Fans and polytopes in tilting theory

(j.w Aoki, Higashitani, Iyama, Kase) 2023/3/24

§ Backgrounds

$$\begin{array}{ccc} & Q : \text{quiver} & \\ \swarrow & & \searrow \\ \text{cluster cat } \mathcal{C}_{(Q,W)} & & \mathcal{A}_Q : \text{cluster alg} \\ \Downarrow & & \Downarrow \\ T = T_1 \oplus \cdots \oplus T_n & \longmapsto & \mathbb{t} = (t_1, \cdots, t_n) \\ \text{cluster tilting object} & & \text{cluster} \end{array}$$

[Adachi-Iyama-Reiten]

$$\boxed{\begin{array}{c} \tau\text{-tilting theory} \\ \text{“}\|\text{”} \\ \text{2-term silting theory} \end{array}} \longleftrightarrow \boxed{\text{cluster tilting theory}}$$

Today

<u>Setting</u> A : f.d. alg , $|A| = n$.

<u>Def</u> $T = (T^i, d^i) \in K^b(\mathrm{proj}A)$

(1) T : presilting $:\Leftrightarrow$ $\mathrm{Hom}(T, T[\ell]) = 0 \quad ({}^\forall \ell > 0)$

(2) T : silting $:\Leftrightarrow$ • T : presilting,
• $\mathrm{thick}(T) = K^b(\mathrm{proj}A)$

(3) T : 2-term $:\Leftrightarrow$ $T^i = 0$ for ${}^\forall i \neq 0, -1$

(Write $T = (\cdots 0 \to \overset{-1}{T^{-1}} \to \overset{0}{T^{0}} \to 0 \cdots) = (T^{-1} \to T^0)$)

2-siltA : set of isoclass of basic 2-term silt cpxes
(2-psiltA) ———— presilt ——

(4) A is g-finite (τ-tilting finite)
$:\Leftrightarrow \# \text{2-silt}A < \infty$

<u>Rem/Thm</u>

- $\mathrm{add}\,T^0 \cap \mathrm{add}\,T^{-1} = \{0\}$ if $T = (T^{-1} \to T^0) \in$ 2-psiltA
(sign-coherent)

- $T = \bigoplus_{i=1}^{\ell} T_i \in$ 2-psiltA is silting $\Leftrightarrow \ell = n$

$\text{2-silt}^i A := \{ T \in \text{2-psilt}A \mid |T| = i \} \quad (1 \leq i \leq n)$.

Def Take $T := \bigoplus_{i=1}^{\ell} T_i \in 2\text{-psilt}A$.

[I] $\Delta(A)$: g-simplicial cpx

$:\Longleftrightarrow$ j-simplices is $2\text{-psilt}^{j+1}A$

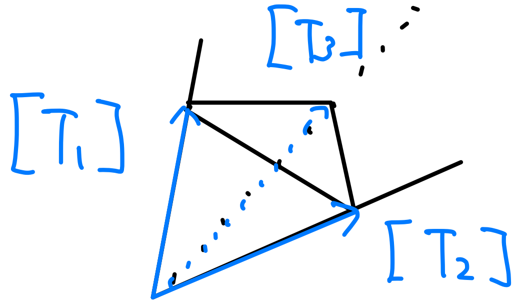
$T_i := (T_i^{-1} \to T_i^{0})$

$\Rightarrow$ • $[T_i] := [T_i^{0}] - [T_i^{-1}] \in K_0(\mathrm{proj}A) = \langle [P_1], \cdots, [P_n] \rangle \cong \mathbb{Z}^n$

(g-vector)

• $C(T) := \left\{ \sum_{i=1}^{\ell} a_i [T_i] \;\middle|\; a_i \geq 0 \right\} \subseteq K_0(\mathrm{proj}A)_{\mathbb{R}} \simeq \mathbb{R}^n$

• $C_{\leq 1}(T) := \left\{ \sum_{i=1}^{\ell} a_i [T_i] \;\middle|\; a_i \geq 0, \sum a_i \leq 1 \right\}$

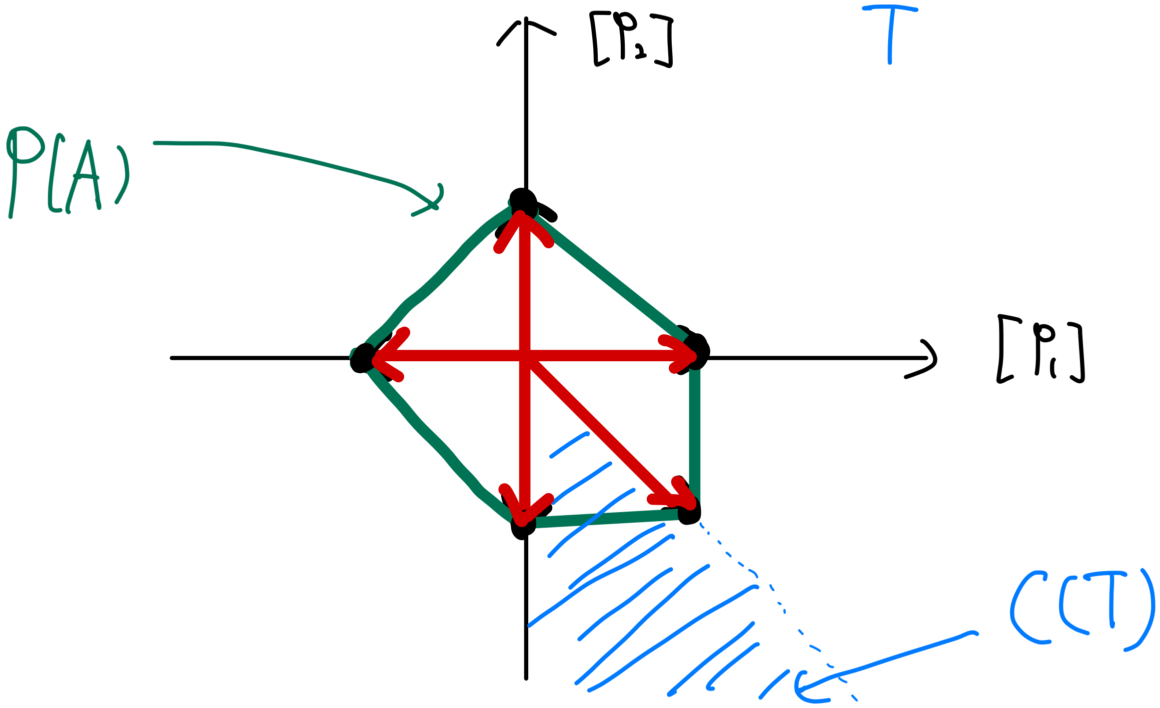


[II] $P(A) := \bigcup_{T \in 2\text{-silt}A} C_{\leq 1}(T)$: g-polytope of A

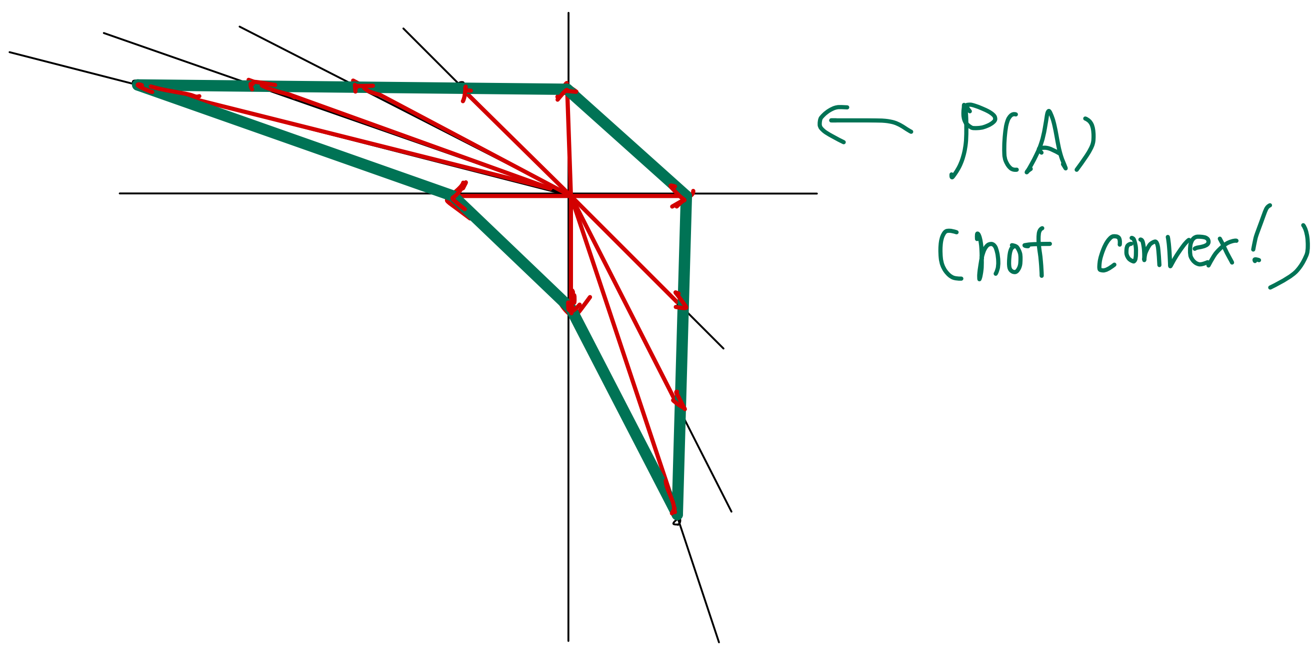
[III] $\Sigma(A) := \{ C(T) \mid T \in 2\text{-psilt}A \}$: g-fan of A

Ex (1) $A = K(1 \to 2)$. $A = e_1 A \oplus e_2 A = P_1 \oplus P_2$

$$2\text{-silt}\, A = \left\{ \begin{matrix} (0 \to P_1) \\ \oplus \\ (0 \to P_2) \end{matrix}, \begin{matrix} (0 \to P_1) \\ \oplus \\ (P_2 \to P_1) \end{matrix}, \underbrace{\begin{matrix} (P_2 \to 0) \\ \oplus \\ (P_2 \to P_1) \end{matrix}}_{T}, \begin{matrix} (P_1 \to 0) \\ \oplus \\ (0 \to P_1) \end{matrix}, \begin{matrix} (P_1 \to 0) \\ \oplus \\ (P_2 \to 0) \end{matrix} \right\}$$

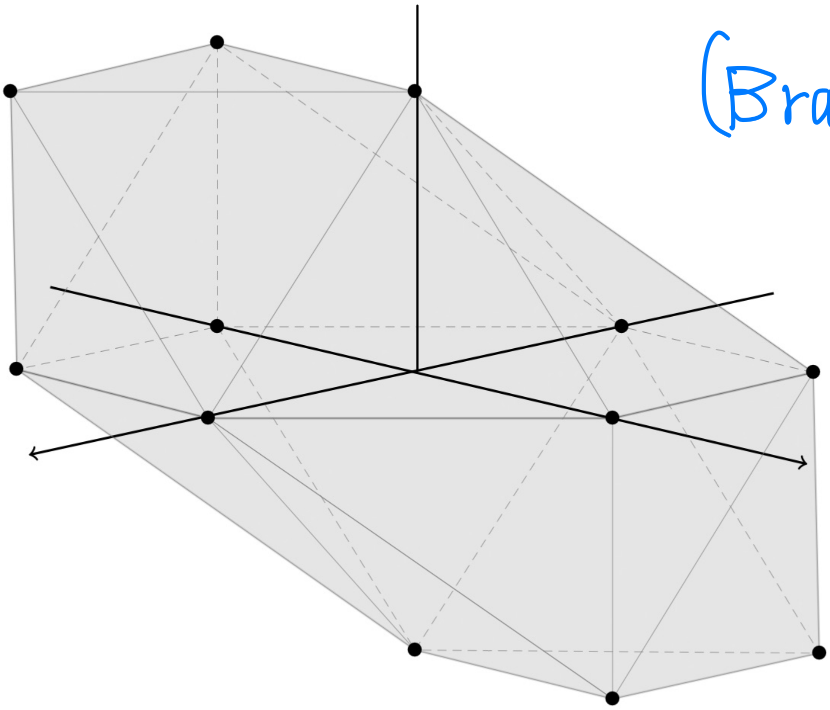


(2)
$$K\left(\begin{tikzcd} 1 \arrow[loop above, "a_2"] \arrow[loop left, "a_3"] \arrow[loop below, "a_4"] \arrow[r, shift left, "a_1"] & 2 \arrow[l, shift left, "b_1"] \arrow[loop above, "b_2"] \arrow[loop right] \arrow[loop right] \arrow[loop below, "b_5"] \end{tikzcd} \right) \Big/ \left\langle \begin{matrix} a_i a_j \,(i-j \neq 1), \\ b_i b_j \,(i-j \neq 1), \\ a_i b_j,\ b_i a_j \end{matrix} \right\rangle$$



(3) $A = K\left(1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3\right) / \langle ab, b^*a^*, aa^*a, aa^*-bb^*, b^*bb^* \rangle$

(Brauer tree alg)



Thm [Demonet-Iyama-Jasso, Hille, $\cdots$]

(1) $\Sigma(A)$ is nonsingular (i.e, maximal cone is generated by $\mathbb{Z}$-basis of $\mathbb{Z}^n$)

(2) $\Sigma(A)$ is complete (i.e, $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$)

$\Longleftrightarrow$ A is g-finite (i.e, $\# 2\text{-silt} A < \infty$)

$\boxed{\begin{array}{ccc} 2\text{-silt} A & \leftrightsquigarrow & \Delta(A), \Sigma(A), P(A) \\ (\text{Rep. theory}) & & (\text{Combinatorics}) \end{array}}$

[I] g-simplicial complex (A: g-finite)

<u>Def</u> • f-vector $(f_{-1}, f_0, f_1, \dots, f_{n-1})$ by

$$f_{-1} := 1, \quad f_j := \# 2\text{-psilt}^{j+1} A.$$

• h-vector $(h_0, h_1, \dots, h_n)$ by

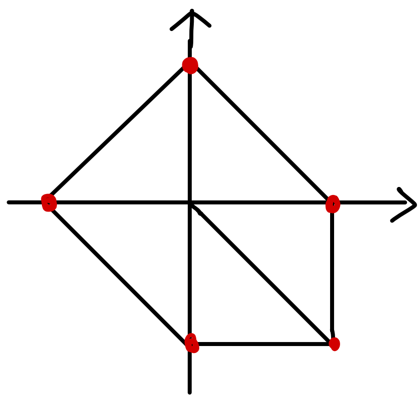
$$h_j := \sum_{i=0}^{j} (-1)^{j-i} \binom{n-i}{j-i} f_{i-1} \qquad (0 \le j \le n)$$

$$\left(\Longleftrightarrow \ f(x) := \sum_{i=0}^{n} f_{i-1} x^{n-i}, \ h(x) := \sum_{i=0}^{n} h_i x^{n-i} \quad f(x-1) = h(x) \right)$$

<u>Ex</u> (1) $A = K(1 \to 2)$

(1, 5, 5): f-vector

(1, 3, 1): h-vector



(2) A: Brauer tree alg

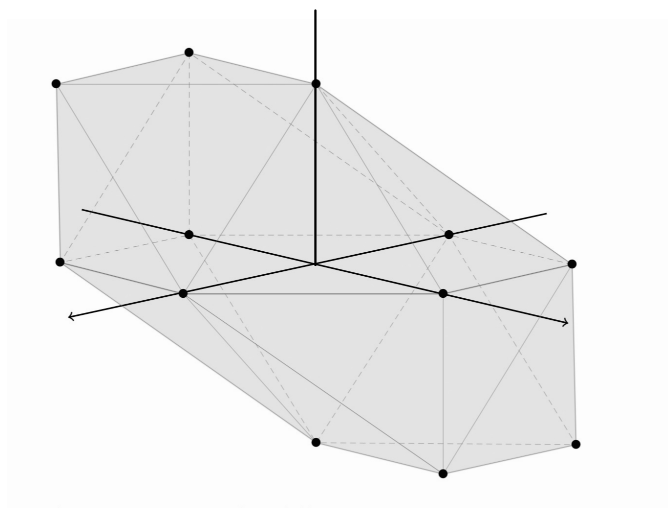
(1, 12, 30, 20): f-vector

(1, 9, 9, 1): h-vector

$$\begin{array}{cccc}
 & & & 1 \\
 & & 1 & 12 \\
 & 1 & 11 & 30 \\
1 & 10 & 19 & 20 \\
\hline
1 & 9 & 9 & 1
\end{array}$$

f-vector

h-vector



Q How to interpret h-vector into Rep. thy?

Thm [König-Yang] $\exists$ bijection

$$2\text{-silt}A \overset{1:1}{\longleftrightarrow} 2\text{-s.m.c}A$$

$$K^b(\mathrm{proj}A) \ni \bigoplus_{i=1}^{n} T_i \qquad\qquad \bigoplus_{i=1}^{n} X_i \in D^b(\mathrm{mod}A)$$

$$\mathrm{Hom}(T_i, X_j[l]) = \begin{cases} k & (i=j \text{ and } l=0) \\ 0 & (\text{o.w}) \end{cases}$$

$$\boxed{\text{Generalization of projectives and simples}}$$

Thm [AHIKM]

$$h_j = \#2\text{-s.m.c}^jA = \{X \in 2\text{-s.m.c}A \mid \#\underbrace{H^0(X)}_{\text{semibrick}} = j\}$$

Cor (1) $\#2\text{-s.m.c}^jA = \#2\text{-s.m.c}^{n-j}A$ (DS-equations)

(2) $\#2\text{-s.m.c}^1A \leq \#2\text{-s.m.c}^2A \leq \cdots \leq \#2\text{-s.m.c}^{\lfloor \frac{n}{2} \rfloor}A$

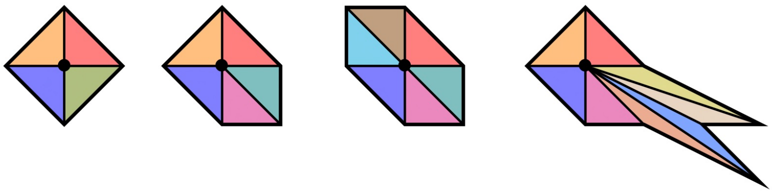
$\#2\text{-s.m.c}^{\lfloor \frac{n}{2} \rfloor}A \geq \cdots \geq \#2\text{-s.m.c}^{n-1}A \geq \#2\text{-s.m.c}^{n}A$

(Unimodality)

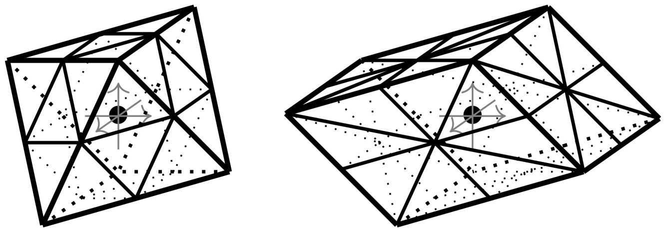
[II] g-polytope

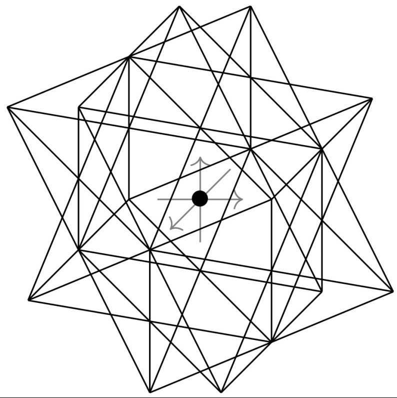
<u>Def</u> A is g-convex $:\Leftrightarrow$ $P(A)$ is convex

$$P(A) = \bigcup_{T\in 2\text{-silt}A} C_{\le 1}(T)$$



$k\times k$ $\quad$ $k(1\xrightarrow{\alpha}2)$ $\quad$ $\dfrac{k\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\right)}{\langle\alpha\beta,\beta\alpha\rangle}$ $\quad$ $\dfrac{k\left(1\xrightarrow{\alpha} 2 \circlearrowleft \beta\right)}{\langle\beta^3\rangle}$





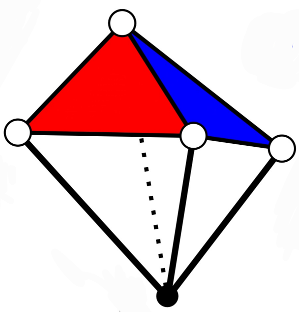
<u>Rem/Thm</u> [Hille, Asashiba-M-Nakashima, AHIKM]

Assume A is g-finite. TFAE

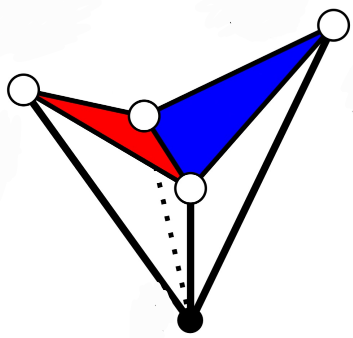
(1) A is g-convex

(2) # of middle term U is at most 2.

for $^\forall$ exchange seq. $T_i \to U \to T_i^* \quad (T=\oplus T_i \in 2\text{-silt}A)$



pairwise convex



non-pairwise convex

<u>Def</u> (1) $V := \mathbb{R}^n \supset P$: convex lattice polytope

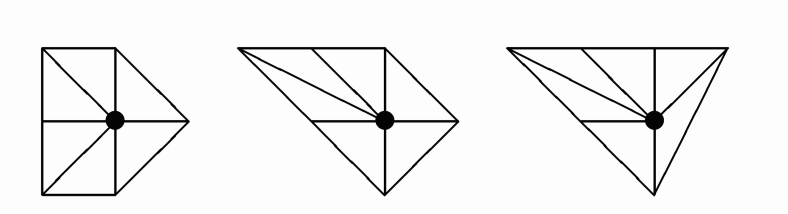
($^\forall$vertex of $P \in \mathbb{Z}^n$, $0 \in P^\circ$)

$(-,-) : V^* \times V \to \mathbb{R}$: natural pairing

$P^* := \{ x \in V^* \mid {}^\forall y \in P,\ (x, y) \leq 1 \}$: dual polytope

which is convex poly and $(P^*)^* = P$

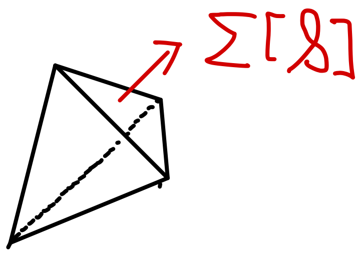
(2) P : reflexive $:\Longleftrightarrow$ P^* is lattice poly.



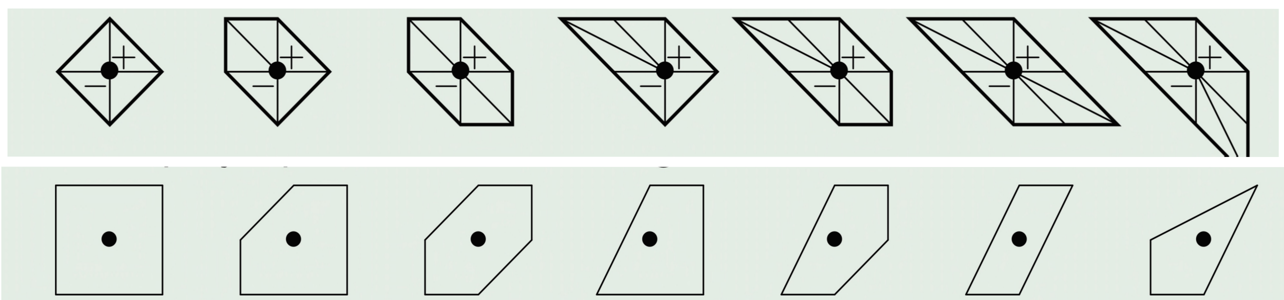
<u>Thm</u> [AHIKM] (1) A : g-convex

$\Rightarrow$ $P(A)$ is reflexive and $P(A) = (P^c(A))^*$

$$P^c(A) = \mathrm{conv}\{ \Sigma[\delta] \mid \delta \in 2\text{-s.m.c}A \}$$



(2) $^\exists$Precisely 7 g-convex of rank 2.



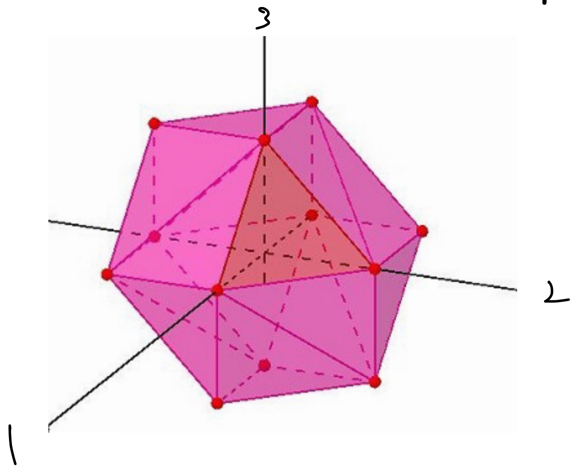
$\updownarrow (-)^*$

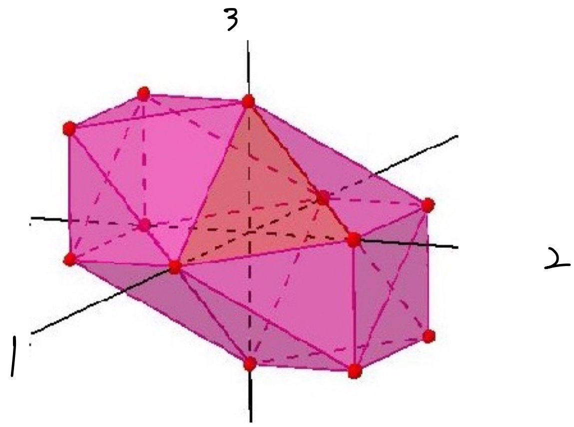
Ex

(1) **Thm** [Asashiba-M-Nakashima]

Brauer tree algs are g-convex.

Moreover, $\# 2\text{-silt} A_G$ only depends on $\# G_0$





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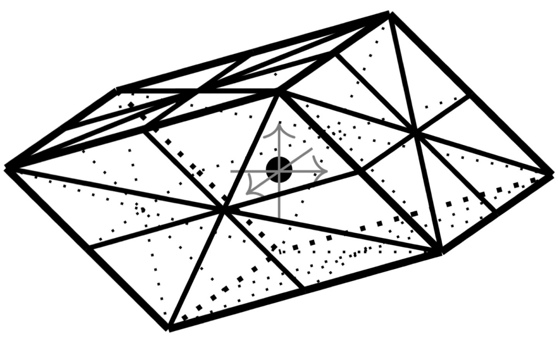
(2) Φ : Dynkin graph. $\Pi := \Pi(\Phi)$: p.p. alg. $W := W(\Phi)$: Coxeter grp

Thm [M, Fu-Geng, AHIKM]

(1) $2\text{-silt}\,\Pi \overset{1:1}{\simeq} W$ and $\Delta(\Pi)$ is Coxeter complex

(2) Π is g-convex $\Leftrightarrow$ Φ is type A or B_n.



In this case,

$$P(\Pi) = (\mathrm{conv}(\Phi_{\mathrm{short}}))^* \underset{P^c(\Pi)}{\overset{\shortparallel}{}}$$

[III] g-fans

Q Which fans can be realized as g-fans?

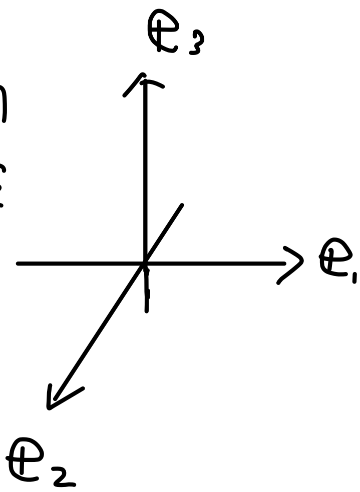
Def Σ : sign-coherent fan :$\Leftrightarrow$

(i) Σ is nonsingular

(ii) $\exists \sigma_+, \underset{\sigma_-}{\underset{\shortparallel}{-\sigma_+}} \in \Sigma_n$ (c.f. $-C(A) = C(A[1]))$

(iii) For each $\sigma \in \Sigma$,

$\exists \varepsilon = (\varepsilon_i) \in \{\pm 1\}^n$ s.t. $\sigma \subseteq \underset{\text{Cone}\{\varepsilon_1 e_1, \cdots, \varepsilon_n e_n\}}{\underset{\shortparallel}{\mathbb{R}^n_\varepsilon}}$



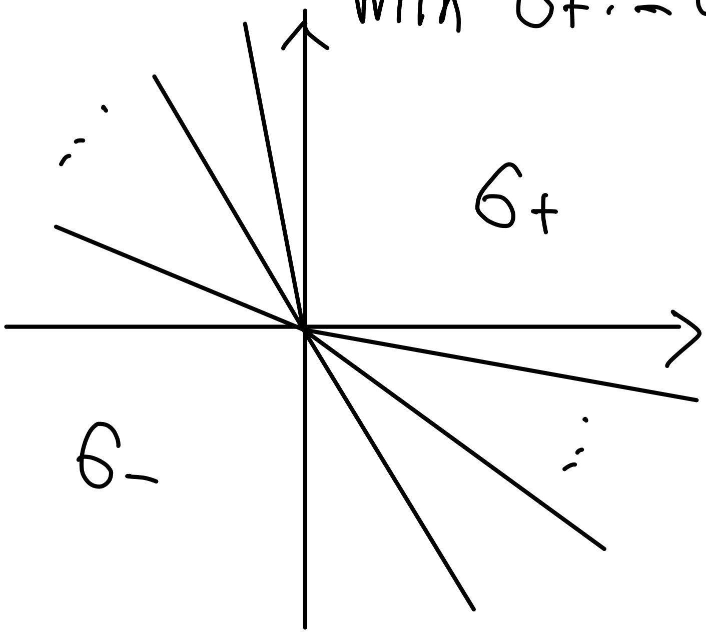
Rmk g-fan is sign-coherent.

Thm [AHIKM] (answer for rank 2)

$\{\text{complete g-fans}\} = \{\text{complete s.c. fans}\}$

$\boxed{\text{Key}}$ Gluing

$\mathrm{Fan}_{s.c}$: Set of complete sign-coherent fans with $\sigma_+ := \mathrm{cone}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$



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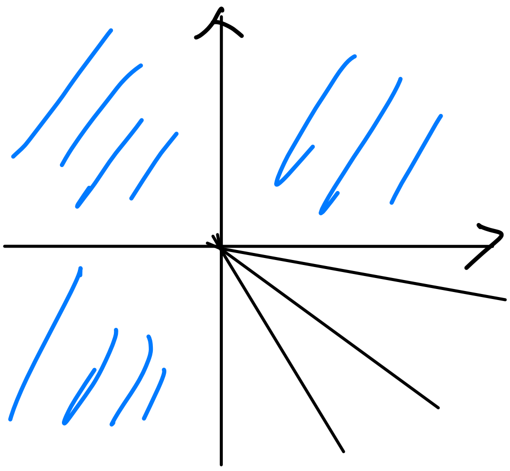

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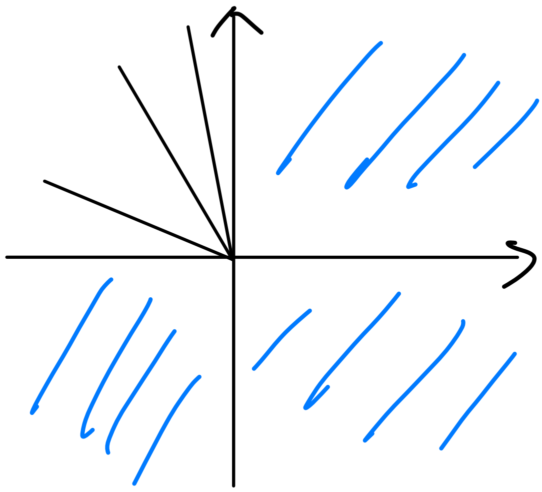
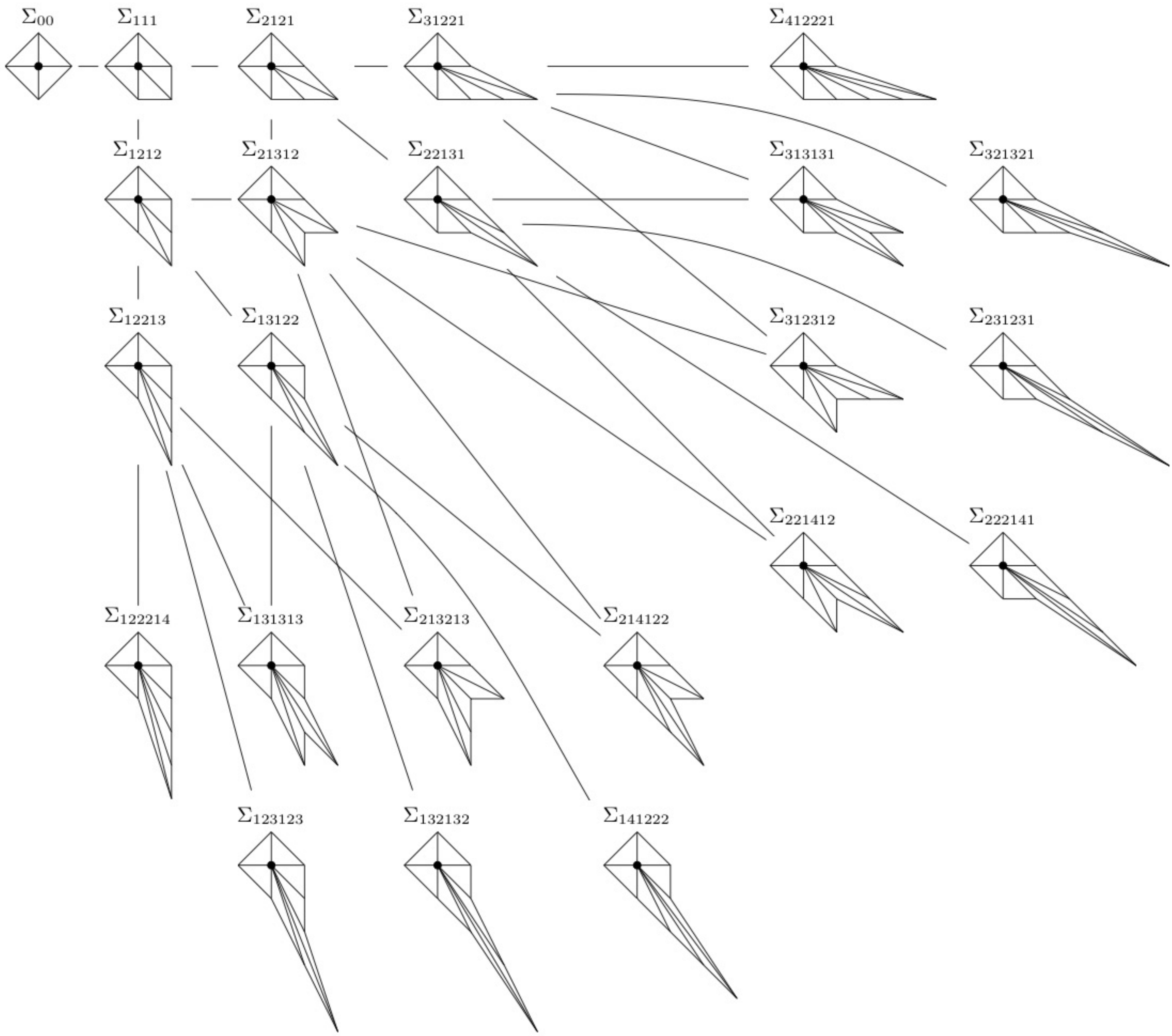


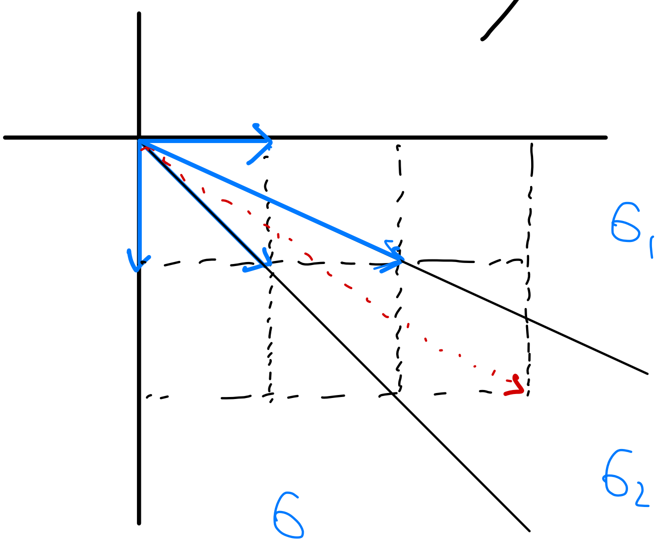
$\in$ $\mathrm{Fan}^{+-}_{s.c}$ $\qquad$ $\in$ $\mathrm{Fan}^{-+}_{s.c}$

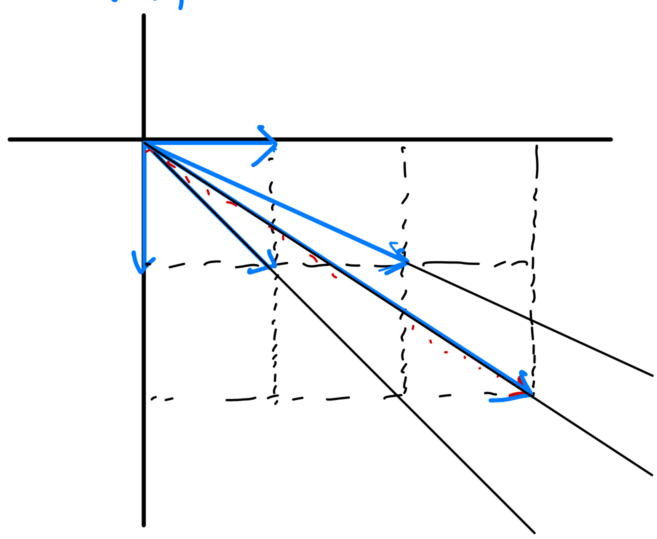
$\Rightarrow$ Study $\mathrm{Fan}^{+-}_{s.c}$ and their realization

EX $\mathrm{Fan}^{+-}_{s.c}$



Connected by subdivision

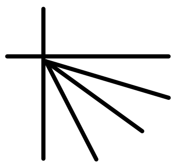
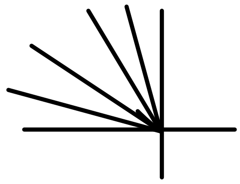


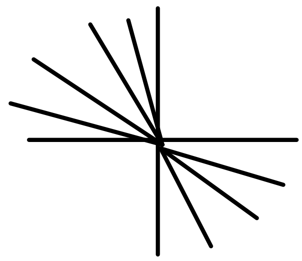


Gluing Thm Λ, Λ' : f.d. alg s.t.

$\Sigma(\Lambda) \in \mathrm{Fan}^{+-}_{s.c}$ and $\Sigma(\Lambda') \in \mathrm{Fan}^{-+}_{s.c}$.

$\Rightarrow {}^{\exists}$ f.d. alg Γ s.t. $\Sigma(\Gamma) = \Sigma(\Lambda) * \Sigma(\Lambda')$

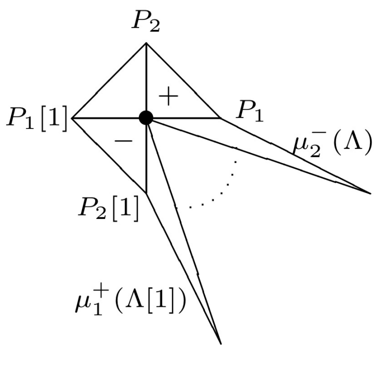
($\Sigma(\Lambda) =$  , $\Sigma(\Lambda') =$ 

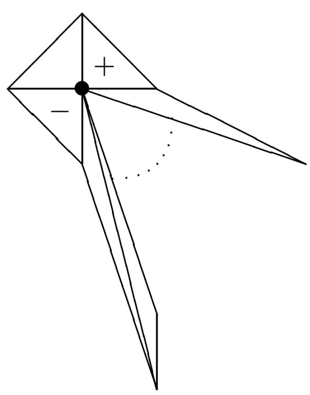
$\Rightarrow \Sigma(\Gamma) =$ )

Subdivision Thm Λ : f.d. alg s.t. $\Sigma(\Lambda) \in \mathrm{Fan}^{+-}_{s.c}$.

$\Rightarrow {}^{\exists}$ f.d. alg Γ s.t. $\Sigma(\Gamma) = D_\sigma(\Sigma(\Lambda))$,

where $\sigma := C(\mu_1^+(\Lambda[1]))$.

($\Sigma(\Lambda) =$ 

$\Sigma(\Gamma) =$ 

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The corresponding algs

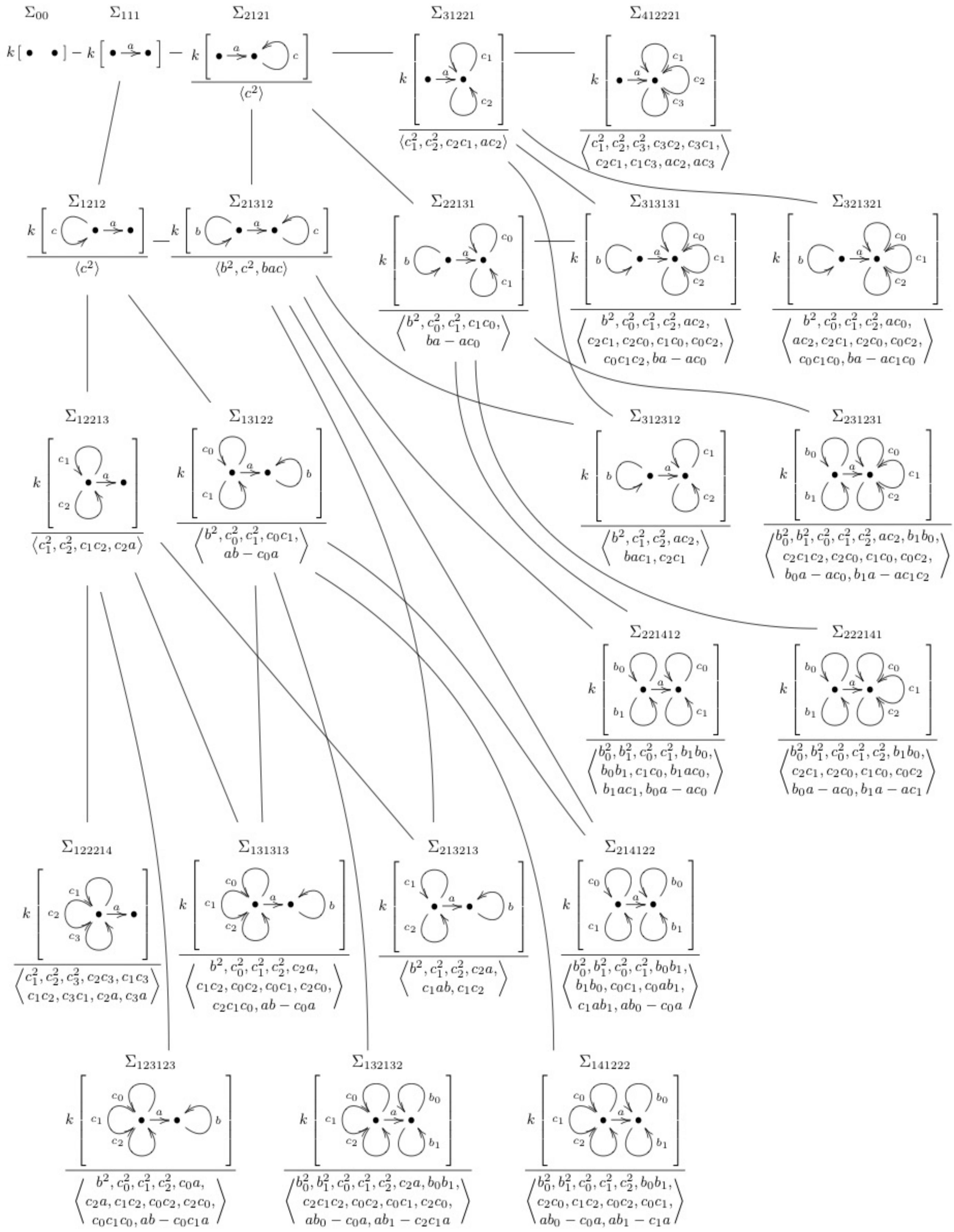

FIGURE 2. Algebras whose g-fans are given in Figure 1

Thank you for your attention!