PBW parametrizations and generalized preprojective algebras

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Contents

Introduction

In representation theory, study of "good basis" of algebras or representations are very fundamental problems.

Example (Poincaré-Birkhoff-Witt)

Let $\mathfrak{g}$ be a Lie algebra. If $\{x_i \mid i = 1, 2, \dots\}$ is a basis of $\mathfrak{g}$, then

$$\{x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \mid n \in \mathbb{Z}_{>0}, r_i \in \mathbb{Z}_{\geq 0}\}$$

form a basis of the universal enveloping algebra $U(\mathfrak{g})$.

However, this basis depends on a choice of $\{x_i \mid i \in I\}$, and we cannot call this basis "canonical". Therefore, we often consider basis with (geometric) characterizations.

Aim of this talk

Comparing two realizations of dual canonical bases (crystal) for finite symmetrizable types including B, C, F, G as a generalization of the work of [Baumann-Kamnitzer-Tingley, 2013]:

1. (generalized) preprojective algebras [Kashiwara-Saito, 1997], [Geiß-Leclerc-Schröer, 2018]
2. Mirković-Villonen polytopes [Kamnitzer, 2010]

$$\begin{tikzcd}
 & B(-\infty) \arrow[dl, "\text{[KS], [GLS]}"'] \arrow[dr, "\text{[Kamnitzer]}"] & \\
\coprod_{\mathbf{r}\in\mathbb{N}^n} \mathsf{Irr}\left(\Pi(\mathbf{r})\right)^{\max} \arrow[rr, "??"] & & \mathcal{MV}
\end{tikzcd}$$

Lusztig has studied some $\mathbb{Q}(q)$-automorphisms T_i of $U_q(\mathfrak{g})$.

Theorem (Lusztig, Saito)

$\mathfrak{g}$: *f.d simple Lie algebra*, $\mathbf{i} = (i_1, \dots, i_l)$: *red. expression of* $w_0 \in W$
$\beta_{\mathbf{i},k} := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ $(k = 1, \dots, l)$

$$\exists \Psi_{\mathbf{i}} : \mathbb{Z}^l_{\geq 0} \xrightarrow{1:1} B(-\infty)$$
$$\mathbf{a} = (a_1, \dots, a_l) \to b_{\mathbf{i},\mathbf{a}} (\equiv F_{\mathbf{i}}(\mathbf{a}) \in U_q^+(\mathfrak{g})),$$

where $F_{\mathbf{i}}(\mathbf{a}) = (F(\beta_{\mathbf{i},1}))^{a_1} \cdots (F(\beta_{\mathbf{i},l}))^{a_l}$ *and*
$F(\beta_{\mathbf{i},k}) = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k})$.

We call $\Psi_{\mathbf{i}}^{-1}(b)$ the $\mathbf{i}$-Lusztig datum of $b \in B(-\infty)$.
How do $\mathbf{i}$-Lusztig data appear in representation theory of generalized preprojective algebra (=: GPA)?

In the quiver representation theory, Gabriel theorem gives a bijection between indecomposable representations of Dynkin quivers and the positive root systems:

Theorem (Gabriel)

Q: Dynkin quiver of type $\mathsf{A}, \mathsf{D}, \mathsf{E}$, kQ: path alg. of Q

$$\mathsf{Ind}kQ \xrightarrow{1:1} \Delta^+(Q)$$

$$M \mapsto \underline{\dim} M$$

- There are many relationships between quiver representation theory and Lie theory. *e.g.* Hall algebras, quiver varieties,
- We seek for a nice generalization containing $\mathsf{B}, \mathsf{C}, \mathsf{F}, \mathsf{G}$ types of this relationship.

In representation theory and study of physics, some kinds of quivers with relations, which are expected to characterize "symmetrizable theory" have appeared:

- Study of quiver varieties with multiplicities [Yamakawa]
- q-characters of Kirillov-Reshetikhin module of quantum loop algebras [Hernandez-Leclerc]
- Additive categorification of cluster algebras [Ladkani]
- $4d$ $N = 2$ quiver gauge theory [Cecotti-Del Zotto]

Can we formulate a "symmetrizable theory" adopting above examples?

In the work of [Geiß–Leclerc-Schröer, 2017], one of candidates for generalizations is introduced.

Definition

$C = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$: symmetrizable GCM, $D = \mathrm{diag}\,(c_1, \ldots, c_n)$ s.t. DC is a symm.
Define quiver $Q(C, D)$ as follows:

- verteces: $\{1, \ldots, n\}$
- arrows: Take an orientation Ω. Only one of (i, j) or (j, i) belongs to Ω iff $c_{ij} \neq 0$.

$$\{\alpha_{ij} \mid (i, j) \in \Omega\} \cup \{\varepsilon_i \mid i = 1, \ldots, n\}.$$

In the work of Geiß-Leclerc-Schröer, a generalization of path algebras of Dynkin quivers is introduced as quivers with relations:

Definition

Define K-algebra $H = H(C, D, \Omega) := KQ/I$ by the quiver Q with relations I generated by (H1), (H2):

(H1) $\varepsilon_i^{c_i} = 0 \ (i \in Q_0)$;

(H2) $\varepsilon_i^{f_{ji}} \alpha_{ij} = \alpha_{ij} \varepsilon_j^{f_{ij}} \ (\forall (i,j) \in \Omega)$.

where $f_{ij} := \frac{|c_{ij}|}{\gcd(|c_{ij}|, |c_{ji}|)}$

Since we deal with roots of the forms $s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)$, we want larger module categories than mod H in a viewpoint of Gabriel theorem.

Define a GPA as a double of Q with relations:

Definition

C: symmetrizable GCM, D: symmetrizer Define $\Pi := \Pi(C, D)$ as a quiver $\overline{Q}$ with the following relations (P1)-(P3):

(P1) $\varepsilon_i^{c_i} = 0 \ (i \in Q_0)$;

(P2) $\varepsilon_i^{f_{ji}} \alpha_{ij} = \alpha_{ij} \varepsilon_j^{f_{ij}} \ (1 \leq g \leq g_{ij}, \forall (i, j) \in \overline{\Omega})$;

(P3) $\sum_{j \in \overline{\Omega}(i)} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i, j) \varepsilon_i^{f} \alpha_{ij} \alpha_{ji} \varepsilon_i^{f_{ji}-1-f} = 0 \ (i \in Q_0)$

$$\overline{\Omega}(i) := \{j \in Q_0 \mid (i, j) \in \overline{\Omega}\} \quad \operatorname{sgn}(i, j) := \begin{cases} 1 & (i, j) \in \Omega, \\ -1 & (i, j) \in \Omega^*. \end{cases}$$

Note that a GPA is finite dimensional self-injective algebra iff C is of finite type. Any GPA does not depend on a choice of orientation up to isomorphisms. Today, we always assume C is connected finite type and D is minimal symmetrizer.

Example

Let $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $D = \mathrm{diag}\,(2d, d)$ $(d \in \mathbb{Z}_{>0})$, $\Omega = \{(1,2)\}$

Then, we have $c_1 = 2d, c_2 = d, f_{12} = 1, f_{21} = 2$.

$\Pi = \Pi\,(C, D)$ is isomorphic to the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] \arrow[r, shift left, "\alpha_{21}"] & 2 \arrow[loop above, "\varepsilon_2"] \arrow[l, shift left, "\alpha_{12}"] \end{tikzcd}$$

with relations (P1) $\varepsilon_1^{2d} = 0, \varepsilon_2^{d} = 0$;

(P2) $\varepsilon_1^2 \alpha_{12} = \alpha_{12}\varepsilon_2, \varepsilon_2\alpha_{21} = \alpha_{21}\varepsilon_1^2$;

(P3) $\alpha_{12}\alpha_{21}\varepsilon_1 + \varepsilon_1\alpha_{12}\alpha_{21} = 0, -\alpha_{21}\alpha_{12} = 0$.

In the representation theory of GLS's algebras, the most fundamental class is locally free modules. Here, let $\Lambda := H$ or Π.

Definition

1. $H_i := e_i\left(K[\varepsilon_i]/\left(\varepsilon_i^{c_i}\right)\right)e_i \cong K[\varepsilon_i]/\left(\varepsilon_i^{c_i}\right) \ (\forall i \in Q_0)$.
2. $M \in \operatorname{mod}\Lambda$ is *locally free* $\Leftrightarrow e_iM$ is a free H_i-module $\forall i \in Q_0$.
3. M:locally free, $a_i := \operatorname{rank}_i e_iM = \operatorname{rank}_{H_i} e_iM$

$$\underline{\operatorname{rank}}\, M = (a_1, \dots, a_n).$$

4. Define a *generalized simple module* $E_i \in \operatorname{mod}_{\mathrm{l.f.}}\Lambda \ (i \in Q_0)$ as

$$e_jE_i \cong \begin{cases} H_i & (j = i) \\ 0 & (j \neq i). \end{cases}$$

In the work of Geiß-Leclerc-Schröer, they assume additional conditions on locally free modules and consider their module categories:

Example

$$\{\text{iso-classes of ind. } \tau\text{-locally free } H(C,D,\Omega)\text{-modules}\} \xleftrightarrow{1:1} \Delta_+(C)$$

$$M \mapsto \underline{\mathrm{rank}}\, M$$

Definition

$M \in \mathrm{mod}_{\mathrm{l.f.}}\,\Pi$: $\mathbb{E}$-filtered :⇔ M has a filtration

$$M = M_m \supsetneq M_{m-1} \supsetneq \cdots \supsetneq M_0 = 0$$

s.t. each $M_{i+1}/M_i \cong E_j$ $(\exists j \in Q_0)$.

1. Introduction
2. quiver with relations and preprojective algebras associated with symmetrizable GCM
3. PBW-parametrizations

Setting

In the work of Geiß-Leclerc-Schröer, they have a geometric realization of crystal $B(-\infty)$:

$$B(-\infty) \overset{1:1}{\longleftrightarrow} \coprod_{\mathbf{r}\in\mathbb{N}^n} \mathsf{Irr}\,(\Pi(\mathbf{r}))^{\max}$$

- $\Pi(\mathbf{r})$: the variety of $\mathbb{E}$-filtered module with $\underline{\mathrm{rank}}\, M = \mathbf{r}$
- $\mathsf{Irr}\,(\Pi(\mathbf{r}))^{\max}$: maximal dimensional irreducible components of $\Pi(\mathbf{r})$

We take a generic module $M \in Z \in \mathsf{Irr}\,(\Pi(\mathbf{r}))^{\max}$, and extract numerical data of crystal from M in terms of Weyl group.

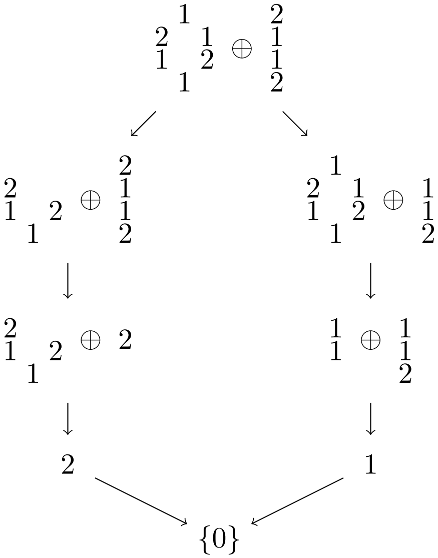
Symmetries of module categories of preprojective algebras Π are often described in terms of Weyl groups. We define $I_i := \Pi(1 - e_i)\Pi$.

Theorem (Buan-Iyama-Reiten-Scott, Fu-Geng)

Let $w \in W$. $\mathbf{i}_w = (i_1, \ldots, i_l)$: a reduced expression of w. The following map is well-defined and bijective:

$$W = \langle s_1, \ldots, s_n \rangle \xrightarrow{1:1} \langle I_1, \ldots, I_n \rangle$$
$$w = s_{i_1}, \ldots, s_{i_l} \mapsto I_{i_1} \cdots I_{i_l} =: I_w$$

We give an example of type B_2:



Definition

- M is a *τ-rigid* Λ-module, if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$;
- M is a *τ-tilting* Λ-module, if M is τ-rigid and $|M| = |\Lambda|$;
- M is a *support τ-tilting* module, if there exists an idempotent $e \in \Lambda$ such that M is τ-tilting $(\Lambda/\langle e \rangle)$-module;

Definition

- A full subcategory $\mathcal{T}$ in $\mathrm{mod}\,\Lambda$ is a *torsion class*, if $\mathcal{T}$ is closed under extensions and taking a factor module of objects.
- A full subcategory $\mathcal{F}$ in $\mathrm{mod}\,\Lambda$ is a *torsion-free class*, if $\mathcal{F}$ is closed under extensions and taking a submodule of objects.

Theorem (Mizuno, Fu-Geng, M)

We have the following bijections:

$$W \xrightarrow{1:1} \mathsf{s\tau\text{-}tilt}\,\Pi \qquad W \xrightarrow{1:1} \mathsf{tors}\,\Pi \qquad W \xrightarrow{1:1} \mathsf{torf}\,\Pi$$
$$w \longmapsto I_w \qquad w \longmapsto \mathsf{Fac}\, I_w \qquad w \longmapsto \mathsf{Sub}\,\Pi/I_w.$$

Here, we denote

- $\mathsf{Fac}\, M := \{N \in \mathsf{mod}\,\Pi \mid \exists M^{\oplus m} \twoheadrightarrow N\}$;
- $\mathsf{Sub}\, M := \{N \in \mathsf{mod}\,\Pi \mid \exists N \hookrightarrow M^{\oplus m}\}$.

For $M \in \mathsf{mod}\,\Pi$ and $w \in W$, we have a unique short exact sequence

$$0 \to M_w \to M \to M/M_w \to 0$$

s.t. $M_w \in \mathsf{Fac}\, I_w$ and $M/M_w \in \mathsf{Sub}\,\Pi/I_w$ up to isomorphisms.

Theorem (M)

$\mathbf{i} = (i_1, \ldots, i_l)$:red. expression of $w_0 \in W$. Let $\Pi_{\mathbf{i}}^{\mathbf{a}}$ be the set of $M \in \Pi(\mathbf{r})$ s.t. M has a filtration

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l = 0$$

with $M_{j-1}/M_j \cong (I_{s_{i_1} \cdots s_{i_{j-1}}} / I_{s_{i_1} \cdots s_{i_j}})^{\oplus a_j}$ for $\exists \mathbf{a} := (a_1, \ldots, a_l) \in \mathbb{Z}^r$. Then, $Z_{\mathbf{i}}^{\mathbf{a}} := \overline{\Pi_{\mathbf{i}}^{\mathbf{a}}}$ is a maximal irreducible component. Any maximal irreducible component has this form.

Idea of Theorem: Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair.

- Prove that M :generic $\Rightarrow M_{\mathcal{T}}, M_{\mathcal{F}} \in \mathbb{E}$-filt$(\Pi)$.
- $\Pi^{\mathcal{T}}(\mathbf{r}) := \{M \in \Pi(\mathbf{r}) \mid M \in \mathcal{T}\}$ and $\Pi^{\mathcal{F}}(\mathbf{r}) := \{M \in \Pi(\mathbf{r}) \mid M \in \mathcal{F}\}$.
- $\Pi^{\mathcal{T}}(\mathbf{r})$ and $\Pi^{\mathcal{F}}(\mathbf{r})$ define Zariski open subset of $\Pi(\mathbf{r})$ by Brenner-Buttler theory and τ-tilting finiteness of Π.

- Consider fiber bundles:

$$\Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f) \xleftarrow{p} E(\mathbf{r}_t, \mathbf{r}_f) \xrightarrow{q} \Pi(\mathbf{r}),$$

where $E(\mathbf{r}_t, \mathbf{r}_f)$: the set of (M, M_t, M_f, f, g) s.t. $M \in \Pi(\mathbf{r})$ and $(M_t, M_f) \in \Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f)$ with an exact sequence

$$0 \to M_t \xrightarrow{f} M \xrightarrow{g} M_f \to 0$$

in $\operatorname{mod} \Pi$. (p, q are canonical projections).

- $\overline{q\left(p^{-1}\left(Z_t \times Z_f\right)\right)}$ is a maximal irreducible component of $\Pi(\mathbf{r})$.
- $Z := \overline{q\left(p^{-1}\left(Z_t \times Z_f\right)\right)}$, M : generic point of Z.
$\Rightarrow \underline{\operatorname{rank}}\, tM = \mathbf{r}_t$ and $(tM, M/tM)$ is a generic point in $Z_t \times Z_f$.
- By induction, we obtain a tuple $\left(M_{s_1 \cdots s_{l-1}}, M_{s_1 \cdots s_{l-2}}/M_{s_1 \cdots s_{l-1}}, \ldots, M/M_{s_1}\right)$ of generic points of maximal irreducible components from M.

Construction of MV-polytopes

Roughly speaking, MV-polytopes are convex polytopes defined on $V := R \otimes_{\mathbb{Z}} \mathbb{R}$ (R is the root lattice) whose verteces are labelled by $w \in W$. They contain $\mathbf{i}$-Lusztig data for any reduced expression of w_0 as "length of edges".

Definition

A convex polytope $P = \{v \in V \mid \langle v, \alpha \rangle \leq \psi_P(\alpha)$ for any $\alpha \in V^*\}$ is a MV-polytope iff

- ψ_P is linear on each Weyl chamber.
- $(\psi_P(\gamma))_{\gamma \in \Gamma}$ satisfies Berenstein-Zelevinsky data, where $\Gamma := \{w\varpi_i \mid w \in W, i \in Q_0\}$. (cf. [Kamnitzer, 2010])

We realize MV-polytopes from generic modules by constructing Harder-Narasimhan polytopes.

Definition

$\mathcal{A}$: Abelian cat., $K_0(\mathcal{A})$:the Grothendieck grp. of $\mathcal{A}$
For $T \in \mathcal{A}$, define $\mathrm{Pol}\,(T)$ as the convex hull in $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ of all of points $[X]$ s.t. $X \subseteq T$. We refer this $\mathrm{Pol}\,(T)$ as the *HN (=Harder-Narasimhan) polytope* of T.

Definition

Let $Z \in \mathrm{Irr}\,(\Pi(\mathbf{r}))^{\max}$, T : generic module in Z

$$P(T) := \{(x_1/c_1, \ldots, x_n/c_n) \mid x = (x_1, \ldots, x_n) \in \mathrm{Pol}\,(T)\}$$

where $\mathrm{diag}\,(c_1, \ldots, c_n)$ is a symmetrizer of C.

Now, $P(T)$ has faces $P_\theta = \left\{x \in \mathbb{R}^n \mid \langle \theta, x \rangle = \psi_{P(T)}(\theta)\right\}$
$(\theta \in (\mathbb{R}^n)^*)$,
where $\psi_{P(T)} : \mathbb{R}^n \to \mathbb{R}$ which maps θ to its maximum on $P(T)$.

Definition

Let $M \in \operatorname{mod}\Lambda$ and let $P \in \operatorname{proj}\Lambda$.

- (M, P) is a *τ-rigid pair*, if M is τ-rigid and $\operatorname{Hom}_\Lambda(P, M) = 0$;
- (M, P) is a *support τ-tilting pair*, if (M, P) is a τ-rigid pair and $|M| + |P| = |\Lambda|$.

Note that (M, P) is a support τ-tilt. pair
$\Leftrightarrow$ M is a τ-tilting $(\Lambda/\langle e\rangle)$-module s.t. $\exists e$:idempotent
add $P =$ add Λe.

Definition

Let $M \in \operatorname{mod}\Pi$, $P_1(M) \to P_0(M) \to M \to 0$: min. proj. pres.
Define the *g-vector of M* by

$$g(M) := (g_1(M), \ldots, g_n(M))^{\mathsf{T}} = [P_0(M)] - [P_1(M)] \in K_0(\operatorname{proj}\Pi).$$

(M, P) : basic support τ-tilt. pair with $M = \bigoplus_{i=1}^{l} M_i$ and $P = \bigoplus_{i=l+1}^{l} P_i$ with M_i and P_i indecomp.
$\Rightarrow g(M_1), \ldots, g(M_l), g(P_{l+1}), \ldots, g(P_n)$ form a basis of $K_0(\text{proj}\,\Lambda)$.

Theorem (Mizuno, Fu-Geng)

$\{\varpi_1, \ldots, \varpi_n\} \subset \mathbb{Z}^n$: standard basis of $\mathbb{Z}^n$.

$$g_i(w) := g(I_w e_i) - (0, \ldots, \delta_{I_w e_i, 0}, \ldots, 0) = w\varpi_i,$$

where $s_i \varpi_j := \begin{cases} \varpi_j - \sum_{i \in Q_0} c_{ij} \varpi_i & (i = j) \\ \varpi_j & (i \neq j). \end{cases}$, *$\sigma$ is the Nakayama permutation ass. with Π. In fact, σ is a trivial permutation iff C is of type* $\mathsf{B}, \mathsf{C}, \mathsf{D}_{2n}, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4, \mathsf{G}_2$.

We denote $C(w) := \{a_1 g_1(w) + \cdots + a_n g_n(w) \mid a_i \in \mathbb{R}_{>0}\}$.

Theorem (Auslander-Reiten)

Λ : *f.d.* K*-algebra*, $\langle -,- \rangle \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$:*natural pairing*,
$M, N \in \operatorname{mod} \Lambda$.

$$\langle g^M, \underline{\dim}\, N \rangle = \dim \operatorname{Hom}_\Pi(M, N) - \dim \operatorname{Hom}_\Pi(N, \tau M).$$

Corollary

$M \in \operatorname{mod}_{\mathrm{l.f.}} \Pi$ *s.t.* $\alpha := \underline{\operatorname{rank}}\, M$,
$\gamma := \sum_{i \in Q_0} a_i g_i(w) \in \mathbb{Z}^n \ (a_i \in \mathbb{Z})$

$$\langle \gamma, \alpha \rangle = \sum_{I_w e_i \neq 0} \frac{a_i}{c_i} (\dim \operatorname{Hom}_\Pi(I_w e_i, M) - \dim \operatorname{Hom}_\Pi(M, \tau(I_w e_i)))$$
$$- \sum_{I_w e_j = 0} \frac{a_j}{c_j} (\dim \operatorname{Hom}_\Pi(\Pi e_{\sigma(j)}, M)).$$

By the work of Brüstle-Smith-Treffinger which developed stability conditions for general f.d. algebras by τ-tilting theory, we know the following corollary by applying their theorem to our setting:
Let $\theta\colon K_0(\mathcal{A}) \to \mathbb{R}$: a group homomorphism.

$$\mathcal{T}_\theta := \{M \in \mathcal{A} \mid \langle \theta, [N] \rangle > 0 \text{ for any quotient } N \text{ of } M\}$$
$$\overline{\mathcal{T}_\theta} := \{M \in \mathcal{A} \mid \langle \theta, [N] \rangle \geq 0 \text{ for any quotient } N \text{ of } M\}$$
$$\mathcal{F}_\theta := \{M \in \mathcal{A} \mid \langle \theta, [L] \rangle < 0 \text{ for any submodule } L \text{ of } M\}$$
$$\overline{\mathcal{F}_\theta} := \{M \in \mathcal{A} \mid \langle \theta, [L] \rangle \leq 0 \text{ for any submodule } L \text{ of } M\}$$

Corollary

Let $w \in W$ and let $\theta \in C(w)$. If we take $\langle \theta, - \rangle$ as the Auslander-Reiten's product, then $\overline{\mathcal{T}}_\theta = \mathsf{Fac}\, I_w$ and $\mathcal{F}_\theta = \mathsf{Sub}\, \Pi/I_w$ for $\theta \in C(w)$.

Theorem (M)

Let $Z \in \mathsf{Irr}\,(\Pi(\mathbf{r}))^{\max}$, T *: generic module of* Z

$$P(T) = \{v \in \mathbb{R}^n \mid \langle \gamma, v \rangle \leq D_\gamma(T), \gamma \in \Gamma\}$$

are MV-polytopes associated with root datum of C^{T} *except for type* G_2*, where we denote*

$$D_{w\varpi_i}(M) := \frac{1}{c_i} \dim \mathrm{Hom}_\Pi(I_w \otimes_\Pi P_i, M)$$

.

Idea of Theorem: $(D_\gamma(T))_{\gamma\in\Gamma}$ describes BZ data.

- $\underline{\mathrm{rank}}\, T_w$ $(w \in W)$ define verteces of $P(T)$.
  (Quotient modules $T_w/T_{ws_i}(\cong (I_w \otimes_\Pi E_i)^{\oplus a_i})$ define "length of edges").
- Let $J = \{i, j\}$ s.t. $l(ws_i) = l(ws_j) > l(w)$.
  2-faces of $P(T)$ are described by $\mathsf{Sub}(\Pi/I_{ww_J}) \cap \mathsf{Fac}\, I_w \cong \mathrm{mod}\ (\Pi/I_{w_J})$.
- We can check $D_\gamma(T)$ satisfy BZ-data by counting multiplicities of layer modules in HN-filtration of generic modules over 2-verteces GPA.

Example

For type B_2, any generic module has a form $X_1^{\oplus a} \oplus X_2^{\oplus b} \oplus X_3^{\oplus c} \oplus X_4^{\oplus d}$ where X_i is one of the 8 Loewy series of modules bellow $(i = 1, j = 2)$ and (X_1, X_2, X_3, X_4) is one of $(X_1, X_2, X_3, X_4) = (P_1, P_2, E_1, T_1)$, (P_1, P_2, E_1, T_2), (P_1, P_2, E_2, T_3), (P_1, P_2, E_2, T_4), (P_1, P_2, T_1, T_3), (P_1, P_2, T_2, T_4).

$$P_1 = \begin{array}{ccc} & 1 & \\ 2 & & 1 \\ 1 & & 2 \\ & 1 & \end{array} ; P_2 = \begin{array}{c} 2 \\ 1 \\ 1 \\ 2 \end{array} ; E_1 = \begin{array}{c} 1 \\ 1 \end{array} ; \quad E_2 = 2 ;$$

$$T_1 = \begin{array}{c} 1 \\ 1 \\ 2 \end{array} ; \quad T_2 = \begin{array}{c} 2 \\ 1 \\ 1 \end{array} ; T_3 = \begin{array}{ccc} & 1 & \\ 2 & & 1 \\ & & 2 \end{array} ; T_4 = \begin{array}{ccc} 2 & & \\ 1 & & 2 \\ & 1 & \end{array} .$$