q-Painleve equations, mutations and reductions

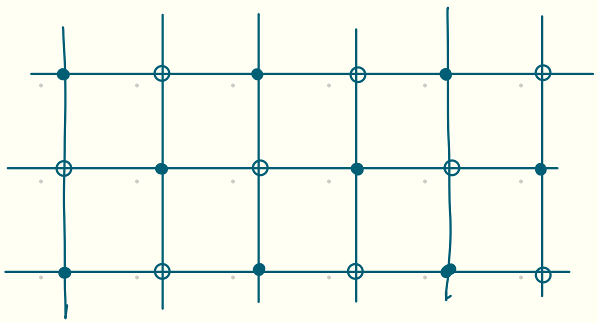
- Joint project (in progress) with Gavrylenko
  Marshakov
  Semenyakin

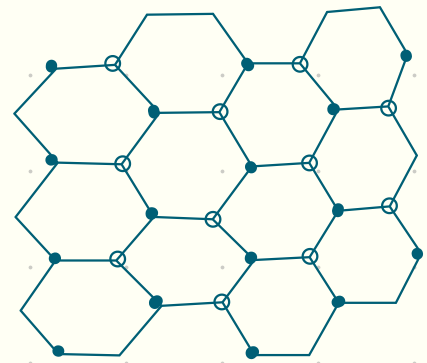
Zoom 24 March 2023

Goncharov-Kenyon integrable systems

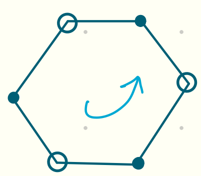
- (good) Bipartate graph on torus

e.g.





- Orient edges  Weight of edge $w(e) \in \mathbb{C}^*$

face  $\prod_{e \in \partial f} w(e) = x_f$ – face variables

$\prod x_i = 1$

A, B cycles – $w(A) = \lambda$, $w(B) = \mu$

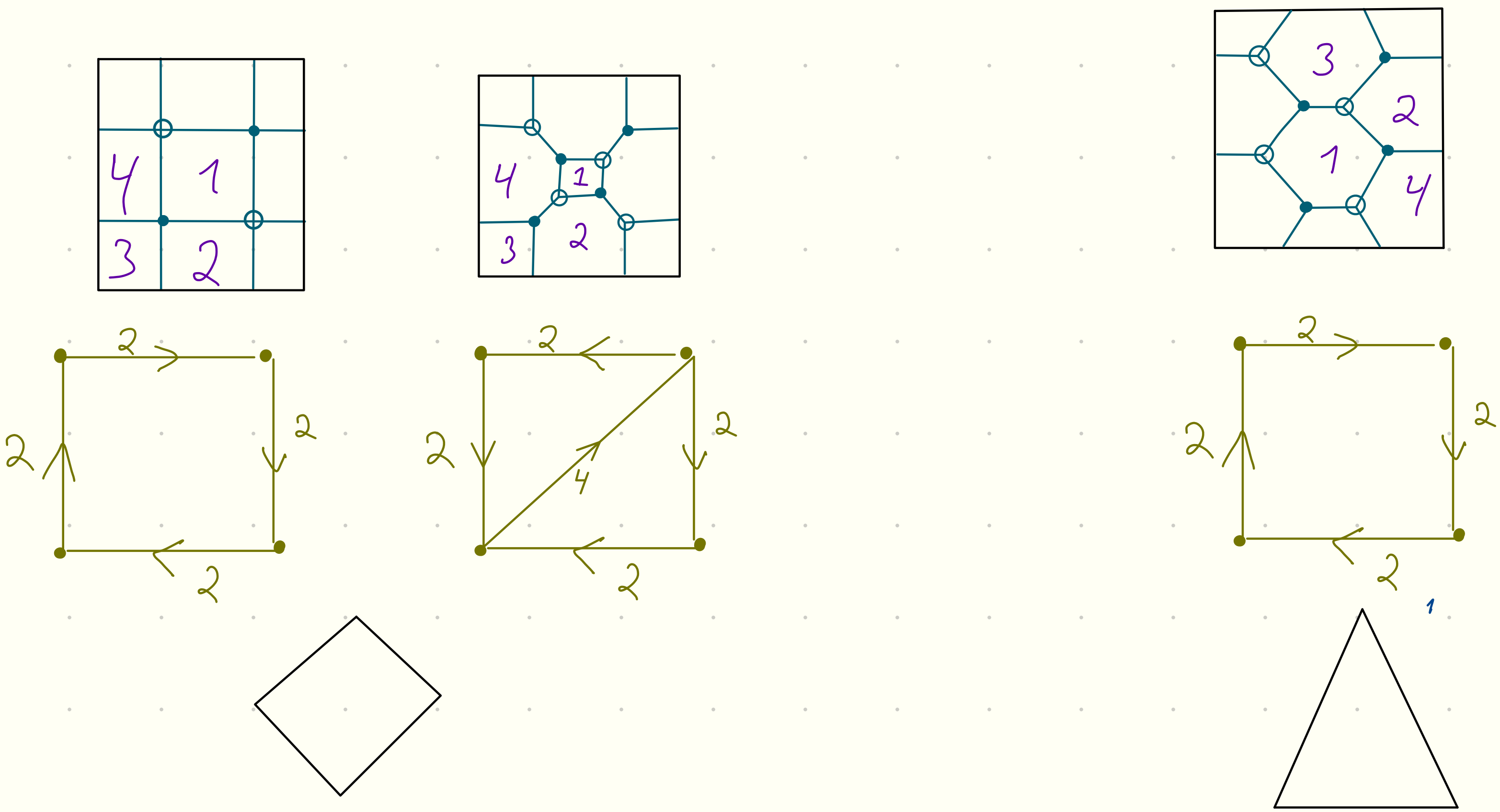
(concrete representatives)

- Quiver

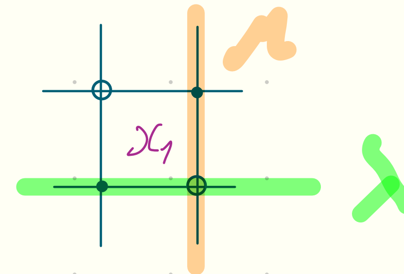
- Poisson structure $\{X_i, X_j\} = \varepsilon_{ij} X_i X_j$

logarithmically constant

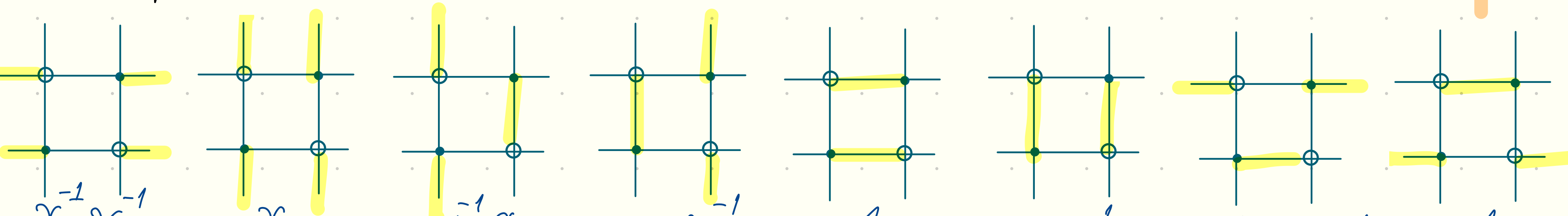
Example



- Dimer configuration – perfect matching



Example



$x_1^{-1}x_2^{-1}$ x_4 $\mu^{-1}x_4$ μx_1^{-1} 1 x_1^{-1} $\lambda x_1^{-1}x_2^{-1}$ λ^{-1}

Partition function $Z = \sum_D \pm w(D-D_0)$

Newton polygon $\Delta_Z = \mathrm{Conv}((a,b))$ s.t $Z = \sum \lambda^a \mu^b f_{ab}(x)$

I = # interiour points

B = # boundary points

$C = \{(\lambda,\mu) \mid Z(\lambda,\mu) = 0\}$

spectral curve

Action of $SA_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$

Th (Goncharov, Kenyon, ...) ① $2\,\text{Area}(\Delta) = \#\,\text{faces} = \dim \mathcal{X}$, where $\#\,\text{faces} = 2I + B - 2$

$\mathcal{X}$ – cluster variety

② $(a,b) \in \partial\Delta$ $\quad f_{ab}$ – Casimirs, (essentially $B-3$ casimirs)

③ $(a,b) \in \Delta \setminus \partial\Delta$ $\quad f_{ab}$ – Hamiltonians

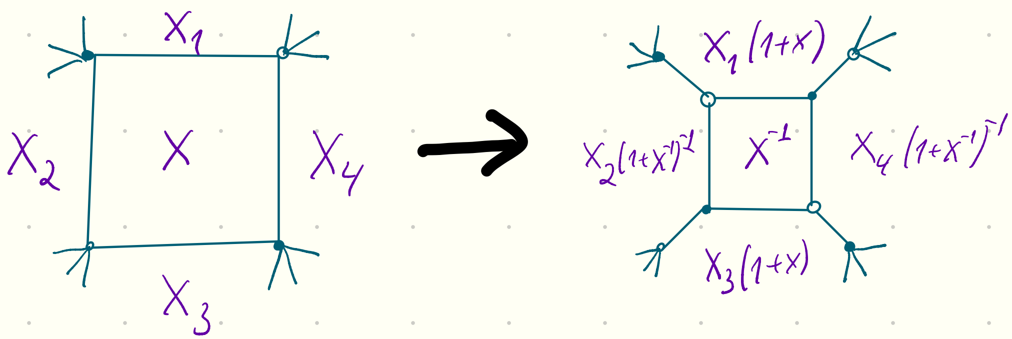
(in particular, $\operatorname{rk}\{\cdot,\cdot\} = 2I$)

$\{\prod x_i = 1\} \subset \mathcal{X}$

$\downarrow$ Jac(C) – integrable system $\quad \prod x_i = 1$, $\prod x_i$ – Casimir

{Coeff Z}

- Face Mutation (spider move)

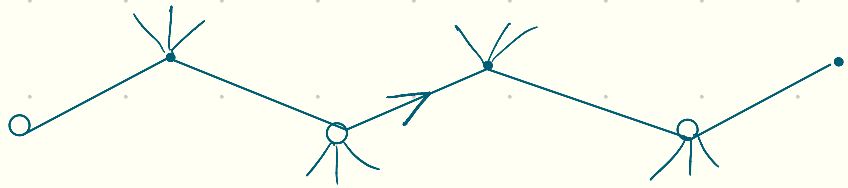


Only 4 valent vertices

<u>Properties</u> Poisson map, Z is preserved

Def Dual surface Σ – inversion of ribbon struct. in black vertices

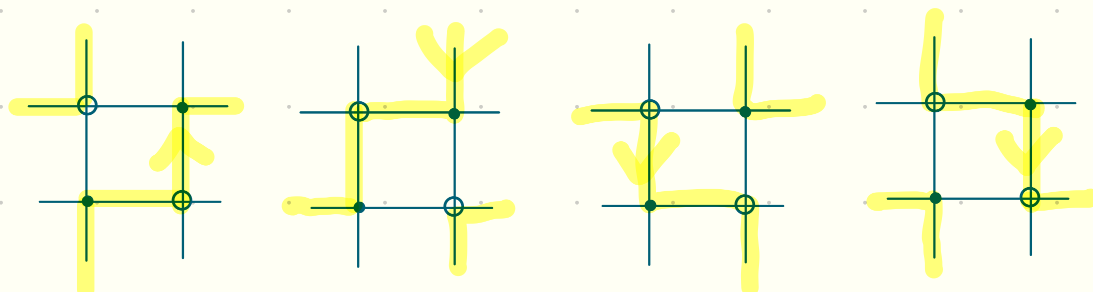
Def Zig-zag path



Black – right
White – left

Prop Faces on Σ – zig-zag paths

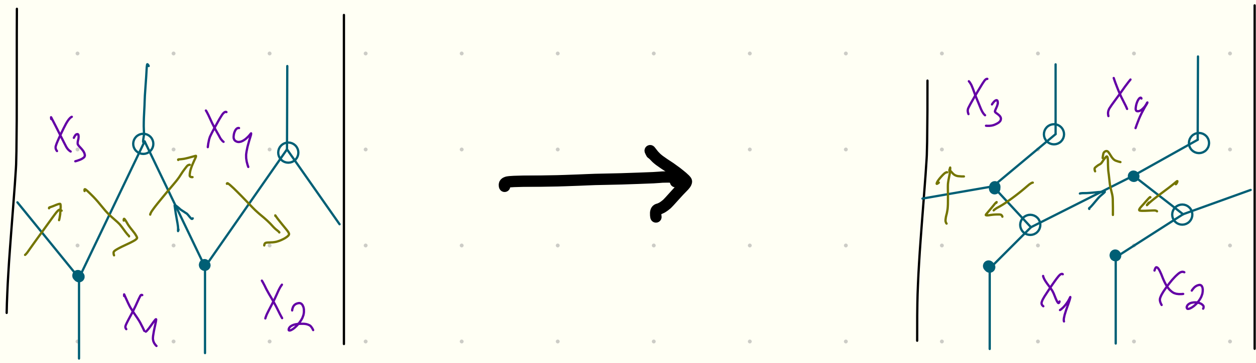
In example above



Th [GK] $\Sigma \simeq C$ (topologically)

- zig-zag mutation

picture on cylinder



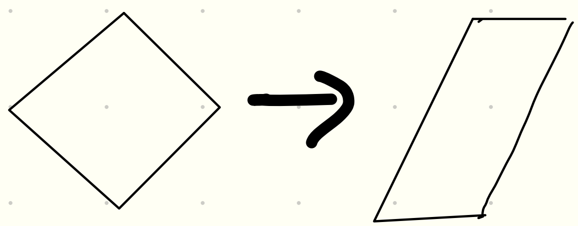
Mutation on dual surface Σ

Q is preserved, Δ, Z are mutated

Definition: Akhtar Coates Galkin Kasprzyk 2012

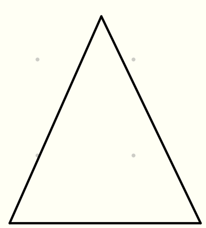
For dimer models Higashitany Nakajima 2021

<u>Example</u>



$$Z(\lambda,\mu)=\mu(\lambda+a)+\beta+\mu^{-1}(\lambda^{-1}+c)d$$

$$\nu=\mu(\lambda+a) \qquad \mu^{-1}=\nu^{-1}(\lambda+a)$$



$$Z(\lambda,\nu)=\nu+\beta+\nu^{-1}(\lambda+a)(\lambda^{-1}+c)d$$

General definition of mutation of polynomial

Def $SL(2,\mathbb{Z})$ action $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts $(\lambda,\mu)\mapsto(\lambda^a\mu^b, \lambda^c\mu^d)$

Def $P(\lambda,\mu)=\mu^h P_h(\lambda)+\mu^{h-1}P_{h-1}(\lambda)+\ldots+P_0(\lambda)+\mu^{-1}P_{-1}(\lambda)+\ldots+\mu^{-h'}P_{-h'}(\lambda)$

if $(\lambda-a)^h \mid P_h(\lambda)$, $(\lambda-a)^{h-1}\mid P_{h-1}(\lambda)$, ... $(\lambda-a)\mid P_1(\lambda)$

then $\nu=\mu(\lambda-a)$, $P \rightsquigarrow P(\lambda,\nu)$ Laurent polynomial

Mutation — composition of such transform and $SL_2(\mathbb{Z})$

Remark ① For $h=1$ no special conditions on P

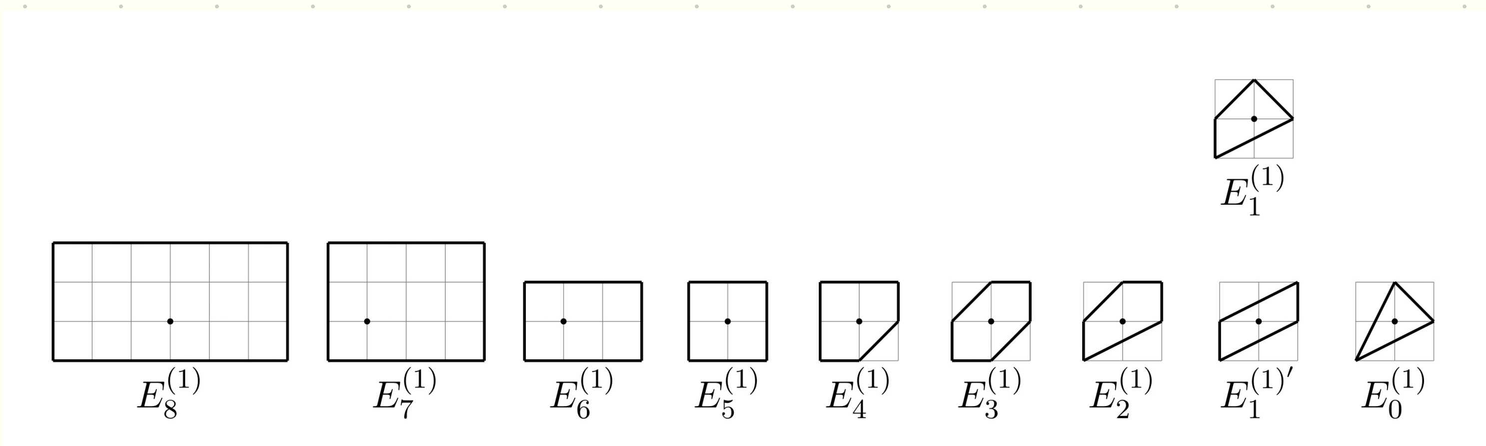
② Integrable system (generated by coefficients of spectral curve) is preserved

- For given h we have
  $h-1$ casimirs
  $\frac{(h-1)(h-2)}{2}$ hamiltonians $\Big\}$ hamiltonian reduction

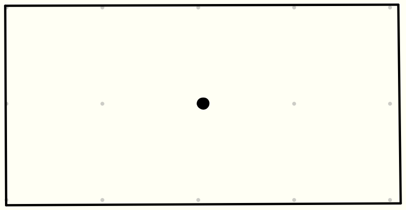
- Reduction case $g = \text{Area}\,\Delta + 1 - \frac{B}{2} - \#\underset{\text{Hamilton}}{\text{reduct}}$

<u>Lemma</u> Let $\exists\; p \in \Delta \cap \mathbb{Z}^2$ s.t. for any side $\frac{\ell_s}{h_s} \in \mathbb{Z}$,
where ℓ_s is integer length of s
h_s is distance from p to s. $\Rightarrow g = 1$

- Mizuno



- Example



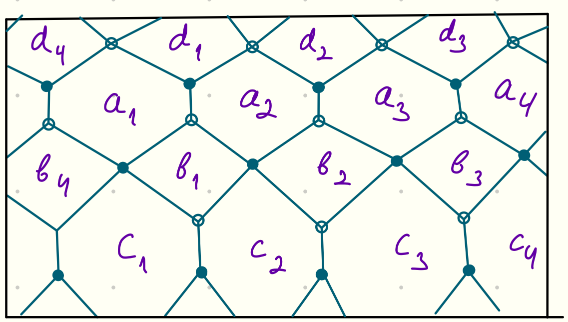
Partition function

$$Z = P_{-2}(\mu)\lambda^{-2} + P_{-1}(\mu)\lambda^{-1} + P_0(\mu) + P_1(\mu)\lambda + P_2(\mu)\lambda^2$$

Reduction condition

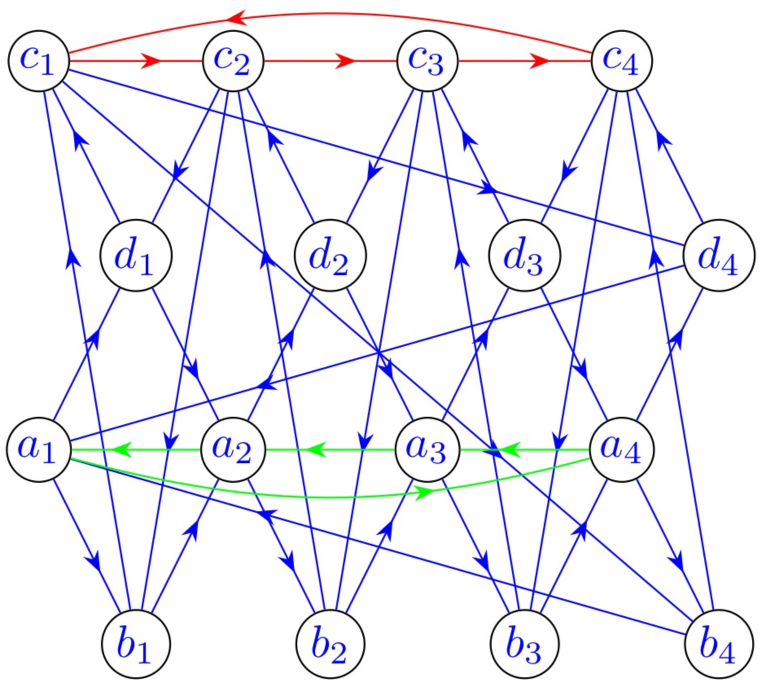
$$\left.\begin{array}{l} P_{-2}(\mu) \sim \mu^{-1}(\mu-a)^2, \\ P_2(\mu) \sim \mu^{-1}(\mu-c)^2 \end{array}\right\} \text{Casimirs} \qquad \left.\begin{array}{l} P_{-1}(a)=0 \\ P_1(c)=0 \end{array}\right\} \text{Hamiltonians}$$

Bipartite graph



$$\left.\begin{array}{l} a_1 a_2 a_3 a_4 = 1 \\ c_1 c_2 c_3 c_4 = 1 \end{array}\right\} \text{Casimirs}$$

$$\left.\begin{array}{l} 1 + a_2(1 + a_3(1 + a_4)) = 0 \\ 1 + c_2(1 + c_3(1 + c_4)) = 0 \end{array}\right\} \text{Hamiltonians}$$

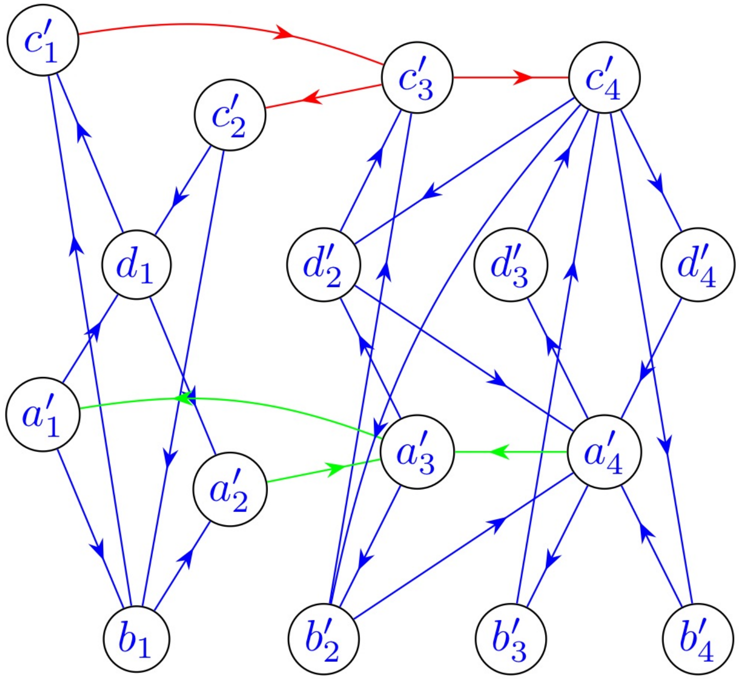


Quiver

$$\left.\begin{array}{l} a_1 a_2 a_3 a_4 = 1 \\ c_1 c_2 c_3 c_4 = 1 \end{array}\right\} \text{Casimirs}$$

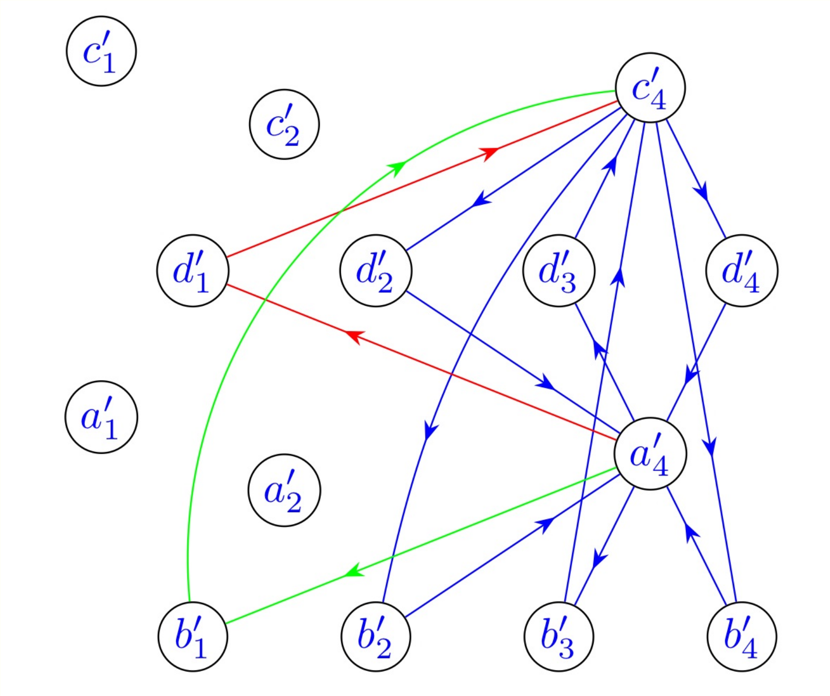
$$\left.\begin{array}{l} 1 + a_2(1 + a_3(1 + a_4)) = 0 \\ 1 + c_2(1 + c_3(1 + c_4)) = 0 \end{array}\right\} \text{Hamiltonians}$$

$\mu_{c_3}\ \mu_{a_3}\ \mu_{c_4}\ \mu_{a_4}$



$a_1' = a_2' = c_1' = c_2' = -1$

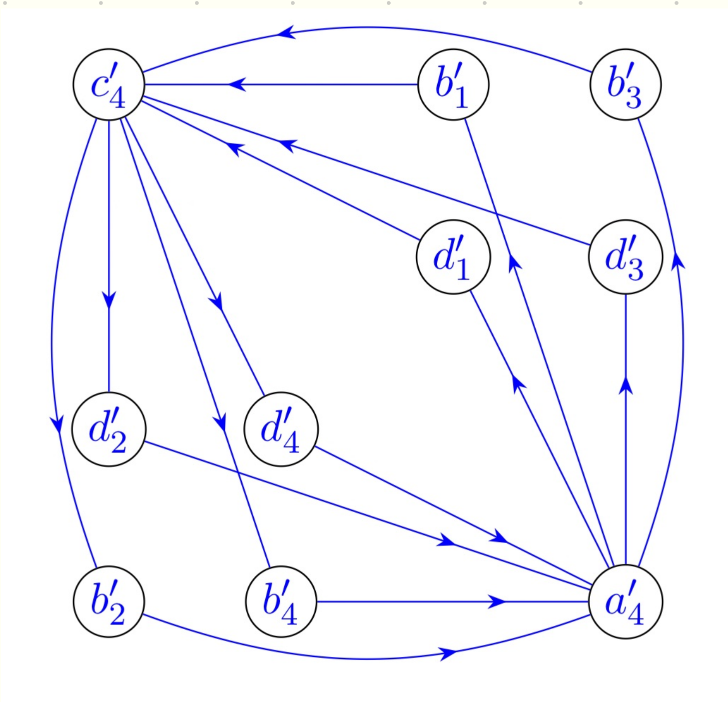
Reduction



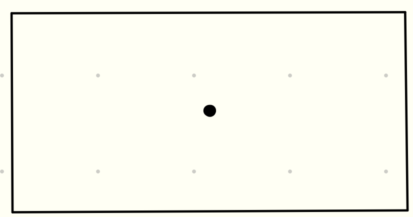
$b_1' = b_1 a_3' c_3'$

$d_1' = d_1 a_3' c_3'$

commute with a_1', a_2', c_1', c_2'



- quiver coincides with quiver for dual surface



- Has $W(E_7^{(1)})$ symmetry

Conclusion and Further questions

- Goncharov Kenyon Integrable systems $\subset$ Hamiltonian reduction (Goncharov Kenyon Integrable systems)

- Zig-zag (polinomial, polygon) mutations : equivalence between reduced GK IS

- q-Painlevé IS $\subset$ reduced GK IS

- Similar story exists for Fock-Goncharov-Shen varieties