

Relative Koszul coresolutions and relative Betti numbers

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k : a field

A : a f.d. k -algebra (basic)

$\text{mod } A :=$ the cat. of (right) A -modules

$D := \text{Hom}_k(-, k)$: the usual k -duality of A

$G \in \text{mod } A$ (basic)

i.e. $G = \bigoplus_{I \in \underline{I}} V_I$, $V_I : \text{ind}$, $I \neq J \text{ in } \underline{I} \Rightarrow V_I \neq V_J$.

G : a generator $\Leftrightarrow A_A \lesssim \bigoplus G$

a cogenerator $\Leftrightarrow (DA)_A \lesssim \bigoplus G$.

$\Lambda := \text{End}_A(G) \rightsquigarrow {}_\Lambda G_A$: a bimodule

$\forall M \in \text{mod } A,$

$\text{add } M := \text{full} \{ X \in \text{mod } A \mid X \not\subseteq M^{(n)}, \exists n \geq 1 \}$

$\mathcal{J} := \text{add } G.$

usually

Q : a finite quiver

$p \leq kQ$ (the path alg of Q)

(Q, p) : a bound quiver

$A := k(Q, p) := kQ/p$

Exm Q : $\begin{array}{ccc} 3 & \xrightarrow{a_3} & 4 \\ b_1 \uparrow & & \uparrow b_2 \\ 1 & \xrightarrow{a_1} & 2 \end{array}, p = \langle b_2 a_1 - a_3 b_1 \rangle$

1. Introduction

[GGNST] Chachólski et. al. gave
 "an effective" way to compute relative
 Betti numbers for modules M over the
 incidence algebra kP of a poset P by using
 the so-called Koszul complex. (upper
semilattice)

In more detail,

$$(1) \text{ if } \dots \longrightarrow \bigoplus_{a \in P} P_a^{\beta_M^1(a)} \longrightarrow \bigoplus_{a \in P} P_a^{\beta_M^0(a)} \longrightarrow M \longrightarrow 0,$$

(P_a is the proj. indecomposable $k\mathbb{P}$ -mod $\leftrightarrow a \in \mathbb{P}$) ^{5/25}
 is a minimal proj. resolution of M , then

$\beta_M^i(a)$: the i -th Betti number for M at a .

(2) Let $K_a(M)$ be a Koszul complex of M at $a \in \mathbb{P}$ that is a complex of vector spaces:

$K_a(M)_0 := M(a)$, $\forall n \geq 1$. $K_a(M)_n := \bigoplus_{S \in \mathcal{U}(a)_n} M(\vee S)$,
 where $\mathcal{U}(a) := \{b \in \mathbb{P} \mid b < a, [b, a] = \{b, a\}\}$

$\mathcal{U}(a)_n := \{S \subseteq \mathcal{U}(a) \mid \#S = n, S \text{ bounded below}\}$

$\partial_n: \bigoplus_{S \in \mathcal{U}(a)_n} M(\wedge S) \rightarrow \bigoplus_{T \in \mathcal{U}(a)_{n-1}} M(\wedge T) := \left[\chi(T \subseteq S) (-1)^{n(S,T)} M(\wedge S \triangleleft \wedge T) \right]_{T,S}$
 $\chi(T \subseteq S) := \begin{cases} 1 & (T \subseteq S) \\ 0 & \text{o/w} \end{cases}$, $S = \{a_0 < a_1 < \dots < a_{n-1}\}$
 $S \setminus T = a_i \iff n(S, T) = i$.

(\prec : another linear order on \mathbb{P})

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Then they proved that

$$\beta_M^d(a) = \dim H_d(K_a(M)).$$

Problem For a minimal \mathcal{I} -resolution

$$\cdots \rightarrow \bigoplus_{I \in \mathcal{I}} V_I^{\beta_M^1(I)} \rightarrow \bigoplus_{I \in \mathcal{I}} V_I^{\beta_M^0(I)} \rightarrow M \rightarrow 0 \text{ of } M,$$

define a complex $K_I(M)$. s.t.

$$\beta_M^d(I) = \dim H_d(K_I(M)).$$

We will define $K_I(M)$ below and call it
the Koszul complex of M . (P. 11)

2. Preliminaries

$$_A(-, -) := \text{Hom}_A(-, -)$$

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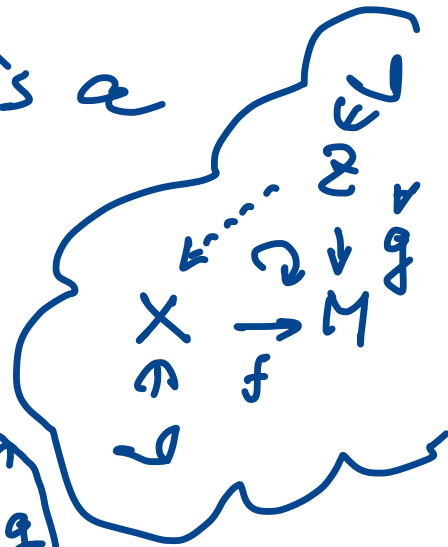
Dfn (right min. \mathcal{J} -approximation) $M \in \text{mod } A$.

(1) A right \mathcal{J} -approximation of M is a mor $f: X \rightarrow M$ with $X \in \mathcal{J}$ s.t.

$$\forall Z \in \mathcal{J}, {}_A(Z, f): {}_A(Z, X) \rightarrow {}_A(Z, M).$$

(G : a generator $\leadsto f$: an epi)

$$\begin{array}{c} Z = A^n \\ \downarrow g \\ M \end{array}$$



(2) $f: X \rightarrow M$ in $\text{mod } A$ is right minimal if $0 \neq \forall X' \hookrightarrow X, X' \not\subseteq \text{Ker } f$.

(3) $f: X \rightarrow M$ is a minimal right \mathcal{J} -appx. if it is a right min. right \mathcal{J} -appx.

Dually, left (min.) \mathcal{J} -appr $g: M \rightarrow Y$ is def^a. 8/25

Dfn (min. \mathcal{J} -resolution) $M \in \text{mod } A$.

An ~~exact sequence~~ chain complex

$$\cdots \xrightarrow{f_{r+1}} X_r \xrightarrow{f_r} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0 \quad (1)$$

is called a minimal \mathcal{J} -resolution of M if

f_i restricts to a minimal right \mathcal{J} -appr

$X_i \rightarrow \text{Ker } f_{i-1}$, $\forall i \geq 0$. (unique up to iso)

• $\forall i \geq 0, \exists! (\beta_M^i(I)) \in \mathbb{Z}_{\geq 0}^{\mathbb{I}}, X_i \cong \bigoplus_{I \in \mathbb{I}} \bigvee_I \beta_M^i(I)$

$\beta_M^i(I)$: \mathcal{J} -relative i -th Betti number of M at I

chain complex

- $A(Z, -)$ sends the ~~ex seq~~ (1) to an ex. seq $\forall Z \in \mathcal{J}$.

- Dually a min. \mathcal{J} -coresolution

$$0 \xrightarrow{g^{-1}} M \xrightarrow{g^0} Y^0 \xrightarrow{g^1} Y^1 \xrightarrow{g^2} \dots \xrightarrow{g^r} Y^r \xrightarrow{g^{r+1}} \dots \quad (2)$$

and the \mathcal{J} -relative i -th co-Betti numbers $\bar{\beta}_M^i(I)$

$(I \in \mathcal{I})$ is defined. $\left(Y^i \cong \bigoplus_{I \in \mathcal{I}} V_I^{\bar{\beta}_M^i(I)} \right)$

co complex

- $A(-, Z)$ sends the ~~ex. seq~~ (2) to an ex seq $\forall Z \in \mathcal{J}$.

3. Relative Koszul coresolution

Fix $I \in \mathbb{I}$, $e_I \in \Lambda : G \xrightarrow{\text{can}} V_I \hookrightarrow G$.

Ntn $M, N \in \text{mod } A$.

- $\text{rad}_A(M, N) :=$ the radical maps $M \rightarrow N$.
- $\text{rad}_A(G, G) = \text{rad } \Lambda$, Jacobson rad of Λ
- ${}_A(V_I, G) = \Lambda e_I : \text{a left } \Lambda\text{-module}$

$$\text{rad}_A(V_I, G) = \text{rad}_A(G, G) \cdot {}_A(V_I, G)$$

${}_I S := A(V_I, G) / \text{rad}_A(V_I, G) : \text{simple lt } \Lambda\text{-mod}$

- Similarly, ${}_A(G, V_I) = e_I \Lambda, \dots, S_I : \text{simple rt } \Lambda\text{-mod}$

Dfn An \mathcal{J} -relative Koszul coresolution

$K^\bullet(V_I)$ of V_I is a sequence in \mathcal{J}

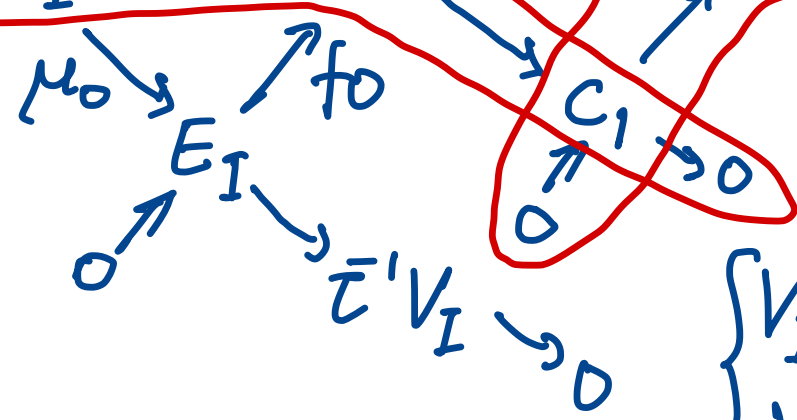
$$0 \rightarrow V_I \xrightarrow{\eta} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \dots \text{ (uniq up to iso)}$$

st.

exact

$$0 \rightarrow V_I \xrightarrow{\eta} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} X^3 \xrightarrow{d^3} \dots$$

min left \mathcal{J} -coresol



$\begin{cases} \mu_0 : \text{a source map} \\ f_0 : \text{min left } \mathcal{J}\text{-appx} \end{cases}$

$\begin{cases} V_I : \text{non-inj} \Rightarrow K^\bullet(V_I) : \text{ex} \\ V_I : \text{inj} \Rightarrow \text{Ker } \eta = \text{soc } V_I \end{cases}$

- $K_I(M) := {}_A[K^\bullet(V_I), M]$: the \mathcal{J} -relative Koszul complex of M

Lem Let $M \in \text{mod } A$.

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$$(1) \quad \dots \xrightarrow{f_{r+1}} X_r \xrightarrow{f_r} \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0 \quad (*1)$$

is a min right \mathcal{I} -resolution, ~~G : a generator~~

$$\Rightarrow \dots \rightarrow (G, X_1) \xrightarrow{(G, f_1)} (G, X_0) \xrightarrow{(G, f_0)} {}_A(G, M) \rightarrow 0$$

is a min proj resol. of ${}_A(G, M)$ in $\text{mod } \Lambda$.

$$(2) \quad 0 \xrightarrow{g^{-1}} M \xrightarrow{g^0} Y^0 \xrightarrow{g^1} Y^1 \xrightarrow{g^2} \dots \xrightarrow{g^r} Y^r \xrightarrow{g^{r+1}} \dots \quad (*2)$$

is a min left \mathcal{I} -coresol. ~~G : a cogenerator~~

$$\Rightarrow \dots \rightarrow (Y^1, G) \xrightarrow{(g^1, G)} (Y^0, G) \xrightarrow{(g^0, G)} (M, G) \rightarrow 0$$

is a min proj resol. of (M, G) in $\text{mod } \Lambda^{\text{op}}$.

Thm Let $M \in \text{mod } A$.

(1) ~~G : a cogenerator~~

\Rightarrow ${}_A(K^\bullet(V_I), G)$ turns out to be a min proj resol.

$\dots \rightarrow (X^2, G) \rightarrow (X^1, G) \rightarrow {}_A(V_I, G) \rightarrow {}_I S \rightarrow 0$
of the simple ${}_I S$.

(2) ~~G : a generator and a cogenerator~~

$\Rightarrow \beta_M^i(I) = \dim H_i(K_I(M)), \forall i \geq 0.$

This gives an answer to Problem.

Cor Let $M \in \text{mod } A$. Then (\Leftarrow) :

$$(1) M \in \mathcal{V}$$

$$(2) H_1(K_I(M)) = 0, \quad \forall I \in \mathcal{I}.$$

$$(3) \dim M = \sum_{I \in \mathcal{I}} \dim H_0(K_I(M)) \dim V_I$$

$$(4) M \cong \bigoplus_{I \in \mathcal{I}} V_I^{\dim H_0(K_I(M))}$$

Pf Each of (1) ~ (4) is eq to saying that f_0 in (*1) is an iso.



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Rmk To compute $H_0(K_I(M))$ in (3),
 enough to know

$$\begin{array}{ccc}
 V_I & \xrightarrow{\eta} & X^1 \\
 \mu_0 \searrow & \curvearrowright & \nearrow f_0 \\
 & E_I &
 \end{array}$$

$$\beta_M^0(I) = \dim_A(V_I, M) - \dim \operatorname{Im}_A(\eta, M).$$

⊖ Enough to know

- $$\left\{ \begin{array}{l}
 \bullet \mu_0: V_I \rightarrow E_I \text{ a source map (Computed)} \\
 \bullet f_0: E_I \rightarrow X^1 \text{ min left } \downarrow\text{-appx } \forall I \in \mathbb{I}.
 \end{array} \right.$$
- ↖ (computable for interval case easier.)

Application \mathbb{P} : a finite poset

$A := k\mathbb{P}$ (incidence alg), $\mathcal{I} :=$ the set of intervals of \mathbb{P}

V_I : interval module $\leftrightarrow I$, $G := \bigoplus_{I \in \mathcal{I}} V_I$: gen + wgen, $\psi = \text{ad} G$.

$\psi \ni M$: interval decomposable.

Cor Let $M \in \text{mod } A$. Then (2):

(1) M : interval decomposable

(2) $H_1(K_I(M)) = 0$, $\forall I \in \mathcal{I}$.

(3) $\dim M = \sum \dim H_0(K_I(M)) \dim V_I$

$$\begin{array}{c} V_I \twoheadrightarrow M \twoheadrightarrow V_I \\ I \in \mathcal{I} \end{array}$$

(4) $M \cong \bigoplus_{\text{as above}} V_I \dim H_0(K_I(M))$

4. Applications and examples

P : a finite poset

$A = kP$: the incidence algebra of P .

$\mathcal{I} :=$ the set of intervals of P (connected & convex)

$\forall I \in \mathcal{I}, V_I$: the interval module defined by I .

$G := \bigoplus_{I \in \mathcal{I}} V_I \leftarrow$ gen & cogen. $\downarrow :=$ add G .
(interval decomposable)

$\left\{ \begin{array}{l} \downarrow\text{-resolution} \leadsto \text{interval resolution} \\ \downarrow\text{-appx} \leadsto \text{interval appx} \end{array} \right.$

right [left] intv. appx \leadsto RIA [LIA]

Lem $I \in \mathcal{I}, S \leq V_I \Rightarrow V_I/S, S \in \mathcal{J}$



Prp Let $f: X \rightarrow M$ (resp. $M \rightarrow X$) in $\text{mod } A$. (12)

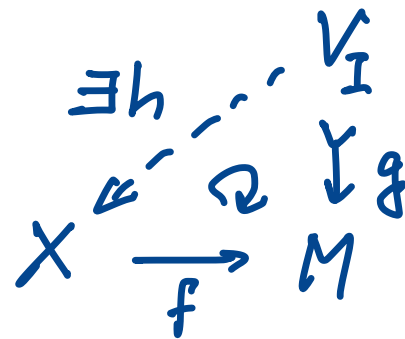
(1) f : a RIA [LIA]

sometimes
max $S_{\text{int}}(M)$ is enough

(2) $\forall I \in S_{\text{int}}(M) := \{I \in \mathcal{I} \mid V_I \xrightarrow{\exists} M\},$

$\forall g: V_I \rightarrow M$ in $\text{mod } A,$

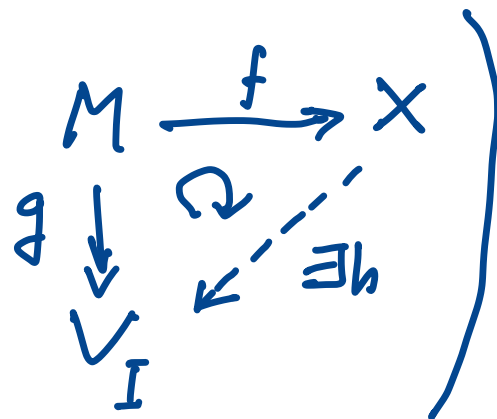
$g = fh, \exists h: V_I \rightarrow X$



(resp. $\forall I \in F_{\text{int}}(M) := \{I \in \mathcal{I} \mid M \xrightarrow{\exists} V_I\}$)

$\forall g: M \rightarrow V_I$ in $\text{mod } A,$

$g = hf, \exists h: X \rightarrow V_I.$



Cor A min. RIA of $M \in \text{mod } A$ is obtained as follows.

(1) $\forall I \in S_{\text{int}}(M)$, choose $\{f_I^{(1)}, \dots, f_I^{(d_I)}\} \subseteq \text{Mon}(V_I, M)$ st. ← monos

$\langle \text{Mon}(V_I, M) \rangle = \langle f_I^{(i)} \rangle_{i=1}^{d_I}$, and set $f_I := (f_I^{(i)})_{i=1}^{d_I} : V_I^{d_I} \rightarrow M$.

(2) Then by Prp, $(f_I)_{I \in S_{\text{int}}(M)} : \bigoplus_{I \in S_{\text{int}}(M)} V_I^{(d_I)} \rightarrow M$ is a RIA.

Set this to be $(g_i)_{i=1}^n : \bigoplus_{i=1}^n W_i \rightarrow M$ ($W_i : \text{intv. mod}$).

(3) $J_0 := \{1, \dots, n\} \ni^{\forall} j$, $J_j := \left\{ \begin{array}{l} J_{j-1} \setminus \{j\} \\ J_{j-1} \end{array} \right\} \begin{array}{l} (g_i)_{i \in J_{j-1} \setminus \{j\}} : \text{RIA} \\ (\text{o/w}) \end{array}$, $J := J_n$.

Then $(g_i)_{i \in J} : \bigoplus_{i \in J} W_i \rightarrow M$ is a min. RIA.

(The dual statement also holds.)

Rmk Let $f: X \rightarrow M$ be a min RIA of $M \in \text{mod } A$. 19/25

Then by Cor,

$\exists X = \bigoplus_{i \in J} W_i$ ($W_i: \text{intv}, \forall i$) st. $\forall i \in J, f|_{W_i}$ is a mon. ---- (*)

Concerning this, Prp [Aoki-Escobar-Toda] states:

$f: X \rightarrow M$: a min RIA

$\Rightarrow \begin{cases} (1) f: \text{epi} & (\text{true for all RIA } \odot G: \text{gen}), \\ (2) \forall X = \bigoplus_{i \in J} W_i \text{ } (W_i: \text{indec}), \forall i \in J, f|_{W_i} \text{ is a mon,} \\ (3) \text{supp } X = \text{supp } M & (\text{by (1), (2)}) \end{cases}$

- (2) is stronger than (*).
- However, (\Leftarrow) does not hold. (e.g. $(f, f|_{W_1}): X \oplus W_1 \rightarrow M$)

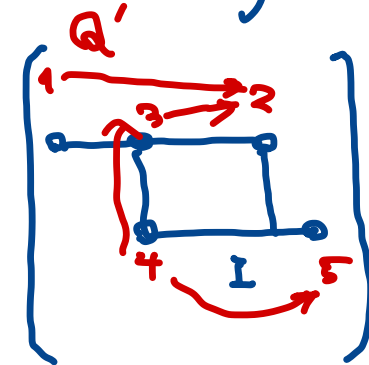
5. Application to compression multiplicity

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Thm [AENY] $\mathbb{P} = A_n \times A_2$ ($A_n: 1 \rightarrow 2 \rightarrow \dots \rightarrow n$)

$\exists \xi = (\xi_I)_{I \in \Pi} : \text{a compression system}$

$\xi_I : Q' := 1 \swarrow^2 \searrow^4 \rightarrow 5 \rightarrow I \hookrightarrow \mathbb{P}$



$\bar{\xi}_I : kQ' \rightarrow k\mathbb{P} = A$, a unique extension

restriction functor by $\bar{\xi}_I$

$R_I : \text{mod } A \rightarrow \text{mod } kQ'$

$M, V_I \mapsto R_I(M), R_I(V_I)$
st.

$\left(\begin{array}{l} \exists r \geq 0 \text{ st. } \beta_M^i = 0 \\ \text{for } \forall i > r \end{array} \right)$

\Rightarrow

$c_M^{\xi}(I) = \sum_{I \leq J} \left(\sum_{i=0}^r (-1)^i \beta_M^i(J) \right) : \text{multiplicity of } R_I(V_2) \text{ in the indecomp decomp. of } R_I(M)$
 $\left[:= d_{R_I(M)}^{R_I(V_I)} \right]$

$$\delta_M^\xi(I) = \sum_{i=0}^r (-1)^i \beta_M^i(I) : \text{Möbius inversion of } c_M^\xi : \mathbb{I} \rightarrow \mathbb{R}$$

$$\delta^\xi(M) = \left[\bigoplus_{\substack{I \in \mathbb{I} \\ \delta_M^\xi(I) > 0}} V_I^{\delta_M^\xi(I)} \right] - \left[\bigoplus_{\substack{I \in \mathbb{I} \\ \delta_M^\xi(I) < 0}} V_I^{-\delta_M^\xi(I)} \right] \quad \text{interval replacement of } M \text{ under } \xi$$

\downarrow
 $M = \bigoplus_{I \in \mathbb{I}} V_I^{d_M(V_I)}$

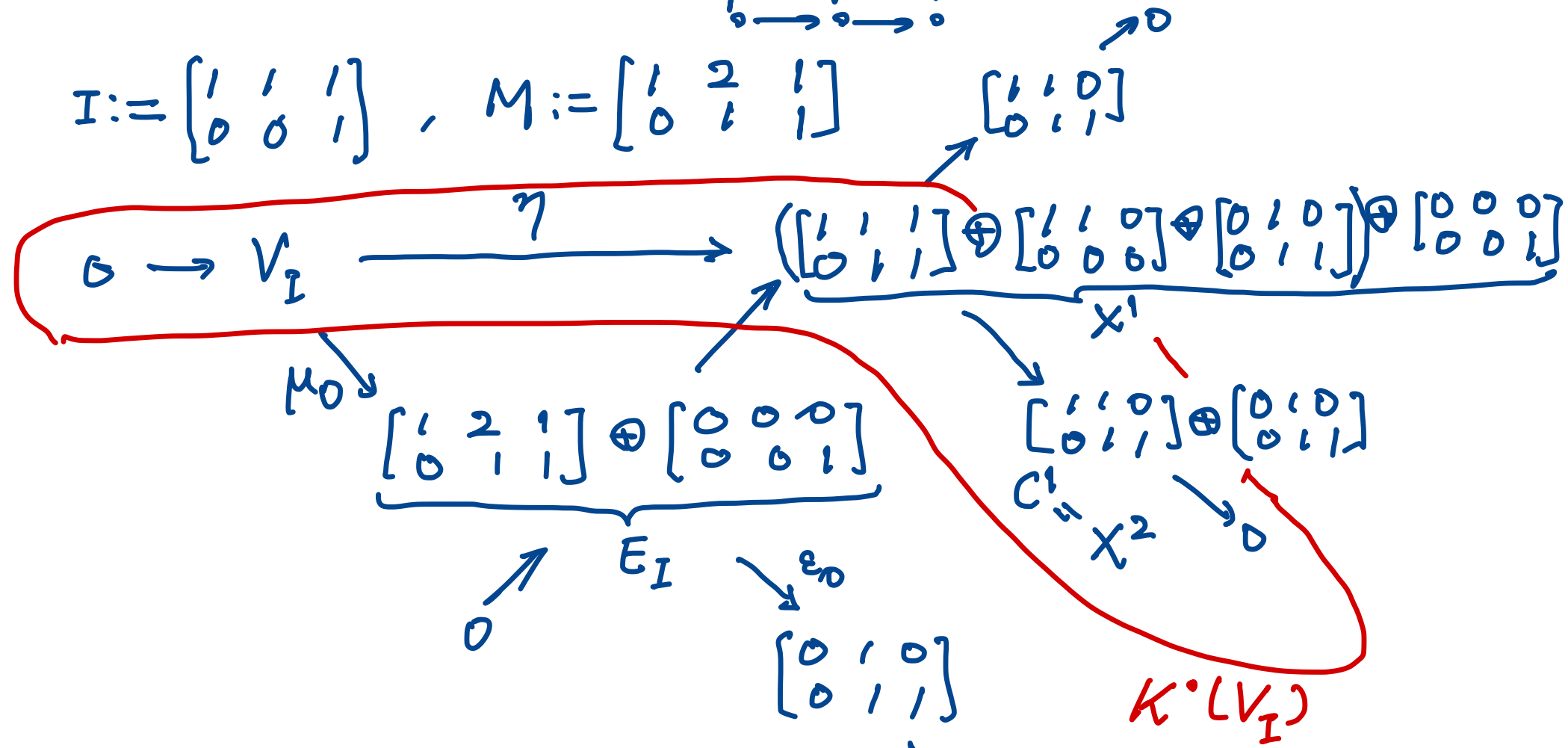
Cor

$$c_M^\xi(I) = \sum_{I \leq J} \left(\sum_{i=0}^r (-1)^i \dim H_i(\underbrace{K_I(M)}) \right)$$

$$\delta_M^\xi(I) = \sum_{i=0}^r (-1)^i \dim H_i(\underbrace{K_I(M)})$$

Exm $P := (A_3 \times A_2) : \begin{array}{ccc} \circ & \rightarrow & \circ & \rightarrow & \circ \\ \uparrow & & \uparrow & & \uparrow \\ \circ & \rightarrow & \circ & \rightarrow & \circ \end{array} \quad A = kP$

$$I := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M := \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$



$$K_M(I)_\bullet : 0 \rightarrow {}_A(V_I, M) \xrightarrow{\quad} (x^1, M) \xrightarrow{\quad} (x^2, M) \rightarrow 0$$

$\cong k \qquad \qquad \qquad = 0 \qquad \qquad \qquad = 0$

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$$\textcircled{c)} \dim H_i(K_M(I).) = \delta_{i,0} \quad (i \geq 0)$$

interval resolution of M :

$$0 \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow M \rightarrow 0$$

$$\beta_M^i(I) = 0 \quad (i \geq 1), \quad \beta_M^0(I) = 1$$

Exm $\mathbb{P} = A_5 \times A_2$, $A = k\mathbb{P}$, $I = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

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$M: \begin{array}{ccccccccc} 0 & \rightarrow & 0 & \rightarrow & k & \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} & k^2 & \xrightarrow{(0,1)} & k \\ \uparrow & & \uparrow & & \uparrow & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \uparrow & \uparrow & & \uparrow \\ 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & k & \xrightarrow{1} & k \end{array} : \text{indecomp. } \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

$K^0(V_I):$ $X^1 = E_I \in \mathcal{V}$

$0 \rightarrow V_I \xrightarrow{\mu_0} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\varepsilon_0} M \rightarrow 0 \quad (*)$

$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ \leftarrow embeddings \leftarrow can epi

(ass) \leftarrow min intv resol of M also

min. intv. coresol. of $M = C^1$:

$0 \rightarrow M \xrightarrow{\mu_1} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\alpha^2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow 0$

$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$ \leftarrow can epis X^2 X^3

$$\textcircled{ii} \quad K^\bullet(V_I) : 0 \rightarrow V_I \xrightarrow{\mu_0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} X^3 \rightarrow 0 \quad \begin{array}{r} 25 \\ \hline 25 \end{array}$$

$$\begin{array}{ccc} & \searrow \varepsilon_0 & \nearrow \mu_1 \\ & M = C^1 & \end{array}$$

$$\textcircled{ii} \quad K_M(I) :$$

$$0 \rightarrow_A (X^3, M) \xrightarrow{(d^2, M)} (X^2, M) \xrightarrow{(d^1, M)} (X^1, M) \xrightarrow{(\mu_0, M)} (V_I, M) \rightarrow 0$$

$\begin{array}{ccccc} & \parallel & & \parallel & \\ & 0 & & 0 & \end{array}$

$\dots = \beta_M^3(I) = 0 = \beta_M^2(I)$

$\text{End}_A(M) \cong k$

$\beta_M^1(I) = 1$

$\beta_M^0(I) = 0$

ex

$$\textcircled{ii} \quad \beta_M^i(I) = \delta_{i,1} \quad (i \geq 0) \quad \text{as } (*) \text{ suggests.}$$

(min intv resol of M)