectoid ring and a perfect $\mathbf{F}_p$-algebra. Since every crystalline prism admits a map from a transversal prism (using q-de Rham prisms), we can reduce to the case of $\mathrm{THH}(R_0;\mathbf{Z}_p)$ when R_0 is a p-torsionfree perfectoid ring.

We want to show that $N_0^1(\sigma) = \sigma^p u_{\lambda_0}^{-(p-1)} = u_0^p u_{\lambda_0}$, which is the generator of $\mathrm{TR}^1_{[p]_\lambda}(R_0;\mathbf{Z}_p)$. Whatever $N(\sigma)$ is, we know that it must satisfy

$$\begin{aligned} FN(\sigma) &= \sigma^p \in \pi_{2p}\mathrm{THH}(R_0;\mathbf{Z}_p)\\ RN(\sigma) &= \psi^p(\sigma) \in \pi_2\mathrm{THH}(R_0;\mathbf{Z}_p)\end{aligned}$$

Since R_0 is p-torsionfree, these properties uniquely determine $N(\sigma)$, so we just need to show that $u_0^p u_{\lambda_0}$ satisfies these. This is clear for F. In our notation, the R map goes

$$A/\xi\phi(\xi)\left\langle u_0^p u_{\lambda_0}\right\rangle = \mathrm{TR}^1_{[p]_\lambda} \xrightarrow{R} \pi_{[p]_\lambda}\mathrm{THH}^{\Phi C_p} = A/\xi\left\langle a_0^{-p}u_{\lambda_0}\right\rangle$$

Thus R sends σ to $\phi^{-1}(\xi)^p \equiv p \bmod \xi$ times the generator of $\pi_2\mathrm{THH}(R_0;\mathbf{Z}_p)$. Since $\psi^p(\sigma) = p\sigma$ by Lemma A.3, this is exactly what we wanted.

Warning 3.12. Although $N\colon \mathrm{TR}_2^n \to \mathrm{TR}^{n+1}_{[p]_{\lambda_n}}$ takes a generator to a generator, it is not a bijection since it is not a ring homomorphism.

Now we identify the norm canonically.

Theorem 3.13 (Prismatic norms for Breuil-Kisin twists)**.** *Let R be a quasiregular semiperfectoid ring, and $\alpha = (d_0, \ldots, d_{n-1}; d_\infty)$ a virtual C_{p^∞}-representation. The norm maps*

$$\begin{tikzcd} \pi_\alpha\mathrm{THH}(R;\mathbf{Z}_p)^{hC_{p^n}} \arrow[r, "N"] \arrow[d] & \pi_{\mathrm{ind}\,\alpha}\mathrm{THH}(R;\mathbf{Z}_p)^{hC_{p^{n+1}}} \arrow[d] \\ \pi_\alpha\mathrm{THH}(R;\mathbf{Z}_p)^{tC_{p^n}} \arrow[r, "N"'] & \pi_{\mathrm{ind}\,\alpha}\mathrm{THH}(R;\mathbf{Z}_p)^{tC_{p^{n+1}}} \end{tikzcd}$$

give rise to prismatic norm maps with domains and codomains as follows (where $d_n = d_\infty$):

$$\begin{tikzcd} \dfrac{\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_n\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R\{d_\infty\}} \arrow[r, "N"] \arrow[d] & \dfrac{\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0}\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_{n+1}\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0+1}\hat{\mathbb{\Delta}}_R\{d_\infty\}} \arrow[d] \\ \dfrac{\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_n\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\hat{\mathbb{\Delta}}_R\{d_\infty\}} \arrow[r, "N"'] & \dfrac{\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_{n+1}\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\hat{\mathbb{\Delta}}_R\{d_\infty\}} \end{tikzcd}$$

When $(\mathbb{\Delta}_R, I)$ is transversal, these are characterized by

$$\begin{tikzcd} \dfrac{\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_n\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R\{d_\infty\}} \arrow[r, "N"] \arrow[dr, "x^p"'] & \dfrac{\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0}\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_{n+1}\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0+1}\hat{\mathbb{\Delta}}_R\{d_\infty\}} \arrow[d, "F"] \\ & \dfrac{\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0}\hat{\mathbb{\Delta}}_R\{d_\infty\}}{I_n\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0+1}\hat{\mathbb{\Delta}}_R\{d_\infty\}} \end{tikzcd}$$

and

$$\begin{tikzcd}
\dfrac{\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0}\hat{\Delta}_R\{d_\infty\}}{I_n\left(\prod_{i=1}^{n-1}\phi^{i-1}(I)^{d_i-d_\infty}\right)\mathcal{N}^{\geq d_0+1}\hat{\Delta}_R\{d_\infty\}} \arrow[r, "N"] \arrow[dr, "p^{d_\infty}\phi"'] & \dfrac{\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0}\hat{\Delta}_R\{d_\infty\}}{I_{n+1}\left(\prod_{i=1}^{n}\phi^{i-1}(I)^{pd_i-d_\infty}\right)\mathcal{N}^{\geq pd_0+1}\hat{\Delta}_R\{d_\infty\}} \arrow[d] \\
& \dfrac{\hat{\Delta}_R\{d_\infty\}}{\phi^n(I)\hat{\Delta}_R\{d_\infty\}}
\end{tikzcd}$$

These norms are given concretely by

???; translate Lemma 3.10 into a nice formula here

Example 3.14. Over the q-de Rham prism, we have a norm map

$$\begin{tikzcd}
\dfrac{(p)_q A}{(p)_q^2 A} \arrow[r, "N"] \arrow[rr, bend left, "p\phi"] \arrow[dr, "f^p"'] & \dfrac{(p^2)_q! A}{(p^2)_q (p^2)_q! A} \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dfrac{(p)_{q^p} A}{(p)_{q^p}^2 A} \arrow[d, "F"] \\
& \dfrac{(p)_q^p A}{(p)_q^{p+1} A} \arrow[r] & \dfrac{A}{(p)_q,\ (p)_{q^p}}
\end{tikzcd}$$

where F divides by $(p)_{q^p}$ as in [BL22, Corollary 2.2.10]. Informally, this says that "$N_1^2((p)_q) \approx (p^2)_q!$"; more generally, we have "$N_1^n((p)_q) \approx (p^{n+1})_q!$", where again $\approx$ means up to units.

figure out the unit — related to a q-analogue of Wilson's theorem $(p-1)! \equiv -1 \bmod p$

this suggests that for understanding the norm, it might be useful to write representations as linear combinations of $[n]_\lambda$

3.3. **Sign representations.** We can also consider the $\mathrm{RO}(C_{p^n})$-graded groups $\mathrm{THH}_\bigstar^{hC_{p^n}}$ and $\mathrm{THH}_\bigstar^{tC_{p^n}}$. When α is a C_{p^∞}-representation, we have (using the notation from Example 3.5)

$$\mathrm{THH}^{hC_{p^n}}_{\operatorname{res}\alpha} = \frac{\mathrm{TC}^-_\alpha}{\mathcal{N}^{\bullet+1}I_n\mathrm{TC}^-_\alpha} = \frac{\mathcal{N}^{\geq d_0}\hat{\Delta}_R\otimes I^{\alpha'}}{I_n\otimes\mathcal{N}^{\geq d_0+1}\hat{\Delta}_R\otimes I^{\alpha'}} \qquad \mathrm{THH}^{tC_{p^n}}_{\operatorname{res}\alpha} = \frac{\mathrm{TP}_\alpha}{I_n} = \frac{\hat{\Delta}_R\otimes I^{\alpha'}}{I_n\otimes\hat{\Delta}_R\otimes I^{\alpha'}}$$

When p is odd, the restrictions of C_{p^∞}-representations exhaust $\mathrm{RO}(C_{p^n})$. When $p=2$, there is an additional *sign representation* ς_n that we must account for. In this section we compute $\mathrm{THH}^{hC_{2^n}}_{\bigstar\pm\varsigma_n}$ and $\mathrm{THH}^{tC_{2^n}}_{\bigstar\pm\varsigma_n}$ (the $\pm$ is redundant but conceptually useful). One can extend the computations to odd p by considering *spoke spheres* as in [HSW22], but we do not consider such a generalization here.

For topological Hochschild homology of arbitrary rings, one would be forced to consider these groups by norm maps such as

$$N_0^1\colon \pi_1\mathrm{THH}\to\mathrm{THH}^{hC_2}_{1+\varsigma_1}$$

For quasiregular semiperfectoid rings, the odd groups vanish so there is no direct "normal" motivation to study these groups. However, in §4.6 we will want to compute norms for $\mathrm{TR}^n(S;\mathbf{Z}_2)$ when S is formally smooth over a perfectoid ring, and $\mathrm{TR}^n_*(S;\mathbf{Z}_2)$ is definitely not even. We will therefore leverage these computations in §4.5 to compute the necessary groups $\mathrm{TR}^n_{\bigstar\pm\varsigma_n}(S;\mathbf{Z}_2)$. Evenness is also what allows us to compute $\mathrm{THH}^{\{h,t\}C_{2^n}}_{\bigstar\pm\varsigma_n}$ from cell structures; applying the cellular method directly to $\mathrm{TR}^n_{\bigstar\pm\varsigma}(S;\mathbf{Z}_2)$ leads to extension problems that we do not otherwise know how to solve.

Proposition 3.15. *Let $\alpha = (d_0, \ldots, d_{n-1}; d_\infty)$ be an orientable C_{2^n}-representation. Then the non-orientably-graded groups $\mathrm{THH}^{\{h,t\}C_{2^n}}_{\alpha\pm\varsigma_n}$ are given canonically by*

$$\mathrm{THH}^{hC_{2^n}}_{\alpha+\varsigma_n} = \frac{I_{n-1} \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_n \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \qquad \mathrm{THH}^{tC_{2^n}}_{\alpha+\varsigma_n} = \frac{I_{n-1} \otimes \hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_n \otimes \hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}$$

$$\mathrm{THH}^{hC_{2^n}}_{\alpha-\varsigma_n} = \frac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{\phi^n(I) \otimes \mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \qquad \mathrm{THH}^{tC_{2^n}}_{\alpha-\varsigma_n} = \frac{\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{\phi^n(I) \otimes \hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}$$

Proof. The cell structure

$$C_{2^n}/C_{2^{n-1}+} \longrightarrow S^0 \xrightarrow{a_{\varsigma_n}} S^{\varsigma_n}$$

gives short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{THH}^{\{h,t\}C_{2^n}}_{\alpha+\varsigma_n} \arrow[r, "a_{\varsigma_n}"] \arrow[d, equal] & \mathrm{THH}^{\{h,t\}C_{2^n}}_{\alpha} \arrow[r, "F"] \arrow[d, equal] & \mathrm{THH}^{\{h,t\}C_{2^{n-1}}}_{\alpha} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \boxed{\dfrac{I_{n-1} \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_n \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}} \arrow[r] & \dfrac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_n \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \arrow[r] & \dfrac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_{n-1} \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{THH}^{\{h,t\}C_{2^{n-1}}}_{\alpha} \arrow[r, "V"] \arrow[d, equal] & \mathrm{THH}^{\{h,t\}C_{2^n}}_{\alpha} \arrow[r, "a_{\varsigma_n}"] \arrow[d, equal] & \mathrm{THH}^{\{h,t\}C_{2^n}}_{\alpha-\varsigma_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_{n-1} \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \arrow[r] & \dfrac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{I_n \otimes \mathcal{N}^{\geq d_0+1}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}} \arrow[r] & \boxed{\dfrac{\mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}{\phi^n(I) \otimes \mathcal{N}^{\geq d_0}\hat{\mathbb{\Delta}}_R \otimes I^{\alpha'}}} \arrow[r] & 0
\end{tikzcd}$$

□

We can run the same argument in the more general context of Theorem 1.14. (Warning: the p-typical ς_n corresponds to the integral $\varsigma_{2^{n-1}}$.)

Proposition 3.16. *Let E be a $\mathbf{T}$-equivariant even $\mathbf{E}_\infty$-ring with associated formal group $\mathbf{G}$, and assume that $\langle n \rangle_{\mathbf{G}}$ is a non-zerodivisor in $A = \pi_0 E^{t\mathbf{T}}$ for all n. Set $\langle n \rangle_{\mathbf{G}^{(m)}} = \frac{\langle mn \rangle_{\mathbf{G}}}{\langle m \rangle_{\mathbf{G}}}$.*

Let α be a complex virtual C_{2n}-representation with dimension sequence $d_k = \dim_{\mathbf{C}} \alpha^{C_k}$ for $k \mid 2n$, and let ς_n be the sign representation of C_{2n} with kernel $C_n \subset C_{2n}$. Using bad notation for consistency with the p-typical case, set

$$I^{\alpha'} = \bigotimes_{\substack{k \mid 2n \\ k \neq 1}} (\Phi_{n,\mathbf{G}} A)^{\otimes d_k}.$$

Then the non-orientably graded groups $E^{\{h,t\}C_{2n}}_{\bigstar\pm\varsigma_n}$ are given canonically by

$$E^{hC_{2n}}_{\alpha+\varsigma_n} = \frac{\langle n \rangle_{\mathbf{G}} A \otimes_A \mathrm{Fil}^{\geq d_1+1} A \otimes_A I^{\alpha'}}{\langle 2n \rangle_{\mathbf{G}} A \otimes_A \mathrm{Fil}^{\geq d_1+1} A \otimes_A I^{\alpha'}} \qquad E^{tC_{2n}}_{\alpha+\varsigma_n} = \frac{\langle n \rangle_{\mathbf{G}} A \otimes_A I^{\alpha'}}{\langle 2n \rangle_{\mathbf{G}} A \otimes_A I^{\alpha'}}$$

$$E^{hC_{2n}}_{\alpha-\varsigma_n} = \frac{\mathrm{Fil}^{\geq d_1} A \otimes_A I^{\alpha'}}{\langle 2 \rangle_{\mathbf{G}^{(n)}} A \otimes_A \mathrm{Fil}^{\geq d_1} A \otimes_A I^{\alpha'}} \qquad E^{tC_{2n}}_{\alpha-\varsigma_n} = \frac{I^{\alpha'}}{\langle 2 \rangle_{\mathbf{G}^{(n)}} A \otimes_A I^{\alpha'}}$$

4. Poly-de Rham-Witt forms

Recall that as a $\mathbf{T}$-spectrum, the representation sphere S^{λ^k} has a 1-step cell structure with associated graded

$$S^0, \quad \mathbf{T}/C_{k+} \otimes S^1.$$

As a C_n-spectrum, it instead has a 2-step cell structure with associated graded

$$S^0, \quad C_n/C_{k+} \otimes S^1, \quad C_n/C_{k+} \otimes S^2.$$

While this is slightly inconvenient, we will see that there is a good reason for it: essentially this is because $\mathrm{W}_k\Omega^*$ is not given by $\mathrm{W}\Omega^*/V^k$, but rather $\mathrm{W}\Omega^*/(V^k + dV^k)$.

4.1. Structure of the de Rham-Witt complex over a perfectoid base.

Example 4.1 (Cartier isomorphism)**.** Let R be a perfectoid ring admitting all p-power roots of unity. Fix a compatible choice of roots $\{\zeta_{p^n}\}$, and define the following elements of $A = \mathrm{A}_{\inf}(R) = \mathrm{W}(R^\flat)$:

$$\begin{aligned} q &= [\epsilon] & \widetilde{\xi}_n &= (p^n)_q & \xi_n &= (p^n)_{q^{1/p^{n+1}}} \\ \mu &= 1 - q & &= \frac{\phi^n(\mu)}{\mu} & &= \frac{\phi^{-1}(\mu)}{\phi^{-(n-1)}(\mu)} \end{aligned}$$

In this context, prismatic cohomology of R-algebras can be computed (via the equivalence $A\Omega = \mathbb{\Delta}^{(1)}$) as a q-de Rham complex. cite me

We work this out for the affine line $S = R[x]_p^\wedge$. The q-de Rham complex is given by

$$A\Omega_S = \widehat{\bigoplus_n} \left[A \langle x^n \rangle \xrightarrow{\nabla_q} A \langle x^n \operatorname{dlog} x \rangle \right]$$

where $\widehat{\bigoplus}$ denotes a p-completed direct sum, and the differential sends $\nabla_q(x^n) = (n)_q x^n \operatorname{dlog} x$.

Let's compute the cohomology of $\widetilde{\Omega}_S = A\Omega_S/\widetilde{\xi} = A\Omega_S/(p)_q$. Now the differential is 0 for exponents divisible by p, and an isomorphism otherwise. We conclude that

$$\mathrm{H}^*(\widetilde{\Omega}_S) = \widehat{\bigoplus_n} A/\widetilde{\xi} \langle x^{pn} \rangle [0] \oplus A/\xi \langle x^{pn} \operatorname{dlog} x \rangle [-1]$$

These are isomorphic to

$$A/\xi \langle x^n \rangle \xrightarrow[\sim]{\phi} A/\widetilde{\xi} \langle x^{pn} \rangle$$

$$A/\xi \langle x^n \operatorname{dlog} x \rangle \otimes \{\xi^{-1}\} \xrightarrow[\sim]{\phi} A/\widetilde{\xi} \langle x^{pn} \operatorname{dlog} x \rangle$$

The twist by ξ^{-1} is necessary in the second line because $\phi(\operatorname{dlog} x) = \widetilde{\xi} \operatorname{dlog} x$. In other words, $\mathrm{H}^*(\widetilde{\Omega}_S) = \Omega_S^*\{-*\}$, in accordance with [BMS18, Theorem 8.7].

Next we compute the cohomology of $\widetilde{\mathrm{W}_1\Omega}_S = A\Omega_S/\widetilde{\xi}_2 = A\Omega_S/(p^2)_q$. We find that

$$\begin{aligned} \mathrm{H}^*(\widetilde{\mathrm{W}_1\Omega}) = & \widehat{\bigoplus_n} A/\widetilde{\xi}_2 \left\langle x^{p^2 n} \right\rangle [0] \oplus A/\widetilde{\xi}_2 \left\langle x^{p^2 n} \operatorname{dlog} x \right\rangle [-1] \\ & \oplus \widehat{\bigoplus_{p \nmid j}} A/\widetilde{\xi} \left\langle \phi(\widetilde{\xi}) x^{pj} \right\rangle [0] \oplus A/\widetilde{\xi} \left\langle x^{pj} \operatorname{dlog} x \right\rangle [-1]. \end{aligned}$$

These are isomorphic to

$$A/\xi_2 \left\langle x^n \right\rangle \xrightarrow[\sim]{\phi^2} A/\widetilde{\xi}_2 \left\langle x^{p^2 n} \right\rangle$$

$$A/\phi^{-1}(\xi) \left\langle \xi x^{j/p} \right\rangle \xrightarrow[\sim]{\phi^2} A/\widetilde{\xi} \left\langle \phi(\widetilde{\xi}) x^{pj} \right\rangle$$

$$A/\xi_2 \left\langle x^n \operatorname{dlog} x \right\rangle \otimes \{\xi_2^{-1}\} \xrightarrow[\sim]{\phi^2} A/\widetilde{\xi}_2 \left\langle x^{p^2 n} \operatorname{dlog} x \right\rangle$$

$$A/\phi^{-1}(\xi) \left\langle x^{j/p} \operatorname{dlog} x \right\rangle \otimes \{\xi_2^{-1}\} \xrightarrow[\sim]{\phi^2} A/\widetilde{\xi} \left\langle x^{pj} \operatorname{dlog} x \right\rangle$$

In other words, $\mathrm{H}^*(\widetilde{\mathrm{W}_1\Omega_S}) = \mathrm{W}_1\Omega^*_S\{-*\}$, in accordance with [BMS18, Theorem 11.1].

do the Nygaard part

explain the construction of β

construct $B_k\mathrm{W}_n\Omega^*_S$ and $Z_k\mathrm{W}_n\Omega^*_S$

c.f. [Ill79, Proposition 3.11]
These will not be needed until §4.3.

Lemma 4.2. *If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then*

$$\ker(F^k \colon \mathrm{W}_n\Omega^j_S \to \mathrm{W}_{n-k}\Omega^j_S) = ???$$
$$\ker(V^k \colon \mathrm{W}_{n-k}\Omega^j_S \to \mathrm{W}_n\Omega^j_S) = ???$$
$$\ker(F^k \colon \mathrm{W}\Omega^j_S \to \mathrm{W}\Omega^j_S) = 0$$
$$\ker(V^k \colon \mathrm{W}\Omega^j_S \to \mathrm{W}\Omega^j_S) = 0$$

If S is formally smooth over a p-torsionfree perfectoid ring R^+, then

$$\ker(F^k \colon \mathrm{W}_n\Omega^j_{S/R^+} \to \mathrm{W}_{n-k}\Omega^j_{S/R^+}) = ???$$
$$\ker(V^k \colon \mathrm{W}_{n-k}\Omega^j_{S/R^+} \to \mathrm{W}_n\Omega^j_{S/R^+}) = 0$$

If $R^+ = \mathcal{O}_C$ for a spherically complete perfectoid field C, then

$$\ker(F^k \colon \mathrm{W}\Omega^j_{S/\mathcal{O}_C} \to \mathrm{W}\Omega^j_{S/\mathcal{O}_C}) = \phi^{-k}(\mu)\mathrm{W}\Omega^j_{S/\mathcal{O}_C}$$
$$\ker(V^k \colon \mathrm{W}\Omega^j_{S/\mathcal{O}_C} \to \mathrm{W}\Omega^j_{S/\mathcal{O}_C}) = 0$$

4.2. **Contiguous range.** In this section we compute $\mathrm{TR}_\bigstar(S\mathbf{Z}_p)$ in the contiguous range. Our method is the homotopy orbits to TR spectral sequence from [AG11, Sul22]. In fact, the computation for $\mathrm{TR}_{\bigstar,j}(S;\mathbf{Z}_p)$ works the same way as the computation for $\mathrm{TR}_\bigstar(R;\mathbf{Z}_p)$. We will first compute using the gold elements, then switch to giving canonical identifications.

Example 4.3. We begin by computing $\mathrm{TR}^1_{\bigstar,j}(S)$ when S is a smooth k-algebra. Let $\alpha = (d_0; d_\infty)$ be a virtual C_p-representation. There are three cases, depending on the signs of d_0 and d_∞ (the result is zero if both are negative).

(I)	(II)
0	(III)

(1) $d_0 < 0$, $d_\infty \geq 0$:

$$\begin{pmatrix} \Omega^j a_0^{-d_0} u_{\lambda_0}^{d_\infty} \longleftarrow & \Omega^j a_0^{-d_0} u_{\lambda_0}^{d_\infty} \\ & \Omega^j a_0^{-d_0} u_{\lambda_0}^{d_\infty} \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \\ & \Omega^j a_0^{-d_0} u_{\lambda_0}^{d_\infty} \end{pmatrix}$$

In total, this region is $a_0\Omega^*_S[a_0, u_{\lambda_0}]$.

(2) $d_0 \geq 0$, $d_\infty \geq 0$:

$$\begin{pmatrix} \Omega^j a_0^{-d_0} \longleftarrow & \mathrm{W}_1\Omega^j u_0^{d_0} \\ & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \Omega^j/p^{d_0}\left\langle a_0^{-d_0}\right\rangle & & \mathrm{Fil}^{1-d_0}\,\mathrm{W}_1\Omega^j u_0^{d_0} \\ & \nwarrow & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \mathrm{Fil}^{1-d_0}\,\mathrm{W}_1\Omega^j u_0^{d_0} \\ & \mathrm{Fil}^{d_0}\,\Omega^j a_0^{-d_0} \end{pmatrix}$$

and the extension is $\mathrm{W}_1\Omega^j u_0^{d_0}$. In total, this region is $\mathrm{W}_1\Omega^*_S[u_0, u_{\lambda_0}]$.

Writing things in terms of Fil may seem weird, but it allows us to express the computation uniformly. If $d_0 = 0$, then the spectral sequence is

$$\begin{pmatrix} \Omega^j \longleftarrow & \mathrm{W}_1\Omega^j \\ & \Omega^j \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \mathrm{Fil}^1\,\mathrm{W}_1\Omega^j \\ & \Omega^j \end{pmatrix}$$

and the extension is $\mathrm{W}_1\Omega^j$. If $d_0 > 0$, then the spectral sequence is

$$\begin{pmatrix} \Omega^j a_0^{-d_0} \longleftarrow & \mathrm{W}_1\Omega^j u_0^{d_0} \\ & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \Omega^j a_0^{-d_0} & & \mathrm{W}_1\Omega^j u_0^{d_0} \\ & \nwarrow & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \mathrm{W}_1\Omega^j u_0^{d_0} \end{pmatrix}$$

(3) $d_0 \geq 0$, $d_\infty < 0$:

$$\begin{pmatrix} \Omega^j a_0^{-d_0} \longleftarrow \mathrm{W}_1\Omega^j u_0^{d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \Omega^j/p^{d_0}\left\langle a_0^{-d_0}\right\rangle & & \mathrm{Fil}^{1-d_0}\,\mathrm{W}_1\Omega^j u_0^{d_0} \end{pmatrix}$$

Of course, this is $\mathrm{Fil}^1\,\mathrm{W}_1\Omega^j$ if $d_0 = 0$ and $\mathrm{W}_1\Omega^j \oplus \Omega^j[-1]$ otherwise. What is the total expression for this region? Can we use the Rees algebra of $\mathrm{Fil}^\bullet\,\mathrm{W}_1\Omega^*_S$?

In total, we get

$$\mathrm{TR}^1_\bigstar(S) = \mathrm{W}_1\Omega^*_S[a_0, u_0, u_{\lambda_0}] \oplus \text{(III)}$$

Example 4.4. Next we compute $\mathrm{TR}^1_{\bigstar,j}(S; \mathbf{Z}_p)$ when S is a formally smooth R^+-algebra.

(1) $d_0 < 0$, $d_\infty \geq 0$: exactly the same as the characteristic p case (in fact, this even works over R)
(2) $d_0 \geq 0$, $d_\infty \geq 0$:

$$\begin{pmatrix} \widetilde{\Omega}^j a_0^{-d_0} \longleftarrow & {}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega}^j u_0^{d_0} \\ & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \widetilde{\Omega}^j/\xi^{d_0}\left\langle a_0^{-d_0}\right\rangle & & {}^{\phi^{-1}}\mathrm{Fil}^1\,\widetilde{\mathrm{W}_1\Omega}^j u_0^{d_0} \\ & \overset{\phi}{\nwarrow} & \Omega^j a_0^{-d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} & {}^{\phi^{-1}}\mathrm{Fil}^1\,\widetilde{\mathrm{W}_1\Omega}^j u_0^{d_0} \\ & \Omega^j u_0^{d_0} \end{pmatrix}$$

(noting that the gold relation in $\mathrm{THH}^{\Phi C_p}$ is $a_0 u_0 = \phi^{-1}(\xi)$), and the extension is

$$0 \to {}^{\phi^{-1}}\mathrm{Fil}^1\,\widetilde{\mathrm{W}_1\Omega^j} u_0^{d_0} \to {}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j} u_0^{d_0} \xrightarrow{\phi^{-1}} \Omega^j u_0^{d_0} \to 0.$$

(3) $d_0 \geq 0$, $d_\infty < 0$:

$$\begin{pmatrix} \widetilde{\Omega}^j a_0^{-d_0} \longleftarrow {}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j} u_0^{d_0} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \widetilde{\Omega}^j/\xi^{d_0}\left\langle a_0^{-d_0}\right\rangle & & {}^{\phi^{-1}}\mathrm{Fil}^1\,\widetilde{\mathrm{W}_1\Omega^j} u_0^{d_0} \end{pmatrix}$$

In total, we get

$$\mathrm{TR}^1_\bigstar(S; \mathbf{Z}_p) = ???$$

Example 4.5. Now we show how to convert from $\mathrm{TF}_\bigstar$ notation to $\mathrm{TR}_\bigstar$ notation.

	$\mathrm{TF}_\bigstar$ naming	$\mathrm{TR}_\bigstar$ naming
C_p	$\widetilde{\Omega}^j \left\langle a_0^{-d_0} \right\rangle \longleftarrow {}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j} \left\langle u_0^{d_0} \right\rangle$ $\Omega^j \left\langle a_0^{-d_0} \right\rangle$	$\Omega^j \left\langle \underline{a}_1^{-\underline{d}_1} \right\rangle \longleftarrow \mathrm{W}_1\Omega^j \left\langle \underline{u}_1^{\underline{d}_1} \right\rangle$ $\Omega^j \left\langle \underline{a}_1^{-\underline{d}_1} \right\rangle$
C_{p^2}	$\widetilde{\mathrm{W}_1\Omega^j} \left\langle a_0^{-d_0} u_1^{d_1} \right\rangle \longleftarrow {}^{\phi^{-1}}\widetilde{\mathrm{W}_2\Omega^j} \left\langle u_0^{d_0} u_1^{d_1} \right\rangle$ $\widetilde{\Omega}^j \left\langle a_0^{-d_0} a_1^{-d_1} \right\rangle \longleftarrow {}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j} \left\langle a_0^{-d_0} u_1^{d_1} \right\rangle$ $\Omega^j \left\langle a_0^{-d_0} a_1^{-d_1} \right\rangle$	$\mathrm{W}_1\Omega^j \left\langle \underline{a}_2^{-\underline{d}_2} \underline{u}_1^{\underline{d}_1} \right\rangle \longleftarrow \mathrm{W}_2\Omega^j \left\langle \underline{u}_2^{\underline{d}_2} \underline{u}_1^{\underline{d}_1} \right\rangle$ $\Omega^j \left\langle \underline{a}_2^{-\underline{d}_2} \underline{a}_1^{-\underline{d}_1} \right\rangle \longleftarrow \mathrm{W}_1\Omega^j \left\langle \underline{a}_2^{-\underline{d}_2} \underline{u}_1^{\underline{d}_1} \right\rangle$ $\Omega^j \left\langle \underline{a}_2^{-\underline{d}_2} \underline{a}_1^{-\underline{d}_1} \right\rangle$

Remark 4.6. The q-gold relations tell us how to rewrite the above modules more canonically. We have

$$\begin{tikzcd}
{}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j} \otimes \{\phi^{-1}(\widetilde{\xi}_2)^{d_\infty}\} \arrow[d, equal] & \\
{}^{\phi^{-1}}\widetilde{\mathrm{W}_1\Omega^j}\sigma^{d_\infty} \arrow[d, equal] & \Omega^j a_{\lambda_0}^{-d_0} u_{\lambda_0}^{d_\infty} \arrow[d, equal] \\
\mathrm{TR}^1_{(d_\infty;d_\infty),j} \arrow[r, "a_0^{d_\infty - d_0}"'] & \mathrm{TR}^1_{(d_0;d_\infty),j}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{W}_1\Omega^j \otimes \{\xi_2^{\underline{d}_0}\} \arrow[d, equal] & \\
\mathrm{W}_1\Omega^j \sigma_1^{\underline{d}_0} \arrow[d, equal] & \Omega^j \underline{u}_{\underline{\lambda}_1}^{\underline{d}_0} \underline{a}_1^{-\underline{d}_1} \arrow[d, equal] \\
\mathrm{TR}^1_{(\underline{d}_0,\underline{d}_0),j} \arrow[r, "\underline{a}_1^{\underline{d}_0 - \underline{d}_1}"'] & \mathrm{TR}^1_{(\underline{d}_0,\underline{d}_1),j}
\end{tikzcd}$$

so we compute

$$(a_0^{-d_0} u_{\lambda_0}^{d_\infty})(a_0^{d_0 - d_\infty}) = \sigma^{d_\infty}\xi^{-d_\infty} = \{\phi(\xi)^{d_\infty}\}$$
$$(\underline{a}_1^{-\underline{d}_1} u_{\underline{\lambda}_1}^{\underline{d}_0})(\underline{a}_1^{\underline{d}_1 - \underline{d}_0}) = \sigma_1^{\underline{d}_0}\phi^{-1}(\xi)^{-\underline{d}_0} = \{\xi^{\underline{d}_0}\}$$

$$(u_0^{d_0} u_{\lambda_0}^{d_\infty})(a_0^{d_0 - d_\infty}) = \xi^{d_0}(\sigma\xi^{-1})^{-d_\infty} = \{\xi^{d_0}\phi(\xi)^{d_\infty}\}$$
$$(u_{\underline{\lambda}_1}^{\underline{d}_0} \underline{u}_1^{\underline{d}_1})(\underline{a}_1^{\underline{d}_1 - \underline{d}_0}) = \phi^{-1}(\xi)^{\underline{d}_1}(\sigma_1\phi^{-1}(\xi)^{-1})^{-\underline{d}_0} = \{\xi^{\underline{d}_0}\phi^{-1}(\xi)^{\underline{d}_1}\}$$

Thus $\mathrm{TR}^1_{\bigstar,j}(S;\mathbf{Z}_p)$ is given canonically by

characteristic p	$\underline{d}_1 < 0$	$\underline{d}_1 \geq 0$	
$\underline{d}_0 \geq 0$	$\Omega^j \otimes \{p^{\underline{d}_0}\}$		$\mathrm{W}_1\Omega^j \otimes \{p^{\underline{d}_0 + \underline{d}_1}\}$
$\underline{d}_0 < 0$		$\Omega^j/p^{\underline{d}_1} \otimes \{p^{\underline{d}_0}\}[-1]$	$\mathrm{Fil}^{1-\underline{d}_1}\mathrm{W}_1\Omega^j \otimes \{p^{\underline{d}_0 + \underline{d}_1}\}$
mixed characteristic	$\underline{d}_1 < 0$	$\underline{d}_1 \geq 0$	
$\underline{d}_0 \geq 0$	$\Omega^j \otimes \{\xi^{\underline{d}_0}\}$		$\mathrm{W}_1\Omega^j \otimes \{\xi^{\underline{d}_0}\phi^{-1}(\xi)^{\underline{d}_1}\}$
$\underline{d}_0 < 0$		$\Omega^j/\phi^{-1}(\xi)^{\underline{d}_1} \otimes \{\xi^{\underline{d}_0}\}[-1]$	$\mathrm{Fil}^1\mathrm{W}_1\Omega^j \otimes \{\xi^{\underline{d}_0}\phi^{-1}(\xi)^{\underline{d}_1}\}$

Example 4.7. We can also compute these as Mackey functors. The basic Mackey functors in play are

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^1\mathrm{W}_1\Omega^j \arrow[r] \arrow[d, bend right, "F"'] & \mathrm{W}_1\Omega^j \arrow[r] \arrow[d, bend right, "F"'] & \Omega^j \arrow[r] \arrow[d, bend right] & 0 \\
0 \arrow[r] & \Omega^j \arrow[r, equal] \arrow[u, bend right, "V"'] & \Omega^j \arrow[r] \arrow[u, bend right, "V"'] & 0 \arrow[r] \arrow[u, bend right] & 0
\end{tikzcd}$$

(we do not need the fact that these fit into a short exact sequence). It remains to identify the action of F on the twists. In our various notation systems, these are:

$\mathrm{TF}_\bigstar$ notation I	$F(\sigma)=\sigma$	$F(u_{\lambda_0})=1$	$F(\sigma u_{\lambda_0}^{-1})=\sigma$
$\mathrm{TF}_\bigstar$ notation II	$F(u_0u_1)=u_0$	$F(u_1)=1$	$F(u_0)=u_0$
$\mathrm{TR}_\bigstar$ notation I	$F(\sigma_1)=\sigma_0$	$F(u_{\underline{\lambda}_1})=1$	$F(\sigma_1 u_{\underline{\lambda}_1}^{-1})=\sigma_0$
$\mathrm{TR}_\bigstar$ notation II	$F(\underline{u}_1\underline{u}_0)=\underline{u}_0$	$F(\underline{u}_0)=1$	$F(\underline{u}_1)=\underline{u}_0$
"invariant"	$F(1\otimes f\xi_2)=1\otimes\phi(f)\xi$	$F(1\otimes f\xi)=1\otimes\phi(f)$	$F(1\otimes f\phi^{-1}(\xi))=1\otimes\phi(f)\xi$

Definition 4.8. Given a poly-representation $\underline{\alpha}=(\underline{d}_0,\underline{d}_1,\dots)$, we let

$$\vartheta_r^{\underline{\alpha}}=\underline{u}_0^{\underline{d}_0}\,\underline{u}_1^{\underline{d}_1}\cdots\underline{u}_r^{\underline{d}_r}\,\underline{a}_{r+1}^{-\underline{d}_{r+1}}\cdots$$

Example 4.9. We will now study $\mathrm{TR}^2_{\bigstar,j}(S)$ in the range $\underline{d}_0$, $\underline{d}_1$, $\underline{d}_2\geq 0$. By Tsalidis' theorem [Tsa98, Theorem 2.4], we know a priori that the result will be isomorphic to $\mathrm{W}_2\Omega^j_S$. We use the identification

$$p^i\operatorname{Fil}^r\mathrm{W}_n\Omega^*=\operatorname{Fil}^{r-i}\mathrm{W}_{n-i}\Omega^*\otimes\{p^i\}.$$

To avoid casework, recall that $\operatorname{Fil}^r\mathrm{W}_n\Omega^*$ is $\mathrm{W}_n\Omega^*$ for $r<0$, and vanishes for $r>n$ or $n<0$. The spectral sequence goes

$$E_0=\boxed{\begin{array}{lcl} \mathrm{W}_1\Omega^j\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle & \longleftarrow & \mathrm{W}_2\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ \mathrm{W}_0\Omega^j\left\langle\vartheta_0^{\underline{\alpha}}\right\rangle & \longleftarrow & \mathrm{W}_1\Omega^j\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle \\ & & \mathrm{W}_0\Omega^j\left\langle\vartheta_0^{\underline{\alpha}}\right\rangle \end{array}}$$

$$E_1=\boxed{\begin{tikzcd}[row sep=tiny] \mathrm{W}_1\Omega^j/p^{\underline{d}_2}\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle & \operatorname{Fil}^{2-\underline{d}_2}\mathrm{W}_2\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ \mathrm{W}_0\Omega^j/p^{\underline{d}_1}\left\langle\vartheta_0^{\underline{\alpha}}\right\rangle & \operatorname{Fil}^{1-\underline{d}_1}\mathrm{W}_1\Omega^j\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle \arrow[ul] \\ & \mathrm{W}_0\Omega^j\left\langle\vartheta_0^{\underline{\alpha}}\right\rangle \arrow[ul] \end{tikzcd}}$$

$$E_2=\boxed{\begin{tikzcd}[row sep=tiny] \mathrm{W}_{-\underline{d}_1}\Omega^j/p^{\underline{d}_2}\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle & \operatorname{Fil}^{2-\underline{d}_2}\mathrm{W}_2\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ & \operatorname{Fil}^{1-(\underline{d}_1+\underline{d}_2)}\mathrm{W}_{1-\underline{d}_2}\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ & \mathrm{W}_{-\underline{d}_1}\Omega^j\left\langle\vartheta_1^{\underline{\alpha}}\right\rangle \arrow[uul] \end{tikzcd}}$$

$$E_\infty=\boxed{\begin{array}{l} \operatorname{Fil}^{2-\underline{d}_2}\mathrm{W}_2\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ \operatorname{Fil}^{1-(\underline{d}_1+\underline{d}_2)}\mathrm{W}_{1-\underline{d}_2}\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \\ \mathrm{W}_{-(\underline{d}_1+\underline{d}_2)}\Omega^j\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle \end{array}}$$

The extension is $\mathrm{W}_2\Omega^j_S\left\langle\vartheta_2^{\underline{\alpha}}\right\rangle$, or more canonically $\mathrm{W}_2\Omega^j_S\otimes\{p^{\underline{d}_0+\underline{d}_1+\underline{d}_2}\}$. The reader may find it instructive to run the spectral sequence for various values of $\underline{d}_1$, $\underline{d}_2$ and see which terms vanish.

Example 4.10. We repeat the previous calculation in the mixed characteristic case. The spectral sequence goes

$$E_0 = \boxed{\begin{tikzcd} \mathrm{W}_1\Omega^j\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle & \mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \arrow[l] \\ \mathrm{W}_0\Omega^j\left\langle \vartheta^{\underline{\alpha}}_0\right\rangle & \mathrm{W}_1\Omega^j\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle \arrow[l] \\ & \mathrm{W}_0\Omega^j\left\langle \vartheta^{\underline{\alpha}}_0\right\rangle \end{tikzcd}}$$

$$E_1 = \boxed{\begin{tikzcd} \mathrm{W}_1\Omega^j/\phi^{-2}(\xi)^{\underline{d}_2}\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle & \mathrm{Fil}^2\,\mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ \mathrm{W}_0\Omega^j/\phi^{-1}(\xi)^{\underline{d}_1}\left\langle \vartheta^{\underline{\alpha}}_0\right\rangle & \mathrm{Fil}^1\,\mathrm{W}_1\Omega^j\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle \arrow[ul] \\ & \mathrm{Fil}^0\,\mathrm{W}_0\Omega^j\left\langle \vartheta^{\underline{\alpha}}_0\right\rangle \arrow[ul] \end{tikzcd}}$$

$$E_2 = \boxed{\begin{tikzcd} \mathrm{W}_0\Omega^j/\phi^{-2}(\xi)^{\underline{d}_2}\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle & \mathrm{Fil}^2\,\mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ & \mathrm{Fil}^1\,\mathrm{W}_1\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ & \mathrm{Fil}^0\,\mathrm{W}_0\Omega^j\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle \arrow[uul] \end{tikzcd}}$$

$$E_\infty = \boxed{\begin{matrix} \mathrm{Fil}^2\,\mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ \mathrm{Fil}^1\,\mathrm{W}_1\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ \mathrm{Fil}^0\,\mathrm{W}_0\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \end{matrix}}$$

The extension is $\mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle$, or more canonically $\mathrm{W}_2\Omega^j \otimes \bigotimes_{i=0}^2\{\phi^{-i}(\xi)^{\underline{d}_i}\}$.

Example 4.11. Now let $\underline{\alpha} = (\underline{d}_0, \underline{d}_1, \underline{d}_2)$ be any $\mathbf{Z}/p^2$-polyrepresentation. Using the same ideas as the previous examples, it is straightforward to evaluate the HOTRSS for $\underline{\alpha}$, provided that $\underline{\alpha}$ is contiguous. The results are listed in Figures 4.2 and 4.2.

We comment on the non-contiguous case $\underline{d}_0 \geq 0$, $\underline{d}_1 < 0$, $\underline{d}_2 \geq 0$. In characteristic p, the E_2 page is

$$\begin{tikzcd} \frac{\mathrm{W}_1\Omega^j}{p^{\underline{d}_2}}\left\langle \vartheta^{\underline{\alpha}}_1\right\rangle & \mathrm{Fil}^{2-\underline{d}_2}\,\mathrm{W}_2\Omega^j\left\langle \vartheta^{\underline{\alpha}}_2\right\rangle \\ & \\ & \Omega^j\left\langle \vartheta^{\underline{\alpha}}_0\right\rangle \arrow[uul] \end{tikzcd}$$

given by $\vartheta^{\underline{\alpha}}_0 \mapsto p^{-\underline{d}_1}\vartheta^{\underline{\alpha}}_1$. The kernel of this differential is

$$p^{(\!(\underline{d}_1+\underline{d}_2)\!)}\vartheta_0 = p^{(\!(\underline{d}_1+\underline{d}_2)\!)}\underline{u}_0^{\underline{d}_0}\underline{a}_1^{-\underline{d}_1}a_2^{-\underline{d}_2}.$$

Recall from [Sul22, Theorem 1.4] that $\mathrm{TF}_\bigstar(k)$ contains the "collision classes"

$$c_i = a_{i-1}^{-1}a_i = u_{i-1}u_i^{-1};$$

in poly-graded notation, this is

$$\underline{c}_i = \underline{a}_{i+1}^{-1}\underline{a}_i = \underline{u}_{i+1}\underline{u}_i^{-1},$$

the relevant case for us being $\underline{c}_1 = \underline{a}_1\underline{a}_2^{-1} = \underline{u}_1^{-1}\underline{u}_2$. The kernel is thus

$$\begin{cases} \Omega^j\left\langle u_0^{\underline{d}_0}\underline{c}_1^{-\underline{d}_1}\underline{u}_2^{\underline{d}_1+\underline{d}_2}\right\rangle & \underline{d}_1+\underline{d}_2 \geq 0 \\ \Omega^j\left\langle u_0^{\underline{d}_0}c_1^{-\underline{d}_1}\underline{a}_2^{-(\underline{d}_1+\underline{d}_2)}\right\rangle & \underline{d}_1+\underline{d}_2 < 0 \end{cases}$$

$\underline{d}_0 \geq 0$	$\underline{d}_2 \geq 0$		$\underline{d}_2 < 0$
$\underline{d}_1 \geq 0$		$\mathrm{Fil}^{2-\underline{d}_2}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{Fil}^{1-(\underline{d}_1+\underline{d}_2)}\, \mathrm{W}_{1-\underline{d}_2}\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_{-(\underline{d}_1+\underline{d}_2)}\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{1-\underline{d}_1}\, \mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_{-\underline{d}_1}\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
		$\mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$	$\mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
$\underline{d}_1 < 0$	$\dfrac{\mathrm{W}_1\Omega^j}{p^{\min(-\underline{d}_1,\underline{d}_2)}} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{2-\underline{d}_2}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\Omega^j \left\langle \vartheta_0 \right\rangle$	$\mathrm{W}_0\Omega^j \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$
		???	

$\underline{d}_0 < 0$	$\underline{d}_2 \geq 0$		$\underline{d}_2 < 0$	
$\underline{d}_1 \geq 0$	$\mathrm{W}_{-\underline{d}_1}\Omega^j/p^{\underline{d}_2} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_0\Omega^j/p^{\underline{d}_1} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{2-\underline{d}_2}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{Fil}^{1-(\underline{d}_1+\underline{d}_2)}\, \mathrm{W}_{1-\underline{d}_2}\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$	$\mathrm{W}_0\Omega^j/p^{\underline{d}_1} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{1-\underline{d}_1}\, \mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
	$\mathrm{W}_0\Omega^j/p^{\underline{d}_1+\underline{d}_2} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{1-(\underline{d}_1+\underline{d}_2)}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		
$\underline{d}_1 < 0$	$\mathrm{W}_1\Omega/p^{\underline{d}_2} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^{2-\underline{d}_2}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		

FIGURE 1. E_∞ pages of the HOTRSS for $\mathrm{TR}^2_\bigstar(S \,|\, \mathbf{S}_{\mathrm{W}(k)})$. Extensions are marked below the dotted lines.

$$\frac{\mathrm{Fil}^{2-\underline{d}_2}\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle \oplus \mathrm{W}\Omega^j \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle}{\left(\begin{aligned} V\omega \otimes \vartheta_0^{\underline{\alpha}} &= V\omega \otimes p^{-\underline{d}_1}\vartheta_2^{\underline{\alpha}} \\ &= V^2F\omega \otimes p^{-(\underline{d}_1+1)}\vartheta_2^{\underline{\alpha}} \\ dV\omega \otimes \vartheta_0^{\underline{\alpha}} &= dV\omega \otimes p^{-\underline{d}_1}\vartheta_2^{\underline{\alpha}} \\ &= dV^2F\omega \otimes p^{-(\underline{d}_1+1)}\vartheta_2^{\underline{\alpha}} \end{aligned} \right)}$$

do the extension

In mixed characteristic, the E_2 page is

$$\begin{tikzcd} \dfrac{\mathrm{W}_1\Omega^j}{\phi^{-2}(\xi)^{\underline{d}_2}} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle & \mathrm{Fil}^2\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle \\ & \Omega^j \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle \arrow[ul] \end{tikzcd}$$

given by $\vartheta_0^{\underline{\alpha}} \mapsto \phi^{-1}(\xi)^{-\underline{d}_1}\vartheta_1^{\underline{\alpha}}$. We do not know how to compute the kernel of this map in general—recall that $\xi\phi^{-1}(\xi) \equiv p^2 \bmod \phi^{-2}(\xi)$ and $\phi^{-1}(\xi) \equiv p \bmod \phi^{-2}(\xi)$. In fact, these congruences hold mod $\phi^{-2}(\xi)^p$, which is the phenomenon of *refraction* discussed for C_p in [Sul23b, Remark 1.2] and generalized to C_{p^n} in [Mao24, Lemma 2.1]. Thus, computing the kernel of this differential requires a very precise understanding of refraction.

Theorem 4.12 (Contiguous range of $\mathrm{TR}_\bigstar$). *Let $\underline{\alpha} = (\underline{d}_0, \ldots, \underline{d}_n)$ be a $\mathbf{Z}/p^n$-polyrepresentation which is non-negatively concentrated in $[-\ell, -k]$.*

$\underline{d}_0 \geq 0$	$\underline{d}_2 \geq 0$		$\underline{d}_2 < 0$	
$\underline{d}_1 \geq 0$		$\mathrm{Fil}^2\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{Fil}^1\, \mathrm{W}_1\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_0\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		$\mathrm{Fil}^1\, \mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_0\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
		$\mathrm{W}_2\Omega \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		$\mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
$\underline{d}_1 < 0$	$\dfrac{\mathrm{W}_1\Omega^j}{\phi^{-1}(\xi)^{-\underline{d}_1}, \phi^{-2}(\xi)^{\underline{d}_2}} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^2\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$???		$\mathrm{W}_0\Omega^j \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$

$\underline{d}_0 < 0$	$\underline{d}_2 \geq 0$		$\underline{d}_2 < 0$	
$\underline{d}_1 \geq 0$	$\mathrm{W}_0\Omega^j/\phi^{-2}(\xi)^{\underline{d}_2} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$ $\mathrm{W}_0\Omega^j/\phi^{-1}(\xi)^{\underline{d}_1} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^2\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$ $\mathrm{Fil}^1\, \mathrm{W}_1\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$	$\mathrm{W}_0\Omega^j/\phi^{-1}(\xi)^{\underline{d}_1} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^1\, \mathrm{W}_1\Omega^j \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$
	$\dfrac{\mathrm{W}_0\Omega^j}{\phi^{-1}(\xi)^{\underline{d}_1}\phi^{-2}(\xi)^{\underline{d}_2}} \left\langle \vartheta_0^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^1\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		
$\underline{d}_1 < 0$	$\mathrm{W}_1\Omega/\phi^{-2}(\xi)^{\underline{d}_2} \left\langle \vartheta_1^{\underline{\alpha}} \right\rangle$	$\mathrm{Fil}^2\, \mathrm{W}_2\Omega^j \left\langle \vartheta_2^{\underline{\alpha}} \right\rangle$		

FIGURE 2. E_∞ pages of the HOTRSS for $\mathrm{TR}^2_{\bigstar}(S|\mathbf{S}_{\mathrm{A}_{\mathrm{inf}}(R^+)})$. Extensions are marked below the dotted lines.

If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then

$$\begin{aligned}
\mathrm{TR}^n_{\underline{\alpha},j}(S \mid \mathbf{S}_{\mathrm{W}(k)}) &= \mathrm{Fil}^{k-(\underline{d}_k+\cdots+\underline{d}_\ell)}\, \mathrm{W}_\ell\Omega^j_S \left\langle \vartheta_\ell^{\underline{\alpha}} \right\rangle \\
&= \mathrm{Fil}^{k-(\underline{d}_k+\cdots+\underline{d}_\ell)}\, \mathrm{W}_\ell\Omega^j_S \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_\ell}\} \\
\mathrm{TR}^n_{\underline{\alpha}-1,j}(S \mid \mathbf{S}_{\mathrm{W}(k)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\underline{d}_k+\cdots+\underline{d}_\ell}} \left\langle \vartheta_{k-1}^{\underline{\alpha}} \right\rangle \\
&= \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\underline{d}_k+\cdots+\underline{d}_\ell}} \otimes \{p^{\underline{d}_0+\cdots+\underline{d}_{(k-1)}}\}
\end{aligned}$$

If S is formally smooth over a p-torsionfree perfectoid ring R^+, then

$$\begin{aligned}
\mathrm{TR}^n_{\underline{\alpha},j}(S \mid \mathbf{S}_{\mathrm{A}_{\mathrm{inf}}(R^+)}) &= \mathrm{Fil}^k\, \mathrm{W}_\ell\Omega^j_{S/R^+} \left\langle \vartheta_\ell^{\underline{\alpha}} \right\rangle \\
&= \mathrm{Fil}^k\, \mathrm{W}_\ell\Omega^j_{S/R^+} \otimes \bigotimes_{i=0}^{\ell} \{\phi^{-i}(\xi)\}^{\underline{d}_i} \\
\mathrm{TR}^n_{\underline{\alpha}-1,j}(S \mid \mathbf{S}_{\mathrm{A}_{\mathrm{inf}}(R^+)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k}\cdots\phi^{-\ell}(\xi)^{\underline{d}_\ell}} \left\langle \vartheta_{k-1}^{\underline{\alpha}} \right\rangle \\
&= \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k}\cdots\phi^{-\ell}(\xi)^{\underline{d}_\ell}} \otimes \bigotimes_{i=0}^{k-1} \{\phi^{-i}(\xi)\}^{\underline{d}_i}
\end{aligned}$$

Corollary 4.13 (Eventually constant contiguous range)**.** *Let $\underline{\alpha}$ be a $\mathbf{Z}_p$-polyrepresentation non-negatively concentrated in $(-\infty, -k]$, and suppose that $\underline{d}_r = \underline{d}_\ell$ for all $r \geq \ell$. If S is formally smooth over $\mathcal{O}_C$, then*

$$\mathrm{TR}_{\underline{\alpha},j}(S \,|\, \mathbf{S}_{\mathrm{A_{inf}}(\mathcal{O}_C)}) = \mathrm{Fil}^k \, \mathrm{W}\Omega^j_{S/\mathcal{O}_C} \otimes \left(\bigotimes_{i=0}^{\ell-1} \{\phi^{-i}(\xi)\}^{\underline{d}_i} \right) \otimes \{\phi^{-\ell}(\mu)\}^{\underline{d}_\ell}$$

$$\mathrm{TR}_{\underline{\alpha}-1,j}(S \,|\, \mathbf{S}_{\mathrm{A_{inf}}(\mathcal{O}_C)}) = \frac{\mathrm{W}_{k-1}\Omega^j_{S/\mathcal{O}_C}}{\phi^{-\ell}(\mu)^{\underline{d}_\ell} \prod_{i=0}^{\ell-1} \phi^{-i}(\xi)^{\underline{d}_i}} \otimes \bigotimes_{i=0}^{k-1} \{\phi^{-i}(\xi)\}^{\underline{d}_i}$$

4.3. **Negative range.** We explore Theorem 4.12 by specializing it to the negative range, which is the range that arises in K-theory calculations. In principle, this range can also be computed using cell structures. In practice, computing the kernels, cokernels, and extensions in the cellular method is much harder than the HOTRSS. However, now that we know the result in advance, we can figure these out, and it gives a very interesting filtration on the $\mathrm{TR}_\bigstar$ groups. We will also need the cellular strategy to handle the non-orientable representations at the prime 2 in §4.5.

Corollary 4.14 (Negative range)**.** *Let $\underline{\alpha} = \sum_{r=0}^{\infty} \underline{k}_r \underline{\lambda}_r$ be an actual poly-representation. Given $i \geq 0$, let ℓ be such that*

$$i - \underline{d}_{\ell+1}(\underline{\alpha}) < 0 \leq i - \underline{d}_\ell(\underline{\alpha}).$$

Then if S is formally smooth over an arbitrary perfectoid ring R, we have

$$\begin{aligned} \mathrm{TR}^n_{2i-\underline{\alpha},j}(S; \mathbf{Z}_p) &= \mathrm{W}_\ell \Omega^j_{S/R} \otimes \bigotimes_{r=0}^{\ell} \{\phi^{-r}(\xi)\}^{i-\underline{d}_r} \\ &= \mathrm{W}_\ell \Omega^j_{S/R} \otimes ??? \\ \mathrm{TR}^n_{(2i-\underline{\alpha})-1,j}(S; \mathbf{Z}_p) &= 0 \end{aligned}$$

Proof. When R is either a perfect $\mathbf{F}_p$-algebra or p-torsionfree, this is a special case of Theorem 4.12. Since the result is the same in both cases, it must be true for arbitrary perfectoid rings using the fracture square of [Bha18, Proposition IV.3.2]. □

Here is an equivalent way to state this using the gold elements.

Theorem 4.15. *Let S be formally smooth over any perfectoid ring R. The non-positive range of $\mathrm{TR}^n_\bigstar(S; \mathbf{Z}_p)$ and negative range of $\mathrm{TR}_\bigstar(S; \mathbf{Z}_p)$ are given by*

$$\begin{aligned} \mathrm{TR}^n_{\bigstar \leq 0}(S; \mathbf{Z}_p) &= \mathrm{W}_n \Omega^*_{S/R}[\sigma_n, a_{\underline{\lambda}_i}, u_{\underline{\lambda}_i}]/\sim \\ \mathrm{TR}_{\bigstar < 0}(S; \mathbf{Z}_p) &= \mathrm{W}\Omega^*_{S/R}[a_{\underline{\lambda}_i}, u_{\underline{\lambda}_i}]/\sim \end{aligned}$$

where $\sim$ denotes the gold relations.

If S is smooth over k, then $\mathrm{TR}_{\bigstar<0}(S)$ agrees with $\mathrm{TR}_{\bigstar\leq 0}(S)$. If S is formally smooth over $\mathcal{O}_C$, then the non-positive range is given by

$$\mathrm{TR}_{\bigstar \leq 0}(S; \mathbf{Z}_p) = \mathrm{W}\Omega^*_{S/\mathcal{O}_C}[\beta, a_{\underline{\lambda}_i}, u_{\underline{\lambda}_i}]/\sim$$

In the case of regular poly-representations, we get an interesting relationship to the prismatic Gamma function.

Lemma 4.16. *We have*

$$\prod_{r=a}^{b} \phi^{-r}(\xi)^{\lceil p^r \nu \rceil} = \phi^{-a}(\xi_{(b-a+1)}) \frac{\phi^{-(b+1)} \Gamma_{\mathbb{A}} \lceil p^{b+1} \nu \rceil}{\phi^{-a} \Gamma_{\mathbb{A}} \lceil p^a \nu \rceil}$$

Proof. We compute

$$\begin{aligned}
\prod_{r=a}^{b} \phi^{-r}(\xi)^{\lceil p^r \nu \rceil} &= \phi^{-b} \prod_{r=a}^{b} \phi^{b-r}(\xi)^{\lceil p^r \nu \rceil} \\
&= \phi^{-(b+1)} \prod_{r=0}^{b-a} \phi^{r}(\widetilde{\xi})^{\left\lceil \frac{\lceil p^{b+1}\nu \rceil}{p^{r+1}} \right\rceil} \\
&= \phi^{-a}(\xi_{(b-a+1)})\phi^{-(b+1)} \prod_{r=0}^{b-a} \phi^{r}(\widetilde{\xi})^{\left\lfloor \frac{\lceil p^{b+1}\nu \rceil - 1}{p^{r+1}} \right\rfloor} \\
&= \phi^{-a}(\xi_{(b-a+1)})\phi^{-(b+1)} \frac{\Gamma_{\mathbb{A}}\lceil p^{b+1}\nu \rceil}{\phi^{b-a+1}\Gamma_{\mathbb{A}}\lceil p^{a}\nu \rceil} \\
&= \phi^{-a}(\xi_{(b-a+1)}) \frac{\phi^{-(b+1)}\Gamma_{\mathbb{A}}\lceil p^{b+1}\nu \rceil}{\phi^{-a}\Gamma_{\mathbb{A}}\lceil p^{a}\nu \rceil}
\end{aligned}$$

□

Corollary 4.17 (Regular poly-representations, negative range; compare [Sul23a, Corollary 4.6])**.** *Let $\nu > 0$ be a slope. Given $i \geq 0$, let $\ell = \ell(\nu, i) = \lfloor \log_p(i/\nu) \rfloor$. If S is formally smooth over an arbitrary perfectoid ring R, we have*

$$\begin{aligned}
\mathrm{TR}_{2i-[\![\nu]\!]_\lambda, j}(S; \mathbf{Z}_p) &= \mathrm{W}_\ell \Omega^j_{S/R} \otimes \left\{ \frac{\xi_\ell^{i-1}\Gamma_{\mathbb{A}}\lceil \nu \rceil}{\phi^{-(\ell+1)}\Gamma_{\mathbb{A}}\lceil p^{\ell+1}\nu \rceil} \right\} \\
\mathbf{TR}_{2i-[\![\nu]\!]_\lambda, j}(S; \mathbf{Z}_p) &= \mathbf{W}_{\lfloor i/\nu \rfloor} \Omega^j_{S/R} \otimes \left\{ \frac{\xi_\ell^{i-1}\Gamma_{\mathbb{A}}\lceil \nu \rceil}{\phi^{-(\ell+1)}\Gamma_{\mathbb{A}}\lceil p^{\ell+1}\nu \rceil} \right\} \\
\mathrm{TR}_{(2i-[\![\nu]\!]_\lambda)-1, j}(S; \mathbf{Z}_p) &= 0 \\
\mathbf{TR}_{(2i-[\![\nu]\!]_\lambda)-1, j}(S; \mathbf{Z}_p) &= 0
\end{aligned}$$

Corollary 4.18.

$\mathbf{Z}_p(i)$ transition maps

Now we compare to the cellular method. To orient the reader, we first explain the calculation for a general polygonic spectrum.

Example 4.19. Let M be an integral polygonic spectrum. We define the boxed entries via the short exact sequences

$$0 \longrightarrow \boxed{\pi_i M[\mathbf{V}^k]} \longrightarrow (\pi_i M)_{k\mathbf{Z}} \xrightarrow{\mathbf{V}^k} \pi_i M \longrightarrow \pi_i M/\mathbf{V}^k \longrightarrow 0$$

$$0 \longrightarrow \boxed{\mathcal{F}^k \pi_i M} \longrightarrow (\pi_i M)^{k\mathbf{Z}} \xrightarrow{d\mathbf{V}^k} \pi_{i+1} M/\mathbf{V}^k \longrightarrow \boxed{\pi_{i+1}^k M} \longrightarrow 0$$

Using the cell structure, $\pi_{*-\lambda^{1/k}} M$ is computed by the spectral sequence

$$E_1 = \begin{tikzcd}
\pi_0 M & \pi_1 M & \pi_2 M & & & \\
& \pi_0 M \arrow[ul, "\mathbf{V}^k"'] & \pi_1 M \arrow[ul] & \pi_2 M \arrow[ul] & \dots & \\
& & \pi_0 M \arrow[ul, "1-\gamma^k"'] & \pi_1 M \arrow[ul] & \pi_2 M \arrow[ul] & \dots
\end{tikzcd}$$

$$E_2 = \begin{tikzcd}
\pi_0^k M & \pi_1 M/\mathbf{V}^k & \pi_2 M/\mathbf{V}^k & \dots & & \\
& \pi_0 M[\mathbf{V}^k] & \pi_1 M[\mathbf{V}^k] & \pi_2 M[\mathbf{V}^k] & \dots & \\
& & (\pi_0 M)^{k\mathbf{Z}} \arrow[uul, "d\mathbf{V}^k"] & (\pi_1 M)^{k\mathbf{Z}} \arrow[uul] & (\pi_2 M)^{k\mathbf{Z}} \arrow[uul] & \dots
\end{tikzcd}$$

$$E_\infty = \begin{tikzcd}
\pi_0^k M & \pi_1^k M & \pi_2^k M & \dots & & \\
& \pi_0 M[\mathbf{V}^k] & \pi_1 M[\mathbf{V}^k] & \pi_2 M[\mathbf{V}^k] & \dots & \\
& & \mathcal{F}^k \pi_0 M & \mathcal{F}^k \pi_1 M & \mathcal{F}^k \pi_2 M & \dots
\end{tikzcd}$$

with the first column being $* = 0$. cite the differentials

Example 4.20. Let S be smooth over a perfect $\mathbf{F}_p$-algebra k. When $M = \mathrm{TR}(S)$, we have $(\pi_i M)[V^k] = 0$ by by [Ill79, Corollaire 3.6] and $\mathcal{F}^{p^k} \pi_i M = F^k \mathrm{W}\Omega_S^i$ by [DM23b, Lemma 2.15]. Example 4.19 thus becomes

$$\begin{matrix}
\mathrm{W}_k\Omega^0 & \mathrm{W}_k\Omega^1 & \mathrm{W}_k\Omega^2 & \dots & & \\
& & F^k\mathrm{W}\Omega^0 & F^k\mathrm{W}\Omega^1 & F^k\mathrm{W}\Omega^2 & \dots
\end{matrix}$$

which gives $\mathrm{TR}_{*-\underline{\lambda}_k}(S) \cong \mathrm{W}_k\Omega_S^* \oplus F^k\mathrm{W}\Omega_S^{*-2}$. By Theorem 4.12, the canonical identification is

$$\mathrm{TR}_{*-\underline{\lambda}_k}(S) = \mathrm{W}_k\Omega_S^* \oplus (\mathrm{W}\Omega_S^{*-2} \otimes \{p^k\})$$

with F^k trivializing the twist.

The finite-level computation $\mathrm{TR}^n_{*-\underline{\lambda}_k}(S)$ is more interesting. For simplicity, let's focus on $\mathrm{TR}^1_{*-\underline{\lambda}_1}(S)$. By cite me, the E_∞ page is

$$\begin{matrix}
\Omega^0 & \Omega^1 & \Omega^2 \oplus (\Omega^0 \otimes \{p^2\}) & \dots \\
& & B\Omega^1 & \dots \\
& & Z\Omega^0 & \dots
\end{matrix}$$

By Theorem 4.12, we should have a canonical identification

$$\mathrm{TR}^1_{2-\underline{\lambda}_1}(S) = \Omega_S^2 \oplus (\mathrm{W}_1\Omega_S^0 \otimes \{p\}),$$

so there is an interesting extension going on. Using $\ker F = \operatorname{im} V$ cite meand cite me, the extension is given by

$$\begin{tikzcd}
\Omega^0 \otimes \{p^2\} \arrow[r, tail] & \operatorname{im} V \otimes \{p\} \arrow[r, tail] \arrow[d, two heads, "Fd"] & \mathrm{W}_1\Omega^0 \otimes \{p\} \arrow[d, two heads] \\
& B\Omega^1 \arrow[r, tail] & (\mathrm{W}_1\Omega^0/p) \otimes \{p\} \arrow[d, two heads, "F"] \\
& & Z\Omega^0
\end{tikzcd}$$

Example 4.21. Let S be formally smooth over a p-torsionfree perfectoid ring R^+. By cite me, in this case $\mathrm{TR}^n_{*-\underline{\lambda}_k}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)})$ is computed by the E_∞ page

$$\begin{array}{cccc}
\mathrm{W}_{k-1}\Omega^0 & \mathrm{W}_{k-1}\Omega^1 & \mathrm{W}_{k-1}\Omega^2 \oplus (\mathrm{W}_{k-1}\Omega^0 \otimes \{\xi_{n+1}\}) & \ldots \\
 & & 0 & \ldots \\
 & & \operatorname{im} F^k|_{\mathrm{W}_{n-k}\Omega^0} & \ldots
\end{array}$$

By Theorem 4.12, we should have a canonical identification

$$\mathrm{TR}^n_{2-\underline{\lambda}_k}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)}) = \mathrm{W}_{k-1}\Omega^2_{S/R^+} \oplus (\mathrm{W}_n\Omega^0_{S/R^+} \otimes \{\xi_k\});$$

the extension is explained by noting that $\ker(F^k|_{\mathrm{W}_n\Omega^0_{S/R^+}}) = \phi^{-k}(\xi_{n-k+1})\mathrm{W}_n\Omega^0_{S/R^+} = \dfrac{\xi_{n+1}}{\xi_k}\mathrm{W}_n\Omega^0_{S/R^+}$ cite me.

Next suppose that $R^+ = \mathcal{O}_C$ for a spherically complete perfectoid field C. Then the E_∞ page computing $\mathrm{TR}_{*-\underline{\lambda}_k}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(\mathcal{O}_C)})$ is

$$\begin{array}{cccc}
\mathrm{W}_{k-1}\Omega^0 & \mathrm{W}_{k-1}\Omega^1 & \mathrm{W}_{k-1}\Omega^2 \oplus (\mathrm{W}_{k-1}\Omega^0 \otimes \{\mu\}) & \ldots \\
 & & 0 & \ldots \\
 & & \operatorname{im} F^k|_{\mathrm{W}\Omega^0} & \ldots
\end{array}$$

By Theorem 4.12, we should have a canonical identification

$$\mathrm{TR}_{2-\underline{\lambda}_k}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(\mathcal{O}_C)}) = \mathrm{W}_{k-1}\Omega^2_{S/\mathcal{O}_C} \oplus (\mathrm{W}\Omega^0_{S/\mathcal{O}_C} \otimes \{\xi_k\});$$

the extension is explained by noting that $\ker(F^k|_{\mathrm{W}\Omega^0_{S/\mathcal{O}_C}}) = \phi^{-k}(\mu)\mathrm{W}\Omega^0_{S/\mathcal{O}_C} = \dfrac{\mu}{\xi_k}\mathrm{W}\Omega^0_{S/\mathcal{O}_C}$ cite me.

Next we show how to mix representations.

Example 4.22. Let $k \le \ell \le n$, and let S be formally smooth over a p-torsionfree perfectoid ring R^+. By cite me, $\mathrm{TR}^n_{*-\underline{\lambda}_k-\underline{\lambda}_\ell}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)})$ is computed by the E_1 page

$$\begin{array}{cccc}
\mathrm{W}_{\ell-1}\Omega^0 & \mathrm{W}_{\ell-1}\Omega^1 & \mathrm{W}_{\ell-1}\Omega^2 \oplus (\mathrm{W}_{\ell-1}\Omega^0 \otimes \{\xi_\ell\}) & \ldots \\
 & & 0 & \ldots \\
 & & \operatorname{im} F^\ell|_{\mathrm{W}\Omega^0} & \ldots
\end{array}$$

We should have

$$\mathrm{TR}^n_{-\underline{\lambda}_k-\underline{\lambda}_\ell}(S;\mathbf{Z}_p) = ???$$
$$\mathrm{TR}^n_{2-\underline{\lambda}_k-\underline{\lambda}_\ell}(S;\mathbf{Z}_p) = ???$$
$$\mathrm{TR}^n_{4-\underline{\lambda}_k-\underline{\lambda}_\ell}(S;\mathbf{Z}_p) = ???$$

4.4. **Positive range.** Next we specialize Theorem 4.12 to the positive range; this is the range that will receive Norm maps from the ($\mathbf{Z}$-graded) de Rham-Witt complex. Once again, we will compare with the cellular method and describe the resulting filtration.

Corollary 4.23 (Positive range)**.** *Let $\underline{\alpha} = \sum_{r=0}^{\infty} \underline{k}_r \underline{\lambda}_r$ be an actual poly-representation. Given $i \in \mathbf{Z}$, let $k = k(i)$ be such that*

$$\underline{d}_{(k-1)}(\underline{\alpha}) - i < 0 \leq \underline{d}_k(\underline{\alpha}) - i.$$

If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then there are canonical identifications

$$\begin{aligned}
\mathrm{TR}^n_{\underline{\alpha}-2i,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= \mathrm{Fil}^{k-((\underline{d}_k-i)+\cdots+(\underline{d}_n-i))}\,\mathrm{W}_n\Omega^j_S \otimes \{p^{(\underline{d}_0-i)+\cdots+(\underline{d}_n-i)}\} \\
&= \mathrm{Fil}^{k-(\underline{d}_k+\cdots+\underline{d}_n)+(n-k+1)i}\,\mathrm{W}_n\Omega^j_S \otimes \left\{\frac{p^{\underline{d}_0+\cdots+\underline{d}_n}}{p^{(n+1)i}}\right\} \\
\mathrm{TR}^n_{\underline{\alpha}-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{(\underline{d}_k-i)+\cdots+(\underline{d}_n-i)}} \otimes \{p^{(\underline{d}_0-i)+\cdots+(\underline{d}_{(k-1)}-i)}\} \\
&= \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\underline{d}_k+\cdots+\underline{d}_n}/p^{(n-k+1)i}} \otimes \left\{\frac{p^{\underline{d}_0+\cdots+\underline{d}_{(k-1)}}}{p^{ki}}\right\}
\end{aligned}$$

This also holds for $n=\infty$ using the first form.

If S is formally smooth over a p-torsionfree perfectoid ring R^+, then there are canonical identifications

$$\begin{aligned}
\mathrm{TR}^n_{\underline{\alpha}-2i,j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)}) &= \mathrm{Fil}^k\,\mathrm{W}_n\Omega^j_{S/R^+} \otimes \bigotimes_{r=0}^{n} \{\phi^{-r}(\xi)\}^{\underline{d}_r-i} \\
&= \mathrm{Fil}^k\,\mathrm{W}_n\Omega^j_{S/R^+} \otimes \left(\bigotimes_{r=0}^{n} \{\phi^{-r}(\xi)\}^{\underline{d}_r}\right) \otimes \{\xi_{n+1}\}^{-i} \\
\mathrm{TR}^n_{\underline{\alpha}-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R^+)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k-i}\cdots\phi^{-n}(\xi)^{\underline{d}_n-i}} \otimes \bigotimes_{r=0}^{k-1} \{\phi^{-r}(\xi)\}^{\underline{d}_r-i} \\
&= \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\phi^{-k}(\xi)^{\underline{d}_k}\cdots\phi^{-n}(\xi)^{\underline{d}_n}/\phi^{-k}(\xi_{n-k+1})^i} \otimes \left(\bigotimes_{r=0}^{k-1} \{\phi^{-r}(\xi)\}^{\underline{d}_r}\right) \otimes \{\xi_k\}^{-i}
\end{aligned}$$

Using the first form, this also holds for $n=\infty$ if $n<\infty$ or if $R^+ = \mathcal{O}_C$.

Corollary 4.24. *Let $\underline{\alpha}$ be an actual poly-representation with $\underline{d}_0(\underline{\alpha}) \geq i$. If S is formally smooth over an arbitrary perfectoid ring R, then there are canonical identifications*

$$\begin{aligned}
\mathrm{TR}^n_{\underline{\alpha}-2i,j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R)}) &= \mathrm{W}_n\Omega^j_{S/R} \otimes \bigotimes_{r=0}^{n} \{\phi^{-r}(\xi)\}^{\underline{d}_r-i} \\
&= \mathrm{W}_n\Omega^j_{S/R} \otimes \left(\bigotimes_{r=0}^{n} \{\phi^{-r}(\xi)\}^{\underline{d}_r}\right) \otimes \{\xi_{n+1}\}^{-i} \\
\mathrm{TR}^n_{\underline{\alpha}-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{A_{inf}}(R)}) &= 0
\end{aligned}$$

Using the first form, this also holds for $n=\infty$ if $n<\infty$ or if $R \in \{k, \mathcal{O}_C\}$.

For regular poly-representations, we again get a relationship to the prismatic Gamma function.

Corollary 4.25 (Regular poly-representations, positive range)**.** *Let $\nu \in \mathbf{Q}_{>0}$ be a slope. Given $i \in \mathbf{Z}$, let*

$$k = k(\nu, i) = ???$$

If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then there are canonical identifications

$$\begin{aligned}
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= \mathrm{Fil}^{k-(\lceil p^k\nu\rceil+\cdots+\lceil p^n\nu\rceil)+(n-k+1)i}\,\mathrm{W}_n\Omega^j_S\otimes\left\{\frac{\Gamma\lceil p^{n+1}\nu\rceil}{p^{(n+1)(i-1)}\Gamma\lceil\nu\rceil}\right\}\\
\mathrm{TR}_{\llbracket\nu\rrbracket_\lambda-2i,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= 0\\
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_S}{p^{\lceil p^k\nu\rceil+\cdots+\lceil p^n\nu\rceil}/p^{(n-k+1)i}}\otimes\left\{\frac{\Gamma\lceil p^k\nu\rceil}{p^{k(i-1)}\Gamma\lceil\nu\rceil}\right\}\\
\mathrm{TR}_{\llbracket\nu\rrbracket_\lambda-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{W}(k)}) &= \mathrm{W}_{k-1}\Omega^j_S\otimes\left\{\frac{\Gamma\lceil p^k\nu\rceil}{p^{k(i-1)}\Gamma\lceil\nu\rceil}\right\}
\end{aligned}$$

If S is formally smooth over a p-torsionfree perfectoid ring R^+, then there are canonical identifications

$$\begin{aligned}
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(R^+)}) &= \mathrm{Fil}^k\,\mathrm{W}_n\Omega^j_{S/R^+}\otimes\left\{\frac{\phi^{-(n+1)}\Gamma_{\Delta}\lceil p^{n+1}\nu\rceil}{\Gamma_{\Delta}\lceil\nu\rceil}\right\}\otimes\{\xi_{n+1}\}^{-(i-1)}\\
\mathrm{TR}_{\llbracket\nu\rrbracket_\lambda-2i,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(\mathcal{O}_C)}) &= \mathrm{Fil}^k\,\mathrm{W}\Omega^j_{S/\mathcal{O}_C}\otimes\left(\bigotimes_{r=0}^{\infty}\{\phi^{-r}(\xi)\}^{\lceil p^r\nu\rceil}\right)\otimes\{\mu\}^{-i}\\
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(R^+)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_{S/R^+}}{\left(\dfrac{\phi^{-(n+1)}\Gamma_{\Delta}\lceil p^{n+1}\nu\rceil}{\phi^{-k}\left(\xi^{i-1}_{n-k+1}\Gamma_{\Delta}\lceil p^k\nu\rceil\right)}\right)}\otimes\left\{\frac{\phi^{-k}\Gamma_{\Delta}\lceil p^k\nu\rceil}{\Gamma_{\Delta}\lceil\nu\rceil}\right\}\otimes\{\xi_k\}^{-(i-1)}\\
\mathrm{TR}_{\llbracket\nu\rrbracket_\lambda-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(\mathcal{O}_C)}) &= \frac{\mathrm{W}_{k-1}\Omega^j_{S/\mathcal{O}_C}}{\phi^{-k}(\xi)^{\lceil p^k\nu\rceil}\cdots/\phi^{-k}(\xi_{n-k+1})^i}\otimes\left\{\frac{\phi^{-k}\Gamma_{\Delta}\lceil p^k\nu\rceil}{\Gamma_{\Delta}\lceil\nu\rceil}\right\}\otimes\{\xi_k\}^{-(i-1)}
\end{aligned}$$

If $\lceil\nu\rceil\geq i$ and S is formally smooth over an arbitrary perfectoid ring R, there are canonical identifications

$$\begin{aligned}
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(R)}) &= \mathrm{W}_n\Omega^j_{S/R}\otimes\left\{\frac{\phi^{-(n+1)}\Gamma_{\Delta}\lceil p^{n+1}\nu\rceil}{\Gamma_{\Delta}\lceil\nu\rceil}\right\}\otimes\{\xi_{n+1}\}^{-(i-1)}\\
\mathrm{TR}^n_{\llbracket\nu\rrbracket_\lambda-2i-1,j}(S\,|\,\mathbf{S}_{\mathrm{A}_{\inf}(R)}) &= 0
\end{aligned}$$

Now we compare to the cellular method. To orient the reader, we first explain the calculation for a general polygonic spectrum.

Example 4.26. Let M be an integral polygonic spectrum. We define the boxed entries via the short exact sequences

$$0\longrightarrow\boxed{\pi_iM[\mathbf{F}^k]}\longrightarrow\pi_iM\xrightarrow{\mathbf{F}^k}\pi_iM$$

$$0\longrightarrow\boxed{\mathcal{Q}^k\pi_iM}\longrightarrow\pi_iM[\mathbf{F}^k]\xrightarrow{\mathbf{F}^kd}(\pi_{i+1}M)_{k\mathbf{Z}}\longrightarrow\boxed{\mathcal{V}^k\pi_{i+1}M}\longrightarrow 0$$

Using the cell structure, $\pi_{*+\lambda^{1/k}}M$ is computed by the spectral sequence

$$E_1 = \boxed{\begin{tikzcd}[column sep=small]
\pi_0 M & \pi_1 M & \pi_2 M & \cdots & & \\
& \pi_0 M \arrow[ul, "1-\gamma^k"'] & \pi_1 M \arrow[ul] & \pi_2 M \arrow[ul] & \cdots & \\
& & \pi_0 M \arrow[ul, "\mathbf{F}^k"'] & \pi_1 M \arrow[ul] & \pi_2 M \arrow[ul] & \cdots
\end{tikzcd}}$$

$$E_2 = \boxed{\begin{tikzcd}[column sep=small]
(\pi_0 M)_{k\mathbf{Z}} & (\pi_1 M)_{k\mathbf{Z}} & (\pi_2 M)_{k\mathbf{Z}} & \cdots & & \\
& (\pi_0 M)^{k\mathbf{Z}}/\mathbf{F}^k & (\pi_1 M)^{k\mathbf{Z}}/\mathbf{F}^k & (\pi_2 M)^{k\mathbf{Z}}/\mathbf{F}^k & \cdots & \\
& & \pi_0 M[\mathbf{F}^k] \arrow[uul, "\mathbf{F}^k d"'] & \pi_1[\mathbf{F}^k] \arrow[uul] & \pi_2 M[\mathbf{F}^k] \arrow[uul] & \cdots
\end{tikzcd}}$$

$$E_\infty = \boxed{\begin{array}{cccccc}
\mathcal{V}^k \pi_0 M & \mathcal{V}^k \pi_1 M & \mathcal{V}^k \pi_2 M & \cdots & & \\
& (\pi_0 M)^{k\mathbf{Z}}/\mathbf{F}^k & (\pi_1 M)^{k\mathbf{Z}}/\mathbf{F}^k & (\pi_2 M)^{k\mathbf{Z}}/\mathbf{F}^k & \cdots & \\
& & \mathcal{Q}^k \pi_0 M & \mathcal{Q}^k \pi_1 M & \mathcal{Q}^k \pi_2 M & \cdots
\end{array}}$$

with the first column being $* = -2$. cite the differentials

Example 4.27. When $M = \mathrm{TR}(S)$, we have $(\pi_i M)[F^k] = 0$ by [Ill79, Corollaire 3.6]. Example 4.26 thus becomes

$$\begin{array}{ccccc}
\mathrm{W}\Omega^0 & \mathrm{W}\Omega^1 & \mathrm{W}\Omega^2 & \cdots & \\
& \mathrm{W}\Omega^0/F^k & \mathrm{W}\Omega^1/F^k & \mathrm{W}\Omega^2/F^k & \cdots
\end{array}$$

The extensions are given by

$$\begin{tikzcd}
& V^k \mathrm{W}\Omega^j + dV^k \mathrm{W}\Omega^{j-1} \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{W}\Omega^j \arrow[r, "V^k"] & \boxed{\mathrm{Fil}^k \mathrm{W}\Omega^j} \arrow[r] \arrow[d, equal] & \mathrm{W}\Omega^{j-1}/F^k \arrow[r] & 0 \\
& & \mathrm{TR}_{\underline{\lambda}_k - 2, j}(S) & &
\end{tikzcd}$$

where the identification $\mathrm{Fil}^k \mathrm{W}\Omega^j / V^k = \mathrm{W}\Omega^{j-1}/F^k$ is [DM23b, Proposition 2.14].

explain why + do the finite-level version

4.5. **Sign representations.** When p is odd, the real representations of C_{p^n} are accounted for by the complex representations $\lambda_0, \ldots, \lambda_{n-1}$ along with the real trivial representation 1. When $p = 2$, there is additionally a sign representation ς_n of C_{2^n}, which has real dimension 1 and fixes $C_{2^{n-1}}$. This does not lift to $\mathbf{T}$, so we did not need to consider it in §3. However, it is compatible under the R maps, so it lifts to a poly-representation ς of $\mathbf{Z}_2$. In this section we compute this additional degree.

Example 4.28. We compute $\mathrm{TR}_{*-\varsigma}(S)$ and $\mathrm{TR}^n_{*-\varsigma}(S)$ for S a smooth k-algebra. We have cell structures

$$(C_{2^n}/C_{2^{n-1}})_+ \to S^0 \to S^{\varsigma_n}$$

of C_{2^n}-spectra, which assemble to a cell structure

$$(\mathbf{Z}_2/2\mathbf{Z}_2)_+ \to S^0 \to S^{\varsigma}$$

of 2-polygonic spectra. Thus, $\mathrm{TR}_{*-\varsigma}(S)$ is computed by the spectral sequence

$$\begin{tikzcd}
\mathrm{W}\Omega^0 & \mathrm{W}\Omega^1 & \mathrm{W}\Omega^2 & \cdots & \\
& \mathrm{W}\Omega^0 \arrow[ul, "V"'] & \mathrm{W}\Omega^1 \arrow[ul] & \mathrm{W}\Omega^2 \arrow[ul] & \cdots \\
0 & 1 & 2 & 3 &
\end{tikzcd}$$

while $\mathrm{TR}^n_{*-\varsigma}(S)$ is computed by the spectral sequence

$$\begin{tikzcd}
\mathrm{W}_n\Omega^0 & \mathrm{W}_n\Omega^1 & \mathrm{W}_n\Omega^2 & \cdots & \\
& \mathrm{W}_{n-1}\Omega^0 \arrow[ul, "V"'] & \mathrm{W}_{n-1}\Omega^1 \arrow[ul] & \mathrm{W}_{n-1}\Omega^2 \arrow[ul] & \cdots \\
0 & 1 & 2 & 3 &
\end{tikzcd}$$

Since V is injective on $\mathrm{W}\Omega^*$, we get $\mathrm{TR}_{*-\varsigma}(S) = \mathrm{W}\Omega^*_S/V$. At finite level, we need to work a bit harder. TODO: figure out the kernel of V

Example 4.29. We repeat the previous example in mixed characteristic. If S is formally smooth over $\mathcal{O}_C$, we get $\mathrm{TR}_{*-\varsigma}(S;\mathbf{Z}_p) = \mathrm{W}\Omega^*_S/V[\beta]$. If S is formally smooth over R, then $\mathrm{TR}^n_{*-\varsigma}(S;\mathbf{Z}_p)$ is

halp

4.6. **Norm maps.**

$$\mathrm{Ind}_n^{n+1}(\underline{d}_0,\ldots,\underline{d}_n) = (\underline{d}_0, p\underline{d}_0, p\underline{d}_1,\ldots,p\underline{d}_n)$$

Lemma 4.30. *We have*

$$N^n_{n-1}(\phi^{-n}(\mu)) = v\phi^{-(n+1)}(\mu)$$

for a unit v satisfying

$$\begin{aligned} v &\equiv 1 \bmod (\xi, p)\\ v &\equiv 1 \bmod \phi^{-1}(\xi_{n-1}) \end{aligned}$$

Proof. In ghost coordinates, we have

$$\begin{aligned}
N^n_{n-1}(\phi^{-n}(\mu)) &= N(\phi^{-n}(\mu), \phi^{-(n-1)}(\mu),\ldots,\phi^{-1}(\mu)))_{\text{👻}}\\
&= (\qquad \phi^{-n}(\mu), \phi^{-n}(\mu)^p, \phi^{-(n-1)}(\mu)^p,\ldots,\phi^{-1}(\mu)^p)_{\text{👻}}\\
\phi^{-(n+1)}(\mu)^p &= (\phi^{-(n+1)}(\mu)^p, \phi^{-n}(\mu)^p, \phi^{-(n-1)}(\mu)^p,\ldots,\phi^{-1}(\mu)^p)_{\text{👻}}
\end{aligned}$$

It thus suffices to show that $\frac{\phi^{-n}(\mu)}{\phi^{-(n+1)}(\mu)^p}$ is congruent to 1 mod p (in A/ξ).

We have

$$\begin{aligned}
\widetilde{\xi} &= \mu^{p-1} + p\frac{\delta(\mu)}{\mu}\\
&= \xi^p + p\delta(\xi)
\end{aligned}$$

so that $\mu^{p-1} \equiv \xi^p \bmod p$. Applying ϕ^{-n}, we get

$$\phi^{-n}(\mu)^{p-1} \equiv \frac{\phi^{-n}(\mu)^p}{\phi^{-(n+1)}(\mu)^p} \bmod p$$

Since p is a non-zerodivisor mod μ, by torsion interchange μ must be a non-zerodivisor mod p. Since ϕ is an isomorphism, $\phi^{-n}(\mu)$ is also a non-zerodivisor mod p. We can thus cancel $\phi^{-n}(\mu)^{p-1}$ from both sides to conclude that

$$\frac{\phi^{-n}(\mu)}{\phi^{-(n+1)}(\mu)^p} \equiv 1 \bmod p. \qquad \square$$

Theorem 4.31 (Norms for the de Rham-Witt complex)**.** *Let R be any perfectoid ring. For a formally smooth R-algebra S, there are maps*

$$N\colon \mathrm{W}\Omega^i_S \to (\mathrm{W}\Omega^i_{S/R} \otimes ???) \oplus \mathrm{Fil}^1 \mathrm{W}\Omega^{ip}_{S/R}$$
$$N\colon \mathrm{W}_n\Omega^i_S\{j\} \to \mathrm{W}_{n+1}\Omega^{ip}_S\{pj\}$$

with the following properties:

- *when $i = 0$, $N(f) = f - V\delta(f)$ is the Norm map of Witt vectors* [Ang15]*;*
- *when $i > 0$, N is given by*

$$N(\omega) = V\frac{\omega^p}{p} + (p - V(1))\frac{\psi^p(\omega)}{p}$$

- *N satisfies the following identities for all ω, τ:*

$$\begin{aligned} FN\omega &= \omega^p \\ NV\omega &= p^{p-2}V^2\omega^p \\ NR\omega &= RN\omega \\ N(\omega\tau) &= N(\omega)N(\tau) \\ N(\omega+\tau) &= N(\omega) + N(\tau) + Vs_1(\omega,\tau) \\ &\qquad \textit{where } (a+b)^p = a^p + b^p + ps_1(a,b) \end{aligned}$$

- *the action on Breuil-Kisin twists is given by "$N(a\{1\}) = N(a)\{p\}$".*

Corollary 4.32. *If S is smooth over a perfect $\mathbf{F}_p$-algebra k, then $N(\omega) = V\frac{\omega^p}{p}$ for $|\omega| > 0$, and $N(\omega) = 0$ for $|\omega|$ odd.*

Proof. Intuitively this is true because $p - V(1) = 0$ in $\mathrm{W}(k)$; however, we must be careful since some of this factor gets absorbed into the twist.

finish

□

[Ang21, Theorem 6.1] for the multiplication rule

Example 4.33. Consider (p-complete) affine $2p$-space $X = (\mathbf{A}^{2p}_R)^\wedge_p = \mathrm{Spf}\, R[x_1, y_1, \ldots, x_p, y_p]^\wedge_p$ over an arbitrary perfectoid ring R. Let

$$\tau = \sum_{i=1}^p dx_i\, dy_i \in \mathrm{W}\Omega^2_X$$
$$\omega = \prod_{i=1}^p dx_i\, dy_i \in \mathrm{W}\Omega^{2p}_X$$

Then

$$N(\tau) = V((p-1)!\omega) + (p - V(1))p\tau$$

When R is a perfect $\mathbf{F}_p$-algebra k, this simplifies to

$$N(\tau) = V((p-1)!\omega)$$

Appendix A. p-primary Adams operations

For an integer ℓ, the natural map
$$\psi^\ell \colon \mathrm{THH}(R) = R^{\otimes \mathbf{T}} \to R^{\otimes \mathbf{T}/C_\ell} = \mathrm{THH}(R)_{\otimes C_\ell}$$
is called the *ℓth Adams operation*. Beware that $\mathrm{THH}(R)$ is equivalent to $\mathrm{THH}(R)_{\otimes C_\ell}$ in Sp, but not in $\mathrm{Sp}^{B\mathbf{T}}$. Therefore, ψ^ℓ may be regarded as an endomorphism of THH, but it does not necessarily pass to TC^- and TP. However, $\mathrm{THH}(R)$ and $\mathrm{THH}(R)_{\otimes C_\ell}$ do become equivalent in $\mathrm{Sp}[1/\ell]^{B\mathbf{T}}$, so that in this case ψ^ℓ may be regarded as an automorphism of TC^- and TP. This holds in particular if we p-complete at a prime p not dividing ℓ (in which case ℓ may be generalized to any element of $\mathbf{Z}_p^\times$).

For this reason, the analysis in the literature has mostly focused on the case $p \nmid \ell$. For example, in this case ψ^ℓ acts as multiplication by ℓ^i on gr^i of $\mathrm{THH}(-;\mathbf{Z}_p)$, $\mathrm{TC}^-(-;\mathbf{Z}_p)$, and $\mathrm{TP}(-;\mathbf{Z}_p)$ by [BMS19, Proposition 9.14] (see also [ABG+18, Theorem 10.5]). However, we will need to study ψ^p in the p-complete setting.

Lemma A.1. *Let R be a perfectoid ring. Then ψ^p acts as multiplication by p^i on $\pi_{2i}\mathrm{THH}(R;\mathbf{Z}_p)$.*

Proof. The proof of [Rak20, Proposition 6.4.12] goes through to show that ψ^p acts as p^i on $\mathrm{gr}^i_{\mathrm{HKR}}\,\mathrm{HH}_{\mathrm{fil}}(B/A)$ for any map $A \to B$ of derived commutative rings (the HC^- and HP parts of his proof do not go through). Applying this to $\mathrm{HH}(R;\mathbf{Z}_p)$, we learn that ψ^p acts as p^i on $\mathrm{HH}_{2i}(R;\mathbf{Z}_p)$. But the natural map $\mathrm{THH}_2(R;\mathbf{Z}_p) \to \mathrm{HH}_2(R;\mathbf{Z}_p)$ is an isomorphism by [NS18, Proposition IV.4.2], so ψ^p acts as p^i on $\mathrm{THH}_{2i}(R;\mathbf{Z}_p) = \mathrm{Sym}^i\,\pi_2\mathrm{THH}(R;\mathbf{Z}_p) = R \otimes \{\xi^i\}$. □

Let S be formally smooth over any perfectoid ring R. In this case, recall from [BMS19, Corollary 6.9] (but essentially due to Hesselholt [Hes96]) that
$$\pi_*\mathrm{THH}(S;\mathbf{Z}_p) = \Omega^*_{S/R} \otimes_R \pi_*\mathrm{THH}(R;\mathbf{Z}_p).$$

Lemma A.2. *Let R be a quasiregular semiperfectoid ring. Then ψ^p acts as multiplication by p^i on $\pi_{2i}\mathrm{THH}(S;\mathbf{Z}_p)$.*

Proof. Choose a perfectoid ring R mapping to S. As in the proofs of [BS22, Proposition 7.10] and [BS22, Lemma 13.2], we can reduce to the case that $S = R\left\langle X^{1/p^\infty}\right\rangle/(X)$. By [BL22, Proposition 2.4.9], the map of prisms $\Delta_S \to \Delta_S \amalg (\mathbf{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, (p)_q)$ is faithfully flat, so we can assume that $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge_p$. In this case, Δ_S is explicitly identified as a q-divided power envelope in [BS22, §12.2]. In particular, its Frobenius is injective, so it injects into its direct-limit perfection [BS22, Lemma 3.9]. Now the claim follows from Lemma A.1. □

now extend to the proper Tate case

Lemma A.3. *Let S be formally smooth over any perfectoid ring R. Then ψ^p acts as multiplication by p^{i+j} on $\Omega^j_{S/R} \otimes \{\xi^i\} \subset \mathrm{THH}_{2i+j}(S;\mathbf{Z}_p)$.*

Proof. Given Lemma A.1, we just need to show that ψ^p acts as p^j on $\Omega^j_{S/R}$, which follows by applying [Rak20, Proposition 6.4.12] to $\mathrm{HH}(S/R;\mathbf{Z}_p)$. □

maybe put the Wilson's theorem stuff in here too

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