Exchange quivers of root systems, generalized path algebras, and cluster algebras

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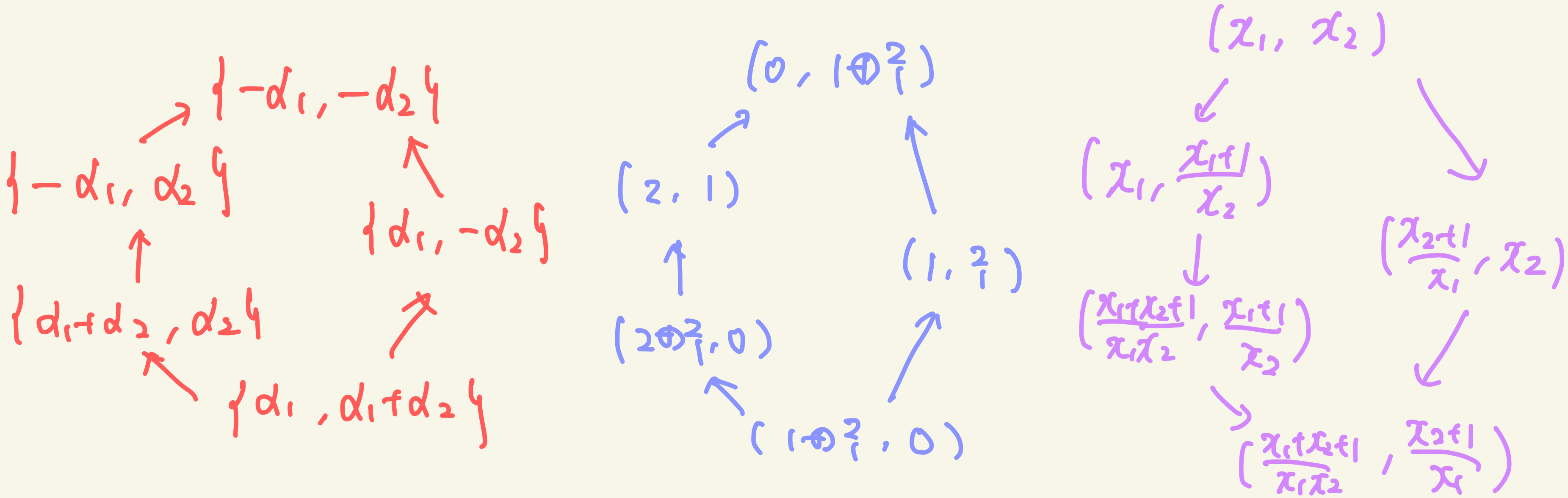


Contents

§1. Introduction

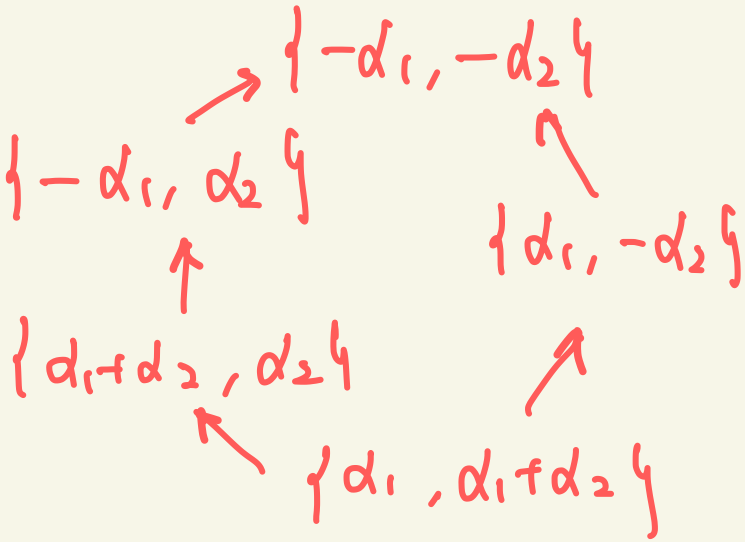
Previous studies [Buan-Marsh-Reineke-Reiten-Todorov]
[Ceballos-Pilaud] [Geiß-Leclerc-Schröer] etc...

Root system Φ of Dynkin type	GLS path alg. KQ/I of Dynkin type	Cluster alg. $\mathcal{A}(B)$ of finite type
{c-clusters}	{τ-tilting pairs}	{clusters}

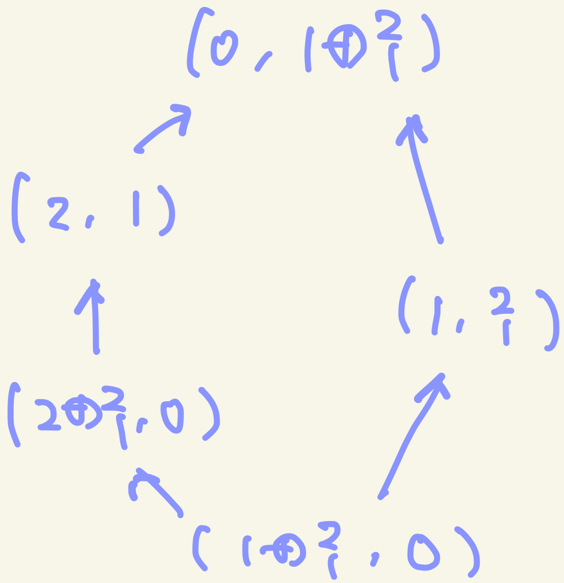
Common Cluster structure = Same exchange graph

Today's talk :

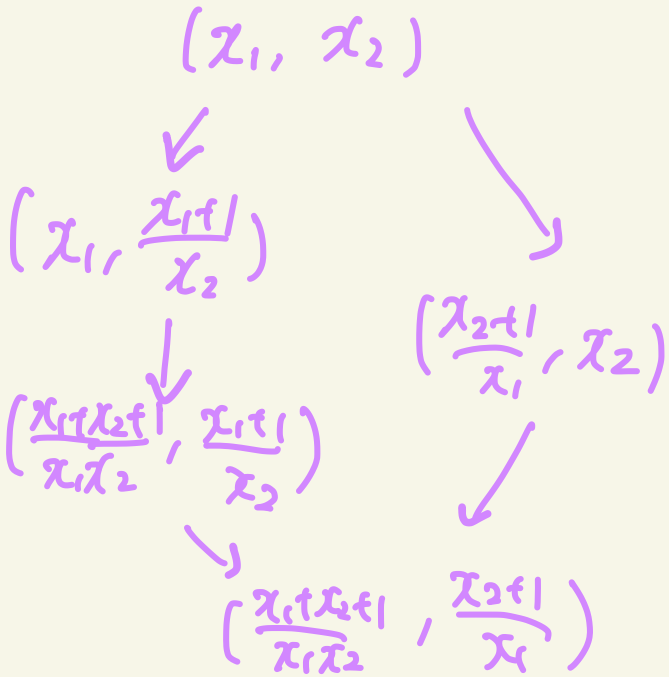
Exchange ~~graph~~ quiver of c-clusters $\vec{\Gamma}(\Phi, c)$



Exchange ~~graph~~ quiver of τ-tilting pairs $\vec{\Gamma}(s\tau\text{-tilt}\,H)$



Exchange ~~graph~~ quiver of clusters $\Gamma(B)$



We introduce "natural orientations" into each graph and see their coincidence.

§2. Exchange quiver of root system

Φ: Dynkin root system, $W(\Phi)$: Weyl group corresponding to Φ

$\{s_i\}_{i\in[1,n]}$: simple reflections of $W(\Phi)$.

Fix a Coxeter element $c = s_{i_1} \cdots s_{i_n}$.

← "almost positive root set"

$\Phi_{\geq -1} := \Phi^+ \cup -\Delta$ (Union of positive roots and simple negative roots)

$\sigma_i : \Phi_{\geq -1} \to \Phi_{\geq -1}$, $\sigma_i(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in -\Delta \setminus \{-\alpha_i\} \\ s_i(\alpha) & \text{otherwise} \end{cases}$

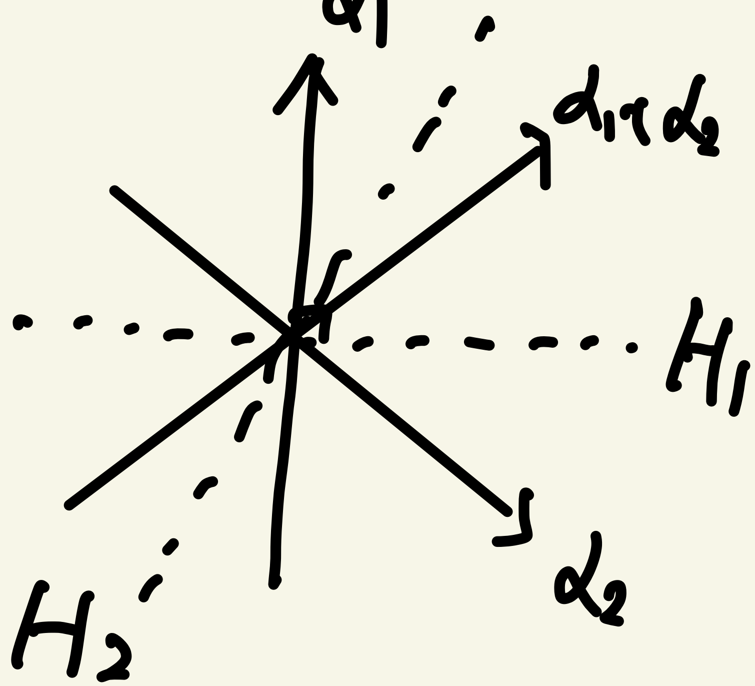
$\tau_c := \sigma_{i_1} \circ \cdots \circ \sigma_{i_n} : \Phi_{\geq -1} \to \Phi_{\geq -1}$

Ex. Φ : A_2 type, $c := s_2 s_1$

$$\Phi_{\geq -1} = \{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1+\alpha_2\}$$

$$\tau_c(\alpha_1) = \sigma_2\sigma_1(\alpha_1) = \sigma_2(-\alpha_1) = -\alpha_1.$$

$$\alpha_1 \xmapsto{\tau_c} -\alpha_1 \longmapsto \alpha_1+\alpha_2 \longmapsto -\alpha_2 \longmapsto \alpha_2 \longmapsto \alpha_1.$$



Construction of c-clusters

Def. (Compatibility)

$(-\|_c-) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ s.t.

(i) $(-\alpha_i \|_c \beta) = \max\{0, b_i\}$ for all $i \in \{1, \dots, n\}$ and $\beta = \sum b_i \alpha_i$

(ii) $(\alpha \|_c \beta) = (\tau_c(\alpha) \|_c \tau_c(\beta))$ for all $\alpha, \beta \in \Phi_{\geq -1}$.

α and β are compatible $:\Leftrightarrow (\alpha \| _c \beta) = (\beta \| _c \alpha) = 0$

$R(\alpha) = R(\beta)$

$C \cup \{\alpha\}$: c-cluster

$C \cup \{\beta\}$: c-cluster

$\Rightarrow \cancel{C \cup \{\alpha, \beta\}}$

c-cluster : maximal pairwise compatible set.

Exchange quiver $\vec{\Gamma}(\Phi, c)$:

$$\begin{cases} \text{vertex} : \text{c-clusters} \\ C_1 \to C_2 :\Leftrightarrow \begin{cases} C_1 = C' \cup \{\alpha\},\ C_2 = C' \cup \{\beta\}, \\ R_c(\alpha) > R_c(\beta) \end{cases} \\ \quad \text{where } R_c(\alpha) := \min\{n \in \mathbb{N} \mid \tau_c^{-n}(\alpha) \in -\Delta\} \end{cases}$$

$$\alpha_1 \overset{\tau_c}{\longmapsto} -\alpha_1 \longmapsto \alpha_1 + \alpha_2 \longmapsto -\alpha_2 \longmapsto \alpha_2 \longmapsto \alpha_1.$$

$$\begin{tikzcd}
 & \{-\alpha_1, -\alpha_2\} & \\
\{-\alpha_1, \alpha_2\} \arrow[ur] & & \\
 & & \{\alpha_1, -\alpha_2\} \arrow[uul] \\
\{\alpha_1+\alpha_2, \alpha_2\} \arrow[uu] & & \\
 & \{\alpha_1+\alpha_2, \alpha_1\} \arrow[ul] \arrow[uur] &
\end{tikzcd}$$

$(\alpha \| \alpha) = (\tau\alpha \| \tau\alpha)$

$\cdots = (-\alpha_j \| -\alpha_j)$

$= \max(0, -1)$

$= 0.$

$(\alpha_1 + \alpha_2 \| \alpha_2) = (\tau(\alpha_1+\alpha_2) \| \tau\alpha_2)$

$= (-\alpha_2 \| \alpha_1) = 0$

§3. Exchange quiver of GLS path algebra

C: Cartan matrix of Φ (rank n), Γ: Dynkin diagram
D: symmetrizer i.e. $D = \mathrm{diag}(d_1, \dots, d_n)$ s.t. DC: symmetric matrix.

Q^c: quiver corresponding to (C, D, c) i.e.
$(Q_0)^c = [1, n]$,
$(Q_1)^c = \{ \alpha_{ij} : i \to j \mid i - j \in \Gamma,\ c = \cdots s_i \cdots s_j \cdots \} \cup \{ \varepsilon_i : i \to i \mid i \in [1, n] \}$

I: ideal of KQ^c satisfying the following relations:
(H1) for $i \in [1, n]$, $\varepsilon_i^{d_i} = 0$,
(H2) for $\alpha_{ij} \in (Q_1)^c$, $\varepsilon_i^{|c_{ji}|} \alpha_{ij} = \alpha_{ij} \varepsilon_j^{|c_{ij}|}$.

$H = H(C,D,c) := KQ^c/I$: GLS-path algebra.

Ex. Φ : A_2 type, $\Gamma : 1 — 2$, $C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & \\ & 2 \end{bmatrix}$, $c = s_2 s_1$

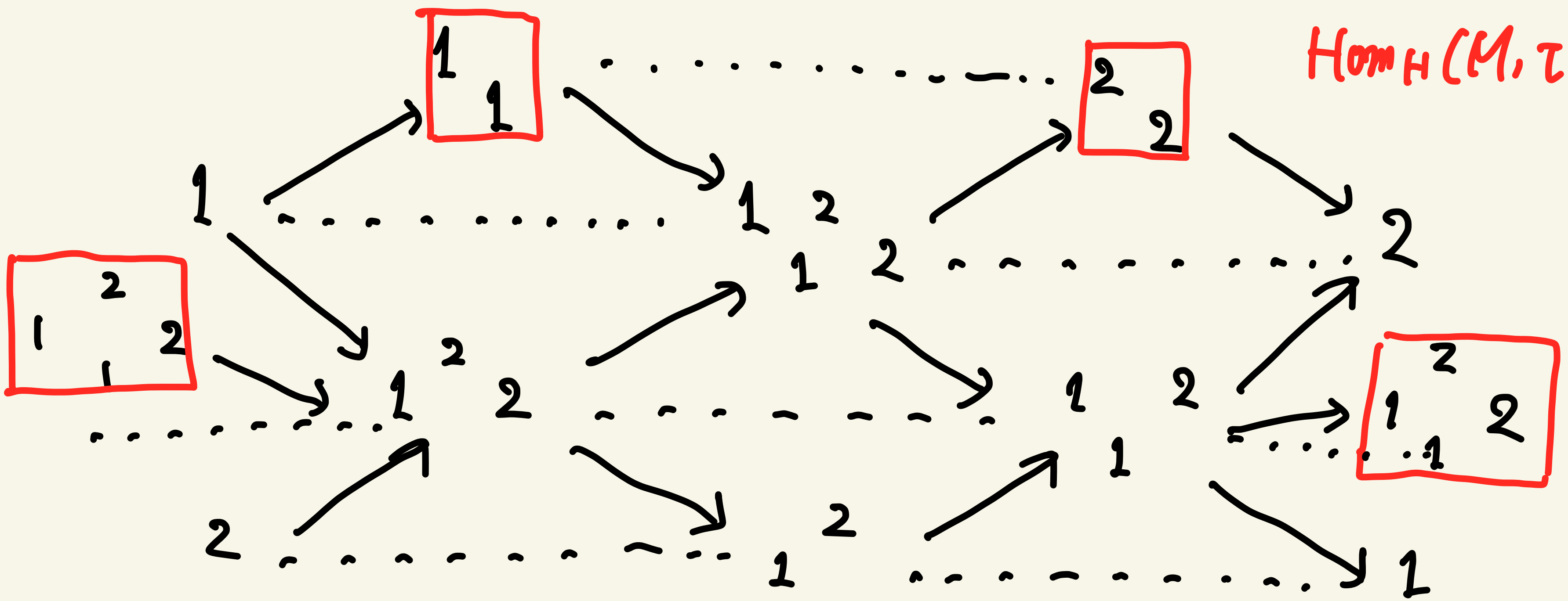
$$\longrightarrow \quad Q^c = \begin{tikzcd} 1 \arrow[loop below, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{21}"'] \arrow[loop below, "\varepsilon_2"] \end{tikzcd}, \quad I = \langle \varepsilon_1^2, \varepsilon_2^2, \varepsilon_2\alpha_{21} - \alpha_{21}\varepsilon_1 \rangle$$

Def. (τ-tilting pair)

$(M, P) \in \operatorname{mod} H \times \operatorname{proj} H$: τ-tilting pair

$$:\Leftrightarrow \begin{cases} \operatorname{Hom}_H(M, \tau M) = 0 \\ \operatorname{Hom}_H(P, M) = 0 \\ |P| + |M| = |H|. \end{cases}$$

AR-quiver of KQ^c/I [GLS, Fig.5]



$\mathrm{Hom}_H(M, \tau M) = 0$

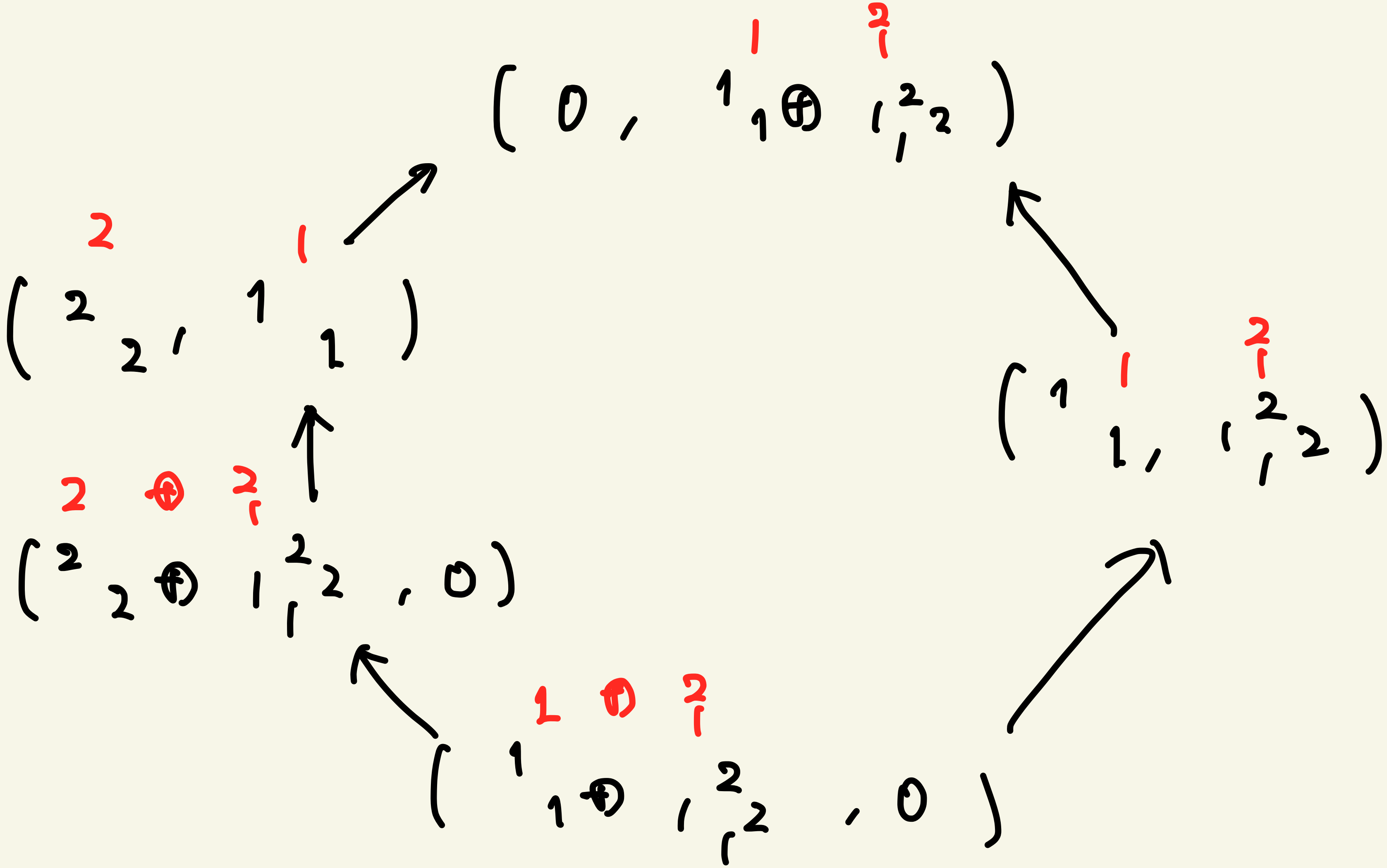
Exchange quiver $\vec{\Gamma}(s\tau\text{-tilt}\,H)$:

vertex : basic τ-tilting pair

$\left({}_{1}\overset{2}{}{}_{1}^{\,2} \oplus {}^{1}{}_{1} \; , \; 0 \right)$: τ-tilting pair.

$$(M,P) \to (M',P') \Leftrightarrow \begin{cases} |M \cap M'| + |P \cap P'| = n-1, \\ \mathrm{Fac}\,M \supset \mathrm{Fac}\,M' \end{cases}$$

Ex.



Rmk. Each modules M or P is locally-free modules, that is, Me_i is a free e_iHe_i module.

Red numbers mean rank vectors of M or P as e_iHe_i-free modules.

§4. Exchange quiver of cluster algebra

$\tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$: $m \times n$ matrix s.t. upper $n \times n$ submatrix B is skew-symmetrizable.

$\mathfrak{X} = (x_1, \dots, x_n)$: cluster, $(\mathfrak{X}, \tilde{B})$: seed

For $k \in \{1, \dots, n\}$, $(\mathfrak{X}', \tilde{B}') := \mu_k(\mathfrak{X}, \tilde{B})$ is obtained as follows :

For $x_i' \in \mathfrak{X}$, $x_i' = \begin{cases} x_i & \text{if } i \neq k \\ \dfrac{\prod_j x_j^{\max(0, b_{kj})} + \prod_j x_j^{\max(0, -b_{kj})}}{x_i} & \text{if } i = k \end{cases}$

For b_{ij}' in $\tilde{B}'$, $b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}\left(|b_{ik}| b_{kj} + b_{ik} |b_{kj}|\right) & \text{otherwise.} \end{cases}$

Ex. $\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x = (x_1, x_2)$

$$
\begin{tikzcd}[column sep=small]
& {(x_1, x_2) \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} \arrow[dl, "2"'] \arrow[dr] \arrow[r, phantom, "="] & {(x_2, x_1)} \arrow[d, dash, "2" , "\text{(mutation \& permutation)}"'] \\
{\left(x_1, \frac{x_1+1}{x_2}\right) \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}} \arrow[d, "1"] & & {\left(x_2, \frac{x_2+1}{x_1}\right) \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}} \arrow[dl, "1"] \\
{\left(\frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}\right) \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}} \arrow[r, "2"'] & {\left(\frac{x_1+x_2+1}{x_1x_2}, \frac{x_2+1}{x_1}\right) \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}} &
\end{tikzcd}
$$

Exchange quiver $\vec{\Gamma}(B)$:

vertex: clusters generated by the initial seed $\left(\mathbb{X}, \begin{bmatrix} B \\ I_n \end{bmatrix}\right)$

$\mathbb{X} \to \mathbb{X}' :\Leftrightarrow \begin{cases} \exists k \in [1, n] \text{ s.t. } \mu_k(\mathbb{X}, \tilde{B}) = (\mathbb{X}', \tilde{B}') \\ \text{when } \tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}, \text{ all entries of } k\text{th column of } C \text{ are non-negative.} \end{cases}$

Fact (sign-coherence)

For any column $\mathbb{c}$ of C in the above, entries of $\mathbb{c}$ are all non-negative or all non-positive.

§5. Main Theorem

Thm (G.)

For given a Cartan matrix C and a Coxeter element,

we define $B(C,c) := \begin{cases} 0 & \text{if } i=j \\ C_{ij} & \text{if } c = \cdots s_j \cdots s_i \cdots \\ -C_{ij} & \text{if } c = \cdots s_i \cdots s_j \cdots \end{cases}$

Then, we have quiver isomorphisms

$$\vec{\Gamma}(B(C,c))^{op} \simeq \vec{\Gamma}(\overline{\Phi}, c) \simeq \vec{\Gamma}(s\tau\text{-tilt}\, H(C,D,c))$$

$$\begin{tikzcd}[column sep=small]
 & \{-\alpha_1, -\alpha_2\} & \\
\{-\alpha_1, \alpha_2\} \arrow[ur] & & \{\alpha_1, -\alpha_2\} \arrow[ul] \\
\{\alpha_1+\alpha_2, \alpha_2\} \arrow[u] & & \\
 & \{\alpha_1, \alpha_1+\alpha_2\} \arrow[ul] \arrow[uur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & (0, 1\oplus \substack{2\\1}) & \\
(2,1) \arrow[ur] & & \\
 & & (1, \substack{2\\1}) \arrow[uul] \\
(2\oplus \substack{2\\1}, 0) \arrow[uu] & & \\
 & (1\oplus \substack{2\\1}, 0) \arrow[ul] \arrow[uur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & (x_1, x_2) \arrow[dl] \arrow[ddr] & \\
(x_1, \frac{x_1+1}{x_2}) \arrow[d] & & \\
(\frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2}) \arrow[dr] & & (\frac{x_2+1}{x_1}, x_2) \arrow[dl] \\
 & (\frac{x_1+x_2+1}{x_1x_2}, \frac{x_2+1}{x_1}) &
\end{tikzcd}$$

$\vec{\Gamma}(\Phi, c)$		$\vec{\Gamma}(\text{st-tilt}\, H)$		$\vec{\Gamma}(B, (C, c))^{op}$
$-\alpha_i$	$\longleftrightarrow$	$(0, P(i))$	$\longleftrightarrow$	x_i
$\sum d_i \alpha_i$	$\longleftrightarrow$	$(M, 0)$ $\operatorname{rank} M = (d_i)_i$	$\longleftrightarrow$	$\dfrac{f(x_1, \dots, x_n)}{\prod x_i^{d_i}}$

The strategy of proof for main theorem

$$
\begin{tikzcd}
\Gamma(\mathcal{B}(C,C)^{op}) & \left(\mathfrak{X}, \begin{bmatrix} B \\ \boxed{C} \end{bmatrix}\right) \arrow[d, dash, "\mu_i"'] \arrow[r, leftrightarrow] & (M, P) \arrow[d, dash, "\substack{\text{Brick} \\ S}"] & \Gamma(s\tau\text{-Tilt}A) \\
& \left(\mathfrak{X}', \begin{bmatrix} B' \\ C' \end{bmatrix}\right) \arrow[r, leftrightarrow] & (M', P') &
\end{tikzcd}
$$

$$(M,P) \to (M',P') \Leftrightarrow \begin{cases} \mathbb{C}_{(M,P)\to(M',P)} = \underline{\dim}_K S \\ \mathbb{C}_{(M',P')\to(M,P)} = -\underline{\dim}_K S. \end{cases}$$

$\mathbb{C}_i$: the ith column of C in the above.

Lem.

We have $\mathbb{C}_i = -\mathbb{C}_{(M,P)\to(M',P')}$

$$\Gamma(B(C,C)) \quad \boxed{\begin{array}{c} X\cup\{x_i\} \\ \mu_i \big| \\ X\cup\{x_i'\}\end{array}} \begin{array}{c}\longleftrightarrow \\ \\ \\ \longleftrightarrow\end{array} \boxed{\begin{array}{c} C\cup\{\alpha\} \\ \big| \\ C\cup\{\beta\}\end{array}} \quad \Gamma(\Phi, C)$$

- We use the "interpretation of τ_c in terms of mutation," that is, we define "$\tau_c(\mathcal{X})$" in $\Gamma(B(C,C))$.

Lem.

$\vec{\Gamma}(B(C,C))^{op} \simeq \vec{\Gamma}(\Phi, C) \Leftrightarrow \tau_c^{-1}$ preserves the order of $\vec{\Gamma}(B(C,C))$ under a certain condition

$$\Gamma(\text{st-tilt}\,H) \ni (M, P) \longleftrightarrow C \cup \{d\} \in \Gamma(\Phi_c)$$

Lem.

We have "$\tau_c \doteqdot \tau$", where τ is AR-translation.

2-term silting

$$\begin{tikzcd}
P_1 \oplus P \arrow[d] & & \nu P_1 \oplus \nu P \arrow[d] \\
P_0 \arrow[d, leftrightarrow] \arrow[rr, "{\nu[-1]}"] & & \nu P_0 \arrow[d, leftrightarrow] \\
(M, P) \arrow[rr, leftrightarrow, "\tau_c"] & & (\tau M \oplus \nu P, M_{pr})
\end{tikzcd}$$

Lem.

$\vec{\Gamma}(\text{st-tilt}\,H) \simeq \vec{\Gamma}(\Phi_c) \Leftrightarrow \tau^{-1}$ preserves the order of $\vec{\Gamma}(\text{st-tilt}\,H)$ under a certain condition.

Application: Generalization of Ingalls-Thomas bijection

Thm (Ingalls-Thomas)

Φ: simply-laced root system, c: Coxeter element

There is a isomorphism between the torsion lattice of path algebra corresponding to (Φ, c) and the Cambrian lattice corresponding to (Φ, c).

Thm (G.)

The above theorem can be generalized to one between the GLS-path algebra and non-simply-laced Cambrian lattice.

Sketch of proof.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\vec{\Gamma}(\text{st-tilt}\, H) \arrow[r, leftrightarrow, "\sim", "\text{[Adachi-Iyama-Reiten]}"'] \arrow[d, leftrightarrow, "\wr", "\text{Main Theorem}"'] & \mathrm{Hasse}(\mathrm{tors}\, H) \\
\vec{\Gamma}(\Phi, c) \arrow[r, leftrightarrow, "\sim", "\text{[Reading-Speyer]}"'] & \mathrm{Hasse}(\mathrm{Camb}(C, c))
\end{tikzcd}$$

Thank you for listening!