Regular Positive Representations

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Advances in Cluster Algebra 2023

Overview

Algebraic set up:

- Quantum group $\mathcal{U}_q(\mathfrak{g})$, semisimple Lie type
- Quantum cluster algebra $\mathcal{O}_q(\mathcal{X}^{\mathbf{Q}})$ associated to initial quiver $\mathbf{Q}$
- Regularity: $\mathcal{U}_q(\mathfrak{g}) \longrightarrow \mathcal{O}_q(\mathcal{X}^{\mathbf{Q}})$

Self-adjoint conditions: $X_i^* = X_i$, $|q| = 1$:

- Polarization of cluster variables $\Longrightarrow$ representations of $\mathcal{O}_q(\mathcal{X}^{\mathbf{Q}})$ on some Hilbert space $\mathcal{H}$ by Heisenberg operators x_i, p_i.
- Induces positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ on $\mathcal{H}$ by positive self-adjoint operators.

Goal: Classify the regular positive representations obtained this way.

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Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Definition

$\mathcal{U}_q(\mathfrak{sl}_2)$= Hopf algebra $\langle \mathbf{E}, \mathbf{F}, \mathbf{K}^{\pm1}\rangle$ over $\mathbb{C}(q)$ such that

$$\mathbf{KE} = q^2\mathbf{EK}, \qquad \mathbf{KF} = q^{-2}\mathbf{FK}, \qquad [\mathbf{E},\mathbf{F}] = \frac{\mathbf{K}-\mathbf{K}^{-1}}{q-q^{-1}}.$$

Coproduct:

$$\Delta(\mathbf{E}) = 1\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{K}, \qquad \Delta(\mathbf{F}) = \mathbf{F}\otimes 1 + \mathbf{K}^{-1}\otimes\mathbf{F},$$
$$\Delta(\mathbf{K}) = \mathbf{K}\otimes\mathbf{K}.$$

(Also counit ε, antipode S.)

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$: $\quad (|q|=1,\ e^{\pi \mathbf{i} b^2},\ b^2\in(0,1)\setminus\mathbb{Q})$

$$\mathbf{E}^* = \mathbf{E}, \qquad \mathbf{F}^* = \mathbf{F}, \qquad \mathbf{K}^* = \mathbf{K}, \qquad q^* = q^{-1}.$$

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Definition of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Definition

$\mathcal{U}_q(\mathfrak{g})$= Hopf-algebra $\langle \mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm 1}\rangle_{i\in I}$ over $\mathbb{C}(q)$ such that

$$\mathbf{K}_i\mathbf{E}_j = q_i^{a_{ij}}\mathbf{E}_j\mathbf{K}_i, \qquad \mathbf{K}_i\mathbf{F}_j = q_i^{-a_{ij}}\mathbf{F}_j\mathbf{K}_i, \qquad [\mathbf{E}_i, \mathbf{F}_j] = \delta_{ij}\frac{\mathbf{K}_i - \mathbf{K}_i^{-1}}{q_i - q_i^{-1}}$$

+ Serre's relations. $\qquad (q_i = q^{d_i})$

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Definition of $\mathcal{D}_q(\mathfrak{g}_{\mathbb{R}})$

Definition

$\mathcal{D}_q(\mathfrak{g})$= *Drinfeld's Double* of $\mathcal{U}_q(\mathfrak{b})$: $\langle \mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm 1}, \mathbf{K}_i'^{\pm 1}\rangle_{i\in I}$

$$\mathbf{K}_i\mathbf{E}_j = q_i^{a_{ij}}\mathbf{E}_j\mathbf{K}_i, \qquad \mathbf{K}_i\mathbf{F}_j = q_i^{-a_{ij}}\mathbf{F}_j\mathbf{K}_i, \qquad [\mathbf{E}_i, \mathbf{F}_j] = \delta_{ij}\frac{\mathbf{K}_i - \mathbf{K}_i'}{q_i - q_i^{-1}}$$

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$$\mathcal{U}_q(\mathfrak{g}) = \mathcal{D}_q(\mathfrak{g})/\langle \mathbf{K}_i\mathbf{K}_i' = 1\rangle_{i\in I}.$$

$$\mathbf{e}_i := -\mathbf{i}(q_i - q_i^{-1})\mathbf{E}_i, \qquad \mathbf{f}_i := -\mathbf{i}(q_i - q_i^{-1})\mathbf{F}_i, \qquad i \in I.$$

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$2\sin \pi b^2 > 0$

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Example: $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

$e^{2\pi b p} f(x) = f(x-ib)$

$e^{\pi b(u-2p)} := q e^{\pi b u} e^{-2\pi b p}$

Theorem (*Faddeev, Teschner (1999)*)

Irreducible representations $\mathcal{P}_\lambda$ of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ parametrized by $\lambda \geq 0$: ($p = \frac{1}{2\pi \mathbf{i}}\frac{d}{du}$)

$$\pi_\lambda(\mathbf{e}) := e^{\pi b(u+2p+2\lambda)} + e^{\pi b(-u+2p-2\lambda)}$$

$$\pi_\lambda(\mathbf{f}) := e^{\pi b(u-2p)} + e^{\pi b(-u-2p)}$$

$$\pi_\lambda(\mathbf{K}) := e^{\pi b(2u+2\lambda)}$$

$u \quad p$

$C^\infty \; L^2(\mathbb{R})$

acting on $L^2(\mathbb{R})$ as (unbounded essentially) positive self-adjoint operators.

(Irreducible: the only operators *strongly commute* with them are the scalars.)

Corollary

The Casimir element

$$\mathbf{C} := \mathbf{fe} - q\mathbf{K} - q^{-1}\mathbf{K}$$

acts on $\mathcal{P}_\lambda$ by the scalar

$$\pi_\lambda(\mathbf{C}) = e^{2\pi b\lambda} + e^{-2\pi b\lambda} \geq 2.$$

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Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Research program started in [Frenkel-I. (2012)]

- Representations $\mathcal{P}_\lambda$ by positive operators on Hilbert space.
- ="Quantization of principal series representations".
- Generalization of Teschner's representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

Theorem (*I. (2012)*)

There exists a family of irreducible representations $\mathcal{P}_\lambda$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$:

- *Parametrized by $\lambda \in \mathbb{R}_{\geq 0}P^+ \simeq \mathbb{R}_{\geq 0}^{n=\mathrm{rank}\ \mathfrak{g}}$.*
- *Positivity: $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i\}$ are represented by positive, essentially self-adjoint (unbounded) operators on $L^2(\mathbb{R}^N)$, $N := \ell(w_0)$.*
- *$\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i$ are expressed in terms of Laurent polynomials of $\{e^{\pi b u_k}, e^{2\pi b p_k}\}_{k=1}^N$ where $[p_j, u_k] = \frac{\delta_{jk}}{2\pi \mathbf{i}}$.*
- *Transcendental relations (compatibility with modular double $b \longleftrightarrow b^{-1}$) (Langland's duality).*

Can be understood in terms of quantum cluster algebra!

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$q = e^{\pi i b^2}$
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Quantum Cluster Variety

Quantum Torus Algebra

"Quantization of cluster $\mathcal{X}$ variety" [Fock-Goncharov]

Definition

Seed $\mathbf{Q} = (Q, Q_0, B, D)$:

- Q = nodes (finite set),
- $Q_0 \subset Q$ = frozen nodes,
- $B = (\varepsilon_{ij})$ exchange matrix (skew-symmetrizable, $\frac{1}{2}\mathbb{Z}$-valued),
- $D = (d_i)$ multipliers (in $\mathbb{Q}$). $d := \min d_i$.

Let $W := DB$ such that $w_{ij} = d_i\varepsilon_{ij} = -w_{ji}$.

The quantum torus algebra $\mathcal{X}_q^{\mathbf{Q}}$ is generated by $\langle X_i \rangle_{i\in Q}$ over $\mathbb{C}[q^{\pm d}]$ with

$$X_iX_j = q^{-2w_{ij}}X_jX_i.$$

X_i = quantum cluster variables.

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- $D = (d_i)$ multipliers (in $\mathbb{Q}$). $d := \min d_i$.

Let $W := DB$ such that $w_{ij} = d_i \varepsilon_{ij} = -w_{ji}$.

The quantum torus algebra $\mathcal{X}_q^{\mathbf{Q}}$ is generated by $\langle X_i \rangle_{i \in Q}$ over $\mathbb{C}[q^{\pm d}]$ with

$$X_i X_j = q^{-2w_{ij}} X_j X_i.$$

X_i = quantum cluster variables.

Quantum Torus Algebra

"Quantization of cluster $\mathcal{X}$ variety" [Fock-Goncharov]

Definition

Seed $\mathbf{Q} = (Q, Q_0, B, D)$:

- $\Lambda_{\mathbf{Q}} = \mathbb{Z}$-Lattice with basis $\{e_i\}_{i\in Q}$,
- $(-,-)$ skew-symmetric form, $(e_i, e_j) := w_{ij}$.

The quantum torus algebra $\mathcal{X}_q^{\mathbf{Q}}$ is generated by $\langle X_\lambda\rangle_{\lambda\in\Lambda_{\mathbf{Q}}}$ over $\mathbb{C}[q^{\pm d}]$ with

$$X_{\lambda+\mu} = q^{(\lambda,\mu)} X_\lambda X_\mu.$$

We use notation

$$X_i := X_{e_i}$$

$$X_{i_1,i_2,\dots,i_k} := X_{e_{i_1}+e_{i_2}+\cdots+e_{i_k}}.$$

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$$X_{i_1, i_2, \ldots, i_k} := X_{e_{i_1} + e_{i_2} + \cdots + e_{i_k}}.$$

$$X_{1,2} = X_{e_1+e_2} := q X_1 X_2$$

Quiver

Cluster seed $\mathbf{Q} \rightsquigarrow$ quiver with adjacency matrix (c_{ij}) where

$$c_{ij} := \begin{cases} \varepsilon_{ij} d_i d_j^{-1} & \text{if } d_j > d_i, \\ \varepsilon_{ij} & \text{otherwise.} \end{cases}$$

An arrow

$$i \longrightarrow j$$

represents the algebraic relation

$$X_i X_j = \begin{cases} q_i^{-2} X_j X_i & \text{if } d_i \geq d_j, \\ q_j^{-2} X_j X_i & \text{if } d_j \geq d_i. \end{cases}$$

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Quiver

We also use **thickness** to indicate commutation relations w.r.t. q, but thickness is not part of the data of $\mathbf{Q}$.

For example,

$$\begin{tikzcd} \underset{d_1=1}{1} \arrow[r] & \underset{d_2=1}{2} \arrow[r, thick] & \underset{d_3=2}{3} \end{tikzcd}$$

represents the relations

$$X_1X_2 = q^{-2}X_2X_1$$
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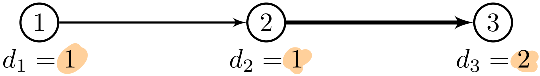

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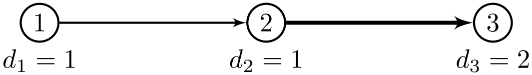

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Polarization of $\mathcal{X}_q^{\mathbf{Q}}$

Recall $q = e^{\pi \mathbf{i} b^2}$ such that $|q| = 1$.

Definition

A polarization of $\mathcal{X}_q^{\mathbf{Q}}$ is a choice of representation of the cluster variables $X_k \in \mathcal{X}_q^{\mathbf{Q}}$ of the form $X_k = e^{2\pi b x_k}$ such that

- x_k is a $\mathbb{Q}$-linear combination of $u_i, p_i = \frac{1}{2\pi\mathbf{i}}\frac{\partial}{\partial u_i}$ and scalars $\lambda_i \in \mathbb{R}$.
- x_k satisfies the Heisenberg algebra relations

$$[x_j, x_k] = \frac{1}{2\pi\mathbf{i}}\varepsilon_{jk},$$

X_k acts on some Hilbert space $\mathcal{H}_{\mathbf{Q}} \simeq L^2(\mathbb{R}^M, du_i)$ as positive operators.

Remark

Modular double $\widetilde{X}_k := X^{\frac{1}{b^2}}$ acts by $e^{2\pi b^{-1} x_k}$ on $\mathcal{H}_{\mathbf{Q}}$.

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- The center of $\mathcal{X}_q^{\mathbf{Q}}$ is generated by $|Q| - \mathrm{rank}(B)$ central monomials.
- A polarization is irreducible if every central monomial acts by a scalar.

Proposition

Any irreducible polarization with the same action of the central monomials are unitarily equivalent.

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Quantum Cluster Mutations

$\mathbf{T}_q^{\mathbf{Q}}$:= (non-commutative) field of fractions of $\mathcal{X}_q^{\mathbf{Q}}$.
Usual cluster mutation μ_k induces $\mu_k^q : \mathbf{T}_q^{\mathbf{Q}'} \longrightarrow \mathbf{T}_q^{\mathbf{Q}}$:

$$\mu_k^q(X_i') := \begin{cases} X_k^{-1} & i = k, \\ X_i \prod_{r=1}^{|\varepsilon_{ki}|}(1+q_i^{2r-1}X_k) & i \neq k, \varepsilon_{ki} < 0, \\ X_i \prod_{r=1}^{\varepsilon_{ki}}(1+q_i^{2r-1}X_k^{-1})^{-1} & i \neq k, \varepsilon_{ki} > 0. \end{cases}$$

Can be rewritten as

$$\mu_k^q = \mu_k^{\#} \circ \mu_k^m,$$

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Definition

$\mathcal{O}_q(\mathcal{X})$ denotes the quantum upper cluster algebra of the cluster variety $\mathcal{X}$.

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Quantum Cluster Variety

S=Riemann surface with marked points on ∂S and punctures.

Fock-Goncharov's $\mathcal{X}_{G,S}$-space= "(framed) local G-system"

- $\mathcal{X}_{G,S}$ has Poisson cluster $\mathcal{X}$ variety structure $\rightsquigarrow$ quantization $\mathcal{X}^q_{G,S}$
- To each triangle of ideal triangulation of S, assign a basic quiver.
- $G = PGL_{n+1}$: "n-triangulation"

(Full generality: [I. (2016)], [Le (2016)], [Goncharov-Shen (2019)])

For any longest word $w_0 \in W \rightsquigarrow \mathbf{Q} \rightsquigarrow \mathcal{X}_q^{\mathbf{Q}}$.

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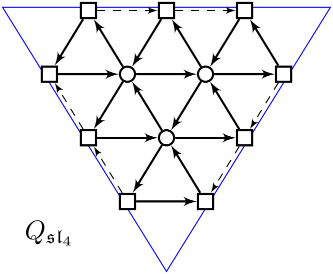

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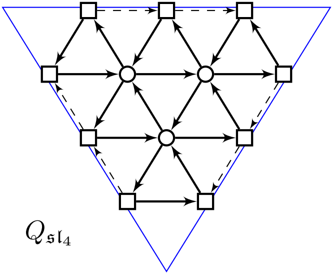

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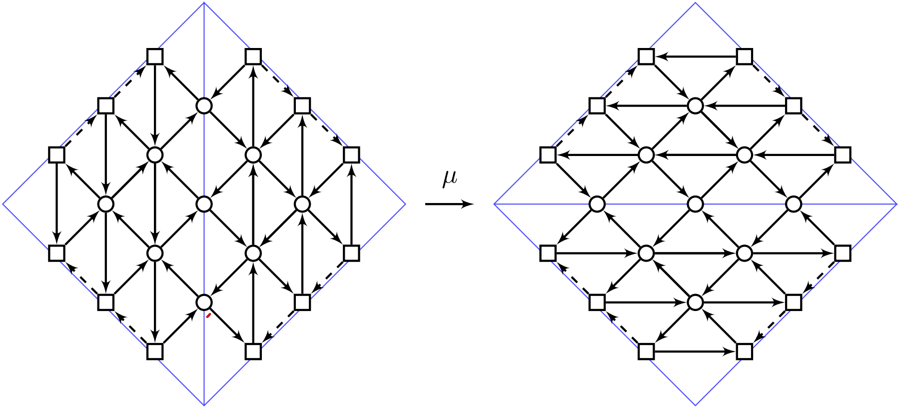
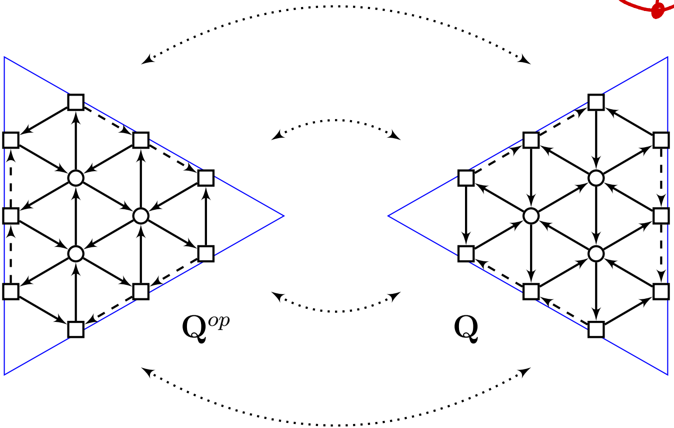
Quantum Cluster Variety

We can flip the triangulation with $\binom{n+2}{3}$ quiver mutations.

[Fock-Goncharov]



Example: Type A_n Case



Amalgamation of 2 quivers

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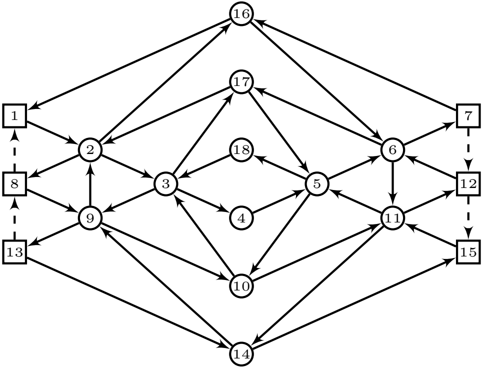

$\mathbf{D}_{\mathfrak{sl}_{n+1}}$-quiver $\rightsquigarrow \mathcal{X}_q^{std}$ [Schrader-Shapiro]

$$\iota : \mathcal{D}_q(\mathfrak{sl}_{n+1}) \hookrightarrow \mathcal{X}_q^{std}$$

$$\mathcal{U}_q(\mathfrak{sl}_{n+1}) \hookrightarrow \mathcal{X}_q^{std} / \langle \iota(\mathbf{K}_i \mathbf{K}_i') = 1 \rangle$$

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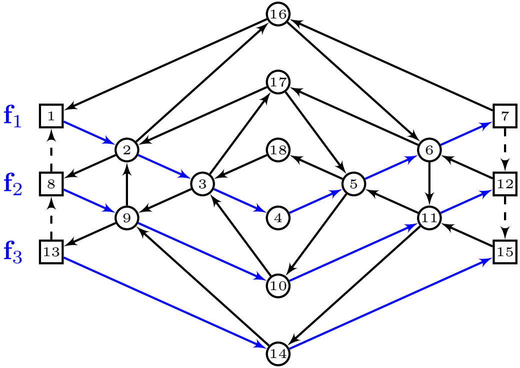

Embedding of $F_i \in \mathfrak{D}_{\mathfrak{sl}_4} \hookrightarrow \mathcal{X}_q^{std}$

$q X_1 X_2$ $\quad q^2 X_1 X_2 X_3$

$$\mathbf{f}_1 = X_1 + X_{1,2} + X_{1,2,3} + X_{1,2,3,4} + X_{1,2,3,4,5} + X_{1,2,3,4,5,6}$$

$$\mathbf{f}_2 = X_8 + X_{8,9} + X_{8,9,10} + X_{8,9,10,11}$$

$$\mathbf{f}_3 = X_{13} + X_{13,14}$$

$$\mathbf{K}'_1 = X_{1,2,3,4,5,6,7} \qquad \mathbf{K}'_2 = X_{8,9,10,11,12} \qquad \mathbf{K}'_3 = X_{13,14,15}$$

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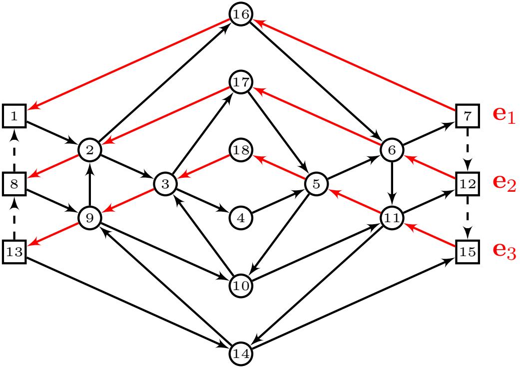


Embedding of $E_i \in \mathfrak{D}_{\mathfrak{sl}_4} \hookrightarrow \mathcal{X}_q^{std}$

$$\mathbf{e}_1 = X_7 + X_{7,16}$$

$$\mathbf{e}_2 = X_{12} + X_{12,6} + X_{12,6,17} + X_{12,6,17,2}$$

$$\mathbf{e}_3 = X_{15} + X_{15,11} + X_{15,11,5} + X_{15,11,5,18} + X_{15,11,5,18,3} + X_{15,11,5,18,3,9}$$

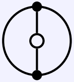
$$\mathbf{K}_1 = X_{7,16,1} \qquad \mathbf{K}_2 = X_{12,6,17,2,8} \qquad \mathbf{K}_3 = X_{15,11,5,18,3,9,13}$$

Positive Representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$

$\forall$ type $\mathfrak{g}$.

Theorem (*Schrader-Shapiro, I. (2016)*)

Given a longest reduced word $\mathbf{i}_0$

- *There exists a quiver* $\mathbf{D}(\mathbf{i}_0)$ *associated to*  *and an embedding*

$$\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{std}.$$

- *We recover the positive representations* $\mathcal{P}_\lambda$ *through a polarization of* $\mathcal{X}_q^{std}$.

Theorem (*I. (2016), Goncharov-Shen (2019)*)

- $\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}'_i$ *are universally Laurent polynomials in* $\mathcal{O}_q(\mathcal{X}^{std})$.
- $\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{std}$ *can be described by paths on the quiver* $\mathbf{D}(\mathbf{i}_0)$.
- $\mathbf{f}_i$ *paths are simple* $\rightsquigarrow$ *Feigin's homomorphism* $\mathcal{U}_q^-(\mathfrak{g}) \hookrightarrow \mathcal{X}_q$.

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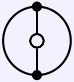
- *There exists a quiver* $\mathbf{D}(\mathbf{i}_0)$ *associated to*  *and an embedding*

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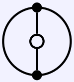
- *There exists a quiver* $\mathbf{D}(\mathbf{i}_0)$ *associated to*  *and an embedding*

$$\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{std}.$$

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Theorem (*I. (2016), Goncharov-Shen (2019)*)

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Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

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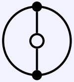
- *There exists a quiver* $\mathbf{D}(\mathbf{i}_0)$ *associated to*  *and an embedding*

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$\mathcal{U}_q(\mathfrak{b}^-)$

Regular positive representations

Definition

A representation $\mathcal{P}$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ is called a *regular positive representation* if $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ acts by a polarization of some quantum upper cluster algebra $\mathcal{O}_q(\mathcal{X})$.

In other words,

- There exists a homomorphism $\iota : \mathcal{D}_q(\mathfrak{g}) \longrightarrow \mathcal{O}_q(\mathcal{X})$.
- i.e. The image of $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}'_i\}$ are universally Laurent. ($\Longleftrightarrow$?)The image of $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}'_i\}$ are universally polynomial. [I.-Ye 2023]
- The representation $\mathcal{P}$ is obtained via an irreducible polarization of $\mathcal{O}_q(\mathcal{X})$ (where the central monomials $\iota(\mathbf{K}_i\mathbf{K}'_i)$ act as 1.)

Main Question: Classify all irreducible regular positive representations!

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Parabolic Positive Representations

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Parabolic induction $\longleftrightarrow$ truncating $\mathbf{i}_J \subset \mathbf{i}_0$ where $\mathbf{i}_J, \mathbf{i}_0$ are the longest word of the Weyl groups $W_J \subset W$.

$$w_0 = w_J \overline{w}$$

$$\overline{w} \longleftrightarrow \bar{\mathbf{i}}$$

Example

$$W_{\mathfrak{sl}_3} \subset W_{\mathfrak{sl}_4}$$

$$\mathbf{i}_0 = (1, 2, 1, 3, 2, 1)$$

Observe that

$$\mathbf{Q}(\mathbf{i}_0) = \mathbf{Q}(\mathbf{i}_J) * \mathbf{Q}(\bar{\mathbf{i}}).$$

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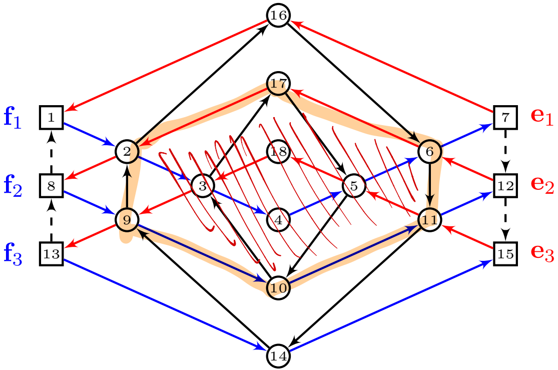
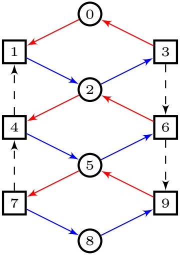
Example: Type A_3 Case



Embedding of $\mathfrak{D}_{\mathfrak{sl}_4} \hookrightarrow \mathcal{X}_q^{std}$

Parabolic Positive Representations



$$\mathbf{D}(\bar{\mathbf{i}}) := \mathbf{Q}(\bar{\mathbf{i}}^{op}) * \mathbf{Q}(\bar{\mathbf{i}}), \qquad \bar{\mathbf{i}} = (3,2,1)$$

$$\mathbf{e}_1 = X_3 + X_{3,0} \qquad \mathbf{K}_1 = X_{3,0,1}$$

$$\mathbf{e}_2 = X_6 + X_{6,2} \qquad \mathbf{K}_2 = X_{6,2,4}$$

$$\mathbf{e}_3 = X_9 + X_{9,5} \qquad \mathbf{K}_3 = X_{9,5,7}$$

$$\mathbf{f}_1 = X_1 + X_{1,2} \qquad \mathbf{K}'_1 = X_{1,2,3}$$

$$\mathbf{f}_2 = X_4 + X_{4,5} \qquad \mathbf{K}'_2 = X_{4,5,6}$$

$$\mathbf{f}_3 = X_7 + X_{7,8} \qquad \mathbf{K}'_3 = X_{7,8,9}$$

$D_1(g) \hookrightarrow O_q(X)$

Parabolic Positive Representation

Parabolic positive representation is regular.

Theorem (*I. (2020), I.-Ye (2023)*)

- *There is a homomorphism*

$$\mathcal{D}_q(\mathfrak{g}) \longrightarrow \mathcal{O}_q(\mathcal{X}^{\mathbf{D}(\bar{\mathbf{i}})})$$

such that the image of $\langle \mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}'_i \rangle$ *are universally polynomials.*

- *A polarization of* $\mathcal{O}_q(\mathcal{X}^{\mathbf{D}(\bar{\mathbf{i}})})$ *induces a family of irreducible representations* $\mathcal{P}^J_\lambda$ *of* $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ *parametrized by* $\lambda \in \mathbb{R}^{|I \setminus J|}$ *as positive self-adjoint operators on* $L^2(\mathbb{R}^{l(\overline{w})})$.

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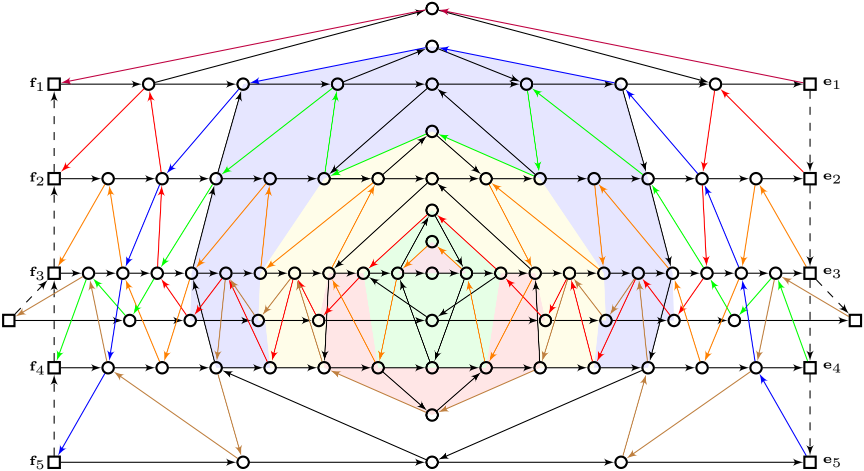
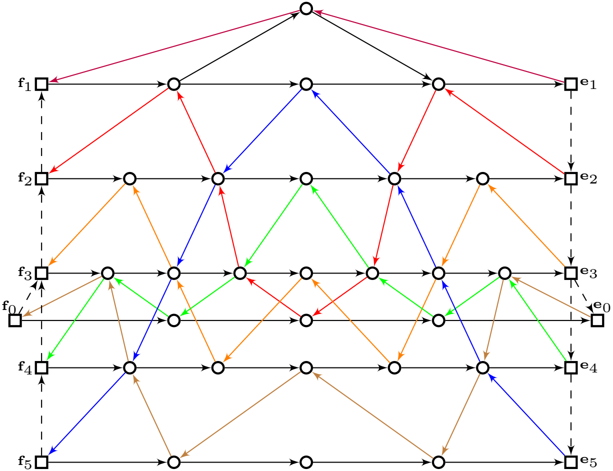
Example: E_6



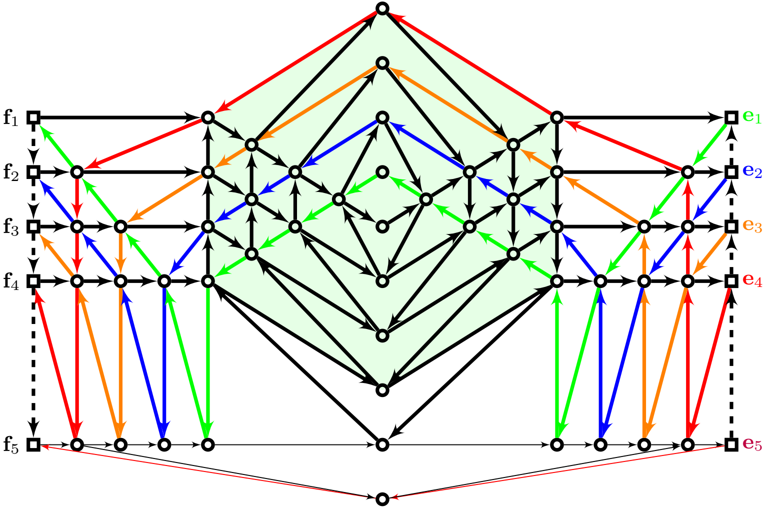
$$A_1 \subset A_2 \subset A_3 \subset D_4 \subset D_5 \subset E_6$$

Example: E_6



$D_5 \subset E_6$

Example: B_5



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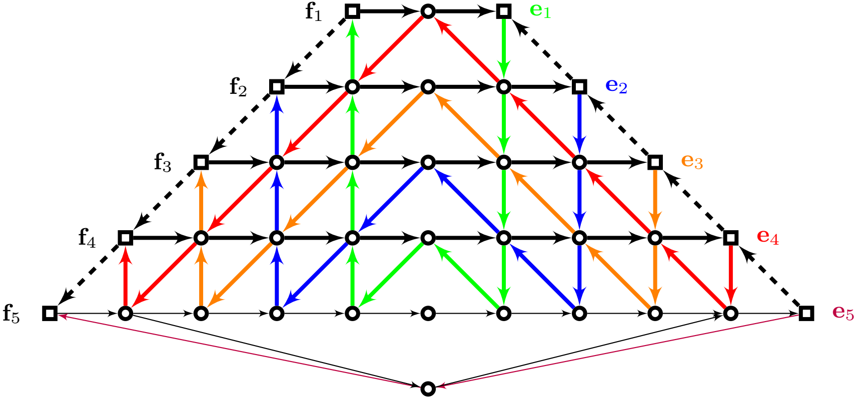

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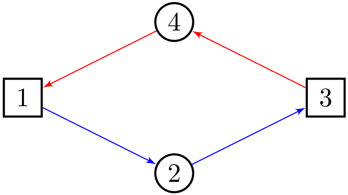
Back to $\mathcal{U}_q(\mathfrak{sl}_2)$

Cluster Realization of $\mathcal{U}_q(\mathfrak{sl}_2)$



We have $\iota : \mathcal{D}_q(\mathfrak{sl}_2) \hookrightarrow \mathcal{X}_q^{std}$:

$$\begin{aligned} \mathbf{e} &\mapsto X_3 + X_{3,4} \\ \mathbf{f} &\mapsto X_1 + X_{1,2} \\ \mathbf{K} &\mapsto X_{3,4,1} \\ \mathbf{K}' &\mapsto X_{1,2,3} \end{aligned}$$

exactly Teschner's rep

- The Casimir is given by $\mathbf{C} \mapsto X_{1,3} + X_{1,2,3,4}$.
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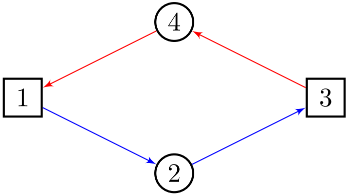

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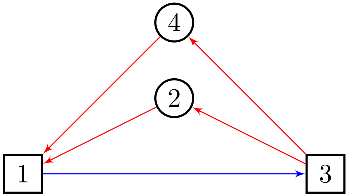
Cluster Realization of $\mathcal{U}_q(\mathfrak{sl}_2)$

Do a mutation at vertex 2:



The vertex 2 and 4 become *symmetric* in $\mathcal{X}_q^{sym}$.

$$\begin{aligned}
\mathbf{e} &\mapsto X_3 + X_{3,2} + X_{3,4} + X_{3,2,4} \\
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\end{aligned}$$

and the Casimir element is given by

$$\begin{aligned}
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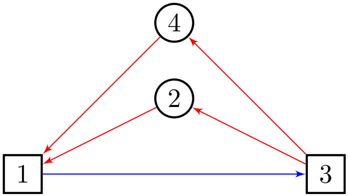

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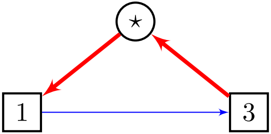
Cluster Realization of $\mathcal{U}_q(\mathfrak{sl}_2)$

Setting $\mathbf{C} = 0 \rightsquigarrow$ folding, where we identify the vertex 2 and 4:

$$X_\star := X_{2,4}$$

and obtain a new *skew-symmetrizable* $\mathcal{X}_q^0$ with $d_\star := 2$.



$$(d_1, d_\star, d_3) = (1, 2, 1)$$

and a homomorphism $\mathcal{D}_q(\mathfrak{sl}_2) \longrightarrow \mathcal{X}_q^0$:

$$\begin{aligned}
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We obtain the degenerate representation $\mathcal{P}^0$ of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ by polarization!

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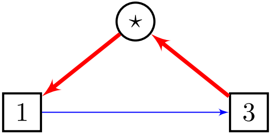

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Positive Representations at Zero Casimir

Casimir Operators

Theorem

The center of $\dot{\mathcal{U}}_q(\mathfrak{g}) := \mathcal{U}_q(\mathfrak{g})[\mathbf{K}_i^{\frac{1}{h}}]$ *is spanned by* $\mathrm{rank}\,\mathfrak{g} = n$ *generalized Casimirs*

$$\mathbf{C}_k := (1 \otimes \mathrm{Tr}|^q_{V_k})(RR_{21}).$$

- $R \in \dot{\mathcal{U}}_q(\mathfrak{g})\widehat{\otimes}\dot{\mathcal{U}}_q(\mathfrak{g})$ = *universal* R *matrix.*
- $\{V_k\}_{k=1}^n$ = *fundamental representations of* $\mathcal{U}_q(\mathfrak{g})$.
- *The quantum trace* $\mathrm{Tr}|^q_V$ *is given by*

$$\mathrm{Tr}|^q_V(x) := \mathrm{Tr}|_V(xu^{-1}), \qquad x \in \dot{\mathcal{U}}_q(\mathfrak{g})$$

where $u \in \dot{\mathcal{U}}_q(\mathfrak{g})$ *invertible such that* $\mathrm{Ad}(u) = S^2$.

To ensure positivity of $\mathbf{C}_k$ in $\mathcal{P}_\lambda$, we choose [I. (2016)]

$$u := \mathbf{K}_{2\rho}\widetilde{\mathbf{K}}_{2\rho} \in \dot{\mathcal{U}}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

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Casimir Operators

Example

In $\mathcal{U}_q(\mathfrak{sl}_2)$, the Casimir is

$$\mathbf{C} = \mathbf{fe} - q\mathbf{K} - q^{-1}\mathbf{K}^{-1}.$$

Example

In $\mathcal{U}_q(\mathfrak{sl}_3)$, the two generalized Casimirs are

$$\begin{aligned}\mathbf{C}_1 =& \mathbf{K}(q^{-2}\mathbf{K}_1\mathbf{K}_2 + \mathbf{K}_1^{-1}\mathbf{K}_2 + q^2\mathbf{K}_1^{-1}\mathbf{K}_2^{-1} - q^{-1}\mathbf{K}_2\mathbf{e}_1\mathbf{f}_1 - q\mathbf{K}_1^{-1}\mathbf{e}_2\mathbf{f}_2 + \mathbf{e}_{21}\mathbf{f}_{12}),\\ \mathbf{C}_2 =& \mathbf{K}^{-1}(q^2\mathbf{K}_1^{-1}\mathbf{K}_2^{-1} + \mathbf{K}_1\mathbf{K}_2^{-1} + q^{-2}\mathbf{K}_1\mathbf{K}_2 - q\mathbf{K}_2^{-1}\mathbf{e}_1\mathbf{f}_1 - q^{-1}\mathbf{K}_1\mathbf{e}_2\mathbf{f}_2 + \mathbf{e}_{12}\mathbf{f}_{21}),\end{aligned}$$

where $\mathbf{K} = \mathbf{K}_1^{\frac{1}{3}}\mathbf{K}_2^{-\frac{1}{3}}$, and

$$\mathbf{e}_{ij} := \frac{q^{\frac{1}{2}}\mathbf{e}_j\mathbf{e}_i - q^{-\frac{1}{2}}\mathbf{e}_i\mathbf{e}_j}{q-q^{-1}}, \qquad \mathbf{f}_{ij} := \frac{q^{\frac{1}{2}}\mathbf{f}_j\mathbf{f}_i - q^{-\frac{1}{2}}\mathbf{f}_i\mathbf{f}_j}{q-q^{-1}}$$

are obtained by the *symmetrized* Lusztig's transformations.

Casimir Operators

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are obtained by the *symmetrized* Lusztig's transformations.

Casimir Operators

Theorem (*I. (2016)*)

The generalized Casimirs $\mathbf{C}_k$ *act on* $\mathcal{P}_\lambda$ *by the scalars*

$$\pi_\lambda(\mathbf{C}_k) = \sum_{\mathcal{V}\subset V_k} \exp\left(-4\pi\mu_{\mathcal{V}}(\overrightarrow{\lambda_{\mathfrak{h}}})\right),$$

over all weight subspaces $\mathcal{V} \subset V_k$ *with weight* $\mu_{\mathcal{V}} \in \mathfrak{h}_{\mathbb{R}}^*$, *and*

$$\overrightarrow{\lambda_{\mathfrak{h}}} = \sum_{i\in I} \lambda_i b_i W_i \in \mathfrak{h}_{\mathbb{R}}$$

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Proposition

In type A_n, $V_k=\Lambda^k V_1$, and the actions of $\mathbf{C}_k$ on $\mathcal{P}_\lambda$ are given by

$$\pi_\lambda(\mathbf{C}_k)=\mathcal{E}_k\left(e^{4\pi b\varpi_0},...,e^{4\pi b\varpi_n}\right).$$

where $\mathcal{E}_k$ are the elementary symmetric polynomials in $n+1$ variables, and

$$\varpi_i:=\frac{1}{n+1}\sum_{k=1}^n k\lambda_k-\sum_{j=n+1-i}^n \lambda_j,\qquad i=0,...,n.$$

Example

In $\mathcal{U}_q(\mathfrak{sl}_3)$, (type A_2) we have

$$\pi_\lambda(\mathbf{C}_1)=e^{\frac{4}{3}\pi b\lambda_1+\frac{8}{3}\pi\lambda_2}+e^{\frac{4}{3}\pi b\lambda_1-\frac{4}{3}\pi\lambda_2}+e^{-\frac{8}{3}\pi b\lambda_1-\frac{4}{3}\pi\lambda_2},$$

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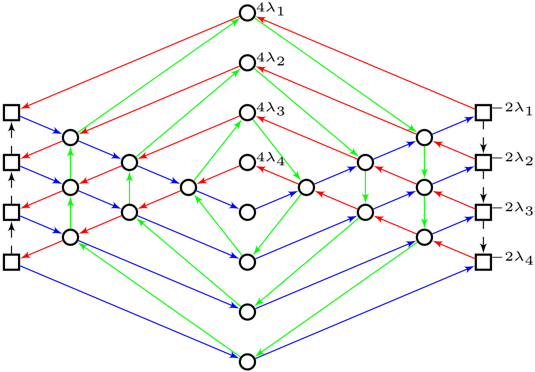
Symmetric Folding

We start with the standard quiver $\mathbf{D}(\mathbf{i}_0)$ for $\mathbf{i}_0 = (1\ 21\ 321\ 4321 \cdots n\ \cdots\ 1)$.



- The center of $\mathcal{X}_q^{std}$ is generated by $\iota(\mathbf{K}_i\mathbf{K}_i')$ and the monodromies Q_i around the puncture.
- The polarization is given by $\pi_\lambda(\mathbf{K}_i\mathbf{K}_i') = 1$ and $\pi_\lambda(Q_i) = e^{4\pi b\lambda_i}$.

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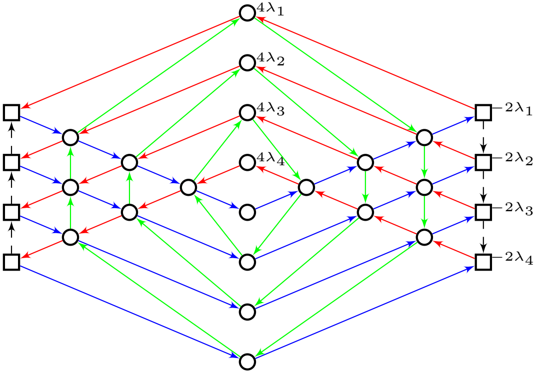

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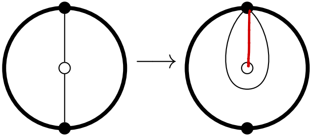
We perform the flip to obtain a self-folded triangulation



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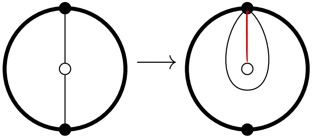

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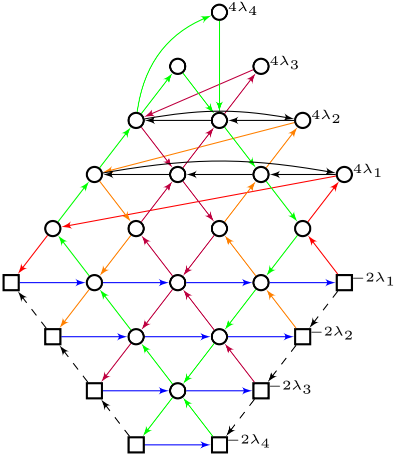

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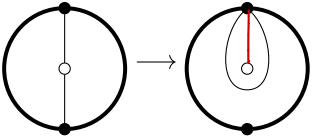

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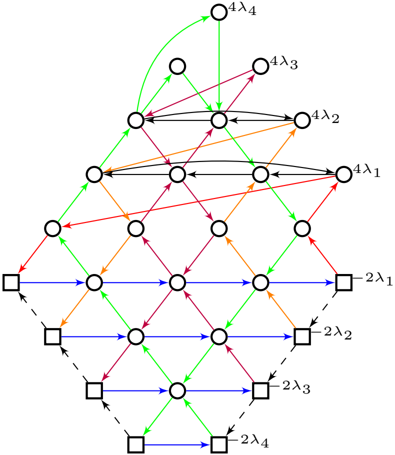

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The central monodromies become

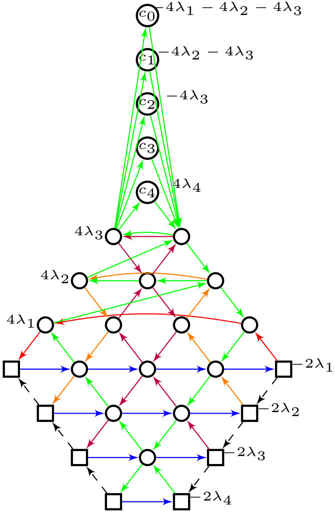
$$Q_i := X_{c_i} X_{c_{i-1}}^{-1}, \qquad \pi_\lambda(Q_i) = e^{4\pi b \lambda_i}$$

The embedding of the generators $\mathbf{f}_1, ..., \mathbf{f}_n$ as well as $\mathbf{e}_1, ..., \mathbf{e}_{n-1}$ does not involve the indices c_i.

However, $\mathbf{e}_n$ is of the form

$$\iota(\mathbf{e}_n) = \sum A_k B_k$$

where $B_k = \mathcal{E}_k(X_{c_0}, ..., X_{c_n})$ which is proportional to $\iota(\mathbf{C}_k)$!



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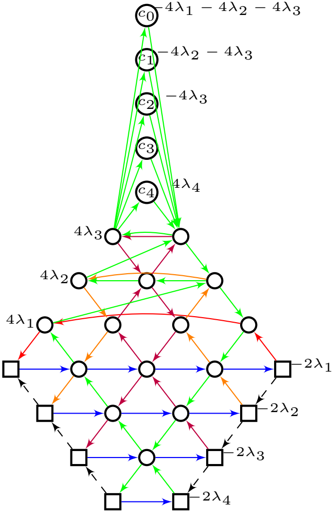

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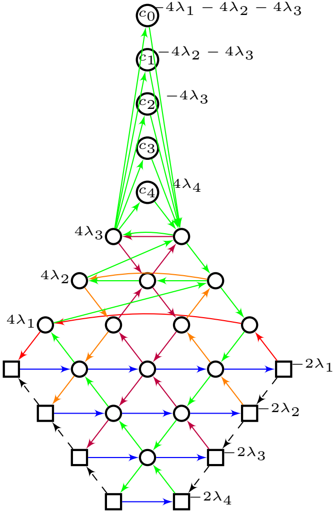

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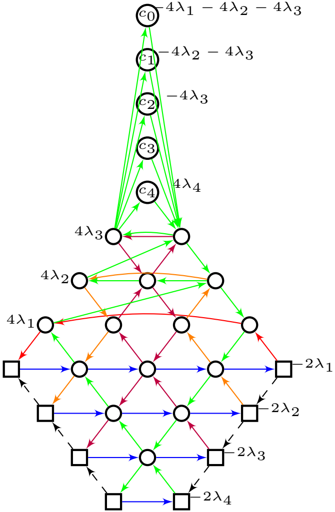

Symmetric Folding

We can now perform the folding of the symmetric variables to get $\mathcal{X}_q^0$, where

$$X_\star := \prod_{k=0}^{n} X_{c_k}$$

and assign $d_\star := n+1$.

- The embeddings of $\mathbf{f}_1, ..., \mathbf{f}_n$ and $\mathbf{e}_1, ..., \mathbf{e}_{n-1}$ are unaffected.
- $\iota(\mathbf{e}_n)$ is modified such that

$$\begin{cases} B_k \mapsto 0 & k = 1, ..., n, \\ B_{n+1} \mapsto X_\star. \end{cases}$$

By construction $\mathbf{C}_k = 0$ in $\mathcal{X}_q^0$.

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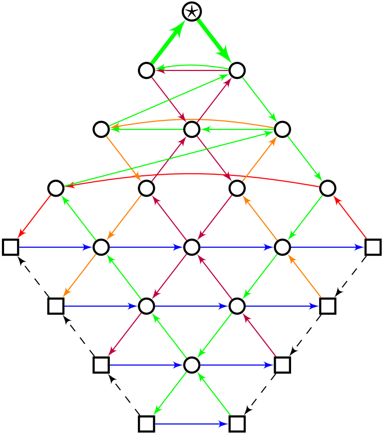

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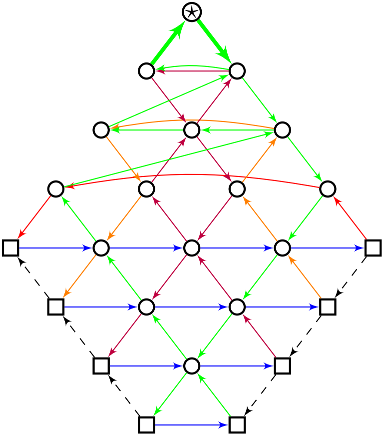

Main Theorem: Type A_n

Theorem (*I.-Man (2022), I.-Ye (2023)*)

There is an embedding

$$\mathcal{D}_q(\mathfrak{sl}_{n+1})/\langle \mathbf{C}_k = 0\rangle \hookrightarrow \mathcal{O}_q(X^0)$$

to a skew-symmetrizable quantum cluster algebra such that

- *The image of* $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}_i'\}$ *are universally polynomials.*
- *We have an irreducible representation* $\mathcal{P}^0$ *of* $\mathcal{U}_q(\mathfrak{sl}(n+1,\mathbb{R}))$ *acting on* $L^2(\mathbb{R}^N)$ *as positive operators, such that* $\pi(\mathbf{C}_k) = 0$.

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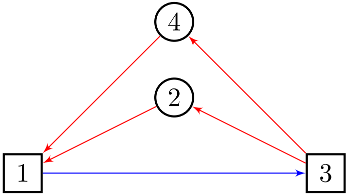
Degenerate Modular Double

Recall

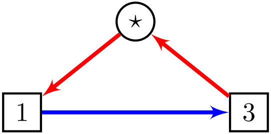


The vertex 2 and 4 are *symmetric* in $\mathcal{X}_q^{sym}$.

$$\begin{aligned}
\mathbf{e} &\mapsto X_3 + X_1^{-1}\mathbf{C} + X_{3,2,4} \\
\mathbf{f} &\mapsto X_1 \\
\mathbf{K} &\mapsto X_{3,2,4,1} \\
\mathbf{K}' &\mapsto X_{1,3}
\end{aligned}$$

Degenerate Modular Double

By reversing the folding, we obtain a new *skew-symmetrizable* $\widetilde{\mathcal{X}}_q^0$



$$(d_1, d_\star, d_3) = (1, \tfrac{1}{2}, 1)$$

and a homomorphism $\mathcal{D}_q(\mathfrak{sl}_2) \longrightarrow \widetilde{\mathcal{X}}_q^0$:

$$\begin{aligned} \mathbf{e} &\mapsto X_3 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})X_{3,\star} + X_{3,\star^2} \\ \mathbf{f} &\mapsto X_1 \\ \mathbf{K} &\mapsto X_{3,\star^2,1} \\ \mathbf{K}' &\mapsto X_{1,3} \end{aligned}$$

We obtain the modular double counterpart of the degenerate representation $\widetilde{\mathcal{P}}^0$ of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ by polarization!

Degenerate representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$

Degenerate Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

- Recall a parabolic subgroup $\longleftrightarrow J \subset I$ provides a decomposition of $\mathcal{X}_q^{std}$.

$$\mathbf{i}_0 = \mathbf{i}_J \mathbf{i}'.$$

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- The number of central parameters are reduced by $|J| = k_1 + \cdots + k_m$.

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For each parabolic subgroup $W_J \subset W$ of type $A_{k_1} \times \cdots \times A_{k_m}$:

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$$\mathcal{D}_q(\mathfrak{g}) \longrightarrow \mathcal{O}_q(\mathcal{X}^0)$$

  to some skew-symmetrizable quantum cluster algebra.
- $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i, \mathbf{K}'_i\}$ *are universally polynomials.*
- *We have a family of irreducible regular positive representations $\mathcal{P}_\lambda^{0,J}$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ parametrized by $\lambda \in \mathbb{R}^{n-|J|}$.*

- The same results hold for the modular double counterpart with $\widetilde{\mathcal{X}}_q^0$, where the multipliers $d_\star$ of any folded vertices are inverted.

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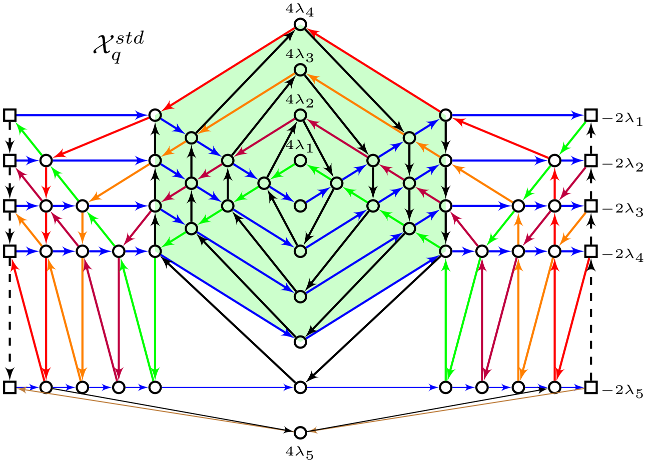
Maximal Degenerate Representations

$\mathbf{i}_0 = \mathbf{i}_{A_4}\mathbf{i}' = (1213214321)(545345234512345)$



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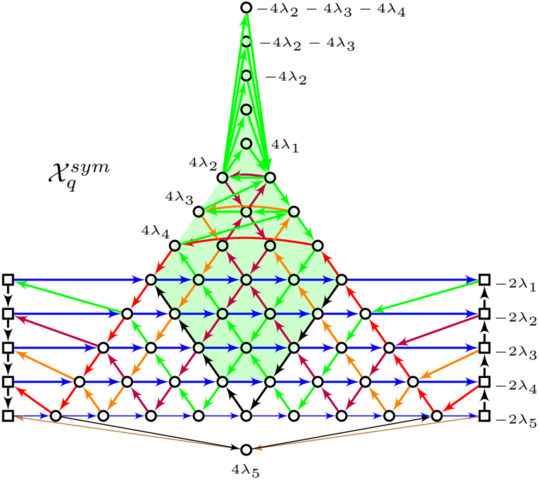

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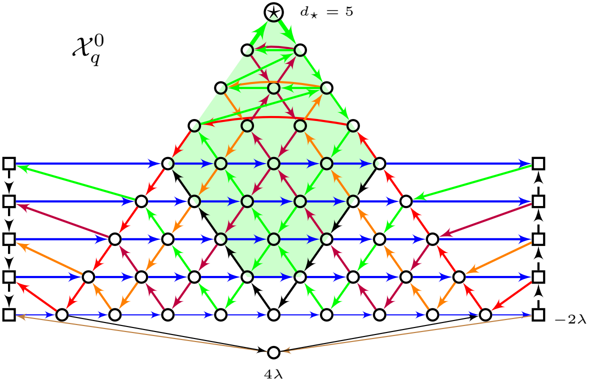

Polarization gives the degenerate representation $P_\lambda^{0,J}$ of $\mathcal{U}_q(\mathfrak{g}_{B_5})$.

Classification of Regular Positive Representations

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We have the following list of irreducible regular positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$:

(1) The standard positive representations $\mathcal{P}_\lambda$, parametrized by n positive scalars.

(2) The parabolic positive representations $\mathcal{P}_\lambda^J$, with respect to parabolic subgroup $W_J \subset W$ parametrized by $n-|J|$ positive scalars.

(3) The degenerate representations $\mathcal{P}_\lambda^{0,J}$ with respect to parabolic subgroup $W_J \subset W$ of type $A_{k_1} \times \cdots \times A_{k_m}$, parametrized by $n-|J|$ positive scalars.

(4) The modular double counterpart of the degenerate representations $\mathcal{P}_{\widetilde{\lambda}}^{0,J}$, also parametrized by $n-|J|$ positive scalars.

(5) A mixture of type (2)–(4) for disconnected subsets of Dynkin index.

Conjecture

The classes of all irreducible regular positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ are classified by the list (1)–(5) above.

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(3) The degenerate representations $\mathcal{P}_\lambda^{0,J}$ with respect to parabolic subgroup $W_J \subset W$ of type $A_{k_1} \times \cdots \times A_{k_m}$, parametrized by $n-|J|$ positive scalars.

(4) The modular double counterpart of the degenerate representations $\mathcal{P}_{\widetilde{\lambda}}^{0,J}$, also parametrized by $n-|J|$ positive scalars.

(5) A mixture of type (2)–(4) for disconnected subsets of Dynkin index.

Conjecture

The classes of all irreducible regular positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ are classified by the list (1)–(5) above.

Example: Type A_2

We illustrate the joint spectrum $(\pi_\lambda(\mathbf{C}_1), \pi_\lambda(\mathbf{C}_2))$ of the Casimirs of the irreducible regular positive representations of $\mathcal{U}_q(\mathfrak{sl}_3)$. We assume $\lambda \geq 0$ and q not a root of unity.

Representations	J	Action of Casimirs
$\mathcal{P}_\lambda$		$\pi_\lambda(\mathbf{C}_1) = e^{\frac{4}{3}\pi b\lambda_1+\frac{8}{3}\pi b\lambda_2} + e^{\frac{4}{3}\pi b\lambda_1-\frac{4}{3}\pi b\lambda_2} + e^{-\frac{8}{3}\pi b\lambda_1-\frac{4}{3}\pi b\lambda_2}$ $\pi_\lambda(\mathbf{C}_2) = e^{\frac{8}{3}\pi b\lambda_1+\frac{4}{3}\pi b\lambda_2} + e^{-\frac{4}{3}\pi b\lambda_1+\frac{4}{3}\pi b\lambda_2} + e^{-\frac{4}{3}\pi b\lambda_1-\frac{8}{3}\pi b\lambda_2}$
$\mathcal{P}_\lambda^J$	$\{1\}$	$\pi_\lambda(\mathbf{C}_1) = e^{-\frac{8}{3}\pi b\lambda} - (q+q^{-1})e^{\frac{4}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{\frac{8}{3}\pi b\lambda} - (q+q^{-1})e^{-\frac{4}{3}\pi b\lambda}$
	$\{2\}$	$\pi_\lambda(\mathbf{C}_1) = e^{\frac{8}{3}\pi b\lambda} - (q+q^{-1})e^{-\frac{4}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{-\frac{8}{3}\pi b\lambda} - (q+q^{-1})e^{\frac{4}{3}\pi b\lambda}$
$\mathcal{P}_\lambda^{0,J}$	$\{1,2\}$	$\pi_\lambda(\mathbf{C}_1) = 0$ $\pi_\lambda(\mathbf{C}_2) = 0$
	$\{1\}$	$\pi_\lambda(\mathbf{C}_1) = e^{\frac{8}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{-\frac{8}{3}\pi b\lambda}$
	$\{2\}$	$\pi_\lambda(\mathbf{C}_1) = e^{-\frac{8}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{\frac{8}{3}\pi b\lambda}$
$\mathcal{P}_{\widetilde{\lambda}}^{0,J}$	$\{1,2\}$	$\pi_\lambda(\mathbf{C}_1) = q^{\frac{2}{3}} + 1 + q^{-\frac{2}{3}}$ $\pi_\lambda(\mathbf{C}_2) = q^{\frac{2}{3}} + 1 + q^{-\frac{2}{3}}$
	$\{1\}$	$\pi_\lambda(\mathbf{C}_1) = e^{\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e^{-\frac{4}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{-\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e^{\frac{4}{3}\pi b\lambda}$
	$\{2\}$	$\pi_\lambda(\mathbf{C}_1) = e^{-\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e^{\frac{4}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e^{-\frac{4}{3}\pi b\lambda}$

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$\mathcal{P}_{\widetilde{\lambda}}^{0,J}$	$\{1,2\}$	$\pi_\lambda(\mathbf{C}_1) = q^{\frac{2}{3}} + 1 + q^{-\frac{2}{3}}$ $\pi_\lambda(\mathbf{C}_2) = q^{\frac{2}{3}} + 1 + q^{-\frac{2}{3}}$
	$\{1\}$	$\pi_\lambda(\mathbf{C}_1) = e^{\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}}+q^{-\frac{1}{2}})e^{-\frac{4}{3}\pi b\lambda}$ $\pi_\lambda(\mathbf{C}_2) = e^{-\frac{8}{3}\pi b\lambda} + (q^{\frac{1}{2}}+q^{-\frac{1}{2}})e^{\frac{4}{3}\pi b\lambda}$
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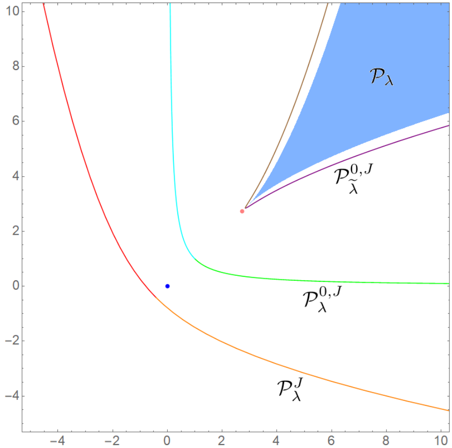
Example: Type A_2

A generic plot of $(x, y) = (\pi_\lambda(\mathbf{C}_1), \pi_\lambda(\mathbf{C}_2))$ with $b \sim 0.5$.



If q is a root of unity, these family of Casimir values can overlap!

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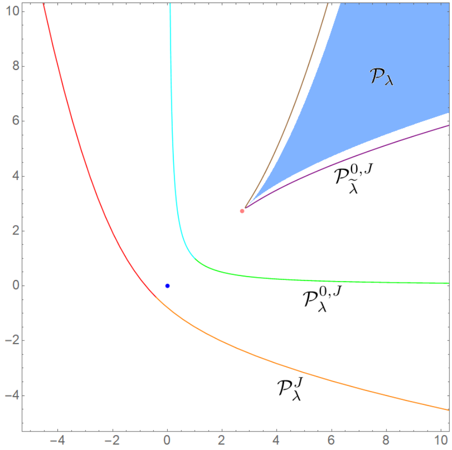

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Thank you for your attention!