Exchange Graphs of partial cluster tilting obj.

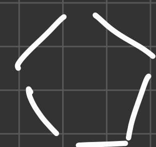
&

Collapsing Subsurfaces

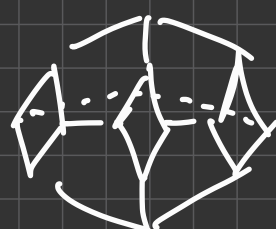
@ Advances in Cluster Algebras 2023

§ Cluster Exchange Graphs

- (Q,W) : a QP (non-degenerated)
- $\underline{CEG}(Q,W) = \begin{cases} \text{vertices} = \text{cluster seeds} \\ \text{edges} = \text{mutation} \end{cases}$

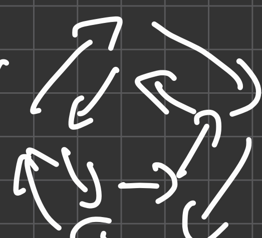
e.g. $\underline{CEG}(A_2) =$ 

A_n associahedron of rank n

A_3 

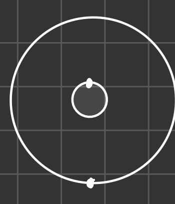
e.g. 

$\mu^2 = \mathrm{id}$

$\pi_1\left(\right.$  $\left. / \, x^2 = y^3 \right) = Br A_2 = Br_3$

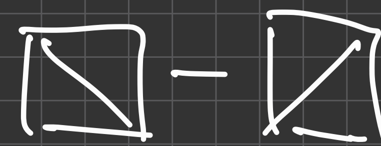
Def. (FST) A marked surface $\mathbb{S} = (S, \mathbb{M})$, $\mathbb{M} \subset \partial S$ $(\mathbb{P} = \emptyset)$

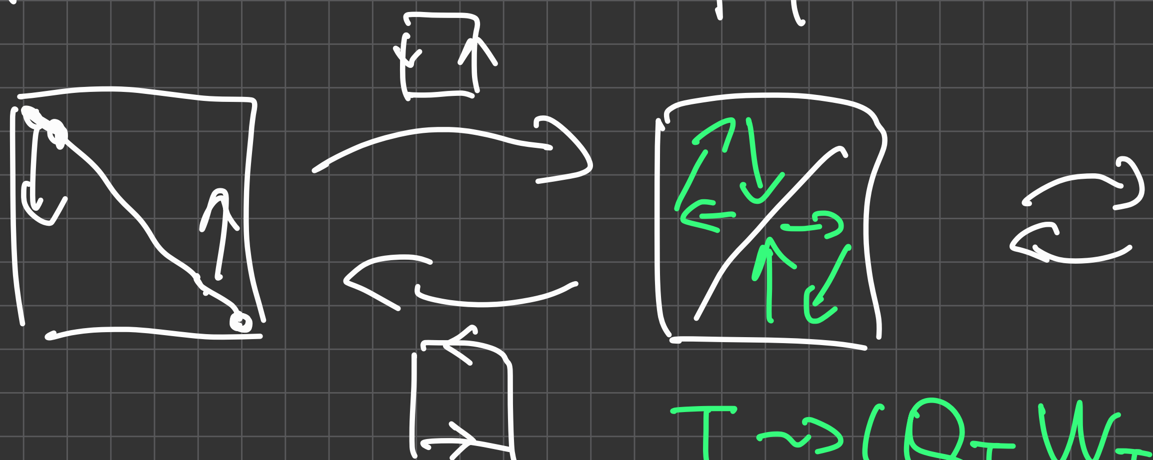
$\left(g,\ b = |\partial S|,\ m = \sum_{i=1}^{b} \underset{\overset{\vee}{0}}{m_i}\right)$

· (open) arcs γ $(\partial\gamma \subset M)$

· Triangulation $= \{\gamma_i\}$

Def. $\underline{EG}(S) = \begin{cases} v = \text{Tri.} \\ e = \text{flip} \end{cases}$ 



$T \rightsquigarrow (Q_T, W_T)$

Thm (FST) $EG(S) \cong CEG(S)$

§ Categorification (additive)

· Q acyclic $(W=0)$

[BMRRT, K] $\mathcal{C}_2(Q) := D^b(kQ) / \tau^{-1}\circ[1]$

· (Q, W)

Amiot: $\mathcal{C}_2(Q,W) := \operatorname{per}\Gamma / \operatorname{pvd}\Gamma$

$\Gamma = \Gamma(Q,W) =$ Ginzburg dga $= 3CY$ of preproj.

FACT. $\mathcal{C}_2$ is $2CY$

Thm $(\dots)$ $\underline{CEG(Q,W) \cong CEG\, \mathcal{C}_2(Q,W)}$

$\Downarrow$

cluster tilting obj := vertices

mutation = edge

$\oplus Y_i = Y \overset{\mu_k^\#}{\rightleftarrows} Y'$ $\quad (\mu_k^\#)^{\overset{m}{2}} = \mathrm{id}$ $\quad (\underset{m}{2}CY)$

$\mathrm{Ext}^1(Y,Y)$

& $\mathcal{C}_2 = Y * Y[1]$

$\boxed{\cdot\ \mathrm{pvd}\,\Gamma \text{ is } 3CY}$

$\cdot$ $\underline{\mathrm{per}\,\Gamma \supset \mathrm{pvd}\,\Gamma}$

∞-cluster tilting

$\parallel$

silting obj. $\tilde{Y} = \oplus Y_i$

$\mathrm{Ext}^{>0}(Y,Y)$

$\pi : \mathrm{per}\,\Gamma \to \mathcal{C}_2(\Gamma)$

silting $\to$ cluster tilting

$$\mathrm{SEG}\,\mathrm{per}\,\Gamma \,/\, \mathrm{ST}\,\Gamma$$
$$\Big\Downarrow$$
$$\mathrm{CEG}\,\mathcal{C}_2(\Gamma)$$

mutation "=" mutation

Thm (KN, KQ)

$$\mathrm{pvd}\,\Gamma \xleftrightarrow[\text{Koszul dual}]{} \mathrm{per}\,\Gamma$$

finite heart $\langle \mathrm{sim}\,\Gamma \rangle$ silting Γ

$$\mathrm{EG}\,\mathrm{pvd}\,\Gamma \overset{\cong}{\longleftrightarrow} \mathrm{SEG}\,\mathrm{per}\,\Gamma$$

Thm (KN, KY, KQ)

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§ Decorated Marked surface

Cor: $\mathrm{CEG}\,\mathcal{C}_2(\mathbb{S}) \cong \mathrm{EG}(\mathbb{S})$ (*) (BZ, QZ)

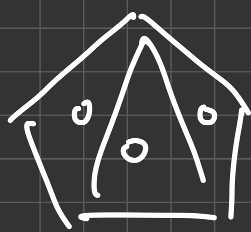
$$\text{ind. obj.} \xleftarrow[X]{} \text{arc in } \mathbb{S}$$

X induces (*) $\pi \longmapsto X(\pi)$

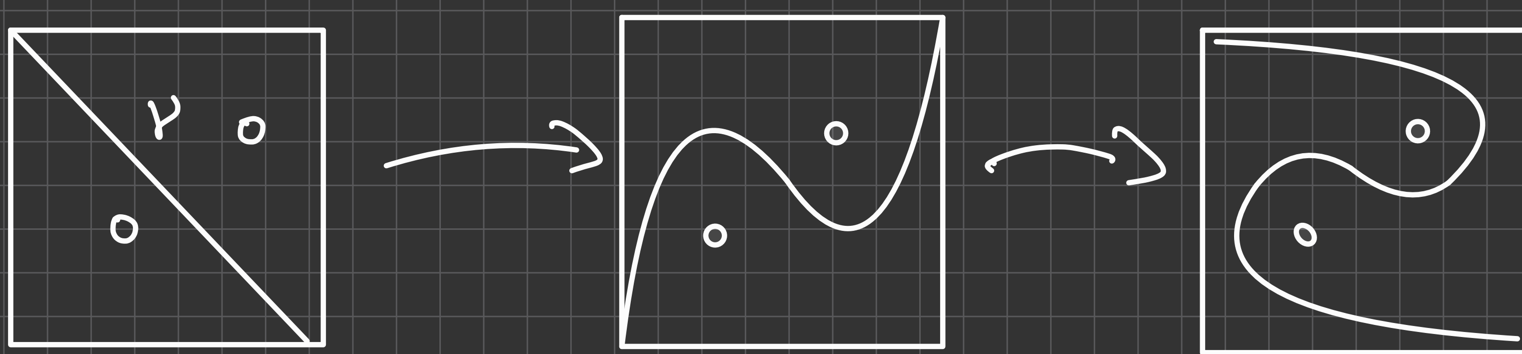
$\mathrm{SEG}\,\mathrm{per}\,\Gamma_{\mathbb{S}}$?

Def (Q): $\mathbb{S} \rightsquigarrow \mathbb{S} + \Delta$ = a set of dec. $\subset \mathbb{S}^{\circ}$

$|\Delta| = \#$ triangles in $\forall\, \mathbb{T}$ of $\mathbb{S}$.



$$EG(\mathbb{S}_\Delta) = \begin{cases} v = \text{dec. triangulation} \\ e = \text{flips} \end{cases}$$



Thm (Q). $EG^{\circ}(\mathbb{S}_\Delta) \cong EG^{\circ} D_3(\mathbb{S}_\Delta)$

(*) $\cong$ $SEG^{\circ}\, \mathrm{per}\, \Gamma_{\mathbb{S}}$

$\cong$

$EG^{\circ} D_3(\mathbb{S}_\Delta) = \mathrm{pvd}\, \Gamma_{\mathbb{S}}$

X : arcs in $\mathbb{S}_\Delta \longrightarrow$ obj in $\mathrm{per}\, \Gamma_{\mathbb{S}}$

induces (*)

$BT \xleftrightarrow{\cong} ST$

$\underline{EG(\mathbb{S})} \cong CEG\, C_2(\mathbb{S})$

$\otimes\mathbb{C}$
$\Longrightarrow$

$\underline{\mathrm{FQuad}^{\circ}(\mathbb{S}_\Delta)} \cong \mathrm{Stab}^\circ \underline{\mathcal{D}_3(\mathbb{S}_\Delta)}$

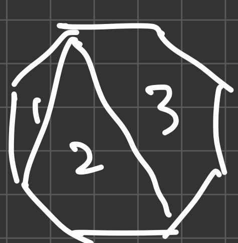
3CY (3-2) order 1 zero = decoration

§. EG of partial CTO / triangulations

$\mathbb{S}_w$ = a weighted DMS.

$\Delta \subset \mathbb{S}^\circ$ $\quad w : \Delta \longrightarrow \mathbb{Z}_{\geq +1}$ (order)

$$|w| = \sum_{Z \in \Delta} w(Z) = \#\text{triangles}$$

e.g. $\mathbb{S} =$ 

$w = (1, 2, 3)$
$\quad\ \ \Delta\ \ \Delta\ \ \Delta$

Def. A mixed-angulation $\mathbb{A}$ of $\mathbb{S}_w$ is a collection of arcs that divides $\mathbb{S}_w$ into polygons with #edges $= \boxed{w+2}$

A mixed angulation of $\underline{w}$-type of $\underline{S}$

· flip

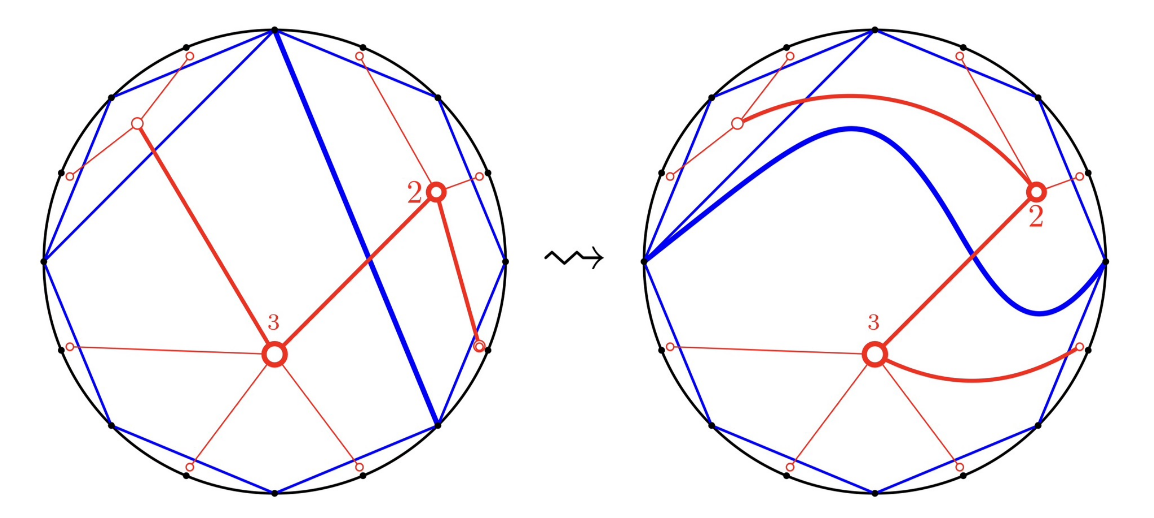


FIGURE 2. Forward flip of **w**-arc systems and their dual

$$EG(S_w)/_{SBr} = EG_w(S)$$

Ex:

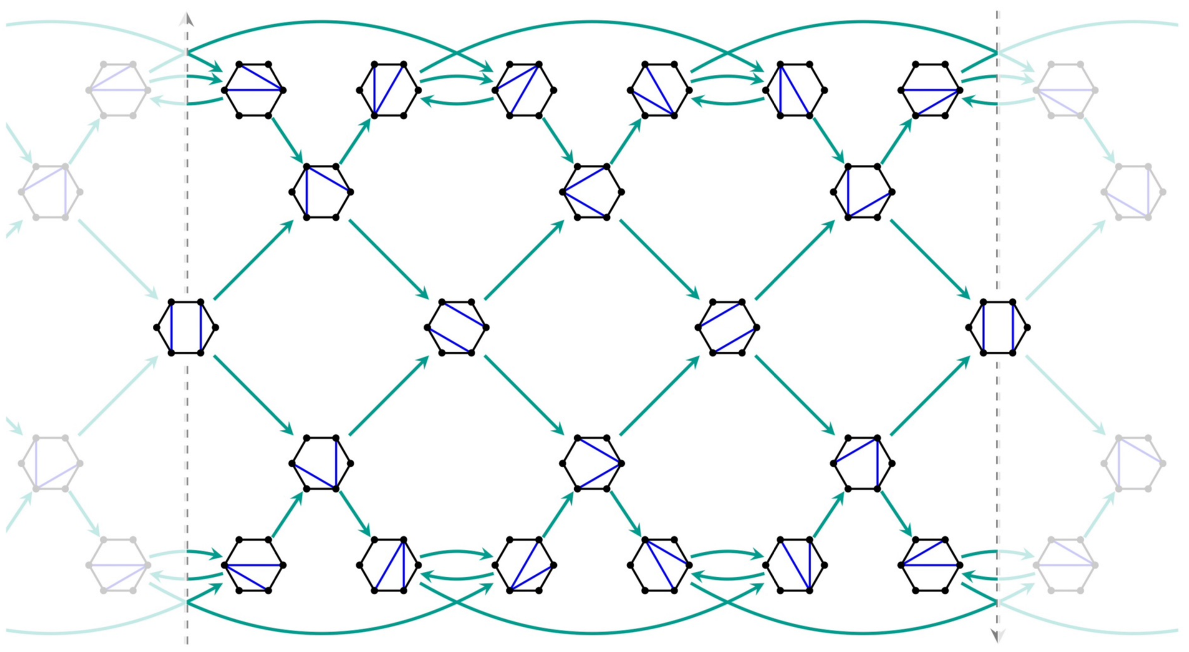
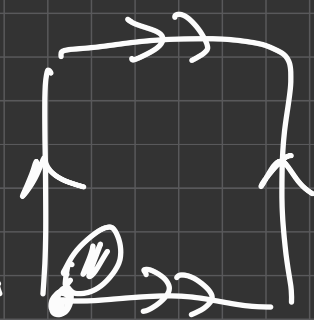


FIGURE 9. Exchange graph $\mathrm{EG}_{\mathbf{w}}(\overline{\mathbf{S}}_{\mathbf{w}}) = \mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\mathbf{w}})/\,\mathrm{SBr}(\overline{\mathbf{S}}_{\mathbf{w}})$ for a hexagon surface $\overline{\mathbf{S}}_{\mathbf{w}}$ with $\mathbf{w} = (1, 1, 2)$

<u>Ex</u>: $\mathbb{S}$ = torus with $b=1$ & $m=1$

$w=3$

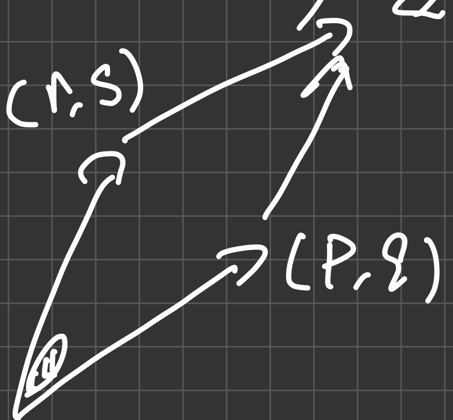
$EG_w(\mathbb{S})$ is not connected.



($EG(\mathbb{S})$ is conn. , even when $\mathbb{P} \neq \emptyset$)

$\pi_0 EG_w(\mathbb{S}) / \mathbb{Z}\langle D_\partial \rangle$ is parametrized by

normal form



$(p,q)+(r,s) \in (\mathbb{Z}/3)^2$

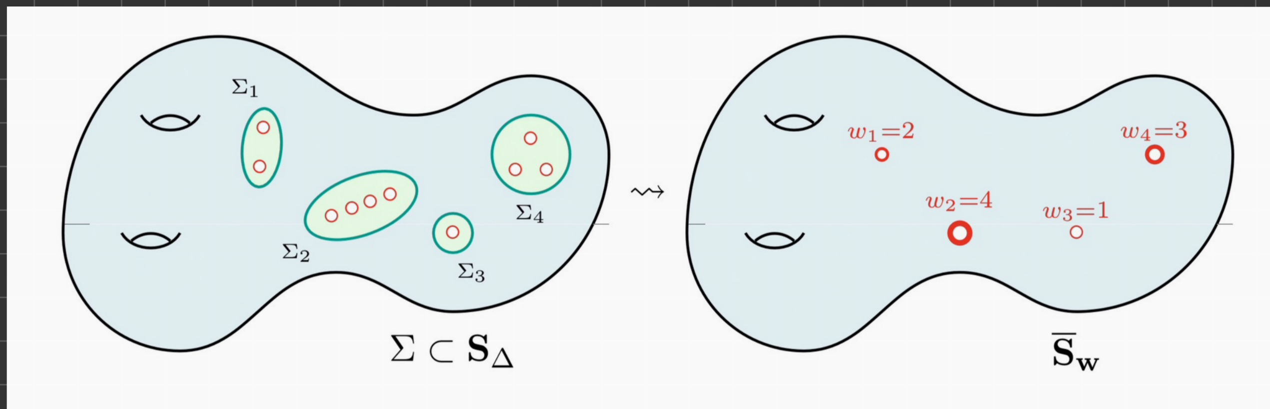
<u>Rem</u>: · EG of τ-tilting.

· Construct cat. to realize EG's

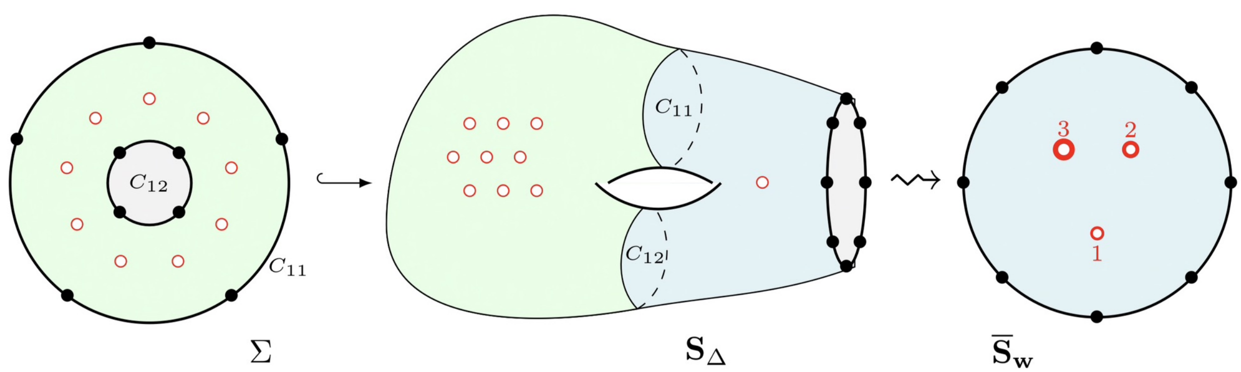
§ Collapsing subsurfaces

$$\Sigma \hookrightarrow \underline{\mathbb{S}_\Delta} \rightsquigarrow \overline{\mathbb{S}}_w$$

collision:



$$\Sigma \longrightarrow \mathbf{S}_\Delta \rightsquigarrow \overline{\mathbf{S}}_{\mathbf{w}}\,. \qquad (4.2)$$



$$\begin{tikzcd}
\underline{\text{Cat}}.\ 0 \arrow[r] & D_3(\Sigma) \arrow[r, hook] & D_3(\mathbb{S}_\Delta) \arrow[r] \arrow[d, equal] & D(\overline{\mathbb{S}}_w) \arrow[r] & 0 \\
V = \text{pvd}\,\Gamma_\Sigma \arrow[rr, hook] \arrow[d, leftrightarrow] & & \text{pvd}\,\Gamma_{\mathbb{S}_\Delta} \arrow[r] \arrow[d, leftrightarrow] & \text{pvd}\,\Gamma_{\mathbb{S}_\Delta \setminus \mathbb{P}} \arrow[d, leftrightarrow] & \\
\text{per}\,\Gamma_\Sigma & & \text{per}\,\Gamma_{\mathbb{S}_\Delta} \arrow[ll] & \text{per}\,\Gamma_{\mathbb{S}_\Delta \setminus \mathbb{P}} \arrow[l] & \\
& & p\mathcal{SEG}_{\perp V} \arrow[r, leftrightarrow, "\simeq"] & \mathcal{SEG}\,\text{per}\,\Gamma_{\mathbb{S}_\Delta \setminus \mathbb{P}} &
\end{tikzcd}$$

is not 3CY

Thm (Bartini–Möller–Q–So)

$$\underline{EG^{\bullet}D(\bar{S}_w)} \cong EG^{\bullet}(\bar{S}_w)$$

$\otimes\mathbb{C}$ $\Downarrow$

$$\mathrm{Stab}^{\bullet}D(\bar{S}_w) \cong \mathrm{FQuad}^{\bullet}(\bar{S}_w)$$

(allow $\forall$ order of zeros)

Rem.

Prop. (Antieau–Gepner–Heller)

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{V} & \to & \mathcal{D} & \to & \mathcal{D}/\mathcal{V} & \to & 0 \\ & & & & \cup & & & & \\ & & \underline{\text{a heart}} & & \mathcal{H} & \mapsto & \underline{\mathcal{H}} & & \end{array}$$

$0 \to A \to \underset{\triangle}{B} \to C \to 0$

- If $\mathcal{H} \cap \mathcal{V}$ is a Serre subcat. of the abelian $\mathcal{H}$

then $\underline{\mathcal{H}} \cong \mathcal{H}/\mathcal{H}\cap\mathcal{V}$ is a heart in $\mathcal{D}/\mathcal{V}$.

- Verdier quotient categorifies subsurfaces collapsing. (with Stab $=$ FQuad)

<u>Thm</u> (Christ–Haiden–Q).

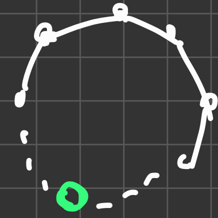
$$\mathrm{Stab}\, D(S_w) \cong \mathrm{FQuad}(S_w)$$

$$\boxed{EG\, D(S_w) \cong EG(S_w)}$$

Construct many classes categories

- perverse schobers

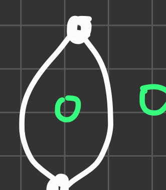
"$w = \infty$"



S_w $\forall$

$w : \Delta \to \mathbb{Z}_{\geq -1}$

$w = 0$



$w = -1$

$\partial =$ relative Ginzburg $\longleftrightarrow$ exceptional singularity in Quad

$\Updownarrow$

forzon vertices in QP