# Quantum tori assoc. w. sequences & their application II.

• **Quantum group**

$\mathfrak{g}$ : KM alg $\quad Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \quad$ ______ $\quad Q^+ = \sum$ ____ $\alpha_i$ .

$Q^- = \sum$ ____ $\alpha_i$

$U_q(\mathfrak{g})$ = quantum group over $\mathbb{Q}(q^{1/2})$ gen'd by $e_i, f_i$ $(i \in I)$ & $q^h$

**+ q-Serre rel.**

$$\sum_{r=0}^{1-c_{ij}} (-1)^r f_i^{(1-c_{ij}-r)} f_j f_i^{(r)} = 0 \qquad \sum_{r=0}^{1-c_{ij}} (-1)^r e_i^{(1-c_{ij}-r)} e_j e_i^{(r)} = 0$$

where $f_i^{(n)} = f_i^n / [n]_i!$, $\quad e_i^{(n)} = e_i^n / [n]_i!$ .

$\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}]$. $\quad U_{\mathbb{A}}^+(\mathfrak{g})$ = $\mathbb{A}$-subalg of $U_q(\mathfrak{g})$ gen'd by ______

• **Quantum unipotent coordinate ring. $A_q(\mathfrak{n})$**

Set $A_q(\mathfrak{n}) = \bigoplus_{\beta \in Q^-}$ ______ where ______ $:= \mathrm{Hom}_{\mathbb{Q}(q^{1/2})}($ ______ $, \mathbb{Q}(q^{1/2}))$.

**Known** $\cdot A_q(\mathfrak{n})$ has also ________.

· The $\mathbb{A}$-sub alg $A_{\mathbb{A}}(\mathfrak{n}) := \{\psi \in A_q(\mathfrak{n}) \mid \psi(U_{\mathbb{A}}^+(\mathfrak{g})) \subset \mathbb{A}\}$ has a unique __________

· $U_q^+(\mathfrak{g})$-acts on $A_q(\mathfrak{n})$ as $e_i \psi(x) =$ ________ $\forall x \in U_q^+(\mathfrak{g}), \psi \in A_q(\mathfrak{n})$.

**Let us choose $w \in W$, take $\mathbf{i} = (i_1 \ldots i_r) \in R(w)$ (red. seq)**

For $\beta = \sum_i n_i \alpha_i \in Q^+$ with $r = \sum n_i$, set • $I^\beta = \{(i_1 \ldots i_r) \in I^r \mid$ ____________ $\}$.

• ______ $= \{\beta_k^{\mathbf{i}} \mid 1 \le k \le r\}$ $\forall \mathbf{i} \in R(w)$ $\subset$ ______ & $\Phi(w) \cap ($ ______ $) =$ ______ .

• ____ $:= \mathrm{Span}_{\mathbb{A}}\{\psi \in A_{\mathbb{A}}(\mathfrak{n}) \mid$ ____ $\psi = 0$ $\forall \beta \in$ , $\forall (i_1 \dots i_r) \in$ ____ $\}$ has an ____

Thm [Kimura] ____ $:=$ $\underline{B^{up} \cap A_{\mathbb{A}}(\mathfrak{n}(w))}$ is an ____

## • i-boxes and unipotent quantum minors.

For $0 \le a \le b \le r$, we set $[a,b] = \{k \in \mathbb{Z} \mid a \le k \le b\}$ & call it ____.

$\boxed{\text{Setting}}$ We have chosen $\mathbf{i} = (i_1 \cdots i_r) \in R(w)$ for $w \in W$.

- $K = [1, r]$  $K = K_{ex} \sqcup K_{fr}$.
- An interval $[a,b] \subseteq [\underline{0}, r]$ is called an ____ if ____ or ____
- ____ $[a_1, b_1]$ & $[a_2, b_2]$ are said to be ____ if ____ or ____ (Recall ____)

For an $\mathbf{i}$-box $[a,b]$ w/ $i_b = i$, we can associated an elt ____ $\in B^{up}(w) \subseteq A_q(\mathfrak{n}(w))$ & called a ____ with $\mathrm{wt}($ ____ $) =$ ____ $\in Q^-$.

In particular, ____ is an ____ vector of wt ____ & $\{$ ____ $\mid 1 \le k \le r\}$ generates $A_{\mathbb{A}}(\mathfrak{n}(w))$.

For elts $x, y \in A_{\mathbb{A}}(\mathfrak{n})$, $x, y$ are ____ if $xy = q^{l} yx$ for some $l \in \mathbb{Z}/2$.

## • Quantum tori of $A_{\mathbb{A}}(\mathfrak{n}(w))$

**Thm** [BZ] If $\mathbf{i}$-boxes $[a_1, b_1]$ & $[a_2, b_2]$ commute, then $D^{\mathbf{i}}[a_1, b_1]$ & $D^{\mathbf{i}}[a_2, b_2]$ ____ s.t $\underline{D^{\mathbf{i}}[a,b]\, D^{\mathbf{i}}[a',b'] \in q^{\mathbb{Z}} B^{up}(w)}$ and

$$l = (w_{\le a_1}\varpi_{i_a} - w_{\le b_1}\varpi_{i_b},\ w_{\le a_2}\varpi_{i_{a_2}} + w_{\le b_2}\varpi_{i_{b_2}}) \text{ if } \underline{a_2^- < a_1 \le b_1 < b_2^+}.$$

In particular $D^{\mathbf{i}} := \{D^{\mathbf{i}}[0,k] \mid 1 \le k \le r\}$ forms a ____, with

$$l_{st} = (\varpi_{i_s} - w_{\le s}\varpi_{i_s},\ \varpi_{i_t} + w_{\le t}\varpi_{i_t}) \quad \text{if } s \le t.$$

$\rightsquigarrow$ **we have a q.torus** of $A_q(\mathfrak{n}(w))$ and q.seed.

$$\mathcal{S}^{\mathbf{i}} := (L^{\mathbf{i}}, \tilde{B}^{\mathbf{i}}, \{\tilde{Z}_k = D^{\mathbf{i}}[0,k]\})$$

**Thm** (Geiß – Leclerc – Schröer, Goodreal – Yakimov, ....)

$$\underline{\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w)) \simeq \mathcal{A}_q(\mathcal{J}^{i})}$$

## • Elements in $\mathbb{B}^{up}$ & Conjectures

$b \in \mathbb{B}^{up}$ is ____ if ____ $\in q^{\mathbb{Z}/2}\mathbb{B}^{up}$

____ if $b$ does not have ____ $b=$ ____ w/ $b_1, b_2 \in \mathbb{B}^{up}$.

**Conjecture.** • {cluster monomials in $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$} $\overset{1-1}{\longleftrightarrow}$ { ____ elts in $\mathbb{B}^{up}$ }

• {cluster variables in $\mathcal{A}_{\mathbb{A}}(\mathfrak{n}(w))$} $\overset{1-1}{\longleftrightarrow}$ { ____ elt in $\mathbb{B}^{up}$ }.

## • Monoidal Categorification

(proposed by Hernandez – Leclerc for Conjectures)

$\mathcal{C}$ : monoidal category with $\underline{\otimes}$, auto-functor $\underline{q}$. ("grading shift" ftor")

$\mathcal{C}$ provide a m. Cat of a q. cluster alg $\mathcal{A}_q$ if

$\boxed{\text{Axioms for m. Cat}}$ ① $\mathcal{A}_q \simeq$ ____ ,

② q. cluster monomials $\longleftrightarrow$ ____ simple objs in $\mathcal{C}$

③ // vars $\longleftrightarrow$ ____ simple objs in $\mathcal{C}$.

$\Rightarrow$ Categorifying cluster variables mutations

In this talk, $\mathcal{C}$ denotes a module Category.

**Consequence.** $x$ : q. cluster var $\longrightarrow$ $M$ : simple

$$[M] = x = \frac{\sum_a \quad}{\quad} = \frac{\sum_a P_a[\quad]}{[\quad]} = \frac{\sum P_a[\quad]}{[\quad]}$$

q. Laurent phenomena.

$\Rightarrow$ $P_a$ : a " ____ " # of [ __ ] in [ __ ].

## • Quiver Hecke algebra. (Khovanov – Landa, Rouquier).

Take $\beta = \sum n_i \alpha_i \in Q^+$ w/ $\sum n_i = r$ & recall $I^{\beta} \subset I^r$.

Fix symmetrizable KM-alg $\mathfrak{g}$

KL and R introduce a ______________ alg $R(\beta)$ gen'd by

$x_k$ $(1 \le k \le r)$, $\tau_s$ $(1 \le s < r)$, $e(\nu)$ $(\nu = (\nu_1, \dots \nu_r) \in I^\beta)$,

subj to certain relations asso w/ $\mathfrak{g}$. ______________.

Here $e(\beta) = \sum_{\nu \in I^\beta}$ ____ is a unit $R(\beta)$

$R(\beta)$-gmod := the category ________ $R(\beta)$-modules

$R\text{-gmod} = \bigoplus_{\beta \in Q^+} R(\beta)\text{-gmod}$ (block-decomposition)

$q : R\text{-gmod} \longrightarrow R\text{-gmod}$ ________ s.t $(qM)_n =$ ____ for $M = \bigoplus_{k \in \mathbb{Z}} M_k \in R\text{-gmod}$.

## • Convolution product (Monoidal structure)

For $M \in R(\beta)\text{-gmod}$, $N \in R(\gamma)\text{-gmod}$.

$$M \circ N := \underline{\qquad\qquad} \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N)$$

where $e(\beta, \gamma) = \sum_{\substack{\nu \in I^\beta \\ \mu \in I^\gamma}}$ ______ ; via non-unital homo $R(\beta) \otimes R(\gamma) \hookrightarrow R(\beta+\gamma)$.
$e(\beta),\ e(\gamma) \longmapsto e(\beta, \gamma)$

$\leadsto$ $(R\text{-gmod}, \circ)$ : ________ category $\Rightarrow$ $K(R\text{-gmod})$ has ________

Thm [① Khovanov-Lauda, Rouquier, ....]

① $\exists$ $\mathbb{A}$-alg iso

$$\Omega : \mathbb{A} \otimes_{\mathbb{Z}[q^{\pm 1}]} \underline{\qquad\qquad} \xrightarrow{\sim} A_{\mathbb{A}}(\mathfrak{n}) \quad (\qquad)$$

② For each $\mathbf{i} = (i_1 \dots i_r) \in \underline{R(w)}$ for $w \in W$ and an $\mathbf{i}$-box $\underline{[a,b]} \subset [0,r]$, $\exists$ ________
________ simple module ________$^{\mathbf{i}}$ $\in R(w_{\le a} \varpi_{i_a} - w_{\le b} \varpi_{i_b})$-gmod s.t

$\Omega($________$^{\mathbf{i}}) =$ ________$^{\mathbf{i}}$ (> initial cluster variables )

In particular we set ____$^{\mathbf{i}}$ = ________$^{\mathbf{i}}$ for $1 \le k \le r$.

Def (subcategory $\mathcal{C}_w$ of R-gmod)

$\mathcal{C}_w$ is the ______________________ of R-gmod

① stable under taking ________, ____ & ________

② containing $\underline{\{\ \ ^{\mathbf{i}} \mid 1 \le k \le r\}}$.

$\rightsquigarrow \Omega(\mathbb{A}\otimes_{\mathbb{Z}[q^{\pm1/2}]} K(\mathcal{C}_w)) \simeq A_{\mathbb{A}}(\mathfrak{n}(w))$ (note $A_{\mathbb{A}}(\mathfrak{n}(w))$ has q.cluster alg str).

Here the def of $\mathcal{C}_w$ does not depend on the choice of $\mathbf{i}$.

**Note** For the def of $R(\beta)$, we need to choose set of "polys".

But we skip it and take it in [KL].

**From now on,** we consider $\mathfrak{g}$ of ______

**R-matrix, Λ-inv** For $M, N \in R\text{-gmod}$ $\exists$ a ______ R-module homo

______ $= q^a M \circ N \longrightarrow N \circ M$.

We call ______ the ______ & denote by ______ $= a$ the ______ of ______

prop [Brundan-Kleshchev-McNamara, Tingley-Webster..]

For $1 \le k \le l \le r$,

$\underline{(\mathbf{i}\ ,\ \mathbf{i})} = -\delta\underline{(\quad)(\mathbf{i}\ ,\ \mathbf{i})} = \underline{i_l\ ,\ )}$

**Rmk** The existence of R-matrix for ______ $\mathfrak{g}$ is quite ______.

**d-invariant** For $M, N \in R\text{-gmod}$, $d(M,N) := \underline{\ (\quad + \quad)}$.

Thm (Kang-Kashiwara-Kim-O) M, N : simple R-module s.t one of them is real.

(MoM simple)

① $d(M,N) \geq$ ______

② $d(M,N) =$ ______ $\Longleftrightarrow$ $M \circ N \simeq$ ______

③ $\mathrm{hd}(M \circ N)$ is ______

(head)

Note For ______ $\mathbf{i}$-boxes $[a,b]$ & $[a',b']$, $d(M^{\mathbf{i}}[a,b], M^{\mathbf{i}}[a',b'] = \_\_)$ &

$\Lambda(M^{\mathbf{i}}[a,b], M^{\mathbf{i}}[a',b']) = -(w_{\le a}\varpi_{i_a} - w_{\le b}\varpi_{i_b},\ w_{\le a'}\varpi_{i_{a'}} + w_{\le b'}\varpi_{i_{b'}})$ ______.

Comparison $A_{\mathbb{A}}(\mathfrak{n}(w)) \longleftrightarrow \mathcal{C}_w$.

$\cup \qquad \cup$

• cluster var

are homo ______ $\longleftrightarrow$

- $D[a,b] \longleftrightarrow$

- $D[a,b]\,D[a',b'] = q^{-l} D[a',b']\,D[a,b] \longleftrightarrow l =$ ____________

- $N^{*} \longleftrightarrow \Lambda$-inv { }. ____________

- exchange matrix $\longleftrightarrow$ ?

<u>Thm</u> (KKKO) For $k \in K_{ex}$, $\exists$ real simple module ___ s.t. • $d(M^{*}[0,k], \_\_) =$ ___ $\ast$

- $0 \to \underset{b_{ik}>0}{\circ}$ ______ ① $\to M[0,k] \circ$ ___ $\to \underset{b_{ki}>0}{\circ}$ ______ ② $\to 0$,

where ①, ② are ______. (Categorifying initial ex matrices and mutation rule)
and comm / with $M[0,i]$ $(i \neq k)$.

<u>Thm</u> (KKKO) $C_w$ provides a ______ of the $q$-cluster alg $A_{q}(n(w))$.
In particular, every cluster monomial corresponds to a ______ $M \in C_w$,
is ______ in ______ $\longleftrightarrow$

Additional Information (Maximal commuting family : Suggested by Kimura as the notation ( ____________ in $B^{up}(w)$).

(Kashiwara-Kim)
<u>Cor</u> $X$ simple in $R$-gmod and $\{M_k\}$ a set of real simple module ____________ $\{\tilde{Z}_k'\}$. If $\circ \simeq \circ$ ______ $\forall\ 1 \le k \le r$, $\exists\, p \in \mathbb{Z}^{k}_{\ge 0}$ ☆
s.t. $X \simeq \circ \cdots \circ$ ______

pf By Laurent phenomenon, $\exists\, a \in \mathbb{Z}^{k}_{\ge 0}$ s.t.

$$[X] \cdot [M^{a}] = \sum_{s} q^{c_s} [M^{b_s}] \qquad \text{———} \ ☆$$

<u>Cor</u> For every simple $M \in C_w$, $[M]$ is $(c)$-pointed.

(pf)