Cluster realization of Weyl groups and q-characters of quantum affine algebras

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Plan

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§1 Prelimininaries

• Cluster mutation [Fomin-Zelevinsky 00]

A seed $t = (\varepsilon, \boldsymbol{A}, \boldsymbol{X})$:

$\varepsilon = (\varepsilon_{ij})_{i,j} \in \mathrm{Mat}_N(\mathbb{Z})$: a skew-symmetrizable matrix

$\exists d = \mathrm{diag}(d_i)_i,\ d_i \in \mathbb{Z}_{>0}$ s.t. εd is skew-symmetric

(with $\gcd\{d_i\}_i = 1$)

$\boldsymbol{A} = (A_1, \dots, A_N)$: cluster variables ($A$-var.)

$\boldsymbol{X} = (X_1, \dots, X_N)$: coefficients ($X$-var.)

The mutation $\mu_k(\varepsilon, \boldsymbol{A}, \boldsymbol{X}) = (\widetilde{\varepsilon}, \widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{X}})$ at $k \in \{1, \cdots, N\}$:

$$\widetilde{X}_i = \begin{cases} X_k^{-1} & i = k \\ X_i \left(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})}\right)^{-\varepsilon_{ik}} & i \neq k \end{cases} \qquad \widetilde{A}_i = \begin{cases} \dfrac{\prod_{j:\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j:\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}}{A_k} & i = k \\ A_i \quad \text{for } i \neq k \end{cases}$$

$$\widetilde{\varepsilon}_{ij} = \begin{cases} -\varepsilon_{ij} & i = k \text{ or } j = k \\ \varepsilon_{ij} + \dfrac{|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|}{2} & \text{ow} \end{cases}$$

(d is the same.)

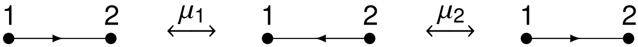
Remark

$(\varepsilon, d) \overset{1:1}{\longleftrightarrow}$ a weighted quiver $Q = (\sigma, d)$

$\mathrm{wt}(i) = d_i$, $\sigma_{ij} = \varepsilon_{ij} \frac{\gcd(d_i, d_j)}{d_i}$

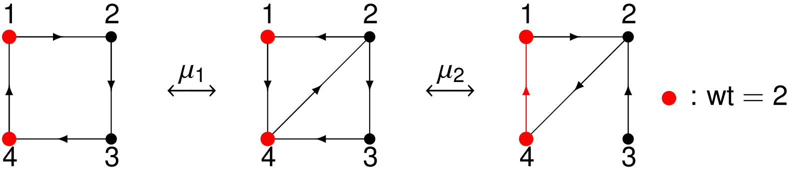
$\sigma_{ij} = \#\{\text{arrows from } i \text{ to } j\} - \#\{\text{arrows from } j \text{ to } i\}$

(Ex)



$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad \begin{pmatrix} (1 + A_2)/A_1 := A_1' \\ A_2 \end{pmatrix} \qquad \begin{pmatrix} A_1' \\ (1 + A_1')/A_2 \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad \begin{pmatrix} 1/X_1 \\ X_2(1 + X_1) := X_2' \end{pmatrix} \qquad \begin{pmatrix} (1 + X_2')/X_1 \\ 1/X_2' \end{pmatrix}$$



• Cluster modular group

$\Gamma_\varepsilon := \{\gamma = s \circ \mu_{i_1} \cdots \mu_{i_\ell} \mid s \in \mathfrak{S}_N,\ \gamma(\varepsilon) = \varepsilon\} / \{\gamma \mid \gamma(t) = t\}$

(Ex) $\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$: $\mu_2\mu_1$, $(1,2)\mu_1$, $(1,2)\mu_1\mu_2\mu_1\mu_2\mu_1 = \mathrm{id} \in \Gamma_\varepsilon$

• Poisson structure on $\mathbb{C}(\boldsymbol{X})$

$\varepsilon \rightsquigarrow \{X_i, X_j\} = \varepsilon_{ij} d_j X_i X_j$

$\gamma \in \Gamma_\varepsilon \Rightarrow \gamma^* \curvearrowright \mathbb{C}(\boldsymbol{X})$ preserving the Poisson structure.

<u>Applications of Γ_ε</u>

· rational maps, integrable systems

Somos 4,5, T-systems, Y-systems, discrete Painlevé eq...

· Dilogarithm identity [Kashaev-Nakanishi 11]

$s \circ \mu_{i_1} \cdots \mu_{i_\ell} = \mathrm{id} \in \Gamma_\varepsilon \Rightarrow \ell$-term identity for dilog. fcn.

· (higher) Teichmüller theory of a marked surface [Foch-Goncharov 03]

mapping class group $\subset \Gamma_\varepsilon$

· $\cdots$

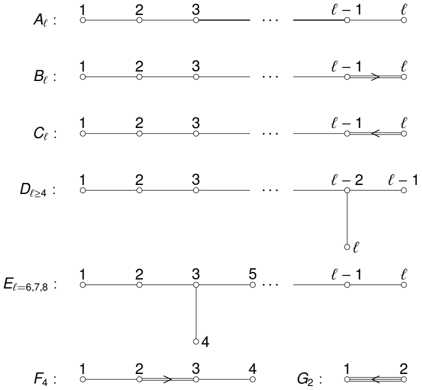
§2 Cluster realization of Weyl groups

• Lie algebras

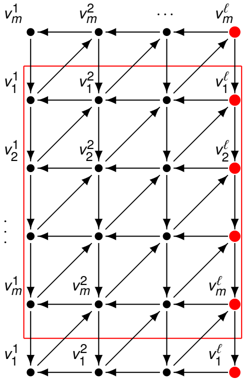
$\mathfrak{g}$: a fin. dim. irr. Lie alg. of rank ℓ, $S := \{1, 2, \ldots, \ell\}$

$\mathbf{C} = (C_{ij})_{i,j \in S}$: the Cartan matrix; $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

$\mathbf{D} = \mathrm{diag}(D_i)_{i \in S}$; $D_i = \frac{(\alpha_i, \alpha_i)}{2}$ (**DC** is symmetric)



- Quivers



<u>Def</u> (quiver $Q'_m(\mathfrak{g})$) [I-Ishibashi-Oya19]

$$\varepsilon = (\varepsilon_{ij})_{i,j\in S};\ \varepsilon_{ij} = \begin{cases} -C_{ji}\ (i > j) \\ C_{ji}\ (i < j) \\ 0\ (i = j) \end{cases},\ \mathrm{wt}(i) = d_i$$

For $m > 1$,

Dynkin quiver $\times m$

$$\rightsquigarrow Q'_m(\mathfrak{g});\ \ I = \{v^i_n;\ \ i \in S,\ \ n \in \mathbb{Z}/m\mathbb{Z}\}$$

(Ex) $\mathfrak{g} = A_\ell$; $\mathrm{wt}(i) = 1\ (i \in S)$

$$\mathfrak{g} = C_\ell;\ \mathrm{wt}(i) = \begin{cases} 1 & i \in S \setminus \{\ell\} \\ 2 & i = \ell \end{cases}$$

For $i \in S$,

P_i: an oriented circle $v^i_1 \to v^i_2 \to \cdots \to v^i_m \to v^i_1$

Def (quiver $Q_m(\mathfrak{g})$) [I 20] (Cf. [Hernandez-Leclerc16])

$\mathfrak{g}$ is simply-laced: $Q_m(\mathfrak{g}) = Q'_m(\mathfrak{g})$

$\mathfrak{g}$ is non-simply-laced ($\mathbf{D} \neq \mathbb{I}_\ell$):

$Q_m(\mathfrak{g})$; $I = \{v^i_n;\ i \in S, n \in D\mathbb{Z}/D'm\mathbb{Z}\}$

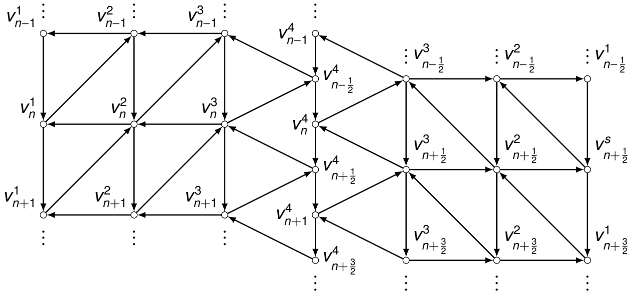
$$D := \min(D_i)_{i\in\ \ell},\ D' := \max(D_i)_{i\in\ \ell},\ \mathrm{wt}(v^i_n) = 1$$

For $i \in S$ and $\gamma_i = 1, 2, \ldots, D_i/D$,

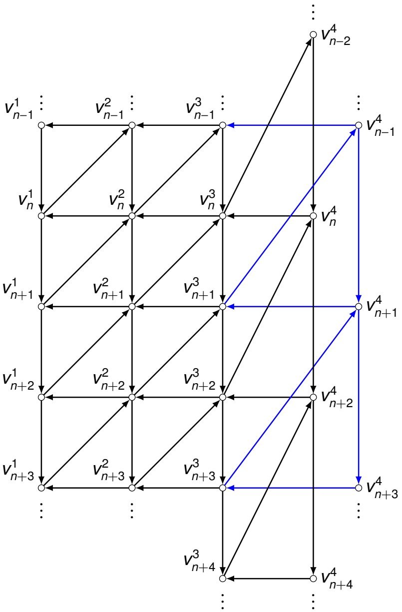
P_{i,γ_i}: an oriented circle in $Q_m(\mathfrak{g})$

For $\mathfrak{g}$, $Q(\mathfrak{g})$; $I = \{v^i_n;\ i \in S, n \in D\mathbb{Z}\}$: an infinite quiver

$Q(B_4)$

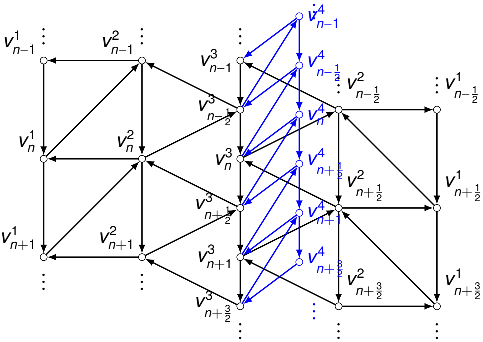


$Q(C_4)$:

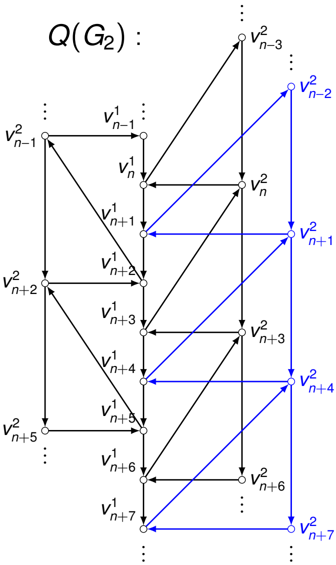


v^4_{n-2}
v^1_{n-1} v^2_{n-1} v^3_{n-1} v^4_{n-1}
v^1_n v^2_n v^3_n v^4_n
v^1_{n+1} v^2_{n+1} v^3_{n+1} v^4_{n+1}
v^1_{n+2} v^2_{n+2} v^3_{n+2} v^4_{n+2}
v^1_{n+3} v^2_{n+3} v^3_{n+3} v^4_{n+3}
v^3_{n+4} v^4_{n+4}

$Q(F_4)$:


v^1_{n-1} v^2_{n-1} v^3_{n-1} v^4_{n-1}
$v^4_{n-\frac{1}{2}}$ $v^3_{n-\frac{1}{2}}$ $v^2_{n-\frac{1}{2}}$ $v^1_{n-\frac{1}{2}}$
v^1_n v^2_n v^3_n v^4_n
$v^4_{n+\frac{1}{2}}$ $v^3_{n+\frac{1}{2}}$ $v^2_{n+\frac{1}{2}}$ $v^1_{n+\frac{1}{2}}$
v^1_{n+1} v^2_{n+1} v^3_{n+1} v^4_{n+1}
$v^4_{n+\frac{3}{2}}$ $v^3_{n+\frac{3}{2}}$ $v^2_{n+\frac{3}{2}}$ $v^1_{n+\frac{3}{2}}$

$Q(G_2)$:


v^2_{n-3} v^2_{n-2}
v^2_{n-1} v^1_{n-1}
v^1_n v^2_n
v^1_{n+1} v^2_{n+1}
v^1_{n+2} v^2_{n+2}
v^1_{n+3} v^2_{n+3}
v^1_{n+4} v^2_{n+4}
v^1_{n+5} v^2_{n+5}
v^1_{n+6} v^2_{n+6}
v^1_{n+7} v^2_{n+7}

• Weyl group action

Def

For an oriented circle P; $v_1 \to v_2 \to \cdots \to v_p \to v_1$,

$$M(P; v_1) := \mu_{p-2} \cdots \mu_2 \mu_1$$
$$R(P; v_1) := M(P; v_1)^{-1} \circ (v_{p-1}, v_p) \circ \mu_p \mu_{p-1} \circ M(P; v_1)$$

$\rightsquigarrow$ $R(P_i, v_n^i)$ for $Q'_m(\mathfrak{g})$

$R(P_{i,\gamma_i}, v_n^i)$ for $Q_m(\mathfrak{g})$; $v_n^i \in P_{i,\gamma_i}$

Remark

$R(P, v_n)$ appeared in [Bucher 14], in studying green sequence.

Thm 1 [IIO 19]

(i) $R(P_i, v_n^i) \in \Gamma_{Q'_m(\mathfrak{g})}$.

(ii) For $n \neq k$, $R(P_i, v_n^i) = R(P_i, v_k^i)$ as elements in $\Gamma_{Q'_m(\mathfrak{g})}$.
$\rightsquigarrow R(i) := R(P_i, v_n^i)$.

(iii) $R(i)$ $(i \in S)$ generate the $W(\mathfrak{g})$-action on $\mathbb{C}(\boldsymbol{X})$ and $\mathbb{C}(\boldsymbol{A})$; the $R(i)$ satisfy $(R(i)R(j))^{m_{ij}} = 1$, where $m_{ii} = 1$ and for $i \neq j$

$C_{ij}C_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Remark

· For $\mathfrak{g} = A_\ell$, Thm 1 appeared in [I-Lam-Pylyavskyy 16].

· $p : \mathbb{C}(\boldsymbol{X}) \to \mathbb{C}(\boldsymbol{A})$; $X_v \mapsto \prod_{v' \in I} A_{v'}^{\varepsilon_{v,v'}}$: the positive map
$\rightsquigarrow R(i)^* p(X_n^j) = p(X_n^j)$ for $v_n^j \in I$.

(Ex) $Q_3(A_3)$ and $Q_3(C_3)$

$$d = \mathbf{D} = \mathrm{diag}(1, 1, 1/2), \quad C(A_3/C_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1/-2 \\ 0 & -1 & 2 \end{pmatrix},$$

$$R(2)^*(A_1^1, A_1^2, A_1^3) = \left(A_1^1, \frac{A_2^1 A_3^2 A_1^3 + A_3^1 A_1^2 A_2^3 + A_1^1 A_2^2 A_3^3}{A_2^2 A_3^2}, A_1^3\right)$$

$$R(3)^*(A_1^1, A_1^2, A_1^3) = \left(A_1^1, A_1^2, \frac{(A_2^2)^{1/2} A_3^3 + (A_3^2)^{1/2} A_1^3 + (A_1^2)^{1/2} A_2^3}{A_2^3 A_3^3}\right)$$

$$R(2)^*(X_1^1, X_1^2, X_1^3) = \left(X_1^1 X_3^2 \frac{1 + X_2^2 + X_2^2 X_1^2}{1 + X_3^2 + X_3^2 X_2^2}, \frac{1 + X_1^2 + X_1^2 X_3^2}{X_3^2(1 + X_2^2 + X_2^2 X_1^2)}, X_1^3 \left(X_1^2 \frac{1 + X_3^2 + X_3^2 X_2^2}{1 + X_1^2 + X_1^2 X_3^2}\right)^{1/2}\right)$$

$$R(3)^*(X_1^1, X_1^2, X_1^3) = \left(X_1^1, X_1^2 X_3^3 \frac{1 + X_2^3 + X_2^3 X_1^3}{1 + X_3^3 + X_3^3 X_2^3}, \frac{1 + X_1^3 + X_1^3 X_3^3}{X_3^3(1 + X_2^3 + X_2^3 X_1^3)}\right)$$

Thm 2 [I 20]

(i) $R(P_{i,\gamma_i}, v_n^i) \in \Gamma_{Q_m(\mathfrak{g})}$.

(ii) For $n \neq k$, $R(P_{i,\gamma_i}, v_n^i) = R(P_{i,\gamma_i}, v_k^i) \in \Gamma_{Q_m(\mathfrak{g})}$.
$R(P_{i,\gamma_i}, v_n^i); \gamma_i = 1, \cdots, D_i/D$ are commutative.

$$\rightsquigarrow R_i := \prod_{\gamma_i = 1, \cdots, D_i/D} R(P_{i,\gamma_i}, v_n^i).$$

(iii) R_i $(i \in S)$ generate the $W(\mathfrak{g})$-action on $\mathbb{C}(\boldsymbol{X})$ and $\mathbb{C}(\boldsymbol{A})$;
the R_i satisfy $(R_i R_j)^{m_{ij}} = 1$.

For P; $v_1 \to v_2 \to \cdots \to v_p \to v_1$, $f_X(P; v_n) := 1 + \sum_{k=0}^{p-2} X_n X_{n-1} \cdots X_{n-k}$

When $D_i \geq D_j$:

$$R_i^*(X_n^j) = \begin{cases} \dfrac{f_X(i,n)}{X_{n-D_j}^j \, f_X(i, n-2D_i)} & i = j, \\ X_n^j \dfrac{X_{n-D_i}^i f_X(i, n-2D_i)}{f_X(i, n-D_i)} & v_n^j \leftarrow v_n^i, \\ X_n^j \dfrac{X_n^i f_X(i, n-D_i)}{f_X(i,n)} & v_n^j \to v_n^i, \quad \text{except for } (i,j) = \begin{cases} (\ell-1, \ell); \ B_\ell, \\ (2,3); \ F_4, \end{cases} \\ X_n^j & \text{o/w} \end{cases}$$

where $f_X(i,n) := f_X(P_{i,\gamma_i}; v_n^i)$ for $v_n^i \in P_{i,\gamma_i}$. When $D_i < D_j$:

$$R_i^*(X_n^j) = \begin{cases} X_n^j \dfrac{X_{n-\frac{1}{2}}^i X_n^i f_X(i, n-1)}{f_X(i,n)} & (i,j) = \begin{cases} (\ell, \ell-1); \ B_\ell, \\ (3,2); \ F_4, \end{cases} \\ X_n^\ell \dfrac{X_n^{\ell-1} X_{n+1}^{\ell-1} f_X(\ell-1, n-1)}{f_X(\ell-1, n+1)} & (i,j) = (\ell-1, \ell); \ C_\ell, \\ X_n^2 \dfrac{X_n^1 X_{n+1}^1 X_{n+2}^1 f_X(1, n-1)}{f_X(1, n+2)} & (i,j) = (1,2); \ G_2. \end{cases}$$

$$R(P_{i,\gamma_i})^*(A_n^j) = \begin{cases} f_A(i,\gamma_i)A_n^j & i=j,\ v_n^i \in P_{i,\gamma_i} \\ A_n^j & \text{o/w} \end{cases}$$

where ($j \lhd i$ means that $j < i$ and $C_{ij} \neq 0$)

$$f_A(i,\gamma_i) = \sum_{n:v_n^i \in P_{i,\gamma_i}} \frac{1}{A_n^i A_{n+d_i}^i} \prod_{j:i \lhd j} A_n^j \cdot \prod_{j:i \rhd j} A_{n+d_i}^j$$

for all cases, except for:

$$f_A(i,\gamma_i) = \sum_{n:v_n^i \in P_{i,\gamma_i}} \frac{A_{n+\frac{1}{2}}^{i+1} A_{n+1}^{i-1}}{A_n^i A_{n+1}^i} \quad i = \ell - 1;\ B_\ell,\ i=2;\ F_4$$

$$f_A(i,1) = \begin{cases} \displaystyle\sum_{n:v_n^\ell \in P_\ell} \frac{A_{n+\frac{1}{2}}^{i-1} A_n^{i-1} A_n^{i+1}}{A_n^i A_{n+\frac{1}{2}}^i} & i=\ell;\ B_\ell,\ i=3;\ F_4 \\ \displaystyle\sum_{n:v_n^{\ell-1} \in P_{\ell-1}} \frac{A_{n+1}^{\ell-2} A_n^\ell A_{n-1}^\ell}{A_n^{\ell-1} A_{n+1}^{\ell-1}} & i=\ell-1;\ C_\ell \\ \displaystyle\sum_{n:v_n^1 \in P_1} \frac{A_n^2 A_{n-1}^2 A_{n-2}^2}{A_n^1 A_{n+1}^1} & i=1;\ G_2 \end{cases}$$

Remarks

$\cdot$ $f_X(i,n)$ corresponds to the F-polynomial at v^i_{n+1} for $R(i)$ and $v^i_{n+D_i}$ for R_i.

$\cdot$ In both cases of $Q'_m(\mathfrak{g})$ and $Q_m(\mathfrak{g})$, the Weyl group action on X-var is analogous to that on simple roots α_j. In fact, the induced action on $\mathbb{X}_j := \prod_{n\in\mathbb{Z}/m\mathbb{Z}} X^j_n$ or $\mathbb{X}_j := \prod_{n\in D'\mathbb{Z}/Dm\mathbb{Z}} X^j_n$ is

$$(R(i) \text{ or } R_i)^*(\mathbb{X}_j) = \begin{cases} \mathbb{X}_i^{-1} & j = i, \\ \mathbb{X}_j \, \mathbb{X}_i^{-C_{ij}} & j \neq i \end{cases}$$

which corresponds to

$$r_i \, \alpha_j = \alpha_j - C_{ij} \, \alpha_i; \;\; i,j \in S.$$

• Green sequenses

Q: a weighted quiver, I: the vertex set, $\boldsymbol{u} = (u_i)_{i\in I}$

Consider the tropical X-var in the tropical semifield:

$\mathbb{P}_{\mathrm{trop}}(\boldsymbol{u}) = (\{\prod_{i\in I} u_i^{m_i}; m_i \in \mathbb{Z}\}, \cdot, \oplus)$;

$$\prod_i u_i^{m_i} \oplus \prod_i u_i^{n_i} = \prod_i u_i^{\min(m_i,n_i)}, \quad \prod_i u_i^{m_i} \cdot \prod_i u_i^{n_i} = \prod_i u_i^{m_i+n_i}.$$

Fact (sign coherence) [Fomin-Zelevinsky 07]

For any sequence $\boldsymbol{i}$ in I, consider $\mu_{\boldsymbol{i}}(Q, \boldsymbol{u}) = (Q', \boldsymbol{X}')$.

$\rightsquigarrow$ each X-variable $X_i' = \prod_j u_j^{m_j}$ satisfies

$(m_j)_{j\in I} \in (\mathbb{Z}_{\geq 0})^{|I|}$ ($X_i' > 0$) or $(m_j)_{j\in I} \in (\mathbb{Z}_{\leq 0})^{|I|}$ ($X_i' < 0$).

Def [Keller 11]

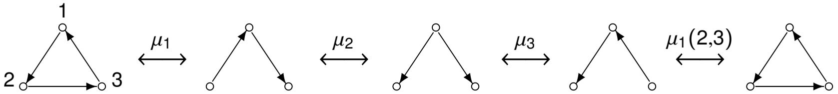
For a sequence of tropical X-seeds of $\boldsymbol{i} = (i_1, \dots, i_p)$ in I:

$$(Q[0], \boldsymbol{X}[0]) := (Q, \boldsymbol{u}) \overset{\mu_{i_1}}{\longmapsto} (Q[1], \boldsymbol{X}[1]) \overset{\mu_{i_2}}{\longmapsto} \cdots \overset{\mu_{i_p}}{\longmapsto} (Q[p], \boldsymbol{X}[p]).$$

$\boldsymbol{i}$ is green $\overset{\text{def}}{\iff} X[k]_{i_{k+1}} > 0$ for all $k = 0, 1, \dots, p-1$.

$\boldsymbol{i}$ is maximal green $\overset{\text{def}}{\iff}$ $\boldsymbol{i}$ is green and $X[p]_i < 0$ for all $i \in I$.

(Ex) $Q_3(A_1)$, $R(1) = \mu_1(2,3)\mu_3\mu_2\mu_1$



$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \begin{pmatrix} u_1^{-1} \\ u_2(1 \oplus u_1) \\ u_3(1 \oplus u_1^{-1})^{-1} \end{pmatrix} = \begin{pmatrix} u_1^{-1} \\ u_2 \\ u_1 u_3 \end{pmatrix} \quad \begin{pmatrix} u_1^{-1} \\ u_2^{-1} \\ u_1 u_3 \end{pmatrix} \quad \begin{pmatrix} u_3 \\ u_2^{-1} \\ (u_1 u_3)^{-1} \end{pmatrix} \quad \begin{pmatrix} u_3^{-1} \\ u_1^{-1} \\ u_2^{-1} \end{pmatrix}$$

$\Rightarrow$ $R(1)$ is maximal green.

Thm 3 [IIO 19], [I 20]

$\cdot\ Q = Q'_m(\mathfrak{g})$:

(i) $R(i)$; $i \in S$ is green. (Cf. [Bucher 14])

(ii) For $w \in W(\mathfrak{g})$ of a reduced expression $w = r_{i_1} r_{i_2} \cdots r_{i_k}$,
$R(w) := R(i_1) R(i_2) \cdots R(i_k)$ is green.
When w is the longest element $w_0 \in W(\mathfrak{g})$, $R(w_0)$ is maximal green.

$\cdot\ Q = Q_m(\mathfrak{g})$: with R_i; $i \in S$, (i) and (ii) hold.

(Ex) $Q'_m(A_2)(= Q_m(A_2))$ and $w_0 = r_1 r_2 r_1$

$$(u_i^1,\, u_i^2)_{i \in \mathbb{Z}_m} \overset{R(1)}{\mapsto} ((u_{i-1}^1)^{-1},\, u_i^1 u_i^2)_{i \in \mathbb{Z}_m}$$
$$\overset{R(2)}{\mapsto} (u_{i-1}^2,\, (u_{i-1}^1 u_{i-1}^2)^{-1})_{i \in \mathbb{Z}_m} \overset{R(1)}{\mapsto} ((u_{i-2}^2)^{-1},\, (u_{i-1}^1)^{-1})_{i \in \mathbb{Z}_m}$$

§3 Application to q-characters

- q-characters for $U_q(\hat{\mathfrak{g}})$ [Frenkel-Reshetikhin 90s]

$U_q(\hat{\mathfrak{g}})$: an affine quantum group; $q \in \mathbb{C}^\times$ (not a root of unity)

χ_q : Rep $U_q(\hat{\mathfrak{g}}) \to \mathbf{Y} := \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i \in S, a_i \in \mathbb{C}^\times}$: q-character

Thm [Frenkel-Reshetikhin 99]

$$\operatorname{Im} \chi_q = \bigcap_{i \in S} \Big(\mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i, a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b \in \mathbb{C}^\times} \Big)$$

where $q_i = q^{D_i}$ and

$$A_{i,a} = Y_{i,aq_i} Y_{i,aq_i^{-1}} \prod_{j : C_{ji} = -1} Y_{j,a}^{-1} \prod_{j : C_{ji} = -2} Y_{j,aq_j}^{-1} Y_{j,aq_j^{-1}}^{-1} \prod_{j : C_{ji} = -3} Y_{j,aq_j^2}^{-1} Y_{j,a}^{-1} Y_{j,aq_j^{-2}}^{-1}.$$

Fix $a \in \mathbb{C}^\times / q^{D\mathbb{Z}}$, and define ($n(i) = i - 1$ for 'many' $i \in S$)

$$\mathbf{Y}_a := \mathbb{C}(Y_{i,aq^{2n+n(i)}}; i \in S, n \in D\mathbb{Z}),$$

$$\mathbf{A}_a := \mathbb{C}(A_{i,aq^{2n+n(i)+D_i}}; i \in S, n \in D\mathbb{Z}) \subset \mathbf{Y}_a.$$

- q-characters on the lattice

Def

$I = \{v^i_n;\ i \in S, n \in D\mathbb{Z}\}$: the vertex set of $Q(\mathfrak{g})$

$$\phi_a : \mathbf{Y}_a \to \mathbb{C}(y) := \mathbb{C}(y_i(n); v^i_n \in I);\ \ Y_{i,aq^{2n+n(i)}} \mapsto y_i(n)$$
$$\beta : \mathbb{C}(\mathbf{X}) \hookrightarrow \mathbb{C}(y);\ \ X^i_n \mapsto \phi_a(A^{-1}_{i,aq^{2n+n(i)+D_i}})$$

$\rightsquigarrow \phi_a(\mathrm{Im}\,\chi_q \cap \mathbf{Y}_a)$ is

$$\mathcal{Y}_{\chi_q} := \bigcap_{i \in S} \mathbb{Z}[y_j(n)^{\pm}; j \neq i, n \in D\mathbb{Z}] \otimes \mathbb{Z}[y_i(n)(1 + X^i_n); n \in D\mathbb{Z}].$$

Prop. [I 20]

β is Poisson map; the Poisson structure on $\mathbb{C}(\mathbf{X})$ (for $Q(\mathfrak{g})$) is compatible with that on $\mathbf{Y}_a$.

• Weyl group action on $\mathbb{C}(y)$

<u>Fact</u> [Frenkel-Mukhin 01]

When q is the root of unity ε, χ_q gives the character map

$$\chi_\varepsilon : \mathrm{Rep}\ U_\varepsilon^{\mathrm{res}}(\hat{\mathfrak{g}}) \to \mathbf{Y} := \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i\in S, a_i\in\mathbb{C}^\times}$$

by setting q equal to ε.

Set $\varepsilon^{2D'm} = 1$ and consider $Q_m(\mathfrak{g})$. $\mathcal{Y}_{\chi_\varepsilon} := \phi_a(\mathrm{Im}\chi_\varepsilon \cap \mathbf{Y}_{a,\varepsilon})$ is $\bigcap_{i\in S} \mathbb{Z}[y_j(n)^\pm; j \neq i, n \in D\mathbb{Z}/D'm\mathbb{Z}] \otimes \mathbb{Z}[y_i(n)(1 + X_n^i); n \in D\mathbb{Z}/D'm\mathbb{Z}]$.

<u>Def</u>

For $i \in S$ and $y_j(n) \in \mathbb{C}(y)$, define $r_i \curvearrowright \mathbb{C}(y)$ by

$$r_i(y_j(n)) = \begin{cases} y_i(n) X^i_{n-D_i} \dfrac{f_X(i, n-2D_i)}{f_X(i, n-D_i)} & j = i, \\ y_j(n) & j \neq i. \end{cases}$$

Thm 4 [I 20]

(i) For $i \in S$, we have a commuting diagram:

$$\begin{tikzcd}
\mathbb{C}(\mathbf{X}) \arrow[r, "\beta"'] \arrow[d, "R_i^*"] & \mathbb{C}(y) \arrow[d, "r_i"] \\
\mathbb{C}(\mathbf{X}) \arrow[r, "\beta"'] & \mathbb{C}(y)
\end{tikzcd}$$

Especially, the r_i ($i \in S$) generate $W(\mathfrak{g})$-action on $\mathbb{C}(y)$.

(ii) $\mathcal{Y}_{\chi_\varepsilon}$ is invariant under the action of $W(\mathfrak{g})$.

- Extension of the action

For generic q, define 'an infinite version' of $f_X(i,n)$:

$$\hat{f}_X(i,n) = 1 + \sum_{k=0}^{\infty} X^i_n X^i_{n-D_i} \cdots X^i_{n-kD_i} \in \mathbb{C}[[\mathbf{X}]]$$

Def For $i \in S$ and $y_j(n) \in \mathbb{C}(y)$, define $\hat{r}_i : \mathbb{C}(y) \to \widehat{\mathbb{C}}_X(y)$ by

$$\hat{r}_i(y_j(n)) = \begin{cases} y_i(n) X^i_{n-D_i} \dfrac{\hat{f}_X(i, n-2D_i)}{\hat{f}_X(i, n-D_i)} & j = i, \\ y_j(n) & j \neq i. \end{cases}$$

$\mathbb{C}[y^{\pm}] := \mathbb{C}[y_i(n), y_i(n)^{-1}; i \in S, n \in D\mathbb{Z}]$

I_X: the ideal of $\mathbb{C}[y^{\pm}]$ generated by X^i_n $(i \in S,\ n \in D\mathbb{Z})$ via β

$\widehat{\mathbb{C}}_X(y)$: the quot field of the completion of $\mathbb{C}[y^{\pm}]$ by $\varprojlim(\mathbb{C}[y^{\pm}]/I_X^k)$

Prop [I 20] $\mathcal{Y}_{\chi_q}$ is invariant under $\hat{r}_i$ $(i \in S)$.

Remarks

$\cdot$ $\hat{r}_i$ is troublesome! We do not know

if $\hat{r}_i$ is realized as a mutation sequence for $Q(\mathfrak{g})$,
if $\hat{r}_i$ generate $W(\mathfrak{g})$.

$\cdot$ Nevertheless we have 'restricted' $W(\mathfrak{g})$-action on tropical X-seed induced from that of $Q_m(\mathfrak{g})$:

$$\hat{R}_i^{\mathrm{trop}\pm} : \mathcal{X}^{i\pm}_{Q(\mathfrak{g})}(\mathbb{P}) \to \mathcal{X}^{i\mp}_{Q(\mathfrak{g})}(\mathbb{P})$$

where $\mathcal{X}^{i+}_{Q(\mathfrak{g})}(\mathbb{P}) := \{\mathbf{x} \in \mathbb{P}^{|I|}; x^i_n > 0\}$, $\mathcal{X}^{i-}_{Q(\mathfrak{g})}(\mathbb{P}) := \{\mathbf{x} \in \mathbb{P}^{|I|}; x^i_n < 0\}$.

$\rightsquigarrow$ $\hat{R}^{\mathrm{trop}+}(w) : \cap_i \mathcal{X}^{i+}_{Q(\mathfrak{g})}(\mathbb{P}) \to \mathcal{X}_{Q(\mathfrak{g})}(\mathbb{P})$ is well defined.

(as $R(w)$ for $Q_m(\mathfrak{g})$ is green)

(Cf.) Braid grp action on the ℓ-integral root lattice [Chari-Moura 05]

Question

What is the $W(\mathfrak{g})$-action (in rep-theoretically) ?

§4 Application to lattice $\mathfrak{g}$-Toda field

• Screening operator $\quad\quad\quad\quad\quad\quad \mathbf{Y} = \mathbb{Z}[Y_{j,a_j}^{\pm 1}]_{j \in S, a_j \in \mathbb{C}^\times}$

$\mathcal{S}_i$: screening operator; $i \in S$

$\mathcal{S}_i : \mathbf{Y} \to \mathbf{YS}_i := \oplus_{b \in \mathbb{C}^\times} \mathbf{Y} \otimes S_{i,b}$; $\; Y_{j,a} \mapsto \delta_{i,j} Y_{i,a} S_{i,a}$ (Leibniz rule)

$\mathbf{YS}_i'$: a quotient of $\mathbf{YS}_i$ with relations $S_{i,aq_i^2} = A_{i,aq_i} S_{i,a}$

Thm [Frenkel-Mukhin 99] $\quad$ $\mathrm{Im}\, \chi_q = \mathrm{Ker} \cap_{i \in S} \mathcal{S}_i$.

$\rightsquigarrow$ lattice version on $Q(\mathfrak{g})$

$\mathbb{C}(s) := \mathbb{C}(s_i(n);\; v_n^i \in I)$

$\mathcal{S}_i : \mathbb{C}(y) \to \mathbb{C}(y)_i := \oplus_{n \in d\mathbb{Z}} \mathbb{C}(y) \otimes s_i(n)$; $\; y_j(n) \mapsto \delta_{i,j} y_i(n) s_i(n)$

$\mathbb{C}(y)_i'$: a quotient of $\mathbb{C}(y)_i$ with relations $s_i(n + D_i) = (X_n^i)^{-1} s_i(n)$

$(\mathcal{Y}_{\chi_q} \subset \mathrm{Ker} \cap_{i \in S} \mathcal{S}_i)$

- The lattice Toda field

Def [IH 00]

Poisson structure on $\mathbb{C}(s)$ ($D_{ij} := \min(D_i, D_j)$):

$$\{s_i(n), s_i(m)\} = s_i(n)s_i(m);\ n \equiv m (\mathrm{mod}\ \ D_i), n < m$$

$$\{s_i(n), s_j(m)\} = -\frac{1}{2} s_i(n)s_j(m);\ C_{ij} \neq 0,\ n \equiv m\,(\mathrm{mod} D_{ij}), n < m$$

$$\{s_i(n), s_j(n)\} = -\frac{1}{2} s_i(n)s_j(n);\ i < j,\ C_{ij} \neq 0$$

$$\mathcal{H}_i := \sum_{n \in D\mathbb{Z}} s_i(n); i \in S \quad \mathcal{H} := \sum_{i \in S} \mathcal{H}_i$$

Prop [I 02]

(i) $\sigma : \mathbb{C}(\mathbf{X}) \to \mathbb{C}(s);\ X_n^i \mapsto \frac{s_i(n)}{s_i(n+D_i)}$ is Poisson.

(ii) It holds that $\mathcal{S}_i \cdot X_n^j = \{\mathcal{H}_i, X_n^j\}$.

Def [IH 02] $\frac{\partial}{\partial t} \log X_n^i := \{\mathcal{H}, \log X_n^i\}$: lattice $\mathfrak{g}$-Toda field equation

$$\frac{\partial}{\partial t}\log X_n^i = \Big(\sum_{j:j<i}\sum_{k=1}^{-C_{ji}} s_j(n+kD_{ij})\Big) - s_i(n) - s_i(n+d_i)$$

$$+\Big(\sum_{j:j>i}\sum_{k=0}^{-C_{ji}-1} s_j(n+kD_{ij})\Big) : \text{lattice } \mathfrak{g}\text{-Toda field eq}$$

$$\longrightarrow \quad \frac{\partial^2}{\partial t\partial z}\log s_i = \sum_{k\in S}\frac{C_{ji}}{d_i}s_j : \mathfrak{g}\text{-Toda field eq}$$

with continuous limit as $\log X_n^i \to -d_i\frac{\partial}{\partial z}\log s_i$

Fact

Let $\tau_i(n); v_n^i \in I$ be the τ-function satisfy

$$D_t\tau_i(n)\cdot\tau_i(n+d_i) = \prod_{j:j<i}\prod_{k=1}^{-C_{ij}}\tau_j(n+kd_{ij})\cdot\prod_{j:j>i}\prod_{k=0}^{-C_{ij}-1}\tau_j(n+kd_{ij}) =: T_i(n).$$

Then $s_i(n) = \frac{T_i(n)}{\tau_i(n)\tau_i(n+d_i)}$ satisfy the lattice $\mathfrak{g}$-Toda eq.

Prop [I 20]

$\phi : \mathbb{C}(\tau) \to \mathbb{C}(\mathbf{A});\ \tau_i(n) \mapsto A^i_{n-d_i}$

$p_\tau : \mathbb{C}(\mathbf{X}) \to \mathbb{C}(\tau);\ X^i_n \mapsto \frac{s_i(n)}{s_i(n+D_i)}$ (with $s_i(n) = \frac{T_i(n)}{\tau_i(n)\tau_i(n+d_i)}$)

Then the following diagram is commutative:

$$\begin{tikzcd} \mathbb{C}(\mathbf{X}) \arrow[r, "p_\tau"] \arrow[dr, "p^*"'] & \mathbb{C}(\tau) \arrow[d, "\phi"] \\ & \mathbb{C}(\mathbf{A}) \end{tikzcd}$$

Especially, A-var for $Q(\mathfrak{g})$ are the τ-function for the lattice $\mathfrak{g}$-Toda eq.

Thank you!