Toric degenerations and Newton-Okounkov bodies of flag varieties arising from cluster structures.

Joint work with Hironori Oya

(arXiv: 2002.09912)

1. Intro

Toric theory

$$(Z(Q), \mathcal{L}(Q)) \longleftrightarrow Q \subseteq \mathbb{R}^N$$

$Z(Q)$: N-dim. irr. normal proj. toric var.

$\mathcal{L}(Q)$: torus-equiv. ample line bundle

Q: N-dim. integral polytope

Want

to apply toric theory to non-toric varieties.

$\longrightarrow$ Degenerations to toric vars. (toric degenerations) are useful

Aim

to study toric degen. using cluster str.

2. Double Bruhat cells

$$G = SL_{n+1}(\mathbb{C})$$

$$\cup \qquad\qquad\qquad \cup$$

$$B^+ := \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \qquad B^- := \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}\right\}$$

$$\cup \qquad\qquad \cup \qquad\qquad \cup \qquad\qquad \cup$$

$$U^+ = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\} \qquad H = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \qquad U^- = \left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\right\}$$

$\longrightarrow G/B^+ \cdots$ the full flag var.

$$= \bigsqcup_{w \in \mathfrak{S}_{n+1}} B^+ \overline{w} B^+ / B^+$$

perm. matrix up to signs

$$G^{w,e} := B^+ \overline{w} B^+ \cap B^- \quad \text{(double Bruhat cell)}$$
$$\cup$$
$$U_w^- := B^+ \overline{w} B^+ \cap U^- \quad \text{(unip. cell)}$$

__Thm__ (Berenstein-Fomin-Zelevinsky 2005)

$\mathbb{C}[G^{w,e}]$ and $\mathbb{C}[U_w^-]$ have upper cluster alg strs.
s.t. $\pi_w : \mathbb{C}[G^{w,e}] \xrightarrow[\text{restr.}]{} \mathbb{C}[U_w^-]$ is given by

some frozen variables $\longmapsto 0$

$s_i := (i\ i+1) \in \mathfrak{S}_{n+1} \quad (1 \leq i \leq n)$

__Def__ Let $w \in \mathfrak{S}_{n+1}$.
$w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is called a reduced expression
if m is minimum.
$\longrightarrow (i_1, i_2, \cdots, i_m)$ is called a reduced word

$R(w)$: the set of reduced words.

__Example__

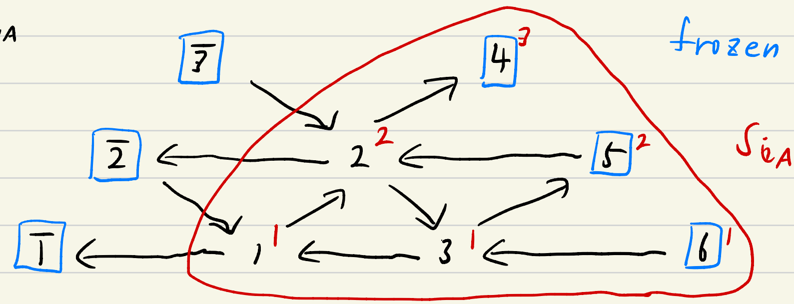
$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n+1 \\ n+1 & n & \cdots & 1 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{i}_A = (1, 2, 1, 3, 2, 1, \cdots, n, n-1, \cdots, 1) \in R(w_0).$$

Each $\boldsymbol{i} \in R(w)$ gives a seed $\tilde{s}_{\boldsymbol{i}}$ for $\mathbb{C}[G^{w,e}]$
and $s_{\boldsymbol{i}}$ for $\mathbb{C}[U_w^-]$.

Example ($G = SL_4(\mathbb{C})$, $\mathbf{i}_A = (1, 2, 1, 3, 2, 1)$)
1 2 3 4 5 6

$\widetilde{S}_{\mathbf{i}_A}$



$J = \{1, \dots, 6\}$, $\widetilde{J} = J \sqcup \{\overline{1}, \overline{2}, \overline{3}\}$,

$\widetilde{J}_{fr} = \{4, 5, 6, \overline{1}, \overline{2}, \overline{3}\}$

S ⋯ a seed for $\mathbb{C}[U^-_w]$

$\widetilde{S}$ ⋯ a seed for $\mathbb{C}[G^{w,e}]$ corr. to S.

3. Superpotential polytopes

alg. Peter-Weyl thm

$$\mathbb{C}[G] = \mathbb{C}[SL_{n+1}(\mathbb{C})] \simeq \bigoplus_{\lambda \in P_+ := \mathbb{Z}^n_{\geq 0}} V(\lambda)^* \otimes V(\lambda)$$

as $G \times G$-modules,

where $G \times G \curvearrowright \mathbb{C}[G]$ is given by

$$\begin{array}{ccc} G \times G & \curvearrowright & G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g_1, g_2) & & g \mapsto g_1 g g_2^{-1} \end{array}$$

and $\{V(\lambda) \mid \lambda \in P_+\}$ is the set of fin.-dim. irr. G-modules.

$$\rightsquigarrow \mathbb{C}[G/U^+] \simeq \mathbb{C}[G]^{\{e\}\times U^+} \simeq \bigoplus_{\lambda \in P_+} V(\lambda)^*$$

$G \times G \curvearrowright \mathbb{C}[G]$ induces $G \times H \curvearrowright \mathbb{C}[G/U^+]$

Let $\mathcal{A}^\vee = \bigcup_{\tilde{s}:\text{seeds}} \underbrace{\mathcal{A}^\vee_{\tilde{s}}}_{\substack{\shortparallel \\ (\mathbb{C}^\times)^{\tilde{J}} \ni (X_{j,\tilde{s}})_{j\in\tilde{J}}}}$ be the Fock-Goncharov dual of $G^{w_0,e}$.

(an X-cluster variety)

gluing map : $\mu_k : \mathcal{A}^\vee_{\tilde{s}} \dashrightarrow \mathcal{A}^\vee_{\mu_k(\tilde{s})}$,

$$X_{j,\mu_k(\tilde{s})} = \begin{cases} X_{j,\tilde{s}} X_{k,\tilde{s}}^{\max\{-\varepsilon_{k,j},0\}} (1+X_{k,\tilde{s}})^{\varepsilon_{kj}} & (j\neq k) \\ X_{j,\tilde{s}}^{-1} & (j=k) \end{cases}$$

($(\varepsilon_{i,l})_{i,l}$ ⋯ exchange matrix of $\tilde{s}$)

$$\rightsquigarrow \mathcal{A}^\vee(\mathbb{R}^T) = \bigcup_{\tilde{s}} \underbrace{\mathcal{A}^\vee_{\tilde{s}}(\mathbb{R}^T)}_{\substack{\shortparallel \\ \mathbb{R}^{\tilde{J}}}}$$

gluing map : μ_k^T

(obtained from μ_k by $\cdot \mapsto +$, $+ \mapsto \max$)

$$\rightsquigarrow \begin{array}{ccccc} \mathcal{A}^\vee(\mathbb{R}^T) & = & \mathcal{A}^\vee_{\tilde{s}}(\mathbb{R}^T) & = & \mathbb{R}^{\tilde{J}} \\ \cup\!\shortmid & & & & \cup\!\shortmid \\ \Delta & & \longmapsto & & \Delta_{\tilde{s}} \end{array}$$ as a set

Let $W = \sum_{j\in\tilde{J}_{fr}} \vartheta_j \in \mathbb{C}[\mathcal{A}^\vee]$ be the GHKK superpotential,

(Gross-Hacking-Keel-Kontsevich 2018)

where $\vartheta_j \in \mathbb{C}[\mathcal{A}^\vee]$ is defined by

$$\vartheta_j|_{\mathcal{A}^\vee_{\tilde{s}}} = X_{j,\tilde{s}} \text{ if } j \text{ is a sink in } \tilde{s}.$$

$\rightsquigarrow \Xi := \{x \in \mathcal{A}^\vee(\mathbb{R}^T) \mid W^T(x) \geq 0\}$

$\rotatebox[origin=c]{90}{$\supseteq$}$

$\Xi_{\mathbb{Z}}$ ⋯ the set of lattice pts.

<u>Thm</u> (GHKK 2018)

(1) $\mathbb{C}[G/U^+]$ has the theta func. basis $\{\theta_g \mid g \in \Xi_{\mathbb{Z}}\}$

(2). The extended g-vector $g_{\tilde{s}}(\theta_g)$ coincides with $g_{\tilde{s}}$

(3). θ_g is a weight vector w.r.t. the $H \simeq \{e\} \times H$-action.

(4). $\Xi_{\mathbb{Z}} \longrightarrow P_+$, $g \longmapsto wt(\theta_g)$, is lifted to an $\mathbb{R}$-linear map $\mathcal{A}^\vee(\mathbb{R}^T) \xrightarrow[\underline{wt}]{} \mathbb{R}^n$

(5). For $\lambda \in P_+$, $\{\theta_g \mid g \in \Xi_{\mathbb{Z}} \cap \underline{wt}^{-1}(\lambda)\}$ is a $\mathbb{C}$-basis of $V(\lambda)^*$.

$$\mathbb{R}^{\tilde{J}} = \mathbb{R}^J \oplus \mathbb{R}^{\{\bar{1}, \dots, \bar{n}\}} \xrightarrow{pr_J} \mathbb{R}^J$$

<u>Def</u>

$$\Xi(\tilde{s}, \lambda) := pr_J\left((\Xi \cap \underline{wt}^{-1}(\lambda))_{\tilde{s}} \right) \subseteq \mathbb{R}^J$$

is called a superpotential polytope

<u>Thm</u> (Bossinger–Fourier 2019, Genz–Koshevoy–Schumann 2020)

$\Xi(\tilde{s}_{\mathbf{i}}, \lambda)$ is unim. equiv. to the string polytope in representation theory.

Aim

to realize $\Xi(\tilde{s},\lambda)$ using only the cluster str. on $\mathbb{C}[U^-_w]$.

Since $G^{w_0,e} \xhookrightarrow[\text{open}]{} G/U^+$, we have

$$g \longmapsto gU^+$$

$$\begin{tikzcd}
\mathbb{C}[G^{w_0,e}] \arrow[r, "\pi_w"] & \mathbb{C}[U^-_{w_0}] \ni \sigma/\tau_\lambda & (\exists\tau_\lambda \in H^0(G/B^+,\mathcal{L}_\lambda)) \\
\mathbb{C}[G/U^+] \arrow[u, phantom, "\cup" rotate=90] & \pi_w(V(\lambda)^*) \arrow[u, phantom, "\cup" rotate=90] & \\
V(\lambda)^* \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, "\sim"] & & H^0(G/B^+, {}^{\exists}\mathcal{L}_\lambda) \arrow[ul, "\wr"'] \ni \sigma \arrow[uul, mapsto]
\end{tikzcd}$$

Borel-Weil theory

line bundle

Prop

$$g_s(\pi_w(\theta_q)) = pr_J(g_{\tilde{s}}(\theta_q))$$

Key Lemma (F.-Oya preprint 2020)

There exists a valuation $v_s : \mathbb{C}[U^-_{w_0}]\setminus\{0\} \to \mathbb{Z}^J$

s.t. $v_s(\pi_w(\theta_q)) = g_s(\pi_w(\theta_q))$

$$\rightsquigarrow \Xi(\tilde{s},\lambda) = \overline{\bigcup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\{v_s(\sigma/\tau_{k\lambda}) \mid \sigma\in H^0(G/B^+,\mathcal{L}_{k\lambda})\setminus\{0\}\}}$$

Newton-Okounkov body

$\Delta(G/B^+,\mathcal{L}_\lambda, v_s)$

<u>Thm</u> (F.-Oya preprint 2020)

(1) $\Delta(G/B^+, \mathcal{L}_\lambda, V_s)$ is naturally extended to compactifications of arbitrary $\mathcal{A}$-cluster varieties with full rank exchange matrices.

(including flag varieties in general Lie type)

(2) These induce a family of toric degenerations of flag varieties in general Lie type

- parametrized by seeds
- includes Caldero's toric degenerations for string polytopes.