Modularity of Nahm sums and periodicity phenomena in cluster algebras
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Outline

Rogers-Ramanujan identities

- The Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty},\quad \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty},$$

  where $(x;q)_n := (1-x)(1-qx)\cdots(1-q^{n-1}x)$, $(x;q)_\infty := \prod_{i=0}^{\infty}(1-q^i x)$, and $(x_1,\dots,x_n;q)_\infty := (x_1;q)_\infty\cdots(x_n;q)_\infty$.
- We have the vector-valued function on the upper-half plane

$$g(\tau)=\left(q^{-1/60}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n},\quad q^{11/60}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}\right)^{\mathsf{T}}$$

  with $q=e^{2\pi i\tau}$
- Modular transformations:

$$g(\tau+1)=\begin{pmatrix}\zeta_{60}^{-1} & 0\\ 0 & \zeta_{60}^{11}\end{pmatrix}g(\tau),\quad g(-\frac{1}{\tau})=\frac{2}{\sqrt{5}}\begin{pmatrix}\sin\frac{2\pi}{5} & \sin\frac{\pi}{5}\\ \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5}\end{pmatrix}g(z).$$

  where $\zeta_N=e^{2\pi i/N}$.

In representation theory

- The q-series $\sum_n \frac{q^{n^2}}{(q;q)_n}$ and $\sum_n \frac{q^{n^2+n}}{(q;q)_n}$ are characters of the irreducible modules for the minimal model $M(5,2)$.
- The Grothendieck ring of the module category is described by the Verlinde formula:

$$x_i x_j = \sum_k N_{ij}^k x_k, \quad N_{ij}^k = \sum_m \frac{S_{im} S_{jm} \bar{S}_{mk}}{S_{0m}}$$

- For $S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix}$, we have

$$x^2 = 1 + x$$

where $1 = x_2$ and $x = x_1$ with $x_0 = x_2$.

Nahm sum

- N: natural number
- $A \in \mathbb{Q}^{N \times N}$, $b \in \mathbb{Q}^N$, and $c \in \mathbb{Q}$.
- Suppose that A is symmetric positive definite.
- $Q(n) := \frac{1}{2} n^\intercal A n + n^\intercal b + c$

Nahm sum

$$f_{A,b,c}(q) := \sum_{n \in \mathbb{N}^N} \frac{q^{Q(n)}}{(q;q)_{n_1} \cdots (q;q)_{n_N}}$$

- Nahm sums appear in conformal field theories, quantum invariants of knots, partition identities, cluster algebras, etc.
- They are q-hypergeometric series.
- For very special (A, b, c), they can also be modular functions.

Example:

- $(A, b, c) = (2, 0, -\frac{1}{60})$ and $(2, 1, \frac{11}{60})$ (sum sides of Rogers-Ramanujan identities)

Nahm sum for symmetrizable matrices

- N: natural number
- $A \in \mathbb{Q}^{N \times N}$, $b \in \mathbb{Q}^N$, $c \in \mathbb{Q}$, and $d \in \mathbb{Z}_{>0}^N$.
- Suppose that AD is symmetric positive definite, where $D = \mathrm{diag}(d)$.
- $Q(n) := \frac{1}{2} n^\intercal ADn + n^\intercal b + c$
- Nahm sum:

Nahm sum

$$f_{A,b,c,d}(q) := \sum_{n \in \mathbb{N}^N} \frac{q^{Q(n)}}{(q^{d_1}; q^{d_1})_{n_1} \cdots (q^{d_N}; q^{d_N})_{n_N}}$$

Mod 9 Rogers-Ramanujan type identities

- Kanade-Russell conjecture [2015] (in the form given by Kurşungöz [2019]):

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_1n_2+3n_2^2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \stackrel{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty},$$

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_1n_2+3n_2^2+n_1+3n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \stackrel{?}{=} \frac{1}{(q^2,q^3,q^6,q^7;q^9)_\infty},$$

$$\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+3n_1n_2+3n_2^2+2n_1+3n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} \stackrel{?}{=} \frac{1}{(q^3,q^4,q^5,q^6;q^9)_\infty}.$$

- The LHS are the Nahm sums for $A = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right) = (\text{Cartan matrix of type } G_2)^{-1}$ with the symmetrizer $d = (1,3)$.
- The linear terms are given by $b = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$.

Mod 9 Rogers-Ramanujan type identities

- Define

$$g(\tau) := \begin{pmatrix} f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-1/18,d} & (q) \\ f_{A,\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right),5/18,d} & (q) \\ f_{A,\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right),11/18,d} & (q) \end{pmatrix}$$

- Assuming Kanade-Russell conjecture, $g(\tau)$ satisfies the following modular transformation formula:

$$g(\tau+1) = \begin{pmatrix} \zeta_{18}^{-1} & 0 & 0 \\ 0 & \zeta_{18}^{5} & 0 \\ 0 & 0 & \zeta_{18}^{11} \end{pmatrix} g(\tau), \quad g(-\frac{1}{\tau}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_2 & -\alpha_4 & -\alpha_1 \\ \alpha_4 & -\alpha_1 & \alpha_2 \end{pmatrix} g(\frac{\tau}{3})$$

where $\alpha_k = \dfrac{1}{2\sqrt{3}\sin\frac{k\pi}{9}}$.

Numerical computations

- The following procedure detects, for a given q-series $f(q)$, whether $q^c f(q)$ is likely modular for some c.
  1. Compute $\phi(N) = N \log f(e^{-1/N})$ for four successive values (say $N = 20, 21, 22, 23$).
  2. Take the third difference of the values computed in (1). If this value is extremely small, then the Nahm sum $f(q)$ is likely modular.
- This is justified by the following asymptotic formula for a modular function:

$$q^c f(q) \sim \beta e^{-\alpha/\tau} + O(\varepsilon^N) \quad \text{for any } N > 0.$$

- We can search for candidates for modular Nahm sums.
- This procedure is efficient and works fine with `float64`.
- In addition, higher precision calculations can predict modular transformation formulas.
  - What is $0.84402962874598535680159510037027$?
  - Integer relation algorithm: This may be $\dfrac{1}{2\sqrt{3}\sin\frac{\pi}{9}}$.

Bloch groups

- F: field
- $Z(F) \coloneqq$ the (additive) free abelian group on F.
- We denote by $[X] \in Z(F)$ the element corresponding to $X \in F$.
- $A(F) \coloneqq \ker d$, where

$$d : Z(F) \to \wedge^2 F^\times, \quad [X] \mapsto X \wedge (1 - X)$$

- The Bloch group of F is the quotient

$$B(F) = A(F)/C(F)$$

where $C(F) \subseteq A(F)$ is the subgroup generated by the pentagon relation:

$$\sum_{i \in \mathbb{Z}/5\mathbb{Z}} [x_i] \text{ satisfying } 1 - x_i = x_{i-1}x_{i+1}.$$

If a sequence $(x_i)_{i\in\mathbb{Z}}$ satisfies $1 - x_i = x_{i-1}x_{i+1}$ for any i, then we have $x_i = x_{i+5}$ for any i.

- $B(F)$ is isomorphic to a certain quotient of the K-group $K_3(F)$ [Suslin].

Bloch groups

Example:

- The fixed point of $1 - x_i = x_{i-1}x_{i+1}$ is obtained by solving the equation $1 - x = x^2$.
- Its solution is $\xi = \frac{-1 \pm \sqrt{5}}{2} \in F$, where $F = \mathbb{Q}(\sqrt{5})$.
- $[\xi] \in B(F)$ since $d[\xi] = \xi \wedge (1 - \xi) = \xi \wedge \xi^2 = 2 \cdot \xi \wedge \xi = 0$.
- $5[\xi] = 0$ in $B(F)$ by the pentagon relation.

More generally,

- Let $A \in \mathbb{Q}^{N \times N}$ be a symmetrizable matrix with symmetrizer d.
- Nahm's equation:

$$1 - x_i = \prod_j x_j^{A_{ij}}$$

- $\xi = \sum_i d_i^{-1}[x_i] \in B(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ since

$$d[\xi] = \sum_i d_i^{-1} x_i \wedge (1 - x_i) = \sum_i d_i^{-1} x_i \wedge \prod_j x_j^{A_{ij}} = \sum_{i,j} d_i^{-1} A_{ij} \cdot x_i \wedge x_j = 0$$

Nahm sums and Bloch groups

- Given (A, b, c, d), we have
  - the Nahm sum $f_{A,b,c,d}(q)$,
  - $\xi_{A,d}$ in the Bloch group for each solution of Nahm's equation.

Nahm's conjecture

Given (A, d), the Nahm sum $f_{A,b,c,d}(q)$ is modular for some b, c iff $\xi_{A,d}$ is a torsion element in the Bloch group.

- This conjecture itself is known to be false [Zagier, Vlasenko-Zwegers].

Theorem [Calegari-Garoufalidis-Zagier 2023]

If the Nahm sum $f_{A,b,c,d}(q)$ is modular for some b, c, then $\xi_{A,d}$ for distingushed solution of Nahm's equation is a torsion element in the Bloch group.

- The Nahm's equation $1 - x_i = \prod_j x_j^{A_{ij}}$ has a unique solution such that $0 < x_i < 1$ for any i.
- Remark: Symmetrizable case are treated in [M. 2305.02267].

Ingredient of the proof in CGZ

1. Asymptotic expansion of Nahm sums at roots of unity
   - Nahm's equations appear in the saddle point method.
   - The constant term is described by the quantum cyclic dilogarithm function.
2. Concrete construction of a Chern class map on $K_3(F)$
   - F: number field,
   - For any integer m, set $F_m := F(\zeta)$ where ζ is a primitive mth root of unity.
   - They defined an injective map

     $$R_\zeta : B(F)/mB(F) \to F_m^\times / F_m^{\times m}$$

     for sufficiently many m.
   - Roughly,

     $$R_\zeta([x]) := D_\zeta(x^{1/m})$$

     where $D(\theta)$ is the quantum cyclic dilogarithm function:

     $$D_\zeta(\theta) := \prod_{k=1}^{m-1} (1 - \zeta^k \theta)^k.$$

Y-systems

- Y-system associated with a pair (r, n):

$$Y_i(u)Y_i(u-r_i) = \prod_{j=1}^{N} \prod_{p \in \mathbb{N}} Y_j(u-p)^{[n_{ij;p}]_+} \big(1 + Y_j(u-p)\big)^{-n_{ij;p}}$$

  where $[x]_+ := \max(0, x)$.
- $r = (r_i)_{1 \le i \le N}$ and $n = (n_{ij;p})_{1 \le i,j \le N, p \in \mathbb{N}}$ are families of integers satisfying
  - $r_i \ge 1$
  - $n_{ij;p} = 0$ unless $0 < p < r_i$
- The pair (r, n) can be expressed as the matrix form:

$$N_0(z) := \mathrm{diag}(1 + z^{r_i})_i, \quad N_\pm(z) := \Bigg(\sum_{p \in \mathbb{N}} [\pm n_{ij;p}]_+ z^p \Bigg)_{i,j} \quad \in \mathbb{N}[z]^{N \times N}$$

- Setting $A_\pm(z) = \big(\sum_p a_{ij;p} z^p \big)_{i,j} := N_0(z) - N_\pm(z)$ and $x_i(u) := \frac{1}{1+Y_i(u)}$, the Y-system is rewritten as

$$\prod_{j,p} (1 - x_j(u-p))^{a^+_{ij;p}} = \prod_{j,p} x_j(u-p)^{a^-_{ij;p}}$$

  which can be regarded as an "affinization" of the Nahm's equation.

Y-systems

Example:

- $A_+(z) = 1 - z + z^2$, $A_-(z) = 1 + z^2$
- Y-system:

$$Y(u)Y(u-2) = \frac{1}{1 + Y(u-1)^{-1}}$$

  or

$$\frac{(1 - x(u))(1 - x(u-2))}{1 - x(u-2)} = x(u)x(u-2)$$

- Periodicity:

$$Y(u) = Y(u+5), \quad x(u) = x(u+5) \quad \text{for any } u.$$

Y-systems

- The pair $A_{\pm}(z)$ satisfies the symplectic property (with a symmetrizer d) if

$$A_+(z) D A_-(z^{-1})^\intercal = A_-(z) D A_+(z^{-1})^\intercal.$$

(Recall: a symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfies $AB^\intercal = BA^\intercal$.)

Theorem (M. 2021)

$A_{\pm}(z)$ satisfies the symplectic property iff the Y-system can be realized as the exchange relations of coefficients in a cluster algebra.

Example:

- $A_+(z) = 1 - z + z^2$, $A_-(z) = 1 + z^2$
- A_2 quiver:

$$0 \longrightarrow 1$$

which is preserved by $(0\ 1) \circ \mu_0$.

From Y-systems to Nahm sums

- Suppose that $A_\pm(z)$ satisfies the symplectic property with a symmetrizer d.
- Define matrix $A \coloneqq A_+(1)^{-1}A_-(1)$.
- Then AD is symmetric by the symplectic property.
- If the Y-system associated with $A_\pm(z)$ is periodic, then AD is positive definite [M. 2021].

Conjecture

The Nahm sum $f_{A,0,c,d}(q)$ is modular for some c.

Example:

- $A_+(z) = 1 - z + z^2$, $A_-(z) = 1 + z^2$
- $A = 2$
- $f_{2,0,-\frac{1}{60}}(q)$ is modular.

Dilogarithm identities

- Periodic Y-systems give dilogarithm identities in the general setting of cluster algebras [Nakanishi 2011].
- Dilogarithm values are related counting signs of tropical mutations (sign-coherence).

Example from Kanade-Russell conjecture:

- Recall: Assuming Kanade-Russell conjecture, the vector-valued function

$$g(\tau) := \begin{pmatrix} f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-1/18,d} & (q) \\ f_{A,\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right),5/18,d} & (q) \\ f_{A,\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right),11/18,d} & (q) \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix},$$

  $g(\tau)$ satisfies the following modular transformation formula:

$$g(-\frac{1}{\tau}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_2 & -\alpha_4 & -\alpha_1 \\ \alpha_4 & -\alpha_1 & \alpha_2 \end{pmatrix} g(\frac{\tau}{3})$$

  where $\alpha_k = \dfrac{1}{2\sqrt{3}\sin\frac{k\pi}{9}}$.

- Now focus on the first row (next page).

Dilogarithm identities

$$
\begin{aligned}
f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-1/18,d}(\tilde{q}) =& \frac{1}{2\sqrt{3}\sin\frac{\pi}{9}} f_{A,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),-1/18,d}(q^{1/3}) \\
&+ \frac{1}{2\sqrt{3}\sin\frac{2\pi}{9}} f_{A,\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right),5/18,d}(q^{1/3}) \\
&+ \frac{1}{2\sqrt{3}\sin\frac{4\pi}{9}} f_{A,\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right),11/18,d}(q^{1/3})
\end{aligned}
$$

with $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$.

- As τ tends to $i\infty$,

$$
\text{LHS} \sim \text{constant} \cdot e^{-\frac{i\tau}{2\pi}\Lambda}, \quad \text{RHS} \sim \frac{1}{2\sqrt{3}\sin\frac{\pi}{9}} e^{-\frac{2\pi i\tau}{54}}, \quad \text{where}
$$

$$
\Lambda = L(1-X_1) + \frac{1}{3}L(1-X_2), \quad (L(X) : \text{Rogers dilogarithm function})
$$

  and $0 < X_1, X_2 < 1$ is a solution of the Nahm's equation.
- Thus we have

$$
L(1-X_1) + \frac{1}{3}L(1-X_2) = \frac{2\pi^2}{27}
$$

Classification

Problem

Classify periodic Y-systems.

Recall: N is the "number of variables" and r is the "order" of the Y-system.

- Fomin-Zelevinsky [2007]: $r_i = 2$, $n_{ij;p} \leq 0$, and $n_{ii;p} = 0$ for any i, j, p. N: general.
  - Cartan-Killing classification ($A_-(1)$ is a Cartan matrix)
- Galashin-Pylyavskyy [2019]: $r_i = 2$ and $n_{ii;p} = 0$ for any i, p. N: general. d: trivial
  - ADE bigraphs of Stembridge [2010] ($A_+(1)$ and $A_-(1)$ are commuting Cartan matrices)
- Fordy-Marsh [2011], M. [2021]: $N = 1$. r: general.
  - $A_\pm(z) = (1 + z^2, 1 + z^2)$, $(1 - z + z^2, 1 + z^2)$, or $(1 + z^2, 1 - z + z^2)$, essentially.
- M. [2301.13239]: $N = 2$. r: general. d: trivial
  - Next page.

Classification of periodic Y-systems of rank 2

$A_+(z)$	$A_-(z)$	h_+	h_-	
$\begin{pmatrix} 1+z^2 & -z \\ -z & 1+z^2 \end{pmatrix}$	$\begin{pmatrix} 1+z^2 & 0 \\ 0 & 1+z^2 \end{pmatrix}$	3	2	(1)
$\begin{pmatrix} 1+z^2 & -z \\ -z-z^5 & 1+z^6 \end{pmatrix}$	$\begin{pmatrix} 1+z^2 & 0 \\ -z^3 & 1+z^6 \end{pmatrix}$	8	6	(2)
$\begin{pmatrix} 1+z^2 & -z \\ -z-z^5-z^9 & 1+z^{10} \end{pmatrix}$	$\begin{pmatrix} 1+z^2 & 0 \\ -z^3-z^7 & 1+z^{10} \end{pmatrix}$	18	10	(3)
$\begin{pmatrix} 1+z^2 & -z \\ -z & 1+z^2 \end{pmatrix}$	$\begin{pmatrix} 1+z^2-z & 0 \\ 0 & 1+z^2-z \end{pmatrix}$	3	3	(4)
$\begin{pmatrix} 1+z^2 & -z \\ -z-z^2 & 1+z^3 \end{pmatrix}$	$\begin{pmatrix} 1+z^2-z & 0 \\ 0 & 1+z^3 \end{pmatrix}$	5	3	(5)
$\begin{pmatrix} 1+z^2 & -z \\ -z & 1+z^2-z \end{pmatrix}$	$\begin{pmatrix} 1+z^2 & 0 \\ 0 & 1+z^2 \end{pmatrix}$	5	2	(6)

- $h_\pm$ are "reddening lengths". The period is some multiple of $h_+ + h_-$.

Quivers for periodic Y-systems of rank 2 (page 1)

$$(1,0) \longrightarrow (2,1) \qquad\qquad (1,1) \longleftarrow (2,0)$$

(1)

$$\begin{tikzcd}
(2,3) \arrow[dr] & & \\
& (1,0) \arrow[r] \arrow[dl] & (2,1) \arrow[ull] \\
(2,5) & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(2,2) \arrow[dd] & & \\
& (1,1) \arrow[ul] & (2,0) \arrow[l] \\
(2,4) \arrow[ur] & &
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& & (2,3) \arrow[ddl] & \\
(2,5) \arrow[dd] & & & \\
& (1,0) \arrow[ul] \arrow[rr] \arrow[ddr] & & (2,1) \arrow[uul] \arrow[dlll] \\
(2,7) \arrow[ur] & & & \\
& & (2,9) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& & (2,2) \arrow[dll] \arrow[dddd] & \\
(2,4) \arrow[dr] & & & \\
& (1,1) \arrow[uur] \arrow[dl] & & (2,0) \arrow[ll] \\
(2,6) \arrow[drr] & & & \\
& & (2,8) \arrow[uul] &
\end{tikzcd}$$

(3)

Quivers for periodic Y-systems of rank 2 (page 2)

$\begin{tikzcd} (1,1) \arrow[d] & (2,0) \arrow[l] \arrow[d] \\ (1,0) \arrow[r] & (2,1) \end{tikzcd}$	(4)
$\begin{tikzcd} & & & (2,1) \arrow[dl] & & & \\ (2,0) \arrow[rr] & & (1,1) \arrow[rr] & & (1,0) \arrow[ul] \arrow[rr] & & (2,2) \end{tikzcd}$	(5)
$(1,0) \longrightarrow (2,1) \longleftarrow (2,0) \longrightarrow (1,1)$	(6)

Future directions

Representation theoretic meaning? Some clues...

- vertex algebra
  - S-matrices and Verlinde formula
  - interpretation of Nahm sums by principal subspaces
- quantum affine algebra
  - level restricted T-systems [Kuniba-Nakanishi-Suzuki,...]
  - categorification of cluster algebras [Hernandez-Leclerc,...]