Super duality for quantum affine algebras of type A

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Contents

Quantum affine superalgebra of type A

- Quantum group of the affine superalgebra $\widehat{\mathfrak{sl}}_{M|N}$ is first introduced by [Yamane '99].
- As in the non-super case, its finite-dimensional representations become accessible once the Drinfeld-type presentation was obtained.
  1. [H.Zhang '14] classifies irreps in terms of the highest ℓ-wt
  2. [H.Zhang '17] studies fundamental representations, their R-matrices and the irreducibility of their tensor products
- However, no direct connection to non-super case has been known.

Generalized quantum group of type A

- [Kuniba-Okado-Sergeev '15] constructs solutions of the Yang-Baxter equation as the reduction of those of the tetrahedron equations.
- The generalized quantum groups, which are first introduced in the context of pointed Hopf algebras, are recognized as a symmetry of the space on which the solutions (R-matrix) act.
- [KOS15] also establishes a close connection to the quantum affine superalgebra via an algebra isomorphism.
- Generalized quantum group of finite type A is also realized as a certain degeneration of the quantum group $U_q(\mathfrak{gl}_{M+N})$ [J.Cheng-Y.Wang-R.B.Zhang '18].

Super duality

- Representation theory of simple Lie superalgebras is not parallel to that of simple Lie algebras.
- Crucial obstructions are:
  - Borel subalgebras are not always conjugate
  - Finite-dimensional modules are not semisimple in general
  - Weyl group is too small to control the representation theory
    (e.g. linkage principle)
- Super duality describes an equivalence of *parabolic* BGG categories over various pairs of Lie (super)algebras at infinite rank.
  e.g. [Cheng-Lam '10] type A, [Cheng-Lam-Wang '11] type BCD

Super duality of type A

Let $\mathfrak{g}$, $\overline{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ be Lie superalgebras of infinite rank given by the following Dynkin diagram:

Let $\mathcal{O}_{m+\infty}$, $\overline{\mathcal{O}}_{m|\infty}$ and $\widetilde{\mathcal{O}}_{\infty|\infty}$ be parabolic BGG categories for $\mathfrak{g}$, $\overline{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ resp.. Then there exists *truncation* functors to connect such categories:

$$
\begin{tikzcd}
 & \widetilde{\mathcal{O}}_{\infty|\infty} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathcal{O}_{m+\infty} \arrow[d] & & \overline{\mathcal{O}}_{m|\infty} \arrow[d] \\
\mathcal{O}_{m+n} & & \overline{\mathcal{O}}_{m|n}
\end{tikzcd}
$$

Main results

1. Relate module categories of generalized quantum groups and quantum affine algebras of type A using truncation functors
2. Explain this connection in terms of generalized quantum affine Schur-Weyl duality and establish a super-duality-type equivalence

Contents

Notations for super analogues

- $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i \in \{0,1\}$, $n \geq 4$.
- $\mathbb{I} = \{1 < 2 < \cdots < n\} = \mathbb{I}_0 \cup \mathbb{I}_1$
- $P = \bigoplus_{i \in \mathbb{I}} \mathbb{Z}\delta_i$: **weight lattice** with $(\delta_i, \delta_j) = (-1)^{\epsilon_i}\delta_{ij}$
- $I = \{0, 1, \ldots, n-1\}$: index of simple roots
- $\alpha_i = \delta_i - \delta_{i+1}$: simple roots $\rightsquigarrow I = I_{\text{even}} \cup I_{\text{odd}}$

$\Rightarrow$ Dynkin diagrams:

- $q_i = (-1)^{\epsilon_i} q^{(-1)^{\epsilon_i}} = \begin{cases} q & \epsilon_i = 0 \\ -q^{-1} & \epsilon_i = 1 \end{cases} \qquad (i \in \mathbb{I})$
- $\mathbf{q}(\mu, \nu) = \prod_{i \in \mathbb{I}} q_i^{a_i b_i}$ where $\mu = \sum a_i \delta_i$, $\nu = \sum b_i \delta_i \in P$

Generalized quantum group of type A

Fix the base field $\Bbbk = \mathbb{Q}(q)$.

Definition (Kuniba-Okado-Sergeev '15)

The **generalized quantum group** $\mathcal{U}(\epsilon)$ of affine type A is the associative $\Bbbk$-algebra with 1, generated by $e_i,\ f_i,\ k_\mu\ (i \in I,\ \mu \in P)$ with relations:

$$k_0 = 1,\ k_{\mu+\mu'} = k_\mu k_{\mu'},\ k_\mu e_i k_{-\mu} = \mathbf{q}(\mu, \alpha_i) e_i,\ k_\mu f_i k_{-\mu} = \mathbf{q}(\mu, \alpha_i)^{-1} f_i, \tag{1}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_{\alpha_i} - k_{-\alpha_i}}{q - q^{-1}}, \tag{2}$$

$$e_i^2 = f_i^2 = 0 \quad \text{for } i \in I_{\mathrm{odd}},$$

$$e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{for } i - j \not\equiv_n \pm 1,$$

$$e_i^2 e_j - (-1)^{\epsilon_i} [2] e_i e_j e_i + e_j e_i^2 = (\ldots f \ldots) = 0 \ \text{ for } i \in I_{\mathrm{even}},\ i - j \equiv_n \pm 1,$$

$$\begin{aligned} & e_i e_{i-1} e_i e_{i+1} - e_i e_{i+1} e_i e_{i-1} + e_{i+1} e_i e_{i-1} e_i - e_{i-1} e_i e_{i+1} e_i \\ & \qquad + (-1)^{\epsilon_i} [2] e_i e_{i-1} e_{i+1} e_i = (\ldots f \ldots) = 0 \ \text{ for } i \in I_{\mathrm{odd}}. \end{aligned}$$

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$$e_i e_{i-1} e_i e_{i+1} - e_i e_{i+1} e_i e_{i-1} + e_{i+1} e_i e_{i-1} e_i - e_{i-1} e_i e_{i+1} e_i$$
$$+ (-1)^{\epsilon_i} [2] e_i e_{i-1} e_{i+1} e_i = (\ldots f \ldots) = 0 \ \text{ for } i \in I_{\mathrm{odd}}.$$

Usual quantum affine algebra occurs: $\mathcal{U}((1^n)) > \langle e_i, f_i, k_{\alpha_i} \rangle_{i \in I} = U'_{-q^{-1}}(A^{(1)}_{n-1})$.

vs Quantum affine superalgebra

Definition (Yamane '99)

The *quantum affine superalgebra* $U(\epsilon)$ of affine type A is the associative $\Bbbk$-superalgebra with 1, generated by e_i, f_i, k_μ ($i \in I$, $\mu \in P$) with relations:

$$k_0 = 1,\ k_{\mu+\mu'} = k_\mu k_{\mu'},\ k_\mu e_i k_{-\mu} = q^{(\mu,\alpha_i)} e_i,\ k_\mu f_i k_{-\mu} = q^{-(\mu,\alpha_i)} f_i,$$

$$e_i f_j - (-1)^{p(i)p(j)} f_j e_i = (-1)^{\epsilon_i} \delta_{ij} \frac{k_{\alpha_i} - k_{-\alpha_i}}{q - q^{-1}},$$

$$e_i^2 = f_i^2 = 0 \quad \text{for } i \in I_{\text{odd}},$$

$$e_i e_j - (-1)^{p(i)p(j)} e_j e_i = f_i f_j - (-1)^{p(i)p(j)} f_j f_i = 0 \quad \text{for } i - j \not\equiv_n \pm 1,$$

$$e_i^2 e_j - [2] e_i e_j e_i + e_j e_i^2 = (\dots f \dots) = 0 \ \text{ for } i \in I_{\text{even}},\ i - j \equiv_n \pm 1,$$

$$e_i e_{i-1} e_i e_{i+1} \pm e_i e_{i+1} e_i e_{i-1} \pm e_{i+1} e_i e_{i-1} e_i \pm e_{i-1} e_i e_{i+1} e_i$$
$$\pm [2] e_i e_{i-1} e_{i+1} e_i = (\dots f \dots) = 0 \ \text{ for } i \in I_{\text{odd}}.$$

Notice that only sign differs in defining relations.

vs Quantum affine superalgebra

Introduce 'sign operators' to relieve the discrepancy [KOS15].

Define $U(\epsilon)^e_{\rm aff}[\sigma]$, $\mathcal{U}(\epsilon)^e_{\rm aff}[\sigma]$ by adjoining involutive elements $\sigma_1, \ldots, \sigma_n$ to extended algebras $U(\epsilon)^e_{\rm aff}$ and $\mathcal{U}(\epsilon)^e_{\rm aff}$:

$$\sigma_j k_\mu = k_\mu \sigma_j, \quad \sigma_j e_i = (-1)^{\epsilon_j(\delta_j, \alpha_i)} e_i \sigma_j, \quad \sigma_j f_i = (-1)^{\epsilon_j(\delta_j, \alpha_i)} f_i \sigma_j.$$

Let $\mathscr{X}$, $\mathscr{Y}$ be certain subalgebras (corresp. to $U_q(\widehat{\mathfrak{sl}}_{M|N})$) respectively.

Theorem (Kwon-L.)

There exists a $\Bbbk$-algebra isomorphism $\tau : \mathscr{X} \to \mathscr{Y}$.

Remark. τ does not respect the coproduct.

Quantum Gabber-Kac theorem

Recall the Lusztig's approach to quantum groups.

1. ${}'\mathbf{f}(\epsilon) = \langle \theta_i, i \in I \rangle$: free $\mathbb{k}$-algebra.
2. $(-,-)$: symm. bilinear form on ${}'\mathbf{f}(\epsilon)$ satisfying
$$(1,1) = 1, \quad (\theta_i, \theta_j) = \delta_{ij}, \quad (x, yy') = (r(x), y \otimes y')$$
and $\mathcal{I}$: its radical.
3. ${}'\mathbf{U}(\epsilon)$: the asso. $\mathbb{k}$-alg. gen. by e_i, f_i, k_μ with relations (1), (2) in $\mathcal{U}(\epsilon)$.
4. Define
$$\mathbf{U}(\epsilon) = \frac{{}'\mathbf{U}(\epsilon)}{\langle h(e_0, \ldots, e_n), h(f_0, \ldots, f_n) | h \in \mathcal{I} \rangle}.$$

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$$\mathbf{U}(\epsilon) = \frac{{}'\mathbf{U}(\epsilon)}{\langle h(e_0, \ldots, e_n), h(f_0, \ldots, f_n) | h \in \mathcal{I} \rangle}.$$

Theorem (Kwon-L.)

There exists a surjective algebra morphism $\pi : \mathcal{U}(\epsilon) \to \mathbf{U}(\epsilon)$.

Furthermore, π is an isomorphism if $M \neq N$.

Remark. When $M \neq N$, we need more relations. e.g. $\mathfrak{sl}(M|M)$ is not simple.

Category $\mathcal{C}(\epsilon)$

- The **weight space** of a $\mathcal{U}(\epsilon)$-module V with weight $\lambda \in P$ is

$$V_\lambda = \{v \in V \mid k_\mu v = \mathbf{q}(\mu, \lambda)v, \ \mu \in P\}$$

Remark. GQG: $k_\mu v = \mathbf{q}(\mu, \lambda)v \underset{\tau}{\longleftrightarrow} k_\mu v = q^{(\mu,\lambda)}v$:QSA

- $P_{\geq 0} = \bigoplus \mathbb{Z}_{\geq 0}\delta_i$
- For $\lambda = \sum \lambda_i \delta_i \in P_{\geq 0}$, $\deg \lambda = \sum \lambda_i$

Definition

- $\mathcal{C}(\epsilon)$: the category of finite-dimensional $\mathcal{U}(\epsilon)$-modules V with

$$V = \bigoplus_{\lambda \in P_{\geq 0}} V_\lambda.$$

- $\mathcal{C}^\ell(\epsilon)$: the subcategory consists of those with weights of degree ℓ.

- $\mathcal{C}(\epsilon) = \bigoplus_{\ell \geq 0} \mathcal{C}^\ell(\epsilon)$.

Fundamental representations $\mathcal{W}_{\ell,\epsilon}(x)$

Set $\mathbb{Z}_+^n(\epsilon) = \{(m_1, \ldots, m_n) \,|\, m_i \in \mathbb{Z}_{\geq 0}$ if $\epsilon_i = 0$, $m_i \in \{0, 1\}$ if $\epsilon_i = 1\}$.

Definition

For $\ell \in \mathbb{Z}_+$ and $x \in \Bbbk^\times$, let $\mathcal{W}_{\ell,\epsilon}(x) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_+^n(\epsilon), |\mathbf{m}| = \ell} \Bbbk \,|\mathbf{m}\rangle$ with $\mathcal{U}(\epsilon)$-action:

$$k_\mu \,|\mathbf{m}\rangle = \mathbf{q}(\mu, \sum_{j \in \mathbb{I}} m_j \delta_j) \,|\mathbf{m}\rangle$$

$$e_i \,|\mathbf{m}\rangle = \begin{cases} x^{\delta_{i,0}} [m_{i+1}] \,|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle & \text{if } \mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon) \\ 0 & \text{otherwise} \end{cases}$$

$$f_i \,|\mathbf{m}\rangle = \begin{cases} x^{-\delta_{i,0}} [m_i] \,|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle & \text{if } \mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1} \in \mathbb{Z}_+^n(\epsilon) \\ 0 & \text{otherwise.} \end{cases}$$

Polynomial representations

- $\mathring{\mathcal{U}}(\epsilon)$: the subalgebra of $\mathcal{U}(\epsilon)$ generated by $e_i,\ f_i,\ k_\mu$ $(i \in I \setminus \{0\},\ \mu \in P)$.
- $\mathcal{V}$: the natural representation of $\mathring{\mathcal{U}}(\epsilon)$
- $\mathscr{P}_{M|N}$: the set of $(M|N)$-hook partitions (i.e. $\lambda_{M+1} \leq N$)
- $\mathcal{V}^{\otimes \ell}$: semisimple with

$$\mathcal{V}^{\otimes \ell} = \bigoplus_{\lambda \in \mathscr{P}_{M|N}} V_\epsilon(\lambda)^{\oplus m_\lambda}$$

where $V_\epsilon(\lambda)$: the irred. h.w. $\mathring{\mathcal{U}}(\epsilon)$-module

m_λ : the number of standard tableaux of shape λ

Example

- $\mathcal{W}_{\ell,\epsilon}(x) \cong_{\mathring{\mathcal{U}}(\epsilon)} V_\epsilon((\ell))$
- $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \cong_{\mathring{\mathcal{U}}(\epsilon)} \bigoplus_{H(l,m)} V_\epsilon((l+m-t,t))$
  where $H(l,m) = \{t \leq \min(l,m) \,|\, (l+m-t,t) \in \mathscr{P}_{M|N}\}$.

Normalized R-matrix

- We can define a universal R-matrix for $\mathcal{U}(\epsilon)$ and a $\mathcal{U}(\epsilon)$-linear map

$$\mathcal{R}^{\mathrm{univ}} : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \longrightarrow \Bbbk \, [\![x/y]\!] \otimes_{\Bbbk[x/y]} (\mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x))$$

- $\mathcal{R}^{\mathrm{norm}}_{l,m}(x/y) = c\mathcal{R}^{\mathrm{univ}}$: the **normalized R-matrix** s.t.

$$\mathcal{R}^{\mathrm{norm}}_{l,m}(x/y)|_{V_\epsilon((\max(l,m),\min(l,m)))} = \mathrm{id}.$$

Spectral decomposition

Theorem (KOS15 ($\epsilon = \epsilon_{M|N}$), Kwon-Yu '21 (arbitrary ϵ))

For l, $m \in \mathbb{Z}_+$, there exists a $\mathcal{U}(\epsilon)$-linear map $R(z) : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \to \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x)$ with spectral decomposition

$$R(z) = \sum_{t \in H(l,m)} \rho_t(z) \mathcal{P}_t^{l,m}, \qquad \rho_t(z) = \prod_{i=t+1}^{\min(l,m)} \frac{1 - q^{l+m-2i+2} z}{z - q^{l+m-2i+2}}$$

where $z = x/y$ and

$$\mathcal{P}_t^{l,m} : \mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y) \cong_{\mathring{\mathcal{U}}} \mathcal{W}_{m,\epsilon}(y) \otimes \mathcal{W}_{l,\epsilon}(x) \twoheadrightarrow V((l+m-t,t)).$$

- $\mathcal{R}_{l,m}^{\mathrm{norm}}(z) = R(z)$ by the irreducibility of $\mathcal{W}_{l,\epsilon}(x) \otimes \mathcal{W}_{m,\epsilon}(y)$.
- The spectral decomposition of $\mathcal{R}_{l,m}^{\mathrm{norm}}(z)$ is independent of ϵ, *except* $H(l,m)$.

Fusion construction

Theorem (Kwon-L.)

Suppose that $c_1,\ldots,c_t \in \Bbbk^\times$ *are given such that* c_i/c_j *is not a pole of* $\mathcal{R}^{\rm norm}_{l_i,l_j}(z_i/z_j)$ *for all* $i<j$*. Then the specialization*

$$R_{\boldsymbol{l},\epsilon}(\mathbf{c}) : \mathcal{W}_{l_1,\epsilon}(c_1)\otimes\cdots\otimes\mathcal{W}_{l_t,\epsilon}(c_t) \longrightarrow \mathcal{W}_{l_t,\epsilon}(c_t)\otimes\cdots\otimes\mathcal{W}_{l_1,\epsilon}(c_1)$$

of $\mathcal{R}^{\rm norm}_{l_1,\ldots,l_t}$ *at* $(c_1,\ldots,c_t)$ *has a simple image* $\mathcal{W}_\epsilon(\boldsymbol{l},\mathbf{c})$ *if it is not zero.*

Proof. Induction on t. The case $t=2$ follows by the argument of [Kang-Kashiwara-Kim-Oh '15]. □

- A complete classification of irreps of $\mathcal{U}(\epsilon)$ is not yet known.
  cf. classification of irreps of $U(\epsilon)$ by highest ℓ-weight [H.Zhang '14]

Contents

Truncation functor

The first connection to modules over quantum affine algebras is due to the truncation, motivated by the super duality for simple Lie superalgebra.

Theorem (KY21)

Let ϵ' be the sequence obtained from ϵ by removing ϵ_i.

Then there is an algebra homomorphism $\phi^{\epsilon}_{\epsilon'} : \mathcal{U}(\epsilon') \longrightarrow \mathcal{U}(\epsilon)$.

Truncation functor

- For $V \in \mathcal{C}(\epsilon)$, take a subspace of V:

$$\mathfrak{tr}^{\epsilon}_{\epsilon'}(V) = \bigoplus_{(\mu,\delta_i)=0} V_\mu$$

$\Rightarrow$ closed under the $\mathcal{U}(\epsilon')$-action by $\phi^{\epsilon}_{\epsilon'}$.

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- For $f : V \to W$, define $\mathfrak{tr}^{\epsilon}_{\epsilon'}(f) : \mathfrak{tr}^{\epsilon}_{\epsilon'}(V) \longrightarrow \mathfrak{tr}^{\epsilon}_{\epsilon'}(W)$ by restriction.

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Definition

The resulting functor $\mathrm{tr}^{\epsilon}_{\epsilon'} : \mathcal{C}(\epsilon) \to \mathcal{C}(\epsilon')$ is called a **truncation**.

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Definition

The resulting functor $\mathfrak{tr}^{\epsilon}_{\epsilon'} : \mathcal{C}(\epsilon) \to \mathcal{C}(\epsilon')$ is called a **truncation**.

- For ϵ and $\epsilon' = \epsilon - \{\epsilon_{i_1}, \cdots, \epsilon_{i_r}\}$, we also define $\mathfrak{tr}^{\epsilon}_{\epsilon'}$ by

$$\mathfrak{tr}^{\epsilon}_{\epsilon'} = \mathfrak{tr}^{\epsilon^{(r-1)}}_{\epsilon^{(r)}} \circ \cdots \circ \mathfrak{tr}^{\epsilon}_{\epsilon^{(1)}}$$

where $\epsilon^{(0)} = \epsilon$ and $\epsilon^{(k)} = \epsilon^{(k-1)} - \epsilon_{i_k}$.

Truncation functor

Lemma (KY21)

The truncation functor $\mathrm{tr}^{\epsilon}_{\epsilon'}$ is exact and monoidal.

Lemma (KY21)

1. $$\mathrm{tr}^{\epsilon}_{\epsilon'} V_{\epsilon}(\lambda) \cong \begin{cases} V_{\epsilon'}(\lambda) & \text{if } \lambda \in \mathscr{P}_{M'|N'} \\ 0 & \text{otherwise.} \end{cases}$$
2. $$\mathrm{tr}^{\epsilon}_{\epsilon'} \mathcal{W}_{l,\epsilon}(x) \cong \begin{cases} \mathcal{W}_{l,\epsilon'}(x) & \text{if } (l) \in \mathscr{P}_{M'|N'} \\ 0 & \text{otherwise.} \end{cases}$$

Truncation functor

Theorem

Let $(\boldsymbol{l}, \mathbf{c})$ *be such that* $\mathrm{Im} R_{\boldsymbol{l},\epsilon}(\mathbf{c}) \neq 0$*. Then*

1. $\mathrm{tr}^{\epsilon}_{\epsilon'}(R_{\boldsymbol{l},\epsilon}(\mathbf{c})) = R_{\boldsymbol{l},\epsilon'}(\mathbf{c})$.
2. $\mathrm{tr}^{\epsilon}_{\epsilon'} \mathcal{W}_{\epsilon}(\boldsymbol{l}, \mathbf{c}) \cong \mathcal{W}_{\epsilon'}(\boldsymbol{l}, \mathbf{c})$.

Proof. Follows from the irreducibility of $(\mathcal{W}_{l,\epsilon})_{\mathrm{aff}} \otimes (\mathcal{W}_{m,\epsilon})_{\mathrm{aff}}$. $\square$

Corollary

Suppose $\mathcal{W}_{\epsilon}(\boldsymbol{l}, \mathbf{c}) \cong_{\mathring{\mathcal{U}}(\epsilon)} \bigoplus_{\lambda} V_{\epsilon}(\lambda)^{\oplus m_{\lambda}^{(\boldsymbol{l},\mathbf{c})}(\epsilon)}$.

If $m_{\lambda}^{(\boldsymbol{l},\mathbf{c})}(\epsilon) \neq 0$ *for some* ϵ*, then we have* $m_{\lambda}^{(\boldsymbol{l},\mathbf{c})}(\epsilon') = m_{\lambda}^{(\boldsymbol{l},\mathbf{c})}(\epsilon)$ *for all* $\epsilon' \geq \epsilon$.

Therefore, $\mathcal{W}_{\epsilon}(\boldsymbol{l}, \mathbf{c})$ is nonzero for sufficiently large ϵ and its character is stable.

Truncation functor

Example

How to compute the classical decomposition of $\mathcal{W}_\epsilon = \mathcal{W}_\epsilon((4,4,4),(q^{-2},1,q^2))$?

Truncation functor

Example

How to compute the classical decomposition of $\mathcal{W}_\epsilon = \mathcal{W}_\epsilon((4,4,4),(q^{-2},1,q^2))$?

1. Take a large $\epsilon' \geq \epsilon$ and set $\epsilon'' = (0^{M'}) \leq \epsilon'$ so that $V_{\epsilon'}((4))^{\otimes 3}$ and $V_{\epsilon''}((4))^{\otimes 3}$ does not lose any component expected by combinatorics:

$$V_{\epsilon'}((4))^{\otimes 3} = \bigoplus_\lambda V_{\epsilon'}(\lambda)^{\oplus K_{\lambda,(4^3)}}, \qquad V_{\epsilon''}((4))^{\otimes 3} = \bigoplus_\lambda V_{\epsilon''}(\lambda)^{\oplus K_{\lambda,(4^3)}}$$

   In other words, so that $K_{\lambda,(4^3)} \neq 0 \Rightarrow \lambda \in \mathscr{P}_{M'|0}$.

2. $\mathcal{W}_{\epsilon''} \cong_{\mathring{\mathcal{U}}(\epsilon'')} V_{\epsilon''}((4^3))$ (the KR module associated with (4^3)).
3. $\mathcal{W}_{\epsilon'} \cong_{\mathring{\mathcal{U}}(\epsilon')} V_{\epsilon'}((4^3))$ by the assumption on ϵ'.
4. Applying $\mathrm{tr}_\epsilon^{\epsilon'}$, we conclude $\mathcal{W}_\epsilon \cong_{\mathring{\mathcal{U}}(\epsilon)} \begin{cases} V_\epsilon((4^3)) & \text{if } (4^3) \in \mathscr{P}_{M|N} \\ 0 & \text{else.} \end{cases}$

Quiver Hecke algebra of type A_∞

Following [Kang-Kashiwara-Kim '18], from the spectral decomp. of $\mathcal{R}^{\mathrm{norm}}_{1,1}$ we introduce:

- $J = \mathbb{Z}$, $Q_{ij}(u,v) = \begin{cases} u - v & \text{if } j = i+1 \\ v - u & \text{if } j = i-1 \end{cases}$ for $i, j \in J$.
- Γ^J $\quad \cdots \longrightarrow \underset{-1}{\bigcirc} \longrightarrow \underset{0}{\bigcirc} \longrightarrow \underset{1}{\bigcirc} \longrightarrow \cdots$: quiver of type A_∞.

Quiver Hecke algebra of type A_∞

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- $\Gamma^J \quad \cdots \longrightarrow \underset{-1}{\bigcirc} \longrightarrow \underset{0}{\bigcirc} \longrightarrow \underset{1}{\bigcirc} \longrightarrow \cdots$: quiver of type A_∞.
- $R^J(\ell) = \langle e(\nu),\, x_k,\, \tau_m \rangle$: **Quiver Hecke algebra** associated with Γ^J.
- $R^J = \bigoplus_{\ell \geq 0} R^J(\ell)$.
- $R^J(\ell)$-mod$_0$: category of fin-dim'l $R^J(\ell)$-modules with nilpotent x_k.

Generalized quantum affine Schur-Weyl duality

Since the data Q_{ij} is determined by the spectral decomposition of $\mathcal{R}_{1,1}^{\mathrm{norm}}$, we obtain a natural analogue of [KKK18] for generalized quantum groups.

Theorem (Kwon-L.)

There is a $\big(\mathcal{U}(\epsilon), R^J(\ell)\big)$-bimodule $V_{\mathbb{O}}^{\otimes \ell}$, inducing an exact functor

$$\mathcal{F}_{\epsilon,\ell} = V_{\mathbb{O}}^{\otimes \ell} \otimes_{R^J(\ell)} - : R^J(\ell)\text{-mod}_0 \longrightarrow \mathcal{C}(\epsilon).$$

Moreover, the sum $\mathcal{F}_\epsilon = \bigoplus_{\ell \geq 0} \mathcal{F}_{\epsilon,\ell}$ is monoidal.

Properties of $\mathcal{F}_{\epsilon,\ell}$

Following [KKK18], we have

- $\mathcal{F}_{\epsilon,\ell}$ sends renormalized R-matrices for QHA to $\mathcal{R}^{\mathrm{norm}}_{\ell_1,\ell_2}$.
  Hence it is compatible with fusion construction.
- $\mathcal{F}_{\epsilon,\ell}(S)$ is either simple or zero, for each simple S in $R^J(\ell)$-mod$_0$.
- Let $\mathcal{C}_J(\epsilon)$ be the full subcategory of $\mathcal{C}(\epsilon)$ whose objects have composition factors appearing in tensor products of $\mathcal{W}_{1,\epsilon}(q^{-2j})$, $j \in J$.
- Let $\mathcal{C}^\ell_J(\epsilon) = \mathcal{C}_J(\epsilon) \cap \mathcal{C}^\ell(\epsilon)$. Then

$$\mathcal{F}_{\epsilon,\ell} : R^J(\ell)\text{-mod}_0 \longrightarrow \mathcal{C}^\ell_J(\epsilon).$$

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Theorem (Kwon-L.)

For any $\epsilon' < \epsilon$, we have $\mathfrak{tr}^{\epsilon}_{\epsilon'} \circ \mathcal{F}_{\epsilon,\ell} \cong \mathcal{F}_{\epsilon',\ell}$.

Equivalence

Theorem (Kwon-L.)

$\mathcal{F}_{\epsilon,\ell}$ is an equivalence of categories whenever $\ell < n$.

Proof. Let $H_\ell^{\mathrm{aff}}(q^2)$ be the affine Hecke algebra.

- [Brundan-Kleshchev '09] $\mathbb{O}H_\ell^{\mathrm{aff}}(q^2) \cong \widehat{R}^J(\ell)$.
- $V_{\mathbb{O}}^{\otimes \ell}$ is a $\big(\mathcal{U}(\epsilon), \mathbb{O}H_\ell^{\mathrm{aff}}(q^2)\big)$-bimodule with the following compatibility:

$$
\begin{tikzcd}
\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)\text{-mod} \arrow[r, leftrightarrow, "\cong"] \arrow[dd, leftrightarrow, "\cong"'] & H_\ell^{\mathrm{aff}}(q^2)\text{-mod}_J \arrow[dr, "\mathcal{F}^*_{\epsilon,\ell} = V_{\mathbb{O}}^{\otimes \ell} \otimes_{\mathbb{O}H_\ell^{\mathrm{aff}}(q^2)} -"] & \\
 & & \mathcal{C}_J^\ell(\epsilon) \\
\widehat{R}^J(\ell)\text{-mod} \arrow[r, leftrightarrow, "\cong"] & R^J(\ell)\text{-mod}_0 \arrow[ur, "\mathcal{F}_{\epsilon,\ell}"'] &
\end{tikzcd}
$$

- $\mathcal{F}^*_{\epsilon,\ell}$ is an equivalence if $\ell < n$ [Chari-Pressley '96]. $\square$

Summary

$$
\begin{tikzcd}
U_q'(\widehat{\mathfrak{gl}}_{M|N})\text{-mod} \arrow[r, leftrightarrow, "\tau"'] & \mathcal{U}(\epsilon)\text{-mod} \arrow[rr, "\mathfrak{tr}"'] & & U_q'(\mathfrak{sl}_N)\text{-mod} \\
& \mathcal{C}_J^\ell(\epsilon) \arrow[u, hook] & & \mathcal{C}_J^\ell \arrow[u, hook] \\
& & R^J(\ell)\text{-mod}_0 \arrow[ul, "\mathcal{F}_{\epsilon,\ell}"] \arrow[ur, "\mathcal{F}_\ell"'] &
\end{tikzcd}
$$

Limit category for infinite rank

- $\epsilon^\infty = (\epsilon_1, \epsilon_2, \dots)$, $\epsilon_i \in \{0, 1\}$ with infinitely many $0, 1$
- $(\epsilon^{(k)})_{k\geq 1}$: ascending chain of fin. subseq. of ϵ^∞

  s.t. $M_k \neq N_k$ and $\epsilon^\infty = \lim_k \epsilon^{(k)}$

Definition

The 'inverse limit category' $\mathcal{C}(\epsilon^\infty) = \varprojlim \mathcal{C}(\epsilon^{(k)})$ is the monoidal, abelian category consisting of

- object : $\mathbb{V} = ((V_k)_{k\geq 1}, (f_k)_{k\geq 1})$ where

$$V_k \in \mathcal{C}(\epsilon^{(k)}), \quad f_k : \mathfrak{tr}_k^{k+1}(V_{k+1}) \xrightarrow{\cong} V_k$$

- morphism from $\mathbb{V} = ((V_k), (f_k))$ to $\mathbb{W} = ((W_k), (g_k))$:

  a sequence $\phi = (\phi_k : V_k \to W_k)_{k\geq 1}$ satisfying a suitable commutativity
- tensor product of $\mathbb{V}$ and $\mathbb{W}$: $\mathbb{V} \otimes \mathbb{W} = ((V_k \otimes W_k), (f_k \otimes g_k))$.

Similary we define $\mathcal{C}_J^\ell(\epsilon^\infty) = \varprojlim \mathcal{C}_J^\ell(\epsilon^{(k)})$ and $\mathcal{C}_J(\epsilon^\infty) = \bigoplus \mathcal{C}_J^\ell(\epsilon^\infty)$.

Super duality

Lemma

Given a category $\mathcal{C}$ with functors $F_k : \mathcal{C} \to \mathcal{C}(\epsilon^{(k)})$ such that $\mathfrak{tr}_k^{k+1} \circ F_{k+1} \cong F_k$, there exists a functor $F := \varprojlim F_k : \mathcal{C} \to \mathcal{C}(\epsilon^{\infty})$ such that $\mathfrak{tr}_k \circ F \cong F_k$.

- If every F_k is exact (resp. monoidal), then so is F.
- If F_k is an equivalence for all $k \geq N$, then F is an equivalence as well.

Theorem (Kwon-L.)

The functor $\mathcal{F}_{\epsilon^\infty,\ell} = \varprojlim \mathcal{F}_{\epsilon^{(k)},\ell}$ is an equivalence of categories, and so is $\mathcal{F}_{\epsilon^\infty} = \bigoplus \mathcal{F}_{\epsilon^\infty,\ell}$.

$$
\begin{tikzcd}
 & R^J(\ell)\text{-mod}_0 \arrow[dl, dashed, "\simeq"'] \arrow[dr, dashed, "\simeq"] \arrow[ddl, "\mathcal{F}_{\epsilon^{(k)},\ell}"] \arrow[ddr] & \\
\mathcal{C}_J^\ell(\epsilon^\infty) \arrow[d, "\mathfrak{tr}"'] & & \mathcal{C}_J^\ell(1^\infty) \arrow[d, "\mathfrak{tr}"] \\
\mathcal{C}_J^\ell(\epsilon^{(k)}) & & \mathcal{C}_J^\ell(1^k)
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & \mathcal{C}_J(\epsilon^\infty) \arrow[dl, "\mathcal{S}_{\infty|0}"'] \arrow[dr, "\mathcal{S}_{0|\infty}"] & \\
\mathcal{C}_J(0^\infty) & & \mathcal{C}_J(1^\infty)
\end{tikzcd}
$$

Application: Grothendieck rings

- Recall that by the theory of q-characters,

$$K_0(\mathcal{C}_J(1^\infty)) \cong \mathbb{Z}[t_{l,a} | l \in \mathbb{N}, a \in 2\mathbb{Z}+l]$$
$$\mathcal{W}_l(q^a) \longmapsto t_{l,a}$$

- The above equivalence yields a ring isom.

$$K_0(\mathcal{C}_J(0^\infty)) \cong K_0(\mathcal{C}_J(\epsilon^\infty)) \cong K_0(\mathcal{C}_J(1^\infty)).$$

Hence, $K_0(\mathcal{C}_J(\epsilon))$ is identified with a quotient of the polynomial ring and this 'character' of an irreducible module is stable.

Application: T-systems

- $\mathcal{W}_{\epsilon}^{r,s}(c)$: Kirillov-Reshetikhin module of shape (r^s)
  i.e. $\mathcal{W}_{\epsilon}^{r,s}(c) = \mathcal{W}_{\epsilon}(\boldsymbol{l}, \mathbf{c})$ for $\boldsymbol{l} = (r, \ldots, r)$, $\mathbf{c} = c(q^{1-s}, q^{3-s}, \ldots, q^{s-1})$.
- Recall the T-system for quantum affine algebras ($\tilde{q} = -q^{-1}$)

$$0 \to \mathcal{W}_{(1^N)}^{r-1,s}(\tilde{q}) \otimes \mathcal{W}_{(1^N)}^{r+1,s}(\tilde{q}) \to \mathcal{W}_{(1^N)}^{r,s}(1) \otimes \mathcal{W}_{(1^N)}^{r,s}(\tilde{q}^2) \to \mathcal{W}_{(1^N)}^{r,s+1}(1) \otimes \mathcal{W}_{(1^N)}^{r,s-1}(\tilde{q}^2) \to 0.$$

- We may lift it to $\mathcal{C}_J(\epsilon^{\infty})$ by taking a large N, and then truncate to $\mathcal{C}_J(\epsilon)$ to obtain the same T-system for $\mathcal{U}(\epsilon)$.

Thank you for listening!

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Simple modules over $R^J(\ell)$

- Segment $=(a,b)$ with integers $a \leq b$, with length $\ell = b-a+1$.
- $L(a,b)$ = 1-dim'l $R^J(\ell)$-module $k \cdot u(a,b)$

$$\text{with action } \begin{cases} x_i u(a,b) = \tau_m u(a,b) = 0 \\ e(\nu) u(a,b) = \delta_{\nu=(a,a+1,\ldots,b)} u(a,b). \end{cases}$$

- [KKK18] There exists a **renormalized** R**-matrix** $\mathbf{r}_{M,N} : M \circ N \to N \circ M$.

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Theorem (Kang-Park '11, Kang-Kashiwara-Kim '18)

1. *There is a one-to-one correspondence*

$$\left\{ \textit{f.d. irred. } R^J(\ell)\textit{-modules} \right\} /\cong \;\longleftrightarrow\; \left\{ ((a_1,b_1) \ge \cdots \ge (a_t,b_t)) \mid \ell = \sum \ell_i \right\}$$

$$\mathrm{hd}\left(L(a_1,b_1)\circ\cdots\circ L(a_t,b_t)\right) \longleftarrow ((a_1,b_1),\ldots,(a_t,b_t)).$$

2. *The composition of renormalized R-matrices*

$$\mathbf{r} : L(a_1,b_1)\circ\cdots\circ L(a_t,b_t) \longrightarrow L(a_t,b_t)\circ\cdots\circ L(a_1,b_1)$$

has a simple image which is isomorphic to $\mathrm{hd}\left(L(a_1,b_1)\circ\cdots\circ L(a_t,b_t)\right)$.

Construction of the functor

- $X : J \to \Bbbk^{\times}$, $X(j) = q^{-2j}$
- $\mathbb{O}_{\nu} = \Bbbk[\![X_1 - X(\nu_1), \dots, X_{\ell} - X(\nu_{\ell})]\!]$, $\mathbb{O}_{\ell} = \bigoplus_{\nu \in J^{\ell}} \mathbb{O}_{\nu} e(\nu)$
  Also $\mathbb{K}_{\nu} = (\mathbb{O}_{\nu} \setminus \{0\})^{-1} \mathbb{O}_{\nu}$, $\mathbb{K}_{\ell} = \bigoplus_{\nu \in J^{\ell}} \mathbb{K}_{\nu} e(\nu)$
- $V_{\mathbb{O}}^{\otimes \ell} = \mathbb{O}_{\ell} \otimes_{\Bbbk[X_1^{\pm 1}, \dots, X_{\ell}^{\pm 1}]} \mathcal{W}_{1,\epsilon}^{\otimes \ell}$, $V_{\mathbb{K}}^{\otimes \ell} = \mathbb{K}_{\ell} \otimes_{\mathbb{O}_{\ell}} V_{\mathbb{O}}^{\otimes \ell}$
- $V_{\mathbb{K}}^{\otimes \ell}$ is a $(\mathcal{U}(\epsilon), \mathbb{K}_{\ell} \otimes \Bbbk[\mathfrak{S}_{\ell}])$-bimodule by R-matrices
- The subspace $V_{\mathbb{O}}^{\otimes \ell}$ is closed under the right action by $R^J(\ell)$, where $R^J(\ell)$ is realized as a subalgebra of $\mathbb{K}_{\ell} \otimes \Bbbk[\mathfrak{S}_{\ell}]$.

$\Rightarrow$ the functor $\mathcal{F}_{\epsilon,\ell} = V_{\mathbb{O}}^{\otimes \ell} \otimes_{R^J(\ell)} -$

Properties

Theorem (Kwon-L. (cf. KKK18))

1. $\mathcal{F}_\epsilon(L(a,b)) \cong \mathcal{W}_{\ell,\epsilon}(q^{-a-b})$.
2. *For* $(a,b) \geq (a',b')$, $\mathcal{F}_\epsilon(\mathbf{r}_{L(a,b),L(a',b')}) \in \Bbbk^\times \mathcal{R}^{\mathrm{norm}}_{\ell,\ell'}$ *except a few anomalies, where it is zero.*
3. *For* $(a_1,b_1) \geq \cdots \geq (a_t,b_t)$ *and* L *the corresp. simple* $R^J(\ell)$*-module,* $\mathcal{F}_\epsilon(L) \cong \mathcal{W}_\epsilon(\mathbf{l},\mathbf{c})$ *holds whenever* $N > \max\{\ell_1,\ldots,\ell_t\}$*, where* $\mathbf{l} = (\ell_1,\ldots,\ell_t)$ *and* $\mathbf{c} = (q^{-a_1-b_1},\ldots,q^{-a_t-b_t})$.