Quantum tori assoc. w. sequences & their application.

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($\mathbb{Q}$: rational #)

z_i : vars $\qquad\qquad$ $\tilde{z}_i$: vars

$\mathbb{Q}(z_1, \dots z_r)$ comm $\qquad\qquad$ $\mathbb{Q}(q^{1/2})(\tilde{z}_1, \dots \tilde{z}_r)$

$\cup \qquad\qquad\qquad\qquad \cup$

$\mathcal{A}$: ____ $\rightsquigarrow$ $\mathcal{A}_q$: ____

(Fomin-Zelevinsky) $\qquad$ q : indeterminate $\qquad$ (Berenstein – Zelevinski)

($q^{1/2}$: formal sq. root of q)

One of Goal of BFZ: ____

(____)

1. Ingradients for quantum cluster alg.

K: index set w/ $K_{ex} \sqcup K_{fr} = K$

exchangeable indices

frozen indices

$L = (l_{ij})_{i,j \in K}$: $\mathbb{Z}$-matrix s.t ____ $\qquad$ ($\mathbb{Z}$: integer)

$\tilde{Z} = \{\tilde{z}_k\}_{k \in K}$ alg. ind. vars.

* $\mathcal{A}_q$ is "related to" ____ determined by ___.

Def [____] $\mathbb{Z}[q^{\pm 1/2}]$-alg. gen. by $\{\tilde{z}_k^{\pm}\}$ s.t

____ = ____ = 1, ____ = ____

① ____ $\simeq \mathbb{Z}[z_k^{\pm}]_{k \in K}$ where $z_k := \tilde{z}_k|_{q=1}$

② ____ $\subset$ ____ field of fraction of ____.

Def [____] An ____ $\tilde{B} = (b_{ij})_{i \in K, j \in K_{ex}}$ is a $\mathbb{Z}$-mtx s.t

$B = (b_{ij})_{i,j \in K_{ex}}$ is ____ i.e $\exists$ $D = \mathrm{diag}(d_i \geq 1)_{i \in K}$ s.t

DB is ____. i.e., $(DB)^t = -DB$.

A pair $(L,\tilde{B})$ is said to be a ________ **CP** if

$$\sum_{k\in K} b_{ki}\, l_{kj} = \_\_\_\_\_\_ \quad \text{for } i\in K_{ex},\ j\in K.$$

(we can define g. cluster when such $(L,\tilde{B})$ is given)

Q: How to ______________________________ ?

2. Kac Moody algebra $\mathfrak{g}$ and Index set I of $\mathfrak{g}$

$\boxed{\text{Setting}}$ $\mathfrak{g}$ = KM alg w/ $C=(c_{ij})_{i,j\in I}$ and I the index set

① $\{\alpha_i\}_{i\in I}$: ________ ② $\{\varpi_i\}_{i\in I}$: ________

Assume we can choose them ① and ② s.t.

$$\_\_\_\_ = \_\_\_\_ + \sum_{j\in I,\ c_{ji}<0} \_\_\_\_$$

③ $\Phi^{\pm}$: ________. $\Phi=\Phi^{+}\sqcup\Phi^{-}$

④ $(\ ,\ )$: ________

⑤ W: ________ $\{s_i\}_{i\in I}$ s.t. $s_i\varpi_j=\varpi_j-\delta_{ij}\alpha_i$

For $w\in W$, $R(w):=\{(i_1\dots i_r)\in I^r \mid s_{i_1}\cdots s_{i_r}$ is a ______ of $w\}$.

Combinatorics on sequence.

$\mathbf{i}=(i_1\dots i_r)$ "______" of I, $1\le k\le r$, $j\in I$

$k^{+}_{\mathbf{i}}(j) =$ ____ $(\{u \mid \quad i_u=j\}\sqcup\{r+1\})$ $\qquad k^{+}_{\mathbf{i}}=k^{+}_{\mathbf{i}}(\_\_)$

$k^{-}_{\mathbf{i}}(j) =$ ____ $(\{u \mid \quad i_u=j\}\sqcup\{0\})$ $\qquad k^{-}_{\mathbf{i}}=k^{-}_{\mathbf{i}}(\_\_)$

$k^{\min}_{\mathbf{i}} =$ ____ $\{u \mid 1\le k\le r,\ i_k=i_u\}$

$w^{\mathbf{i}}_{\le k}=$ ______ (for $k\le r$). $\qquad 0^{-}:=-1$.

We set $K=[1,r]=\{1,2,\dots,r\}\subseteq\mathbb{Z}$, $K_{fr}=\{k\in K \mid$ ______ $\}$ $K_{ex}=K\setminus K_{fr}$.

↑ the index set for $(L,\tilde{B})$

(Ex) $\mathfrak{g}=A_3$

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}
1	2	1	2	1	3	3	1	2	3	2

$\Rightarrow$

$K=$

$4^{+}=$

$4^{-}=$

$4^{+}(1)=$ $\qquad 4^{-}(3)=$

$K_{fr}=$

$w_{\le 4}=$

Let us fix a sequence (________)

$\mathbf{i}=(i_1,\dots,i_r)$

until the end of today talk, and we skip it frequently in notations!

Def\Thm (BZ, ..., Fujita-Hernandez-O-Oya,) $(L^{\mathbf{i}}, \tilde{B}^{\mathbf{i}})$ defined below is a **CP**.

$$\tilde{B}^{\mathbf{i}} = (b^{\mathbf{i}}_{st}) \text{ s.t } b^{\mathbf{i}}_{st} = \begin{cases} \underline{\pm 1} & \text{if } \underline{\qquad\qquad} \\ \underline{c_{i_s, i_t}} & \text{if } \underline{\qquad\qquad} \\ \underline{-c_{i_s, i_t}} & \text{if } \underline{\qquad\qquad} \\ \underline{0} & \text{o.w} \end{cases}$$

$$L^{\mathbf{i}} = (\ell^{\mathbf{i}}_{st}) \text{ s.t } \underline{(\varpi_{i_s} - w^{\mathbf{i}}_{\leq s}\varpi_{i_s}, \varpi_{i_t} + w^{\mathbf{i}}_{\leq t}\varpi_{i_t})} \text{ for } s \leq t.$$

$\uparrow$ complicated!!

3. Isomorphism of two quantum tori.

Set $\beta^{\mathbf{i}}_k := \underline{\qquad\qquad} \in \Phi$ for $1 \leq k \leq r$. Define a skew-sym pairing $N^{\mathbf{i}}$ on $K \times K$ ($K = [1, r] = \{1 \dots r\}$)

$$N^{\mathbf{i}}(a, b) = \underline{\qquad\qquad\qquad\qquad}$$

Here for statement P.

$$\delta(P) = \begin{cases} 1 & \text{if } P \text{ is true.} \\ 0 & \text{o.w} \end{cases}$$

Let $\{\tilde{X}_k\}_{1 \leq k \leq r}$: other alg. ind. vars.

Def [q. torus $\mathcal{T}(\Lambda^{\mathbf{i}})$] $\mathbb{Z}[q^{\pm 1/2}]$-alg gen'd by $\{\tilde{X}_k\}$ s.t

$$\tilde{X}_k \tilde{X}_k^{-1} = 1 = \tilde{X}_k^{-1}\tilde{X}_k, \quad \underline{\qquad\qquad\qquad}$$

Set $-\mathrm{wt}_{\mathbf{i}}(\tilde{X}_k) = \underline{\quad}$ ($\Leftrightarrow \mathrm{wt}_{\mathbf{i}}(\tilde{X}_k) = \underline{\quad}$)

$\uparrow$ technical reason. we put $-$

Thm (Hernandez-Leclerc, Fujita-O, Kashiwara-O, Fujita-Hernadez-O-OYA)

$$\mathcal{T}(L^{\mathbf{i}}) \xrightarrow{\sim} \mathcal{T}(\Lambda^{\mathbf{i}})$$

$$\underline{\qquad} \longmapsto \underline{\quad} \qquad (1 \leq k \leq r)$$

(equivalently $\underline{\quad} \longmapsto \underline{\qquad\qquad}$)

4. Definition of (quantum) cluster algebra.

For a while, let us forget $\mathbf{i}$-combinatorics!

$(L,\tilde{B})$: compatible pair, $\{\tilde{Z}_k\}_{k\in K}$ = q-comm family controlled by L.

The triple $\mathcal{S}=(L,\tilde{B},\{\tilde{Z}_k\})$ called a ________ & $\{\tilde{Z}_k\}$ a ________
____

Set $|K|=r$. For $a=(a_k)\in\mathbb{Z}^K$, $\tilde{Z}^a :=$ ________.

• Mutation at $k\in K_{ex}$.

Set $[a]_+=$ ________ for $a\in\mathbb{Z}$.

For $k\in K_{ex}$ and $(L=(l_{ij}),\ \tilde{B}=(b_{ij}))$ cp

$$\mu_k(\tilde{B})=\tilde{B}'=(b'_{ij}) \text{ where } b'_{ij}=\begin{cases}-b_{ij} & \text{if } i=k \text{ or } j=k\\ b_{ij}+b_{ik}[b_{kj}]_+ + [-b_{ik}]_+ b_{kj} & \text{if } i,j\neq k\end{cases}$$

$$\mu_k(L)=L'=(l'_{ij}) \text{ where } l'_{ij}=\begin{cases}0 & \text{if } i=j\\ -l_{kj}+\sum_{t\in K}[-b_{tk}]_+ l_{tj} & \text{if } i=k,\ j\neq k\\ -l_{ik}+\sum_{t\in K}[-b_{tk}]_+ l_{it} & \text{if } i\neq k,\ j=k\\ l_{ij} & \text{o.w.}\end{cases}$$

We define $a'_i=\begin{cases}-1 & \text{if } i=k\\ [b_{ik}]_+ & \text{if } i\neq k\end{cases}$ $a''_i=\begin{cases}-1 & \text{if } i=k\\ [-b_{ik}]_+ & \text{if } i\neq k\end{cases}$. and $a'=(a'_i)$, $a''=(a''_i)\in\mathbb{Z}^K$.

We set $\tilde{Z}'_i=\begin{cases}\text{\_\_\_\_\_\_} & \text{if } i\neq k\\ & \text{if } i=k.\end{cases}$
____

$\mu_k(\mathcal{S})=(\mu_k(L),\mu_k(\tilde{B}),\mu_k(\{\tilde{Z}_i\}):=\{\tilde{Z}'_i\})$ ________ of $\mathcal{S}$ at $k\in K_{ex}$.

Thm (BZ, FZ) $\mu_k(\mathcal{S})$ is also a ________ . i.e

① $(\mu_k(L),\mu_k(\tilde{B}))$ ________, $\{\tilde{Z}'_k\}$: ________

② $\tilde{Z}'_k\tilde{Z}'_s=$ ________.

We call $\{\tilde{Z}'_k\}$ also a ________. A monomial $\tilde{Z}'^a_k$ in a cluster $\{\tilde{Z}'_k\}$ a ________
q-cluster var.
________.

Finally, the def of q-cluster alg: The q-cluster alg $\underline{A_q(\mathcal{S})}$ asso. w/ q-seed $\mathcal{S}$ is $\mathbb{Z}[q^{\pm 1/2}]$-alg gen'd by all ________ in ________ obtained from $\mathcal{S}$ by ________. $\underline{A_q(\mathcal{S})|_{q=1}=A(\mathcal{S})}$ ________.

In the def. we can "only" guarantee that $\underline{A_q(\mathcal{S})\subset\mathbb{F}(L)}$!!

Thm [BZ, FZ : **quantum Laurent phenomenon**] ______ Indeed!

$\Rightarrow$ Hence we can understand $\mathcal{A}_q$ as a ______ of ______

Conjecture **[quantum Laurent positivity conjecture]**. Moreover, every coefficient of a q-cluster var resides in ______.

up to now, $\{\tilde{z}_k\}$, $\{\tilde{x}_k\}$ appear. ($\tilde{Y}$?)

For $k \in K_{ex}$, ______ $:=$ ______ where $\{e_k\}$ is a natural basis of $\mathbb{Z}^{K_{ex}}$.

(-variable)

Let us comback to i-combinatorics.

$$\mathcal{S}^{\mathbf{i}} = (L^{\mathbf{i}}, \tilde{B}^{\mathbf{i}}, \{\tilde{z}_k\}).$$

Recall $T(L^{\mathbf{i}}) \longrightarrow T(\Lambda^{\mathbf{i}})$ Then $wt(\tilde{z}_k) = \beta_k + \beta_{k^-} + \cdots + \beta_{k^{min}}$.

$z_k \longrightarrow \tilde{x}_k \tilde{x}_{k^-} \cdots \tilde{x}_{k^{min}}$

$\boxed{\text{proposition}}$ $wt_{\mathbf{i}}(\tilde{y}_k)$ ______

(pf) $\boxed{1}$ $u \in [1, r]$ with $u^- \in [1, r]$ & $i = i_u$

$$w_{\le u} \varpi_i - w_{\le u^-} \varpi_i = w_{\le u}(\varpi_i - s_i \varpi_i) = w_{\le u}(\alpha_i) \qquad (\because s_j \varpi_i = \varpi_i - \delta_{ij} \alpha_i)$$

$$= w_{\le u}(\qquad\qquad) = \qquad\qquad$$

$\Rightarrow$ $=0$ — ⊛

⊛ – twice

$$w_{\le u^+} \varpi_i - w_{\le u^-} \varpi_i + \sum_{j \sim i} c_{ji} (w_{\le (u^+)^-(j)} \varpi_j - w_{\le u^-(j)} \varpi_j) = 0 \text{ — } \boxed{⊛}$$

$\boxed{2}$ By def of $\tilde{B}^{\mathbf{i}}$, $\tilde{y}_u = \tilde{z}_{u^-} \cdot \tilde{z}_{u^+}^{-1} \cdot \prod_{j \sim i} (\tilde{z}_{u^-(j)} \cdot \tilde{z}_{(u^+)^-(j)}^{-1})^{c_{ji}}$

$$\Rightarrow wt_{\mathbf{i}}(\tilde{y}_u) = wt_{\mathbf{i}}(\tilde{z}_{u^-}) - wt_{\mathbf{i}}(\tilde{z}_{u^+}) + \sum_{j \sim i} c_{ji} (wt_{\mathbf{i}}(\tilde{z}_{u^-(j)}) - wt_{\mathbf{i}}(\tilde{z}_{(u^+)^-(j)}))$$

Then the assertion follow from $wt_i(\tilde{z}_k) =$ ______ .

5. pointed & co-pointed elts in $T(L)$.

<u>Def</u> $x \in T(L)$ is ______ if it is of the form

$$x = q^a \underline{\tilde{z}^{\bar{g}}} + \sum_{c \in \mathbb{Z}^{k_{ex}}_{\geq 0} \setminus \{0\}} p_c \, \underline{\tilde{z}^{\bar{g}+\tilde{B}c}}$$

$$= q^a \underline{\tilde{z}^{\bar{g}}} \left(\underline{1 + \sum_c p_c \tilde{z}^{\tilde{B}c}} \right)$$ for some $a \in \mathbb{Z}/2$, $p_c \in \mathbb{Z}[q^{\pm 1/2}]$ & $\bar{g} \in \mathbb{Z}^k$.

We call $\underline{\bar{g}}$ the ______ of a ______ elt $x \in T(L)$.

$x \in T(L)$ is ______ if it is of the form

$$x = q^a \underline{\tilde{z}^{\underline{g}}} + \sum_{d \in \mathbb{Z}^{k_{ex}}_{\leq 0} \setminus \{0\}} p_d \, \underline{\tilde{z}^{\underline{g}+\tilde{B}d}}$$

$$= q^a \underline{\tilde{z}^{\underline{g}}} \left(\underline{1 + \sum_d p_d \tilde{z}^{\tilde{B}d}} \right)$$ for some $a \in \mathbb{Z}/2$, $p_d \in \mathbb{Z}[q^{\pm 1/2}]$ & $\underline{g} \in \mathbb{Z}^k$.

We call $\underline{g}$ the ______ of a ______ elt $x \in T(L)$.

<u>Consequence</u> Every (co)-pointed elt can be written as
$q^a \tilde{X}^{\bar{g}}$ (______) (resp $q^a \tilde{X}^{\underline{g}} \cdot$ (______)) (not Laurent!!)

<u>Thm</u> [Tran, BZ] Every q-cluster var of $A_q(S)$ asso. w/ $S = (L, \tilde{B}, \{\tilde{z}_k\})$ is ______ . w.r.t $T(L)$.

$\Rightarrow$ Mul. str of A_q is ______ by the ones among ______

<u>Corollary</u> Every q-cluster var of $A_q(S^{i})$ is ______ w.r.t i ______

Recall • $k_i^+(j)$, k_i^+, $k_i^-(j)$, k_i^-, Hence the mul. str. of $A_q(S)$ is
______ by the ones among ___*___ .

Lemma The mul. str among ______ $\star$ is given as follows: (cf Hernandez, Kashiwara–O)

$$\tilde{x}_p \tilde{y}_s = \begin{cases} q^{\underline{\quad}} \tilde{y}_s \tilde{x}_p & \text{if } \underline{\quad} \\ q^{\underline{\quad}} \tilde{y}_s \tilde{x}_p & \text{if } \underline{\quad} \\ \underline{\quad} & \text{o.w} \end{cases}$$

$$\tilde{y}_t \tilde{y}_u = \begin{cases} q^{\underline{\quad}} \tilde{y}_u \tilde{y}_* & \text{if } \underline{\quad}. \\ q^{\underline{\quad}} y_u \tilde{y}_* & \text{if } \underline{\quad\quad\quad\quad} \\ q^{\underline{\quad}} y_u \tilde{y}_* & \text{if } \underline{\quad\quad\quad\quad} \\ \underline{\quad} & \underline{\text{o.w}}. \end{cases}$$

DAY I