

T2

由于 U_1, \dots, U_n 为在 $(0, 1)$ 中均匀分布的独立随机变量 (U_1, \dots, U_n 同分布)

$$\mu_X(t) = E[X(t)] = \frac{1}{n} \sum_{k=1}^n E[I(t, U_k)] = E[I(t, U_1)] = 1 \times P(U_1 \leq t) + 0 \times P(U_1 > t) = P(U_1 \leq t) = t, \quad 0 \leq t \leq 1$$

对于协方差有

$$R_X(t, s) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n Cov[I(t, U_i), I(s, U_j)]$$

注意到对于 $i \neq j$

$$E[I(t, U_i)I(s, U_j)] = 1 \times P(U_i \leq t, U_j \leq s) = P(U_i \leq t)P(U_j \leq s) = ts$$

故

$$Cov[I(t, U_i), I(s, U_j)] = E[I(t, U_i)I(s, U_j)] - E[I(t, U_i)]E[I(s, U_j)] = ts - ts = 0$$

而对于 $i = j (= k)$

$$E[I(t, U_k)I(s, U_k)] = 1 \times P(U_k \leq t, U_k \leq s) = P(U_k \leq \min\{t, s\}) = \min\{t, s\}$$

从而

$$Cov[I(t, U_k), I(s, U_k)] = E[I(t, U_k)I(s, U_k)] - E[I(t, U_k)]E[I(s, U_k)] = \min\{t, s\} - ts$$

故

$$R_X(t, s) = \frac{1}{n^2} \sum_{k=1}^n Cov[I(t, U_k), I(s, U_k)] = \frac{1}{n} Cov[I(t, U_1), I(s, U_1)] = \frac{1}{n} (\min\{t, s\} - ts) \quad 0 \leq t, s \leq 1$$

T4

$$\mu_X(t) = E[X(t)] = E[X(t) - X(0)] = \lambda(t - 0) = \lambda t$$

$$R_X(t, s) = E[X(t)X(s)] - E[X(t)]E[X(s)]$$

其中(不妨 $s > t$)

$$\begin{aligned} E[X(t)X(s)] &= E[(X(t) - X(0))(X(s) - X(t) + X(t) - X(0))] \\ &= E[(X(t) - X(0))^2] + E[(X(t) - X(0))(X(s) - X(t))] \end{aligned}$$

又注意到

$$E[(X(t) - X(0))^2] = Var[X(t) - X(0)] + (E[X(t) - X(0)])^2 = (\lambda t - 0) + (\lambda t - 0)^2 = \lambda t + (\lambda t)^2$$

故

$$E[X(t)X(s)] = \lambda t + (\lambda t)^2 + \lambda(t - 0) \times \lambda(s - t) = \lambda t(\lambda s + 1)$$

(注意由先前假设 $s > t$, 所以实际上 $E[X(t)X(s)] = \lambda \min\{t, s\}(\lambda \max\{t, s\} + 1)$)

从而

$$R_X(t, s) = \lambda t(\lambda s + 1) - \lambda t \times \lambda s = \lambda t = \lambda \min\{t, s\}$$

由定义知不是宽平稳。

T5

$$\mu_Y(t) = E[Y(t)] = E[X(t+1) - X(t)] = \lambda(t+1 - t) = \lambda$$

$$\begin{aligned} R_Y(t, s) &= Cov[Y(t), Y(s)] = Cov[(X(t+1) - X(t))(X(s+1) - X(s))] \\ &= Cov[(X(t+1), X(s+1))] - Cov[(X(t+1), X(s))] - Cov[(X(t), X(s+1))] + Cov[(X(t), X(s))] \\ &= \lambda \min\{t+1, s+1\} - \lambda \min\{t+1, s\} - \lambda \min\{t, s+1\} + \lambda \min\{t, s\} \end{aligned}$$

从而

$$R_Y(t,s)=\begin{cases} 0, & 0\leq t\leq s-1 \text{ 或 } t\geq s+1 \\ \lambda(t-s+1), & s-1<t<s \\ \lambda(s-t+1), & s\leq t<s+1 \end{cases}$$

考虑二阶矩

$$Var[Y(t)]=R_Y(t,t)=\lambda<\infty$$

故是宽平稳的