T-systems and Y-systems in cluster algebras

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Outline

1. Background: Nahm's problem
2. T-systems in cluster algebras
3. Characterization of T-systems
4. Relation to Nahm's problem

Background: Nahm's problem

- $A \in \mathbb{Q}^{r\times r}$, symmetric and positive definite
- $B \in \mathbb{Q}^r$
- $C \in \mathbb{Q}$

Find (A, B, C) such that

$$f_{A,B,C}(q) := \sum_{n\in\mathbb{N}^r} \frac{q^{Q(n)}}{(q)_{n_1}\cdots(q)_{n_r}}$$

is a modular function.

- $Q(n) = \frac{1}{2}n^{\mathsf{T}}An + n^{\mathsf{T}}B + C$,
- $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

In this talk, we assume that $B = 0$.

This problem is motivated by:

- Rogers-Ramanujan type identities
- Fermionic formulae of characters in conformal field theories

If $r = 1$, $f_{A,0,C}(q)$ is modular iff

$$\begin{array}{c|ccc} A & 2 & 1 & 1/2 \\ C & -1/60 & -1/48 & -1/40 \end{array}.$$

The "if part" follows from Rogers-Ramanujan type identities, and the "only if part" follows from asymptotic calculations of $f_{A,0,C}(q)$ [Terhoeven, Zagier].

The cases for $r \geq 2$ is not well understood.
Candidates for $r = 2$ [Terhoeven, Zagier]:

$$
\begin{array}{c|ccccc}
A & \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \\
C & -1/24 & -1/32 & -5/96 & -1/42 & -5/84
\end{array}
$$

$$
\begin{array}{c|cccc}
A & \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & -1/2 \\ -1/2 & 3/4 \end{bmatrix} & \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} & \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \\
C & -5/168 & -3/56 & -1/30 & -1/20
\end{array}
$$

Where do these matrices come from? Nahm conjectured that they are related to torsion elements in Bloch groups (algebraic K-theory).

Motivation

Develop an approach to Nahm's problem using cluster algebras.

1. Bloch groups are related to cluster ensembles [Fock-Goncharov].
2. The modular S-transformation is related to the modular double structure of quantum cluster varieties [Fadeev, Fock-Goncharov].
3. Some torsion elements in Bloch groups are constructed from Zamolodchikov's Y-systems, which are special cases of the coefficient dynamics in cluster algebras [Fomin-Zelevinsky].

T-systems in cluster algebras

Cluster algebras are commutative rings equipped with a combinatorial structure: cluster variables and exchange relations.

- I: finite index set
- $\mathcal{F}$: field of rational functions over $\mathbb{Q}$ in $|I|$ variables

Definition

An (I-labeled) *seed* in $\mathcal{F}$ is a pair (B, x), where

- $B = (B_{ij})_{i,j \in I}$ is an $I \times I$ skew-symmetrizable integer matrix,
- $x = (x_i)_{i \in I}$ is an I-tuple of elements in $\mathcal{F}$ forming a free generating set.

If B is skew-symmetric, it is often represented as a *quiver* without loops nor 2-cycles.

Definition

Let Q be a quiver, and let k be a vertex of Q. The *quiver mutation* $\mu_k : Q \mapsto Q'$ is defined as follows:

1. For each length two path $i \to k \to j$, add a new arrow $i \to j$.
2. Reverse all arrows incident to the vertex k.
3. Remove all 2-cycles.

$$\begin{tikzcd} & & b \arrow[dll] \\ a \arrow[r] & k \arrow[ur] \arrow[d, shift left] \arrow[d, shift right] & \\ & c & \end{tikzcd} \quad \xmapsto{\mu_k} \quad \begin{tikzcd} & & b \arrow[dl] \\ a \arrow[dr, shift left] \arrow[dr, shift right] & k \arrow[l] & \\ & c \arrow[u, shift left] \arrow[u, shift right] & \end{tikzcd}$$

Definition

Let (B, x) be an I-labeled seed and let $k \in I$. The *seed mutation* $\mu_k : (B, x) \mapsto (B', x')$ is defined as follows:

- $B' = (B'_{ij})_{i,j \in I}$ is given by the quiver mutation
- $x' = (x'_i)_{i \in I}$ is given by $x'_i = x_i$ if $i \neq k$, and

$$x_k x'_k = \prod_{j \in I} x_j^{[B_{jk}]_+} + \prod_{j \in I} x_j^{[-B_{jk}]_+},$$

where $[-]_+ := \max(0, -)$.

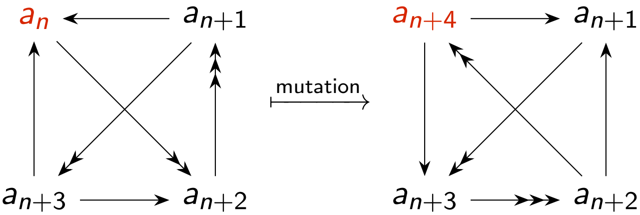
Given a sequence of quiver mutations that preserves the quiver, we obtain a finite set of algebraic relations. Such a set of algebraic relations is called a *T-system*.

Example 1: Somos-4 sequence

Somos-4 sequence $(a_n)_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} a_n a_{n+4} = a_{n+1}a_{n+3} + a_{n+2}^2, & \text{(recurrence)},\\ a_0 = a_1 = a_2 = a_3 = 1, & \text{(initial values)}. \end{cases}$$

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 88813, \ldots$$



- Mutation: $(2\pi/4 \text{ rotation}) + (a_n \mapsto a_{n+1})$
- Exchange relation: $a_n a_{n+4} = a_{n+1} a_{n+3} + a_{n+2}^2$

The Laurent phenomenon of cluster algebras implies $a_n \in \mathbb{Z}$.

Example 2: bipartite belt

Γ: Dynkin diagram of a bipartite symmetrizable generalized Cartan matrix

$$\begin{tikzcd}
 & & 5 \arrow[d, dash] & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4
\end{tikzcd} \qquad \text{(type } D_5\text{)}$$

$Q(\Gamma)$: quiver associated with Γ

$$\begin{tikzcd}
 & & 5 \arrow[d] & \\
1 & 2 \arrow[l] \arrow[r] & 3 & 4 \arrow[l]
\end{tikzcd}$$

The sequence of mutations $\mu_{\mathrm{red}} \circ \mu_{\mathrm{blue}}$ preserves the quiver $Q(\Gamma)$. Exchange relations:

$$T_a(u)T_a(u+2) = 1 + \prod_b T_b(u+1)^{n_{ba}},$$

where $N = (n_{ab})$ is the adjacency matrix of the Dynkin diagram Γ.

Theorem (Fomin-Zelevinsky, Cluster algebras IV)

The T-system

$$T_a(u)T_a(u+2) = 1 + \prod_b T_b(u+1)^{n_{ba}}$$

is periodic if and only if Γ *is of finite type. If* Γ *is indecomposable, then* $2(h+2)$ *is a period, where* h *is the Coxeter number of* Γ*.*

Characterization of T-systems

Let $r \in \mathbb{Z}_{>0}$, and $T_a(u)$ $(a \in [1, r], u \in \mathbb{Z})$ be indeterminates.
Any T-system can be written (at least) as the following expression:

$$\prod_{b=1}^{r} \prod_{p \in \mathbb{Z}} T_b(u+p)^{n^0_{ba;p}} = \prod_{b=1}^{r} \prod_{p \in \mathbb{Z}} T_b(u+p)^{n^+_{ba;p}} + \prod_{b=1}^{r} \prod_{p \in \mathbb{Z}} T_b(u+p)^{n^-_{ba;p}}$$

$$(a \in [1, r], u \in \mathbb{Z})$$

(N1) $n^0_{ab;p} = \delta_{ab}\delta_{p0} + \delta_{a\sigma(b)}\delta_{pp_a}$ for some $\sigma \in \mathfrak{S}_r$ and $p_a \in \mathbb{Z}_{>0}$,

(N2) $n^+_{ab;p} \geq 0$ and $n^-_{ab;p} \geq 0$ for any a, b, p,

(N3) $n^+_{ab;p} = 0$ and $n^-_{ab;p} = 0$ unless $0 < p < p_a$,

(N4) $n^+_{ab;p} n^-_{ab;p} = 0$ for any a, b, p.

(N1): (LHS) $= T_a(u) T_{\sigma(a)}(u + p_{\sigma(a)})$.

(N2): (RHS) $=$ (monomial) $+$ (monomial).

(N3): Any $T_a(u)$ can be written as a rational function in the initial variables $(T_a(p))_{(a,p)\in R_{\mathrm{in}}}$, where $R_{\mathrm{in}} = \{(a, p) \in [1, r] \times \mathbb{Z} \mid 0 \leq p < p_a\}$.

(N4): The two monomials in the right-hand side do not have common divisors.

We define the following matrices in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$:

$$N_0 := \left(\sum_{p\in\mathbb{Z}_{\geq 0}} n^0_{ab;p} z^p\right)_{a,b\in[1,r]},$$

$$N_+ := \left(\sum_{p\in\mathbb{Z}_{\geq 0}} n^+_{ab;p} z^p\right)_{a,b\in[1,r]},$$

$$N_- := \left(\sum_{p\in\mathbb{Z}_{\geq 0}} n^-_{ab;p} z^p\right)_{a,b\in[1,r]}.$$

Somos-4:

$$T_1(u)\,T_1(u+4) = T_1(u+2)^2 + T_1(u+1)\,T_1(u+3).$$

(N_0, N_+, N_-) is given by

$$N_0 = \left[1+z^4\right], \quad N_+ = \left[2z^2\right], \quad N_- = \left[z+z^3\right].$$

Bipartite belt:

$$T_a(u)\,T_a(u+2) = 1 + \prod_{b=1}^{r} T_b(u+1)^{n_{ba}}.$$

(N_0, N_+, N_-) is given by

$$N_0 = \mathrm{diag}(1+z^2, \ldots, 1+z^2), \quad N_+ = O, \quad N_- = zN.$$

Main result

Definition

We say that a triple of matrices $\alpha = (A_+, A_-, D)$ is a *T-datum* of size r if $A_\pm$ can be written as $A_\pm = N_0 - N_\pm$ by a triple of matrices (N_0, N_+, N_-) in $\mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$ satisfying (N1)–(N4), and D is a positive integer diagonal matrix satisfying the following conditions:

1. $N_0 D = D N_0$,
2. $D^{-1} N_\pm D \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[z])$,
3. $A_+ D A_-^\dagger = A_- D A_+^\dagger$,

where $A_\pm^\dagger := (A_\pm|_{z=z^{-1}})^\mathsf{T}$.

Theorem

Let $\alpha = (A_+, A_-, D)$ be a T-datum. Then there exists an $R_{\mathrm{in}} \times R_{\mathrm{in}}$ skew-symmetrizable integer matrix B_α such that the algebraic relations associated with (N_0, N_+, N_-) is a T-system for a sequence of mutations on B_α. Moreover, all T-systems in cluster algebras are realized in this way.

Explicit formula

The *Langlands dual* T-datum $\alpha^\vee = (A_+^\vee, A_-^\vee, D^\vee)$ is defined by

$$A_\pm^\vee = D^{-1} A_\pm D, \quad D^\vee = D^{-1}\,\mathrm{lcm}(D)\,\mathrm{gcd}(D).$$

$B_\alpha = \big(B_{(a,p)(b,q)}\big)_{(a,p),(b,q)\in R_{\mathrm{in}}}$ is given by

$$\begin{aligned} B_{(a,p)(b,q)} &= -n^+_{ab;p-q} + n^-_{ab;p-q} + \check{n}^+_{ba;q-p} - \check{n}^-_{ba;q-p} \\ &+ \sum_{c=1}^{r} \sum_{v=0}^{\min(p,q)} \big(n^+_{ac;p-v}\check{n}^-_{bc;q-v} - n^-_{ac;p-v}\check{n}^+_{bc;q-v} \big). \end{aligned}$$

The triples for Somos-4 recurrence

$$A_+ = \left[1 - 2z^2 + z^4\right], \quad A_- = \left[1 - z - z^3 + z^4\right], \quad D = \left[1\right]$$

is a T-datum since

$$\begin{aligned} A_+ D A_-^\dagger &= (1 - 2z^2 + z^4)(1 - z^{-1} - z^{-3} + z^{-4}), \\ A_- D A_+^\dagger &= (1 - z - z^3 + z^4)(1 - 2z^{-2} + z^{-4}). \end{aligned}$$

Then we have

$$B_\alpha = \begin{tikzcd} (1,0) \arrow[dr] & (1,1) \arrow[l] \arrow[dl] \\ (1,3) \arrow[u] \arrow[r] & (1,2) \arrow[u, shift left=1.5] \arrow[u] \arrow[u, shift right=1.5] \end{tikzcd} .$$

$r = 1$ case: (A_+, A_-, D) is a T-datum iff $A_\pm$ are palindromic polynomial [Fordy-Marsh].

Bipartite belts give T-data by setting $D =$ (right symmetrizer) since

$$A_+ D A_-^\dagger - A_- D A_+^\dagger = (z + z^{-1})(-DN^\mathsf{T} + ND) = 0.$$

Type D_5:

$$A_+ = \begin{bmatrix} 1+z^2 & 0 & 0 & 0 & 0 \\ 0 & 1+z^2 & 0 & 0 & 0 \\ 0 & 0 & 1+z^2 & 0 & 0 \\ 0 & 0 & 0 & 1+z^2 & 0 \\ 0 & 0 & 0 & 0 & 1+z^2 \end{bmatrix}$$

$$A_- = \begin{bmatrix} 1+z^2 & -z & 0 & 0 & 0 \\ -z & 1+z^2 & -z & 0 & 0 \\ 0 & -z & 1+z^2 & -z & -z \\ 0 & 0 & -z & 1+z^2 & 0 \\ 0 & 0 & -z & 0 & 1+z^2 \end{bmatrix}$$

$D =$ identity matrix of size 5

$$B_\alpha = \begin{tikzcd} & & (5,0) \arrow[d] & & & & (5,1) & \\ (1,0) & (2,1) \arrow[l] \arrow[r] & (3,0) & (4,1) \arrow[l] & (1,1) \arrow[r] & (2,0) & (3,1) \arrow[l] \arrow[u] \arrow[r] & (4,0) \end{tikzcd}$$

Y-systems in cluster algebras

The results on T-systems discussed so far can be generalized to T-systems *with coefficients*. The dynamics of coefficients of T-system with coefficients associated with α is governed by a *Y-system*, and this is described by the Langlands dual T-datum $\alpha^\vee = (A_+^\vee, A_-^\vee, D^\vee)$:

$$\prod_{b=1}^{r}\prod_{p\geq 0} Y_b(u-p)^{\check{n}^0_{ab;p}} = \frac{\prod_{b=1}^{r}\prod_{p\geq 0}\big(1\oplus Y_b(u-p)\big)^{\check{n}^-_{ab;p}}}{\prod_{b=1}^{r}\prod_{p\geq 0}\big(1\oplus Y_b(u-p)^{-1}\big)^{\check{n}^+_{ab;p}}}.$$

Periodic T-systems

We say that α is of *finite type* if the T-system associated with α is periodic.

Examples

1. finite type Cartan matrices [Zamolodchikov, Fomin-Zelevinsky],
2. tensor products of pairs of finite type Cartan matrices [Ravanini-Valleriani-Tateo, Keller]
3. untwisted quantum affine algebras [Kuniba-Nakanishi, Inoue-Iyama-Keller-Kuniba-Nakanishi]
4. the sine-Gordon Y-systems and the reduced sine-Gordon Y-systems, which are associated with continued fractions [Tateo, Nakanishi-Stella]
5. admissible ADE bigraphs [Galashin-Pylyavskyy]

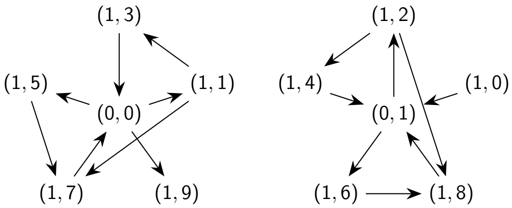
Remark

In the above list, N_0 is always diagonal.

Example for size 2

$$A_+ = \begin{bmatrix} 1+z^2 & -z \\ -z-z^5-z^9 & 1+z^{10} \end{bmatrix}, \; A_- = \begin{bmatrix} 1+z^2 & 0 \\ -z^3-z^7 & 1+z^{10} \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} T_1(u)T_1(u+2) &= T_2(u+1)T_2(u+5)T_2(u+9) + T_2(u+3)T_2(u+7) \\ T_2(u)T_2(u+10) &= T_1(u+1)+1 \end{aligned}$$

$B_\alpha =$ 

This T-system is period with period 28.

Criterion for periodicity

Theorem

Let $\alpha = (A_+, A_-, D)$ be a T-datum. If α is of finite type, then there exists a vector $v > 0$ such that $\mathring{A}_+^\mathsf{T} v > 0$ and $\mathring{A}_-^\mathsf{T} v > 0$, where $\mathring{A}_\pm = A_\pm|_{z=1}$.

(cf. symmetrizable generalized Cartan matrix A is of finite type if and only if there exists $v > 0$ such that $A^\mathsf{T} v > 0$.)

Relation to Nahm's problem

Suppose that α is of finite type and N_0 is diagonal.

Proposition

Let K be the $r \times r$ matrix defined by $K = (\mathring{A}_+)^{-1}\mathring{A}_-$. Then KD is a positive definite symmetric matrix.

Conjecture

Let $\mathcal{Z}_\alpha(q)$ be the q-series defined by

$$\mathcal{Z}_\alpha(q) := \sum_{n \in \mathbb{N}^r} \frac{q^{\frac{1}{2} n^{\mathsf{T}} K^\vee D^\vee n}}{(q^{d_1^\vee})_{n_1} \cdots (q^{d_r^\vee})_{n_r}}.$$

Then $q^C \mathcal{Z}_\alpha(q)$ is a modular function for some $C \in \mathbb{Q}$.

Evidences

1. For many α, $\mathscr{Z}_\alpha(q)$ coincides with the q-series in Rogers-Ramanujan type identities, fermionic formulae, or the candidates given by Terhoeven and Zagier.

$$\begin{aligned} K^\vee D^\vee &= \begin{bmatrix} 1+z^2 & -z \\ -z-z^5-z^9 & 1+z^{10} \end{bmatrix}^{-1} \begin{bmatrix} 1+z^2 & 0 \\ -z^3-z^7 & 1+z^{10} \end{bmatrix} \Bigg|_{z=1} \\ &= \begin{bmatrix} \frac{1+z^2-z^4-z^8+z^{10}+z^{12}}{1-z^6+z^{12}} & \frac{z+z^{11}}{1-z^6+z^{12}} \\ \frac{z+z^{11}}{1-z^6+z^{12}} & \frac{1+z^2+z^{10}+z^{12}}{1-z^6+z^{12}} \end{bmatrix} \Bigg|_{z=1} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

2. This conjecture is consistent with the asymptotic behavior of $\mathscr{Z}_\alpha(q)$ by virtue of the dilogarithm identities in cluster algebras that was proved by Nakanishi.

Summary and Conclusion

1. Characterization of T-systems and Y-systems in cluster algebras
   - T-datum $(A_+, A_-, D) \mapsto$ element in a cluster modular group
2. Relation to Nahm's problem
   - Fermionic formula
   - Rogers-Ramanujan type identities
   - Quantum Langlands modular double