Singularities of R-matrices, Graded Quiver Varieties and Generalized Quantum Affine Schur-Weyl Duality

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<u>Plan</u>

1. Motivation
2. Singularities of R-matrices
3. Relation to quiver representations
4. Relation to graded quiver varieties
5. Generalized quantum affine Schur-Weyl duality

1. Motivation   Classical Theory

$\mathfrak{g}$: f.d. simple Lie alg / $\mathbb{C}$

$C := (c_{ij})_{i,j \in I}$: Cartan matrix

$\rightsquigarrow \quad \mathscr{C}_0 := U(\mathfrak{g})\text{-mod}_{fd}$: monoidal abelian category

symmetric   semisimple

i.e. $V \otimes W \cong W \otimes V$

$\rightsquigarrow$ Grothendieck ring

$$K(\mathscr{C}_0) \cong \mathbb{Z}[X_i \mid i \in I]$$

fundamental modules $\quad \underset{[V_i]}{\Psi} \longleftrightarrow X_i$

Recall : Grothendieck ring

$$K(\mathcal{C}_0) = \bigoplus_{V \in \mathrm{Irr}\,\mathcal{C}} \mathbb{Z}[V] \quad \text{free abelian group}$$

product $[U]\cdot[W] = [U \otimes W]$

$$= \sum_{V \in \mathrm{Irr}\,\mathcal{C}_0} m_V^{U,W} [V]$$

where $m_V^{U,W}$ = multiplicity of V in $U \otimes W$

Jordan-Hölder

Quantum affinization

$\mathfrak{g}$: f.d. simple Lie alg / $\mathbb{C}$ of type ADE

$C := (c_{ij})_{i,j\in I}$: Cartan matrix

$\xrightarrow{\text{"affinize"}}$ $L\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$: loop alg

$\xrightarrow{\text{"quantize"}}$ $U_q(L\mathfrak{g})$: quantum loop alg (Hopf alg / $\mathbb{k} := \mathbb{Q}(q)$)

$\mathcal{C} := U_q(L\mathfrak{g})\text{-mod}_{fd}$: monoidal abelian category

non-semisimple

non-symmetric

$\simeq U_q'(\hat{\mathfrak{g}})\text{-mod}_{fd}$

$V \otimes W \not\cong W \otimes V$ in general.

($\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d \supset \hat{\mathfrak{g}}' = L\mathfrak{g} \oplus \mathbb{C}K$)

Grothendieck ring is commutative !! [Frenkel-Reshetikhin]

$$\begin{array}{ccccc} & K(\mathscr{C}) & \cong & \mathbb{Z}[X_{i,a} \mid (i,a) \in I \times \mathbb{k}^\times] \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{fundamental modules} & [V_i(a)] & \longleftrightarrow & X_{i,a} \end{array}$$

Basic questions

- What is the structure of $V_i(a) \otimes V_j(b)$?
- When do they (strongly) commute ?

$$V_i(a) \otimes V_j(b) \overset{?}{\cong} V_j(b) \otimes V_i(a)$$

"Monoidal categorification of Cluster alg." (Hernandez-Leclerc 2010)

Expectation :

$$\begin{array}{ccc} K(\mathcal{C}) & \overset{?}{\cong} & \text{Cluster alg} \\ \cup & & \cup \\ \operatorname{Irr}\mathcal{C} & \hookleftarrow & \{\text{cluster monomials}\} \end{array}$$

$\boxed{\text{(max'l) comm. family of simple modules}}$ $\overset{?}{\longleftrightarrow}$ $\boxed{\text{cluster}}$

$\boxed{\text{failure of } V \otimes W \cong W \otimes V}$ $\overset{?}{\longleftrightarrow}$ $\boxed{\text{mutation}}$

2. Singularities of R-matrices

Fix $i \in I$, $\{V_i(a)\}_{a \in \Bbbk^\times}$ forms a continuous family.

$\ni v_i$: highest weight vector

z_1, z_2 : formal parameters

$$\exists!\ R_{i,j}(u) : \begin{array}{ccc} V_i(z_1) \otimes V_j(z_2) & \xrightarrow{\sim} & V_j(z_2) \otimes V_i(z_1) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v_i \otimes v_j & \longmapsto & v_j \otimes v_i \end{array}$$

R-matrix (normalized)

a matrix-valued rational function in $u = z_2/z_1$

$\rightsquigarrow$ $d_{i,j}(u) \in \Bbbk[u]$: denominator of $R_{i,j}(u)$

Example $\quad \mathfrak{g} = \mathfrak{sl}_2$

$$V_1(z) = \mathbb{k}(z)^{\oplus 2}$$

$$R_{1,1}(z_2/z_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1-q^2}{z_2/z_1 - q^2} & \dfrac{q(z_2/z_1 - 1)}{z_2/z_1 - q^2} & 0 \\ 0 & \dfrac{q(z_2/z_1 - 1)}{z_2/z_1 - q^2} & \dfrac{(1-q^2)\, z_2/z_1}{z_2/z_1 - q^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \quad d_{1,1}(u) = u - q^2$$

Th'm (Chari, Kashiwara, Varagnolo-Vasserot, Frenkel-Mukhin, ...)

TFAE:

(1) $V_i(a) \otimes V_j(b)$ is irreducible;

(2) $V_i(a) \otimes V_j(b) \cong V_j(b) \otimes V_i(a)$;

(3) $d_{i,j}(b/a) \neq 0$, $d_{j,i}(a/b) \neq 0$.

Rem Explicit computations of $\{d_{i,j}(u)\}_{i,j \in I}$ are known:

- type A by Date-Okado (1994)
- type D by Kang-Kashiwara-Kim (2015)
- type E by Oh-Scrimshaw (2019)

A unified denominator formula

Def. (Quantum Cartan matrix)

$$C(z) := \left([c_{ij}]_z \right)_{i,j \in I} = z^{-1}\left(\mathrm{id} + \text{higher deg.}\right) \in GL_I(\mathbb{Z}(\!(z)\!))$$

$$[c_{ij}]_z = \begin{cases} z + z^{-1} & i = j \\ c_{ij} & i \neq j \quad (0 \text{ or } -1) \end{cases}$$

$$\rightsquigarrow \quad \widetilde{C}(z) := C(z)^{-1}$$

$$\widetilde{C}_{ij}(z) = \sum_{\ell \geq 0} \widetilde{c}_{ij}(\ell)\, z^{\ell} \in \mathbb{Z}[\![z]\!]$$

Example

type A_2

$$C(z) = \begin{bmatrix} z+z^{-1} & -1 \\ -1 & z+z^{-1} \end{bmatrix}$$

$$\rightsquigarrow \tilde{C}(z) = \frac{1}{z^2+1+z^{-2}} \begin{bmatrix} z+z^{-1} & 1 \\ 1 & z+z^{-1} \end{bmatrix}$$

$$= \frac{1}{z^3-z^{-3}} \begin{bmatrix} z^2-z^{-2} & z-z^{-1} \\ z-z^{-1} & z^2-z^{-2} \end{bmatrix}$$

$$\frac{1}{z^3-z^{-3}} = -z^3 \sum_{k\geq 0} z^{6k}, \qquad 2h = 6$$

Properties. h : Coxeter number of $\mathfrak{g}$

$i \mapsto i^*$ involution

$w_0 \alpha_i = -\alpha_{i^*}$

(1) $\widetilde{c}_{ij}(\ell) = \widetilde{c}_{ji}(\ell)$

(2) $\widetilde{c}_{ij}(\ell + h) = -\widetilde{c}_{i^*j}(\ell) \quad \forall \ell \geq 1$

(3) $\widetilde{c}_{ij}(\ell + 2h) = \widetilde{c}_{ij}(\ell) \quad \forall \ell \geq 1$

(4) $\widetilde{c}_{ij}(kh) = 0 \quad \forall k \in \mathbb{Z}_{\geq 0}$

(5) $\widetilde{c}_{ij}(\ell) \geq 0 \quad$ if $\quad 0 \leq \ell \leq h$

Th'm 1. (A unified denominator formula)

$\forall i, j \in I$

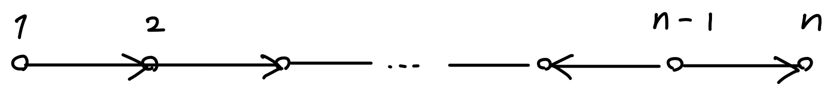
$$d_{i,j}(u) = \prod_{\ell=0}^{h} \left(u - q^{\ell+1}\right)^{\widetilde{c}_{ij}(\ell)}$$

3. Relation to quiver representations

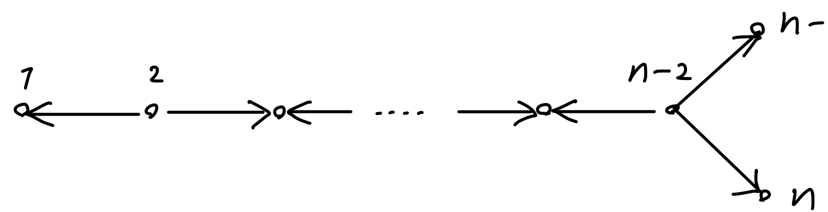
Fix $Q = (\underbrace{Q_0}_{= I}, Q_1)$ a Dynkin quiver (type $\mathfrak{g}$)

Examples

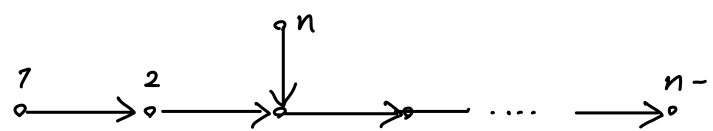
- type A_n



- type D_n



- type E_n $(n = 6, 7, 8)$



$\mathrm{Rep}(Q)$: category of f.d. rep's of Q / $\mathbb{C}$

<u>Th'm</u> (Gabriel 1972)

$$\begin{array}{ccc}
\mathrm{Ind}(\mathrm{Rep}(Q)) & \overset{1:1}{\longleftrightarrow} & R^+ := \{\text{positive roots of } \mathfrak{g}\} \\
M & \longmapsto & \underline{\dim}\, M = \sum_{i \in I} (\dim_{\mathbb{C}} M_i) \cdot \alpha_i \\
M_\alpha & \longleftarrow\!\!\shortmid & \alpha
\end{array}$$

$$\left[\begin{array}{ccc}
S_i \ (\text{simple}) & \longmapsto & \alpha_i \\
\downarrow & & \\
I_i \ (\text{injective}) & \longmapsto & \displaystyle\sum_{\substack{\exists\ \text{path} \\ j \to \cdots \to i \text{ in } Q}} \alpha_j
\end{array}\right]$$

Fix $\xi : I \longrightarrow \mathbb{Z}$ "height function"

s.t. $\xi_i = \xi_j + 1$ if $\overset{i}{\circ} \longrightarrow \overset{j}{\circ} \in Q_1$

<u>Examples</u> ξ

$$Q = \left(\overset{1}{\circ} \longrightarrow \overset{0}{\circ} \longrightarrow \overset{-1}{\circ} \right), \qquad Q = \left(\overset{1}{\circ} \longrightarrow \overset{0}{\circ} \longleftarrow \overset{1}{\circ} \right)$$

$$Q = \left(\begin{tikzcd}[row sep=small] & & \overset{-1}{\circ} \\ \overset{1}{\circ} \arrow[r] & \overset{0}{\circ} \arrow[ur] \arrow[dr] & \\ & & \underset{-1}{\circ} \end{tikzcd} \right)$$

<u>Rem</u> ξ is unique up to constant difference

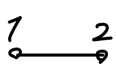
Def. (Repetition quiver)

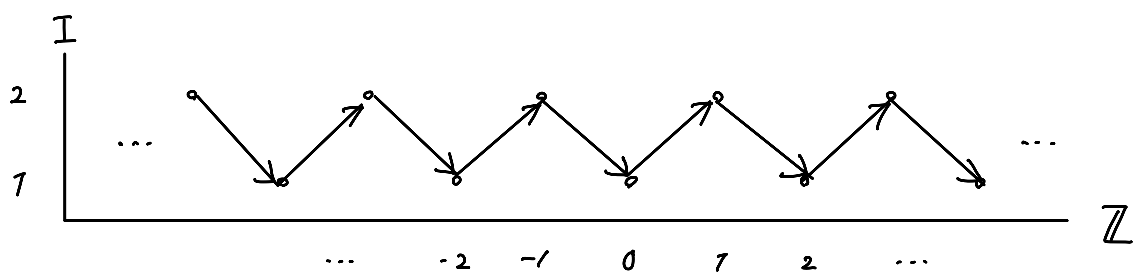
$$\Delta = (\Delta_0, \Delta_1)$$

$$\left(\begin{array}{l} \Delta_0 := \{ (i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z} \}, \\ \Delta_1 := \{ (i,p) \to (j,p+1) \mid (i,p),(j,p+1) \in \Delta_0,\ c_{ij} = -1 \} \end{array} \right.$$

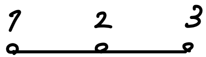
Examples

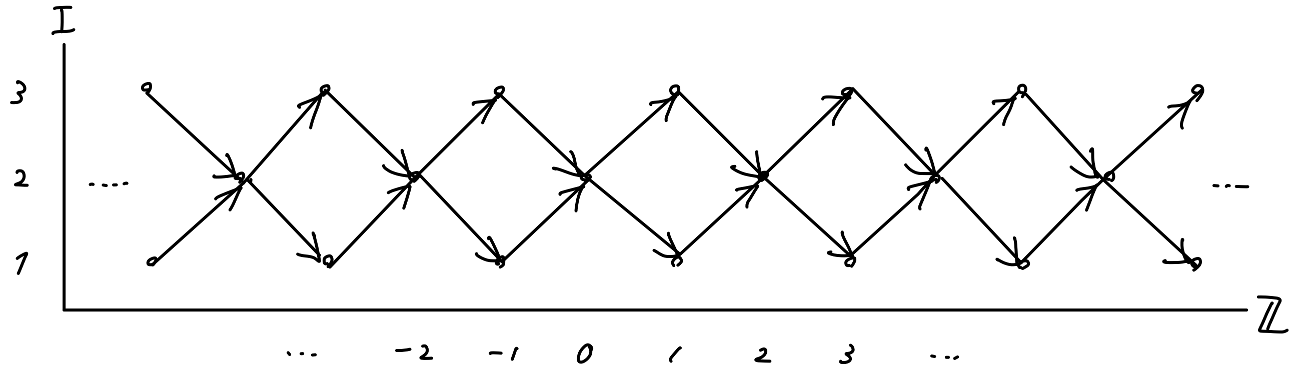
· type A_2



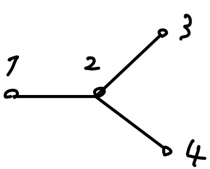


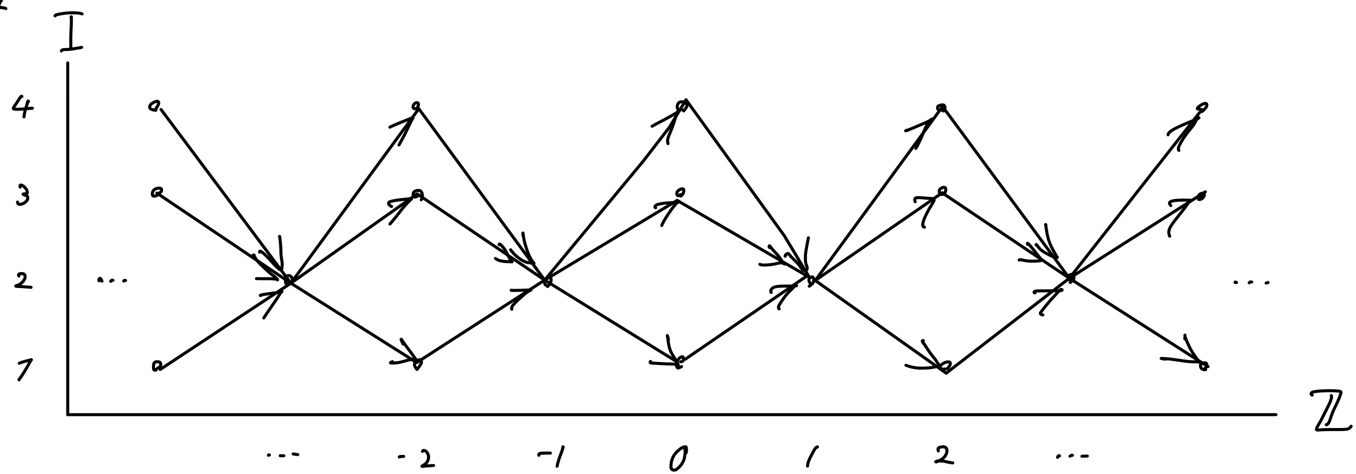
· type A_3





· type D_4





Th'm (Happel 1987)

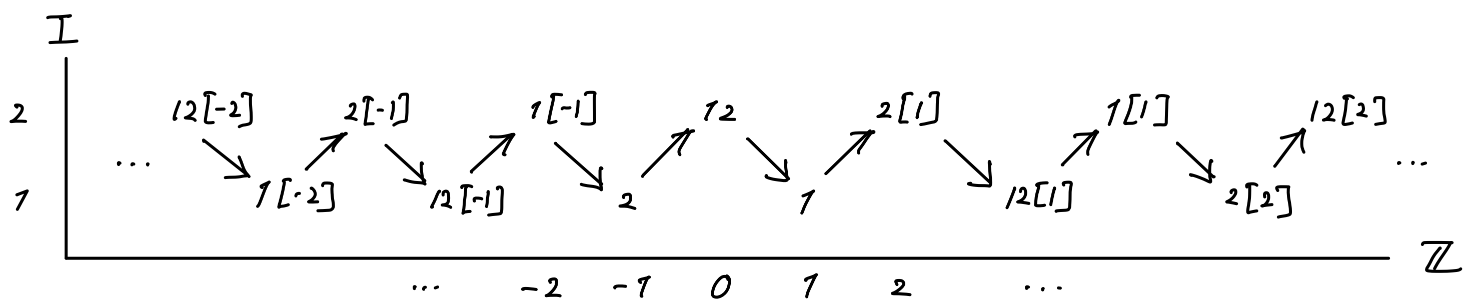
$$\exists\ H_Q : \Delta \xrightarrow{\ \sim\ } \text{AR-quiver of } \mathcal{D}_Q := D^b(\mathrm{Rep}(Q))$$

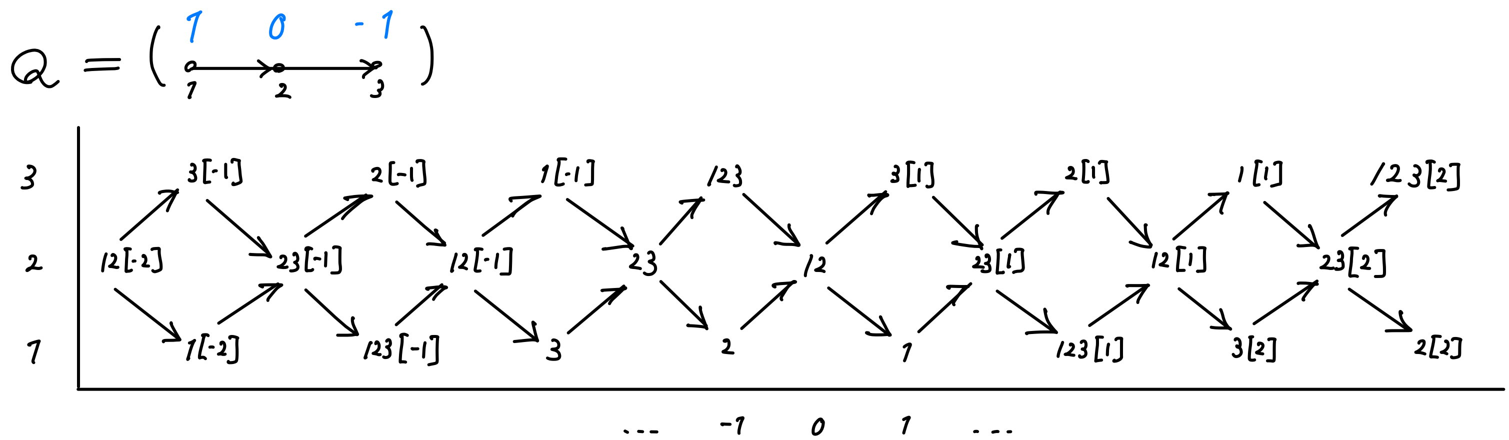
$$\Delta_0 \xleftrightarrow{\ 1:1\ } \mathrm{Ind}(\mathcal{D}_Q) = \left\{ M_\alpha[k] \;\middle|\; \begin{array}{l} \alpha \in R^+ \\ k \in \mathbb{Z} \end{array} \right\}$$

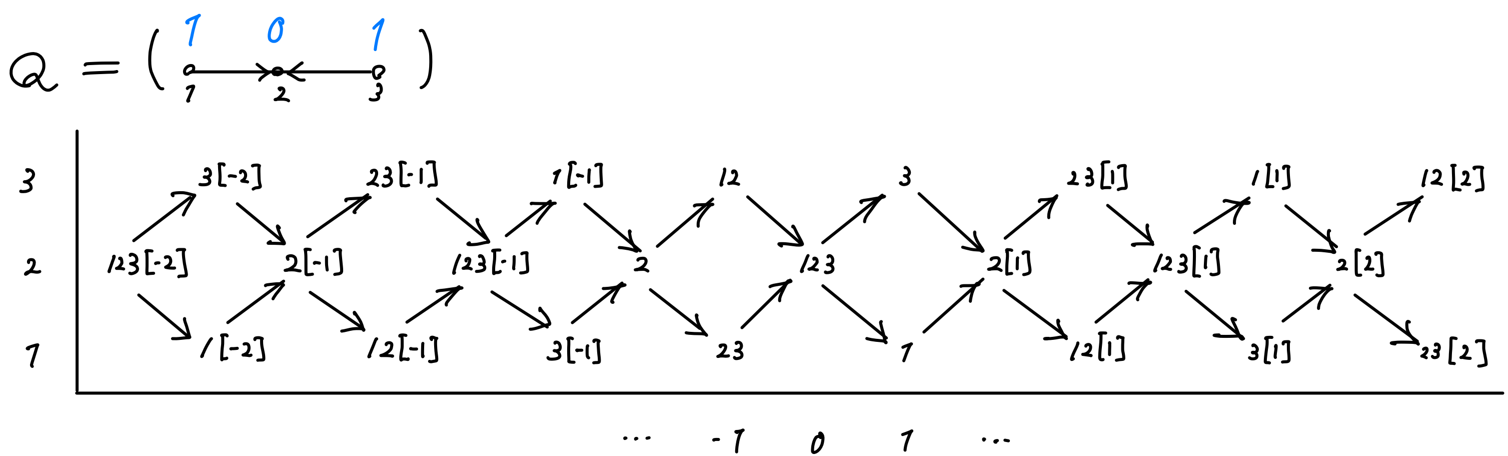
$$(i, p) \longmapsto \underbrace{\tau}_{\text{AR translation}}{}^{(\xi_i - p)/2}(\underbrace{I_i}_{\text{injective}})$$

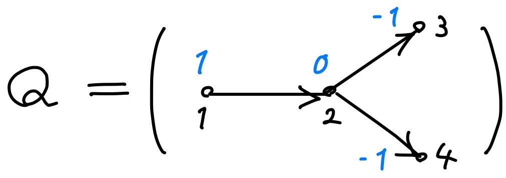
Examples of Δ

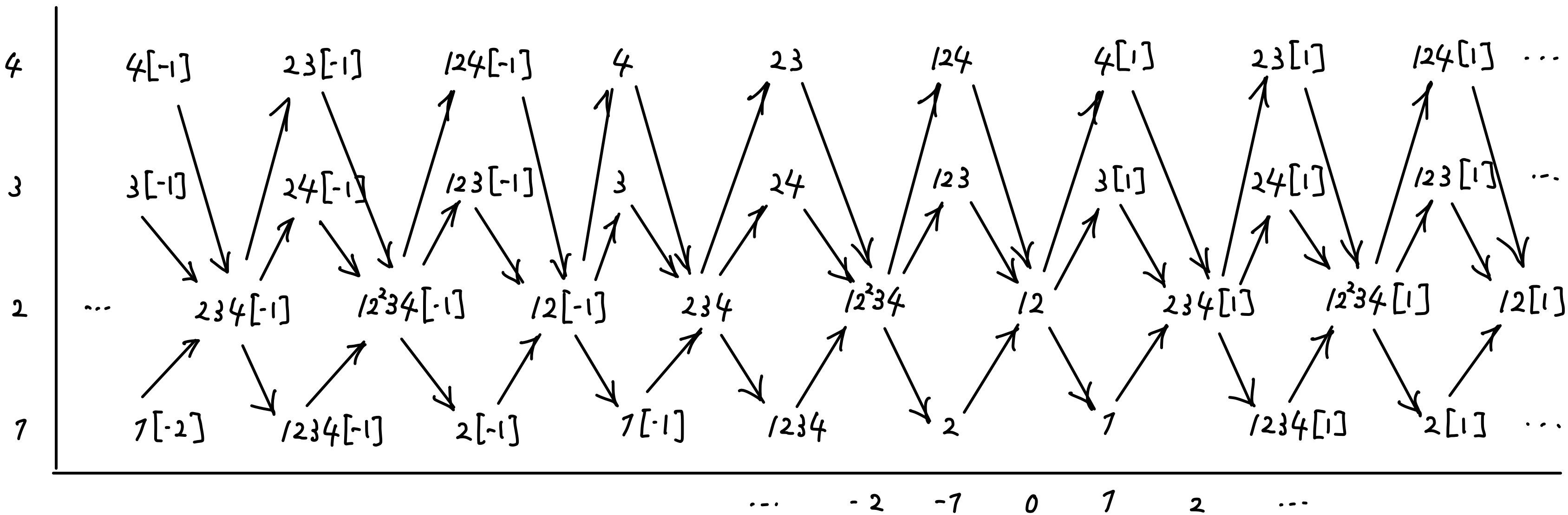
$$Q = \left(\overset{1}{\underset{1}{\circ}} \longrightarrow \overset{0}{\underset{2}{\circ}} \right) \rightsquigarrow R^+ = \{ 1, 2, 12 \}$$




Q =
1 0 -1
1 2 3
3 2 1
3[-1] 2[-1] 1[-1] 123 3[1] 2[1] 1[1] 123[2]
12[-2] 23[-1] 12[-1] 23 12 23[1] 12[1] 23[2]
1[-2] 123[-1] 3 2 1 123[1] 3[2] 2[2]
... -1 0 1 ...


Q =
1 0 1
1 2 3
3 2 1
3[-2] 23[-1] 1[-1] 12 3 23[1] 1[1] 12[2]
123[-2] 2[-1] 123[-1] 2 123 2[1] 123[1] 2[2]
1[-2] 12[-1] 3[-1] 23 1 12[1] 3[1] 23[2]
... -1 0 1 ...


Q =
1
0
-1
-1
1
2
3
4


4
3
2
1
4[-1]
23[-1]
124[-1]
4
23
124
4[1]
23[1]
124[1]
3[-1]
24[-1]
123[-1]
3
24
123
3[1]
24[1]
123[1]
234[-1]
12²34[-1]
12[-1]
234
12²34
12
234[1]
12²34[1]
12[1]
1[-2]
1234[-1]
2[-1]
1[-1]
1234
2
1
1234[1]
2[1]
... -2 -1 0 1 2 ...

$\underline{\text{Prop.}}$ (Hernandez - Leclerc 2015) $i, j \in I$, $\ell \in \mathbb{Z}_{\geq 0}$

$$\widetilde{C}_{ij}(\ell) = \begin{cases} (\varpi_i, \underline{\dim} H_Q(j, \xi_i - \ell - 1)) & \text{if } \xi_i - \ell - 1 \equiv \xi_j \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\cdot$ $\underline{\dim} M_\alpha[k] = (-1)^k \alpha$

$\cdot$ $(\varpi_i, \alpha_j) = \delta_{ij}$

computable from AR-quiver of $\mathcal{D}_Q$!!

Cor. (F.)

$(i,p),(j,r) \in \Delta_0$

$$\dim \mathrm{Ext}^1\big(H_Q(j,r), H_Q(i,p)\big) = \begin{cases} \tilde{c}_{ij}(r-p-1) & \text{if } 0 \leqslant r-p-1 \leqslant h \\ 0 & \text{otherwise} \end{cases}$$

Thm 2 ($\overset{\text{equiv.}}{\Longleftrightarrow}$ unified denominator formula)

$(i,p),(j,r) \in \Delta_0$

$$\underset{u=q^r/q^p}{\mathrm{pole}}\, R_{i,j}(u) = \dim \mathrm{Ext}^1\big(H_Q(j,r), H_Q(i,p)\big)$$

pole order

Notation $x = (i, p) \in \Delta_0$

$$H_Q(x) \in \operatorname{Ind} \mathcal{D}_Q \quad \leftsquigarrow \quad x \quad \rightsquigarrow \quad V(x) := V_i(q^p) \in \operatorname{Irr} \mathcal{C}$$

Cor. of Th'm 2. $x, y \in \Delta_0$

TFAE:

(1) $V(x) \otimes V(y)$ is irreducible ;

(2) $V(x) \otimes V(y) \cong V(y) \otimes V(x)$ in $\mathcal{C}$;

(3) $\operatorname{Ext}^1(H_Q(x), H_Q(y)) = 0 = \operatorname{Ext}^1(H_Q(y), H_Q(x))$

<u>Rem.</u> ("Additive categorification of Cluster alg.")

$\mathcal{D}$: triangulated category ("cluster category")

rough philosophy

$$\mathrm{Obj}\,\mathcal{D} \xrightarrow{\text{"CC"}} \text{Cluster alg.}$$

$\boxed{\text{max'l } \mathrm{Ext}^1\text{-free family of indecomp. obj (cluster tilting obj)}} \overset{?}{\longleftrightarrow} \boxed{\text{cluster}}$

$\boxed{(T \to ? \to T' \xrightarrow{+1}) \neq 0 \in \mathrm{Ext}^1_{\mathcal{D}}(T', T)} \overset{?}{\longleftrightarrow} \boxed{\text{mutation}}$

4. Relation to graded quiver varieties

Usual Nakajima quiver variety

$$\bar{W} = \bigoplus_{i \in I} \bar{W}_i \ , \quad \bar{V} = \bigoplus_{i \in I} \bar{V}_i \quad : \ I\text{-gr. } \mathbb{C}\text{-vec sp}$$

$$\rightsquigarrow \ \mathbb{N}(\bar{V}, \bar{W}) := \bigoplus_{i \to j \in Q_1} \mathrm{Hom}(\bar{V}_i, \bar{V}_j) \oplus \bigoplus_{i \in I} \mathrm{Hom}(\bar{W}_i, \bar{V}_i)$$

$$\circlearrowleft \ G_{\bar{V}} \times G_{\bar{W}} \times \mathbb{C}^\times$$

$$\begin{array}{ccc} \rightsquigarrow \ \mathfrak{M}(\bar{W}) = & \bigsqcup_{\bar{V}} & T^*\mathbb{N}(\bar{V}, \bar{W}) /\!/\!/_{gen} G_{\bar{V}} \\ \downarrow & & \downarrow \\ \mathfrak{M}_0(\bar{W}) = & \bigcup_{\bar{V}} & T^*\mathbb{N}(\bar{V}, \bar{W}) /\!/\!/_{0} G_{\bar{V}} \end{array} \quad \circlearrowleft \ G_{\bar{W}} \times \mathbb{C}^\times$$

Graded quiver varieties

$W = \bigoplus_{x \in \Delta_0} W_x$ Δ_0-gr. $\mathbb{C}$-vec. sp. $\rightsquigarrow$ $\overline{W}$: underlying I-gr. vec. sp.

$$\begin{array}{ccc} \mathfrak{M}(\overline{W}) & & \\ \downarrow & \circlearrowleft \; G_{\overline{W}} \times \mathbb{C}^\times & \supset \; G_W = \prod_x GL(W_x) \\ \mathfrak{M}_0(\overline{W}) & & \supset \; T_W = \left\{ \left(\bigoplus_{(i,p) \in \Delta_0} t^p \cdot \mathrm{id}_{W_{(i,p)}}, \; t \right) \middle| \; t \in \mathbb{C}^\times \right\} \end{array}$$

T_W-fixed part $\rightsquigarrow$

$$\begin{array}{ccc} \mathfrak{M}^\bullet(W) & := & \mathfrak{M}(\overline{W})^{T_W} \\ \downarrow & & \qquad \circlearrowleft \; G_W \\ \mathfrak{M}_0^\bullet(W) & := & \mathfrak{M}_0(\overline{W})^{T_W} \end{array}$$

Nakajima $\exists$ alg hom $U_q(L\mathfrak{g}) \longrightarrow \widehat{K}^{G_W}\left(\mathfrak{M}^\bullet(W) \times_{\mathfrak{M}_0^\bullet(W)} \mathfrak{M}^\bullet(W) \right)_{\Bbbk}$

Keller-Scherotzke's embedding

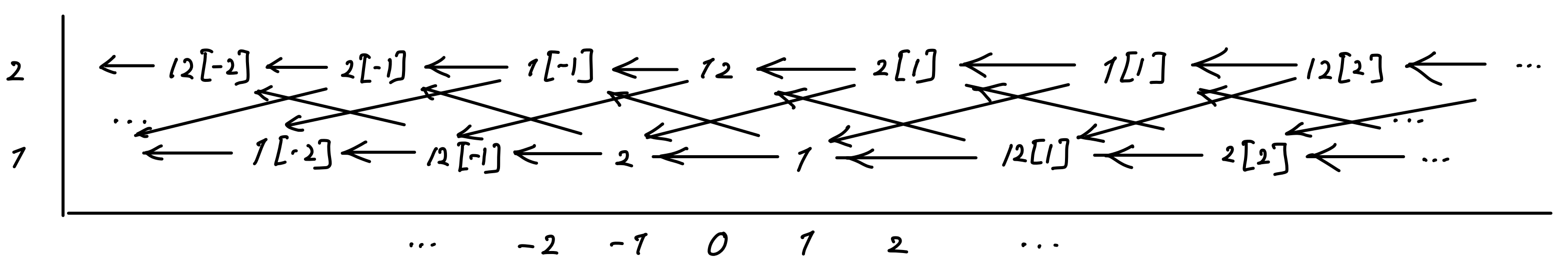
Def. ("Ext^1 quiver")

$$\Gamma = (\Gamma_0, \Gamma_1)$$

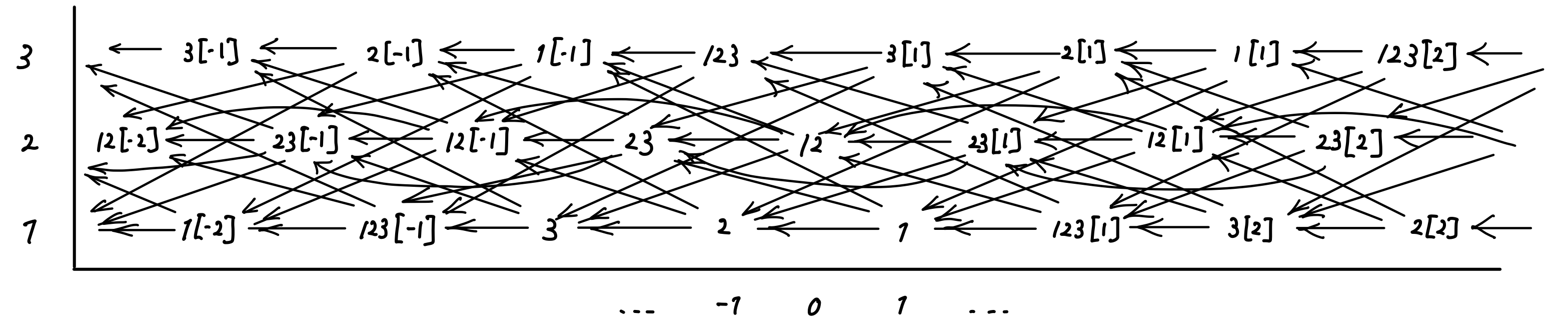
$$\begin{cases} \Gamma_0 := \Delta_0 \\ \Gamma_1 : \ \#\{x \longrightarrow y\} := \dim \mathrm{Ext}^1(H_Q(x), H_Q(y)) \end{cases}$$

Examples of Γ

$$Q = \left(\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \right)$$



$$Q = \left(\begin{tikzcd} \underset{1}{\circ} \arrow[r] & \underset{2}{\circ} \arrow[r] & \underset{3}{\circ} \end{tikzcd} \right)$$



Th'm (Keller - Scherotzke 2016)

$\exists$ G_W - equiv. closed embedding

$$\mathfrak{M}_0^\bullet(W) \hookrightarrow \mathrm{rep}(\Gamma, W) := \bigoplus_{(x \to y) \in \Gamma_1} \mathrm{Hom}(W_x, W_y)$$

+ [Nakajima]

$\rightsquigarrow$ $\left\{ \begin{array}{l} \bullet \text{ Proof of Th'm 2 } (\Leftrightarrow \text{ denominator formula}) \\ \bullet \text{ Geometric interpretation of} \\ \quad \text{generalized quantum affine Schur - Weyl duality} \end{array} \right.$

5. Generalized quantum affine Schur-Weyl duality

classical: $\mathfrak{sl}_n \curvearrowright (\mathbb{C}^n)^{\otimes d} \curvearrowleft \mathfrak{S}_d$

$\rightsquigarrow$

affine Hecke alg

quantum affine: $U_q(L\mathfrak{sl}_n) \curvearrowright (V_1(z))^{\otimes d} \curvearrowleft H^{\mathrm{aff}}(\mathfrak{S}_d)$

[Cherednik, Chari - Pressley, Ginzburg - Reshetikhin - Vasserot ...]

$\rightsquigarrow$

generalized: $U_q(L\mathfrak{g}) \curvearrowright \hat{V}^{\otimes \nu} \curvearrowleft$ Quiver Hecke alg

[Kang - Kashiwara - Kim]

Kang-Kashiwara-Kim (2018)

Given a family $\{V^j\}_{j\in J}$ of f.d. simple $U_q(L\mathfrak{g})$-modules

$\rightsquigarrow$ Define a quiver $\Gamma^J = (\Gamma^J_0, \Gamma^J_1)$ by

$$\begin{cases} \Gamma^J_0 := J \\ \Gamma^J_1 : \ \#\{i \to j\} := \operatorname{pole}_{u=1} R_{V^j, V^i}(u) \end{cases}$$

$\rightsquigarrow$ $\forall\, \nu \in \mathbb{N}J$

Quiver Hecke alg. $H_\nu(\Gamma^J) = \langle \overset{\text{generators}}{x_k, \tau_\ell, \dots} \rangle_{\text{gr. } \Bbbk\text{-alg}} / \text{rel.}$

Thm (KKK 2018) $\forall \nu \in \mathbb{N}J \ \exists$ bimodule

$$U_q(L\mathfrak{g}) \curvearrowright \hat{V}^{\otimes \nu} \curvearrowleft \hat{H}_\nu(\Gamma^J)$$

$$\hat{V}^{\otimes \nu} \stackrel{\text{def}}{=} \bigoplus_{j_1 + \cdots + j_{|\nu|} = \nu} \hat{V}^{j_1} \otimes \cdots \otimes \hat{V}^{j_{|\nu|}}$$

$$\rightsquigarrow F^J : \bigoplus_\nu \hat{H}_\nu(\Gamma^J)\text{-mod}_{fd} \longrightarrow \mathcal{C}$$

monoidal functor

$$M \longmapsto \hat{V}^{\otimes \nu} \otimes_{\hat{H}_\nu(\Gamma^J)} M$$

"generalized quantum affine Schur-Weyl duality functor"

Special case

Assume $\forall V^j$ is fundamental

$\rightsquigarrow \exists x : J \hookrightarrow \Delta_0$ s.t. $V^j \cong V(x(j))$ $\forall j \in J$

By Thm 2.

$$\Gamma^J = \Gamma|_J \overset{\text{full sub}}{\subset} \Gamma$$

$\rightsquigarrow \forall W$: J-gr. $\mathbb{C}$-vec.sp

Lusztig's quiver-flag variety

$$\left(\begin{tikzcd} \mathfrak{M}^\bullet(W) \arrow[dr] & & & \mathcal{F}\ell \arrow[dl] \\ & \mathfrak{M}^\bullet_0(W) \arrow[r, hook, "{[KS]}"] & E := \mathrm{rep}(\Gamma^J, W) & \end{tikzcd}\right) \circlearrowleft G_W$$

<u>Thm 3.</u> $\exists$ comm. diagram

$$\begin{tikzcd}
U_q(L\mathfrak{g}) \arrow[d, "{[\text{Nakajima}]}"'] & \curvearrowright & \hat{V}^{\otimes \nu} \arrow[d, "\wr"] & \curvearrowleft & \hat{H}_\nu(\Gamma^J) \arrow[d, "\wr", "{[\text{Varagnolo - Vasserot}]}"'] \\
\hat{K}^{G_W}(\mathfrak{M}(W) \underset{E}{\times} \mathfrak{M}(W))_{\Bbbk} & \curvearrowright & \hat{K}^{G_W}(\mathfrak{M}(W) \underset{E}{\times} \mathcal{F}\ell)_{\Bbbk} & \curvearrowleft & \hat{K}^{G_W}(\mathcal{F}\ell \underset{E}{\times} \mathcal{F}\ell)_{\Bbbk}
\end{tikzcd}$$

Example 1.

Define $x : J = I \hookrightarrow \Delta_0$ by

$$i \longmapsto H_Q^{-1}(S_i)$$

$$\dim \operatorname{Ext}^1(S_i, S_j) = \#\{ i \to j \text{ in } Q \}$$

$\rightsquigarrow \quad \Gamma^J = Q$

$\mathcal{M}_0^\bullet(W) \cong \operatorname{rep}(Q, W)$ ← recover [HL15]

$\rightsquigarrow \quad F^J : \bigoplus_{\nu \in \mathbb{N}I} \hat{H}_\nu(Q)\text{-mod}_{fd} \xrightarrow{\sim} \mathcal{C}_Q \subset \mathcal{C}$ "supported on $\operatorname{Rep} Q$ $\cap$ $\mathcal{D}_Q$"

$\rightsquigarrow \quad \mathbb{C}[N] \xrightarrow{\sim} K(\mathcal{C}_Q)$

max'l unip. group

dual canonical basis $\xleftrightarrow{1:1}$ Irr

Example 2.

Assume $\exists\, Q' = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \circ \to \overset{N-1}{\circ} \right) \overset{\text{full}}{\subset} Q \qquad (N \geq 1)$

Define $x : J := \mathbb{Z} \hookrightarrow \Delta_0$ by

$$x(j) := \begin{cases} H_Q^{-1}(S_i[-2k]) & \text{if } j = i + kN,\ 1 \leq i < N \\ H_Q^{-1}(M_\theta[-2k+1]) & \text{if } j = kN \end{cases}$$

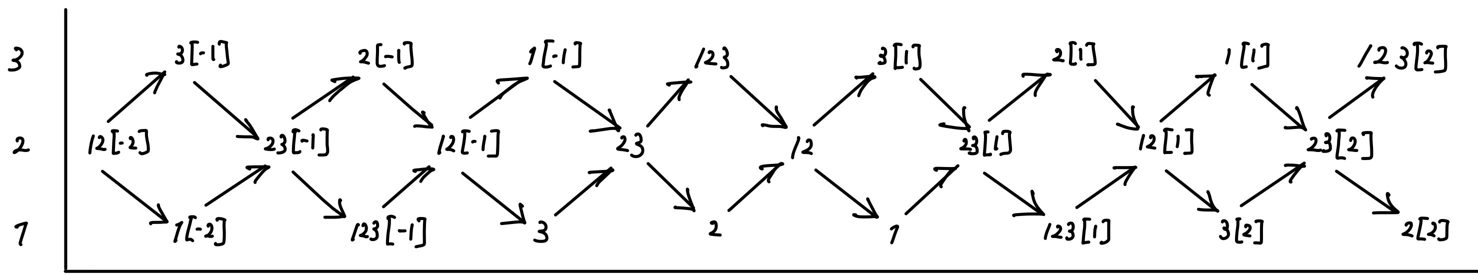
where $\theta := \alpha_1 + \cdots + \alpha_{N-1}$

$$\rightsquigarrow \quad \Gamma^J = \left(\cdots \to \overset{-2}{\circ} \to \overset{-1}{\circ} \to \overset{0}{\circ} \to \overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \right)$$

$$\hat{H}_\nu(\Gamma^J) \cong \hat{H}^{\mathrm{aff}}(\mathfrak{S}_{|\nu|})$$

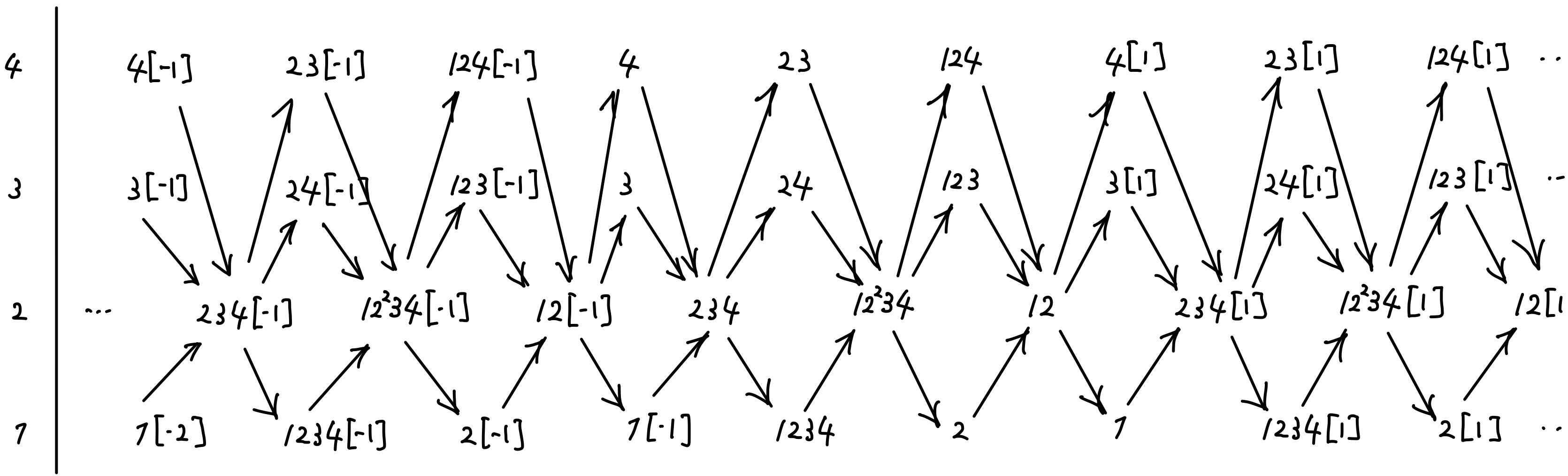
<u>In type A</u>, it goes back to usual quantum affine SW duality

$$Q = \left(\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \right)$$



Type D

$$Q = \left(\begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd} \right)$$



__For general case__ $Q' = (\overset{1}{\circ} \to \circ \cdots \to \circ \to \overset{N-1}{\circ}) \subset Q$

$$\mathfrak{M}^{\bullet}_0(W) \cong \{ X^N = 0 \} \subset \mathrm{rep}(\Gamma^J, W)$$

"graded nilpotent orbits (type A)"

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{F}^J : \bigoplus_{d \geq 0} \hat{H}^{\mathrm{aff}}(\mathfrak{S}_d)\text{-mod}_{\mathrm{fd}} \arrow[r] \arrow[d, "\text{"localization"}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathscr{C} \\ \mathcal{T}_N \arrow[r] & \mathscr{C}_{\mathfrak{D}_{Q'}} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

" supp on $\mathfrak{D}_{Q'} \subset \mathfrak{D}_Q$ "

$\rightsquigarrow$ Grothendieck ring isomorphism

[KKOP 19] $\quad K(\mathcal{T}_N) \xrightarrow{\sim} K(\mathscr{C}_{\mathfrak{D}_{Q'}})$

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