1. The leapfrog scheme has the discrete form of

$$f_j^{n+1} - f_j^{n-1} = \alpha \left(f_{j+1}^n - f_{j-1}^n \right)$$

where $\alpha = -v\,\mathrm{d}t\,/\,\mathrm{d}x$. Substitute in $f(x,t) = \xi^t \exp(\mathrm{i}kx)$ which gives $f_j^n = \xi^n \exp(\mathrm{i}kj)$, we get

$$\xi^{n+1}\mathrm{e}^{\mathrm{i}kj} - \xi^{n-1}\mathrm{e}^{\mathrm{i}kj} = \alpha \left[\xi^n \mathrm{e}^{\mathrm{i}k(j+1)} - \xi^n \mathrm{e}^{\mathrm{i}k(j-1)} \right]$$

then

$$\xi - \xi^{-1} = \alpha \left(\mathrm{e}^{\mathrm{i}k} - \mathrm{e}^{-\mathrm{i}k} \right) = 2\mathrm{i}\alpha \sin k.$$

We can solve for ξ to find

$$\xi = \mathrm{i}\alpha \sin k \pm \sqrt{-\alpha^2 \sin^2 k + 1}.$$

Note that if the CFL condition is satisfied, i.e. $|\alpha| < 1$, we have $-\alpha^2 \sin^2 k + 1 > 0$ and

$$|\xi| = -\alpha^2 \sin^2 k + 1 + \alpha^2 \sin^2 k = 1,$$

i.e. the energy is conserved.

2. a) The potential of a point charge in 2D is given by

$$V(x,y) = c \ln \sqrt{x^2 + y^2}$$

for some constant c and the density is

$$\rho(x,y) = V(x,y) - \frac{V(x+1,y) + V(x-1,y) + V(x,y+1) + V(x,y-1)}{4}$$

in discrete form. Generated $V(x,y)$ with $c = 1$ and $\rho(x,y)$ are normalized such that $\rho(0,0) = 1$ by applying a multiplicative constant. Then we add an additive constant to $V(x,y)$ such that $V(0,0) = 1$. With these transformation, we get

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V(0, 0) = 1.0
V(1, 0) = -0.0
V(2, 0) = -0.49999999999999994
V(5, 0) = -1.1609640474436809
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b) First we note that $V(x,y)$ obtained in a), is precisely the Green's function $\mathcal{G}$. We define a box at the center with side length 20 and assign an initial condition of $V(x,y) = 1$ in this box with $V(x,y) = 0$ elsewhere. Using conjugate gradient, we solve $V = \mathcal{G} * \rho$, restricting $\rho(x,y) = 0$ outside the box. The final distribution of $\rho(x,y)$ is shown in Figure 1. The plot the ρ along one side of the box is shown in Figure 2.

c) Perform another convolution of $\mathcal{G} * \rho$ allows use to find the terminal $V(x,y)$, which is plotted in Figure 3. $V(x,y)$ inside the box is almost constant with

```
Mean of V(x,y) inside the box is 1.000000000005315
Stdev of V(x,y) inside the box is 3.9711454026612064e-08
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and shown in Figure 4. Using $\boldsymbol{E} = -\nabla V$, we also find the $\boldsymbol{E}$ field which is shown in Figure 5. We can see that field is stronger near the corners of the box, which is what we expected since the charge density there is large. We can also see that $\boldsymbol{E}(x,y)$ just outside the box are all pointing to the edge, since the edge of the box is a equipotential surface with negative charge. There is also negligible $\boldsymbol{E}$ field inside the box.

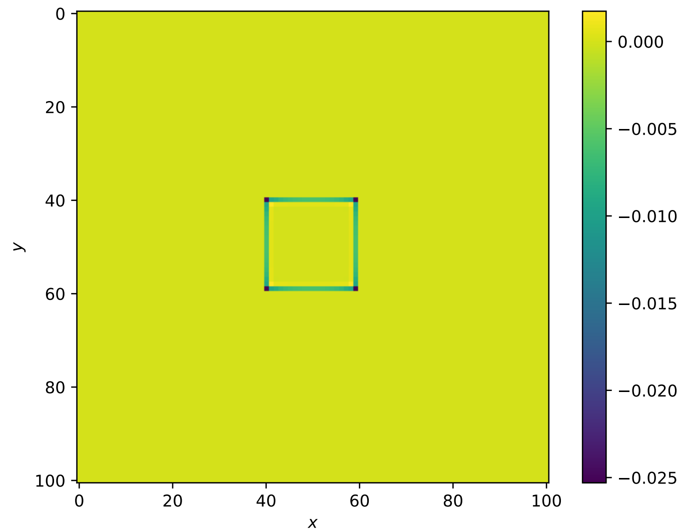

Figure 1: Plot of $\rho(x, y)$.

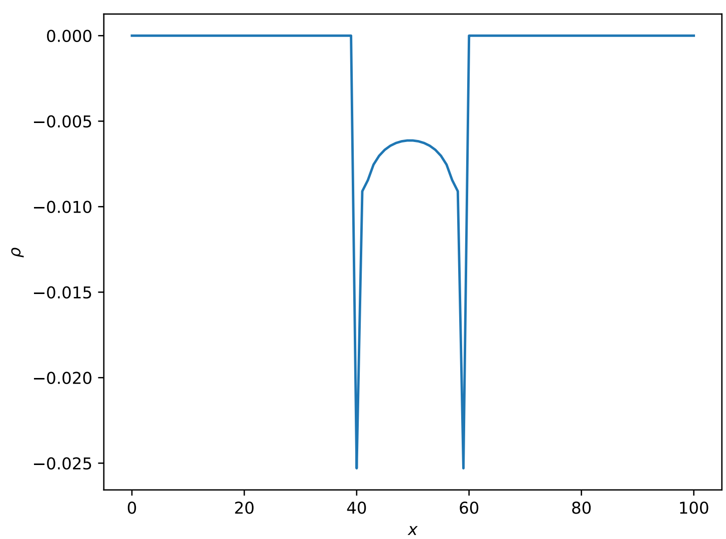

Figure 2: Plot of $\rho(x, y)$ along one side of the box.

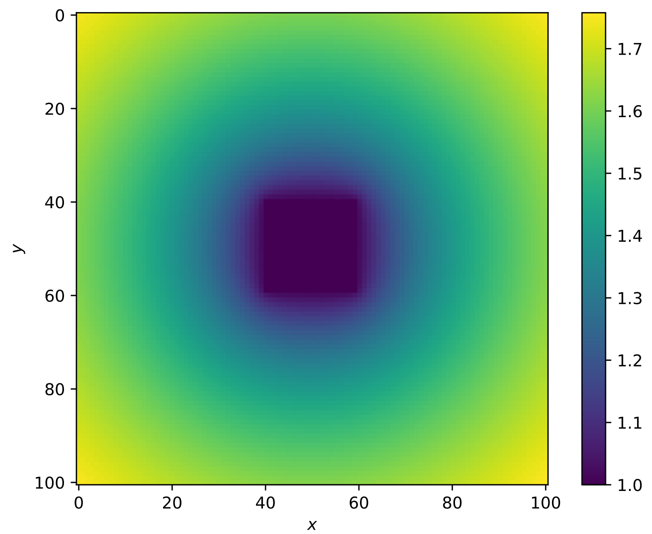

Figure 3: Plot of $V(x, y)$.

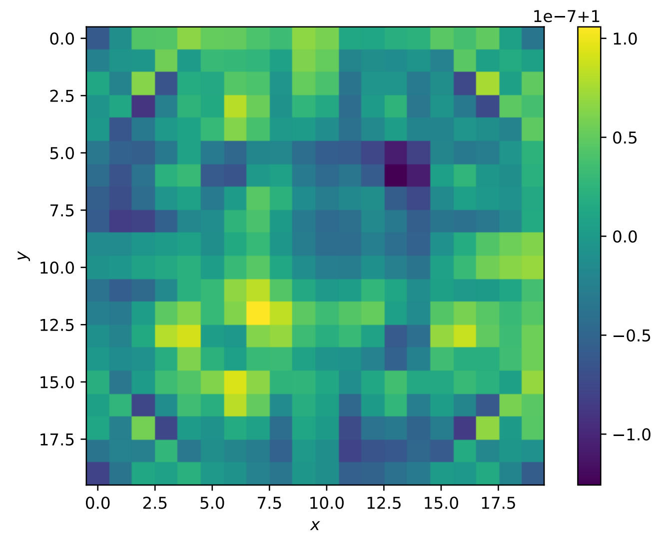

Figure 4: Plot of $V(x, y)$ inside the box. Note that the scale of fluctuations is significantly smaller than the variations of $V(x, y)$ outside the box.

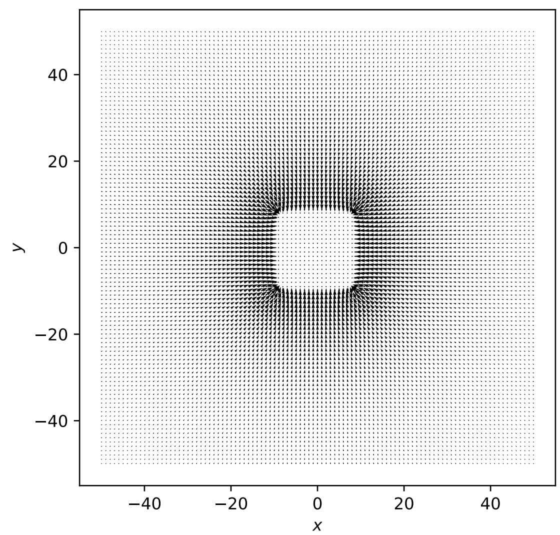

Figure 5: Plot of $\boldsymbol{E}(x, y)$.