Inverse K-Chevalley formula for type A semi-infnite flag manifolds

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Introduction —Semi-infinite flag manifolds—

G: a connected, simply-connected simple algebraic group
$H \subset G$: a maximal torus
$N \subset G$: a unipotent radical

Semi-infinite flag manifolds

$\mathbf{Q}_G^{\mathrm{rat}}$: the semi-infinite flag manifold
a reduced ind-scheme such that $\mathbf{Q}_G^{\mathrm{rat}}(\mathbb{C}) = G(\mathbb{C}((z)))/(H(\mathbb{C}) \cdot N(\mathbb{C}((z))))$.

$\rightarrow$ the semi-infinite analog of the flag manifold $G/B = G/(HN)$.

$L(\lambda)$: irreducible highest weight module of G of highest weight $\lambda \in P^+$
We regard as $L(\lambda+\mu) \subset L(\lambda) \otimes_{\mathbb{C}} L(\mu)$.

$L(\lambda)[\![z]\!] := L(\lambda) \otimes_{\mathbb{C}} \mathbb{C}[\![z]\!]$

Drinfeld-Plücker data

$\{\ell_\lambda\}_{\lambda \in P^+}$: a Drinfeld-Plücker (DP) datum $\Leftrightarrow$

- $\ell_\lambda \subset L(\lambda)[\![z]\!]$: 1-dim. $\mathbb{C}$-subspace for each $\lambda \in P^+$
- $\ell_\lambda \otimes_{\mathbb{C}} \ell_\mu = \ell_{\lambda+\mu}$ for all $\lambda, \mu \in P^+$ (Plücker relation)

Semi-infinite flag manifolds

$\mathbf{Q}_G$ is the set of all DP data.

Introduction —Semi-infinite flag manifolds—

Fact

There exists an embedding

$$\mathbf{Q}_G \hookrightarrow \mathbb{P} := \prod_{i \in I} \mathbb{P}(L(\varpi_i)[\![z]\!]),$$

where ϖ_i, $i \in I$, is a fundamental weight.

An embedding $i_\beta : \mathbb{P} \to \mathbb{P};\ \{[\mathbb{C}u_{\varpi_i}]\}_{i \in I} \mapsto \{[\mathbb{C}z^{\langle \varpi_i, \beta \rangle} u_{\varpi_i}]\}_{i \in I}$, $\beta \in Q^{\vee,+}$, induces an embedding $i_\beta : \mathbf{Q}_G \to \mathbf{Q}_G$.

$\to$ Obtaining an inductive system $((Q_\alpha), (i_{\alpha,\beta}))$
where $Q_\alpha = \mathbf{Q}_G$ for $\alpha \in Q^{\vee,+}$,
and $i_{\alpha,\beta} = i_{\beta-\alpha}$, $\alpha, \beta \in Q^{\vee,+}$ s.t. $\alpha \leq \beta$

Semi-infinite flag manifold

We set $\mathbf{Q}_G^{\mathrm{rat}} := \varinjlim Q_\alpha$.

Introduction —Semi-infinite Schubert varieties—

$\mathbf{I}$: the Iwahori subgroup of $G(\mathbb{C}[\![z]\!])$
(preimage of $B \subset G$ under $\mathrm{ev}_0 : G(\mathbb{C}[\![z]\!]) \to G;\ z \mapsto 0$)

Fact

The set of $\mathbf{I}$-orbits in $\mathbf{Q}_G^{\mathrm{rat}}$ is in bijection with W_{af} (the affine Weyl group).

Semi-infinite Schubert varieties

$\mathbf{Q}_G(x)$: a semi-infinite Schubert variety
the closure of $\mathbf{I}$-orbit in $\mathbf{Q}_G^{\mathrm{rat}}$ corresponding to $x \in W_{\mathrm{af}}$.

$\mathbf{Q}_G = \mathbf{Q}_G(e)$: also called the semi-infinite flag manifold

Introduction —Line bundles—

Fact

There exists an embedding

$$\mathbf{Q}_G \hookrightarrow \mathbb{P} := \prod_{i \in I} \mathbb{P}(L(\varpi_i)[\![z]\!]),$$

where ϖ_i, $i \in I$, is a fundamental weight.

Line bundles

For $\lambda = \sum_{i \in I} m_i \varpi_i \in P$, the ($G(\mathbb{C}[\![z]\!]) \rtimes \mathbb{C}^*$-equivariant) line bundle $\mathcal{O}(\lambda)$ on $\mathbf{Q}_G$ is the pull-back of $\boxtimes_{i \in I} \mathcal{O}(m_i)$ on $\mathbb{P}$.

Introduction —Equivariant K-group—

$q \in R(\mathbb{C}^*)$: the character of loop rotation ($q(a) := a^{-1}$, $a \in \mathbb{C}^*$)

Definition

$$K_{\mathbf{I} \rtimes \mathbb{C}^*}(\mathbf{Q}_G) := \left\{ f = \sum_{\lambda \in P} f_\lambda [\mathcal{O}(\lambda)] \;\middle|\; f_\lambda \in \mathbb{Z}[\![q^{-1}]\!][P] \text{ and } (\#) \right\} \Big/ \sim,$$

where

$$(\#) : \sum_{\lambda \in P} |f_\lambda| \operatorname{gch} \underbrace{H^0(\mathbf{Q}_G, \mathcal{O}(\lambda + \mu))}_{H \times \mathbb{C}^*\text{-module}} \in \mathbb{Z}_{\geq 0}[\![q^{-1}]\!][P] \text{ for all } \mu \in P,$$

and

$$(\sim) : f \sim 0 \iff \sum_{\lambda \in P} f_\lambda \operatorname{gch} H^0(\mathbf{Q}_G, \mathcal{O}(\lambda + \mu)) = 0 \text{ for all } \mu \text{``} \gg 0\text{''}.$$

Introduction —Equivariant K-group—

$\beta \in Q^{\vee,+}$

Fact

An embedding $i_\beta : \mathbf{Q}_G \to \mathbf{Q}_G$ *induces a* $\mathbb{Z}[\![q^{-1}]\!][P]$*-linear injection*

$$(i_\beta)_* : K_{\mathbf{I}\rtimes\mathbb{C}^*}(\mathbf{Q}_G) \to K_{\mathbf{I}\rtimes\mathbb{C}^*}(\mathbf{Q}_G)$$

s.t. $(i_\beta)_*([\mathcal{O}(\lambda)]) = q^{\langle\lambda,\beta\rangle}[\mathcal{O}_{\mathbf{Q}_G(t_{-w_\circ\beta})} \otimes \mathcal{O}(\lambda)]$, $\lambda \in P$.

$\to$ Obtaining an inductive system $((K_\alpha), (i_{\alpha,\beta}))$
where $K_\alpha = K_{\mathbf{I}\rtimes\mathbb{C}^*}(\mathbf{Q}_G)$, $\alpha \in Q^{\vee,+}$,
and $i_{\alpha,\beta} = (i_{\beta-\alpha})_*$, $\alpha, \beta \in Q^{\vee,+}$ s.t. $\alpha \leq \beta$

Definition

The $\mathbf{I} \rtimes \mathbb{C}^*$-equivariant K-group $K_{\mathbf{I}\rtimes\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ of $\mathbf{Q}_G^{\mathrm{rat}}$:

$$K_{\mathbf{I}\rtimes\mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}}) := \mathbb{Z}(\!(q^{-1})\!)[P] \otimes_{\mathbb{Z}[\![q^{-1}]\!][P]} \varinjlim K_\alpha.$$

Introduction —Equivariant K-group—

We can show that $[\mathcal{O}_{\mathbf{Q}_G(x)}] \in K_{\mathbf{I} \rtimes \mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$, $x \in W_{\mathrm{af}}$.
$W_{\mathrm{af}}^{\geq 0} = W \times \{t_\xi \mid \xi \in Q^{\vee,+}\} \subset W_{\mathrm{af}}$

Definition

The $H \times \mathbb{C}^*$-equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ of $\mathbf{Q}_G \overset{\mathrm{def}}{\Leftrightarrow}$
the $\mathbb{Z}[q, q^{-1}][P]$-submodule of $K_{\mathbf{I} \rtimes \mathbb{C}^*}(\mathbf{Q}_G^{\mathrm{rat}})$ consisting of all (formal) sum

$$\sum_{x \in W_{\mathrm{af}}^{\geq 0}} f_x [\mathcal{O}_{\mathbf{Q}_G(x)}], \; f_x \in \mathbb{Z}[q, q^{-1}][P]$$

s.t.

$$\sum_{x \in W_{\mathrm{af}}^{\geq 0}} |f_x| \in \mathbb{Z}_{\geq 0}[q, q^{-1}][P].$$

Introduction —Chevalley formula—

Chevalley formula

The Chevalley formula is an explicit expansion formula

$$[\mathcal{O}_{\mathbf{Q}_G(x)} \otimes \mathcal{O}(\lambda)] = \sum_{\mu \in P,\ y \in W_{\mathrm{af}}} c_{\mu,y} e^{\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(y)}],$$

where $c_{\mu,y} \in \mathbb{Z}[q, q^{-1}]$ for $\mu \in P$ and $y \in W_{\mathrm{af}}$.

λ: dominant $\rightarrow$ Kato-Naito-Sagaki (2017)
λ: anti-dominant $\rightarrow$ Naito-Orr-Sagaki (2018)
λ: arbitrary $\rightarrow$ Lenart-Naito-Sagaki (2019)

Introduction —Inverse Chevalley formula—

Our goal

To obtain an explicit formula of the following form:

$$e^{\mu} \cdot [\mathcal{O}_{\mathbf{Q}_G(y)}] = \sum_{x \in W_{\mathrm{af}},\ \lambda \in P} d_{x,\lambda} [\mathcal{O}_{\mathbf{Q}_G(x)} \otimes \mathcal{O}(\lambda)],$$

where $d_{x,\lambda} \in \mathbb{Z}[q, q^{-1}]$ for $x \in W_{\mathrm{af}}$ and $\lambda \in P$.

Theorem (K.-Naito-Orr-Sagaki)

If G is of type A_n (i.e., $G = SL_{n+1}$) and $\mu = y\varpi_1$ for $y \in W$, we have an explicit description of $\{d_{x,\lambda}\}_{x \in W_{\mathrm{af}}, \lambda \in P}$.

Introduction —Equivariant quantum K-theory—

$QK_H(G/B) = K_H(G/B) \otimes \mathbb{C}[\![Q^{\vee,+}]\!]$: the H-equivariant quantum K-group of G/B

$(\mathbb{C}[\![Q^{\vee,+}]\!] = \mathbb{C}[\![Q_i \mid i \in I]\!]$, $Q^{\xi} = \prod_{i \in I} Q_i^{k_i}$ if $\xi = \sum_{i \in I} k_i \alpha_i^{\vee})$

$\star$: the quantum multiplication

$[\mathcal{O}_{X_w}]$: the Schubert class corresponding to $w \in W$

$\mathcal{O}_{G/B}(\mu)$: the line bundle on G/B associated to $\mu \in P$

Theorem (K.-Naito-Orr-Sagaki)

If G is of type A_n (i.e., $G = SL_{n+1}$) and $\mu = y\varpi_1$ for $y \in W$, we have an explicit description of the expansion formula for $e^{\mu} \cdot [\mathcal{O}_{X_w}]$.

Quantum Bruhat graph

W: the (finite) Weyl group of G

Definition

The quantum Bruhat graph $\mathrm{QBG}(W)$:

- Vertices: W
- Labels: Δ^+ (the set of positive roots)
- Edges: $x \xrightarrow{\alpha} y \Leftrightarrow y = xs_\alpha$ and
  - $\ell(y) = \ell(x) + 1$ (Bruhat edge) or
  - $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$ (quantum edge).

$\mathbf{p} : x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_s} x_s$: a directed path in $\mathrm{QBG}(W)$

$$\mathrm{wt}(\mathbf{p}) := \sum_{\substack{1 \le i \le s \\ x_{i-1} \to x_i \text{ is a quantum edge}}} \gamma_i^\vee$$

Quantum Bruhat graph

$\mathbf{p} : x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_s} x_s$: a directed path in QBG(W)

$$\mathrm{wt}(\mathbf{p}) := \sum_{\substack{1 \le i \le s \\ x_{i-1} \to x_i \text{ is a quantum edge}}} \gamma_i^\vee .$$

$x, y \in W$
$\to$ There exists a shortest directed path $\mathbf{p}$ from x to y in QBG(W).

$$\mathrm{wt}(x \Rightarrow y) := \mathrm{wt}(\mathbf{p}).$$

Remark

$\mathrm{wt}(x \Rightarrow y)$ does not depend on the choice of $\mathbf{p}$.

Extremal weight modules and Demazure submodules

$\mathfrak{g}_{\mathrm{af}}$: the (untwisted) affine Lie algebra associated to $\mathfrak{g} = \mathrm{Lie}(G)$
$U_{\mathrm{v}}(\mathfrak{g}_{\mathrm{af}})$: the quantum affine algebra associated to $\mathfrak{g}_{\mathrm{af}}$
$F_i, E_i \in U_{\mathrm{v}}(\mathfrak{g}_{\mathrm{af}})$, $i \in I_{\mathrm{af}}$: Chevalley genarators

Definition

M: an integrable $U_{\mathrm{v}}(\mathfrak{g}_{\mathrm{af}})$-module.

$v \in M$ is an extremal weight vector of weight $\lambda \in P_{\mathrm{af}} \overset{\mathrm{def}}{\Leftrightarrow}$

- v is a weight vector of weight λ, and
- there exists a (unique) family $\{v_x\}_{x \in W_{\mathrm{af}}} \subset M$ s.t.
  - $v_e = v$,
  - for $i \in I_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \geq 0$, then $E_i v_x = 0$ and $F_i^{(\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$,
  - for $i \in I_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \leq 0$, then $F_i v_x = 0$ and $E_i^{(-\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$.

Extremal weight modules and Demazure submodules

Definition

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  - for $i \in I_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \geq 0$, then $E_i v_x = 0$ and $F_i^{(\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$,
  - for $i \in I_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \leq 0$, then $F_i v_x = 0$ and $E_i^{(-\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$.

Definition

$\lambda \in P_{\mathrm{af}}$

$V(\lambda)$: the extremal weight module over $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{af}})$ of extremal weight $\lambda \overset{\mathrm{def}}{\Leftrightarrow}$

- Generator: v_λ (single vector)
- Relation: v_λ is an extremal weight vector of weight λ

Extremal weight modules and Demazure submodules

Definition

$\lambda \in P_{\mathrm{af}}$

$V(\lambda)$: the extremal weight module over $U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{af}})$ of extremal weight $\lambda \overset{\text{def}}{\Leftrightarrow}$

- Generator: v_λ (single vector)
- Relation: v_λ is an extremal weight vector of weight λ

$\rightarrow$ Unique family $\{v_x\}_{x \in W_{\mathrm{af}}}$ s.t. $v_e = v_\lambda$.

$U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{af}}) = \langle F_i \rangle_{i \in I_{\mathrm{af}}} \subset U_{\mathsf{v}}(\mathfrak{g}_{\mathrm{af}})$.

Definition

$\lambda \in P_{\mathrm{af}}$, $x \in W_{\mathrm{af}}$

$V_x^-(\lambda)$: the Demazure submodule $\overset{\text{def}}{\Leftrightarrow}$

$$V_x^-(\lambda) := U_{\mathsf{v}}^-(\mathfrak{g}_{\mathrm{af}}) v_x.$$

Extremal weight modules and Demazure submodules

Remark

$V_x^-(\lambda)$ has a weight space decomposition:

$$V_x^-(\lambda) = \bigoplus_{\gamma \in Q,\ k \in \mathbb{Z}} V_x^-(\lambda)_{\lambda+\gamma+k\delta}.$$

with $\dim(V_x^-(\lambda)_{\lambda+\gamma+k\delta}) < \infty$ for all $\gamma \in Q$ and $k \in \mathbb{Z}$.

Definition

The graded character of $V_x^-(\lambda) \overset{\text{def}}{\Leftrightarrow}$

$$\mathrm{gch}\, V_x^-(\lambda) := \sum_{k \in \mathbb{Z}} \left(\sum_{\gamma \in Q} \dim(V_x^-(\lambda)_{\lambda+\gamma+k\delta}) e^{\lambda+\gamma} \right) q^k$$

Inverse Chevalley formula

G: type A_n

Goal

To obtain an explicit formula of the following form:

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{w \in W_{\mathrm{af}},\ \mu \in P} d_{w,\mu} [\mathcal{O}_{\mathbf{Q}_G(w)} \otimes \mathcal{O}(\mu)],$$

$x, y \in W$, where $d_{w,\mu} \in \mathbb{Z}[q, q^{-1}]$ for $w \in W_{\mathrm{af}}$ and $\mu \in P$.

Strategy

Consider the cohomology, and reduce the problem to that about Demazure submodules.

Classes of equivariant sheves v.s. graded characters

$\mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) := \{\Phi : P \to \mathbb{C}((q^{-1}))[P]\}$: $\mathbb{C}[q, q^{-1}][P]$-module

$$\mathrm{Fun}_P^{\mathrm{neg}}(\mathbb{C}((q^{-1}))[P]) := \left\{ \Phi \in \mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) \;\middle|\; \begin{array}{l} \exists \gamma \in P \text{ s.t. } \Phi(\mu) = 0 \\ \text{for all } \mu \in \gamma + P^+ \end{array} \right\}$$

$$\mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}((q^{-1}))[P]) := \mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) / \mathrm{Fun}_P^{\mathrm{neg}}(\mathbb{C}((q^{-1}))[P]).$$

Proposition (Kato-Naito-Sagaki, 2017)

There exists a $\mathbb{C}[q, q^{-1}][P]$*-linear injection* $\Psi : K_{H \times \mathbb{C}^*}(\mathbf{Q}_G) \to \mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}((q^{-1}))[P])$ *s.t.*

$$\Psi([\mathcal{E}]) = \left[\lambda \mapsto \sum_{i=0}^{\infty} (-1)^i \operatorname{gch} \underbrace{H^i(\mathbf{Q}_G, \mathcal{E} \otimes \mathcal{O}(\lambda))}_{H \times \mathbb{C}^*\text{-module}} \right]$$

for $[\mathcal{E}] \in K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$.

Cohomology of semi-infinite Schubert varieties

Proposition (Kato-Naito-Sagaki, 2017)

For $x \in W_{\mathrm{af}}^{\geq 0}$ and $\lambda \in P$, we have

$$\mathrm{gch}\, H^i(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(x)} \otimes \mathcal{O}(\lambda)) = \begin{cases} \mathrm{gch}\, V_x^-(-w_\circ \lambda) & \textit{if } \lambda \in P^+ \textit{ and } i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Inverse Chevalley formula

Let $S \subset W_{\mathrm{af}} \times P$ s.t. $\# S < \infty$.

$\lambda \in P^+$ is said to be sufficiently dominant (w.r.t. S) if $\lambda + \mu \in P^+$ for all $\mu \in P$ s.t. there exists $w \in W_{\mathrm{af}}$ for which $(w, \mu) \in S$.

Corollary

The following are equivarent:

- *For $x, y \in W$,*

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{(w,\mu) \in S} d_{w,\mu} [\mathcal{O}_{\mathbf{Q}_G(w)} \otimes \mathcal{O}(\mu)].$$

- *For $x, y \in W$ and sufficiently dominant $\lambda \in P^+$*

$$e^{y\varpi_1} \operatorname{gch} V_x^-(-w_\circ \lambda) = \sum_{(w,\mu) \in S} d_{w,\mu} \operatorname{gch} V_w^-(-w_\circ(\lambda + \mu)).$$

Inverse Chevalley formula in terms of Demazure submodule

G: type A_n

Goal

To obtain an explicit formula of the following form:

$$e^{y\varpi_1} \operatorname{gch} V_x^-(\lambda) = \sum_{w \in W_{\mathrm{af}},\ \mu \in P} d_{w,\mu} \operatorname{gch} V_w^-(\lambda + \mu),$$

$x, y \in W$ and $\lambda \in P^+$, where $d_{w,\mu} \in \mathbb{Z}[q, q^{-1}]$ for $w \in W_{\mathrm{af}}$ and $\mu \in P$.

Settings (1/3)

Set $y_k := s_k \cdots s_2 s_1$, $k = 0, 1, \ldots, n$.
$\to W^{\varpi_1} (= W/W_{\varpi_1}) = \{e = y_0, y_1, \ldots, y_n\}$

Setting 1

$x, y \in W$. Define $m = m(x, y) \in \mathbb{Z}$ by $x^{-1} y \varpi_1 = y_m \varpi_1$.

Set $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

Setting 2

Take a total order (reflection order) $\lhd_k$, $k = 1, \ldots, n$, on Δ^+ s.t.

$$\cdots\cdots \underbrace{\lhd_k \alpha_{k,n} \lhd_k \alpha_{k,n-1} \lhd_k \cdots \lhd_k \alpha_{k,k}}_{\langle y_{k-1}\varpi_1, -^\vee \rangle = 1}.$$

Settings (2/3)

Setting 2

Take a total order (reflection order) $\lhd_k$, $k = 1, \dots, n$, on Δ^+ s.t.

$$\cdots\cdots \underbrace{\lhd_k \alpha_{k,n} \lhd_k \alpha_{k,n-1} \lhd_k \cdots \lhd_k \alpha_{k,k}}_{\langle y_{k-1}\varpi_1, -^\vee\rangle = 1}.$$

$w \in W$, $k = 1, \dots, n$

Setting 3

$$\mathbf{DP}_w^{\lhd_k} := \left\{ \mathbf{p} : \text{directed path in } \mathrm{QBG}(W) \;\middle|\; \begin{array}{l} \mathbf{p} : \text{starting from } w, \\ \mathbf{p} : \text{label-increasing}, \\ \text{label} = \alpha_{k,n}, \dots, \alpha_{k,k} \end{array} \right\}.$$

Settings (3/3)

$k = 0, \ldots, m$

Setting 4

$\mathbf{S}_k$: the set of directed paths in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p} : x = x_0 \xrightarrow{\alpha_{i_1+1, i_0}} x_1 \xrightarrow{\alpha_{i_2+1, i_1}} \cdots \xrightarrow{\alpha_{i_p+1, i_{p-1}}} x_p$$

with $m = i_0 > i_1 > \cdots > i_p = k$.

Identify $\mathbf{p} \leftrightarrow (i_0, \ldots, i_p)$.
$\to$ $\mathbf{S}_k$ is considered as a poset under the lexicographic order.
$\to$ $v_k(x) := \mathrm{end}(\min(\mathbf{S}_k))$: the end point of $\min(\mathbf{S}_k)$.

Character identity of inverse Chevalley type

Theorem (K.-Naito-Orr-Sagaki)

For $\lambda \in P^+$ such that $\lambda + y_k \varpi_1 \in P^+$ for all $0 \le k \le m$, we have the following identity.

$$e^{y\varpi_1} \operatorname{gch} V_x^-(\lambda) = \sum_{k=0}^{m} q^{\langle y_k \varpi_1, \mathrm{wt}(x \Rightarrow v_k(x)) \rangle} \times \sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd k+1}} (-1)^{\ell(\mathbf{p})} \operatorname{gch} V^-_{\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})}} (\lambda + y_k \varpi_1).$$

Remark

RHS is a finite sum.

Special case: $x = w_\circ$

Corollary

Let $k \in \{1, 2, \ldots, n+1\}$. For a sufficiently dominant $\lambda \in P^+$, we have

$$
\begin{aligned}
&e^{\varepsilon_k} \operatorname{gch} V_{w_\circ}^-(-w_\circ \lambda) = \\
&\operatorname{gch} V_{w_\circ}^-(-w_\circ(\lambda - \varepsilon_k)) - \mathbf{1}_{\{m<n+1\}} q \operatorname{gch} V_{w_\circ t_{-w_\circ(\alpha_k^\vee)}}^-(-w_\circ(\lambda - \varepsilon_{k+1})) \\
&+ \sum_{\emptyset \neq \{i_1 < \cdots < i_a\} \subset [1,k-1]} (-1)^a \operatorname{gch} V_{(i_1\ i_2\ \cdots\ i_a\ k)^{-1} w_\circ t_{-w_\circ(\alpha_{i_1,k-1}^\vee)}}^-(-w_\circ(\lambda - \varepsilon_k)) \\
&+ q \sum_{\emptyset \neq \{j_1 < j_2 < \cdots < j_b\} \subset [k+1,n+1]} (-1)^{b-1} \times \\
&\qquad \operatorname{gch} V_{(k\ j_1\ j_2\ \cdots\ j_b)^{-1} w_\circ t_{-w_\circ(\alpha_{k,j_b-1}^\vee)}}^-(-w_\circ(\lambda - \varepsilon_{j_b})),
\end{aligned}
$$

where $\mathbf{1}_{\{m<n+1\}}$ is 1 (resp. 0) if $m < n+1$ (resp. $m = n+1$).

Example —type A_3—

$G = SL_4$: of type A_3
Let $x = s_1$ and $y = s_2 s_1$.
$\to x^{-1} y \varpi_1 = s_2 s_1 \varpi_1 = y_2 \varpi_1 \to m = 2$.

Setting 4

$\mathbf{S}_k$: the set of directed paths in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p} : x = s_1 = x_0 \xrightarrow{\alpha_{i_1+1,i_0}} x_1 \xrightarrow{\alpha_{i_2+1,i_1}} \cdots \xrightarrow{\alpha_{i_p+1,i_{p-1}}} x_p$$

with $m = 2 = i_0 > i_1 > \cdots > i_p = k$.

Identify $\mathbf{p} \leftrightarrow (i_0, \ldots, i_p)$.
$\to \mathbf{S}_k$ is considered as a poset under the lexicographic order.
$\to v_k(x) := \mathrm{end}(\min(\mathbf{S}_k))$: the end point of $\min(\mathbf{S}_k)$.

Example —type A_3—

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Identify $\mathbf{p} \leftrightarrow (i_0, \dots, i_p)$.
$\rightarrow$ $\mathbf{S}_k$ is considered as a poset under the lexicographic order.
$\rightarrow$ $v_k(x) := \mathrm{end}(\min(\mathbf{S}_k))$: the end point of $\min(\mathbf{S}_k)$.

$k = 2$: $\min(\mathbf{S}_2) = [x]$, $v_2(x) = x = s_1$.
$k = 1$: $\min(\mathbf{S}_1) = [x \xrightarrow{\alpha_2} xs_2] = (2,1)$, $v_1(x) = xs_2 = s_1 s_2$.
$k = 0$: $\min(\mathbf{S}_0) = [x \xrightarrow{\alpha_{1,2}} xs_{\alpha_{1,2}}] = (2,0)$, $v_0(x) = xs_{\alpha_{1,2}} = s_2 s_1$.
$\mathbf{S}_0 = \{(2,0),(2,1,0)\}$.
Attention: $\mathrm{wt}(x \Rightarrow v_k(x)) = 0$.

Example —type A_3—

Setting 4

$\mathbf{S}_k$: the set of directed paths in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p} : x = s_1 = x_0 \xrightarrow{\alpha_{i_1+1,i_0}} x_1 \xrightarrow{\alpha_{i_2+1,i_1}} \cdots \xrightarrow{\alpha_{i_p+1,i_{p-1}}} x_p$$

with $m = 2 = i_0 > i_1 > \cdots > i_p = k = 1$.

Identify $\mathbf{p} \leftrightarrow (i_0, \ldots, i_p)$.
$\to$ $\mathbf{S}_k$ is considered as a poset under the lexicographic order.
$\to$ $v_k(x) := \mathrm{end}(\min(\mathbf{S}_k))$: the end point of $\min(\mathbf{S}_k)$.

$k = 2$: $\min(\mathbf{S}_2) = [x]$, $v_2(x) = x = s_1$.
$k = 1$: $\min(\mathbf{S}_1) = [x \xrightarrow{\alpha_2} xs_2] = (2,1)$, $v_1(x) = xs_2 = s_1s_2$.
$k = 0$: $\min(\mathbf{S}_0) = [x \xrightarrow{\alpha_{1,2}} xs_{\alpha_{1,2}}] = (2,0)$, $v_0(x) = xs_{\alpha_{1,2}} = s_2s_1$.
$\mathbf{S}_0 = \{(2,0),(2,1,0)\}$.
Attention: $\mathrm{wt}(x \Rightarrow v_k(x)) = 0$.

Example —type A_3—

Setting 4

$\mathbf{S}_k$: the set of directed paths in $\mathrm{QBG}(W)$ of the form:

$$\mathbf{p} : x = s_1 = x_0 \xrightarrow{\alpha_{i_1+1,i_0}} x_1 \xrightarrow{\alpha_{i_2+1,i_1}} \cdots \xrightarrow{\alpha_{i_p+1,i_{p-1}}} x_p$$

with $m = 2 = i_0 > i_1 > \cdots > i_p = k = 0$.

Identify $\mathbf{p} \leftrightarrow (i_0, \ldots, i_p)$.
$\rightarrow$ $\mathbf{S}_k$ is considered as a poset under the lexicographic order.
$\rightarrow$ $v_k(x) := \mathrm{end}(\min(\mathbf{S}_k))$: the end point of $\min(\mathbf{S}_k)$.

$k = 2$: $\min(\mathbf{S}_2) = [x]$, $v_2(x) = x = s_1$.
$k = 1$: $\min(\mathbf{S}_1) = [x \xrightarrow{\alpha_2} xs_2] = (2, 1)$, $v_1(x) = xs_2 = s_1 s_2$.
$k = 0$: $\min(\mathbf{S}_0) = [x \xrightarrow{\alpha_{1,2}} xs_{\alpha_{1,2}}] = (2, 0)$, $v_0(x) = xs_{\alpha_{1,2}} = s_2 s_1$.
$\mathbf{S}_0 = \{(2, 0), (2, 1, 0)\}$.
Attention: $\mathrm{wt}(x \Rightarrow v_k(x)) = 0$.

Example —type A_3—

Settings 2, 3

$$\cdots\cdots \lhd_k \alpha_{k,n} \lhd_k \alpha_{k,n-1} \lhd_k \cdots \lhd_k \alpha_{k,k}.$$

$$\mathbf{DP}_w^{\lhd_k} := \left\{ \mathbf{p} : \text{directed path in } \mathrm{QBG}(W) \;\middle|\; \begin{array}{l} \mathbf{p} : \text{starting from } w, \\ \mathbf{p} : \text{label-increasing}, \\ \text{label} = \alpha_{k,n}, \dots, \alpha_{k,k} \end{array} \right\}.$$

$k = 2$: $v_2(x) = s_1$. $\mathbf{DP}_{v_2(x)}^{\lhd_3} = \{s_1, s_1 \xrightarrow{\alpha_3} s_1 s_3\}$.

$k = 1$: $v_1(x) = s_1 s_2$.
$\mathbf{DP}_{v_1(x)}^{\lhd_2} = \{s_1 s_2, s_1 s_2 \xrightarrow{\alpha_2} s_1, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2 \xrightarrow{\alpha_2} s_1 s_3\}$.

$k = 0$: $v_0(x) = s_2 s_1$.
$\mathbf{DP}_{v_0(x)}^{\lhd_1} = \{s_2 s_1, s_2 s_1 \xrightarrow{\alpha_1} s_2, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1 \xrightarrow{\alpha_1} s_3 s_2\}$.

Example —type A_3—

Settings 2, 3

$$\cdots\cdots \lhd_3 \alpha_{3,3}.$$

$$\mathbf{DP}^{\lhd_3}_{v_2(x)} := \left\{ \mathbf{p} : \text{directed path in } \mathrm{QBG}(W) \;\middle|\; \begin{array}{l} \mathbf{p} : \text{starting from } v_2(x), \\ \mathbf{p} : \text{label-increasing}, \\ \text{label} = \alpha_{3,3} \end{array} \right\}.$$

$k = 2$: $v_2(x) = s_1$. $\mathbf{DP}^{\lhd_3}_{v_2(x)} = \{s_1, s_1 \xrightarrow{\alpha_3} s_1 s_3\}$.

$k = 1$: $v_1(x) = s_1 s_2$.

$$\mathbf{DP}^{\lhd_2}_{v_1(x)} = \{s_1 s_2, s_1 s_2 \xrightarrow{\alpha_2} s_1, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2 \xrightarrow{\alpha_2} s_1 s_3\}.$$

$k = 0$: $v_0(x) = s_2 s_1$.

$$\mathbf{DP}^{\lhd_1}_{v_0(x)} = \{s_2 s_1, s_2 s_1 \xrightarrow{\alpha_1} s_2, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1 \xrightarrow{\alpha_1} s_3 s_2\}.$$

Example —type A_3—

Settings 2, 3

$$\cdots\cdots \lhd_{2} \alpha_{2,3} \lhd_{2} \alpha_{2,2}.$$

$$\mathbf{DP}^{\lhd_2}_{v_1(x)} := \left\{ \mathbf{p} : \text{directed path in } \mathrm{QBG}(W) \;\middle|\; \begin{array}{l} \mathbf{p} : \text{starting from } v_1(x), \\ \mathbf{p} : \text{label-increasing}, \\ \text{label} = \alpha_{2,3}, \alpha_{2,2} \end{array} \right\}.$$

$k = 2$: $v_2(x) = s_1$. $\mathbf{DP}^{\lhd_3}_{v_2(x)} = \{s_1, s_1 \xrightarrow{\alpha_3} s_1 s_3\}$.

$k = 1$: $v_1(x) = s_1 s_2$.
$\mathbf{DP}^{\lhd_2}_{v_1(x)} = \{s_1 s_2, s_1 s_2 \xrightarrow{\alpha_2} s_1, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2 \xrightarrow{\alpha_2} s_1 s_3\}$.

$k = 0$: $v_0(x) = s_2 s_1$.
$\mathbf{DP}^{\lhd_1}_{v_0(x)} = \{s_2 s_1, s_2 s_1 \xrightarrow{\alpha_1} s_2, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1 \xrightarrow{\alpha_1} s_3 s_2\}$.

Example —type A_3—

Settings 2, 3

$$\cdots\cdots \lhd_1 \alpha_{1,3} \lhd_1 \alpha_{1,2} \lhd_1 \alpha_{1,1}.$$

$$\mathbf{DP}^{\lhd_1}_{v_0(x)} := \left\{ \mathbf{p} : \text{directed path in } \mathrm{QBG}(W) \;\middle|\; \begin{array}{l} \mathbf{p} : \text{starting from } v_0(x), \\ \mathbf{p} : \text{label-increasing}, \\ \text{label} = \alpha_{1,3}, \alpha_{1,2}, \alpha_{1,1} \end{array} \right\}.$$

$k = 2$: $v_2(x) = s_1$. $\mathbf{DP}^{\lhd_3}_{v_2(x)} = \{s_1, s_1 \xrightarrow{\alpha_3} s_1 s_3\}$.

$k = 1$: $v_1(x) = s_1 s_2$.
$\mathbf{DP}^{\lhd_2}_{v_1(x)} = \{s_1 s_2, s_1 s_2 \xrightarrow{\alpha_2} s_1, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2, s_1 s_2 \xrightarrow{\alpha_{2,3}} s_1 s_3 s_2 \xrightarrow{\alpha_2} s_1 s_3\}$.

$k = 0$: $v_0(x) = s_2 s_1$.
$\mathbf{DP}^{\lhd_1}_{v_0(x)} = \{s_2 s_1, s_2 s_1 \xrightarrow{\alpha_1} s_2, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1, s_2 s_1 \xrightarrow{\alpha_{1,3}} s_3 s_2 s_1 \xrightarrow{\alpha_1} s_3 s_2\}$.

Example —type A_3—

$$\mathbf{DP}^{\lhd_1}_{v_0(x)} = \{s_2s_1,\, s_2s_1 \xrightarrow{\alpha_1} s_2,\, s_2s_1 \xrightarrow{\alpha_{1,3}} s_3s_2s_1,\, s_2s_1 \xrightarrow{\alpha_{1,3}} s_3s_2s_1 \xrightarrow{\alpha_1} s_3s_2\}.$$

$$\mathbf{DP}^{\lhd_2}_{v_1(x)} = \{s_1s_2,\, s_1s_2 \xrightarrow{\alpha_2} s_1,\, s_1s_2 \xrightarrow{\alpha_{2,3}} s_1s_3s_2,\, s_1s_2 \xrightarrow{\alpha_{2,3}} s_1s_3s_2 \xrightarrow{\alpha_2} s_1s_3\}.$$

$$\mathbf{DP}^{\lhd_3}_{v_2(x)} = \{s_1,\, s_1 \xrightarrow{\alpha_3} s_1s_3\}.$$

Result

$$\begin{aligned} e^{s_2s_1\varpi_1}\,\mathrm{gch}\, V^-_{s_1}(\lambda) = {} & \mathrm{gch}\, V^-_{s_2s_1}(\lambda+\varpi_1) - \mathrm{gch}\, V^-_{s_2t_{\alpha_1^\vee}}(\lambda+\varpi_1) \\ & - \mathrm{gch}\, V^-_{s_3s_2s_1}(\lambda+\varpi_1) + \mathrm{gch}\, V^-_{s_3s_2t_{\alpha_1^\vee}}(\lambda+\varpi_1) \\ & + \mathrm{gch}\, V^-_{s_1s_2}(\lambda+y_1\varpi_1) - \mathrm{gch}\, V^-_{s_1t_{\alpha_2^\vee}}(\lambda+y_1\varpi_1) \\ & - \mathrm{gch}\, V^-_{s_1s_3s_2}(\lambda+y_1\varpi_1) + \mathrm{gch}\, V^-_{s_1s_3t_{\alpha_2^\vee}}(\lambda+y_1\varpi_1) \\ & + \mathrm{gch}\, V^-_{s_1}(\lambda+y_2\varpi_1) - \mathrm{gch}\, V^-_{s_1s_3}(\lambda+y_2\varpi_1). \end{aligned}$$

Character identity of inverse Chevalley type

Theorem (K.-Naito-Orr-Sagaki)

For $\lambda \in P^+$ such that $\lambda + y_k \varpi_1 \in P^+$ for all $0 \leq k \leq m$, we have the following identity.

$$e^{y\varpi_1} \operatorname{gch} V_x^-(\lambda) = \sum_{k=0}^{m} q^{\langle y_k \varpi_1, \mathrm{wt}(x \Rightarrow v_k(x)) \rangle} \times \sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd_{k+1}}} (-1)^{\ell(\mathbf{p})} \operatorname{gch} V^-_{\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})}} (\lambda + y_k \varpi_1).$$

Remark

RHS is a finite sum.

The sketch of the proof —Chevalley formula—

Using the Chevalley formula for dominant/anti-dominant integral weights.

Reflection order 1

$$\begin{aligned}
\alpha_{m+1} = \alpha_{m+1,m+1} &\prec \alpha_{m+1,m+2} \prec \cdots \prec \alpha_{m+1,n} \\
&\prec \alpha_{m,m+1} \prec \alpha_{m,m+2} \prec \cdots \prec \alpha_{m,n} \\
&\prec \cdots\cdots \\
&\prec \alpha_{1,m+1} \prec \alpha_{1,m+2} \prec \cdots \prec \alpha_{1,n} \prec \underbrace{\cdots\cdots\cdots\cdots}_{\Delta^+_{I \setminus \{m+1\}}}.
\end{aligned}$$

$$\mathbf{D}_w^{\prec} := \{\mathbf{p} : w = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} w_s \text{ in } \mathrm{QBG}(W) \mid \gamma_1 \prec \cdots \prec \gamma_s \preceq \alpha_{1,n}\}.$$

The sketch of the proof —Chevalley formula—

Using the Chevalley formula for dominant/anti-dominant integral weights.

Reflection order 2

$$\begin{aligned}
\underbrace{\cdots\cdots\cdots\cdots}_{\Delta^+_{I\setminus\{m\}}} &\lhd \alpha_{1,n} \lhd \alpha_{1,n-1} \lhd \cdots \lhd \alpha_{1,m+1} \\
&\lhd \alpha_{2,n} \lhd \alpha_{2,n-1} \lhd \cdots \lhd \alpha_{2,m+1} \\
&\lhd \cdots\cdots \\
&\lhd \alpha_{m,n} \lhd \alpha_{m,n-1} \lhd \cdots \lhd \alpha_{m,m+1} \\
&\lhd \alpha_{1,m} \lhd \alpha_{2,m} \lhd \cdots \lhd \alpha_{m,m} = \alpha_m.
\end{aligned}$$

$$\mathbf{E}_w^{\lhd} := \{\mathbf{p} : w = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} w_s \text{ in } \mathrm{QBG}(W) \mid \alpha_{1,n} \unlhd \gamma_1 \lhd \cdots \lhd \gamma_s\}.$$

The sketch of the proof —Chevalley formula—

Theorem (Kato-Naito-Sagaki, 2017)

Let $\lambda \in P^+$. For $x \in W$, we have

$$\operatorname{gch} V_x^-(\lambda + \varpi_{m+1}) = \sum_{\mathbf{p} \in \mathbf{D}_x^{\prec}} \sum_{i=0}^{\infty} e^{\operatorname{end}(\mathbf{p})\varpi_{m+1}} q^{-\langle \varpi_{m+1}, \operatorname{wt}(\mathbf{p})\rangle - i} \operatorname{gch} V^-_{\operatorname{end}(\mathbf{p}) t_{\operatorname{wt}(\mathbf{p}) + i\alpha^\vee_{m+1}}}(\lambda).$$

Theorem (Naito-Orr-Sagaki, 2018)

Let $\lambda \in P^+$ be such that $\lambda - \varpi_m \in P^+$. For $w \in W$, we have

$$\operatorname{gch} V_x^-(\lambda - \varpi_m) = e^{-x\varpi_m} \sum_{\mathbf{p} \in \mathbf{E}_x^{\lhd}} (-1)^{\ell(\mathbf{p})} \operatorname{gch} V^-_{\operatorname{end}(\mathbf{p}) t_{\operatorname{wt}(\mathbf{p})}}(\lambda).$$

The sketch of the proof —Chevalley formula—

$$\overline{\mathbf{DP}}_w^{\prec} := \{\mathbf{p} : w = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} w_s \text{ in } \mathrm{QBG}(W) \mid \gamma_1 \prec \cdots \prec \gamma_s \preceq \alpha_{m+1,n}\},$$

$$\mathbf{F}_w^{\lhd} := \{\mathbf{p} : w = w_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_s} w_s \text{ in } \mathrm{QBG}(W) \mid \alpha_{1,m} \unlhd \gamma_1 \lhd \cdots \lhd \gamma_s\}.$$

Theorem

Let $\lambda \in P^+$ be such that $\lambda + y_m\varpi_1 \in P^+$. For $w \in W$, we have

$$\mathrm{gch}\, V_w^-(\lambda + y_m\varpi_1) = \frac{1}{1 - q^{-\langle \lambda + \varpi_{m+1}, \alpha_{m+1}^\vee \rangle}} \times \sum_{\mathbf{d}' \in \overline{\mathbf{DP}}_w^{\prec}} \sum_{\mathbf{d}'' \in \mathbf{F}_{\mathrm{end}(\mathbf{d}')}^{\lhd}} e^{\mathrm{end}(\mathbf{d}')(-\varpi_m + \varpi_{m+1})} q^{\langle \varpi_m - \varpi_{m+1}, \mathrm{wt}(\mathbf{d}') \rangle} \times (-1)^{\ell(\mathbf{d}'')}\, \mathrm{gch}\, V_{\mathrm{end}(\mathbf{d}'') t_{\mathrm{wt}(\mathbf{d}') + \mathrm{wt}(\mathbf{d}'')}}^-(\lambda).$$

The sketch of the proof —Base step—

We use the induction on $m = m(x, y)$.
$m = 0$: An easy consequence of the Chevalley formula for anti-dominant weights.

Theorem (Naito-Orr-Sagaki, 2018)

Let $\lambda \in P^+$ be such that $\lambda - \varpi_1 \in P^+$. For $w \in W$, we have

$$\operatorname{gch} V_x^-(\lambda - \varpi_1) = e^{-x\varpi_1} \sum_{\mathbf{p} \in \mathbf{E}_x^{\lhd}} (-1)^{\ell(\mathbf{p})} \operatorname{gch} V_{\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(\mathbf{p})}}^-(\lambda).$$

Multiplying $e^{x\varpi_1}$ to both sides, and replacing $\lambda - \varpi_1$ with λ.
$\rightarrow$ Obtaining the desired identity.

The sketch of the proof —Inductive step—

$m > 0$: Expand the (RHS) of the formula, and calculate all cancellation.

Desired identity

$$e^{y\varpi_1} \operatorname{gch} V_x^-(\lambda) = \sum_{k=0}^{m} q^{\langle y_k\varpi_1, \mathrm{wt}(x \Rightarrow v_k(x))\rangle} \times$$
$$\sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd_{k+1}}} (-1)^{\ell(\mathbf{p})} \operatorname{gch} V^-_{\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})}}(\lambda + y_k\varpi_1).$$

$0 \le k \le m-1$: induction hypothesis
$k = m$: the Chevalley formula

Inverse Chevalley formula —Review—

Let $S \subset W_{\mathrm{af}} \times P$ s.t. $\# S < \infty$.

$\lambda \in P^+$ is said to be sufficiently dominant (w.r.t. S) if $\lambda + \mu \in P^+$ for all $\mu \in P$ s.t. there exists $w \in W_{\mathrm{af}}$ for which $(w, \mu) \in S$.

Corollary

The following are equivarent:

- *For $x, y \in W$,*

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{(w,\mu) \in S} d_{w,\mu} [\mathcal{O}_{\mathbf{Q}_G(w)} \otimes \mathcal{O}(\mu)].$$

- *For $x, y \in W$ and sufficiently dominant $\lambda \in P^+$*

$$e^{y\varpi_1} \operatorname{gch} V_x^-(-w_\circ \lambda) = \sum_{(w,\mu) \in S} d_{w,\mu} \operatorname{gch} V_w^-(-w_\circ(\lambda + \mu)).$$

Inverse Chevalley formula

Corollary (K.-Naito-Orr-Sagaki)

For $x, y \in W$, it holds that

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{k=0}^{m} q^{\langle y_k\varpi_1, \mathrm{wt}(x \Rightarrow v_k(x))\rangle} \times \sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd_{k+1}}} (-1)^{\ell(\mathbf{p})} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})})} \otimes \mathcal{O}(-w_\circ y_k \varpi_1)].$$

Quantum K-theory of the flag manifold

$QK_H(G/B)$: the H-equivariant quantum K-group of G/B

$K_H(\mathbf{Q}_G)$: the H-equivariant K-group of $\mathbf{Q}_G$

(We specialize $K_{H\times\mathbb{C}^*}(\mathbf{Q}_G)$ at $q = 1$)

Proposition

There exists a $\mathbb{C}[P]$-module isomorphism $QK_H(G/B) \xrightarrow{\simeq} K_H(\mathbf{Q}_G)$ s.t.

- [Kato, 2018] $[\mathcal{O}_{X_{ww_\circ}}]Q^{-w_\circ\xi} \mapsto [\mathcal{O}_{\mathbf{Q}_G(wt_\xi)}]$ *for* $w \in W$ *and* $\xi \in Q^{\vee,+}$,
- [K.-Naito-Orr-Sagaki] *if G is of type A_n, then* $-\star(1/(1-Q_{i+1}))[\mathcal{O}_{G/B}(y_i\varpi_1)]$ *corresponds to* $-\otimes[\mathcal{O}(y_i\varpi_1)]$,

here, $Q_{n+1} := 0$.

Quantum K-theory of the flag manifold

Corollary (K.-Naito-Orr-Sagaki)

For $x, y \in W$, it holds in $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ that

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{k=0}^{m} q^{\langle y_k \varpi_1, \mathrm{wt}(x \Rightarrow v_k(x)) \rangle} \times$$

$$\sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd k+1}} (-1)^{\ell(\mathbf{p})} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})})} \otimes \mathcal{O}(-w_\circ y_k \varpi_1)].$$

Specialize at $q = 1$ $\to$ We obtain an identity in $K_H(\mathbf{Q}_G)$.

Quantum K-theory of the flag manifold

Corollary (K.-Naito-Orr-Sagaki)

For $x, y \in W$, it holds in $K_H(\mathbf{Q}_G)$ that

$$e^{y\varpi_1} \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{k=0}^{m} \sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd k+1}} (-1)^{\ell(\mathbf{p})} [\mathcal{O}_{\mathbf{Q}_G(\mathrm{end}(\mathbf{p}) t_{\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p})})} \otimes \mathcal{O}(-w_\circ y_k \varpi_1)].$$

Quantum K-theory of the flag manifold

Corollary (K.-Naito-Orr-Sagaki)

For $x, y \in W$, it holds in $QK_H(G/B)$ that

$$e^{y\varpi_1} \cdot [\mathcal{O}_{X_{xw_\circ}}] = \sum_{k=0}^{m} \sum_{\mathbf{p} \in \mathbf{DP}_{v_k(x)}^{\lhd_{k+1}}} (-1)^{\ell(\mathbf{p})} \times$$

$$[\mathcal{O}_{X_{\mathrm{end}(\mathbf{p})w_\circ}}] \star [\mathcal{O}_{G/B}(-w_\circ y_k \varpi_1)] \frac{Q^{-w_\circ(\mathrm{wt}(x \Rightarrow v_k(x)) + \mathrm{wt}(\mathbf{p}))}}{1 - Q_{n-k}},$$

here, $Q_0 := 0$.