Tame algebras have dense g-vector fans

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Motivation

- Λ : a finite dimensional algebra over an algebraic closed field k.
- $K^b(\operatorname{proj}\Lambda)$: the homotopy category of bounded complexes of finitely generated projective right Λ-modules.

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- Λ : a finite dimensional algebra over an algebraic closed field k.
- $K^b(\operatorname{proj}\Lambda)$: the homotopy category of bounded complexes of finitely generated projective right Λ-modules.

A 2-term presilting object in $K^b(\operatorname{proj}\Lambda)$ has a numerical invariant, g-vector, in $\mathbb{Z}^n$ ($\simeq$ Grothendieck group of $K^b(\operatorname{proj}\Lambda)$).
There is a simplicial polyhedral fan $\mathcal{F}^{\mathbf{g}}(\Lambda)$, g-vector fan, whose

- ray is generated by the $\mathbf{g}$-vector of an indecomposable 2-term presilting object;
- maximal cone is generated by the $\mathbf{g}$-vectors of direct summands of a 2-term silting object.

In this talk, we identify $\mathcal{F}^{\mathbf{g}}(\Lambda)$ with its geometric realization.

Relation to other subjects

(1) There are bijections between the following objects:

- Iso. classes of basic 2-term silting objects in $K^b(\operatorname{proj}\Lambda)$;
- Iso. classes of basic support τ-tilting modules in $\operatorname{mod}\Lambda$;
- Functorially finite torsion classes in $\operatorname{mod}\Lambda$;
- Iso. classes of 2-term simple-minded collections in $\mathcal{D}^b(\operatorname{mod}\Lambda)$;
- Intermediate t-structures with length heart in $\mathcal{D}^b(\operatorname{mod}\Lambda)$...

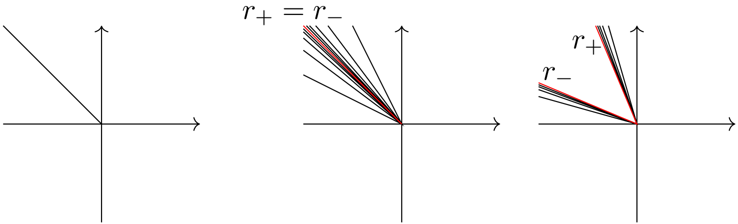
(2) The g-vector fans are related to various subjects:

- normal fans of generalized associahedrons [Chapoton et al., 2002];
- Cambrian fans [Reading and Speyer, 2009];
- tropical cluster $\mathcal{X}$-variety [Fock and Goncharov, 2009];
- cluster/stability scattering diagrams [Bridgeland, 2017, Gross et al., 2018]...

Let $m \in \mathbb{Z}_{\geq 1}$ and K_m be an m-Kronecker quiver, that is,

$$K_m := [\ 1 \overset{m}{\underset{\vdots}{\rightrightarrows}} 2\ ].$$

In particular, K_1 is of type A_2 and K_2 is a Kronecker quiver. The $\mathbf{g}$-vector fan $\mathcal{F}^{\mathbf{g}}(kK_m)$ is well known as follows:



$\mathcal{F}^{\mathbf{g}}(kK_1)$ $\mathcal{F}^{\mathbf{g}}(kK_2)$ $\mathcal{F}^{\mathbf{g}}(kK_{\geq 3})$

For $m \geq 2$, $\mathcal{F}^{\mathbf{g}}(kK_m)$ contains infinitely many rays converging to the rays $r_{\pm}$. If $m = 2$, then $r_+ = r_-$. If $m \geq 3$, then $r_+ \neq r_-$ and the interior of the cone spanned by r_+ and r_- is the complement of the closure $\overline{\mathcal{F}^{\mathbf{g}}(kK_m)}$.

Theorem ([Asai, 2019, Demonet et al., 2019])

The following are equivalent:
(1) $\mathcal{F}^{\mathbf{g}}(\Lambda) = \mathbb{R}^n$; *(2)* $\#\{2\text{-}\textit{term (pre)silting objects for } \Lambda\} < \infty$.

Theorem ([Asai, 2019, Demonet et al., 2019])

The following are equivalent:
(1) $\mathcal{F}^{\mathbf{g}}(\Lambda) = \mathbb{R}^n$*;* *(2)* $\#\{2$*-term (pre)silting objects for* $\Lambda\} < \infty$*.*

This naturally leads the following question.

Question

When does Λ satisfy $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)} = \mathbb{R}^n$?

Today's talk

1. g-vector fans and main theorem
2. Two ingredients of proof
3. Sketch of proof
4. Application to cluster algebras

Notations

- $\Lambda = \bigoplus_{i=1}^{n} P_i$: a decomposition of Λ as direct sum of pairwise non-isomorphic indecomposable projective right Λ-modules.
- $K_0(\operatorname{proj}\Lambda)$: the Grothendieck group of $K^b(\operatorname{proj}\Lambda)$
- $[X]$: the image of an object X in $K_0(\operatorname{proj}\Lambda)$

Then $K_0(\operatorname{proj}\Lambda)$ is a free abelian group with basis $[P_1], \ldots, [P_n]$, thus it gives $K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$.

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- $K^{[-1,0]}(\operatorname{proj}\Lambda)$: the full subcategory of $K^b(\operatorname{proj}\Lambda)$ whose objects are complexes concentrated in degrees -1 and 0, that is $P = P^{-1} \xrightarrow{f} P^0$. We identify P with $f \in \operatorname{Hom}_\Lambda(P^{-1}, P^0)$.

Definition

The g-vector of $P \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ is $[P] \in K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$.

2-term silting complexes and g-vector fan

Definition

An object $X \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ is presilting if $\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(X, \Sigma X) = 0$, where Σ is the shift functor. It is silting if, moreover, it generates $K^b(\operatorname{proj}\Lambda)$.

$2\text{-silt}\,\Lambda = \{$iso. classes of basic silting objects in $K^{[-1,0]}(\operatorname{proj}\Lambda)\}$

Theorem ([Adachi et al., 2014])

There is a simplicial polyhedral fan $\mathcal{F}^{\mathbf{g}}(\Lambda)$ whose

- *ray is generated by the* g*-vectors of an indecomposable presilting object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$;*
- *maximal cone is a positive cone generated by $[S_1], \ldots, [S_n]$ for $\bigoplus_{i=1}^{n} S_i \in 2\text{-silt}\,\Lambda$.*

$\mathcal{F}^{\mathbf{g}}(\Lambda)$: the (2-term silting) g-vector fan of Λ

g-tame algebras

Definition

The algebra Λ is g-tame if $\overline{\mathcal{F}^{\mathbf{g}}(\Lambda)} = \mathbb{R}^n$.

Note that the g-tameness is already known for

- path algebras of extended Dynkin quivers [Hille, 2006];
- Jacobian algebras associated with triangulated surfaces [Y, 2020];
- complete preprojective algebras of extended Dynkin graphs [Kimura and Mizuno, 2019];
- complete special biserial algebras [Aoki and Y, 2020].

Main theorem

Definition (1970s)

The algebra Λ is tame if for any dimension vector $\mathbf{d}$, there are $k[t]$-Λ-bimodules $M_1, \ldots, M_{m(\mathbf{d})}$ such that

(1) each M_i is free of finite rank as a $k[t]$-module;

(2) all but finitely many indecomposable Λ-modules of dimension vector $\mathbf{d}$ have the form

$$k[t]/(t-\lambda) \otimes_{k[t]} M_i$$

with $i \in \{1, \ldots, m(\mathbf{d})\}$ and $\lambda \in k$.

Main theorem

Tame algebras are g-tame.

Remark that there is a non-tame algebra which is g-tame.

Two ingredients of proof for the main theorem

1. Generic decomposition (notation)

For $\mathbf{g} \in K_0(\operatorname{proj}\Lambda)$, let $P^{\mathbf{g}+}$ and $P^{\mathbf{g}-}$ be the unique finitely generated projective modules without common non-zero direct summands such that $\mathbf{g} = [P^{\mathbf{g}+}] - [P^{\mathbf{g}-}]$.

Let $\mathbf{g}, \mathbf{g}' \in K_0(\operatorname{proj}\Lambda)$. We denote by $e(\mathbf{g}, \mathbf{g}')$ the minimal value of

$$\dim \operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(P, \Sigma P'),$$

where $P, P' \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $[P] = \mathbf{g}$ and $[P'] = \mathbf{g}'$.

1. Generic decomposition

Theorem ([Derksen and Fei, 2015],[Plamondon, 2013])

Any $\mathbf{g} \in K_0(\operatorname{proj} \Lambda)$ *can be written as*

$$\mathbf{g} = \mathbf{g}_1 + \ldots + \mathbf{g}_r,$$

where for each $i, j \in \{1, \ldots, r\}$ *with* $i \neq j$,

(1) *a general element of* $\operatorname{Hom}_\Lambda(P^{(\mathbf{g}_i)_-}, P^{(\mathbf{g}_i)_+})$ *is indecomposable;*

(2) $e(\mathbf{g}_i, \mathbf{g}_j) = 0$.

Moreover, $\mathbf{g}_1, \ldots, \mathbf{g}_r$ *are unique for these properties.*

The decomposition in Theorem is the generic decomposition of $\mathbf{g}$.

1. Generic decomposition : tame algebras

Theorem ([Geiss et al., 2020])

Let Λ be a tame algebra, and let $\mathbf{g} \in K_0(\operatorname{proj} \Lambda)$. Then the generic decomposition of $\mathbf{g}$ has the form

$$\mathbf{g} = \mathbf{g}_1 + \ldots + \mathbf{g}_r + \mathbf{h}_1 + \ldots + \mathbf{h}_s,$$

where $r, s \geq 0$ and

(1) *a general element of $\operatorname{Hom}_\Lambda(P^{(\mathbf{g}_i)-}, P^{(\mathbf{g}_i)+})$ is presilting;*

(2) *there is a dense open subset $\mathcal{U}$ of $\operatorname{Hom}_\Lambda(P^{(\mathbf{h}_j)-}, P^{(\mathbf{h}_j)+})$ such that the cokernels of morphisms in $\mathcal{U}$ are indecomposable Λ-modules which are bricks and are isomorphic to their own Auslander-Reiten translate.*

2. Cylinders

To save space, we use the notations

$$\mathrm{Hom}_{K^b(\mathrm{proj}\,\Lambda)}(X,Y) = \mathrm{Hom}(X,Y) = (X,Y).$$

Definition

For $U, X \in K^b(\mathrm{proj}\,\Lambda)$, we choose a basis $(f_1, \ldots, f_d)$ of the space $\mathrm{Hom}(U,X)$ and a triangle

$$\Sigma^{-1}X^d \to \mathrm{Cyl}_X\, U \to U \xrightarrow{f} X^d, \text{ where } f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_d \end{bmatrix}.$$

The object $\mathrm{Cyl}_X\, U$ is the cylinder of U with respect to X.

The cylinder is only defined up to isomorphism.

Commuting cylinders

Lemma (Commuting cylinders)

Let X and Y be non-isomorphic indecomposable objects of $K^b(\operatorname{proj}\Lambda)$, and let $U \in K^b(\operatorname{proj}\Lambda)$. We assume that the following hold:

(1) $\operatorname{Hom}(X, \Sigma Y) = \operatorname{Hom}(Y, \Sigma X) = 0$;

(2) for any $\phi \in \operatorname{Hom}(U, X)$ and $\psi \in \operatorname{Hom}(X, Y)$, then $\psi\phi = 0$;

(3) for any $\phi' \in \operatorname{Hom}(U, Y)$ and $\psi' \in \operatorname{Hom}(Y, X)$, then $\psi'\phi' = 0$.

Then $\operatorname{Cyl}_X \operatorname{Cyl}_Y U \simeq \operatorname{Cyl}_Y \operatorname{Cyl}_X U$.

Sketch of proof for (Commuting cylinders)

The triangles defining $\mathrm{Cyl}_X U$ and $\mathrm{Cyl}_Y U$ are

$$\Sigma^{-1}X^d \to \mathrm{Cyl}_X U \xrightarrow{x} U \xrightarrow{f} X^d\ , \quad \Sigma^{-1}Y^e \to \mathrm{Cyl}_Y U \xrightarrow{y} U \xrightarrow{g} Y^e.$$

By (1) (2), $\mathrm{Hom}(U,Y) \xrightarrow{x^*} \mathrm{Hom}(\mathrm{Cyl}_X U, Y)$ is an isomorphism.
By (1) (3), $\mathrm{Hom}(U,X) \xrightarrow{y^*} \mathrm{Hom}(\mathrm{Cyl}_Y U, X)$ is an isomorphism.
Thus, by the octahedral axiom, there is a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Cyl}_Y U \arrow[r, equal] \arrow[d, "y"] & \mathrm{Cyl}_Y U \arrow[d, "fy"] \\
\mathrm{Cyl}_X U \arrow[r, "x"] \arrow[d, equal] & U \arrow[r, "f"] \arrow[d, "g"] & X^d \arrow[d] \\
\mathrm{Cyl}_X U \arrow[r, "gx"] & Y^e \arrow[r] & ?
\end{tikzcd}$$

$\leftarrow$ defining $\mathrm{Cyl}_Y \mathrm{Cyl}_X U$

$\uparrow$ defining $\mathrm{Cyl}_X \mathrm{Cyl}_Y U$

g-vectors of cylinders

Lemma

Let H be an indecomposable object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $\operatorname{Hom}(H,\Sigma H)$ is one-dimensional, and let $U \in K^{[-1,0]}(\operatorname{proj}\Lambda)$. Then $\operatorname{Cyl}^m_{\Sigma H} U$ is in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ for any $m \in \mathbb{Z}_{>0}$, and

$$[\operatorname{Cyl}^m_{\Sigma H} U] = [U] + md[H],$$

where $d = \dim \operatorname{Hom}(U,\Sigma H)$.

Proof By the triangle $H^d \to \operatorname{Cyl}_{\Sigma H} U \to U \to \Sigma H^d$, we have $[\operatorname{Cyl}_{\Sigma H} U] = [U] + d[H]$ and

$$(\Sigma H^d,\Sigma H) \twoheadrightarrow (U,\Sigma H) \to (\operatorname{Cyl}_{\Sigma H} U,\Sigma H) \to (H^d,\Sigma H) \to 0,$$

thus $\dim(\operatorname{Cyl}_{\Sigma H} U,\Sigma H) = d$. Repeating the cylinder with respect to ΣH, the desired equality is obtained.

g-vectors of cylinders

Lemma (g-vectors of cylinders)

Let $H_1, \ldots, H_s$ be indecomposable objects of $K^{[-1,0]}(\operatorname{proj} \Lambda)$ such that

- for each $i \in \{1, \ldots, s\}$, $\operatorname{Hom}(H_i, \Sigma H_i)$ is one-dimensional;
- for each pair of disctinct $i, j \in \{1, \ldots, s\}$, the objects $X = \Sigma H_i$ and $Y = \Sigma H_j$ satisfy the hypotheses of (Commuting cylinders) for any $U \in K^{[-1,0]}(\operatorname{proj} \Lambda)$.

Let $d_i = \dim \operatorname{Hom}(U, \Sigma H_i)$, and $a_1, \ldots, a_s \in \mathbb{Z}_{>0}$. Then $\operatorname{Cyl}^{a_s}_{\Sigma H_s} \cdots \operatorname{Cyl}^{a_1}_{\Sigma H_1} U$ is in $K^{[-1,0]}(\operatorname{proj} \Lambda)$, and

$$[\operatorname{Cyl}^{a_s}_{\Sigma H_s} \cdots \operatorname{Cyl}^{a_1}_{\Sigma H_1} U] = [U] + \sum_{i=1}^{s} a_i d_i [H_i].$$

Presilting cylinders

Lemma (Presilting cylinders)

Let H be an indecomposable object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $\operatorname{Hom}(H,\Sigma H)$ is one-dimensional, and let $U \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfying the following:

(1) U is presilting (i.e. $\operatorname{Hom}(U,\Sigma U)=0$);

(2) $\operatorname{Hom}(H,\Sigma U)=0$;

(3) for any non-zero $g \in \operatorname{Hom}_{\mathcal{D}\Lambda}(\Sigma H,\nu H)$ the induced morphism

$$\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(U,\Sigma H) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{D}\Lambda}(U,\nu H)$$

is injective, where $\nu = - \otimes^L_\Lambda D\Lambda$ is the Nakayama functor.

Then $\operatorname{Cyl}_{\Sigma H} U$ is in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ and also satisfies (1)–(3).

Sketch of proof for (Presilting cylinders) : only (1) and (2)

Let $\operatorname{Cyl}_{\Sigma H} U \to U \xrightarrow{f} \Sigma H^d$ be the triangle defining $\operatorname{Cyl}_{\Sigma H} U$.
There is a commutative diagram:

$$\begin{tikzcd}
 & & (C,U) \arrow[r] \arrow[d] & (H^d,U) \arrow[r] \arrow[d] & (\Sigma^{-1}U,U) \arrow[d] \\
(\Sigma H^d,\Sigma H^d) \arrow[r, "f^*"] \arrow[d] & (U,\Sigma H^d) \arrow[r] \arrow[d] & (C,\Sigma H^d) \arrow[r] \arrow[d] & (H^d,\Sigma H^d) \arrow[r] \arrow[d] & (\Sigma^{-1}U,\Sigma H^d) \arrow[d] \\
(\Sigma H^d,\Sigma C) \arrow[r] \arrow[d] & (U,\Sigma C) \arrow[r] \arrow[d] & (C,\Sigma C) \arrow[r] \arrow[d] & (H^d,\Sigma C) \arrow[r] \arrow[d] & (\Sigma^{-1}U,\Sigma C) \arrow[d] \\
(\Sigma H^d,\Sigma U) \arrow[r] & (U,\Sigma U) \arrow[r] & (C,\Sigma U) \arrow[r] & (H^d,\Sigma U) \arrow[r] & (\Sigma^{-1}U,\Sigma U)
\end{tikzcd}$$

where $C = \operatorname{Cyl}_{\Sigma H} U$.

Sketch of proof for (Presilting cylinders) : only (1) and (2)

Let $\operatorname{Cyl}_{\Sigma H} U \to U \xrightarrow{f} \Sigma H^d$ be the triangle defining $\operatorname{Cyl}_{\Sigma H} U$.
There is a commutative diagram:

$$
\begin{tikzcd}
 & & (C,U) \arrow[r] \arrow[d] & (H^d,U) \arrow[r] \arrow[d, two heads, "(3)"] & 0 \arrow[d] \\
(\Sigma H^d,\Sigma H^d) \arrow[r, two heads, "f^*"] \arrow[d] & (U,\Sigma H^d) \arrow[r] \arrow[d] & (C,\Sigma H^d) \arrow[r] \arrow[d] & (H^d,\Sigma H^d) \arrow[r] \arrow[d] & 0 \arrow[d] \\
(\Sigma H^d,\Sigma C) \arrow[r] \arrow[d] & (U,\Sigma C) \arrow[r] \arrow[d] & (C,\Sigma C) \arrow[r] \arrow[d] & (H^d,\Sigma C) \arrow[r] \arrow[d] & 0 \arrow[d] \\
(\Sigma H^d,\Sigma U) \arrow[r] & 0 \arrow[r, "(1)"] & (C,\Sigma U) \arrow[r] & 0 \arrow[r, "(2)"] & 0
\end{tikzcd}
$$

where $C = \operatorname{Cyl}_{\Sigma H} U$. Thus C satisfies (1) and (2).

Sketch of proof for the main theorem

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Let Λ be a tame algebra with $K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$ ($\mathcal{F}^{\mathbf{g}}(\Lambda) \subseteq \mathbb{R}^n$).
We only need to prove $\mathbf{g} \in \overline{\mathcal{F}^{\mathbf{g}}(\Lambda)}$ for any $\mathbf{g} \in \mathbb{Z}^n$.

Sketch of proof for the main theorem

Let Λ be a tame algebra with $K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$ $(\mathcal{F}^{\mathbf{g}}(\Lambda) \subseteq \mathbb{R}^n)$. We only need to prove $\mathbf{g} \in \overline{\mathcal{F}^{\mathbf{g}}(\Lambda)}$ for any $\mathbf{g} \in \mathbb{Z}^n$. We consider the generic decomposition

$$\mathbf{g} = \mathbf{g}_1 + \ldots + \mathbf{g}_r + a_1\mathbf{h}_1 + \ldots + a_s\mathbf{h}_s,$$

where $\mathbf{h}_i \neq \mathbf{h}_j$ for $i \neq j$ and $a_i \in \mathbb{Z}_{>0}$. Then

- there is a presilting object G of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $\mathbf{g}$-vector $[G] = \mathbf{g}_1 + \ldots + \mathbf{g}_r$.
- there are indecomposable objects $H_i \in K^{[-1,0]}(\operatorname{proj}\Lambda)$ with $\mathbf{g}$-vector $\mathbf{h}_i$ such that $H_1, \ldots, H_s$ satisfy the hypotheses of ($\mathbf{g}$-vectors of cylinders).

If $s = 0$, then $\mathbf{g} = [G] \in \mathcal{F}^{\mathbf{g}}(\Lambda)$ and there is nothing to prove. Assume that $s > 0$, then G is not silting.

Let G' be its *Bongartz co-completion*, defined by the triangle

$$\Lambda \to G'' \to G' \to \Sigma\Lambda,$$

where the left-most morphism is a left $(\operatorname{add} G)$-approximation of Λ. Then $G \oplus G' \in 2\text{-silt}\,\Lambda$.

Lemma

Taking $U = G'$ and $H = H_i$ for $i \in \{1, \ldots, s\}$, conditions (1)–(3) of (Presilting cylinders) are satisfied. Also, $\mathrm{Hom}(G', \Sigma H_i) \neq 0$.

For each $i \in \{1, \ldots, s\}$, let $d_i = \dim \mathrm{Hom}(G', \Sigma H_i) \neq 0$. Let $d = \prod_{i=1}^{s} d_i$, and let $e_i = \frac{d}{d_i}$ for each i.
In the same way as (Presilting cylinders), we get that

$$\mathrm{Cyl}_{\Sigma H_s}^{a_s e_s} \cdots \mathrm{Cyl}_{\Sigma H_1}^{a_1 e_1} G'$$

is a presilting object of $K^{[-1,0]}(\operatorname{proj}\Lambda)$.

Moreover, it is easy to show that

$$G^{\oplus d} \oplus \mathrm{Cyl}^{a_s e_s}_{\Sigma H_s} \cdots \mathrm{Cyl}^{a_1 e_1}_{\Sigma H_1} G'$$

is presilting. Since $H_1, \ldots, H_s$ satisfy the hypotheses of ($\mathbf{g}$-vectors of cylinders), we get

$$\begin{aligned}
[G^{\oplus d} \oplus \mathrm{Cyl}^{a_s e_s}_{\Sigma H_s} \cdots \mathrm{Cyl}^{a_1 e_1}_{\Sigma H_1} G'] &= d[G] + [G'] + \sum_{i=1}^{s} a_i e_i d_i [H_i] \\
&= d([G] + \sum_{i=1}^{s} a_i [H_i]) + [G'] \\
&= d\mathbf{g} + [G'].
\end{aligned}$$

Similarly, for any $m \in \mathbb{Z}_{>0}$, we have that

$$G^{\oplus dm} \oplus \mathrm{Cyl}_{\Sigma H_s}^{ma_s e_s} \cdots \mathrm{Cyl}_{\Sigma H_1}^{ma_1 e_1} G'$$

is a presilting object with $\mathbf{g}$-vector $md\mathbf{g} + [G']$. Thus

$$\mathbf{g} \in \bigcup_{m=1}^{\infty} \mathbb{R}_{>0}(md\mathbf{g} + [G']).$$

Since each $md\mathbf{g} + [G']$ is the $\mathbf{g}$-vector of a presilting objects, these vectors are in the fan $\mathcal{F}^{\mathbf{g}}(\Lambda)$. Thus $\mathbf{g} \in \overline{\mathcal{F}^{\mathbf{g}}(\Lambda)}$. This finishes the proof of the main theorem.

Application to cluster algebras

g-vectors of cluster algebras

Q : a quiver without loops and 2-cycles
$\mathcal{A}(Q)$: the cluster algebra associated with Q

Mutation of quivers

For a quiver R, the mutation $\mu_k R$ at a vertex k is a quiver obtained from R by the following steps:

(1) For any path $i \to k \to j$, add an arrow $i \to j$;
(2) Reverse all arrows incident to k;
(3) Remove a maximal set of disjoint 2-cycles.

Q^{prin} : the quiver obtained by adding a vertex i' and an arrow $i' \to i$ for every vertex i of Q
$(\mathbf{e}_1, \ldots \mathbf{e}_n)$: the standard basis of $\mathbb{Z}^n$
$\left(Q^{\mathrm{prin}}, (\mathbf{e}_1, \ldots, \mathbf{e}_n)\right)$: the initial g-vector seed of $\mathcal{A}(Q)$

g-vectors of cluster algebras

Definition-Proposition ([Fomin and Zelevinsky, 2007])

All $\mathbf{g}$*-vector seeds of* $\mathcal{A}(Q)$ *are obtained from* $\big(Q^{\mathrm{prin}}, (\mathbf{e}_1, \dots, \mathbf{e}_n)\big)$ *by the following mutation rule:*

For a $\mathbf{g}$*-vector seed* $\big(R, (\mathbf{g}_1, \dots, \mathbf{g}_n)\big)$*, the mutation* $\mu_k\big(R, (\mathbf{g}_1, \dots, \mathbf{g}_n)\big) = \big(\mu_k R, (\mathbf{g}'_1, \dots, \mathbf{g}'_n)\big)$ *at* $k \in \{1, \dots, n\}$ *is also a* $\mathbf{g}$*-vector seed, where*

$$\mathbf{g}'_\ell = \begin{cases} \mathbf{g}_\ell & \text{if } \ell \neq k; \\ -\mathbf{g}_k + \sum_{i=1}^n [b_{ik}]_+ \mathbf{g}_i - \sum_{j=1}^n [b_{jk}]_+ (b_{ij})_{i=1}^n & \text{if } \ell = k, \end{cases}$$

with $b_{ij} = \#\{i \to j \text{ in } R\} - \#\{j \to i \text{ in } R\}$, $[z]_+ = \max(z, 0)$.

The vectors $\mathbf{g}_i$ *in* $\mathbf{g}$*-vector seeds are the* $\mathbf{g}$*-vectors of* $\mathcal{A}(Q)$.

Cluster $\mathbf{g}$-vector fan

Theorem ([Derksen et al., 2010])

There is a simplicial polyhedral fan $\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$ whose

- *ray is generated by a $\mathbf{g}$-vector;*
- *maximal cone is generated by all $\mathbf{g}$-vectors in a $\mathbf{g}$-vector seed.*

$\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$: the cluster $\mathbf{g}$-vector fan of Q

Cluster g-vector fan

Theorem ([Derksen et al., 2010])

There is a simplicial polyhedral fan $\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$ *whose*

- *ray is generated by a* $\mathbf{g}$*-vector;*
- *maximal cone is generated by all* $\mathbf{g}$*-vectors in a* $\mathbf{g}$*-vector seed.*

$\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$: the cluster g-vector fan of Q

Definition

We say that Q is

- cluster-g-dense if $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)} = \mathbb{R}^n$;
- half cluster-g-dense if $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)}$ and $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q^{op})}$ are closed half-spaces in $\mathbb{R}^n$.

Mutation-finite quivers

We say that Q is

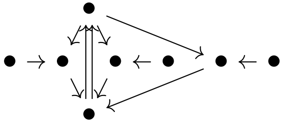
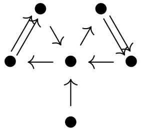
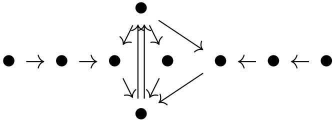
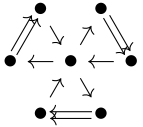
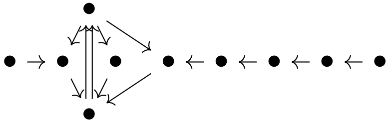
- mutation equivalent to Q' if Q is obtained from Q' by a sequence of mutations;
- mutation-finite if there are only finitely many quivers mutation equivalent to Q.

Theorem ([Felikson et al., 2012])

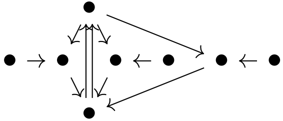
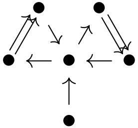
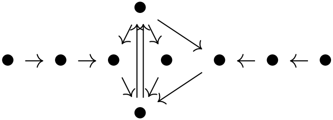
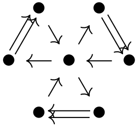
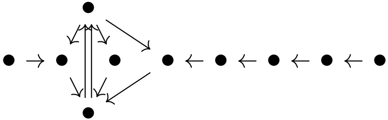
A mutation-finite quiver Q is one of the followings:

- *an m-Kronecker quiver K_m with $m \geq 3$;*
- *a quiver defined from a triangulated surface [Fomin et al., 2008];*
- *a quiver mutation equivalent to one of the quivers E_i, $\tilde{E}_i$, $E_i^{(1,1)}$, X_6 and X_7 for $i \in \{6, 7, 8\}$.*

Mutation-finite quivers

$E_6^{(1,1)}$		X_6	
$E_7^{(1,1)}$		X_7	
$E_8^{(1,1)}$			

Mutation-finite quivers

$E_6^{(1,1)}$		X_6	
$E_7^{(1,1)}$		X_7	
$E_8^{(1,1)}$			

Lemma ([Muller, 2016])

If Q is not mutation-finite, then Q is neither cluster-$\mathbf{g}$-dense nor half cluster-$\mathbf{g}$-dense.

Additive categorification of cluster algebras

The mutation of objects in $2\text{-silt}\,\Lambda$ is also defined.

- $2\text{-silt}^+\,\Lambda \subseteq 2\text{-silt}\,\Lambda$: the subset consisting of objects obtained from Λ by sequences of mutations.
- $2\text{-silt}^-\,\Lambda \subseteq 2\text{-silt}\,\Lambda$: the subset consisting of objects obtained from $\Sigma\Lambda$ by sequences of mutations.

They induce the subfans $\mathcal{F}^{\mathbf{g}}_{+}(\Lambda)$ and $\mathcal{F}^{\mathbf{g}}_{-}(\Lambda)$ of $\mathcal{F}^{\mathbf{g}}(\Lambda)$, respectively.

A potential W of Q is a linear combination of cycles in Q. A non-degenerate potential W of Q defines a Jacobian algebra $J(Q,W)$ [Derksen et al., 2008]. The potential W is Jacobi-finite if $J(Q,W)$ is finite dimensional.

Additive categorification of cluster algebras

Theorem (Additive categorification of cluster algebras)

Let Q be a quiver without loops and 2-cycles. Let W be a non-degenerate Jacobi-finite potential of Q.

(1) *There is a bijection*

$$2\text{-silt}^+ J(Q,W) \leftrightarrow \{\mathbf{g}\text{-}\mathit{vector\ seeds\ of\ } Q\}$$

commuting with mutations, and $\mathcal{F}^{\mathbf{g}}_+(J(Q,W)) = \mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$.

(2) *There is a bijection*

$$2\text{-silt}^- J(Q,W) \leftrightarrow \{\mathbf{g}\text{-}\mathit{vector\ seeds\ of\ } Q^{\text{op}}\}$$

commuting with mutations, and $\mathcal{F}^{\mathbf{g}}_-(J(Q,W)) = -\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q^{\text{op}})$.

Additive categorification : mutation-finite case

Theorem ([Geiss et al., 2016])

A quiver Q is a mutation-finite one that is not mutation equivalent to one of the quivers X_6, X_7 and K_m with $m \geq 3$ if and only if there is a non-degenerate Jacobi-finite potential W of Q such that $J(Q,W)$ is tame.

In this case, our main theorem implies that $\overline{\mathcal{F}^{\mathbf{g}}(J(Q,W))} = \mathbb{R}^n$.

Additive categorification : mutation-finite case

Theorem ([Barot et al., 2010], [Buan et al., 2006], [Y, 2020])

Suppose that Q is mutation-finite except for mutation equivalence classes of X_6 and X_7. Let W be a non-degenerate Jacobi-finite potential of Q and $J = J(Q,W)$. Then

(1) *if Q is not defined from a closed surface with exactly one puncture, then $2\text{-silt}\, J = 2\text{-silt}^+ J$ and thus* $\mathcal{F}^{\mathbf{g}}(J) = \mathcal{F}^{\mathbf{g}}_+(J) = \mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)$;

(2) *otherwise, $2\text{-silt}\, J = 2\text{-silt}^+ J \sqcup 2\text{-silt}^- J$ and thus* $\mathcal{F}^{\mathbf{g}}(J) = \mathcal{F}^{\mathbf{g}}_+(J) \sqcup \mathcal{F}^{\mathbf{g}}_-(J) = \mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q) \sqcup (-\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q^{\text{op}}))$.

Therefore, we get

(1) $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)} = \mathbb{R}^n$, that is, Q is cluster-g-dense;

(2) $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q) \sqcup (-\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q^{\text{op}}))} = \mathbb{R}^n$. In fact, it was given in [Y, 2020] that $\overline{\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(Q)}$ is a closed half-space in $\mathbb{R}^n$, that is, Q is half cluster-g-dense.

Classification of (half) cluster-$\mathbf{g}$-dense quivers

Corollary

Suppose that Q is not mutation equivalent to one of the quivers X_6, X_7 and K_m with $m \geq 3$. Then

- *Q is cluster-$\mathbf{g}$-dense or half cluster-$\mathbf{g}$-dense if and only if it is mutation-finite;*
- *it is half cluster-$\mathbf{g}$-dense if and only if it is defined from a closed surface with exactly one puncture.*

On the other hand, K_m is not (half) cluster-$\mathbf{g}$-dense for $m \geq 3$.

Conjecture for X_6 and X_7

- [Mills, 2017] $\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(X_6) = -\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(X_6^{\text{op}})$;
- [Seven, 2014] $\mathcal{F}^{\mathbf{g}}_{\text{cluster}}(X_7)$ is contained in some open half-space in $\mathbb{R}^n$.

Therefore, the following seems natural.

Conjecture

(1) The quiver X_6 is cluster-g-dense.

(2) The quiver X_7 is half cluster-g-dense.

Remark that the Jacobian algebras associated with X_6 and X_7 are not tame [Geiss et al., 2016].

Thank you for your attention!

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