Background

§ 昔々 $\mathbb{C}$

$\Downarrow$

$q = \exp\left(\frac{2\pi\sqrt{-1}}{e}\right)$

Kazhdan-Lusztig $\mathcal{O}$ $\quad \mathcal{O}_{KL}(\widehat{\mathfrak{gl}_n})$

$\wr|\ \boxed{KL}$ eq (as $\otimes$ cat.) $\quad \dot{\otimes} \leftrightarrow \otimes$

divided power Lusztig $\quad U_q^L(\mathfrak{gl}_n)\text{-mod} \simeq \left\langle \rightsquigarrow, \det_n^{\otimes k} \right\rangle_{\otimes}$, $k \in \mathbb{Z}_{<0}$

$\cup$

Recollement setting

Minamoto talk

polynomial rep. subcat $\quad \bigoplus_{r\geq 0} S_q(n,r)\text{-mod}$

Green

Schur functor

q-Schur

$\mathrm{End}_{\mathcal{H}_q(r)}(V_n^{\otimes r})$

$\uparrow \quad \rotatebox{90}{$\subset$}$

$U_q^L(\mathfrak{gl}_n) \to \mathrm{Im}\,\tau$

(τ, V_n): nec. rep. $/U_q^L$

$\mathcal{H}_q(r) =$ Hecke alg w/ q & $\mathfrak{S}_r$

rank ∞ stable limit

$S_q(m,r)$-mod

$\wr|$

$S_q(r,r)$-mod

$\forall m \geq r$

Ariki's talk

$\bigoplus_{r\geq 0} S_q(r,r)\text{-mod}$ level 1

$\wr|\ \boxed{\text{Rouquier}}$

$\mathcal{O}_{\frac{1}{e}}(\mathfrak{S}_r)$

cat. $\mathcal{O}$ of rational double affine Hecke alg (DAHA) assoc w/ $\mathfrak{S}_r$ & $c = (c_{ij}) \longleftrightarrow \frac{1}{e}$ ref'n

Decategorify

$K_0(\,-\,)\otimes\mathbb{C} = [\,-\,]$

$[\text{simple}] = (\text{canonical}) \text{ base}$ — $\boxed{\text{computable!}}$ KL algo

Kazhdan-Lusztig parabolic

$[\mathcal{O}_{KL}(\widehat{\mathfrak{sl}}_n)] \rightleftharpoons$ affine Hecke alg a.w. $\mathbb{Z} \wr S_n$

$\|$

$[\qquad -\text{mod}]$

$\cup$

$\boxed{\text{contragradient dual "}\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})\text{" で定まる}}$ f

[orderがあるので leading term にあたるものを取り出す]

$[\bigoplus \qquad]$

level 1

で決まる

gradingが入っていることが肝.

$\left[\bigoplus_{r\geq 0} S_q(r,r)\text{-mod}\right] =$ Fock rep. / affine $\widehat{\mathfrak{gl}}_e$

$\|$

$\left[\bigoplus_r \mathcal{O}_{\frac{1}{e}}(RS_r)\right]$

$[\text{simple}] = \underset{\sim\sim\sim}{\text{canonical base}} / U_v(\widehat{\mathfrak{gl}}_e)$

(VV using KT)

another route

(Ariki + Leclerc Schiffmann)

Leclerc-Thibon conjecture

$\boxed{\text{computable! LLT algo}}$

ハミヂ法 と Heisenberg action の renormalize

[小休止

e は Fong-Srinivasan '80s からの記号 (皆様生前)

それは (q modulo p) の位数

〜素数べき]

もっと $e=p$ and 1

$$G = GL_n(\overline{\mathbb{F}}_p)$$

Lusztig conjecture

p > explicit bound $\Rightarrow$ [$K_0(G\text{-mod}) \simeq K_0(U_q^L(\mathfrak{gl}_n)\text{-mod})$

$\boxed{[\mathrm{Irr}]}$ $\longleftarrow$ KL base] ?

[Andersen – Jantzen – Soergel

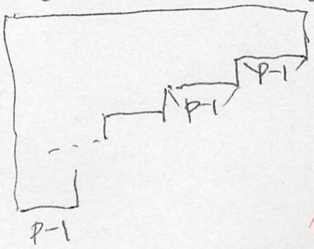
$p \gg 0$ 十分大 $\Rightarrow$ true]

[Fiebig $p >$ exponential bound but explicit $\Rightarrow$ true] ← ☆

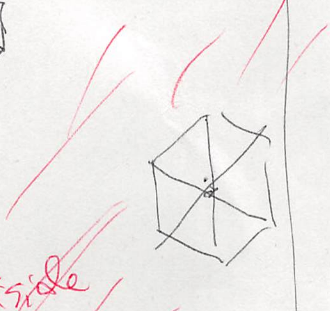
Schocking

[Williamson : NOT TRUE]

around Steinberg wt



☆ is a sort of best possible.



outside treat as junks

[around Steinberg wt $U_q(\mathfrak{sl}_n)/\mathbb{C}$ / modulo junks

$\wr\!|$ Andersen etc

$\mathcal{O}(\mathfrak{gl}_n)$

Kodera's talk.

BGG cat. $\mathcal{O}$]

$K_0(G\text{-mod})$に

不成立

§ Set up

Players

Ariki-Koike alg (cyclotomic Hecke alg)

cyclotomic q-Schur alg

Rational Double Affine Hecke Algebra

refln

cpx ref sp

$\langle S \rangle$

ref. rep.

R DAHA assoc w/ $W \subset GL(\mathfrak{h})$

&

charge $c : S \to \mathbb{C}$ ($\supset \mathbb{R}$)

(cls function)

coff. ry

thm. def

cord. ry

$$\mathbb{H}_c(W, \mathfrak{h}) = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]$$

as vec. sp.

$\mathbb{C}[\mathfrak{h}] \supset \mathfrak{h}^*$, $\mathbb{C}[\mathfrak{h}^*] \supset \mathfrak{h}$

$\triangle$ decomp.

$x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$

$$[y, x] = \langle x, y \rangle - \sum_{s \in S} c(s) \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle s$$

nat. pairy

$\langle \, , \, \rangle : \mathfrak{h}^* \times \mathfrak{h}$ on

α_s : root assoc w/ s

$\alpha_s^\vee$: coroot assoc. w/ s

~ "(non commutative) Cartan"

~ "Borel"

GGOR $\mathcal{O}$

$\mathcal{O}_c = \mathcal{O}_c(W, \mathfrak{h})$ $\left\{ \begin{array}{l} \text{f.g } \mathbb{C}[\mathfrak{h}]\text{-module} \\ \mathfrak{h} : \text{locally nilpotent} \end{array} \right.$ $\subset \ddot{H}_c(W, \mathfrak{h})$-Mod

$\cap$

$\mathbb{C}[\mathfrak{h}^*]$

$\lambda \in \mathrm{Irr}\, W$

$$\Delta(\lambda) := \mathrm{Ind}^{\ddot{H}}_{\underbrace{W \ltimes \mathbb{C}[\mathfrak{h}^*]}_{\text{Borel}}} \left(\mathrm{Inf}^{\underline{\text{Borel}}}_{W} (\lambda) \right)$$

$\text{Borel} \twoheadrightarrow \mathbb{C}W$

$\xleftarrow[\mathrm{Inf}]{}$ -mod

std. module

$\mathcal{O}_c$ = highest wt cat. w/ poset $\mathrm{Irr}\, W$

order is determined by

c

Lusztig's '80

a, A, b, B functions

c $\longleftrightarrow$ a+A function

有木，和田

Leclerc - Thibon Many others P. Shan.

竹村

$U_{\mathfrak{gl}_\infty}$ Fock sp $= \bigoplus_{n \geq 0} K_0(\mathcal{O}_c((\mathbb{Z}/\ell\mathbb{Z})^n \rtimes S_n, \mathfrak{h}_n)) \otimes \mathbb{C}$

w/ charge c

level ℓ

$\circlearrowleft$ i-ind $= f_i$

i-res $= e_i$

/ $\widehat{\mathfrak{sl}}_e$ $e =$ $c((1,2))$ の ~~分子~~ 分母

互換

$\boxed{\text{quantized}} \longleftrightarrow \boxed{\text{graded}} \longleftarrow$ turn tobe true [later VV RSVV etc.

categorification business

$\xleftarrow{K_0}$

wt sp $\longleftrightarrow$ block

"std" base $<$ std / costd

~~simple~~ $\longleftrightarrow$ ~~dual canonical~~

dual canonical $\longleftrightarrow$ simple

canonical base $\longleftrightarrow$ tilting object. (indec. summand of Miyashita tilting)

$-$ operation $\longleftrightarrow$ contragredient dual

in some survey written differently.

§ level-rank

$\mathfrak{sl}_e$ の nat. rep

$V_e = \mathbb{C}^e$ の $-\infty, \infty$ tensor

$= \mathbb{C}^e \otimes \mathbb{C}[t^{-1}, t]$

$\widehat{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}c \oplus \mathbb{C}d$

affine Lie alg

$\widehat{\mathfrak{sl}}_e$

$\nu = (\nu_1, \nu_2, \ldots, \nu_l) \in \mathbb{Z}_{\geq 0}^l$

$\widehat{\mathfrak{gl}}_e$

wedge $\Lambda^\nu(V_e) = \bigotimes_{i=1}^{l} \Lambda^{\nu_i}(V_e)$

$\mathfrak{gl}_e$

$\mathbb{C}^e / \mathbb{C}(\sum_{i=1}^{e} \varepsilon_i)$

$=$ wt of $\mathfrak{sl}_e$

$\widehat{\mathfrak{sl}}_e$

$V_{e,l} = \mathbb{C}^e \otimes \mathbb{C}^l \otimes \mathbb{C}[t^{-1}, t]$

point

view $\mathfrak{t}$ $\mathfrak{gl}$ wt.

$\widehat{\mathfrak{sl}}_e$ + Heisenberg $= \widehat{\mathfrak{gl}}_e$

level l

rank e

$\Lambda^\nu(V_e)_\mu = (\Lambda^N(V_{e,l}))_{\mu,\nu} = \Lambda^\mu(V_l)_\nu$

level e

rank l

lin. map LR

Level rank duality

categorification (lift)

charge $\hat{c}_\nu$

block specified by μ

What's $\widehat{LR}$

charge $\hat{c}_\mu$

block specified by ν

category B^ν_μ $\mathcal{O}$ / RDAHA

$(\mathbb{Z}/l\mathbb{Z}) \wr S_m$

B^μ_ν category $\mathcal{O}$ / RDAHA

$(\mathbb{Z}/e\mathbb{Z}) \wr S_{m'}$

Chuang-M conjecture
(ICRA à Tokyo, Rep. Alg. & Qua. à Nagoya)

$$\widehat{LR} = \underline{(\text{Ringel dual}) \cdot (\overset{\text{quadratic}}{\text{Koszul dual}})}$$

$$K$$

$$B\text{-grmod} \xrightarrow{K} \mathrm{Ext}^{\bullet}_{B}\Big(\bigoplus_{\substack{L:\text{simple} \\ B\text{-mod}/\sim}} L,\ \bigoplus \_\_ \Big)\text{-grmod}$$

Mazorchuk dual
(Balanced p.h. alg)

Th. (Rouquier - Shan - Varagnolo - Vasserot)
TRUE (Major part)
by proving VV conj.

①

affine Lie alg の parabolic category $\mathcal{O}$ で
~~これ~~ これに相当すべきことを
Shan - Varagnolo - Vasserot
が示す。

$\widehat{W}$ affine Weyl gp $\widehat{W}/W_\mu$ にふさわしい幾何学化

- $\overset{\text{truncation}}{\mathcal{O}}{}^{\mu} \simeq \mathrm{Perv}(\overline{X}_{\mu,w})$ partial flag $\widehat{W}/W_\mu$
- Yoneda alg の grading をこめた計算や、KL poly との比較
に Moment graph (Soergel, Fiebig)

- Koszul dual の相手が指定されていることも大きい。(当たりと比較可能) ——(☆)

- Soergel functor (Yoneda alg の比較による)
(structure)
$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}}$ ("big proj.", —)
の full, faithful は、 $G(\mathbb{C}((t)))$
旗多様体 partial flag $\mathcal{G}/P_\mu$

- [Moment graph combinatorics において
当たりのラベルの BM-sheaf と dual は同型で
(☆) が check

Thm (Shan VV)
Chuang-M の affine Lie alg $\hat{\mathfrak{gl}}_e$ の parabolic category $\mathcal{O}$ で類似が成立。

(SVV by "More geometric")

cf Bezrukavnikov-Yun

想像をまじえて. VV conj について.

有限性
に関する ~~[illegible]~~
岩洋分解 $\mathfrak{sl}_n \mathbb{C}[t]$ について

$\mathfrak{p} \oplus \mathfrak{sl}_n \otimes \mathbb{C}[t]t$

$\mathfrak{gl}_n \oplus \mathfrak{sl}_n \mathbb{C}[t]t$
integrable over

$\mathcal{O}_{\boxed{\mathrm{Levi}}}(\hat{\mathfrak{sl}}_n)$ $\quad$ $\mathcal{O}_{\mathfrak{gl}_n}(\hat{\mathfrak{gl}}_n)$ $\quad$ $\exists\ \hat{V}_n$ $\quad$ $\hat{V}_n^{\dot{\otimes} r}$

$\mathcal{O}_{KL}(\hat{\mathfrak{sl}}_n)$

なにかある
~~simple~~ std
~~Hmod~~
Hmod $\quad \Delta(\nu_c)$

$\wr\!\mid$ $\otimes$-cat eq.

$U_q(\mathfrak{sl}_n)$-mod $\quad V_n \quad V_n^{\otimes r}$

$\mathrm{End}\left(\Delta(\nu_c)\dot{\otimes}\hat{V}_n^{\dot{\otimes} r}\right)$

$\mathrm{End}_{U_q}(V_n^{\otimes r})$

$\mathrm{End}_{\mathfrak{g}}(V_n^{\otimes r})$ $\quad$ $H_{q^2}(\mathfrak{S}_r)$

cyclotomic Hecke
Ariki Koike $= \mathcal{H}_q c(l, r)$ $\begin{pmatrix} V_n^{\otimes r} \\ \text{polynomial} \end{pmatrix}$ $r \leq n$ なら iso

n = 十分大なら 同型 になりそう.

~~VV conj~~ $\quad \mathcal{O}_{c,r} := \mathcal{O}_{\text{“Levi”}}(\hat{\mathfrak{sl}}_n)$ の中で
$\Delta(\nu_c)\dot{\otimes}\hat{V}_n^{\dot{\otimes} r} = M_c^r$
の comp. factor

で 出来る ~~[illegible]~~
うち highest wt.
M_c^r が proj. obj. になる

$\boxed{\text{VV conj}}$

$\boxed{\begin{array}{c}\text{eg}\\ \mathbb{Z}_{l\mathbb{Z}}^r \rtimes \mathfrak{S}_r\end{array}}$

$$\begin{array}{ccc} \mathcal{O}_c(\cancel{\mathfrak{g}_n}) & \longrightarrow & \mathcal{O}_{c,r} \\ & {}_{KZ}\searrow \quad \circlearrowright \quad \swarrow_{\mathrm{Hom}(M_c^r, -)} & \\ & \mathcal{H}_q c(l, r)\text{-mod} & \end{array}$$

schur. functor

$l = 2$ に帰着する
technique をもってった。

RSVV

Rouquier:

cover theory から一気に[illegible]ことを

[illegible]たかった…

<u>Difficulty</u>

$$\begin{tikzcd}
\mathcal{O}_c(\mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_r) \arrow[dr, "KZ"'] & & \mathcal{O}_{c,r} \arrow[dl] \\
& \mathcal{H}_q c(\ell, r)\text{-mod} &
\end{tikzcd}$$

$$\langle T_0, T_1, \dots, T_{r-1} \rangle \Big/ \begin{array}{c} (T_0 - q^{c_1}) \cdots (T_0 - q^{c_\ell}) \\ (T_i - q)(T_i + 1) \end{array}$$

体上からの lift をねらうと

$q=-1$ とか

$q^{c_i} = q^{c_j}$ とか 2つそろうと

parameterが…

失敗

$$A\text{-mod} \xrightarrow{F} B\text{-mod}$$

$B = \mathrm{End}_A(P)^{op}$

$T(\lambda) = \mathrm{Hom}_A(A, T(\lambda))$ 小

$\neq \mathrm{Hom}_B(FA, FT(\lambda))$ 大

equality が cover theory になる

$A = k \oplus k$ $\quad \begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix} \oplus \begin{smallmatrix} | \\ 0 \end{smallmatrix}$

$T(1) = \begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}$

$T(0) = 0$

$P = \begin{smallmatrix} k \\ k \end{smallmatrix} \quad \begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}$ $\quad B = k[x]/(x^2)$

RSVV

係数を2次元のanalytic alg に変更

$\mathbb{C} \rightsquigarrow R$

Rouquier の cover theory 使用

- small rank
  (とくに cyclic)

  over $\mathbb{C}$ $k[x]/(x^\ell)$ $\rightsquigarrow$ over R $R[x]/\left(\prod_{i=1}^{\ell}(x-z_i)\right)$

  を直にR上で O.K. なことを check.

- 「係数を手にRに」というのが

  Sugawara construction
  KL tensor product $\dot{\otimes}_R$
  KZ functor.

  check pts が多数

- "~~rank~~ level 2" に帰着する方法