@ Summer school on cluster algebras 2023

Cluster K_2-structure on the moduli space of decorated twisted G-local systems

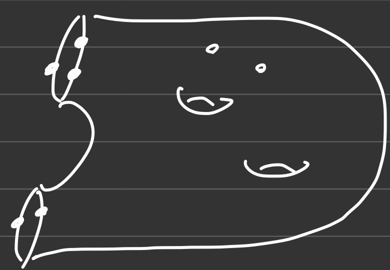
§1. Introduction

- G : simply-conn. semisimple alg. group$/\mathbb{C}$

  e.g. $G = SL_n$ (type A_{n-1})

- $(\Sigma, \mathbb{M})$: marked surface

  $\overset{\shortparallel}{\Sigma}$



$\rightsquigarrow$ $\mathcal{A}_{G,\Sigma}$: moduli sp. of decorated twisted G-local systems on Σ [FG'06]

$$\mathcal{A}_{G,\Sigma} \xrightarrow{\exists} \left[\mathrm{Hom}(\pi_1(\Sigma), G) / G \right]$$

★ $\mathcal{A}_{G,\Sigma}$ admits a natural <u>cluster K_2-str.</u> :

[FG'06, Le'19, GS'19]

$\exists$ a collection $\{ i = (\{A_i\}_{i\in I}, \varepsilon) \}$ of seeds in $\mathcal{K}(\mathcal{A}_{G,\Sigma})$

- giving $\psi_i : \mathcal{A}_{G,\Sigma} \xrightarrow{\sim} (\mathbb{C}^\times)^I$

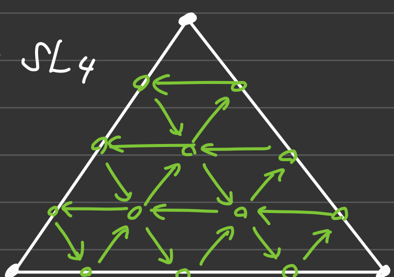
(cluster coord's)

- $\psi_{i'} \circ \psi_i^{-1}$ being cluster K_2-transf's.

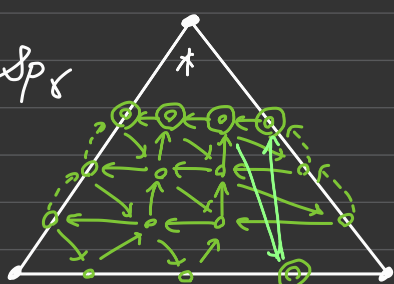
$A_k \cdot A_k' = \prod A_i + \prod A_i$

$\exists$ explicit construction i_Δ for a decorated triangulation Δ of Σ.

$G = SL_4$



$G = Sp_6$



★ $\mathcal{A}_{SL_2,\Sigma}(\mathbb{R}_{>0}) \cong \widetilde{\mathcal{T}}(\Sigma)$: decorated Teichmüller sp.

[Penner]

* Fock–Goncharov duality : $\mathcal{A}_{G,\Sigma} \longleftrightarrow \mathcal{P}_{G^\vee,\Sigma}$

<u>Theorem</u> (I.–Oya–Shen'22 [IOS'22])

Σ : connected, $\mathbb{M} \subset \partial\Sigma$, $|\mathbb{M}| \geq 2$, $G \neq E_8, F_4, G_2$

$\Rightarrow \underset{\text{cluster alg.}}{A_{\mathfrak{g},\Sigma}} = \underset{\text{upper cluster alg.}}{U_{\mathfrak{g},\Sigma}} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$

"generic" part

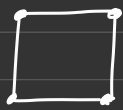
<u>Strategy</u>: $A_{\mathfrak{g},\Sigma} \subseteq U_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma}) \subseteq A_{\mathfrak{g},\Sigma}$

- $U_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}_{|\mathbf{i}_\Delta|})$,

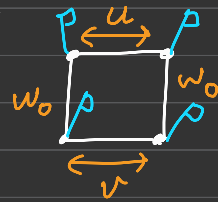
  where $\mathcal{A}_{|\mathbf{i}_\Delta|} = \bigcup_{\mathbf{i} \sim \mathbf{i}_\Delta} T_{\mathbf{i}}$: cluster K_2-variety

We need a comparison between $\mathcal{A}^\times_{G,\Sigma}$ & $\mathcal{A}_{|\mathbf{i}_\Delta|}$ up to codim. 2

* "stratifications" of $\mathcal{A}_{G,\Sigma}$ and braid varieties.

e.g. $\Sigma =$  $\qquad \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} \cong H^2 \times G^{u,v}$

$$\begin{array}{c} \mathcal{A}^{\boldsymbol{w}}_{G,\Sigma} \\ \cap \\ \mathcal{A}_{G,\Sigma} \end{array}$$



$\boldsymbol{w} = (u, w_0, v, w_0)$

$\boxed{\text{Goal}}$

- Geometry of the "generic part" $\mathcal{A}^{\times}_{G,\Sigma}$ §2.
- Cluster structure §3. (§6. §7)
- Stratifications of $\mathcal{A}_{G,\Sigma}$ & braid varieties §4. §5

§2. Geometry of $\mathcal{A}_{G,\Sigma}$

§3. Cluster structure

§4. Interpolation of (decorated) flags

§5. Relation to the braid varieties

§6. Cluster structure on the square

§7. Proof of $\mathcal{A}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$

§2. Geometry of $\mathcal{A}_{G,\Sigma}$

Notation from Lie theory

- G : simply-conn. semisimple alg. group $/\mathbb{C}$

Fix $B^{\pm} < G$: a pair of opposite Borel subgroups

- $H := B^+ \cap B^-$: Cartan subgroup
- $U^{\pm} := [B^{\pm}, B^{\pm}]$: unipotent radical
- $W := N_G(H)/H$: Weyl group

Fix $\{e_s, f_s, \alpha_s^\vee\}_{s \in S}$: Chevalley generators of $\mathfrak{g} = \mathrm{Lie}(G)$

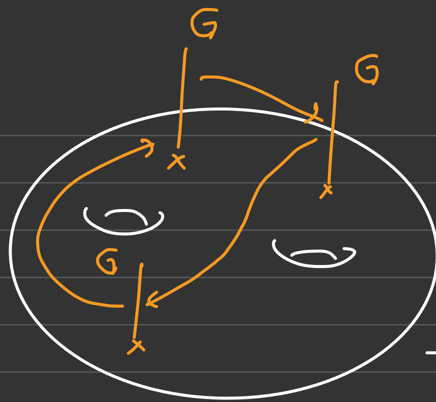
$\rightsquigarrow \varphi_s : SL_2 \longrightarrow G.$ $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto e_s \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto f_s$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \alpha_s^\vee$

- $\overline{r_s} := \varphi_s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N_G(H)$ $\rightsquigarrow$ $r_s := \overline{r_s} H \in W$

- For $w = r_{s_1} \cdots r_{s_m} \in W$, let $\boxed{\overline{w} := \overline{r_{s_1}} \cdots \overline{r_{s_m}}}$

III (Twisted) local systems.



<u>Fact</u> (monodromy corresp.)

$$\left\{\text{flat } G\text{-b'dl on } M\right\}\Big/\cong \;\xrightarrow{1:1}\; \mathrm{Hom}(\pi_1(M), G)\big/G$$

$$\mathcal{L} \longleftrightarrow \rho$$

$(\Sigma, \mathbb{M})$: a marked surface $\quad (\mathbb{M} = \mathbb{M}_\circ \sqcup \mathbb{M}_\partial)$ int. ∂.

$\rightsquigarrow \Sigma^* := \Sigma \setminus \mathbb{M}_\circ$

$\rightsquigarrow T'\Sigma^* := T\Sigma^* \setminus (0\text{-section})$

punctured tangent b'dl

$$0 \longrightarrow \underset{\langle \delta \rangle}{\underset{\parallel}{\pi_1(S^1)}} \longrightarrow \pi_1(T'\Sigma^*) \longrightarrow \pi_1(\Sigma^*) \longrightarrow 1 \quad (\text{exact})$$

<u>Def</u> A twisted G-local system on Σ

$:\Longleftrightarrow$ a G-local system $\mathcal{L}$ on $T'\Sigma^*$ s.t. $\underline{\rho(\delta) = s_G}$.

Here, $s_G := \overline{w_0}^{\,2} \in Z(G)$.

e.g. $s_{SL_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -1$, $\quad s_{SL_n} = (-1)^{n-1}$

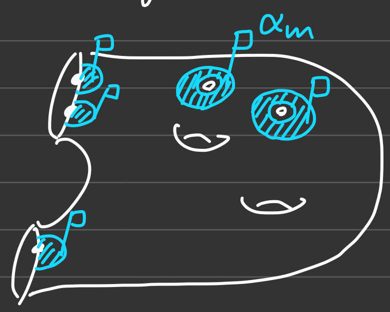
... "signs"

$$\rho(\quad) = s_G \cdot \rho(\quad)$$

<u>Def</u> A decoration of $\mathcal{L}$

$:\Longleftrightarrow$ a flat section α of $\mathcal{L} \times_G (G/U^+)$

defined on a nbd. of $\mathbb{M}$.



<u>What is G/U^+ ?</u>

① $\mathcal{O}(G/U^+) \cong \bigoplus_\lambda V_\lambda$

(Peter-Weyl)

② $SL_n/U^+ \cong \{(V_\bullet, v_\bullet)\}$, where

$$\begin{cases} V_\bullet : 0 < V_1 < V_2 < \cdots < V_n = \mathbb{C}^n & \dim V_i = i \\ v_\bullet : v_i \in \Lambda^i V_i \setminus \{0\} & (v_n = \mathrm{vol}) \end{cases}$$

③ $\pi : G/U^+ \xrightarrow{H} G/B^+$ — the flag variety

<u>Def</u> $\mathcal{A}_{G,\Sigma}$ parametrizes the pairs $(\mathcal{L}, \alpha)$ up to isomorphisms.

It has a presentation $\mathcal{A}_{G,\Sigma} = [A_{G,\Sigma}/G]$ w/

$$A_{G,\Sigma} \simeq \left\{ (\rho, (A_m)) \in \mathrm{Hom}^{tw}(\pi_1(T'\Sigma^*), G) \times (G/U^+)^{\mathbb{M}} \text{ s.t. } \rho(\gamma_m).A_m = A_m \text{ for } m \in \mathbb{M}_0 \right\}$$

— a quasi-affine variety.

$$\mathcal{O}(\mathcal{A}_{G,\Sigma}) = \mathcal{O}(A_{G,\Sigma})^G.$$

From now on, we assume :

$\mathbb{M} \neq \phi$, $\forall C$: ∂-comp. $\mathbb{M} \cap C \neq \phi$,

$\underline{-2\chi(\Sigma^*) + |\mathbb{M}_\partial|} > 0.$

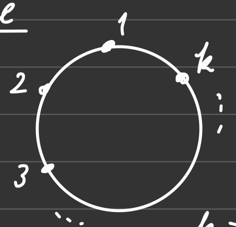
└ # of triangles in any ideal triangulation Δ.

Rem In particular, $\partial\Sigma \neq \emptyset$ or $\mathbb{M}_0 \neq \emptyset$.

$\Rightarrow \pi_1(\Sigma^*)$ is a free group.

$$\Rightarrow \mathcal{A}_{G,\Sigma} \overset{\text{closed}}{\subset} G^{-\chi(\Sigma^*)+1} \times (G/U^+)^{\mathbb{M}}$$

Example

$\Sigma =$ 

$k \geq 3$

$$\Rightarrow \mathcal{A}_{G,\Sigma} \simeq \left[(G/U^+)^k / G \right] =: \mathrm{Conf}_k\, G/U^+$$

$$\boxed{\mathcal{O}(\mathrm{Conf}_k\, G/U^+) \simeq \bigoplus_{\lambda_1,\dots,\lambda_k} (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k})^G}$$

- Relative position : $\mathrm{Conf}_2\, G/U^+$
- Cluster coordinates : $\mathrm{Conf}_3\, G/U^+$
- Flips (mutation-equiv.) : $\mathrm{Conf}_4\, G/U^+$

⑪ Relative position (cf. [GS'19, §3.1.6])

Lemma Any pair $(A_1, A_2) \in (G/U^+)^2$ can be translated into a position $(h.[U^+], \overline{w}.[U^+])$ for unique $h \in H$ & $w \in W$.

(∵ Bruhat decomp. $G = \bigsqcup_{w \in W} U^- H \overline{w} U^+$)

$$\begin{cases} h(A_1, A_2) := h & (h\text{-distance}) \\ w(A_1, A_2) := w & (w\text{-distance}) \end{cases}$$

depends only on $\pi(A_i) \in G/B^+$

Def (A_1, A_2) : generic $\overset{\cdot}{\Leftrightarrow}$ $w(A_1, A_2) = w_0$

Example $SL_2/U^+ \cong \mathbb{C}^2 \setminus \{0\}$ $[U^+] \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\overline{w_0}.[U^+] \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

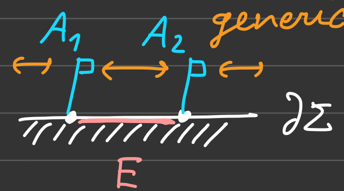
(A_1, A_2) : generic $\Leftrightarrow$ $[A_1] \neq [A_2]$ in $\mathbb{P}^1$.

"pinnings"

<u>Lem/Def</u> ([GS19, §3.1.7])

$$G \xrightarrow{\sim} \{(A_1, B_2) \in G/U^+ \times G/B^+ \mid w(A_1, B_2) = w_0\}$$

$$g \longmapsto g.p_{std}, \quad p_{std} := ([U^+], B^-) \qquad =: \mathcal{P}_G$$

<u>Def</u> $\mathcal{A}^\times_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ subspace s.t. 

$$\mathbb{B} := \pi_0(\partial\Sigma \setminus \mathbb{M}_\partial) = \{\text{boundary interval}\}$$

For $(\mathcal{L}, \alpha) \in \mathcal{A}^\times_{G,\Sigma}$, a section $p_E := (A_1, \pi(A_2))$

of $\mathcal{L} \times_G \mathcal{P}_G$ is associated w/ $E \in \mathbb{B}$.

(III) Wilson lines

Let $[c]: E_1 \longrightarrow E_2$ be $\overbrace{\text{a path in } T'\Sigma^*}^{\text{homotopy class of}}$.

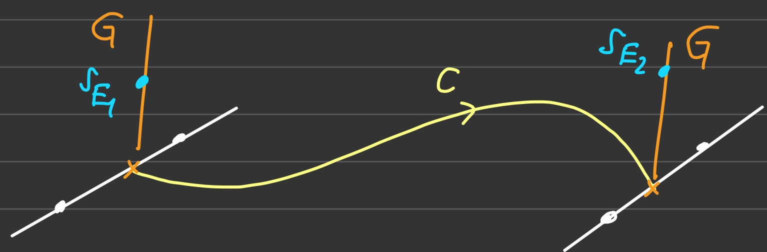
For $(\mathcal{L}, \alpha) \in \mathcal{A}^\times_{G,\Sigma}$, get $p_{E_i} = g_i.p_{std} \in \mathcal{P}_G$

$(i = 1, 2)$

Def $g^{tw}_{[C]}(\mathcal{L}, \alpha) := g_1^{-1} g_2$ twisted Wilson line

$g_{[C]}(\mathcal{L}, \alpha) := g_1^{-1} g_2 \overline{w_0}$ Wilson line

Topologically:



$\mathcal{L} \cong \mathcal{L} \times_G P_G$

$s_E \longleftrightarrow p_E$

$g^{(tw)}_{[C]} : \mathcal{A}^{\times}_{G,\Sigma} \longrightarrow G$ morphism of stacks.

Prop ([IOS'22]) If $M_0 = \phi$,

$g^{tw}_{\bullet} : \mathcal{A}^{\times}_{G,\Sigma} \underset{\text{closed}}{\hookrightarrow} \mathrm{Hom}(\underline{\Pi_1(T'\Sigma^*, \mathbb{B})}, G)$

fund. groupoid w/ obj. $= \mathbb{B}$.

— an affine variety

Cor $\mathcal{M}_0 = \phi \Rightarrow \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is generated by matrix coefficients of Wilson lines.

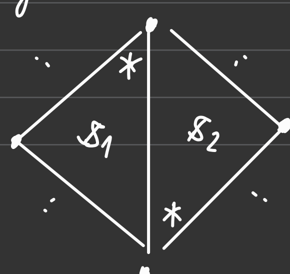
$$\bigoplus_{\lambda} V_\lambda^* \otimes V_\lambda \xrightarrow{\sim} \mathcal{O}(G)$$

$$f \otimes v \longmapsto (c^{\lambda}_{f,v} : g \longmapsto \langle f, g.v \rangle_{V_\lambda})$$

§3. Construction of cluster charts (on the generic part) [GS'19]

A decorated triangulation $\boldsymbol{\Delta} = (\Delta, m_\Delta, s_\Delta)$ of Σ consists of:

1) Δ : an ideal triangulation of Σ
2) $m_\Delta = (m_T)_{T \in t(\Delta)}$: choice of corners
3) $s_\Delta = (s_T)_{T \in t(\Delta)}$: reduced words of w_0



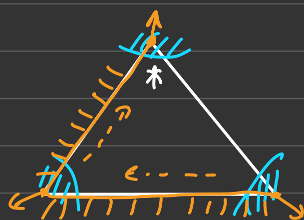
Then we get:

1) Restriction morphism $\psi_\Delta : \mathcal{A}_{G,\Sigma} \longrightarrow \prod_T \mathcal{A}_{G,T}$.

$\exists$ a reconstruction from $\mathcal{A}^\times_{G,T}$.

2) Isomorphism $f_{m_T} : \mathcal{A}_{G,T} \xrightarrow{\sim} \mathrm{Conf}_3\, G/U^+$

w.r.t. the corner m_T.



3) Cluster chart $A_{s_T} : \mathrm{Conf}_3\, G/U^+ \longrightarrow \mathbb{C}^{l(w_0)+2r}$

+ (weighted) quiver

Then the collection

$$A_\Delta := \bigcup_T \psi_\Delta^* f_{m_T}^* A_{s_T} =: \{A_i\}_{i \in I}$$

defines a cluster chart $A_\Delta : \mathcal{A}_{G,\Sigma} \xrightarrow{\sim} (\mathbb{C}^*)^I$

(iii) Cluster chart on $\mathrm{Conf}_3\, G/U^+$ [GS'19, §9.1]

Recall : $\mathcal{O}(\mathrm{Conf}_3\, G/U^+) = \bigoplus_{\lambda,\mu,\nu} (V_\lambda \otimes V_\mu \otimes V_\nu)^G$

<u>Idea</u> : Pick up $A_{\lambda,\mu,\nu} \in \underline{(V_\lambda \otimes V_\mu \otimes V_\nu)}$

for triples (λ,μ,ν) s.t. dim. $(\;\cdot\;) = 1$

<u>Lemma</u> $\dim (V_\lambda \otimes V_\mu \otimes V_\nu)^G = 1$

if $w.\lambda = \nu^* - \mu$ for some $w \in W$

Given a reduced word $\mathbf{s} = (s_1, \cdots, s_N)$ of w_0,

Set $w_{>k} := r_{s_N} \cdots r_{s_{k+1}} \in W$

$$I(\mathbf{s}) := \left\{ (\varpi_s, [w_{>k}.\varpi_s]_-, [w_{>k}.\varpi_s]_+^*) \;\middle|\; \begin{matrix} s \in S \\ k = 0, \cdots, N \end{matrix} \right\}$$

$$\cup \; \{ (0, \varpi_s, \varpi_s^*) \mid s \in S \}$$

Here for $\lambda = \sum_{s \in S} m_s \varpi_s$, $\lambda_\pm := \sum_{s \in S} [\pm m_s]_+ \varpi_s$.

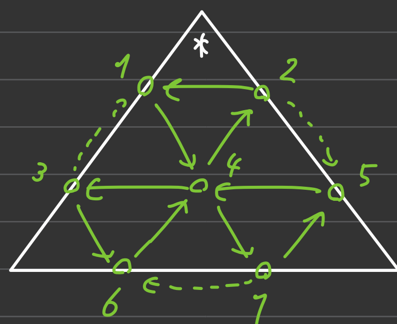
$= \lambda_+ - \lambda_-$

For $(\lambda,\mu,\nu)\in I(\delta)$, $\dim\underline{(V_\lambda\otimes V_\mu\otimes V_\nu)^G}=1$.

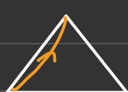
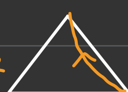
Choose $A_{\lambda,\mu,\nu}\in$ //// w/ a normalization.

Example

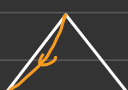
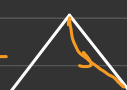
1) Type A_2, $\delta=(1,2,1)$



$I(\delta)=\{(\varpi_2,\varpi_1,0),\ (\varpi_2,0,\varpi_1),$

A_1  A_2 

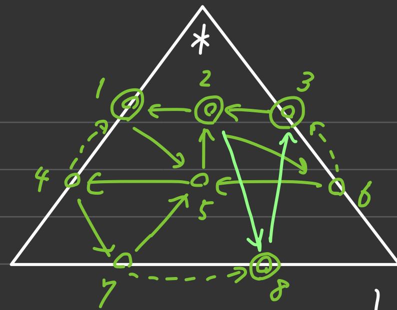
$(\varpi_1,\varpi_2,0),\ (\varpi_1,\varpi_1,\varpi_1),\ (\varpi_1,0,\varpi_2),$

A_3  A_4  A_5 

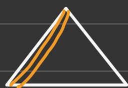
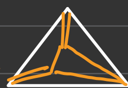
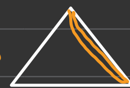
$(0,\varpi_2,\varpi_1),\ (0,\varpi_1,\varpi_2)$

A_6  A_7  $\}$

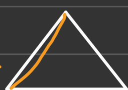
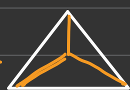
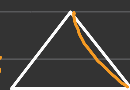
2) Type C_2 , $\underline{s} = (1,2,1,2)$



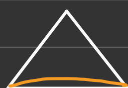
$$I(\underline{s}) = \{ (\varpi_2,\varpi_2,0),\ (\varpi_2,\varpi_2,2\varpi_1),\ (\varpi_2,0,\varpi_2)$$

A_1  A_2  A_3 

$$(\varpi_1,\varpi_1,0),\ (\varpi_1,\varpi_2,\varpi_1),\ (\varpi_1,0,\varpi_1)$$

A_4  A_5  A_6 

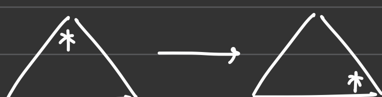
$$(0,\varpi_2,\varpi_2),\ (0,\varpi_1,\varpi_1) \}$$

A_7  A_8 

... cluster str. is not symmetric under rotations !

Need to show the mutation-equivalence for :

1) flips 

2) rotations 

3) changes of words $s_T \longrightarrow s_T'$