Auslander-Buchweitz approximations and cotilting modules (by Ryo Kanda)

k : field. A : fin. dim. k-alg.

subcat = full subcat (cat = category)

additive subcat = subcat closed under $\overset{\text{finite}}{\oplus}$, $\underset{\text{(direct summand)}}{\langle\!\oplus}$.

§1 Approximations and cotorsion pairs

<u>Def</u> $\mathcal{B}$: additive cat. $\mathcal{X} \subset \mathcal{B}$.

(1) $M \in \mathcal{B}$.

$\underset{\mathcal{X}}{\underset{\in}{X}} \xrightarrow{f} M$: right $\mathcal{X}$-approximation (of M)

$:\Leftrightarrow {}^{\forall}\underset{\mathcal{X}}{\underset{\in}{X'}} \xrightarrow{{}^{\forall}f'} M$,

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "\exists"'] & \circlearrowright & M \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

(2) $\mathcal{X}$: contravariantly-finite (in $\mathcal{B}$)

$:\Leftrightarrow$ Every $M \in \mathcal{B}$ has a right $\mathcal{X}$-approx.

(3) Assume $\mathcal{B}$: Krull-Schmidt. (e.g. mod A)

$L \xrightarrow{f} M$: right minimal

$$:\Leftrightarrow \left[\begin{tikzcd} L \arrow[dr, "f"] \arrow[d, "g"'] & \\ L \arrow[r, phantom, "\circlearrowright" near start] \arrow[r, "f"'] & M \end{tikzcd} \Rightarrow g : \text{iso} \right]$$

$$\Leftrightarrow \left[{}^{\forall}\underset{0}{\underset{\neq}{L'}} \langle\!\oplus L, \quad \begin{tikzcd} L' \arrow[d, phantom, sloped, "\subset"] \arrow[dr, "\times^{0}"] & \\ L \arrow[r, "f"'] & M \end{tikzcd} \right].$$

Dually, left approx, covariantly-finite, left minimal are defined. (functorially-finite :$\Leftrightarrow$ cov-fin and contrav-fin)

<u>Prop 1.1</u> $\left.\begin{array}{l} X \xrightarrow{f} M \\ X' \xrightarrow{f'} M \end{array}\right\}$ right min'l $\underset{\text{(right)}}{\mathcal{X}}$-approxs

$$\Rightarrow \begin{tikzcd}
X \arrow[dr, "f"] \arrow[d, "\exists \wr"'] & \\
X' \arrow[r, phantom, "\circlearrowleft", near start, shift left=3ex] \arrow[r, "f'"'] & M
\end{tikzcd}.$$

(∵)

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[d, "\exists h"'] \arrow[dd, bend right=60, "\circlearrowleft"] & \\
X' \arrow[r, "f'"] \arrow[d, "\exists h'"'] \arrow[r, phantom, "\circlearrowleft", near start, shift left=3ex] \arrow[r, phantom, "\circlearrowleft", near start, shift right=3ex] & M \\
X \arrow[ur, "f"'] &
\end{tikzcd}$$

iso (∵ f: right min'l). Similarly hh': iso. //

<u>Ex 1.2</u> A: fin.dim k-alg. $M \in \operatorname{mod} A$.

$P \xrightarrow{f} M$: right proj A-approx $\Leftrightarrow$ $P \in \operatorname{proj} A$, f: epi.

$P \xrightarrow{f} M$: right min'l proj A-approx $\Leftrightarrow$ f: proj cover.

$\operatorname{proj} A \subset \operatorname{mod} A$: contrav-fin.

What does the word "contravariantly-finite" mean:

Prop 1.3 $\underset{\mathcal{X}}{\overset{\cap}{X}} \xrightarrow{f} M$: right $\mathcal{X}$-approx

$\iff \mathrm{Hom}(-,X)|_{\mathcal{X}} \to \mathrm{Hom}(-,M)|_{\mathcal{X}} \to 0$: exact

(as functors $\mathcal{X}^{op} \to \mathrm{Mod}\,\mathbb{Z}$.)

$\mathcal{X}$: contrav-fin $\iff \forall M \in \mathcal{B}$, $\mathrm{Hom}(-,M)|_{\mathcal{X}}$: finitely generated.

Let $\mathcal{A}$: abelian cat.

Ext groups and exactness are always considered in $\mathcal{A}$ (not in a subcat of $\mathcal{A}$).

Thm 1.4 (Wakamatsu's lemma)

Assume $\mathcal{A}$: Krull-Schmidt.

$\mathcal{X} \subset \mathcal{A}$: closed under extensions.

(i.e. $0 \to \underset{\mathcal{X}}{\overset{\cap}{L}} \to M \to \underset{\mathcal{X}}{\overset{\cap}{N}} \to 0 \Rightarrow \underset{\mathcal{X}}{\overset{\cap}{M}}$).

Then

$$0 \to Y \to \underbrace{X \to M}_{\substack{\text{right min'l} \\ \mathcal{X}\text{-approx}}} \Rightarrow \mathrm{Ext}^1(\mathcal{X}, Y) = 0.$$

(i.e. $\forall X' \in \mathcal{X}$, $\mathrm{Ext}^1(X', Y) = 0$)

(∵) Let $0 \to Y \to X \to C \to 0$ by taking the cokernel.

Then easy to see $X \to C$ is also a right min'l $\mathcal{X}$-approx.

We will see that every $0 \to Y \to N \to \underset{\mathcal{X}}{\overset{\cap}{X'}} \to 0$ splits.

By taking the pushout
$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & N \arrow[d, dashed] \\ X \arrow[r, dashed] & E \end{tikzcd}$$
,

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & N \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d] & X' \arrow[r] & 0 \\ & C \arrow[r, equal] \arrow[d] & C \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$
.

Since $\mathcal{X}$: closed under exts, $E \in \mathcal{X}$.

$\therefore$
$$\begin{tikzcd} X \arrow[d, "(approx)"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[l, "\exists"'] \arrow[d] \\ C \arrow[r, equal] & C \end{tikzcd}$$
. By minimality,
$$\begin{tikzcd} X \arrow[dr] \arrow[dd, "\wr"'] & \\ & E \arrow[dl] \\ X & \end{tikzcd}$$
.

$\therefore 0 \to X \to E \to X' \to 0$ splits. $(\because X \to C : \mathcal{X}\text{-approx})$

$$0 \to \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X', X) \twoheadrightarrow \mathrm{Hom}(X', C)$$
$$\to \mathrm{Ext}^1(X', Y) \to \mathrm{Ext}^1(X', X)$$
$$\begin{array}{ccc} \in & & \in \\ (0 \to Y \to N \to X' \to 0) & \longmapsto & (0 \to X \to E \to X' \to 0) = 0 \end{array}$$
(i.e. split)

$\therefore 0 \to Y \to N \to X' \to 0$ splits. //

Prop 1.5 $\mathcal{X} \subset \mathcal{A}$. $0 \to Y \to \underset{\mathcal{X}}{\overset{}{X}} \xrightarrow{f} M \to 0$, $\mathrm{Ext}^1(\mathcal{X}, Y) = 0$

$\Rightarrow f$: right $\mathcal{X}$-approx.

(∵) $\forall X' \in \mathcal{X}$,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(X',Y) \arrow[r] & \mathrm{Hom}(X',X) \arrow[r] & \mathrm{Hom}(X',M) \arrow[r] & \mathrm{Ext}^1(X',Y) \arrow[d, equal] \\ & & \ddot{\text{epi}} \arrow[d, Rightarrow] & & 0 \arrow[ll, Leftarrow] \\ & & X \to M : \text{approx}. & & \end{tikzcd}$$

//

<u>Def</u> $\mathcal{B} \subset \mathcal{A}$: additive. $\mathcal{X}, \mathcal{Y} \subset \mathcal{B}$: additive.

$(\mathcal{X}, \mathcal{Y})$: cotorsion pair in $\mathcal{B}$

$:\Leftrightarrow \begin{cases} \bullet\ \mathrm{Ext}^{>0}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) = 0 \quad \left(\text{i.e. } \forall i > 0, \forall X \in \mathcal{X}, \forall Y \in \mathcal{Y},\ \mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0 \right) \\ \bullet\ \forall B \in \mathcal{B}, \end{cases}$

$$\left.\begin{array}{l} \exists\ 0 \to \underset{\overset{\cup}{\mathcal{Y}}}{Y_B} \to \underset{\overset{\cup}{\mathcal{X}}}{X_B} \to B \to 0 \\ \exists\ 0 \to B \to \underset{\underset{\mathcal{Y}}{\cap}}{Y^B} \to \underset{\underset{\mathcal{X}}{\cap}}{X^B} \to 0 \end{array}\right) \text{exact in } \underset{\sim\sim}{\mathcal{A}}.$$

$\mathcal{X}^{\perp} := \{ M \in \mathcal{A} \mid \mathrm{Ext}^{>0}_{\mathcal{A}}(\mathcal{X}, M) = 0 \}$

$\mathcal{X}^{\perp_1} := \{ \text{——} \mid \mathrm{Ext}^1_{\mathcal{A}} \text{ ——} \}$

${}^{\perp}\mathcal{Y} := \{ M \in \mathcal{A} \mid \mathrm{Ext}^{>0}_{\mathcal{A}}(M, \mathcal{Y}) = 0 \}$

${}^{\perp_1}\mathcal{Y} := \{ \text{——} \mid \mathrm{Ext}^1_{\mathcal{A}} \text{ ——} \}$.

<u>Prop 1.6</u> Let $(\mathcal{X}, \mathcal{Y}) \subset \mathcal{B}$: cotorsion pair, $\omega := \mathcal{X} \cap \mathcal{Y}$. Then

(1) $\mathcal{Y} = \mathcal{X}^{\perp} \cap \mathcal{B} = \mathcal{X}^{\perp_1} \cap \mathcal{B}$.

$\mathcal{X} = {}^{\perp}\mathcal{Y} \cap \mathcal{B} = {}^{\perp_1}\mathcal{Y} \cap \mathcal{B}$.

$\omega = \mathcal{X} \cap \mathcal{X}^{\perp} = {}^{\perp}\mathcal{Y} \cap \mathcal{Y}$

(kernels of epimorphisms)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$

(2) $\mathcal{X}$ is closed under extensions, epikernels in $\mathcal{B}$.

$$\left(\begin{array}{l} \text{i.e. (ext) } 0 \to \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{L}} \to \underset{\mathcal{B}}{\underset{\rotatebox{90}{$\in$}}{M}} \to \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{N}} \to 0 \Rightarrow \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{M}}, \\ \quad\text{(epiker) } 0 \to \underset{\mathcal{B}}{\underset{\rotatebox{90}{$\in$}}{L}} \to \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{M}} \to \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{N}} \to 0 \Rightarrow \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{L}}. \end{array}\right)$$

(3) $\mathcal{Y}$ is closed under exts, monocokernels in $\mathcal{B}$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$
(cokernels of monomorphisms)

(4) $X_B \to B$: right $\mathcal{X}$-approx. $\mathcal{X} \subset \mathcal{B}$: contrav-fin.
$B \to Y^B$: left $\mathcal{Y}$-approx. $\mathcal{Y} \subset \mathcal{B}$: cov-fin.

(5) ω : ($\mathrm{Ext}^{>0}$-) injective in $\mathcal{X}$.
(i.e. $\mathrm{Ext}^{>0}(\mathcal{X}, \omega) = 0$)
ω : cogenerator of $\mathcal{X}$.
(i.e. $\forall X \in \mathcal{X}, \exists\, 0 \to X \to \underset{\omega}{\underset{\rotatebox{90}{$\in$}}{W}} \to \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{X'}} \to 0$.)

(6) $\omega \subset \mathcal{Y}$: projective generator.

(7) $\forall f : \underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{X}} \to \underset{\mathcal{Y}}{\underset{\rotatebox{90}{$\in$}}{Y}}$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\exists"'] & \circlearrowright & Y \\ & {}^{\exists}\underset{\omega}{\underset{\rotatebox{90}{$\in$}}{W}} \arrow[ur, "\exists"'] & \end{tikzcd}$$

. i.e. $f = 0$ in $\dfrac{\mathcal{A}}{\omega}$.

(8) X_B, Y^B are uniquely determined in $\dfrac{\mathcal{A}}{\omega}$.
(up to isom)

Summary:

$$\begin{tikzcd}
 & \mathcal{A} & \\
 & \mathcal{B} \arrow[u, phantom, sloped, "\subset"] & \\
\mathcal{X} \arrow[ur, phantom, sloped, "\subset", "\text{gen}"] & & \mathcal{Y} \arrow[ul, phantom, sloped, "\subset", "\text{cogen}"'] \\
 & \omega \arrow[ul, phantom, sloped, "\subset", "\text{inj cogen}"'] \arrow[ur, phantom, sloped, "\subset", "\text{proj gen}"] &
\end{tikzcd}$$

(∵)(1) Obviously $\mathcal{Y} \subset \mathcal{X}^{\perp} \cap \mathcal{B} \subset \mathcal{X}^{\perp_1} \cap \mathcal{B}$.

$\forall B \in \mathcal{X}^{\perp_1} \cap \mathcal{B}$, $0 \to B \to Y^B \to X^B \to 0$ splits.

(∵ $\mathrm{Ext}^1(X^B, B) = 0$)

$B \mathrel{\langle\!\oplus} Y^B$. $B \in \mathcal{Y}$.

$\omega = \mathcal{X} \cap \underbrace{\mathcal{Y}}_{\overset{\parallel}{\mathcal{X}^{\perp} \cap \mathcal{B}}} = \mathcal{X} \cap \mathcal{X}^{\perp}$. $\lrcorner$

(2) Apply $\mathrm{Ext}^{>0}(-, \underset{\mathcal{Y}}{\underset{\cap}{Y}})$ to $0 \to L \to M \to N \to 0$. $\lrcorner$

(4) Prop 1.5. $\lrcorner$

(5) Note: $\omega \subset \mathcal{X}^{\perp}$.

$\forall X \in \mathcal{X}$, $\exists\, 0 \to X \to \underset{\mathcal{Y}}{\underset{\cap}{Y^X}} \to \underset{\mathcal{X}}{\underset{\in}{X^X}} \to 0$.

Since $\mathcal{X}$ $\underset{\wedge}{:}$ ext, $Y^X \in \mathcal{X}$. $\therefore Y^X \in \omega$. $\lrcorner$

(is closed under)

(7)
$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] \arrow[rd, phantom, "\circlearrowleft"] & \exists \overset{\in\omega}{W} \arrow[r] \arrow[ld, dashed, "\exists"] & \exists \overset{\in\mathcal{X}}{X'} \arrow[r] & 0 \\
 & Y & & &
\end{tikzcd}$$

$(\because X \to W : \mathcal{Y}\text{-approx})$ ⌋

(8) Take two:
$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & Y' \arrow[r] & X' \arrow[r] & B \arrow[r] & 0 . \\
 & \mathcal{Y} \arrow[u, phantom, sloped, "\in"] & \mathcal{X} \arrow[u, phantom, sloped, "\in"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\exists f"'] \arrow[rd, phantom, "\circlearrowleft"] & B \arrow[d, equal] \\
X' \arrow[r] \arrow[d, "\exists g"'] \arrow[rd, phantom, "\circlearrowleft"] & B \arrow[d, equal] \\
X \arrow[r] & B
\end{tikzcd}$$

$(\because (7))$
$$\begin{tikzcd}
 & \exists W \arrow[d, dashed, "\exists"'] \arrow[rd, phantom, "\circlearrowleft"] & X \arrow[l, dashed, "\exists"'] \arrow[ld, dashed, "\exists"] \arrow[d, "1-gf"] \arrow[rd, bend left, "0"] & & \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

$\therefore 1-gf=0$ in $\frac{\mathcal{A}}{\omega}$.

Similarly $1-fg=0$ in $\frac{\mathcal{A}}{\omega}$.

$X \underset{g}{\overset{f}{\rightleftarrows}} X'$ ($\simeq$) in $\frac{\mathcal{A}}{\omega}$. ⌋ //

§2 Auslander-Buchweitz approximations

Setting $\mathcal{A}$: abelian cat.

$\mathcal{X} \subset \mathcal{A}$: additive, closed under exts, epikers (in $\mathcal{A}$)

$\omega \subset \mathcal{X}$: additive, inj cogen (of $\mathcal{X}$)

$$\hat{\mathcal{X}} := \{ M \in \mathcal{A} \mid \exists\, 0 \to \underset{\mathcal{X}}{X_n} \to \cdots \to \underset{\mathcal{X}}{X_0} \to M \to 0 \ (\exists n) \}$$

($\hat{\omega}$: in the same way)

(Here we consider

$$\begin{tikzcd}
& \mathcal{A} & \\
& \hat{\mathcal{X}} = \mathcal{B} \arrow[u, phantom, "\cup" description] & \\
\mathcal{X} \arrow[ur, phantom, "\subset" description] & & \hat{\omega} = \mathcal{Y} \arrow[ul, phantom, "\supset" description] \\
& \omega \arrow[ul, phantom, "\cup" description] \arrow[ur, phantom, "\subset" description] &
\end{tikzcd}$$

)

Thm 2.1 [Auslander–Buchweitz 1989]

(1) $(\mathcal{X}, \hat{\omega}) \subset \hat{\mathcal{X}}$: cotorsion pair

(2) $$\hat{\mathcal{X}} = \{ M \in \mathcal{A} \mid {}^{\exists} 0 \to \underset{\substack{\cap \\ \mathcal{Y}}}{Y_M} \to \underset{\substack{\cap \\ \mathcal{X}}}{X_M} \to M \to 0 \} \quad \cdots (*)$$

$$= \{ M \in \mathcal{A} \mid {}^{\exists} 0 \to M \to \underset{\substack{\cap \\ \mathcal{Y}}}{Y^M} \to \underset{\substack{\cap \\ \mathcal{X}}}{X^M} \to 0 \} \quad \cdots (**)$$

: additive, closed under exts, epikers, monocoks.
(in $\mathcal{A}$)

(3) $\hat{\omega} \subset \mathcal{A}$: additive, closed under exts, monocoks
(in $\mathcal{A}$)

(4) $\omega = \mathcal{X} \cap \mathcal{X}^{\perp}$.

(i.e. ω in Setting is uniquely determined.)

(∵) (1)(2) Ext-orthogonality : $\omega \subset \mathcal{X}^{\perp}$, $\underset{\text{monocok}}{\therefore}$ $\hat{\omega} \subset \mathcal{X}^{\perp}$, ✓

Existence of approxs :

Let $M \in \hat{\mathcal{X}}$. $\exists\ 0 \to X_n \to \cdots \to X_0 \to M \to 0$.

Induction on n.

(i) $n=0$

$$0 \to \underset{\substack{\cap \\ \hat{\omega}}}{0} \to \underset{\substack{\cap \\ \mathcal{X}}}{M} \to M \to 0. \quad \exists\ 0 \to M \to \underset{\substack{\cap \\ \omega}}{W} \to \underset{\substack{\cap \\ \mathcal{X}}}{X} \to 0.$$

$(\because \omega \subset \mathcal{X} : \text{cogen})$

(ii) $n \geqq 1$.

$$0 \to \underset{n-1}{K} \to X_0 \to \underset{n}{M} \to 0$$

By induction hypothesis, $\exists\ 0 \to K \to Y^K \to X^K \to 0$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PO}"] & Y^K \arrow[r] \arrow[d] & X^K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X^K \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

$E \in \mathcal{X}$ $(\because \mathcal{X} : \text{ext})$

← $\mathcal{X}$-approx of M.

$$\exists\ 0 \to E \to \underset{\substack{\cap \\ \omega}}{W} \to \underset{\substack{\cap \\ \mathcal{X}}}{X} \to 0 \quad \cdots (\#)$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y^{\kappa} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[rd, phantom, "PO"] & M \arrow[r] \arrow[d] & 0 & \leftarrow \mathcal{X}\text{-approx we got.} \\
0 \arrow[r] & Y^{\kappa} \arrow[r] & W \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \in \hat{\omega} \cdots (\#\#\#) \\
 & & X \arrow[r, equal] \arrow[d] & X \arrow[d] & & \\
 & & 0 & 0 & & \leftarrow \hat{\omega}\text{-approx of } M.\ \checkmark
\end{tikzcd}$$

$\hat{\mathcal{X}} = (*) = (**)$:

We have shown $\hat{\mathcal{X}} \subset (*) \underset{\substack{\uparrow \\ \text{See above argument.}}}{\subset} (**)$.

Similarly $(**) \subset (*)$.

$\Big($ $\because$ $(\#) \leftarrow \omega \subset \hat{\omega}$: generator by the def of $\hat{\omega}$.

$(\#\#\#) \leftarrow \mathcal{X}$: epikers. $\Big)$

If $M \in (*)$, then

$$\begin{tikzcd}
 & & & & & 0 & & \\
 & & & 0 \arrow[r] & Y_M \arrow[r] \arrow[ur] & X_M \arrow[r] & M \arrow[r] & 0 \\
 & & & W_0 \arrow[ur] & & \rotatebox[origin=c]{-90}{$\in$} & & \\
 & & \cdots \arrow[ur] & & & \mathcal{X} & & \therefore M \in \hat{\mathcal{X}}. \\
 & W_n \arrow[ur] & & \in \omega \subset \mathcal{X} & & & & \\
0 \arrow[ur] & & & & & & &
\end{tikzcd}$$

$\therefore \hat{\mathcal{X}} = (*) = (**)$. ✓

$(\hat{\mathcal{X}} : \text{ext})$:

Let
$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 . \\
 & \hat{\mathcal{X}} \arrow[u, phantom, sloped, "\in"] & & \hat{\mathcal{X}} \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\exists\, 0 \arrow[r] & Y_{M_1} \arrow[r] & X_{M_1} \arrow[r] & M_1 \arrow[r] & 0 . \\
 & \mathcal{Y} \arrow[u, phantom, sloped, "\in"] & \mathcal{X} \arrow[u, phantom, sloped, "\in"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\exists\, 0 \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] & M_2 \arrow[r] & 0 \\
 & & & K \arrow[ur, bend right] & & &
\end{tikzcd}$$
$\underbrace{X_n \to \cdots \to X_1 \to X_0}_{\in \mathcal{X}}$. Induction on n.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & \exists E \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & X_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 .
\end{tikzcd}$$
$\cdots (4)$

$$\begin{tikzcd}
\mathrm{Ext}^1(X_0, Y_{M_1}) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1(X_0, X_{M_1}) \arrow[r] & \mathrm{Ext}^1(X_0, M_1) \arrow[r] & \mathrm{Ext}^2(X_0, Y_{M_1}) \arrow[d, equal] \\
0 \arrow[r, Rightarrow] & \therefore \text{iso} & & 0 \arrow[ll, Rightarrow]
\end{tikzcd}$$

$$\begin{tikzcd}
\exists\, 0 \arrow[r] & X_{M_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & X \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & E \arrow[r] & X_0 \arrow[r] & 0
\end{tikzcd}$$
$X \in \mathcal{X}$ ($\because \mathcal{X}$: exts) $\cdots (44)$

Combining (4) and (44),

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\hat{\mathcal{X}} \supset \hat{\omega} \arrow[r, phantom, "\ni"] & 0 \arrow[r] & Y_{M_1} \arrow[r] \arrow[d] & \exists K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & X_{M_1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

If $n=0$, then $K=0$.
$K' \cong Y_{M_1} \in \hat{\mathcal{X}}$.
$\therefore M \in \hat{\mathcal{X}}$.
If $n \geqq 1$, then by ind. hyp.,
$K' \in \hat{\mathcal{X}}$. $\therefore M \in \hat{\mathcal{X}}$. ✓

($\hat{\mathcal{X}}$: epikers):

Let
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & M_0 \arrow[r] & M_1 \arrow[r] & 0. \\ & & \hat{\mathcal{X}} \arrow[u, phantom, "\rotatebox{-90}{$\in$}"] & \hat{\mathcal{X}} \arrow[u, phantom, "\rotatebox{-90}{$\in$}"] & \end{tikzcd}$$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & Y_{M_1} \arrow[r, equal] \arrow[d] & Y_{M_1} \arrow[d] & \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & X_{M_1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & M_0 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0. \\ & & 0 & 0 & \end{tikzcd}$$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ (\because \mathcal{X}\text{: epikers})\ \mathcal{X} \arrow[dr, phantom, "\ni"] & Y_E \arrow[r, equal] \arrow[d] & Y_E \arrow[d] & & \\ 0 \arrow[r] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & X_E \arrow[r] \arrow[d] & X_{M_1} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & X_{M_1} \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

↖ $\mathcal{X}$-approx of M. $\therefore M \in \hat{\mathcal{X}}$. ✓

($\hat{\mathcal{X}}$: monocoks):

Let
$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & M \arrow[r] & 0. \\ & \hat{\mathcal{X}} \arrow[u, phantom, "\rotatebox{-90}{$\in$}"] & \hat{\mathcal{X}} \arrow[u, phantom, "\rotatebox{-90}{$\in$}"] & & \end{tikzcd}$$

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
(\because \hat{\mathcal{X}}:\text{ats}) & Y_{M_0} \arrow[d] \arrow[r, equal] & Y_{M_0} \arrow[d] & & \\
\overset{\cap}{\hat{\mathcal{X}}} \arrow[r] & E & & & \\
0 \arrow[r] & E \arrow[r] \arrow[d] \arrow[rd, phantom, "PB"] & X_{M_0} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

$\boxed{0 \to E \to X_{M_0} \to M \to 0}$ $\therefore M \in \hat{\mathcal{X}}$. ✓.

$(\hat{\mathcal{X}}:\text{additive})$:

Easy to see $\hat{\mathcal{X}} : \overset{\text{finite}}{\oplus}$.

n is fixed here.

$\underbrace{{}^{\exists} 0 \to \underset{\sim}{X_n} \to \cdots \to X_0 \to \overset{M_1}{\underset{M_2}{\oplus}} \to 0}_{}$

Let $M_1 \oplus M_2 \in \hat{\mathcal{X}}$. We show $M_1, M_2 \in \hat{\mathcal{X}}$.

$$\begin{tikzcd}
X_0 \arrow[r, two heads] \arrow[d, equal] & M_1 \oplus M_2 . \\
X_0''
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & K_i \arrow[r] & X_0'' \arrow[r] \arrow[rd, two heads] & M_i \arrow[r] & 0 \ (i=1,2). \\
& & & M_1 \oplus M_2 \arrow[u] \arrow[ul, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$$0 \to \underset{\hat{\mathcal{X}}}{\underset{\cap}{\overset{K_1}{\underset{K_2}{\oplus}}}} \to \overset{X_0''}{\underset{X_0''}{\oplus}} \to \overset{M_1}{\underset{M_2}{\oplus}} \to 0 .$$

$(\because \hat{\mathcal{X}}:\text{epikers})$.

By repeating this, ${}^{\exists} 0 \to N_i \to \underbrace{X''_{n-1} \to \cdots \to X''_0}_{\in \mathcal{X}} \to M_i \to 0$,

$$\underbrace{0 \to \overset{N_1}{\underset{N_2}{\oplus}} \to \overset{X''_{n-1}}{\underset{X''_{n-1}}{\oplus}} \to \cdots \to \overset{X_0''}{\underset{X_0''}{\oplus}} \to \overset{M_1}{\underset{M_2}{\oplus}} \to 0}_{\in \hat{\mathcal{X}}} .$$

ETS: $N_1 \oplus N_2 \in \mathcal{X}$. $(\because \Rightarrow N_1, N_2 \in \mathcal{X})$
enough to show

<u>Claim</u> $0 \to \underbrace{X_n \to \cdots \to X_0}_{\in \mathcal{X}} \to M \to 0$,

$$0 \to K \to \underbrace{X'_{n-1} \to \cdots \to X'_0}_{\in \mathcal{X}} \to M \to 0$$

$\Rightarrow K \in \mathcal{X}$.

$(\because)$

$$\begin{tikzcd}
0 \arrow[r] & X_n \arrow[r] \arrow[d, phantom, "\|"] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, two heads] & X_0 \arrow[r] \arrow[d, two heads] & M \arrow[r] \arrow[d, phantom, "\|"] & 0. \\
& B_n \arrow[ur, hook] & & & B_1 \arrow[ur, hook] & B_0 & B_0 &
\end{tikzcd}$$

$\mathrm{Ext}^{>0}(B_n, \hat{\omega}) = 0$.

By $0 \to B_n \to \underset{\substack{\cap \\ \mathcal{X}}}{X_{n-1}} \to B_{n-1} \to 0$ and $\mathrm{Ext}^{>0}(\mathcal{X}, \hat{\omega}) = 0$,

$\mathrm{Ext}^{>1}(B_{n-1}, \hat{\omega}) = 0$.

Repeating this, $\mathrm{Ext}^{>n}(\underset{\substack{\| \\ M}}{B_0}, \hat{\omega}) = 0$.

Similarly, using $0 \to K \to X'_{n-1} \to \cdots \to X'_0 \to M \to 0$,

$\mathrm{Ext}^{>n}(M, \hat{\omega}) = 0 \rightsquigarrow \mathrm{Ext}^{>0}(K, \hat{\omega}) = 0$.

$0 \to \underset{\substack{\cap \\ \hat{\omega}}}{Y_K} \to \underset{\substack{\cap \\ \mathcal{X}}}{X_K} \to K \to 0$ splits. $K \,(\!\oplus \underset{\substack{\cap \\ \mathcal{X}}}{X_K}$.

Since $\mathcal{X}$: additive, $K \in \mathcal{X}$. ⌋

$\hat{\omega} = \mathcal{X}^{\perp} \cap \hat{\mathcal{X}}$ has also shown in the proof of Claim.

$\therefore\ \hat{\omega}$: additive. $\lrcorner$

(3)(4) Prop 1.6. $\lrcorner$ //

Ex2.2 R: Iwanaga-Gorenstein ring.

(i.e. R_R, ${}_RR$: noeth, $\underset{\substack{\| \\ \text{inj.dim}}}{\mathrm{id}}\, R_R$, $\mathrm{id}\, {}_RR < \infty$)

($\Rightarrow \mathrm{id}\, R_R = \mathrm{id}\, {}_RR$)

$\mathcal{A} := \mathrm{mod} R$.

$\mathcal{X} := \mathrm{CM}(R) := {}^{\perp}R = \{M \in \mathrm{mod} R \mid \mathrm{Ext}^{>0}(M,R) = 0\}$

: Cohen-Macaulay modules.

$\omega := \mathrm{proj}\, R$.

$(-)^* := \mathrm{Hom}_R(-,R)$.

$$\begin{tikzcd}
\mathrm{CM}(R) \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\sim"] & \mathrm{CM}(R^{op}) \arrow[l, shift left, "(-)^*"] \\
\mathrm{proj}\, R \arrow[u, phantom, "\cup" rotate=90] \arrow[r, shift left] \arrow[r, phantom, "\sim"] & \mathrm{proj}\, R^{op} \arrow[u, phantom, "\cup" rotate=90] \arrow[l, shift left]
\end{tikzcd}.$$

(∵) $M \in \mathrm{CM}(R)$. We show $M^* \in \mathrm{CM}(R^{op})$ and $M \xrightarrow{\sim} M^{**}$.

$$\exists\, 0 \to \underset{\substack{\nearrow \\ \text{“syzygy of } M\text{”}}}{\Omega M} \to \underset{\substack{\cap \\ \mathrm{proj}\, R}}{P_0} \to M \to 0 \rightsquigarrow \Omega M \in \mathrm{CM}(R).$$

$$0 \to M^* \to \underset{\substack{\cap \\ \mathrm{proj}\, R^{op}}}{P_0^*} \to (\Omega M)^* \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1(M,R)}.$$

$$\therefore \operatorname{Ext}^i(M^*, R) \cong \operatorname{Ext}^{i+1}((\Omega M)^*, R), \quad \forall i>0$$

$$\cong \operatorname{Ext}^{i+2}((\Omega^2 M)^*, R)$$

$$\cong \cdots \cong 0 \quad (\because \operatorname{id}\, {}_R R < \infty)$$

$\therefore M^* \in \mathrm{CM}(R^{op})$.

$$\begin{tikzcd}
 & & & \Omega M \arrow[dr, hook] & & & \\
0 \arrow[r] & \Omega^2 M \arrow[r] & P_1 \arrow[rr] \arrow[ur] \arrow[d, "\wr"] & & P_0 \arrow[r] \arrow[d, "\wr"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\Omega^2 M)^{**} \arrow[r] & P_1^{**} \arrow[rr] \arrow[dr] & & P_0^{**} \arrow[r] & M^{**} \arrow[r] & 0 \\
 & & & (\Omega M)^{**} \arrow[ur, hook] & & &
\end{tikzcd}$$

(Exactness comes from $M \in \mathrm{CM}(R)$.)

$\therefore M \xrightarrow{\simeq} M^{**}$. //

$\mathrm{proj}\, R \subset \mathrm{CM}(R)$: inj cogen.

$(\because)$ $\forall M \in \mathrm{CM}(R)$, $M^* \in \mathrm{CM}(R^{op})$.

$$0 \to \underset{\mathrm{CM}(R^{op})}{\underset{\cap}{\Omega(M^*)}} \to \underset{\mathrm{proj}\, R^{op}}{\underset{\cap}{P_0'}} \to M^* \to 0.$$

$$0 \to M^{**} \to \underset{\mathrm{proj}\, R}{\underset{\cap}{P_0'^{*}}} \to \underset{\mathrm{CM}(R)}{\underset{\cap}{(\Omega(M^*))^*}} \to 0. \quad //$$

$\hat{\mathcal{X}} = \mathrm{mod}\, R$.

$(\because)$ $n := \operatorname{id} R_R < \infty$. $\operatorname{Ext}^{>n}(M, R) = 0$.

$\operatorname{Ext}^{>n-1}(\Omega M, R) = 0$.

$\vdots$

$$\mathrm{Ext}^{>0}(\Omega^n M, R) = 0.$$

$$\therefore\ 0 \to \underset{\mathrm{CM}(R)}{\underset{\cap}{\Omega^n M}} \to \underbrace{P_{n-1} \to \cdots \to P_0}_{\in \operatorname{proj} R \subset \mathrm{CM}(R)} \to M \to 0. \quad //$$

$$\hat{\omega} = \{M \in \operatorname{mod} R \mid \underset{\underset{\text{proj dim}}{\|}}{\operatorname{pd}} M < \infty\}.$$

$\therefore$ $(\mathrm{CM}(R), \{\operatorname{pd} < \infty\}) \subset \operatorname{mod} R$: cotorsion pair.

$\mathrm{CM}(R) \cap \{\operatorname{pd} < \infty\} = \operatorname{proj} R$.

<u>Ex 2.3</u> R: comm noeth local Cohen-Macaulay ring with a canonical module ω_R

i.e. $\begin{cases} \bullet\ \omega_R \in \operatorname{mod} R \\ \bullet\ \mathrm{Ext}^{>0}(\omega_R, \omega_R) = 0 \\ \bullet\ \operatorname{id} \omega_R < \infty \\ \bullet\ R \xrightarrow{\sim} \operatorname{End}(\omega_R). \end{cases}$

$$\Rightarrow (\underset{\underset{\{M \in \operatorname{mod} R \mid \mathrm{Ext}^{>0}(M, \omega_R) = 0\}.}{\|}}{\mathrm{CM}(R)}, \{\operatorname{pd} < \infty\}) \subset \operatorname{mod} R : \text{cotorsion pair}$$

$\mathrm{CM}(R) \cap \{\operatorname{pd} < \infty\} = \operatorname{add} \omega_R$.

§3 Resolving subcats

A: fin. dim k-alg.

Def : $\mathcal{X} \subset \mathrm{mod}\,A$: resolving

:$\Leftrightarrow$ $\mathcal{X}$: additive, closed under exts, epikers,

proj $A \subset \mathcal{X}$.

Coresolving subcats are defined dually.

Thm 3.1 [Auslander–Reiten 1991 "Applications of ⋯"]

$$\left\{\mathcal{X} \subset \mathrm{mod}A : \begin{smallmatrix}\text{contrav-fin}\\ \text{resolu}\end{smallmatrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\mathcal{Y} \subset \mathrm{mod}A : \begin{smallmatrix}\text{cov-fin}\\ \text{coresolu}\end{smallmatrix}\right\}$$

$$\mathcal{X} \longmapsto \mathcal{X}^{\perp} = \mathcal{X}^{\perp_1}$$

$${}^{\perp_1}\mathcal{Y} = {}^{\perp}\mathcal{Y} \longleftarrow\!\!\shortmid\ \mathcal{Y}.$$

(∵) Let $\mathcal{X} \subset \mathrm{mod}\,A$: contrav-fin resolu.

$\mathcal{X}^{\perp} = \mathcal{X}^{\perp_1}$.

(∵) $\mathcal{X}^{\perp} \subset \mathcal{X}^{\perp_1}$ is obvious.

$$\forall N \in \mathcal{X}^{\perp_1}, \forall M \in \mathcal{X},\ 0 \to \underset{\underset{\sim}{\mathcal{X}}}{\underset{\rotatebox{90}{$\in$}}{\Omega M}} \to \underset{\mathrm{proj}A}{\underset{\rotatebox{90}{$\in$}}{P}} \to M \to 0.$$

$$\mathrm{Ext}^{i}(M,N) \cong \mathrm{Ext}^{i-1}(\Omega M, N) \cong \cdots \cong \mathrm{Ext}^{1}(\underset{\mathcal{X}}{\underset{\rotatebox{90}{$\in$}}{\Omega^{i-1}M}}, \underset{\mathcal{X}^{\perp_1}}{\underset{\rotatebox{90}{$\in$}}{N}}) = 0.$$

$(i > 0)$

$\therefore N \in \mathcal{X}^{\perp}$. ⌟

Easy to see $\mathcal{X}^{\perp}$: coresolu.

$\mathcal{X}^{\perp}$: cov-fin.

(∵) Let $M \in \mathrm{mod}A$. $\exists\, 0 \to M \to \underset{\underset{\mathcal{X}^{\perp}}{\cap}}{\underset{\mathrm{inj}A}{\underset{\rotatebox{90}{$\in$}}{I}}} \to C \to 0$.

$$\begin{tikzcd}
\exists\ 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & C \arrow[r] & 0 \\
& \mathcal{X}^{\perp_1} \arrow[u, phantom, sloped, "\in"] \arrow[d, equal] & \underbrace{\hphantom{X \to C}}_{\text{minil } \mathcal{X}\text{-approx}} & & \\
& \mathcal{X}^{\perp} & & &
\end{tikzcd}$$

$\mathcal{X}^{\perp}$ ($\because$ Wakamatsu's lemma)

(Rem: Since $\operatorname{proj} A \subset \mathcal{X}$, $\exists\, P \twoheadrightarrow C$.

$$\begin{tikzcd}
P \\
\operatorname{proj} A \arrow[u, phantom, sloped, "\in"] \\
\mathcal{X} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

Then easy to check every $\mathcal{X}$-approx $X \to C$ should be an epimorphism.)

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & M \arrow[r, equal] \arrow[d] & M \arrow[d, "\in \mathcal{X}^{\perp}"'] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[rd, phantom, "PB"] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}$$

with $\mathcal{X}^{\perp} \ni Y$.

$$\begin{tikzcd}
& \mathcal{X}^{\perp} & \mathcal{X} \subset {}^{\perp}(\mathcal{X}^{\perp}) & \\
0 \arrow[r] & \underbrace{M \arrow[r] & E}_{\mathcal{X}^{\perp}\text{-approx}} \arrow[u, phantom, sloped, "\in"] \arrow[r] & X \arrow[u, phantom, sloped, "\in"] \arrow[r] & 0 .
\end{tikzcd}$$

$\mathcal{X}^{\perp}$-approx ($\because$ Prop 1.5). $\therefore\ \mathcal{X}^{\perp}$: cov-fin. ⌋

$\forall M \in {}^{\perp}(\mathcal{X}^{\perp})$, $\exists\ 0 \to Y \to X \to M \to 0$. This splits.

$$\begin{tikzcd}
Y & X \\
\mathcal{X}^{\perp} \arrow[u, phantom, sloped, "\in"] & \mathcal{X} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$M \in \mathcal{X}$. $\therefore\ \mathcal{X} = {}^{\perp}(\mathcal{X}^{\perp})$. //

For $M_1,\cdots,M_n\in \mathrm{mod}A$,

$$\mathcal{F}(M_1,\cdots,M_n):=\{M\in\mathrm{mod}A \mid {}^\exists 0=L_0\subset\cdots\subset L_m=M,\ \frac{L_{i+1}}{L_i}\cong M_{k(i)},\ {}^\exists k(i)\}.$$

Thm 3.2 [AR] $\mathcal{X}\subset\mathrm{mod}A$: resolv, $S_1,\cdots,S_n\in\mathrm{mod}A$: all simples $(/\cong)$

Then

$\mathcal{X}$: contrav-fin $\Leftrightarrow {}^\forall i, {}^\exists X_i\to S_i$: right $\mathcal{X}$-approx.

If this holds, then

$\mathcal{X}=\{X\in\mathrm{mod}A \mid X <\!\oplus\ {}^\exists\tilde{X}\in\mathcal{F}(X_1,\cdots,X_n)\}$.

(∵) Let $0\to M'\to M\to M''\to 0$: exact s.t.

M', M'' have right $\mathcal{X}$-approxs.

$$\begin{tikzcd}
\text{Then } {}^\exists 0 \arrow[r] & Y' \arrow[r] \arrow[d, phantom, sloped, "\in"] & \underbrace{X' \arrow[r] & M'}_{\text{min'l }\mathcal{X}\text{-approx}} \arrow[r] & 0 & (\because \text{Wakamatsu's lemma}) \\
 & \mathcal{X}^\perp & & & & \\
{}^\exists 0 \arrow[r] & Y'' \arrow[r] \arrow[u, phantom, sloped, "\in"] & \overbrace{X'' \arrow[r] & M''} \arrow[r] & 0. &
\end{tikzcd}$$

Similarly to Thm 2.1,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y' \arrow[r, "\exists"] \arrow[d] & Y_{\in\mathcal{X}^\perp} \arrow[r] \arrow[d, "\exists"'] & Y'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X_{\in\mathcal{X}} \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

, $X\to M$: $\mathcal{X}$-approx.

Therefore, if $\exists X_i \to S_i$: right $\mathcal{X}$-approx, then every $M \in \mathrm{mod}A$ has a right $\mathcal{X}$-approx $\widetilde{X} \to M$, (with $\mathrm{Ker} \in \mathcal{X}^{\perp}$) where $\widetilde{X} \in \mathcal{F}(X_1, \cdots, X_n)$.

If $M \in \mathcal{X}$, then $0 \to \underset{\mathcal{X}^{\perp}}{\underset{\wedge}{\widetilde{Y}}} \to \widetilde{X} \to M \to 0$ splits.

$\therefore M \overset{\oplus}{<} \widetilde{X} \in \mathcal{F}(X_1, \cdots, X_n)$. //

§4 Cotilting modules.

$D := \mathrm{Hom}_k(-, k)$. $\quad \mathrm{mod}A \underset{D}{\overset{D}{\underset{\sim}{\rightleftarrows}}} \mathrm{mod}A^{op}$. (duality)

<u>Def</u> $T \in \mathrm{mod}A$: cotilting (of $\underset{\text{inj dim}}{\mathrm{id}} \leq d$)

$$:\Leftrightarrow \begin{cases} \cdot\ \mathrm{Ext}^{>0}(T, T) = 0 \\ \cdot\ \mathrm{id}\, T \leq d < \infty \\ \cdot\ \mathrm{inj}A \subset \widehat{\mathrm{add}T}. \text{ i.e. } \exists 0 \to \underbrace{T_d \to \cdots \to T_0}_{\in \mathrm{add}T} \to DA \to 0 \end{cases}$$

Tilting modules are defined dually.

<u>Thm4.1</u> [AR] $T \in \mathrm{mod}A$: cotilting

$\Rightarrow ({}^{\perp}T, \widehat{\mathrm{add}T}) \subset \mathrm{mod}A$: cotorsion pair.

${}^{\perp}T \cap \widehat{\mathrm{add}T} = \mathrm{add}T$.

($\because$) ${}^{\perp}T \subset \mathrm{mod}A$: resolv. $\mathrm{add}T \subset {}^{\perp}T$: injective

$\widehat{{}^{\perp}T} = \mathrm{mod}A$ can be shown similarly to Ex2.2.

ETS: $\mathrm{add}T \subset {}^{\perp}T$: cogen.

Let $M \in {}^{\perp}T$.

$\exists M \hookrightarrow \underset{\mathrm{add}T}{\underset{\cap}{T'}}$.

$(\because)$ $\exists 0 \to M \to \underset{\mathrm{inj}A}{\underset{\cap}{I}} \to C \to 0$ with

$$\begin{tikzcd} 0 \arrow[r] & T'_d \arrow[r] & \cdots \arrow[r] & T'_1 \arrow[r] \arrow[d, two heads] & T'_0 \arrow[r] & I \arrow[r] & 0 \\ & & & L \arrow[ur, hook, bend right] & & & \end{tikzcd}$$

$(\because T\text{: cotilting})$

$\underbrace{T'_d \to \cdots \to T'_0}_{\underset{\mathrm{add}T}{\cap}}$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d, "\in {}^{\perp}T"] & \\ 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & T_0 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

$L \in \widehat{\mathrm{add}T}$

: splits. $\left(\because \text{Easy to check } \underset{=0}{\mathrm{Ext}^{>0}({}^{\perp}T, \widehat{\mathrm{add}T})}\right)$

$\therefore M \mathrel{<\!\!\oplus} E \hookrightarrow T_0$. ⌋

$\mathrm{add}T \subset \mathrm{mod}A$: functorially-finite (i.e. cov-fin and contrav-fin).

$(\because)$ $\forall M' \in \mathrm{mod}A$, take $f_1, \cdots, f_n \in \mathrm{Hom}(T, M')$: k-basis.

Then $T^{\oplus n} \xrightarrow[{[f_1 \cdots f_n]}]{} M'$: right $\mathrm{add}T$-approx. ⌋

$\therefore$ By Wakamatsu's lemma, $\exists 0 \to \underbrace{M \to T'}_{\text{min'l add}T\text{-approx}} \to \underset{\in {}^{\perp_1}T}{N} \to 0$.

Since $M, T' \in {}^{\perp}T$, $N \in {}^{\perp}T$. ⌋ //

Thm 4.2 [AR]

$$\begin{tikzcd}
\{\mathcal{X} \subset \mathrm{mod}A : \text{contrav-fin resolu} \mid \hat{\mathcal{X}} = \mathrm{mod}A\} \arrow[d, leftrightarrow, "1-1"] & {}^{\perp}T & \mathcal{X} \arrow[dd, mapsto] & {}^{\perp}\mathcal{Y} \\
\{T \in \mathrm{mod}A : \text{basic cotilting}\}/\cong \arrow[d, leftrightarrow, "1-1"] & T \arrow[u, mapsto] \arrow[d, mapsto] & & \\
\{\mathcal{Y} \subset \mathrm{mod}A : \text{cov-fin coresolu} \mid \mathcal{Y} \subset \{\mathrm{id} < \infty\}\} & \widehat{\mathrm{add}T} & \mathcal{X}^{\perp} & \mathcal{Y} \arrow[uu, mapsto]
\end{tikzcd}$$

$\mathcal{X} \cap \mathcal{Y} = \mathrm{add}\,T.$

$(\because)$ $\hat{\mathcal{X}} = \mathrm{mod}\,A \Rightarrow \mathcal{X}^{\perp} \subset \{\mathrm{id} \leqq d\}$ $({}^{\exists}d)$.

$(\because)$ ${}^{\forall}S_i$: simple, ${}^{\exists}0 \to \underbrace{X^i_{d_i} \to \cdots \to X^i_0}_{\in \mathcal{X}} \to S_i \to 0$.
$(i=1,\cdots,n)$

$d := \max\{d_1,\cdots,d_n\}$.

$\mathrm{Ext}^{>0}(\mathcal{X}, \mathcal{X}^{\perp}) = 0 \rightsquigarrow \mathrm{Ext}^{>d}(S_i, \mathcal{X}^{\perp}) = 0$.

$\rightsquigarrow \mathrm{Ext}^{>d}(\mathrm{mod}A, \mathcal{X}^{\perp}) = 0$. ⌋

$\mathcal{Y} \subset \{\mathrm{id} < \infty\} \Rightarrow \widehat{{}^{\perp}\mathcal{Y}} = \mathrm{mod}\,A$.

$(\because)$ ${}^{\forall}S_i$: simple, ${}^{\exists}S_i \longrightarrow Y_i$: min'l $\mathcal{Y}$-approx.

By Thm3.2, $\mathcal{Y} = \mathrm{add}\,\mathcal{F}(Y_1,\cdots,Y_n)$.

$d := \underset{<\infty}{\max\{\mathrm{id}Y_1,\cdots,\mathrm{id}Y_n\}}$. Then $\mathcal{Y} \subset \{\mathrm{id} \leqq d\}$.

${}^{\forall}M \in \mathrm{mod}A$, $\mathrm{Ext}^i(\Omega^d M, \underset{\substack{\cap \\ \mathcal{Y}}}{Y}) \cong \mathrm{Ext}^{i+d}(M, Y) = 0$
$({}^{\forall}i > 0)$.

$\therefore$ $0 \to \underbrace{\Omega^d M \longrightarrow P_{d-1} \to \cdots \to P_0}_{\in {}^{\perp}\mathcal{Y}} \to M \to 0$. $M \in \widehat{{}^{\perp}\mathcal{Y}}$. ⌋

Let $\underset{\text{(See Thm3.1)}}{\mathcal{X} \leftrightarrow \mathcal{Y}}$. $\underset{\text{(∵ Thm4.1)}}{\text{ETS}: \exists T: \text{cotilting}}$, $\mathcal{X} \cap \mathcal{Y} = \operatorname{add} T$, $\widehat{\operatorname{add} T} = \mathcal{Y}$.

$\omega := \mathcal{X} \cap \mathcal{Y}$.

$\forall S_i$: simple, $\exists X_i \to S_i$: min'l $\mathcal{X}$-approx.

By Thm3.2, $\mathcal{X} = \operatorname{add} \mathcal{F}(X_1, \cdots, X_n)$.

$\exists\, 0 \to \underbrace{X_i \to T_i}_{\text{left min'l } \mathcal{Y}\text{-approx}} \to \underset{\substack{\cap \\ {}^{\perp}\mathcal{Y} = \mathcal{X}\ \text{(∵ Wakamatsu)}}}{X_i'} \to 0$. Then $T_i \in \mathcal{X} \cap \mathcal{Y} = \omega$.

Similarly to Thm3.2,

$\omega = \operatorname{add} \mathcal{F}(T_1, \cdots, T_n) = \operatorname{add} \underbrace{T_1 \oplus \cdots \oplus T_n}_{=:\, T}$ (∵ $\operatorname{Ext}^{>0}(\omega, \omega) = 0$).

$\operatorname{Ext}^{>0}(T, T) = 0$. $\operatorname{id} T < \infty$ (∵ $T \in \mathcal{Y}$).

We show $\mathcal{Y} = \widehat{\operatorname{add} T}$. Let $Y \in \mathcal{Y}$.

By the above argument, $\exists d$, $\mathcal{Y} \subset \{\operatorname{id} \leqq d\}$.

$\exists\, 0 \to \underset{\substack{\cap \\ \mathcal{Y}}}{Y'} \to \underbrace{\underset{\substack{\cap \\ \omega}}{W_0} \to Y}_{\text{right min'l } \mathcal{X}\text{-approx}} \to 0$. Repeating this,

$\exists\, 0 \to \underset{\substack{\cap \\ \mathcal{Y}}}{K} \to \underbrace{W_{d-1} \to \cdots \to W_0}_{\in\, \omega\, \subset\, {}^{\perp}\mathcal{Y}} \to Y \to 0$.

$\operatorname{Ext}^{>0}(K, \underset{\substack{\cap \\ \mathcal{Y}}}{Y''}) \cong \operatorname{Ext}^{>d}(Y, Y'') = 0$. $K \in \underset{\substack{\shortparallel \\ \mathcal{X}}}{{}^{\perp}\mathcal{Y}} \cap \mathcal{Y} = \omega$.

∴ $\mathcal{Y} \subset \hat{\omega} = \widehat{\operatorname{add} T}$. In particular $\operatorname{inj} A \subset \widehat{\operatorname{add} T}$.

T : cotilting.

Since $\mathcal{Y}$: coresolv, $\widehat{\mathrm{add}T} \subset \mathcal{Y}$. $\therefore \mathcal{Y} = \widehat{\mathrm{add}T}$.

$\mathcal{X} = {}^{\perp}T$. //

<u>Cor 4.3</u> If gl.dim $A < \infty$, then

$$\begin{tikzcd}
\{T \in \mathrm{mod}A : \text{basic } \underset{\underset{\text{tilting}}{\parallel}}{\text{cotilting}}\}/\cong \arrow[d, leftrightarrow, "1-1"] & \ni & T \arrow[d, mapsto] \\
\{(\mathcal{X}, \mathcal{Y}) \subset \mathrm{mod}A : \text{cotorsion pair}\} & \ni & ({}^{\perp}T, \underset{\underset{T^{\perp}}{\parallel}}{\widehat{\mathrm{add}T}})
\end{tikzcd}$$

($\because \mathcal{X} = {}^{\perp}(\mathcal{X} \cap \mathcal{Y}), \mathcal{Y} = \widehat{\mathcal{X} \cap \mathcal{Y}}$. Duals hold.)

<u>Fact 4.4</u> [AR]

$T \in \mathrm{mod}A$: cotilting $\Rightarrow {}^{\perp}T$: functorially-finite

(i.e. both contrav-fin and cov-fin).

($\because$) Use the equivalence below, which is involved by T. //

More generally,

<u>Fact 4.5</u> [Krause-Solberg 2003]

$\mathcal{X} \subset \mathrm{mod}A$: contrav-fin resolv $\Rightarrow \mathcal{X}$: cov-fin.

<u>Fact 4.6</u> [Miyashita 1986, Happel 1987]

$T \in \mathrm{mod}A$: tilting of pd $\leqq d$. $B := \mathrm{End}_A(T)$. Then

(1) $T \in \mathrm{mod}B^{op}$: tilting of pd $\leqq d$, $A \cong \mathrm{End}_B(T)^{op}$.

($DT \in \mathrm{mod}B$: cotilting of id $\leqq d$, $A \cong \mathrm{End}_B(DT)$)

(2) $0 \leqq {}^{\forall}i \leqq d$,

$$\begin{tikzcd}
\{M \in \mathrm{mod}\,A \mid \mathrm{Ext}_A^{\neq i}(T,M)=0\} \arrow[d, shift right=2, "\mathrm{Ext}_A^i(T,-)"'] \arrow[d, phantom, "\wr"] \\
\{N \in \mathrm{mod}\,B \mid \mathrm{Ext}_B^{\neq i}(N,DT)=0\}. \arrow[u, shift right=2, "D\mathrm{Ext}_B^i(N,DT)\ (\simeq \mathrm{Tor}_i^B(-,T))"']
\end{tikzcd}$$

$$\left(\overset{\shortparallel}{D\mathrm{Tor}^B_{\neq i}(T,N)} \right)$$

In particular, letting $i=0$,

$$\begin{tikzcd}
\mathrm{mod}\,A \arrow[r, shift left=2, "\mathrm{Hom}_A(T,-)"] \arrow[r, phantom, "\text{adjoint}"] \arrow[d, phantom, "\bigcup"] & \mathrm{mod}\,B \arrow[l, shift left=2, "{D\mathrm{Hom}_B(-,DT) \ (\overset{\shortparallel}{- \otimes_B T})}"] \arrow[d, phantom, "\bigcup"] \\
T^{\perp} \arrow[r, shift left=2] \arrow[r, phantom, "\sim"] & {}^{\perp}DT \arrow[l, shift left=2]
\end{tikzcd}$$

cov-fin contrav-fin.

(3) $$\begin{tikzcd} K^b(\mathrm{proj}\,A) \arrow[r, shift left=2, "\mathrm{Hom}_A(T,-)"] \arrow[r, phantom, "\sim"] & K^b(\mathrm{proj}\,B). \arrow[l, shift left=2, "{D\mathrm{Hom}_B(-,DT)}"] \end{tikzcd}$$

(4) $|A|=|T|=|B|$.

($|M|:=$ # of noniso indec summands of M)

§5 Applications to quasi-hereditary algs.

A: fin. dim k-alg.

<u>Setting</u> $\Theta=\{\Theta(1),\cdots,\Theta(n)\}\subset \mathrm{mod}\,A$

s.t. $\mathrm{Ext}^1(\Theta(i),\Theta(j))\neq 0 \Rightarrow i<j$.

Lem 5.1 $\mathcal{X} \subset \mathrm{mod}\, A$: closed under exts,

$\forall M \in \mathrm{mod} A$, $\exists\, 0 \to M \to \underset{\mathcal{X}^{\perp_1}}{\underset{\cap}{Y}} \to \underset{\mathcal{X}}{\underset{\cap}{X}} \to 0$

$\Rightarrow \mathcal{X}$: contrav-fin. ($\mathcal{X}^{\perp_1}$: cov-fin by Prop 1.5)

(∵) $\forall M \in \mathrm{mod} A$, $\exists\, M' \subset M$: largest satisfying $\exists\, \underset{\mathcal{X}}{\underset{\cap}{X}} \twoheadrightarrow M'$.

Let $0 \to K \to \underset{\mathcal{X}}{\underset{\cap}{X}} \to M' \to 0$.

$\exists\, 0 \to K \to \underset{\mathcal{X}^{\perp_1}}{\underset{\cap}{Y^K}} \to \underset{\mathcal{X}}{\underset{\cap}{X^K}} \to 0$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & X \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y^K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M' \arrow[r] & 0 \\
 & X^K \arrow[r, equal] \arrow[d] & X^K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

$E \in \mathcal{X}$

$$\begin{tikzcd}
E \arrow[rr, "\mathcal{X}\text{-approx.}"] \arrow[dr, "\mathcal{X}\text{-approx.}"'] & \circlearrowleft & M & \mathcal{X} \arrow[l] \\
 & M' \arrow[ur] & &
\end{tikzcd}$$

//

Lem 5.2 $M \in \mathrm{mod} A$, $\mathrm{Ext}^1(\Theta(>t), M) = 0$

$\Rightarrow \exists\, 0 \to M \to M' \to \underset{\Theta(t)^{\oplus}}{\underset{\parallel}{Q}} \to 0$, $\mathrm{Ext}^1(\Theta(\geq t), M') = 0$.

(∵) If $\mathrm{Ext}^1(\Theta(t), M) \neq 0$, then

$\exists\, 0 \to M \to M_1 \to \Theta(t) \to 0$: nonsplit.

$0 \to \mathrm{Hom}(\Theta(t), M) \to \mathrm{Hom}(\Theta(t), M_1) \to \mathrm{Hom}(\Theta(t), \Theta(t))$

$\xrightarrow[\underset{0}{\ast}]{} \underset{\text{fin.dim}}{\mathrm{Ext}^1(\Theta(t), M)} \underset{\ncong}{\twoheadrightarrow} \underset{\text{fin.dim}}{\mathrm{Ext}^1(\Theta(t), M_1)} \to 0$

Similarly $\mathrm{Ext}^1(\Theta(>t), M_1)=0$.

Repeating this, $\exists\, 0 \to M \to M' \to \underset{\mathcal{F}(\Theta(t))}{\underset{\cap}{Q}} \to 0$, $\mathrm{Ext}^1(\Theta(\geqq t), M')=0$

$\mathrm{Ext}^1(\Theta(t), \Theta(t))=0$.

$Q = \Theta(t)^{\oplus}$. //

<u>Lem 5.3</u> $M \in \mathrm{mod}\,A$, $\mathrm{Ext}^1(\Theta(>t), M)=0$

$$\Rightarrow \exists\, 0 \to M \to \underset{\mathcal{F}(\Theta)^{\perp_1}}{\underset{\cap}{Y}} \to \underset{\mathcal{F}(\Theta(\leqq t)) \subset \mathcal{F}(\Theta)}{\underset{\cap}{X}} \to 0.$$

(∵) Lem 5.2. Note $\mathcal{F}(\Theta)^{\perp_1} = \Theta^{\perp_1}$. //

<u>Thm 5.4</u> [Ringel 1989]

$\mathcal{F}(\Theta) \subset \mathrm{mod}\,A$: functorially-finite.

(∵) By Lem 5.3 and Lem 5.1, $\mathcal{F}(\Theta)$: contrav-fin.

$D\Theta := \{D\Theta(n), \cdots, D\Theta(1)\} \subset \mathrm{mod}\,A^{op}$

$\rightsquigarrow \underset{D\mathcal{F}(\Theta)}{\underset{\parallel}{\mathcal{F}(D\Theta)}} \subset \mathrm{mod}\,A^{op}$: contrav-fin

$\mathcal{F}(\Theta) \subset \mathrm{mod}\,A$: cov-fin. //

<u>Fact 5.5</u> [Auslander–Smalø 1981]

$\mathcal{C} \subset \mathrm{mod}\,A$: func-fin, closed under $\oplus$

$\Rightarrow \mathcal{C}$ has almost split seqs.

(Hence $\mathrm{add}\,\mathcal{F}(\Theta)$ has almost split seqs.).

A : quasi-hereditary alg.

$S(1), \cdots, S(n)$: $\overset{\text{(all)}}{\text{simples}}$ $(/\cong)$

$P(1), \cdots, P(n)$: indec projs.

$I(1), \cdots, I(n)$: indec injs.

$\Delta(1), \cdots, \Delta(n)$: standard mods. $\Delta := \{\Delta(1), \cdots, \Delta(n)\}$

$\nabla(1), \cdots, \nabla(n)$. costandard mods. $\nabla := \{\nabla(1), \cdots, \nabla(n)\}$.

Recall (1) $P(i) \twoheadrightarrow \Delta(i)$: largest quot s.t. $[\Delta(i) : S(>i)] = 0$.

$I(i) \supset \nabla(i)$: largest sub s.t. $[\nabla(i) : S(>i)] = 0$.

$[\Delta(i) : S(i)] = 1$.

$[\nabla(i) : S(i)] = 1$.

$P(i) \in \mathcal{F}(\Delta(i), \cdots, \Delta(n))$

$I(i) \in \mathcal{F}(\nabla(i), \cdots, \nabla(n))$.

(2) $\mathrm{gl.dim} A < \infty$.

Prop 5.6 (1) $\mathrm{Hom}(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$.

(2) $\mathrm{Ext}^1(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j$.

(3) $\mathrm{Hom}(\Delta(i), \nabla(j)) \neq 0 \Rightarrow i = j$.

(4) $\mathrm{Ext}^1(\Delta, \nabla) = 0$.

(∵) (2) $0 \to K(i) \to P(i) \to \Delta(i) \to 0$.

$$\begin{tikzcd}
\mathrm{Hom}(K(i),\Delta(j)) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\neq$}"] & \mathrm{Ext}^1(\Delta(i),\Delta(j)) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\neq$}"] & \mathrm{Ext}^1(P(i),\Delta(j)) \arrow[d, phantom, "\rotatebox{90}{$=$}"] \\
0 & 0 \arrow[l, Rightarrow] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
K(i) \arrow[rr, "\neq 0", "\circlearrowleft"'] \arrow[dr, two heads] & & \Delta(j). \\
& B \arrow[ur] \arrow[d, phantom, "\rotatebox{90}{$\neq$}"] & \\
& 0 &
\end{tikzcd}$$

Since $\underset{\substack{\uparrow \\ P(i)}}{\Delta(i)}$: largest quot s.t. $[\Delta(i) : S(>i)] = 0$,

$[B : S(>i)] \neq 0$.

Since $[\Delta(j) : S(>j)] = 0$, $[B : S(>j)] = 0$.

$\therefore i < j$. ⌋

(3)
$$\begin{tikzcd} \Delta(i) \arrow[rr, "\neq 0"] \arrow[dr, two heads] & & \nabla(j) \\ & B \arrow[ur, hook] & \end{tikzcd}$$

((1) Similar. ⌋)

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$[B : S(>i)] = 0$ $\qquad$ $[B : S(>j)] = 0$

$[B : S(i)] \neq 0$ $\qquad$ $[B : S(j)] \neq 0$. $\quad \therefore i = j$. ⌋

(4) $0 \to K(i) \to P(i) \to \Delta(i) \to 0$.

$0 \to \mathrm{Hom}(\Delta(i), \nabla(j)) \to \mathrm{Hom}(P(i), \nabla(j)) \to \mathrm{Hom}(K(i), \nabla(j))$

$\to \mathrm{Ext}^1(\Delta(i), \nabla(j)) \to 0$.

$j \leqq i \Rightarrow \mathrm{Hom}(K(i), \nabla(j)) = 0 \Rightarrow \mathrm{Ext}^1(\Delta(i), \nabla(j)) = 0$.

($\because$ top $K(i) \in \mathrm{add}\{S(i+1), \cdots, S(n)\}$)

Assume $i < j$. $\forall f : K(i) \to \nabla(j)$,

$$\begin{tikzcd} 0 \arrow[r] & K(i) \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & P(i) \arrow[r] \arrow[d, "\exists"] \arrow[dr, phantom, "\circlearrowright"] & \Delta(i) \arrow[r] \arrow[d, dashed, "\exists g"] & 0 \\ 0 \arrow[r] & \nabla(j) \arrow[r] & I(j) \arrow[r] & C(j) \arrow[r] & 0 \end{tikzcd}$$

$g = 0$. ($\because$ soc $C(j) \in \mathrm{add}\{j+1, \cdots, n\}$).

$$\begin{tikzcd} K(i) \arrow[r] \arrow[d, "f"'] & P(i) \arrow[d] \arrow[dl, "\exists"'] \arrow[dl, phantom, shift left=3, "\circlearrowright"] \\ \nabla(j) \arrow[r, hook] & I(j) \end{tikzcd} \quad . \quad \begin{tikzcd} K(i) \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & P(i) \arrow[dl] \\ \nabla(j) & \end{tikzcd}$$

$\therefore$

$$\therefore \operatorname{Hom}(P(i), \nabla(j)) \twoheadrightarrow \operatorname{Hom}(K(i), \nabla(j)) \to \underset{\substack{/\!/ \\ 0}}{\operatorname{Ext}^1(\Delta(i), \nabla(j))} \to 0$$

0. ⌋ //

Lem 5.7 (1) $\mathcal{F}(\Delta)$: closed under epikers.

(2) $\operatorname{Ext}^{>0}(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$.

(∵)(1) Let $0 \to M' \to \underbrace{M \to M''}_{\in \mathcal{F}(\Delta)} \to 0$.

ETS: $M' \in \mathcal{F}(\Delta)$ when $M'' = \Delta(i)$ $(\exists i)$

Since $\operatorname{Ext}^1(\Delta(j), \Delta(\leqq j)) = 0$,

$\exists\, 0 = L_n \subset \cdots \subset L_0 = M$ s.t. $\frac{L_{j-1}}{L_j} \cong \Delta(j)^{\oplus}$.

$\operatorname{Hom}(\Delta(>i), \Delta(i)) = 0$. $L_i \in \mathcal{F}(\Delta(>i))$.

$$\therefore \begin{tikzcd} 0 \arrow[r] & L_i \arrow[r] \arrow[dr, "0"'] & M \arrow[r] \arrow[d, two heads] \arrow[dl, phantom, shift right=3, "\circlearrowright"] \arrow[dr, phantom, shift left=2, "\circlearrowright"] & \frac{M}{L_i} \arrow[r] \arrow[dl, dashed, "\exists"] & 0 \\ & & \Delta(i) & & \end{tikzcd}$$

$$\begin{tikzcd} \Delta(i)^{\oplus} \arrow[dr, phantom, sloped, "="] & & & & \\ 0 \arrow[r] & \frac{L_{i-1}}{L_i} \arrow[r] \arrow[dr, "0 \neq"'] & \frac{M}{L_i} \arrow[r] \arrow[d, two heads] \arrow[dl, phantom, shift right=3, "\circlearrowright"] & \frac{M}{L_{i-1}} \arrow[r] & 0 \\ & & \Delta(i) & & \end{tikzcd}$$

(∵ Since $[\frac{M}{L_{i-1}} : S(i)] = 0$, $\frac{M}{L_{i-1}} \to \Delta(i)$ cannot be epi.)

$$\begin{tikzcd}
\cdot\cdot\ 0 \arrow[r] & {}^{\exists}\Delta(i) \arrow[r] \arrow[dr, "\neq 0"'] & \frac{M}{L_i} \arrow[r] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & N \arrow[r] & 0 \\
 & & \Delta(i) & \mathcal{F}(\Delta) \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

epi. $\left(\because \text{If not, } \begin{tikzcd} \Delta(i) \arrow[r] \arrow[dr, "{=0}"'] & \Delta(i) \\ & \mathrm{rad}\Delta(i) \arrow[u, phantom, sloped, "\subset"] \arrow[u, phantom, shift left=3, "\circlearrowleft"] \end{tikzcd} \text{ since } [\mathrm{rad}\Delta(i) : S(i)] = 0 \right)$

$\therefore$ iso.

$\therefore$ This seq splits. $\mathrm{Ker}\left[\begin{array}{c} \frac{M}{L_i} \\ \downarrow \\ \Delta(i) \end{array}\right] \cong N.$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L_i \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_i \arrow[r] & M \arrow[r] \arrow[d] & \frac{M}{L_i} \arrow[r] \arrow[d] & 0 \\
 & & \Delta(i) \arrow[r, equal] \arrow[d] & \Delta(i) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$L_i, N \in \mathcal{F}(\Delta)$. $\therefore M' \in \mathcal{F}(\Delta)$. ⌟

(2) $\mathcal{F}(\Delta)$: closed under exts, epiker,
proj $A \subset \mathcal{F}(\Delta)$. (resolu except $\oplus$).
Since $\mathrm{Ext}^1(\Delta, \nabla) = 0$,
$\mathcal{F}(\nabla) \subset \mathcal{F}(\Delta)^{\perp_1} = \mathcal{F}(\Delta)^{\perp}$. ⌟//

Thm 5.8 [R] $(\mathcal{F}(\Delta),\mathcal{F}(\nabla)) \subset \mathrm{mod}\, A$: cotorsion pair.

(∵) We show $\mathcal{F}(\Delta) = {}^\perp\mathcal{F}(\nabla)$.

$\forall M \in {}^\perp\mathcal{F}(\nabla)$,

let $M \twoheadrightarrow \frac{M}{M_i}$: largest quot s.t. $[\frac{M}{M_i} : S(>i)] = 0$.

We show $\frac{M_{i-1}}{M_i} \cong \Delta(i)^\oplus$ by the induction on i.

By ind hyp, $\frac{M}{M_{i-1}} \in \mathcal{F}(\Delta) \subset {}^\perp\mathcal{F}(\nabla)$.

$$0 \to \underset{\cap}{M_{i-1}} \to \underbrace{M \to \frac{M}{M_{i-1}}} \to 0.$$

$${}^\perp\mathcal{F}(\nabla) \Longleftarrow \in {}^\perp\mathcal{F}(\nabla)$$

Since top $\frac{M_{i-1}}{M_i} = S(i)^\oplus$, $\exists$

$$\begin{tikzcd} P(i)^\oplus \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & \frac{M_{i-1}}{M_i} \\ \Delta(i)^\oplus \arrow[ur, two heads] & \end{tikzcd}$$

: proj cover

$\left(\because [\frac{M_{i-1}}{M_i} : S(>i)] = 0\right)$

$$0 \to K \to \Delta(i)^\oplus \to \frac{M_{i-1}}{M_i} \to 0.$$

Assume $K \neq 0$. Then $\exists j$, $K \twoheadrightarrow S(j) \hookrightarrow \nabla(j)$.

$$\begin{tikzcd} K \arrow[r, two heads] \arrow[rr, bend right, "\neq 0"'] & S(j) \arrow[r, hook] & \nabla(j) \end{tikzcd}$$

$[\overset{\overset{\mathrm{rad}\,\Delta(i)^\oplus}{\cup}}{K} : S(\geq i)] = 0$. $\therefore j < i$.

$$0 \to \mathrm{Hom}(\tfrac{M_{i-1}}{M_i}, \nabla(j)) \to \overset{\overset{0}{\|}}{\mathrm{Hom}(\Delta(i)^\oplus, \nabla(j))} \to \overset{\overset{0}{\#}}{\mathrm{Hom}(K, \nabla(j))}$$

$$\xrightarrow{\sim} \underset{\underset{0}{\#}}{\mathrm{Ext}^1(\tfrac{M_{i-1}}{M_i}, \nabla(j))} \to \underset{\underset{0}{\|}}{\mathrm{Ext}^1(\Delta(i)^\oplus, \nabla(j))}$$

$\therefore$

$$0 \to \mathrm{Hom}(\tfrac{M_{i-1}}{M_i}, \nabla(j)) \to \mathrm{Hom}(M_{i-1}, \nabla(j)) \to \mathrm{Hom}(M_i, \nabla(j))$$

$$\to \underset{\underset{0}{\#}}{\mathrm{Ext}^1(\tfrac{M_{i-1}}{M_i}, \nabla(j))} \to \underset{\underset{0}{\|}}{\mathrm{Ext}^1(M_{i-1}, \nabla(j))} \quad \therefore \underset{0}{\#}.$$

$(\because M_{i-1} \in {}^\perp\mathcal{F}(\nabla))$

Since top $M_i \in \mathrm{add}\{S(i+1), \cdots, S(n)\}$, $\lightning$.

$\therefore K = 0$. $\frac{M_{i-1}}{M_i} \cong \Delta(i)^{\oplus}$, $\mathcal{F}(\Delta) = {}^{\perp}\mathcal{F}(\nabla)$.

$\therefore \mathcal{F}(\Delta)$: $\langle\oplus$. resolving, contrav-fin.

By Wakamatsu's lemma,

$$\left.\begin{array}{c} (\mathcal{F}(\Delta), \mathcal{F}(\Delta)^{\perp}) \\ \Vert \qquad \Vert \\ ({}^{\perp}\mathcal{F}(\nabla), \mathcal{F}(\nabla)) \end{array}\right\} \text{: cotorsion pair in mod } A. \;/\!/$$

§6 Ringel dual.

<u>Cor6.1</u> $\exists\, T \in \mathrm{mod}\,A$: $\overset{\text{basic}}{\text{tilting}}$, cotilting, $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathrm{add}\,T$.

$|T| = |A|$, $\mathcal{F}(\Delta) = {}^{\perp}T$, $\mathcal{F}(\nabla) = T^{\perp}$.

(T is called the characteristic module.)

($\because$) Since gl.dim $A < \infty$, this follows from Cor4.3, Fact4.6(4). //

<u>Prop 6.2</u> $T = \bigoplus_{i=1}^{n} T(i)$ s.t.

$$\exists\; 0 \to \underbrace{\Delta(i) \to T(i)}_{\text{left min'l } \mathcal{F}(\nabla)\text{-approx}} \to \underset{\mathcal{F}(\Delta(<i))}{\underset{\cap}{X(i)}} \to 0,$$

$$\exists\; 0 \to \underset{\mathcal{F}(\nabla(<i))}{\underset{\cap}{Y(i)}} \to \underbrace{T(i) \to \nabla(i)}_{\text{right min'l } \mathcal{F}(\Delta)\text{-approx}} \to 0.$$

($\because$) Since $\mathrm{Ext}^1(\Delta(\geq i), \Delta(i)) = 0$, by Lem5.3,

$$\exists\; 0 \to \Delta(i) \to \underset{\underset{\mathcal{F}(\nabla)}{\Vert}}{\underset{\mathcal{F}(\Delta)^{\perp_1}}{\underset{\cap}{Y}}} \to \underset{\mathcal{F}(\Delta(<i))}{\underset{\cap}{X}} \to 0 \;.\quad Y \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathrm{add}\,T.$$

$$[Y : S(i)] = \underset{1}{\underset{\shortparallel}{[\Delta(i) : S(i)]}} + \underset{0}{\underset{\shortparallel}{[X : S(i)]}} = 1 .$$

$\therefore\ Y = {}^{\exists}Y_1 \oplus \underset{\text{indec}}{\cdots} \oplus Y_*$, $[Y_1 : S(i)] = 1$. $[Y_{>1} : S(i)] = 0$.

$\mathrm{Hom}(\Delta(i), Y_2 \oplus \cdots \oplus Y_*) = 0$.

$\therefore$ Removing $Y_2 \oplus \cdots \oplus Y_*$, we can assume $Y = Y_1$.

($\because$ Direct summands of X belongs to $\mathcal{F}(\Delta)$, and "$\in \mathcal{F}(\Delta(<i))$" can be shown by considering $[- : S(\geq i)]$.)

$T(i) := Y = Y_1$: indec

$[T(i) : S(i)] = 1$, $[T(i) : S(>i)] = 0$.

$\therefore\ i \neq j \Rightarrow T(i) \not\cong T(j)$. $\quad T = \bigoplus_{i=1}^{n} T(i)$ since $|T| = |A|$. //

Def $A' := \mathrm{End}_A(T)$: Ringel dual of A.

Thm 6.3 [R]

A' : quasi-hereditary with $\Delta' = \{\mathrm{Hom}(T, \nabla(n)), \cdots, \mathrm{Hom}(T, \nabla(1))\}$.

($\because$) Since $T \in \mathrm{mod}\,A$: tilting of pd$<\infty$,

$$\begin{tikzcd} \mathrm{mod}\,A & & \mathrm{mod}\,A' \\ \cup & & \cup \\ T^{\perp} \arrow[rr, "\sim", "\mathrm{Hom}_A(T,-)"'] & & {}^{\perp}DT \\ & \shortparallel & \end{tikzcd} \qquad \mathrm{add}\,T \xrightarrow{\sim} \mathrm{proj}\,A'$$

F preserves exactness ($\because$ Ext^1 vanishes.)

$i' := n+1-i$.

$P'(i) := F(T(i')) \in \mathrm{proj}\,A'$.

$\Delta'(i) := F(\nabla(i'))$

$$\exists\ 0 \to Y(i') \to T(i') \to \nabla(i') \to 0.$$

$$\begin{array}{l} Y(i') \\ \rotatebox{90}{$\in$} \\ \mathfrak{F}(\nabla(<i')) \\ \parallel \\ \mathfrak{F}(\nabla(1),\dots,\nabla(i'-1)) \\ \parallel \\ \mathfrak{F}(\nabla((i+1)'),\dots,\nabla(n')) \end{array}$$

$$0 \to F(Y(i')) \to P'(i) \to \Delta'(i) \to 0.$$

$$\begin{array}{l} F(Y(i')) \\ \rotatebox{90}{$\in$} \\ \mathfrak{F}(\Delta'(i+1),\dots,\Delta'(n)) = \mathfrak{F}(\Delta'(>i)) \end{array}$$

$S'(i) := \mathrm{top}\, P'(i)$.

$[\Delta'(i) : S'(j)] \neq 0 \Rightarrow \mathrm{Hom}(P'(j), \Delta'(i)) \neq 0$

$\Rightarrow \mathrm{Hom}(T(j'), \nabla(i')) \neq 0$

$\Rightarrow [T(j') : S(i')] \neq 0$

$\Rightarrow i' \leqq j' \Rightarrow j \leqq i$.

If $[\mathrm{rad}\,\Delta'(i) : S'(i)] \neq 0$, then

$$\begin{tikzcd} P'(i) \arrow[d, two heads] \arrow[dr, "\exists \neq 0"] \arrow[dr, phantom, bend right=20, "\circlearrowleft"'] & \Delta'(i) \\ \Delta'(i) \arrow[r, dashed, "\exists"'] & \mathrm{rad}\,\Delta'(i) \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

$\therefore \Delta'(i) \xrightarrow[\nexists]{\neq 0} \Delta'(i)$. $\nabla(i') \xrightarrow[\nexists]{\exists \neq 0} \nabla(i')$. ↯.

$(\because [\nabla(i') : S(i')] = 1)$

$\therefore [\mathrm{rad}\,\Delta'(i) : S'(\geqq i)] = 0$.

$\mathrm{mod}\, A'$: highest weight cat with Δ'.

$\therefore A'$: quasi-hered with Δ'. //

<u>Prop 6.4</u> $F(I(i')) = T'(i)$.

(∵) $F(I(i'))$: indec. $I(i') \in \mathcal{F}(\nabla) \rightsquigarrow F(I(i')) \in \mathcal{F}(\Delta')$.

$\mathrm{Ext}^1(\mathcal{F}(\nabla), I(i')) = 0 \rightsquigarrow \mathrm{Ext}^1(\mathcal{F}(\Delta'), F(I(i'))) = 0$.

$F(I(i')) \in \mathcal{F}(\Delta') \cap \mathcal{F}(\Delta')^{\perp_1} = \mathrm{add}\, T'$.

$\mathrm{Hom}(P'(i), F(I(i'))) = \mathrm{Hom}(T(i'), I(i')) \neq 0$.

$\therefore [F(I(i')) : S'(i)] \neq 0$.

$\mathrm{Hom}(P'(>i), F(I(i'))) = \mathrm{Hom}(T(<i'), I(i')) = 0$.

$\therefore [F(I(i')) : S'(>i)] = 0$.

$\therefore F(I(i')) = T'(i)$. //

Thm 6.5 [R]

$\mathrm{mod}\, A \cong \mathrm{mod}\, A''$. In particular, if A : basic, then $A \cong A''$.

(∵)
$$\begin{tikzcd}
\mathrm{mod}\, A \arrow[r, "\mathrm{Hom}_A(T,-)"] & \mathrm{mod}\, A' \arrow[r, "\mathrm{Hom}_{A'}(T',-)"] & \mathrm{mod}\, A'' \\
\mathrm{inj}\, A \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & \mathrm{add}\, T' \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & \mathrm{proj}\, A'' \arrow[u, phantom, "\cup"] \\
\mathrm{proj}\, A \arrow[u, "\wr"', "\nu := D\mathrm{Hom}_A(-,A)"] & &
\end{tikzcd}$$

$\therefore \mathrm{proj}\, A \xrightarrow{\sim} \mathrm{proj}\, A''$. $\mathrm{mod}\, A \xrightarrow{\sim} \mathrm{mod}\, A''$.

(If A : basic, then

$$\mathrm{Hom}_{A'}(T', \mathrm{Hom}_A(T,-))|_{\mathcal{F}(\nabla)} = \mathrm{Hom}_{A'}(\mathrm{Hom}_A(T, DA), \mathrm{Hom}_A(T,-))|_{\mathcal{F}(\nabla)}$$
$$= \mathrm{Hom}_A(DA,-)|_{\mathcal{F}(\nabla)} = \nu^{-1}|_{\mathcal{F}(\nabla)}.$$
) //

Cor 6.6 $\mathcal{F}(\nabla') \xrightarrow{\sim} \mathcal{F}(\Delta'') \cong \mathcal{F}(\Delta)$.

Ex 6.7 $\Bbbk$: field. Q: acyclic quiver. $A=\Bbbk Q$.

$Q_0=\{1,\cdots,n\}$. Use this labeling.

Assume $i \xrightarrow{\exists} j \Rightarrow i>j$. Then

$\Delta(i)=P(i)$, $\nabla(i)=S(i)$.

$\mathcal{F}(\Delta)=\operatorname{proj}A$, $\mathcal{F}(\nabla)=\operatorname{mod}A$.

$\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)=\operatorname{proj}A$. $T=A$.

$A'=\operatorname{End}_A(T)=A$

$\Delta'(i)=\operatorname{Hom}(T,S(i'))=S(i')$.

$\nabla'(i)=I(i')$.

$\mathcal{F}(\Delta')=\operatorname{mod}A'$, $\mathcal{F}(\nabla')=\operatorname{inj}A'$.

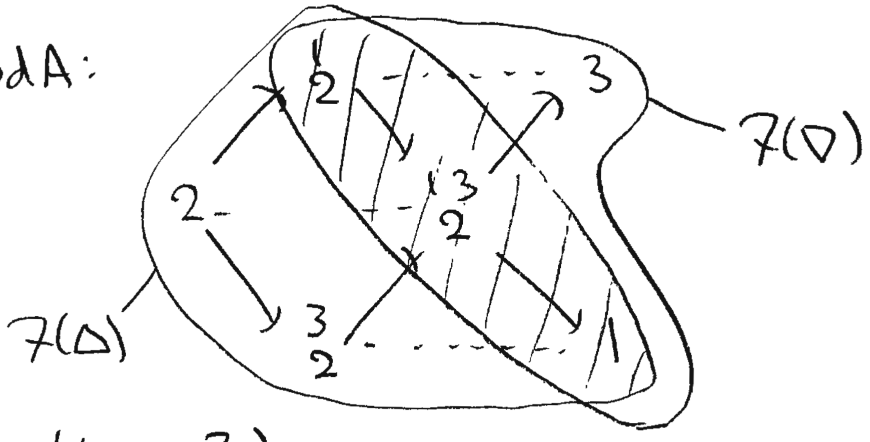
$\mathcal{F}(\Delta')\cap\mathcal{F}(\nabla')=\operatorname{inj}A'$. $T'=DA'$.

$A''=\operatorname{End}_{A'}(T')=\operatorname{End}_{A'}(A')=A'=A$.

Ex 6.8 $\Bbbk$: field.

$Q=[1\longrightarrow 2\longleftarrow 3]$. $A=\Bbbk Q$.

mod A:



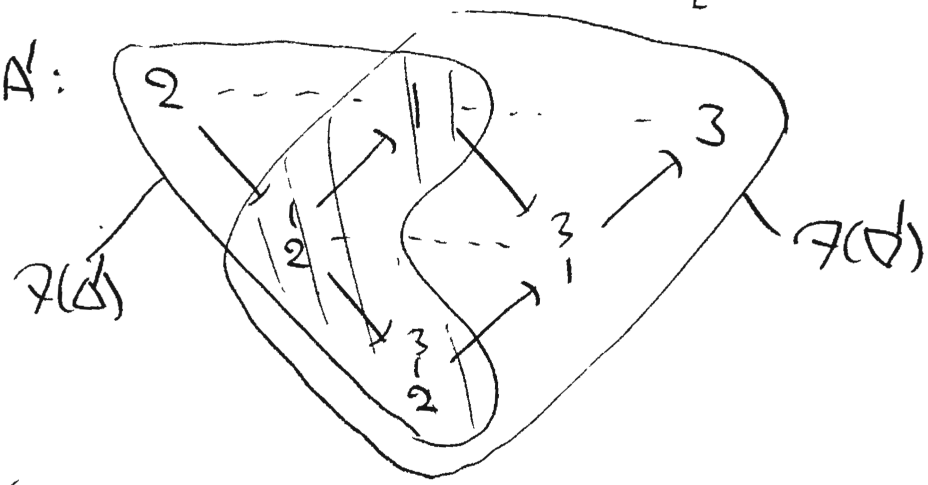
$\Delta=\{1,2,\begin{smallmatrix}3\\2\end{smallmatrix}\}$

$\nabla=\{1,\begin{smallmatrix}1\\2\end{smallmatrix},3\}$

$T=\underset{T(1)}{1}\oplus\underset{T(2)}{\begin{smallmatrix}1\\2\end{smallmatrix}}\oplus\underset{T(3)}{\begin{smallmatrix}1\,3\\2\end{smallmatrix}}$

addT:
$$\begin{tikzcd} T(2) \arrow[dr] & & \\ & T(3) \arrow[dr] & \\ & & T(1) \end{tikzcd} \qquad A' = k\left[\begin{tikzcd} 2 & & \\ & 1 \arrow[ul] & \\ & & 3 \arrow[ul] \end{tikzcd}\right]$$

modA':



$$\Delta' = \left\{1,\ 2,\ \begin{smallmatrix}3\\1\\2\end{smallmatrix}\right\}$$
$$\nabla' = \left\{1,\ \begin{smallmatrix}1\\2\end{smallmatrix},\ 3\right\} \qquad T' = \underset{T'(1)}{1} \oplus \underset{T'(2)}{\begin{smallmatrix}1\\2\end{smallmatrix}} \oplus \underset{T'(3)}{\begin{smallmatrix}3\\1\\2\end{smallmatrix}}.$$

addT':
$$\begin{tikzcd} & T'(1) \\ T'(2) \arrow[ur] \arrow[dr] & \\ & T'(3) \end{tikzcd} \qquad A'' = k\underbrace{\left[\begin{tikzcd} & 3 \arrow[dl] \\ 2 & \\ & 1 \arrow[ul] \end{tikzcd}\right]}_{\overset{\shortparallel}{Q}}$$