T2

由于$U_1,\cdots,U_n$为在$(0,1)$中均匀分布的独立随机变量（$U_1,\cdots,U_n$同分布）

$$\mu_X(t)=E[X(t)]=\frac{1}{n}\sum_{k=1}^{n}E[I(t,U_k)]=E[I(t,U_1)]=1\times P(U_1\leq t)+0\times P(U_1>t)=P(U_1\leq t)=t,\qquad 0\leq t\leq 1$$

对于协方差有

$$R_X(t,s)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov[I(t,U_i),I(s,U_j)]$$

注意到对于$i\neq j$

$$E[I(t,U_i)I(s,U_j)]=1\times P(U_i\leq t,U_j\leq s)=P(U_i\leq t)P(U_j\leq s)=ts$$

故

$$Cov[I(t,U_i),I(s,U_j)]=E[I(t,U_i)I(s,U_j)]-E[I(t,U_i)]E[I(s,U_j)]=ts-ts=0$$

而对于$i=j(=k)$

$$E[I(t,U_k)I(s,U_k)]=1\times P(U_k\leq t,U_k\leq s)=P(U_k\leq min\{t,s\})=min\{t,s\}$$

从而

$$Cov[I(t,U_k),I(s,U_k)]=E[I(t,U_k)I(s,U_k)]-E[I(t,U_k)]E[I(s,U_k)]=min\{t,s\}-ts$$

故

$$R_X(t,s)=\frac{1}{n^2}\sum_{k=1}^{n}Cov[I(t,U_k),I(s,U_k)]=\frac{1}{n}Cov[I(t,U_1),I(s,U_1)]=\frac{1}{n}(min\{t,s\}-ts)\qquad 0\leq t,s\leq 1$$

T4

$$\mu_X(t)=E[X(t)]=E[X(t)-X(0)]=\lambda(t-0)=\lambda t$$

$$R_X(t,s)=E[X(t)X(s)]-E[X(t)]E[X(s)]$$

其中(不妨$s>t$)

$$\begin{aligned}E[X(t)X(s)]&=E[(X(t)-X(0))(X(s)-X(t)+X(t)-X(0))]\\&=E[(X(t)-X(0))^2]+E[(X(t)-X(0))(X(s)-X(t))]\end{aligned}$$

又注意到

$$E[(X(t)-X(0))^2]=Var[X(t)-X(0)]+(E[X(t)-X(0)])^2=(\lambda(t-0))+(\lambda(t-0))^2=\lambda t+(\lambda t)^2$$

故

$$E[X(t)X(s)]=\lambda t+(\lambda t)^2+\lambda(t-0)\times\lambda(s-t)=\lambda t(\lambda s+1)$$

（注意由先前假设$s>t$，所以实际上$E[X(t)X(s)]=\lambda min\{t,s\}(\lambda max\{t,s\}+1)$）

从而

$$R_X(t,s)=\lambda t(\lambda s+1)-\lambda t\times\lambda s=\lambda t=\lambda min\{t,s\}$$

由定义知不是宽平稳。

T5

$$\mu_Y(t)=E[Y(t)]=E[X(t+1)-X(t)]=\lambda(t+1-t)=\lambda$$

$$\begin{aligned}R_Y(t,s)&=Cov[Y(t),Y(s)]=Cov[(X(t+1)-X(t))(X(s+1)-X(s))]\\&=Cov[(X(t+1),X(s+1)]-Cov[(X(t+1),X(s)]-Cov[(X(t),X(s+1)]+Cov[(X(t),X(s)]\\&=\lambda min\{t+1,s+1\}-\lambda min\{t+1,s\}-\lambda min\{t,s+1\}+\lambda min\{t,s\}\end{aligned}$$

从而

$$R_Y(t,s)=\begin{cases}0, & 0\le t\le s-1\text{ 或 }t\ge s+1\\ \lambda(t-s+1), & s-1<t<s\\ \lambda(s-t+1), & s\le t<s+1\end{cases}$$

考虑二阶矩，由于

$$Var[Y(t)]=R_Y(t,t)=\lambda$$

$$E[Y(t)^2]=Var[Y(t)]+E[Y(t)]^2=\lambda+\lambda^2<\infty$$

故是宽平稳的

T9

易知$f(x,y)=\frac{1}{\pi}$，故

$$P(X>Y)=\iint_{x>y,x^2+y^2\le1}f(x,y)dxdy=\frac{1}{2}$$

$$P(X^2+Y^2\ge\frac{3}{4},\ X>Y)=\iint_{x>y,\frac{3}{4}\le x^2+y^2\le1}f(x,y)dxdy=\frac{1}{8}$$

$$P(X^2+Y^2\ge\frac{3}{4}\mid X>Y)=\frac{P(X^2+Y^2\ge\frac{3}{4},X>Y)}{P(X>Y)}=\frac{1}{4}$$

T14

首先考察$Poission$随机变量的矩母函数

$$g(t)=E[e^{tX}]=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{kt}=e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}=e^{\lambda e^t-\lambda}$$

然后由于X_1,X_2相互独立，注意到X_1+X_2的矩母函数为

$$g_{X_1+X_2}(t)=g_{X_1}(t)g_{X_2}(t)=e^{\lambda_1e^t-\lambda_1}\times e^{\lambda_2e^t-\lambda_2}=e^{(\lambda_1+\lambda_2)e^t-(\lambda_1+\lambda_2)}$$

由于矩母函数唯一地确定了分布，故$X_1+X_2\sim P(\lambda_1+\lambda_2)$

又

$$P(X_1+X_2=n,\ X_1=m)=P(X_1=m,\ X_2=n-m)=P(x_1=m)P(X_2=n-m)=\frac{\lambda_1^m\lambda_2^{n-m}}{m!(n-m)!}e^{-\lambda_1-\lambda_2}$$

$$P(X_1+X_2=n)=\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}$$

$$P(X_1=m\mid X_1+X_2=n)=\frac{P(X_1+X_2=n,\ X_1=m)}{P(X_1+X_2=n)}=\binom{n}{m}(\frac{\lambda_1}{\lambda_1+\lambda_2})^m(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-m}$$

T15

服从指数分布的矩母函数为$g_X(t)=\frac{\lambda}{\lambda-t}$（上课证过，亦可参见附表）

利用例`1.12`有

$$\begin{aligned}g_Y(t)&=E[(g_X(t))^N]\\&=E[(\frac{\lambda}{\lambda-t})^N]\\&=\sum_{n=1}^{\infty}(\frac{\lambda}{\lambda-t})^n\beta(1-\beta)^{n-1}\\&=\frac{\beta}{1-\beta}\sum_{n=1}^{\infty}[\frac{\lambda(1-\beta)}{\lambda-t}]^n\\&=\frac{\beta}{1-\beta}\times\frac{\frac{\lambda(1-\beta)}{\lambda-t}}{1-\frac{\lambda(1-\beta)}{\lambda-t}}\\&=\frac{\lambda\beta}{\lambda\beta-t}\end{aligned}$$

从而$Y \sim exp(\lambda\beta)$