Non-rigid regions of real Grothendieck groups

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Motivation

Let A be a fin. dim. K-algebra over a field K.

- $K_0(\operatorname{proj} A)_{\mathbb{R}} := K_0(\operatorname{proj} A) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.
- Each $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ gives an $\mathbb{R}$-linear form

$$\theta \colon K_0(\operatorname{mod} A)_{\mathbb{R}} \to \mathbb{R}$$

  via the Euler form $K_0(\operatorname{proj} A)_{\mathbb{R}} \times K_0(\operatorname{mod} A)_{\mathbb{R}} \to \mathbb{R}$.

By using this duality, the following notions were introduced:

- θ-semistable modules $M \in \operatorname{mod} A$ by [King]
  $\to$ Wall-chamber structures on $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by [BST, Bridgeland].
- Two numerical torsion pairs in $\operatorname{mod} A$ for each θ by [BKT]
  $\to$ TF equivalence on $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by [A].

These two are strongly related to each other.
To study them, silting theory is useful.

TF equiv. classes by presilting complexes

Let $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ be 2-term presilting with U_i: indec.
We set the presilting cone of U by

$$C^+(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i] \subset K_0(\mathrm{proj}\, A)_{\mathbb{R}}.$$

Theorem [Brüstle-Smith-Treffinger, Yurikusa, (A)]

For each $U \in$ 2-psilt A, $C^+(U)$ is a TF equivalence class.

However, presilting cones do not give all TF equivalence classes if A is not τ-tilting finite [Zimmermann-Zvonareva].

Non-rigid regions

We set the non-rigid region of $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by

$$\mathrm{NR} := K_0(\operatorname{proj} A)_{\mathbb{R}} \setminus \bigcup_{U \in \text{2-psilt} A} C^+(U).$$

In these talks, I will explain two approaches to study NR.

(1) Canonical decomp. $\theta = \bigoplus_{i=1}^{m} \theta_i$ in $K_0(\operatorname{proj} A)$ by [Derksen-Fei] give TF equivalence classes $\sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$ if A is E-tame.

- We can obtain construct TF equivalence classes in NR.
- Representation-tame algebras are always E-tame [GLFS].

(2) The non-rigid region NR can be described in terms of 2-term presilting complexes and the purely non-rigid region R_0.

- R_0 is a certain closed subset of $K_0(\operatorname{proj} A)_{\mathbb{R}}$.
- I have determined R_0 in the case A is a special biserial algebra.

Canonical decompositions

We use the presentation space for each $\theta \in K_0(\operatorname{proj} A)$:

$$\operatorname{Hom}(\theta) := \operatorname{Hom}_A(P_1^\theta, P_0^\theta),$$

where $\theta = [P_0^\theta] - [P_1^\theta]$ and $\operatorname{add} P_0^\theta \cap \operatorname{add} P_1^\theta = \{0\}$.
Each $f \in \operatorname{Hom}(\theta)$ defines a 2-term complex

$$P_f := (P_1^\theta \xrightarrow{f} P_0^\theta) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A).$$

[Derksen-Fei] defined direct sums in $K_0(\operatorname{proj} A)$:

$$\bigoplus_{i=1}^m \theta_i :\Longleftrightarrow \begin{bmatrix} \text{For general } f \in \operatorname{Hom}(\sum_{i=1}^m \theta_i), \\ \exists f_i \in \operatorname{Hom}(\theta_i),\ P_f \cong \bigoplus_{i=1}^m P_{f_i} \end{bmatrix}.$$

This is called a canonical decomposition if each θ_i is indecomposable.

Theorem [DF, Plamondon]

Any $\theta \in K_0(\operatorname{proj} A)$ admits a unique canon. decomp. $\bigoplus_{i=1}^m \theta_i$.

Our results

We introduced E-tame algebras in our study:

$$A\text{: E-tame} :\Longleftrightarrow \forall \theta \in K_0(\operatorname{proj} A),\ \theta \oplus \theta.$$

All representation-tame algebras are E-tame [GLFS].

Main theorem of 1st talk [AI]

Assume that A is hereditary or E-tame.
Let $\theta = \bigoplus_{i=1}^m \theta_i$ be a canon. decomp. in $K_0(\operatorname{proj} A)$.
Then, $C^+(\theta) := \sum_{i=1}^m \mathbb{R}_{>0}\theta_i$ is a TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

If $\theta_i \neq \theta_j$ for any $i \neq j$ in above, then $\theta_1, \ldots, \theta_m$ are lin. independent.

Setting

Let A be a fin. dim. algebra over an alg. closed field K.

- proj A: the category of fin. gen. projective A-modules.
- $P_1, P_2, \ldots, P_n$: the non-iso. indec. proj. modules.
- $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$: the homotopy cat. of bounded complexes over proj A.
- mod A: the category of fin. dim. A-modules.
- $S_1, S_2, \ldots, S_n$: the non-iso. simple modules
  (we may assume there exists a surj. $P_i \longrightarrow S_i$).
- $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$: the derived cat. of bounded complexes over mod A.
- $K_0(C)$: the Grothendieck group of C.
- $K_0(C)_{\mathbb{R}} := K_0(C) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.

The Euler form

$K_0(\text{proj}\, A)$ and $K_0(\text{mod}\, A)$ are free abelian groups.

Proposition (see [Happel])

(1) $K_0(\text{proj}\, A) = K_0(\mathsf{K}^{\mathrm{b}}(\text{proj}\, A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$.

(2) $K_0(\text{mod}\, A) = K_0(\mathsf{D}^{\mathrm{b}}(\text{mod}\, A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$.

(3) $\langle [P_i], [S_j] \rangle = \delta_{i,j}$, where

$$\langle \cdot, \cdot \rangle \colon K_0(\text{proj}\, A) \times K_0(\text{mod}\, A) \to \mathbb{Z}$$

is the Euler form.

These are naturally extended to the real Grothendieck groups.
Via the Euler form, each $\theta \in K_0(\text{proj}\, A)_{\mathbb{R}}$ induces the $\mathbb{R}$-linear form

$$\theta := \langle \theta, \cdot \rangle \colon K_0(\text{mod}\, A)_{\mathbb{R}} \to \mathbb{R}.$$

Wall-chamber structures

Definition [King]

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.

(1) $M \in \operatorname{mod} A$: θ-semistable $:\Longleftrightarrow$
$\theta(M) = 0$ and $\theta(N) \geq 0$ for any quotient N of M.

(2) $\mathcal{W}_\theta := \{\text{all } \theta\text{-semistable modules}\} \subset \operatorname{mod} A$.

Definition [Brüstle-Smith-Treffinger, Bridgeland]

(1) For $M \in \operatorname{mod} A \setminus \{0\}$, set $\Theta_M := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta\}$.

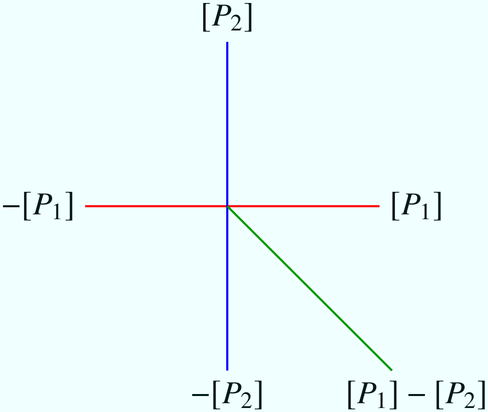
(2) We consider the wall-chamber structure on $K_0(\operatorname{proj} A)_{\mathbb{R}}$
whose walls are Θ_M for all $M \in \operatorname{mod} A \setminus \{0\}$.

Remark

To get the wall-chamber structure,
it suffices to consider indec. modules.

Example of walls

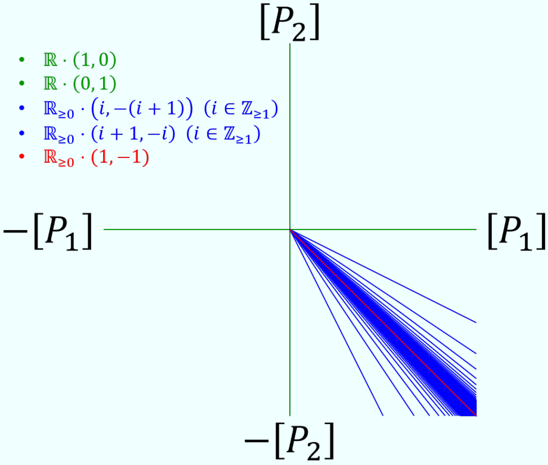
Let $A = K(1 \to 2)$, then the indec. modules are S_2, P_1, S_1.



There are 5 chambers.

Example of walls

Let $A = K(\ 1 \rightrightarrows 2\ )$.



There are infinitely many chambers.

TF equivalence

Definition [Baumann-Kamnitzer-Tingley]

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.
We define numerical torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\operatorname{mod} A$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

Definition

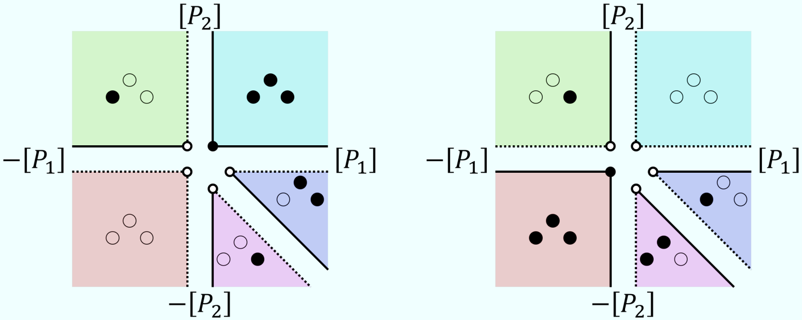
$\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ are TF equivalent $:\Longleftrightarrow$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'}), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'}).$$

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{smallmatrix} & P_1 & \\ S_2 & & S_1 \end{smallmatrix}$ are the indec. A-modules.

Then, $\overline{\mathcal{T}}_\theta$ and $\overline{\mathcal{F}}_\theta$ are given as follows.

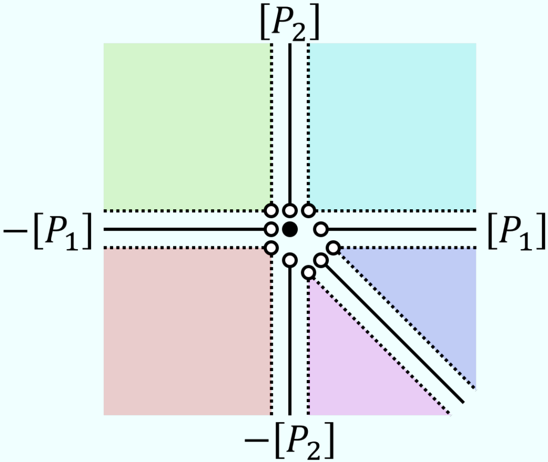


(●: belong, ○: not belong)

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{smallmatrix} & P_1 & \\ S_2 & & S_1 \end{smallmatrix}$ are the indec. A-modules.

There are exactly 11 TF equivalence classes.



Walls and TF equiv. classes

Proposition [A]

Let $\theta \neq \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, then TFAE.

(a) θ and θ' are TF equivalent.

(b) $\mathcal{W}_{\theta''}$ is constant for $\theta'' \in [\theta, \theta']$.

(c) $\nexists S \in \operatorname{brick} A$, $[\theta, \theta'] \cap \Theta_S$ is one point.

Example

If $A = K(\ 1 \rightrightarrows 2\ )$, then the TF equivalence classes are

- $\{0\}$,
- $\mathbb{R}_{>0}(i, -(i+1)), \mathbb{R}_{>0}(i+1, -i)$,
- $\mathbb{R}_{>0}(i, -(i+1)) + \mathbb{R}_{>0}(i+1, -(i+2)), \mathbb{R}_{>0}(i+1, -i) + \mathbb{R}_{>0}(i+2, -(i+1))$,
- $\mathbb{R}_{>0}(1, -1)$

where we consider all $i \in \mathbb{Z}_{\geq 0}$.

Presilting complexes

Definition [Keller-Vossieck]

Let $U = (U^{-1} \to U^0) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ be a 2-term complex.

(1) U: presilting $:\Longleftrightarrow$ $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(U, U[1]) = 0$.

(2) U: silting $:\Longleftrightarrow$ U: presilting, $\operatorname{thick}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)} U = \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

2-psilt $A := \{$basic 2-term presilting complexes$\}/\cong$.
2-silt $A := \{$basic 2-term silting complexes$\}/\cong$.

Proposition [(1) Aihara, (2) Adachi-Iyama-Reiten]

(1) $\forall U \in$ 2-psilt A, $\exists T \in$ 2-silt A s.t.
U is a direct summand of T.

(2) $U \in$ 2-silt A $\Longleftrightarrow$ $U \in$ 2-psilt A, $|U| = n$.

Presilting and func. fin. torsion pairs

For each $U \in$ 2-psilt A, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^{\perp}H^{-1}(\nu U), \mathrm{Sub}\, H^{-1}(\nu U)),$$
$$(\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathrm{Fac}\, H^0(U), H^0(U)^{\perp}).$$

Then, $\mathcal{T}_U \subset \overline{\mathcal{T}}_U$ and $\mathcal{F}_U \subset \overline{\mathcal{F}}_U$.

Theorem [Smalø, Auslander-Smalø, AIR]

Let $U \in$ 2-psilt A.

(1) $(\overline{\mathcal{T}}_U, \mathcal{F}_U)$, $(\mathcal{T}_U, \overline{\mathcal{F}}_U)$ are func. fin. torsion pairs.

(2) All func. fin. torsion(-free) classes are obtained in this way.

Presilting cones

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i: indec.

Proposition [Aihara-Iyama]

$[U_1], \ldots, [U_m] \in K_0(\mathrm{proj}\, A)$ are linearly independent.
If $U \in$ 2-silt A, they are a $\mathbb{Z}$-basis of $K_0(\mathrm{proj}\, A)$.

Definition

We define the presilting cone $C^+(U)$ in $K_0(\mathrm{proj}\, A)_{\mathbb{R}}$ by

$$C^+(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i].$$

Proposition [Demonet-Iyama-Jasso]

If $U \neq U' \in$ 2-psilt A, then $C^+(U) \cap C^+(U') = \emptyset$.

Presilting cones are TF equiv. classes

Theorem (⇒): [Yurikusa, Brüstle-Smith-Treffinger], (⇐): [A]

Let $U \in$ 2-psilt A.
Then, $C^+(U)$ is a TF equiv. class such that

$$\eta \in C^+(U) \iff \overline{\mathcal{T}}_\eta = \overline{\mathcal{T}}_U,\ \overline{\mathcal{F}}_\eta = \overline{\mathcal{F}}_U.$$

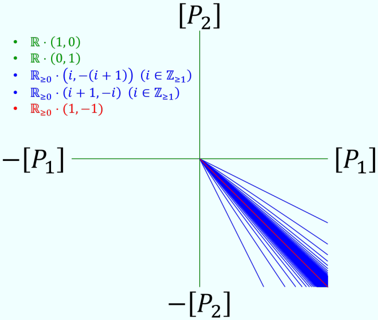
Theorem [A]

The following sets coincide.

- The set of chambers in the wall-chamber structures.
- The set of TF equiv. classes whose interiors are nonempty.
- $\{C^+(T) \mid T \in \text{2-silt}\, A\}$.

Example of presilting and TF equiv. classes

Let $A = K(\ 1 \rightrightarrows 2 \ )$.



The TF equivalence classes in $K_0(\text{proj}\, A)_{\mathbb{R}}$ are

- $C^+(U)$ for all $U \in$ 2-psilt A,
- $\mathbb{R}_{>0}(1, -1)$ (this does not come from 2-psilt A).

Presentation spaces

Definition [Derksen-Fei]

Let $\theta \in K_0(\text{proj}\, A)$.

(1) Take $P_+, P_- \in \text{proj}\, A$ (unique up to iso.) such that $\theta = [P_+] - [P_-]$ and $\text{add}\, P_+ \cap \text{add}\, P_- = \{0\}$.

(2) $\text{Hom}(\theta) := \text{Hom}_A(P_-, P_+)$: the presentation space of θ.

(3) For each $f \in \text{Hom}(\theta)$, set $P_f := (P_- \xrightarrow{f} P_+) \in \mathsf{K}^{\mathrm{b}}(\text{proj}\, A)$ (the terms except -1st and 0th ones vanish).

$\text{Hom}(\theta)$ is an irreducible algebraic variety.

Convention

"Any general $f \in \text{Hom}(\theta)$ satisfies (P)" means
"there exists $X \subset \text{Hom}(\theta)$: nonempty and open (thus dense) such that any $f \in X$ satisfies (P)".

Direct sums in $K_0(\operatorname{proj} A)$

Definition [DF]

We say a direct sum $\bigoplus_{i=1}^m \theta_i$ holds in $K_0(\operatorname{proj} A)$ if

$$\text{for general } f \in \operatorname{Hom}\left(\sum_{i=1}^m \theta_i\right), \exists f_i \in \operatorname{Hom}(\theta_i), P_f \cong \bigoplus_{i=1}^m P_{f_i}.$$

In this case, we also write $\sum_{i=1}^m \theta_i = \bigoplus_{i=1}^m \theta_i$.

This condition can be checked pairwisely.

Proposition [DF]

$\bigoplus_{i=1}^m \theta_i \iff \forall i \neq j,\ \exists (f, g) \in \operatorname{Hom}(\theta_i) \times \operatorname{Hom}(\theta_j)$,

$$\operatorname{Hom}(P_f, P_g[1]) = 0, \quad \operatorname{Hom}(P_g, P_f[1]) = 0.$$

Canonical decompositions

Definition

θ: indecomposable in $K_0(\operatorname{proj} A)$ $:\Longleftrightarrow$
for any general $f \in \operatorname{Hom}(\theta)$, $P_f \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is indec.

Theorem [DF, Plamondon]

Any $\theta \in K_0(\operatorname{proj} A)$ admits a decomposition unique up to reordering

$$\theta = \bigoplus_{i=1}^{m} \theta_i \quad (\theta_i\text{: indecomposable}).$$

We call it the canonical decomposition of θ.

Direct sums and TF equiv. classes

Theorem 1 [AI] (with Demonet)

Let $\bigoplus_{i=1}^{m} \theta_i$ in $K_0(\operatorname{proj} A)$. Then,

$$\eta \in \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i \Longrightarrow \overline{\mathcal{T}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{T}}_{\theta_i},\ \overline{\mathcal{F}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{F}}_{\theta_i}.$$

Thus, for any i, $\mathcal{T}_{\theta_i} \subset \mathcal{T}_\eta \subset \overline{\mathcal{T}}_\eta \subset \overline{\mathcal{T}}_{\theta_i}$, $\mathcal{F}_{\theta_i} \subset \mathcal{F}_\eta \subset \overline{\mathcal{F}}_\eta \subset \overline{\mathcal{F}}_{\theta_i}$.

We can recover the following sign-coherence.

Proposition [Plamondon]

Let $\theta \oplus \theta'$ in $K_0(\operatorname{proj} A)$, $\theta = \sum_{i=1}^{n} a_i[P_i]$ and $\theta' = \sum_{i=1}^{n} a_i'[P_i]$.
Then, $a_i a_i' \geq 0$ for all i.

$\because$ If $a_i > 0$ and $a_i' < 0$, then $S_i \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \subset \mathcal{T}_{\theta+\theta'} \cap \mathcal{F}_{\theta+\theta'} = \{0\}$.

Canon. decomp. and TF equiv. classes

By Theorem 1, if $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$, then

$$C^+(\theta) := \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$$

is contained in some TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.
Is $C^+(\theta)$ really a TF equiv. class?

Theorem 2 [AI]

Assume that

- A is a hereditary algebra; or
- A is E-tame, i.e. $\theta \oplus \theta$ holds for any $\theta \in K_0(\operatorname{proj} A)$.

If $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$,
then $C^+(\theta)$ is a TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

E-tame algebras

Though it is not easy to check the E-tameness, we have the following.

Theorem [Geiss-Labardini-Fragoso-Schröer, (Plamondon-Yurikusa)]

Let A be representation-finite or tame.
Then, A is E-tame.

Why did we assume E-tameness?
Because our proof of Theorem 2 uses the following result.

Theorem [Fei]

If $\theta \in K_0(\text{proj } A)$ and $M \in \text{mod } A$, then TFAE.

(a) $M \in \overline{\mathcal{F}}_\theta$.

(b) $\exists l \in \mathbb{Z}_{\geq 1}, \exists f \in \text{Hom}(l\theta), \text{Hom}_A(\text{Coker } f, M) = 0$.

Moreover, we may let $l = 1$ if $\theta \oplus \theta$.

Example of Theorem 2

Let Q be an extended Dynkin quiver, and $A := KQ$.

- Consider an indec. module $M \in \operatorname{mod} A$ in a regular homog. tube.
- Take the min. proj. resol. $P_1^M \to P_0^M \to M \to 0$, and set $\eta := [P_0^M] - [P_1^M]$.
- $E := \{U \in \text{2-psilt}\, A \mid [U] \oplus \eta\}$.
  - $[U] \oplus \eta \iff [U] \in \Theta_M \iff H^0(U), H^{-1}(\nu U)$ are regular.

Proposition

Under the setting above, the TF equiv. classes in $K_0(\operatorname{proj} A)_{\mathbb{R}}$ are

- $C^+(U)$ for all $U \in \text{2-psilt}\, A$ and
- $C^+([U] \oplus \eta) = C^+(U) + \mathbb{R}_{>0}\eta$ for all $U \in E$.

In particular, all TF equiv. classes come from canon. decomp.

Remark on Theorem 2

In general, even if A is E-tame,
Theorem 2 does not necessarily give all TF equiv. classes.

- We cannot obtain any TF equiv. class $X \subset K_0(\text{proj}\, A)_{\mathbb{R}}$ such that $X \cap K_0(\text{proj}\, A) = \emptyset$ from Theorem 2.
- The following gentle algebra admits a TF equiv. class $\mathbb{R}_{>0}(1-t, -1+2t, -t)$ for each $t \in [0,1] \setminus \mathbb{Q}$:

$$A = K\Big(1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3 \Big)/\langle \alpha\delta, \beta\gamma \rangle.$$

Non-rigid regions

Recall that the non-rigid region of $K_0(\text{proj}\, A)_{\mathbb{R}}$ is

$$\text{NR} = K_0(\text{proj}\, A)_{\mathbb{R}} \setminus \bigcup_{U \in \text{2-psilt}\, A} C^+(U).$$

My 2nd talk deals with a nice decomposition of NR.

Strategy

We will define $R_U \supset C^+(U)$ for each $U \in$ 2-psilt A such that

$$K_0(\text{proj}\, A)_{\mathbb{R}} = \coprod_{U \in \text{2-psilt}\, A} R_U,$$

$$\text{NR} = \coprod_{U \in \text{2-psilt}\, A} (R_U \setminus C^+(U)).$$

Nice subsets including presilting cones

For $U \in$ 2-psilt A, we define $N_U, R_U \supset C^+(U)$ by

$$N_U := \{\theta \in K_0(\mathrm{proj}\, A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\},$$
$$R_U := N_U \setminus \bigcup_{V \in \text{2-psilt}_U A \setminus \{U\}} N_V,$$

where $\text{2-psilt}_U A := \{V \in \text{2-psilt}\, A \mid U \text{ is a direct summand of } V\}$.
We call R_0 the purely non-rigid region.

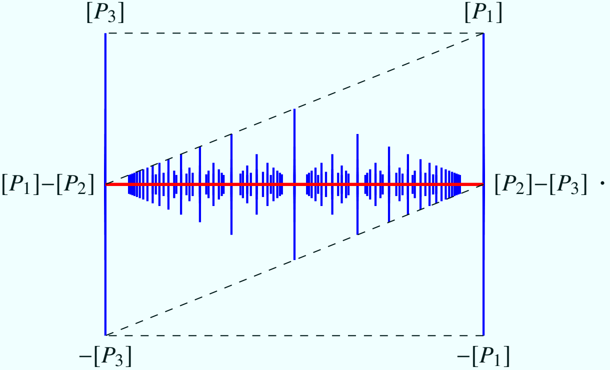
Main theorem of 2nd talk [AI]

We have

$$\mathrm{NR} = \coprod_{U \in \text{2-psilt}\, A} (R_U \setminus C^+(U))$$
$$= \coprod_{U \in \text{2-psilt}\, A} (C^+(U) + ((\overline{N_U} \cap R_0) \setminus \{0\})).$$

Example of non-rigid regions

For $A = K(\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3\ )/\langle \alpha\delta, \beta\gamma \rangle$, NR is described as



The red line is R_0.
Each blue segment is $R_U \setminus C^+(U)$ for some indec. $U \in$ 2-psilt A (the upper or the lower endpoint is $C^+(U)$).

Open neighborhoods of presilting cones

Definition

For any $U \in 2\text{-psilt}\, A$, we set

$$N_U := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\}.$$

This is related to τ-tilting reduction by [Jasso].

Lemma

Let $U, V \in 2\text{-psilt}\, A$.

(1) N_U is a union of TF equiv. classes.

(2) N_U is an open neighborhood of $C^+(U)$.

(3) $\overline{N_U} = \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \overline{\mathcal{T}}_\theta,\ \mathcal{F}_U \subset \overline{\mathcal{F}}_\theta\}$.

(4) $U \oplus V$: 2-term presilting $\iff N_U \cap N_V \neq \emptyset \iff [V] \in \overline{N_U}$.
In this case, $N_U \cap N_V = N_{U \oplus V}$.

(5) $U \in \operatorname{add} V \iff N_U \supset N_V$.

Purely non-rigid regions

$\text{2-psilt}_U\, A := \{V \in \text{2-psilt}\, A \mid U \text{ is a direct summand of } V\}$.

Definition

For $U \in \text{2-psilt}\, A$, we set

$$R_U := N_U \setminus \bigcup_{V \in \text{2-psilt}_U A \setminus \{U\}} N_V.$$

In particular, we call R_0 the purely non-rigid region:

$$R_0 = K_0(\mathrm{proj}\, A)_{\mathbb{R}} \setminus \bigcup_{V \in \text{2-psilt}\, A \setminus \{0\}} N_V.$$

- R_0 is a closed set, and $0 \in R_0$.
- $R_0 = \{0\} \iff \mathrm{NR} = \emptyset \iff A$ is τ-tilting finite.
- $(R_U)_{U \in \text{2-psilt}\, A}$ is a stratification of $K_0(\mathrm{proj}\, A)_{\mathbb{R}}$.

Decompositions of non-rigid regions

Theorem 3 [AI]

(1) Let $U \in$ 2-psilt A and $\theta \in R_U$.
Then, there uniquely exist $\theta_1 \in C^+(U)$ and $\theta_2 \in \overline{N_U} \cap R_0$ such that $\theta = \theta_1 + \theta_2$.

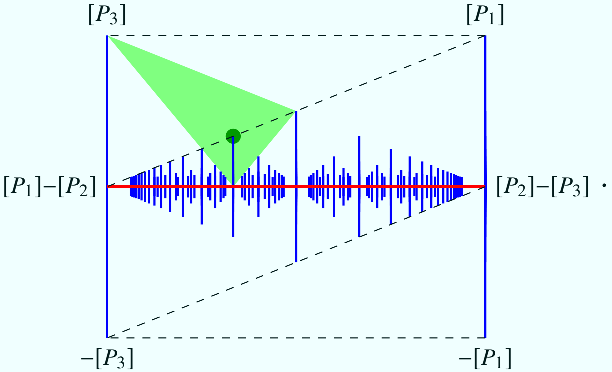
(2) We have

$$\begin{aligned} \mathrm{NR} &= \coprod_{U \in \text{2-psilt } A} (R_U \setminus C^+(U)) \\ &= \coprod_{U \in \text{2-psilt } A} (C^+(U) + ((\overline{N_U} \cap R_0) \setminus \{0\})). \end{aligned}$$

Thus, the non-rigid region is determined
by the 2-term presilting complexes and the purely non-rigid region.

Example of Theorem 3

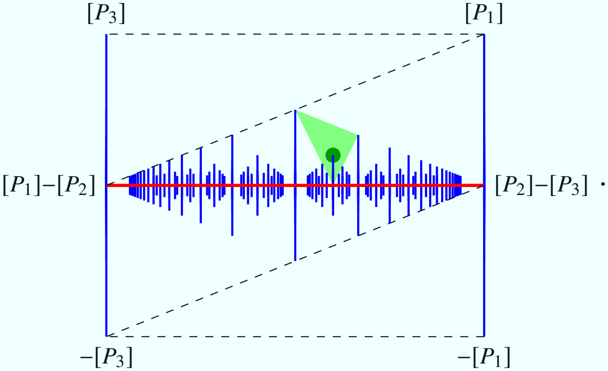
Let $A = K\left(1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \overset{\gamma}{\underset{\delta}{\rightrightarrows}} 3 \right)/\langle \alpha\delta, \beta\gamma \rangle$.



The red line is R_0, and the blue is the rest non-rigid region.
For $U \in$ 2-psilt A with $[U] = (3, -2, 0)$,
N_U is the green triangle, and $C^+(U)$ is the point in it.

Example of Theorem 3

Let $A = K(\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3\ )/\langle \alpha\delta, \beta\gamma \rangle$.



The red line is R_0, and the blue is the rest non-rigid region.
For $U \in$ 2-psilt A with $[U] = (3, 0, -2)$,
N_U is the green triangle, and $C^+(U)$ is the point in it.

Example of Theorem 3

Let $A = K\Big(1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \overset{\gamma}{\underset{\delta}{\rightrightarrows}} 3 \Big)/\langle \alpha\delta, \beta\gamma \rangle$.

Proposition

(1) $R_0 = \mathbb{R}_{\geq 0}(1,-1,0) + \mathbb{R}_{\geq 0}(0,1,-1)$.

(2) $R_{P_3} = \mathbb{R}_{>0}(0,0,1) + \mathbb{R}_{\geq 0}(1,-1,0)$, $R_{P_3[1]} = \mathbb{R}_{>0}(0,0,-1) + \mathbb{R}_{\geq 0}(1,-1,0)$,
$R_{P_1} = \mathbb{R}_{>0}(1,0,0) + \mathbb{R}_{\geq 0}(0,1,-1)$, $R_{P_1[1]} = \mathbb{R}_{>0}(-1,0,0) + \mathbb{R}_{\geq 0}(0,1,-1)$.

(3) For any $k < l \in \mathbb{Z}_{\geq 1}$ with $\gcd(k,l) = 1$,
there exist $U_+, U_- \in$ 2-psilt A such that

$$\begin{aligned} [U_\pm] &= (l-k\pm 1, -l+2k\mp 1, -k\pm 1), \\ \overline{N_{U_\pm}} \cap R_0 &= \mathbb{R}_{\geq 0}(l-k, -l+2k, -k), \\ R_{U_\pm} &= \mathbb{R}_{>0}(l-k\pm 1, -l+2k\mp 1, -k\pm 1) + \mathbb{R}_{\geq 0}(l-k, -l+2k, -k). \end{aligned}$$

(4) For the other $U \in$ 2-psilt A, $R_U = C^+(U)$.

Relationship with canon. decomp.

Definition

Let $\theta \in K_0(\mathrm{proj}\, A)$.

- We say θ is rigid if $\exists U \in$ 2-psilt A, $\theta \in C^+(U)$.
- We set θ_{ri} as the max. rigid direct summand of θ.

For any $\theta \in K_0(\mathrm{proj}\, A)$ and $U \in$ 2-psilt A,

- $\theta \in N_U \iff \exists l \in \mathbb{Z}_{\geq 1}$, $[U]$ is a direct summand of $l\theta$.
- $\theta \in \overline{N_U} \iff \exists l \in \mathbb{Z}_{\geq 1}$, $[U] \oplus l\theta$.

Corollary

Let $\theta \in K_0(\mathrm{proj}\, A)$.
Then, $\exists l \in \mathbb{Z}_{\geq 1}, \forall m \in \mathbb{Z}_{\geq 1}, (ml\theta)_{\mathrm{ri}} = m \cdot (l\theta)_{\mathrm{ri}}$.
Moreover, we can let $l = 1$ if A is E-tame.

τ-tilting reduction

Let $U \in$ 2-psilt A, and take its Bongartz completion $T \in$ 2-silt A.
Set $B = B_U := \mathrm{End}_A(H^0(T))/[H^0(U)]$, then $|B| + |U| = |A|$.

Theorem [Jasso]

There exists a bijection red: 2-psilt$_U$ $A \longrightarrow$ 2-psilt B.

Proposition

There exists an $\mathbb{R}$-linear surj. $\pi\colon K_0(\mathrm{proj}\, A)_{\mathbb{R}} \longrightarrow K_0(\mathrm{proj}\, B)_{\mathbb{R}}$ such that

$$\pi(C^+(V)) = C^+(\mathrm{red}(V)),\ \pi(N_V) = N_{\mathrm{red}(V)},\ \pi(R_V) = R_{\mathrm{red}(V)}$$

in $K_0(\mathrm{proj}\, B)_{\mathbb{R}}$ for any $V \in$ 2-psilt$_U$ A.

In particular, $\pi(R_U) = R_0(B)$, so

$$R_U = C^+(U) \iff B \text{ is } \tau\text{-tilting fin.}$$

Special biserial algebras

- $\widehat{KQ}$: The complete path algebra of a fin. quiver $Q = (Q_0, Q_1)$.
- $I \subset \langle Q_1 \rangle^2 \subset \widehat{KQ}$: a two-sided ideal of $\widehat{KQ}$.
- The arguments before are valid for $A = \widehat{KQ}/I$ [Yuta Kimura, van Garderen].

Definition

$A = \widehat{KQ}/I$ is called a complete special biserial algebra if

(a) I is generated by a finite set of paths and $p - q$ (p, q: paths).

(b) For each $i \in Q_0$, there exist at most two arrows starting at i.

(c) For each $i \in Q_0$, there exist at most two arrows ending at i.

(d) For each $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ s.t. $\alpha\beta \notin I$.

(e) For each $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ s.t. $\beta\alpha \notin I$.

We want to determine R_0 for complete special biserial algebras.

Gentle algebras

Definition

$A = \widehat{KQ}/I$ is called a complete gentle algebra if

(a) $A = \widehat{KQ}/I$ is a complete special biserial algebra.

(b) I is generated by paths of length 2.

(c) For each $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $\alpha\beta$ is a path in Q and $\alpha\beta \in I$.

(d) For each $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $\beta\alpha$ is a path in Q and $\beta\alpha \in I$.

If $A = \widehat{KQ}/I$ is a complete special biserial algebra, we can choose $\widetilde{I} \subset I$ such that $\widetilde{A} = \widehat{KQ}/\widetilde{I}$ is a complete gentle algebra.
Then, A is a quotient algebra of $\widetilde{A}$, so $R_0(A) \subset R_0(\widetilde{A})$.

Maximal nonzero paths

Definition

Let $A = \widehat{KQ}/I$ be a complete gentle algebra.

- $\mathrm{MP}(A) := \{\text{paths } p \notin I \text{ of length} \geq 1 \text{ s.t. } \forall \alpha \in Q_1, \alpha p, p\alpha \in I\}$.
- $\overline{\mathrm{MP}}(A) := \mathrm{MP}(A) \cup \{e_i \mid i \in Q_0 \text{ satisfying } (*)\}$;
  $(*)$: at most one arrow starting at i, and at most one arrow ending at i.
- $\mathrm{Cyc}(A) := \{\text{minimal cycles } c \text{ s.t. } \forall m \geq 1,\ c^m \notin I\}$.

For any path $p \notin I$ in Q, a string module $M(p) \in \operatorname{mod} A$ is defined.

Theorem 4 [A]

Let $A = \widehat{KQ}/I$ be a complete gentle algebra.
Then, $R_0 = \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid \text{(a), (b)}\}$.

(a) $\forall p \in \overline{\mathrm{MP}}(A)$, $M(p) \in \mathcal{W}_\theta$.

(b) $\forall c \in \mathrm{Cyc}(A)$, $\theta(M(c)/\operatorname{soc} M(c)) = 0$.

Example of Theorem 4

Let $A = K(\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[r, shift left, "\gamma"] \arrow[r, shift right, "\delta"'] & 3 \end{tikzcd})/\langle \alpha\delta, \beta\gamma \rangle$.

In this case,

$$\overline{\mathsf{MP}}(A) = \{\alpha\gamma, \beta\delta\},\ \mathsf{Cyc}(A) = \emptyset.$$

Thus, for $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$,

$$\begin{aligned} \theta \in R_0 &\iff M(\alpha\gamma), M(\beta\delta) \in \mathcal{W}_\theta \\ &\iff \theta \in \mathbb{R}_{\geq 0}(1, -1, 0) + \mathbb{R}_{\geq 0}(0, 1, -1). \end{aligned}$$

Example of Theorem 4

Let $A = K\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3 \right)/\langle \alpha\gamma, \delta\beta \rangle$.

In this case,

$$\overline{\mathsf{MP}}(A) = \{e_1, e_3\},\ \mathsf{Cyc}(A) = \{\alpha\beta, \beta\alpha, \gamma\delta, \delta\gamma\}.$$

We can use a complete representative set of $\mathsf{Cyc}(A)/\{\text{cyc. perm.}\}$ instead of $\mathsf{Cyc}(A)$ in Theorem 4.
Thus, for $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$,

$$\begin{aligned} \theta \in R_0 &\iff M(e_1), M(e_3) \in \mathcal{W}_\theta,\ \theta(M(\alpha)) = \theta(M(\gamma)) = 0 \\ &\iff \theta = 0. \end{aligned}$$

Therefore, $R_0 = \{0\}$, and $\#\,\text{2-silt}\,A < \infty$ (A is "τ-tilting finite").

- For any complete special biserial algebra A, 2-silt $A \to$ 2-silt$(A/\langle \mathsf{Cyc}(A) \rangle)$ is a bij. [Yuta Kimura].

Example of Theorem 4

Let $A = K\left(\begin{tikzcd} 1 \arrow[loop left, "\lambda"] \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\delta"] \arrow[loop right, "\mu"] \end{tikzcd}\right)/\langle \alpha\gamma, \delta\beta, \lambda^2, \mu^2 \rangle$.

In this case,

$$\overline{\mathsf{MP}}(A) = \emptyset,\ \mathsf{Cyc}(A) = \{\alpha\beta\lambda, \beta\lambda\alpha, \lambda\alpha\beta, \delta\gamma\mu, \gamma\mu\delta, \mu\delta\gamma\}.$$

Thus, for $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$,

$$\begin{aligned} \theta \in R_0 &\iff \theta(M(\alpha\beta)) = \theta(M(\delta\gamma)) = 0 \\ &\iff \theta = \mathbb{R}(1, -2, 1). \end{aligned}$$

Main result for special biserial algebras

Let $A = \widehat{KQ}/I$ be a complete special biserial algebra.
Fix $\widetilde{I} \subset I$: an ideal of $\widehat{KQ}$ such that $\widetilde{A} = \widehat{KQ}/\widetilde{I}$ is complete gentle.
Define $\widetilde{\mathcal{W}_\theta} \subset \operatorname{mod} \widetilde{A}$ by

$$\widetilde{\mathcal{W}_\theta} := \operatorname{Filt}_{\widetilde{A}} \mathcal{W}_\theta \quad (\mathcal{W}_\theta \subset \operatorname{mod} A).$$

For any path $\widetilde{p}$ admitted in $\widetilde{A}$, $M(\widetilde{p}) \in \widetilde{\mathcal{W}_\theta}$ if and only if
$\exists q_1, \ldots, q_m$: paths admitted in A, $\exists \alpha_1, \ldots, \alpha_{m-1} \in Q_1$,

$$\widetilde{p} = q_1 \alpha_1 \cdots q_{m-1} \alpha_{m-1} q_m, \quad \forall i, M(q_i) \in \mathcal{W}_\theta.$$

Theorem 5 [A]

In above, we have $R_0 = \{\theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid \text{(a), (b)}\}$.

(a) $\forall \widetilde{p} \in \overline{\mathsf{MP}}(\widetilde{A})$, $M(\widetilde{p}) \in \widetilde{\mathcal{W}_\theta}$.

(b) $\forall \widetilde{c} \in \operatorname{Cyc}(\widetilde{A})$, $\exists \widetilde{d}$: a cyc. perm. of $\widetilde{c}$ s.t. $M(\widetilde{d})/\operatorname{soc} M(\widetilde{d}) \in \widetilde{\mathcal{W}_\theta}$.

Example of Theorem 5

Let $A = K\left(1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \overset{\gamma}{\underset{\delta}{\rightrightarrows}} 3 \right)/\langle \alpha\delta, \beta\gamma, \alpha\gamma, \beta\delta \rangle$.

Take the gentle algebra $\widetilde{A} = K\left(1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \overset{\gamma}{\underset{\delta}{\rightrightarrows}} 3 \right)/\langle \alpha\delta, \beta\gamma \rangle$.

In this case,

$$\overline{\mathsf{MP}}(\widetilde{A}) = \{\alpha\gamma, \beta\delta\}, \ \mathsf{Cyc}(\widetilde{A}) = \emptyset.$$

Thus, for $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$,

$$\begin{aligned} \theta \in R_0 &\iff M(\alpha\gamma), M(\beta\delta) \in \widetilde{\mathcal{W}}_\theta \\ &\iff \begin{bmatrix} M(\alpha), M(e_3) \in \mathcal{W}_\theta \text{ or } M(e_1), M(\gamma) \in \mathcal{W}_\theta; \\ M(\beta), M(e_3) \in \mathcal{W}_\theta \text{ or } M(e_1), M(\delta) \in \mathcal{W}_\theta \end{bmatrix} \\ &\iff \theta \in \mathbb{R}_{\geq 0}(1, -1, 0) \cup \mathbb{R}_{\geq 0}(0, 1, -1). \end{aligned}$$

In this case, R_0 is not convex.

R_0 and the sum of the simple modules

Set $h := \sum_{i=1}^{n} [S_i] \in K_0(\mathrm{mod}\, A)$.
If A is a complete gentle algebra, then we can check

$$2h \in \sum_{p \in \overline{\mathsf{MP}}(A)} \mathbb{Z}[M(p)] + \sum_{c \in \mathsf{Cyc}(A)} \mathbb{Z}[M(c)/\mathrm{soc}\, M(c)].$$

Corollary

Let A be a complete special biserial algebra.
Then, R_0 is contained in the hyperplane $\mathrm{Ker}\langle \cdot, h \rangle \subset K_0(\mathrm{proj}\, A)_{\mathbb{R}}$.

Remark

If A is complete gentle, then R_0 is a rational polyhedral cone.
If A is complete special biserial,
then R_0 is a union of finitely many rational polyhedral cones.

Connection with τ-tilting reduction

Let $A = \widehat{KQ}/I$ be a (fin. dim.) special biserial algebra.
Fix $U \in$ 2-psilt A, and consider the algebra $B = B_U$.
Then, $\mathcal{W}_U := \overline{\mathcal{T}}_U \cap \overline{\mathcal{F}}_U$ is equiv. to mod B_U [Jasso].

Proposition

B_U is a (fin. dim.) special biserial algebra.

Set $h_U := \sum_{X \in \mathrm{sim}\, \mathcal{W}_U} [X] \in K_0(\mathrm{mod}\, A)$.

Corollary [A]

$R_U \cap \mathrm{NR}$ is contained in $\mathrm{Ker}\langle \cdot, h_U \rangle \subset K_0(\mathrm{proj}\, A)_{\mathbb{R}}$.

Since 2-psilt A is at most a countable set,
NR is contained in a union of countably many hyperplanes of codim. 1.
Thus, the interior of NR is empty, i.e. A is g-tame.

Application to Brauer graph algebras

Let A be the Brauer graph algebra of $G = (V, E, m)$.
The simple A-modules are S_e for all $e \in E$.
For each $v \in V$, take the cyclic order $e_1, \dots, e_l \in E$ around v, and set $x_v := \sum_{i=1}^{l} [S_{e_i}] \in K_0(\mathrm{mod}\, A)$.

Corollary [A]

In above, $R_0 = \bigcap_{v \in V} \mathrm{Ker}\langle \cdot, x_v \rangle$.

Thus, if $R_0 = \{0\}$, then $\#V \leq \#E$, so G contains at most one cycle.
Any vertex with only one half-edge does not matter whether $R_0 = \{0\}$.

- If G is an odd cycle, then $R_0 = \{0\}$.
- If G is an even cycle, then $R_0 = \mathbb{R}(1, -1, 1, -1, \dots, 1, -1)$.

Recovered Theorem [Adachi-Aihara-Chan]

A is τ-tilting finite if and only if
G contains at most one odd cycle and no even cycle.

Thank you for your attention.

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