長瀬さん　　Recollements and Hochschild cohomology algebras ①

Aim compare $HH(A)$ and $HH(eAe)$

§ Hochschild cohomology

• $k = \bar{k}$, A : f.d. k-alg. (module = left module)

$$HH^n(A) := \mathrm{Ext}^n_{A^e}(A, A) \quad \text{where } A^e = A \otimes_k A^{op}$$

$$HH(A) := \bigoplus_{n \geq 0} HH^n(A)$$ is a graded commutative k-alg with Yoneda product.

Rmk

$\exists$ $$\to A \otimes_k A \otimes_k A \to A \otimes_k A \to A \to 0$$: proj resol. in A^e-mod.

(But difficult to compute ⋯)

Rmk

In general, $\cdots \to P_1 \to P_0 \to {}_A X \to 0$: minimal proj resol.

$$\Rightarrow P_i \cong \bigoplus (Ae_j)^{\dim_k \mathrm{Ext}^i_A(X, S_j)} \qquad \left(S_j := \frac{Ae_j}{Je_j} \right)$$

$\rightsquigarrow$ $\cdots \to P_1 \to P_0 \to {}_A A_A \to 0$: minimal proj resol in A^e-mod

$$\Rightarrow P_i \cong \bigoplus (Ae_j \otimes e_l A)^{\dim_k \mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(S_l, S_j))}$$

$$\left(\leftarrow \mathrm{Hom}_k(S_l, S_j) \cong \frac{Ae_j \otimes e_l A}{\mathrm{rad}(Ae_j \otimes e_l A)} \right)$$

and, $$\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(S_l, S_j)) \cong \mathrm{Ext}^i(S_l, S_j)$$

$\therefore$ • $\mathrm{gl.dim}\, A < \infty \Rightarrow \mathrm{Ext}^{\gg 0}_A(S_l, S_j) = 0 \Rightarrow HH^{\gg 0}(A) = 0$

$\Downarrow$

$HH(A)$: fin.dim alg.

• $\mathrm{gl.dim}\, A = \mathrm{pd}_{A^e}(A)$.

Example

$A = kQ$, Q : quiver without oriented cycles, path, arrow ($\Leftrightarrow$ arrow $i \to j$ $\dim e_j A e_i = 1$)

$$\Rightarrow \mathrm{pd}_{A^e}(A) = \mathrm{gl.dim}\, A \leq 1$$

$0 \to P_1 \to P_0 \to A \to 0$: proj resol.

$$P_0 = \bigoplus (Ae_j \otimes e_l A)^{\dim_k \mathrm{Hom}_A(S_l, S_j)}$$

$$= \bigoplus_{j \in Q_0} (Ae_j \otimes e_j A)$$

$$P_1 = \bigoplus (Ae_j \otimes e_l A)^{\underbrace{\dim_k \mathrm{Ext}^1_A(S_l, S_j)}_{= \#\{\text{arrow } l \to j\} \leq 1}}$$

$$= \bigoplus_{j \leftarrow l \text{ in } Q_1} (Ae_j \otimes e_l A).$$

$$0 \to \mathrm{Hom}_{A^e}(A,A) \to \mathrm{Hom}_{A^e}\Big(\bigoplus_{j\in Q_0}(Ae_j\otimes e_jA),\ A\Big)$$
$$\qquad \mathrm{Hom}_{A^e}(A,A) = HH^0(A)$$
$$\longrightarrow \mathrm{Hom}_{A^e}\Big(\bigoplus_{l\to j} Ae_j\otimes e_lA,\ A\Big) \to \mathrm{Ext}^1_{A^e}(A,A) \to 0.$$
$$\qquad \mathrm{Ext}^1_{A^e}(A,A) = HH^1(A)$$

$\Downarrow$

$$0 \to HH^0(A) \to \bigoplus_{j\in Q_0} e_jAe_j \to \bigoplus_{l\to j} e_jAe_l \to HH^1(A) \to 0$$

$HH^0(A) = c(A)$: center of A $\simeq k^d$, $d = \#\{\text{conn. component of } Q\}$

$\bigoplus_{j\in Q_0} e_jAe_j$: $\dim = \#Q_0$

$\bigoplus_{l\to j} e_jAe_l$: $\dim = \#Q_1$.

$$\Rightarrow \quad \dim HH^0(A) - \#Q_0 + \#Q_1 - \underbrace{\dim HH^1(A)}_{=\#\{\text{cycles in } \bar{Q}\}} = 0$$

(forget orientation !)

$\therefore$ Q : connected tree $\Rightarrow$ $HH(kQ) = HH^0(kQ) \simeq k$.

§ Stratifying ideal.

Def $A \ni e = e^2$: idempotent.

AeA : stratifying ideal $:\Longleftrightarrow$ $\begin{cases} \text{(i)} & Ae\otimes_{eAe} eA \xrightarrow{\text{mult.}} AeA \text{ is isom.} \\ \text{(ii)} & \mathrm{Tor}_i^{eAe}(Ae, eA) = 0 \quad \forall i>0 \end{cases}$

$\Longleftrightarrow$ $\mathrm{Ext}^i_{A/AeA}(X,Y) \to \mathrm{Ext}^i_A(X,Y)$: isom $\forall i\geq 0$.

$\Longleftrightarrow$ $D(A/AeA\text{-Mod}) \to D(A\text{-Mod})$: fully faithful.

Rmk

AeA : stratifying $\Rightarrow$ $D\left(\frac{A}{AeA}\right) \rightleftarrows D(A) \rightleftarrows D(eAe)$: recollement.

$HH\left(\frac{A}{AeA}\right) \qquad HH(A) \qquad HH(eAe)$

Thm [König-N]

AeA : stratifying $\Rightarrow$ $\exists$ $\to \mathrm{Ext}^n_{A^e}\left(\frac{A}{AeA}, AeA\right) \to HH^n(A) \to HH^n\left(\frac{A}{AeA}\right)\oplus HH^n(eAe)$
$\to \mathrm{Ext}^{n+1}_{A^e}\left(\frac{A}{AeA}, AeA\right)$

: long exact seq

(1) $\mathrm{Ext}^n_{A^e}\left(\frac{A}{AeA}, AeA\right) = 0 \quad \forall n\geq 0 \implies HH(A) \simeq HH\left(\frac{A}{AeA}\right)\times HH(eAe)$ as alg

$\Downarrow$

$0 = \mathrm{Ext}^1\left(A/AeA, AeA\right) \ni [0\to AeA \to A \to A/AeA \to 0]$

$\therefore$ $A \simeq eAe \times \frac{A}{AeA}$ as alg (hence trivial)

(2) $\mathrm{Ext}^n_{A^e}\left(\frac{A}{AeA}, AeA\right) = 0$ $(\forall n \gg 0)$ $\Rightarrow$ $HH^{\gg 0}(A) \cong HH^{\gg 0}\left(\frac{A}{AeA}\right) \times HH^{\gg 0}(eAe)$

$\therefore$ $HH(A)$: noeth. $\Leftrightarrow$ $HH\left(\frac{A}{AeA}\right)$, $HH(eAe)$: noeth.

(3) $\mathrm{pd}_{A^e}\left(\frac{A}{AeA}\right) < \infty$ $\Rightarrow$ case (2)

$\Updownarrow$ (AeA : stratifying)

$\mathrm{pd}_A \underbrace{\mathrm{top}\left(\frac{A}{AeA}\right)}_{\bigoplus_{eS_i=0} S_i} < \infty$ and $\mathrm{id}_A \mathrm{top}\left(\frac{A}{AeA}\right) < \infty$

(AeA : str.) $\Updownarrow$

$\begin{cases} \mathrm{gl.dim}\, \frac{A}{AeA} < \infty \\ \mathrm{pd}_{eAe}(eA) < \infty \end{cases}$ $\Longrightarrow$ $HH^{\gg 0}\left(\frac{A}{AeA}\right) = 0$ $\therefore$ $HH^{\gg 0}(A) \cong HH^{\gg 0}(eAe)$

Rmk

e : idempotent $A^e\text{-Mod} \longrightarrow (eAe)^e\text{-Mod}$: exact

$X \longmapsto eXe$

$\Rightarrow$ $\exists\, HH(A) \longrightarrow HH(eAe)$ (even if AeA : not stratifying)

When is this map isom (for $\forall i \gg 0$) ?

Def [PSS]

$e : \mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,eAe$: eventually homological isom.

$:\Leftrightarrow$ $\exists\, n \geq 0$ s.t. $\mathrm{Ext}^i_A(X, Y) \to \mathrm{Ext}^i_{eAe}(eX, eY)$: isom

for $\forall i \geq n$, $\forall X, Y \in \mathrm{Mod}\,A$.

Thm [PSS] $e : \mathrm{Mod}\,A \to \mathrm{Mod}\,eAe$. $X \longmapsto eX$. TFAE :

(1) e : even. homol. isom.

(2) (α) : $\mathrm{id}_A\left(\mathrm{top}\,\frac{A}{AeA}\right) < \infty$ and (β) : $\mathrm{pd}_{eAe}(eA) < \infty$.

(3) (α') : $\mathrm{pd}_A\left(\mathrm{top}\,\frac{A}{AeA}\right) < \infty$ and (β') : $\mathrm{pd}_{(eAe)^{op}}(Ae) < \infty$.

Rmk

$HH^n(A) = \mathrm{Ext}^n_{A^e}(A, A) \xrightarrow{\sim} \mathrm{Ext}^n_{(eAe)^e}(eAe, eAe) = HH^n(eAe)$ $n \gg 0$

if e : eventually homological isom.

$$\begin{array}{ccc}
\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}_k(S_\ell, S_j)) & \simeq & \mathrm{Ext}^n_A(S_\ell, S_j) \\
 & & \big\downarrow \wr \quad n \gg 0 \quad (e : \text{even. hom, iso}) \\
\mathrm{Ext}^n_{(eAe)^e}(eAe, \mathrm{Hom}_k(eS_\ell, eS_j)) & \simeq & \mathrm{Ext}^n_{eAe}(eS_\ell, eS_j)
\end{array}$$