2016/8/26

1. Semisimple Lie algebras

Def of Lie alg

A Lie algebra is a $\mathbb{C}$-vector space $\mathfrak{g}$ equipped with a bilinear map

$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

- $[x, y] = -[y, x]$ $\quad (\Leftrightarrow [x, x] = 0)$
- $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

Jacobi identity //

Notion of homomorphism, subalg, ideal are obviously defined.

A Lie alg $\mathfrak{g}$ is abelian

$\underset{\text{def}}{\Leftrightarrow}$ ${}^{\forall} x, y \in \mathfrak{g}$ $[x, y] = 0$

Ex

(1) A : associative $\mathbb{C}$-alg

$[x, y] := xy - yx$

$\to (A, [\ ,\ ])$ Lie alg

In particular, $\mathrm{End}_{\mathbb{C}}(V)$ for a $\mathbb{C}$-vec sp. V is a Lie alg.

Denote the Lie alg $\mathrm{End}(\mathbb{C}^n)$ by $\mathfrak{gl}_n(\mathbb{C})$ (general linear Lie alg)

(2)

$\mathfrak{sl}_n(\mathbb{C}) := \{ x \in \mathfrak{gl}_n(\mathbb{C}) \mid \mathrm{tr}\, x = 0 \}$

is a Lie subalg of $\mathfrak{gl}_n(\mathbb{C})$

(special linear Lie alg)

$\because$

$x, y \in \mathfrak{sl}_n(\mathbb{C}) \Rightarrow \mathrm{tr}[x, y] = \mathrm{tr}\, xy - \mathrm{tr}\, yx$

$= 0$ $\square$

$$\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$$

$$= \underbrace{\mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e} \oplus \underbrace{\mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{f} \oplus \underbrace{\mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{h}$$

$$\begin{cases} [e,f] = ef - fe = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = h \\ [h,e] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2e \\ [h,f] = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2f \end{cases}$$

<u>Universal enveloping alg</u>

$$\begin{tikzcd} \text{Category of ass } \mathbb{C}\text{-alg} \arrow[r, shift left, "\text{Forget}"] & \text{Category of Lie alg} / \mathbb{C} \arrow[l, shift left] \end{tikzcd}$$

$\exists\, U$ left adjoint

i.e. $\mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathrm{For}(A))$

$$\cong \mathrm{Hom}_{\mathbb{C}\text{-alg}}(U(\mathfrak{g}), A)$$

Construction of $U(\mathfrak{g})$:

$$T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \quad \text{tensor alg}$$

$$U(\mathfrak{g}) := T(\mathfrak{g}) \Big/ \langle x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g} \rangle$$

two-sided

Denote the image of $x_1 \otimes \cdots \otimes x_n \in \mathfrak{g}^{\otimes n}$ & by $x_1 \cdots x_n \in U(\mathfrak{g})$.

$U(\mathfrak{g}) = \mathrm{span}_{\mathbb{C}} \{ x_1 \cdots x_n \mid {n \geq 0 \atop x_i \in \mathfrak{g}} \}$

<u>Ex</u>

$U(sl_2) = \mathrm{span}_{\mathbb{C}} \{ x_1 \cdots x_n \mid {n \geq 0 \atop x_i = e, f, h} \}$

We can rewrite a product w.r.t. the order f, h, e.

e.g. $\underline{ef}he$

$= ([e,f] + fe) he \qquad [e,f] = h$

$= h^2 e + f\underline{eh}e \qquad eh = [e,h] + he = -2e + he$

$= h^2 e - 2fe^2 + fhe^2$ //

$\therefore U(sl_2) = \mathrm{span}_{\mathbb{C}} \{ f^k h^l e^m \mid k, l, m \geq 0 \}$

In fact $f^k h^l e^m$ $(k, l, m \in \mathbb{Z}_{\geq 0})$ are linearly independent.

$\therefore$ It gives a $\mathbb{C}$-basis of $U(sl_2)$.

$\mathfrak{g}$: Lie alg

Define a filtration $F^\bullet U(\mathfrak{g})$ of $U(\mathfrak{g})$ by

$$F^k U(\mathfrak{g}) := \mathrm{Im}\left(\bigoplus_{n=0}^{k} \mathfrak{g}^{\otimes n} \longrightarrow U(\mathfrak{g}) \right)$$

Thm (Poincaré - Birkoff - Witt)

$$\mathrm{gr} F^\bullet U(\mathfrak{g}) \xleftarrow{\simeq} S(\mathfrak{g}) \quad \text{symmetric alg}$$

Cor 1

$(x_1, x_2, \ldots)$: an ordered basis of $\mathfrak{g}$

$\Rightarrow \left\{ x_1^{k_1} \cdots x_n^{k_n} \;\middle|\; \begin{array}{l} n \geq 0 \\ k_i \geq 0 \end{array} \right\}$ gives a $\mathbb{C}$-basis of $U(\mathfrak{g})$.

Cor 2

$\mathfrak{g}$: fin. dim'l $/\mathbb{C}$

$\Rightarrow U(\mathfrak{g})$ is left and right noetherian.

Representation

A representation of $\mathfrak{g}$ is a pair of a $\mathbb{C}$-vec sp M and a Lie alg hom

$\varphi : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}}(M)$.

We simply say that M is a rep of $\mathfrak{g}$ or M is a $\mathfrak{g}$-module.

Rem

The category of $\mathfrak{g}$-modules and the category of left $U(\mathfrak{g})$-modules are isomorphic.

$\because \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathrm{End}(M)) \cong \mathrm{Hom}_{\text{alg}}(U(\mathfrak{g}), \mathrm{End}(M))$

We will not distinguish $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules.

Ex

(1) $M = \{0\}$

(2) $M = \mathbb{C}$ $\quad \forall x \in \mathfrak{g}$ acts by 0

trivial rep

(3) $M = \mathfrak{g}$

$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ adjoint rep

$x \longmapsto [x, -]$

* ad is a Lie alg hom $\Leftrightarrow$ Jacobi identity

$\because$ $\mathrm{ad}\,[x,y] \overset{?}{=} [\mathrm{ad}x, \mathrm{ad}y]$

$\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad \mathrm{ad}x\,\mathrm{ad}y - \mathrm{ad}y\,\mathrm{ad}x$

LHS: $\mathrm{ad}[x,y](z) = [[x,y],z]$

RHS: $[\mathrm{ad}x, \mathrm{ad}y](z)$

$= [x,[y,z]] - [y,[x,z]]$ $\square$

Rep of $\mathfrak{sl}_2$

- $L(0) = \mathbb{C}$ $\quad$ trivial rep
- $L(1) = \mathbb{C}^2$ $\quad \mathfrak{sl}_2 \subset \mathfrak{gl}_2 = \mathrm{End}\,\mathbb{C}^2 \curvearrowright \mathbb{C}^2$

  acts as matrices

  vector rep

- $L(2) = \mathfrak{sl}_2 \cong \mathbb{C}^3$ adjoint rep

What does $L(\underline{n})$ mean?

- $L(1) = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$

eigenvalue

$h\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \underline{1}$

$h\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$ $\quad -1$

eigenvalue of h is called <u>weight</u>.

$e\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$

$\begin{pmatrix}1\\0\end{pmatrix}$ is called highest weight vector

1 is called highest weight

- $L(2) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$

wt $\quad 2 \quad 0 \quad -2$

$([h,e] = 2e, \quad [h,f] = -2f)$

$[e,e] = 0$

$\leadsto$ e is a h.w. vector

with h.w. $\underline{2}$

Thm

$\{$ fin. dim. simple $\mathfrak{sl}_2$-modules $\}/\cong$

$= \{ L(n) \mid \begin{matrix} (n+1)\text{-dim'l} \\ \text{h.w. } n \in \mathbb{Z}_{\geq 0} \end{matrix} \}$ //

Notice:

M : $\mathfrak{sl}_2$-module

Suppose $v \in M$ has wt $m \in \mathbb{C}$ i.e. $hv = mv$.

Then ev has wt $m+2$ if $ev \neq 0$

fv has wt $m-2$ if $fv \neq 0$

$\because$ $hev = ([h,e] + eh)v$ $\quad [h,e] = 2e$

$= 2ev + mev$

$= (m+2)ev$ $\square$

$\boxed{\text{Important}}$

- $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$
- eigen space decomposition w.r.t. h-action (eigenvalue = weight)
- e raises wt by 2
  f lowers wt by 2.

Semisimple Lie alg

A Lie alg $\mathfrak{g}$ is simple

$\overset{\Longleftrightarrow}{\text{def}}$ • $\mathfrak{g}$ is not abelian

• $\mathfrak{g}$ does not have nontrivial ideals //

"Def"
semisimple Lie algebra
= direct product of simple Lie algebras

(ordinary def:
s.s. $\overset{\text{def}}{\Longleftrightarrow}$ radical $= 0$ (maximal solvable ideal)

$\Longleftrightarrow$ direct prod of simples
Thm
analog of Wedderburn thm)

Ex
$sl_n(\mathbb{C}) = \{x \in gl_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n) \mid \mathrm{tr}\, x = 0\}$
is simple.

$gl_n(\mathbb{C})$ is not simple

but reductive = semisimple × center //

Assume $\mathfrak{g}$ = fin. dim. simple Lie alg
in the sequel.

Root space decomposition

$\mathfrak{h} \subset \mathfrak{g}$ Lie subalg $\to$ $\mathfrak{h} \curvearrowright \mathfrak{g}$ adjoint action

$\mathfrak{h}$ is a Cartan subalg

$\underset{\text{def}}{\Leftrightarrow}$ (1) The action of any element of $\mathfrak{h}$ on $\mathfrak{g}$ is semisimple i.e. diagonalizable

(2) $\mathfrak{h}$ is maximal among subalgebras satisfying (1). //

Prop

$\mathfrak{h}$: CSA $\Rightarrow$ $\mathfrak{h}$ is maximal abelian.

Ex

$sl_n(\mathbb{C}) \supset \{\text{diagonal matrix}\}$ is a Cartan subalg.

For $\alpha \in \mathfrak{h}^*$ define

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h} \ [h, x] = \alpha(h) x\}$$

Then

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{h} = \mathfrak{g}_0$$

($\because$ max. abelian)

$\Delta := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\} \setminus \{0\}$

We call an element of Δ a <u>root</u>.

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ root sp decomposition

<u>Ex</u> $\mathfrak{g} = sl_3(\mathbb{C})$

basis: $\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ upper triangular

$h_1 = \begin{pmatrix} 1 && \\ & -1 & \\ && 0 \end{pmatrix}$ $h_2 = \begin{pmatrix} 0 && \\ & 1 & \\ && -1 \end{pmatrix}$ diag

$\begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&1&0 \end{pmatrix}$ $\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}$ lower tri

$\mathfrak{h} := \mathbb{C}h_1 \oplus \mathbb{C}h_2$ is a CSA.

$[h_1, E_{ij}] = [E_{11}, E_{ij}] - [E_{22}, E_{ij}]$

$= (\delta_{1i} E_{1j} - \delta_{1j} E_{i1}) - (\delta_{2i} E_{2j} - \delta_{2j} E_{i2})$

$$= \begin{cases} (i,j) = & (1,2) & (2,3) & (1,3) \\ & 2E_{12} & -E_{23} & E_{13} \\ & (2,1) & (3,2) & (3,1) \\ & -2E_{21} & E_{32} & -E_{31} \end{cases}$$

$[h_2, E_{ij}]$

$$= \begin{cases} -E_{12} & 2E_{23} & E_{13} \\ E_{21} & -2E_{32} & -E_{31} \end{cases}$$

$$\left(\begin{aligned} &[E_{22}, E_{ij}] - [E_{33}, E_{ij}] \\ &= (\delta_{2i} E_{2j} - \delta_{2j} E_{i2}) - (\delta_{3i} E_{3j} - \delta_{3j} E_{i3}) \end{aligned} \right)$$

Define $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathfrak{h}^*$ by

$\varepsilon_i : \mathfrak{h} \ni h \longmapsto (i,i)$-entry of h.

$(\longrightarrow \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0)$

$\longrightarrow [h, E_{ij}] = (\varepsilon_i - \varepsilon_j)(h) E_{ij}$

$\therefore \Delta = \{ \varepsilon_i - \varepsilon_j \ (i \neq j) \}$

Similar for $\mathfrak{sl}_n(\mathbb{C})$

$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{pmatrix}, \ldots, h_{n-1} = \begin{pmatrix} \ddots & & \\ & 1 & \\ & & -1 \end{pmatrix}$, $\varepsilon_1, \ldots, \varepsilon_n \in \mathfrak{h}^*$

In this example

- $\alpha = \varepsilon_i - \varepsilon_j \in \Delta \Rightarrow \mathfrak{g}_\alpha = \mathbb{C} E_{ij}$ 1-dim'l

- We can divide Δ into

$\Delta^+ = \{ \varepsilon_i - \varepsilon_j \ (i > j) \}$ and $\Delta^- = \{ \varepsilon_i - \varepsilon_j \ (i < j) \}$.

$\Leftrightarrow$ lower tri

$\Leftrightarrow$ uppe tri.

These hold for general s.s. Lie alg

<u>Prop</u> $\dim \mathfrak{g}_\alpha = 1$ for $\forall \alpha \in \Delta$

<u>Thm</u> $n := \dim \mathfrak{g}$

$\exists \Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Delta \subset \mathfrak{g}^*$ s.t.

- Π is a $\mathbb{C}$-basis of $\mathfrak{g}^*$
- $\forall \alpha \in \Delta$ can be written as

$$\alpha = \sum_{i=1}^{n} m_i \alpha_i \quad \text{with} \begin{cases} m_i \in \mathbb{Z}_{\geq 0} \ \forall i & -(+) \\ \text{or} & \\ m_i \in \mathbb{Z}_{\leq 0} \ \forall i & -(-) \end{cases}$$

//

We call an element of

$$\begin{cases} \Pi & \text{a simple root} \\ (+) & \text{a positive root} \\ (-) & \text{a negative root.} \end{cases}$$

$\to \quad \underset{}{\Delta} = \underset{\substack{\text{the set of} \\ \text{pos. roots}}}{\Delta^+} \sqcup \underset{\substack{\text{the set of} \\ \text{neg. roots}}}{\Delta^-}$

We have $\Delta^- = -\Delta^+$

(property of roots: $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$)

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

$$= \underbrace{\Big(\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha\Big)}_{\mathfrak{n}^-} \oplus \mathfrak{h} \oplus \underbrace{\Big(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha\Big)}_{\mathfrak{n}^+}$$

triangular decomp.

(cf. $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$)

$\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is called a Borel subalg

(Borel $\underset{\text{def}}{\Leftrightarrow}$ maximal solvable subalg)

<u>Ex</u> $\mathfrak{sl}_n(\mathbb{C})$

We can take

$\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$

$\Rightarrow \quad \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$

We can also take

$\Pi = \{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots, \varepsilon_n - \varepsilon_{n-1}\}$ for example.

$\Rightarrow$ pos. roots and neg. roots are interchanged.

Rem

Notion of roots depends on the choice of CSA.

Notion of $\begin{matrix}\text{pos.}\\ \text{neg.}\end{matrix}$ roots depends on the

choice of simple roots.

Killing form, Weyl group, Cartan matrix

$\mathrm{ad}: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$

Define $(\ ,\ ): \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$(x, y) := \mathrm{tr}\, \mathrm{ad}x\, \mathrm{ad}y$ Killing form

$(\ ,\ )$ is a nondegenerate sym bil. form.

(For $\mathfrak{g}$ = fin. dim. Lie alg
$\mathfrak{g}$ = s.s. $\Leftrightarrow$ Killing form is nondeg)

$(\ ,\ )|_{\mathfrak{h}}$ is nondeg.

$\rightsquigarrow \quad \nu: \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$

$(\ ,\ )$ on $\mathfrak{h}^*$ is defined via ν.

For $\alpha \in \Delta$, define the coroot $\alpha^\vee \in \mathfrak{h}$

by $\quad \alpha^\vee := \dfrac{2}{(\alpha,\alpha)} \nu^{-1}(\alpha)$

$\left(\rightarrow \quad \langle \alpha^\vee, \beta \rangle = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \qquad \alpha,\beta \in \Delta \right)$

For each $\alpha \in \Delta$, we can take

$e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ satisfying $\mathfrak{sl}_2$-rel:

$$[e_\alpha, f_\alpha] = \alpha^\vee, \quad [\alpha^\vee, e_\alpha] = 2e_\alpha, \quad [\alpha^\vee, f_\alpha] = -2f_\alpha$$

For $\alpha \in \Delta$, define $s_\alpha \in \mathrm{End}(\mathfrak{h}^*)$ by

$$s_\alpha(\beta) := \beta - \langle \alpha^\vee, \beta \rangle \alpha$$

The <u>Weyl group</u> W is a ~~finite~~ subgroup of $GL(\mathfrak{h}^*)$ gen by $\{ s_\alpha \mid \alpha \in \Delta \}$.

<u>Ex</u> $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \Rightarrow W \cong \mathfrak{S}_n$ //

W is gen by $S = \{ s_{\alpha_i} \mid \alpha_i \in \Pi \}$.

(W, S) is a finite Coxeter group.

Classification of s.s. Lie algebras?

$(\mathfrak{g}, \mathfrak{h}) \rightsquigarrow \Delta$ & satisfies the axioms of "root system"

$\longrightarrow$ reduced to classification of root systems.

$\Pi = \{\alpha_i \mid i \in I\} \subset \Delta$ simple roots

$$a_{ij} := \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

$(a_{ij})_{i,j \in I}$: Cartan matrix of $\mathfrak{g}$.

Semisimple Lie algebras are determined by their Cartan matrices.

<u>Ex</u> $\mathfrak{sl}_n(\mathbb{C})$, $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \longleftrightarrow \text{Dynkin diagram}$$

$$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\circ} \cdots \circ - \underset{n-1}{\circ}$$

type A_{n-1}

$\{\text{simple Lie algebras}\}/\cong \overset{1:1}{\longleftrightarrow} A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ //

name of root system

2. Representation theory

Fix a triangular decomp $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$

$\Rightarrow U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \underset{\mathbb{C}}{\otimes} U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$
PBW
as $\mathbb{C}$-vec sp

For a $\mathfrak{g}$-module M and $\lambda \in \mathfrak{h}^*$ define

$$M_\lambda := \{ v \in M \mid \forall h \in \mathfrak{h} \ \ hv = \lambda(h) v \}$$

λ is called a weight of M if $M_\lambda \neq 0$.

<u>Prop</u>

M: fin. dim. $\Rightarrow \mathfrak{h} \curvearrowright M$ semisimple

i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ //

<u>Rem</u>

$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ does not hold in general.

Notice:

$v \in M_\lambda$, $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta)$

$\Rightarrow xv \in M_{\lambda+\alpha}$ (similar as $\mathfrak{sl}_2$-case)

$\because$ $[h, x] = \alpha(h) x$

$$\begin{aligned} hxv &= [h,x]v + xhv \\ &= \alpha(h)xv + \lambda(h)xv \\ &= (\lambda+\alpha)(h)xv \qquad \square \end{aligned}$$

Prop

M: fin. dim. simple $\mathfrak{g}$-module

$\Rightarrow \exists 1\ \lambda \in \mathfrak{h}^*$ s.t. • $\dim M_\lambda = 1$
• $\mathfrak{n}^+ M_\lambda = 0$.

Proof

existence:

M: fin. dim. $\Rightarrow$ the set of weights is finite

$\Rightarrow \exists \lambda$ s.t. $M_\lambda \neq 0$ and $\forall \alpha \in \Delta^+$, $M_{\lambda+\alpha} = 0$.

$\Rightarrow \mathfrak{n}^+ M_\lambda = 0$.

Take $v_{\neq 0} \in M_\lambda \Rightarrow M = U(\mathfrak{g})v$
since M is simple

$$M = U(\mathfrak{n}^-) \underbrace{U(\mathfrak{h})\, U(\mathfrak{n}^+) v}_{\text{acts by scalar}}$$

$$= U(\mathfrak{n}^-) v$$

Recall $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ $\quad \dim \mathfrak{g}_\alpha = 1$

Let $\Delta^- = \{\beta_1, \ldots, \beta_r\}$ and take nonzero $f_{\beta_i} \in \mathfrak{g}_{\beta_i}$ for each β_i.

$\to \{f_{\beta_1}^{m_1} \cdots f_{\beta_r}^{m_r} \mid m_i \geq 0\}$ is a $\mathbb{C}$-basis of $U(n^-)$

$\therefore U(n^-)v = \mathrm{span}_{\mathbb{C}} \{ \underbrace{f_{\beta_1}^{m_1} \cdots f_{\beta_r}^{m_r}}_{\in M_{\lambda + \sum_{j=1}^{r} m_j \beta_j}} v \mid m_j \geq 0 \}$

$\therefore \dim M_\lambda = 1$ //

uniqueness:

Suppose λ_1 and λ_2 satisfy the condition.

$\Rightarrow \lambda_2 = \lambda_1 + \sum_{i=1}^{r} m_j \beta_j \quad \forall m_i \in \mathbb{Z}_{\geq 0}$

$\beta_i \in \sum_{i=1}^{n} \mathbb{Z}_{\leq 0} \alpha_j$

$\lambda_2 - \lambda_1 \in \sum_{i=1}^{n} \mathbb{Z}_{\leq 0} \alpha_j$

We also have $\lambda_1 - \lambda_2 \in \sum_{j=1}^{n} \mathbb{Z}_{\leq 0} \alpha_j$

$\therefore \lambda_1 = \lambda_2 \quad \square$

<u>highest weight module</u>

Looking at the proof of Prop.

"the set of weights is finite"

$\rightsquigarrow$ "bounded from above" is enough.

Def

Define a partial order $\geqq$ on $\mathfrak{h}^*$ by

$\lambda \geqq \mu \underset{\text{def}}{\Leftrightarrow} \lambda - \mu \in \sum_{i=1}^{n} \mathbb{Z}_{\geq 0} \alpha_i$ //

M: $\mathfrak{g}$-module

$v_{\neq 0} \in M$ is maximal vector of wt λ

$\underset{\text{def}}{\Leftrightarrow}$ $v \in M_\lambda$ and $\mathfrak{n}^+ v = 0$.

M is a highest weight module with highest weight λ

$\underset{\text{def}}{\Leftrightarrow}$ $M = U(\mathfrak{g}) v$
for a maximal vector v of wt λ

v is called a highest weight vector. //

Prop → fin. dim. simple $\mathfrak{g}$-module is a h.w. module.

<u>Prop</u>

Assume M: h.w. module with h.w. λ

(1) $$M = \bigoplus_{\mu \leq \lambda} M_\mu$$

$\dim M_\mu < \infty \ \forall \mu$ and $\dim M_\lambda = 1$

(2) M has a unique simple quotient (i.e. simple top)

In particular, M is indecomposable.

<u>Proof</u>

(1) Similar as fin. dim'l case

(2) Let $M' \subsetneqq M$ be a proper submodule

$\Rightarrow M'_\lambda = 0$

$\left(\because \dim M_\lambda = 1, \quad \begin{aligned} & M'_\lambda = M_\lambda \cap M' \neq 0 \\ & \Rightarrow M_\lambda \subset M' \\ & \Rightarrow M = U(\mathfrak{g}) M_\lambda \subset M' \\ & \Rightarrow M = M' \end{aligned} \right)$

$\therefore \displaystyle\sum_{\substack{\text{all proper} \\ \text{sub}}} M'$ is also a proper submodule.

This gives a unique maximal submodule.

$\square$

Q. Are there h.w. modules other than fin. dim'l ones?

$\longrightarrow$ Yes.

<u>Verma module</u>

$\lambda \in \mathfrak{h}^* \rightsquigarrow \mathbb{C}_\lambda$: 1-dim'l rep of $\mathfrak{h}$

$\mathbb{C}_\lambda \cong \mathbb{C} \ni 1 \qquad h1 = \lambda(h)1$

Extend $\mathbb{C}_\lambda$ to a $\mathfrak{b}$-module by

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \twoheadrightarrow \mathfrak{h} \curvearrowright \mathbb{C}_\lambda$

i.e. $\mathfrak{n}^+ 1 = 0$.

$$M(\lambda) := U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \ni 1 \otimes 1 =: v_\lambda$$

Verma module

$M(\lambda)$ is a h.w. module with h.w. vector v_λ

PBW: $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \underset{\mathbb{C}}{\otimes} U(\mathfrak{b})$

$\longrightarrow M(\lambda) \cong U(\mathfrak{n}^-)$ as $\mathbb{C}$-vec sp.

$M(\lambda)$ is <u>universal</u> among h.w. modules

i.e. If M: h.w. module with h.w. vector v of wt λ.

$\Rightarrow$ $\exists 1$ surjective $\mathfrak{g}$-module hom

$$M(\lambda) \longrightarrow M$$
$$v_\lambda \longmapsto v$$

$\because$ $\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M)$

$\cong \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M)$

$\cong \{m \in M_\lambda \mid \mathfrak{n}^+ m = 0\}$ $\square$

Denote the unique simple quotient of $M(\lambda)$ by $L(\lambda)$.

<u>Thm</u>

(1) $L(\lambda)$ is fin. dim'l.

$\Leftrightarrow$ $\forall \alpha \in \Delta^+$ $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$

This λ is said to be dominant integral

(2) $\{$fin. dim. simple $\mathfrak{g}$-module$\}/\cong$ $\overset{1:1}{\longleftrightarrow}$ $\{\lambda$: dominant integral$\}$

$L(\lambda) \longleftarrow\!\!\shortmid \lambda$

(3) The category of fin. dim'l $\mathfrak{g}$-modules is semisimple.

Ex $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$

$\alpha := \varepsilon_1 - \varepsilon_2 \quad \Delta = \{\pm\alpha\} \supset \Pi = \{\alpha\}$

$h = \alpha^\vee$

$\mathfrak{h} = \mathbb{C}h \longleftrightarrow \mathfrak{h}^* = \mathbb{C}\varpi \cong \mathbb{C}$

dual basis $\quad \varpi \longleftrightarrow 1$

$\alpha = 2\varpi \longleftrightarrow 2$

order: $\lambda, \mu \in \mathbb{C} \quad \lambda \geq \mu \Leftrightarrow \lambda - \mu \in 2\mathbb{Z}_{\geq 0}$

$$U(\mathfrak{n}^-) = \mathbb{C}[f] = \bigoplus_{k \geq 0} \mathbb{C}f^k$$

$$M(\lambda) = \bigoplus_{k \geq 0} \mathbb{C}\underbrace{f^k v_\lambda}_{} \quad \text{wt } \lambda - 2k$$

$\circledast$ $k \geq 1$, condition for $ef^k v_\lambda = 0$?

$$ef^k v_\lambda = [e, f^k] v_\lambda + \underbrace{f^k e v_\lambda}_{=0}$$

$$[e, f^k] = k f^{k-1} h - k(k-1) f^{k-1}$$

$$[e, f^k] v_\lambda = k(\lambda - (k-1)) f^{k-1} v_\lambda$$

$$= 0 \Leftrightarrow \lambda = k - 1$$

$\because$

$\lambda \notin \mathbb{Z}_{\geq 0} \Rightarrow M(\lambda)$ is simple

$\lambda \in \mathbb{Z}_{\geq 0} \Rightarrow M(\lambda) \underset{\neq}{\supset} U(\mathfrak{g}) f^{\lambda+1} v_\lambda$

max proper submodule

$L(\lambda) = M(\lambda) / \underbrace{U(\mathfrak{g}) f^{\lambda+1} v_\lambda}_{\cong L(-\lambda-2)}$ $\lambda + 1$ - dim'l

$f^{\lambda+1} v_\lambda$

wt $\lambda - 2(\lambda+1)$
$= -\lambda - 2$

$0 \to L(-\lambda-2) \to M(\lambda) \to L(\lambda) \to 0$

if $\lambda \in \mathbb{Z}_{\geq 0}$ //

In general

$[M(\lambda) : L(\lambda)] = 1$

$[M(\lambda) : L(\mu)] \neq 0 \Rightarrow M(\lambda)_\mu \neq 0$

$\quad \| $
$\quad D_{\lambda\mu} \qquad \Rightarrow \mu \leq \lambda$

($M(\lambda)$: finite length)

$D = (D_{\lambda\mu})$ upper triangular matrix

In Grothendieck grp

$$[M(\lambda)] = \sum_{\mu \leq \lambda} D_{\lambda\mu} [L(\mu)]$$

$$\rightsquigarrow [L(\lambda)] = \sum_{\mu \leq \lambda} (D^{-1})_{\lambda\mu} [M(\mu)]$$

Kazhdan–Lusztig conjecture:
(for integral weights)

$[M(\lambda) : L(\mu)] = P_{y,w}(1)$

where $P_{y,w}(q) \in \mathbb{Z}[q]$ is KL polynomial attached to some elements $y, w \in W$.

y, w are determined by λ, μ.

KL conj: 1979 (introduce KL poly via Hecke alg)

Solved by Beilinson–Bernstein
Brylinski–Kashiwara 1981

Use D-modules
perverse sheaves) on flag variety

(2012–) 2014 algebraic proof by
Elias–Williamson

(inspired by geometric method)

Category $\mathcal{O}$

The category of all $\mathfrak{g}$-modules is too big.
Hopeless to classify simples etc.
$\to$ Study a good full subcategory.

Def (Bernstein-Gelfand-Gelfand '76)
$\mathcal{O} \subset \mathfrak{g}$-modules full subcat.

$M \in \mathcal{O} \overset{\text{def}}{\Leftrightarrow}$
- finitely generated over $U(\mathfrak{g})$
- $\mathfrak{h}$-semisimple
- $\forall v \in M$ $\dim U(\mathfrak{n}^+)v < \infty$ (locally $\mathfrak{n}^+$-finite) //

$\mathcal{O}$ is closed by taking sub, quot, fin. direct sum.
(sub: & $U(\mathfrak{g})$: noetherian)
$\to$ abelian category

Note: $\mathcal{O}$ is not closed by extension.
$\uparrow$ failure of $\mathfrak{h}$-semisimplicity.

Prop

(1) $\forall$ h.w. module $\in \mathcal{O}$

In particular $M(\lambda), L(\lambda) \in \mathcal{O}$

(2) $M \in \mathcal{O}$

$\Rightarrow \exists \lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$

weights of $M \subset \bigcup_{i=1}^{r} \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_i\}$

(3) $\{\text{simples in } \mathcal{O}\}/\cong \ \xleftrightarrow{1:1}\ \mathfrak{h}^*$

$L(\lambda) \longleftarrow\!\shortmid \ \lambda$

Proof

(1) • fin. gen $\leftarrow$ gen by h.w. vec

• $\mathfrak{h}$-s.s. $\leftarrow$ done

• loc $\mathfrak{n}^+$-fin.

$M = \bigoplus_{\mu \leq \lambda} M_\mu \quad \lambda$: h.w.

$v \in M_\mu \Rightarrow$ wt of $U(\mathfrak{n}^+)v$

satisfy $\mu \leq \circledast \leq \lambda$

# of such wts is finite

& $\forall$ wt sp are fin. dim'l $\to$ OK //

(2) Take a finite set of generators

$S \subset M$

loc $\mathfrak{n}^+$-fin. $\Rightarrow$ $\dim U(\mathfrak{n}^+)S < \infty$

We can take maximal weights

$\lambda_1, \ldots, \lambda_r$ of $U(\mathfrak{n}^+)S$ w.r.t. $\geq$.

$$\Rightarrow \quad M = U(\mathfrak{g})S$$
$$= U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)S$$
$$\mathrm{wt} \subset \bigcup_{i=1}^{r} \{\mu \leq \lambda_i\} \quad //$$

(3) Let M be a simple in $\mathcal{O}$.

(2) $\Rightarrow$ M has a maximal vector v of wt λ

simple $\Rightarrow$ $M = U(\mathfrak{g})v$

$\therefore$ M is a h.w. module with h.w. λ

$$\therefore \quad M(\lambda) \twoheadrightarrow M \cong L(\lambda) \quad \square$$

Prop

(1) $\mathcal{O}$ is finite length

(2) $\mathcal{O}$ has enough projectives

(Use central character etc. for a proof)

Rem

$\mathcal{O}$ has infinitely many simples.

But each block has finitely many simples.

Hence each block of $\mathcal{O}$ is Morita equiv to some fin. dim. alg.

$P(\lambda) :=$ projective cover of $L(\lambda)$.

Thm $\forall \lambda \in \mathfrak{h}^*$

$\exists$ $\mathfrak{g}$-module filtration

$$P(\lambda) = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_\ell = 0$$

s.t. $\quad M_0/M_1 \cong M(\lambda)$

$\quad M_i/M_{i+1} \cong M(\lambda_i) \quad \exists \lambda_i > \lambda$

$\quad i = 1, \ldots, \ell-1$ //

i.e. $\mathcal{O}$ is a h.w. cat. whose standard modules are Verma modules.

Its poset is $(\mathfrak{h}^*, \geq)$.
By CPS thm, the fin. dim alg which is Morita equiv to a block of $\mathcal{O}$ is quasi-hereditary.

<u>BGG reciprocity</u>

duality:

$\exists$ exact contravariant functor $\vee : \mathcal{O} \to \mathcal{O}$ s.t.

- $M^{\vee\vee} \cong M$
- $L(\lambda)^\vee \cong L(\lambda)$

$\rightsquigarrow$ $M(\lambda)^\vee$ is a costandard module of $\mathcal{O}$.

$[M(\lambda)^\vee : L(\mu)] = [M(\lambda) : L(\mu)]$ holds.

<u>Cor of Thm</u>

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$$

BGG reciprocity

<u>Rem</u>

The reciprocity for a general h.w. cat is $(P(\lambda):M(\mu)) = [M(\mu)^{\vee} : L(\lambda)]$. //

$$C_{\lambda\mu} := [P(\lambda):L(\mu)]$$
$$D_{\lambda\mu} := [M(\lambda):L(\mu)]$$

$C = (C_{\lambda\mu})$ Cartan matrix

$D = (D_{\lambda\mu})$ decomposition matrix & upper tri.

$$C_{\lambda\mu} = \sum_{\nu} \underbrace{(P(\lambda):M(\nu))}_{\overset{\text{BGG}}{=}\,[M(\nu):L(\lambda)]} [M(\nu):L(\mu)]$$

$\therefore\ C = {}^{t}D \cdot D$ In particular symmetric matrix //

Ex $\mathfrak{sl}_2$

$\mathcal{O}_0 :=$ the block containing the trivial rep $L(0)$ $\subset \mathcal{O}$

simples $\quad L(0) \quad L(-2)$

$\quad\quad\quad 0 > -2$

$M(0)$: basis $\quad v_0 \quad \underbrace{fv_0 \quad f^2 v_0 \quad \cdots}_{\text{sub} \cong L(-2) \cong M(-2)}$

$[M(0) : L(0)] = 1$

$[M(0) : L(-2)] = 1$

$[M(-2) : L(0)] = 0$

$[M(-2) : L(-2)] = 1$

projectives?

0 is maximal in $\{0, -2\}$

$\rightarrow \quad P(0) \cong M(0)$

$P(-2)$: $\quad (P(-2) : M(-2)) = 1$

By BGG $\quad (P(-2) : M(0)) = [M(0) : L(-2)]$
$= 1$

$\therefore \; 0 \to M(0) \to P(-2) \to M(-2) \to 0$

$P := P(0) \oplus P(-2)$ proj generator

$\mathrm{End}_{\mathcal{O}} P = ?$

- $\mathrm{Hom}(\underset{M(0)}{\underset{\shortparallel}{P(0)}}, P(-2)) = \mathbb{C}a$ $\qquad M(0) \hookrightarrow P(-2)$

- $\mathrm{Hom}(P(-2), P(0)) = \mathbb{C}b$ $\qquad P(-2) \twoheadrightarrow M(-2) \cong L(-2) \hookrightarrow M(0)$

$\longrightarrow \quad b \circ a = 0$

- $\mathrm{Hom}(\underset{M(0)}{\underset{\shortparallel}{P(0)}}, \underset{M(0)}{\underset{\shortparallel}{P(0)}}) = \mathbb{C}e_0$ $\qquad$ corresp. to idempotent

- $\mathrm{Hom}(P(-2), P(-2)) = \mathbb{C}e_2 \oplus \mathbb{C}a \circ b$

— (*)

$\dim \mathrm{End}_{\mathcal{O}} P = 5$

$$\begin{tikzcd} 0 \arrow[r, bend left, "a"] & -2 \arrow[l, bend left, "b"] \end{tikzcd}$$

rel: $ab = 0$

$(\Leftrightarrow b \circ a = 0)$

$\mathcal{O}_0 \cong \text{mod-}\mathrm{End}_{\mathcal{O}} P$

cat. of fin. dim. right modules

(*)

Apply $\mathrm{Hom}(P(-2), -)$ to

$$0 \to M(0) = P(0) \to P(-2) \to M(-2) \to 0$$

$$0 \to \mathrm{Hom}(P(-2), P(0)) = \mathbb{C}b$$

$$\to \mathrm{Hom}(P(-2), P(-2)) \qquad a \circ b$$

$$\to \mathrm{Hom}(P(-2), M(-2)) = \mathbb{C}[P(-2) \twoheadrightarrow M(-2)]$$

$$\to 0 \qquad \therefore \dim \mathrm{Hom}(P(-2), P(-2)) = 2 \qquad \square$$

$\mathfrak{g}$: fin. dim. s.s.

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

$\mathcal{O}_0$:= the block of $\mathcal{O}$ containing $L(0)$

$\to$ simples of $\mathcal{O}_0$ are $L(-w\rho-\rho)$
$w \in W$

We often use "Bruhat order" on W instead of $\geq$.

KL conj :

$$[M(-w\rho-\rho) : L(-y\rho-\rho)] = P_{w_0 w, w_0 y}(1)$$

w_0 : longest element of W

Soergel : 90

$$P := \bigoplus_{w \in W} P(-w\rho-\rho) \ , \ L := \bigoplus_{w \in W} L(-w\rho-\rho)$$

$$\mathrm{End}_{\mathcal{O}} P \cong \mathrm{Ext}^{\bullet}_{\mathcal{O}}(L, L)$$

and it is Koszul.

<u>quiver & rel</u>

vertex set $= W$

$\dim \mathrm{Ext}^1_{\mathcal{O}}(L(-y\rho-\rho), L(-w\rho-\rho))$

$=$ coeff of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}(q)$

if $y < w$ ($\ell(w) =$ length of w)

($\because$ radical layer of Verma modules is captured by KL poly)

By duality, $\mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\mu))$
$\simeq \mathrm{Ext}^1_{\mathcal{O}}(L(\mu), L(\lambda))$

$\therefore$ $y \rightleftarrows w$ in the quiver.

rel : $\exists$ algorithm to compute relations

by Stroppel 03
Vybornov 07

using Soergel's work.