Algebraic entropy of sign-stable mutation loops

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joint work with Shunsuke Kano
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Outlines

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Cluster algebra

Cluster algebra is a combinatorial theory introduced by Fomin–Zelevinsky, motivated by the study of Lusztig's total positivity. The central operation of study is the **seed mutation** $\mu_k : ((b_{ij})_{i,j\in I}, (A_i)_{i\in I}, (X_i)_{i\in I}) \to ((b'_{ij})_{i,j\in I}, (A'_i)_{i\in I}, (X'_i)_{i\in I})$:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+ & \text{otherwise,} \end{cases}$$

$$A'_i = \begin{cases} A_k^{-1}(\prod_{j\in I} A_j^{[b_{kj}]_+} + \prod_{j\in I} A_j^{[-b_{kj}]_+}) & \text{if } i = k, \\ A_i & \text{if } i \neq k, \end{cases}$$

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\mathrm{sgn}(b_{ik})})^{-b_{ik}} & \text{if } i \neq k. \end{cases}$$

Such a transformation appears in many branches of mathematics, including the representation theory, integrable systems, and the **(higher) Teichmüller theory**.

Cluster variety

The geometric aspect of the cluster algebra is formulated by Fock–Goncharov. Corresponding to the cluster A- and X-transformations, we have two schemes called the **cluster** $\mathcal{A}$- and $\mathcal{X}$-**variety** (defined later):

$$\text{Seed pattern } \boldsymbol{s} \rightsquigarrow \text{Cluster varieties } (\mathcal{A}_{\boldsymbol{s}}, \mathcal{X}_{\boldsymbol{s}}) \circlearrowleft \Gamma_{\boldsymbol{s}}$$

They have a common symmetry group $\Gamma_{\boldsymbol{s}}$, called the **cluster modular group**. It consists of 'mutation loops', which are sequence of seed mutations (and permutations) s.t. the matrix B returns to the initial one.

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> Aim: Study the dynamical systems on $\mathcal{A}_{\boldsymbol{s}}$ and $\mathcal{X}_{\boldsymbol{s}}$ induced by a mutation loop.

In some sense, as opposed to the study of 'integrable' mutation loops, we are going to study 'non-integrable'ones.

Cluster algebra and the Teichmüller–Thurston theory

- Teichmüller theory: study of complex structures on a surface Σ. (cf. Moduli space of Riemann surfaces)
- When $\chi(\Sigma) < 0$, complex structures are equivalent to hyperbolic structures $\rightarrow$ Teichmüller–Thurston theory

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Teichmüller space of Σ		$\mathbb{R}_{>0}$-points of cluster varieties
mapping class group of Σ		cluster modular group
curves (laminations) on Σ		$\mathbb{R}^T$-points of cluster varieties

[Fock–Goncharov07], [Fomin–Shapiro–Thurston08]

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Goal: 'Algebraization' of the Teichmüller–Thurston theory on MCG dynamical systems.

Ideas from Teichmüller-Thurston theory

Σ: an oriented surface of genus g and p punctures. Assume $p \geq 1$ and $\chi(\Sigma) = 2 - 2g - p < 0$.

- The Teichmüller space $T(\Sigma)$ is the set of 'marked' Riemann surfaces ($\leftrightarrow$ marked hyperbolic surfaces).
- The mapping class group $MC(\Sigma) := \pi_0(\mathrm{Homeo}^+(\Sigma, P))$ naturally acts on $T(\Sigma)$. The orbifold $T(\Sigma)/MC(\Sigma)$ is the moduli space of Riemann surfaces.
- As a certain completion of the set of homotopy classes of weighted simple closed curves, we get the space $\mathcal{ML}(\Sigma)$ of 'measured laminations' (or equivalently, 'measured foliations').

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The PL manifold $\mathcal{ML}(\Sigma)$ can be used to compactify the Teichmüller space to a topological disk $\overline{T(\Sigma)} := T(\Sigma) \cup \mathcal{PML}(\Sigma)$, where $\mathcal{PML}(\Sigma) := (\mathcal{ML}(\Sigma) \setminus \{0\})/\mathbb{R}_{>0}$.
It captures the large-scale geometry of the Teichmüller space.

Cluster structure: global coordinates on $T(\Sigma)$

An ideal triangulation $\triangle$ of Σ determines a coord. system $X_\triangle : T(\Sigma) \hookrightarrow \mathbb{R}_{>0}^{\triangle}$ by cross ratio.

There is a larger Teichmüller space $\widehat{T}(\Sigma) \supset T(\Sigma)$ to which $X_\triangle$ extends and gives a bijection

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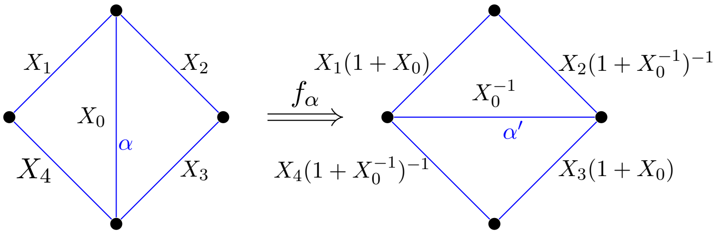
Then the coord. transformation for a flip gives an example of the <u>cluster $\mathcal{X}$-transformation</u>: [Fock–Goncharov07]



Coordinate expression of the MCG action

The action of $MC(\Sigma)$ on $\widehat{T}(\Sigma)$ is expressed as follows.
For $\triangle$ and $\phi \in MC(\Sigma)$, take a seq. of flips

$$f_\phi : \triangle \xrightarrow{f_{\alpha_1}} \triangle_1 \xrightarrow{f_{\alpha_2}} \dots \xrightarrow{f_{\alpha_m}} \triangle_m = \phi^{-1}(\triangle).$$

Let $f_\phi^x : \mathbb{R}_{>0}^{\triangle} \to \mathbb{R}_{>0}^{\phi^{-1}(\triangle)}$ be the composition of coord. transf's.

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Then the following diagram commutes: e.g. [Penner]

$$\begin{tikzcd}
\mathbb{R}_{>0}^{\triangle} \arrow[r, "f_\phi^x"] & \mathbb{R}_{>0}^{\phi^{-1}(\triangle)} \arrow[r, "\phi_*"] & \mathbb{R}_{>0}^{\triangle} \\
\widehat{T}(\Sigma) \arrow[u, "X_\triangle"] \arrow[r, equal] & \widehat{T}(\Sigma) \arrow[u, "X_{\phi^{-1}(\triangle)}"'] \arrow[r, "\phi"'] & \widehat{T}(\Sigma). \arrow[u, "X_\triangle"']
\end{tikzcd}$$

Thus the rational map $\phi^x := \phi_* \circ f_\phi^x$ gives the coord. expression of the action ϕ.

The space of measured laminations as a tropical analogue

Similarly extending the space $\mathcal{ML}(\Sigma)$, we get a PL manifold $\widehat{\mathcal{ML}}(\Sigma)$. Given $\triangle$, it has also a PL coord. system ("shear coordinates")

$$x_\triangle : \widehat{\mathcal{ML}}(\Sigma) \xrightarrow{\sim} \mathbb{R}^\triangle.$$

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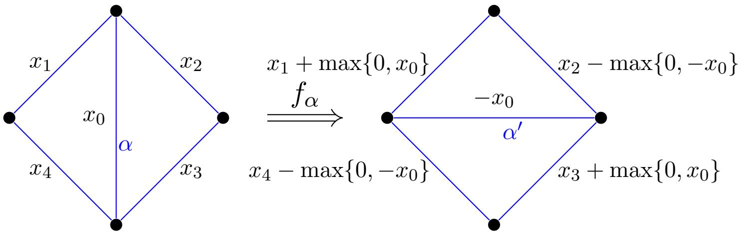
The coord. transf's are given by the **tropical analogues** of the cluster $\mathcal{X}$-transformation: [Fock–Goncharov07]



Now we have the following generalization:

Teichmüller-Thurston theory	$\rightsquigarrow$	Cluster algebra
$\widehat{T}(\Sigma)$		$\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}_{>0})$
$MC(\Sigma) \ni$ mapping class		$\Gamma_{\boldsymbol{s}} \ni$ mutation loop
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We are interested in a special kind of mapping classes.

Nielsen–Thurston classification e.g., [Fathi–Laudenbach–Poénaru]

A mapping class is either periodic, reducible (fixes a multicurve) or **pseudo-Anosov**.

A pA mapping class ϕ has North-South dynamics on the space $\widehat{\mathcal{PML}}(\Sigma)$, whose 'strength' of attraction/repulsion is called the **stretch factor** $\lambda_\phi > 1$.
Today we introduce **sign stability** for mutation loops, as an analogue of the pseudo-Anosov property.

Main Theorem (Rough Statement) [I.–Kano19]

For a sign-stable mutation loop $\phi \in \Gamma_{\boldsymbol{s}}$, both the algebraic entropies $\mathcal{E}_\phi^a$ and $\mathcal{E}_\phi^x$ of the cluster transf's are given by the log of the **cluster stretch factor** $\lambda_\phi \geq 1$.

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Here the **Algebraic entropy** $\mathcal{E}_\varphi$ of a rational map $\varphi : \mathbb{G}_m^N \dashrightarrow \mathbb{G}_m^N$ is defined as follows. [Bellon–Viallet99]

Let $\varphi^* : \mathbb{Q}(u_1, \dots, u_N) \to \mathbb{Q}(u_1, \dots, u_N)$ be the induced homomorphism on the field of rational functions, and $\varphi_i := \varphi^*(u_i)$. Then $\deg \varphi$ is the max of the degrees $\deg \varphi_i$, $i = 1, \dots, N$. Define

$$\mathcal{E}_\varphi := \limsup_{n \to \infty} \frac{1}{n} \log(\deg(\varphi^n)).$$

In the above theorem, $\mathcal{E}_\phi^a := \mathcal{E}_{\phi^a}$ and $\mathcal{E}_\phi^x := \mathcal{E}_{\phi^x}$ where ϕ^a, ϕ^x denote the cluster transf's induced by ϕ.

Remark: Connection to other areas

Top: Our main theorem is a CA analogue of the theorem by Thurston (e.g. [FLP]): the topological entropy of a pseudo-Anosov mapping class $\phi \in MC(\Sigma)$ is given by

$$\mathcal{E}_\phi^{\mathrm{top}} = \log \lambda_\phi.$$

Int: It is widely believed that $\mathcal{E}_\varphi = 0$ corresponds to the 'discrete integrability' of a dynamical system φ. e.g. [Bellon99], [Fordy–Hone14]

Rep: For a symmetrizable Kac–Moody Lie alg $\mathfrak{g}$, there is a family $\boldsymbol{s}_m(\mathfrak{g})$ $(m \in \mathbb{Z}_{\geq 2})$ s.t. $W(\mathfrak{g}) \subset \Gamma_{\boldsymbol{s}_m(\mathfrak{g})}$. [Inoue–I.–Oya19]
For $\mathfrak{g}$ infinite type, a Coxeter element $c \in W(\mathfrak{g})$ gives a 'two-sided' sign-stable mutation loop whose cluster stretch factor is the spectral radius of the **Coxeter transformation**.

[I.] in prep.

Both the stretch factors of pA mapping classes and the spectral radii of Coxeter transformations are known to attain interesting algebraic integers, *e.g.*, Salem numbers. [Pankau17], [McMullen01]

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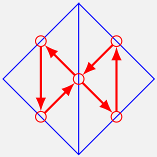
Seed patterns

Seed pattern $\boldsymbol{s} \rightsquigarrow$ Cluster ensemble $(\mathcal{A}_{\boldsymbol{s}}, \mathcal{X}_{\boldsymbol{s}}) \circlearrowleft \Gamma_{\boldsymbol{s}}$.

I: a finite set. (N, Q) is called a seed if

- $N = \bigoplus_{i \in I} \mathbb{Z} e_i$ is a lattice with a fixed basis,
- Q is a (weighted) quiver with $V(Q) = I$, no  , .

Example (Seed from an ideal triangulation $\triangle$)

Given $\triangle$, let $N^{\triangle} := \bigoplus_{\alpha \in \triangle} \mathbb{Z} e_\alpha$ and $Q^{\triangle} =$ 

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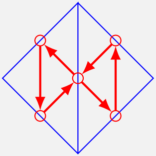
Given $\triangle$, let $N^{\triangle} := \bigoplus_{\alpha \in \triangle} \mathbb{Z} e_\alpha$ and $Q^{\triangle} =$ 

Q defines a skew-form $b_{ij} = \{e_i, e_j\} := \#\{i \to j\} - \#\{j \to i\}$ on N. Conversely a skew-sym. matrix $B = (b_{ij})_{i,j \in I}$ determines Q.

Let $\mathbb{T}_I$ be a regular $|I|$-valent tree, each edge labeled by an index $k \in I$ so that labels are distinct around each vertex.
A seed pattern is an assignment $\boldsymbol{s} : \mathbb{T}_I \ni t \mapsto (N^{(t)}, Q^{(t)})$ of a seed to each vertex t, related by the **matrix mutation** rule:

$$b_{ij}^{(t')} = \begin{cases} -b_{ij}^{(t)} & \text{if } i = k \text{ or } j = k, \\ b_{ij}^{(t)} + [b_{ik}^{(t)}]_+ [b_{kj}^{(t)}]_+ - [-b_{ik}^{(t)}]_+ [-b_{kj}^{(t)}]_+ & \text{otherwise} \end{cases}$$

for each edge $t \overset{k}{\text{——}} t'$. Here $[a]_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.

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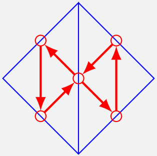
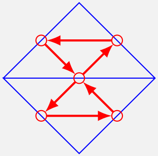
Example (Flips of ideal triangulations)

$Q^{\triangle} =$  $\xrightarrow{f_\alpha} Q^{\triangle'} =$ 

Other examples: square moves of dimer models, braid moves of wiring diagrams, etc.

Cluster $\mathcal{X}$-transformations and the cluster $\mathcal{X}$-variety

From $(N^{(t)}, Q^{(t)})$, we get an alg. torus $\mathcal{X}_{(t)} := \mathrm{Hom}(N^{(t)}, \mathbb{G}_m)$ with

- characters $X_i^{(t)} : \mathcal{X}_{(t)} \to \mathbb{G}_m$, $\psi \mapsto \psi(e_i^{(t)})$, and
- Poisson bracket $\{X_i^{(t)}, X_j^{(t)}\} := b_{ij}^{(t)} X_i^{(t)} X_j^{(t)}$.

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For $t \overset{k}{\text{———}} t'$, define $\mu_k^x : \mathcal{X}_{(t)} \to \mathcal{X}_{(t')}$ by

$$(\mu_k^x)^* X_i^{(t')} = \begin{cases} (X_k^{(t)})^{-1} & \text{if } i = k, \\ X_i^{(t)}(1 + (X_k^{(t)})^{-\mathrm{sgn}(b_{ik}^{(t)})})^{-b_{ik}^{(t)}} & \text{if } i \neq k, \end{cases}$$

which is a birational Poisson map.

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which is a birational Poisson map. Patching the alg. tori $\mathcal{X}_{(t)}$, $t \in \mathbb{T}_I$ by these maps, we get the **cluster $\mathcal{X}$-variety**

$$\mathcal{X}_{\boldsymbol{s}} := \bigcup_{t \in \mathbb{T}_I} \mathcal{X}_{(t)},$$

which is a (possibly non-separated) Poisson scheme. [Fock–Goncharov09]

Aside: Cluster $\mathcal{A}$-transformations and the cluster $\mathcal{A}$-variety

Let $M^{(t)} := \mathrm{Hom}(N^{(t)}, \mathbb{Z})$, and $(f_i^{(t)})_{i\in I}$ the dual basis of $(e_i^{(t)})_{i\in I}$. We get an alg. torus $\mathcal{A}_{(t)} := \mathrm{Hom}(M^{(t)}, \mathbb{G}_m)$ with

- characters $A_i^{(t)} : \mathcal{A}_{(t)} \to \mathbb{G}_m$, $\psi \mapsto \psi(f_i^{(t)})$, and
- closed 2-form $\omega_{\mathcal{A}} := \sum_{i,j\in I} b_{ij}^{(t)} \mathrm{d}\log A_i^{(t)} \wedge \mathrm{d}\log A_j^{(t)}$.

For $t \overset{k}{\text{---}} t'$, define $\mu_k^a : \mathcal{A}_{(t)} \to \mathcal{A}_{(t')}$ by

$$(\mu_k^a)^* A_i^{(t')} = \begin{cases} (A_k^{(t)})^{-1}(\prod_{j\in I}(A_j^{(t)})^{[b_{kj}^{(t)}]_+} + \prod_{j\in I}(A_j^{(t)})^{[-b_{kj}^{(t)}]_+}) & \text{if } i = k, \\ A_i^{(t)} & \text{if } i \neq k, \end{cases}$$

which is a birational map preserving $\omega_{\mathcal{A}}$. Similarly we get the **cluster $\mathcal{A}$-variety**

$$\mathcal{A}_{\boldsymbol{s}} := \bigcup_{t\in\mathbb{T}_I} \mathcal{A}_{(t)},$$

which is a (possibly non-separated) presymplectic scheme. There is an **ensemble map** $p : \mathcal{A}_{\boldsymbol{s}} \to \mathcal{X}_{\boldsymbol{s}}$ given by $p^* X_k^{(t)} = \prod_i (A_i^{(t)})^{b_{ki}^{(t)}}$.

Mutation loops (horizontal ones)

A 'mutation sequence' is given by an edge path in the tree $\mathbb{T}_I$.
A path $\gamma : t \to t'$ *represents* a mutation loop if $B^{(t')} = B^{(t)}$. Let $\mu_\gamma^x : \mathcal{X}_{(t)} \to \mathcal{X}_{(t')}$ be the composite of associated cluster transformations. Then it induces:

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$$\begin{tikzcd}
\mathcal{X}_{(t)} \arrow[r, "\mu_\gamma^x"] \arrow[d, hook] & \mathcal{X}_{(t')} \arrow[r, "i^x_{t',t}"] \arrow[d, hook] & \mathcal{X}_{(t)} \arrow[d, hook] \\
\mathcal{X}_{\boldsymbol{s}} \arrow[r, equal] & \mathcal{X}_{\boldsymbol{s}} \arrow[r, red, "\gamma^x"'] & \mathcal{X}_{\boldsymbol{s}},
\end{tikzcd}$$

where $i^x_{t',t}$ maps $X_k^{(t)} \mapsto X_k^{(t')}$ for all $k \in I$, which is a Poisson isomorphism.
Two paths $\gamma_\nu : t_\nu \to t'_\nu$ for $\nu = 1, 2$ are equivalent if $\gamma_1^x = \gamma_2^x$. An equivalence class $\phi = [\gamma]_{\boldsymbol{s}}$ is called a **mutation loop**.

Remark: the Cluster modular group

- In general, one considers a path $\gamma : t \to t'$ with a weaker periodicity $b^{(t')}_{\sigma(i),\sigma(j)} = b^{(t)}_{ij}$ for some permutation σ, and then defines mutation loops.
  All the mutation loops form a group $\Gamma_{\boldsymbol{s}}$, which we call the cluster modular group. It acts on $\mathcal{X}_{\boldsymbol{s}}$ and $\mathcal{A}_{\boldsymbol{s}}$.
  However, thanks to the property $\mathcal{E}_{\varphi^m} = m\mathcal{E}_\varphi$ of the alg. entropy, it suffices to consider the one with $\sigma = \mathrm{id}$ today.
- In the surface case,

$$\Gamma_{\boldsymbol{s}} \cong \begin{cases} MC(\Sigma) & \text{if } p = 1, \\ MC(\Sigma) \ltimes (\mathbb{Z}/2)^p & \text{otherwise.} \end{cases}$$

[Fomin–Shapiro–Thurston08]

- One could also define the equivalence as $\gamma_1^a = \gamma_2^a$. Both definitions turn out to be equivalent. [Nakanishi20]
- One could also formulate a mutation sequence as a path in the 'exchange graph' which is more economical.

Important examples: Kronecker quivers

Let us begin with the seed $(N^{(t_0)}, Q^{(t_0)}) := (\mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \overset{1}{\circ} \xrightarrow{k} \overset{2}{\circ})$ and consider the path $\gamma : t_0 \overset{1}{—} t_1 \overset{2}{—} t_2$. Let us compute:

$$(Q^{(t_0)}; \mathbf{X}^{(t_0)}) = (\overset{1}{\circ} \xrightarrow{k} \overset{2}{\circ}; X_1, X_2),$$

$$(Q^{(t_1)}; \mu_1^* \mathbf{X}^{(t_1)}) = (\overset{1}{\circ} \xleftarrow{k} \overset{2}{\circ}; X_1^{-1}, X_2(1+X_1)^k),$$

$$(Q^{(t_2)}; \mu_1^* \mu_2^* \mathbf{X}^{(t_2)}) = (\overset{1}{\circ} \xrightarrow{k} \overset{2}{\circ}; X_1^{-1}(1+X_2(1+X_1)^k)^k, X_2^{-1}(1+X_1)^{-k}).$$

Thus $Q^{(t_2)} = Q^{(t_0)}$, and γ represents a mutation loop ϕ. The action of ϕ on $\mathcal{X}_s$ is given by

$$(X_1, X_2) \mapsto (X_1^{-1}(1+X_2(1+X_1)^k)^k, X_2^{-1}(1+X_1)^{-k}).$$

When $k \geq 2$, ϕ has infinite order.

Tropicalizations (semifield-valued points)

Let $\mathbb{P} = (\mathbb{P}, \oplus, \cdot)$ be a semifield: $(\mathbb{P}, \cdot)$ is an abelian group, $(\mathbb{P}, \oplus)$ is an abelian semigroup and distributive.

Tropicalizations (semifield-valued points)

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For each $t \in \mathbb{T}_I$, define the set of $\mathbb{P}$-points as

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Then the set of $\mathbb{P}$-points of the cluster $\mathcal{X}$-variety is defined by

$$\mathcal{X}_{\boldsymbol{s}}(\mathbb{P}) := \bigcup_{t \in \mathbb{T}_I} \mathcal{X}_{(t)}(\mathbb{P}).$$

It may more than just a set: for example, $\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}_{>0})$ is a real-analytic manifold and $\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}^{\mathrm{trop}})$ is a PL manifold.

Here we use $\mathbb{R}^{\mathrm{trop}} := (\mathbb{R}, \min, +)$ below.

Outlines

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Presentation matrices

Seed pattern $\boldsymbol{s} \rightsquigarrow (\mathcal{A}_{\boldsymbol{s}} \xrightarrow{p} \mathcal{X}_{\boldsymbol{s}}) \overset{\bullet(\mathbb{P})}{\rightsquigarrow} (\mathcal{A}_{\boldsymbol{s}}(\mathbb{P}) \xrightarrow{p} \mathcal{X}_{\boldsymbol{s}}(\mathbb{P})) \circlearrowleft \Gamma_{\boldsymbol{s}}$

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Cosider $\mathbb{P} = \mathbb{R}^{\mathrm{trop}}$. Let us study the PL action on $\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}^{\mathrm{trop}})$ of a mutation loop. Each cluster transf. is:

$$x_i^{(t')}(\mu_k(w)) = \begin{cases} -x_k^{(t)}(w) & \text{if } i = k, \\ x_i^{(t)}(w) + [\mathrm{sgn}(x_k^{(t)}(w)) b_{ik}^{(t)}]_+ x_k^{(t)}(w) & \text{if } i \neq k, \end{cases}$$

which is linear on the interior of the half-space

$$\mathcal{H}_{k,\epsilon}^{(t)} := \{ w \in \mathcal{X}_{(t)}(\mathbb{R}^{\mathrm{trop}}) \mid \epsilon x_k^{(t)}(w) \geq 0 \}$$

for $\epsilon \in \{+,-\}$.

Let $E_{k,\epsilon}^{(t)}$ be the presentation matrix on $\mathcal{H}_{k,\epsilon}^{(t)}$. $E_{k,\epsilon}^{(t)} = \begin{pmatrix} 1 & \cdots & 0 & * & 0 & \cdots & 0 \\ & \ddots & & \vdots & & & \\ \vdots & & 1 & * & 0 & & \vdots \\ & & 0 & -1 & 0 & & \\ \vdots & & 0 & * & 1 & & \vdots \\ & & & \vdots & & \ddots & \\ 0 & \cdots & 0 & * & 0 & \cdots & 1 \end{pmatrix}$.

Sign stability (1)

The **sign** of a path $\gamma : t_0 \overset{k_0}{\text{---}} t_1 \overset{k_1}{\text{---}} \cdots \overset{k_{h-1}}{\text{---}} t_h$ at a point $w \in \mathcal{X}_{(t_0)}(\mathbb{R}^{\mathrm{trop}})$ is the seq. $\boldsymbol{\epsilon}_\gamma(w) = (\epsilon_0, \dots, \epsilon_{h-1}) \in \{+, 0, -\}^h$ defined by

$$\epsilon_i := \mathrm{sgn}(x_{k_i}^{(t_i)}(\mu_{\gamma_{\leq i}}(w)))$$

for $i = 0, \dots, h-1$. Here $\gamma_{\leq i} : t_0 \xrightarrow{(k_0,\dots,k_{i-1})} t_i$ and $\gamma_{\leq 0}$ is the constant path.

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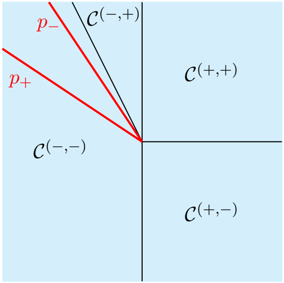
Example (Kronecker quiver)

Recall the path $\gamma : t_0 \overset{1}{-\!\!-} t_1 \overset{2}{-\!\!-} t_2$ and $\phi = [\gamma]_{\boldsymbol{s}}$, which acts on $\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}^{\mathrm{trop}})$ as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 + k\min\{0, x_2 + k\min\{0, x_1\}\} \\ -x_2 - k\min\{0, x_1\} \end{pmatrix}.$$



Sign stability (2)

Suppose that a path $\gamma : t_0 \xrightarrow{k_0} t_1 \xrightarrow{k_1} \cdots \xrightarrow{k_{h-1}} t_h$ represents a mutation loop ϕ.

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Definition

For an $\mathbb{R}_{>0}$-invariant set $\{0\} \neq \Omega \subset \mathcal{X}_{(t_0)}(\mathbb{R}^{\mathrm{trop}})$, we say that γ is sign-stable on Ω if there exists a seq $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega} \in \{+,-\}^h$ of strict signs s.t. for each $w \in \Omega \setminus \{0\}$, there exists an integer $n_0 \in \mathbb{N}$ satisfying

$$\boldsymbol{\epsilon}_\gamma(\phi^n(w)) = \boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega}$$

for all $n \geq n_0$. We call $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega}$ the *stable sign* of γ on Ω.

The previous example is sign-stable on $\Omega = \mathcal{X}_{\boldsymbol{s}}(\mathbb{R}^{\mathrm{trop}})$ for $k \geq 3$, and the stable sign is $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega} = (-,-)$.
In general, if Ω contains a ray $\mathbb{R}_{\geq 0} w$ fixed by a power ϕ^{m_0} and $\boldsymbol{\epsilon}_\gamma(w)$ is non-strict, then γ is not sign-stable on Ω.

A Perron-Frobenius property

Sign stability implies that the presentation matrix *stabilizes* to a common one $E^{\mathrm{stab}}_{\gamma,\Omega} := E^{(t_{h-1})}_{k_{h-1},\epsilon_{h-1}} \dots E^{(t_1)}_{k_1,\epsilon_1} E^{(t_0)}_{k_0,\epsilon_0}$, where $\boldsymbol{\epsilon}^{\mathrm{stab}}_{\gamma,\Omega} = (\epsilon_0, \dots, \epsilon_{h-1})$.

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Proposition

If a path as above is sign-stable on Ω, then there exists a strictly convex cone $\mathcal{C} \subset \mathcal{X}_{(t_0)}(\mathbb{R}^{\mathrm{trop}})$ which is ϕ-invariant. In particular, the spectral radius $\rho(E^{\mathrm{stab}}_{\gamma,\Omega})$ is realized by a positive eigenvalue.

Remark

It happens that $\phi = [\gamma_1]_{\boldsymbol{s}} = [\gamma_2]_{\boldsymbol{s}}$ and γ_1 is sign-stable, but γ_2 is not.

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The sign stability on the following set is basic and sufficient to determine the algebraic entropy:

$$\Omega = \Omega^{\mathrm{can}}_{(t_0)} := \operatorname{int} \mathcal{C}^{+}_{(t_0)} \cup \operatorname{int} \mathcal{C}^{-}_{(t_0)},$$

where $\mathcal{C}^{\pm}_{(t_0)} := \left\{ w \in \mathcal{X}_{(t_0)}(\mathbb{R}^{\mathrm{trop}}) \mid \pm x_i^{(t_0)}(w) \geq 0 \text{ for } i = 1, \dots, N \right\}$.

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We write $\lambda_\phi := \rho(E^{\mathrm{stab}}_{\gamma}) \geq 1$, and call it the **cluster stretch factor**. It is an algebraic integer of degree $\leq |I|$.
(Note: this is independent of a sign-stable path representing ϕ.)

Main Theorem

Recall the algebraic entropy

$$\mathcal{E}_\varphi := \limsup_{n\to\infty} \frac{1}{n} \log(\deg(\varphi^n))$$

of a rational map φ. We can effectively compute $\mathcal{E}_\phi^a$ and $\mathcal{E}_\phi^x$ if a mutation loop ϕ admits a sign-stable representation path.

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Theorem [I.–Kano19]

Suppose that a path $\gamma : t_0 \to t_h$ represents a mutation loop ϕ, and sign-stable on $\Omega^{\mathrm{can}}_{(t_0)}$. Then we have

$$\log \rho(\check{E}_\gamma^{\mathrm{stab}}) \leq \mathcal{E}_\phi^a \leq \log R_\gamma,$$
$$\log \rho(E_\gamma^{\mathrm{stab}}) \leq \mathcal{E}_\phi^x \leq \log R_\gamma.$$

Here $R_\gamma := \max\{\rho(E_\gamma^{\mathrm{stab}}), \rho(\check{E}_\gamma^{\mathrm{stab}})\}$ and $\check{A} := (A^\mathsf{T})^{-1}$.

Spectral duality conjecture

Conjecture

For any mutaton loop $\phi = [\gamma]_{\boldsymbol{s}}$ and a sign seq $\boldsymbol{\epsilon}$, the characteristic polynomial $P(\nu)$ of the pres mat $E_\gamma^{\boldsymbol{\epsilon}}$ is (anti-)palindromic: $P(\nu^{-1}) = \pm \nu^{-N} P(\nu)$.

It implies that $R_\gamma = \rho(E_\gamma^{\mathrm{stab}}) = \rho(\check{E}_\gamma^{\mathrm{stab}})$, and hence $\mathcal{E}_\phi^a = \mathcal{E}_\phi^x = \log \lambda_\phi$ as stated in the beginning.

- The conjecture holds true if the matrix $B^{(t_0)}$ is invertible, since $B^{(t_0)} E_\gamma^{\boldsymbol{\epsilon}} = \check{E}_\gamma^{\boldsymbol{\epsilon}} B^{(t_0)}$.
- The conjecture holds true for the seed obtained from a marked surface. [I.–Kano] in prep.
- The conjecture holds true for the seed pattern $\boldsymbol{s}_m(\mathfrak{g})$ and the tropical sign. [I.] in prep.

Strategy of the proof (1)

Our basic tool is the following **separation formulae**, which is a fundamental theorem in the cluster algebra: [Fomin–Zelevinsky07]

> Fix a vertex $t_0 \in \mathbb{T}_I$ and write $A_i := A_i^{(t_0)}$ and $X_i := X_i^{(t_0)}$. Then for each path $\gamma : t_0 \to t$, we have
>
> $$\mu_\gamma^* A_i^{(t)} = \prod_{j=1}^{N} A_j^{g_{ij}^{(t)}} \cdot F_i^{(t)}(p^* X_1, \ldots, p^* X_N),$$
>
> $$\mu_\gamma^* X_i^{(t)} = \prod_{j=1}^{N} X_j^{c_{ij}^{(t)}} F_j^{(t)}(X_1, \ldots, X_N)^{b_{ji}^{(t)}}.$$

Here the data $g_{ij}^{(t)}, c_{ij}^{(t)} \in \mathbb{Z}$ and $F_i^{(t)}(y_1, \ldots, y_N) \in \mathbb{Z}_{\geq 0}[y_1, \ldots, y_N]$ have their own mutation rules, involving the matrices $E_{k,\epsilon}^{(t)}$ and $\check{E}_{k,\epsilon}^{(t)}$:

Strategy of the proof (2)

Letting $C_t^{\boldsymbol{s};t_0} := (c_{ij}^{(t)})_{i,j\in I}$, $G_t^{\boldsymbol{s};t_0} := (g_{ij}^{(t)})_{i,j\in I}$, and $F_t^{\boldsymbol{s};t_0} := \deg_{y_j}(F_i^{(t)})_{i,j\in I}$, we have

$$C_{t'}^{\boldsymbol{s};t_0} = E_{k,\epsilon_k^{(t)}}^{(t)} C_t^{\boldsymbol{s};t_0}, \quad G_{t'}^{\boldsymbol{s};t_0} = \check{E}_{k,\epsilon_k^{(t)}}^{(t)} G_t^{\boldsymbol{s};t_0},$$ [Nakanishi–Zelevinsky12]

$$F_{t'}^{\boldsymbol{s};t_0} = \check{E}_{k,\epsilon_k^{(t)}}^{(t)} F_t^{\boldsymbol{s};t_0} + [\epsilon_k^{(t)} C_t^{-\boldsymbol{s},t_0}]_+^{k\bullet}.$$ []Fujiwara–Gyoda19]

Here $[M]^{k\bullet} := \mathrm{diag}(0,\dots,0,\overset{k}{1},0,\dots,0)\cdot M$, and $\epsilon_k^{(t)}$ is the **tropical sign**, which coincides with the sign at a point in $\mathrm{int}\,\mathcal{C}_{(t_0)}^+$. The sign stability implies that these mutation rules stabilize to linear and affine recurrences, and thus controled by the stable presentation matrix.

Thank you for your attention !!

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