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Series Editors: M. Grazia Speranza · José Fernando Oliveira

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Risk Management for Pension Funds

A Continuous Time Approach with Applications in R

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Chapter 1
Introduction

Since the seminal papers of Markowitz (1952) and Merton (1969, 1971), the literature about optimal asset allocation and risk management has been developing fast and now takes into account many possible frameworks and applications. Here, we deal with the application of the asset allocation problem to a more recent topic: the pension funds.

Unlike analyses dedicated to non-actuarial institutional investors (a general framework can be found, for instance, in Lioui and Poncet 2001), the case of a pension fund requires the introduction of two new characteristics: (1) the different behaviour of the fund wealth during the accumulation phase (hereafter APh) when contributions are paid by the member and the distribution phase (hereafter DPh) when the pension is paid to the member, and (2) the mortality risk.

The link between contributions and pensions can be established inside one of the two following frameworks: the so-called defined-benefit pension plan (hereafter DB) or the so-called defined-contribution pension plan (hereafter DC). In a DB plan benefits are fixed in advance by the sponsor and contributions are set in order to maintain the fund in balance. In a DC plan contributions are fixed and benefits depend on the returns on fund portfolio (of course, also mix structures are allowed in the real world). DC pension schemes are becoming more and more important in the pension systems of most industrialised countries and are replacing the DB schemes that were more frequent in the past. It is well known that the financial risk is mainly faced by the sponsor in DB schemes and by the member in DC schemes (see, for instance, Knox 1993).

Contributions and pensions must be linked by a so-called "feasibility" condition guaranteeing it is convenient for both the pension fund and the member to underwrite the pension scheme. Such a condition is also present, for instance, in Sundaresan and Zapatero (1997) and Josa-Fombellida and Rincón-Zapatero (2001). The latter work, in particular, examines the problem of a firm which must pay both wages (before its workers retire) and pensions (after they retire). Thus, a "feasibility" condition implies the equality between the total expected value of

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wages and pensions paid with the total expected value of worker productivity (according to the usual economic rule equating the optimal wage with the marginal product of labour).

The demographic dimension is introduced via a survival probability for the member of the pension fund whose time of death τ is stochastic. Furthermore, in order to take into take into account a longevity risk, we also assume that the force of mortality of τ is stochastic itself (in a so-called double stochastic framework). Let us remark that the mortality risk supported by a single subscriber is much more important than the one supported by the fund, and, nevertheless, the longevity risk cannot be effectively managed through a mutualisation technique. The only effective tool for hedging against longevity risk is a derivative on human life (like, for instance, a longevity bond).

Here, we take into account a fully funded pension fund where pensions are paid using only the contributions paid by each member. This means that we do not take into account any overlapping generation (see, for instance, Haberman and Sung 1994).

The existing literature dealing with the asset allocation problem for a pension fund in a mixed actuarial and financial framework, mainly neglects the longevity risk, while some works deal with the mortality risk (see, for instance, the seminal paper of Richard 1975 and Charupat and Milevsky 2002 and Young and Zariphopoulou 2002 for a more recent reference).

The optimal investment strategy in the accumulation phase (i.e. prior to retirement) in a DC framework has been derived in the literature with a variety of both objective functions (mainly maximisation of expected utility of final wealth) and financial market structures (see, among others, Boulier et al. 2001; Vigna and Haberman 2001; Haberman and Vigna 2002; Deelstra et al. 2003; Devolder et al. 2003; Battocchio and Menoncin 2004; Xiao et al. 2007; Battocchio et al. 2007; Gao 2008; Di Giacinto et al. 2011).

There is relatively little literature on mean-variance portfolio selection in long-term investment planning and in pension funds (see also Steinbach 2001). Mean-variance problems for DC plans are solved in He and Liang (2013), Yao et al. (2013), Yao et al. (2014), and Vigna (2014); for DB plans they are solved in Josa-Fombellida and Rincón-Zapatero (2008) and Delong et al. (2008). In continuous time the mean-variance optimisation problem has been solved for the first time analytically by Richardson (1989), and then by Bajeux-Besnainou and Portait (1998), both through the martingale approach. The quadratic utility presents some drawbacks that will be shown in this work while describing how to represent agent's preferences.

Despite the relevant and increasing hedging need of pension funds and annuity providers, the market for longevity risk, i.e. the risk of unexpected changes in the mortality of a group of individuals, is not sufficiently liquid yet.

Many reasons may have contributed to undermine a rapid development of the market, such as the lack of standardisation, informational asymmetries, and basis risk. Nevertheless, recent developments provide a sound technology for modelling the systematic randomness in mortality (see e.g. Lee and Carter 1992), for designing and evaluating hedging instruments (Blake et al. 2006 and Denuit et al. 2007) and for managing longevity risk (Barrieu et al. 2012).

Furthermore, the transfer of longevity risk from pension funds to re-insurers has become more and more common, although on an over-the-counter basis. For instance, the volume of outstanding UK longevity swaps has reached 50 billion pounds as of the end of 2014, with a prevalence of very large deals, such as the 16 billion pounds swap between BT Pension Scheme and Prudential and the 12 billion Euros Delta Lloyd/RGA Re index-based transaction. Investment banks have been also actively in the transactions. Between 2008 and 2014, alongside reinsurance specialists, JP Morgan, Credit Suisse, Goldman Sachs, Deutsche Bank and Société Générale were involved in longevity deals (Luciano and Regis 2014).

Longevity-linked products should be of interest to asset managers for at least two reasons: their low correlation to other asset classes (at least in the short run, see Loeys et al. 2007), and their effectiveness in hedging individual investors against the unexpected fluctuations of their subjective discount factors, which take into account lifetime uncertainty (Yaari 1965, Merton 1971, Huang et al. 2012).

The aim of this work is to present and study the optimal portfolio choices of a pension fund subject to longevity risk during both the APh and the DPh. The fund can invest in a friction-less, arbitrage free, and complete financial market where both traditional assets (bonds and stocks) and a longevity bond are listed. We consider a fixed deterministic retirement age, in contrast with Farhi and Panageas (2007) and Dybvig and Liu (2010) for instance, who consider an endogenous retirement choice.

An extensive literature has explored consumption and investment decisions when mortality contingent claims are present. In particular, Huang and Milevsky (2008) analyse the decisions of families in the presence of income risk and life insurance. Explicit solutions are also obtained by Pirvu and Zhang (2012) with stochastic asset prices drifts and inflation risk and by Kwak and Lim (2014) with constant relative risk aversion (CRRA) preferences. All these papers consider a deterministic force of mortality, while we model it as a stochastic process. We describe longevity risk by means of a doubly stochastic process whose intensity follows a continuous-time diffusion (as in Dahl 2004). This process may be correlated with the other state variables. With stochastic mortality, both individuals and annuity/life insurance sellers are exposed to unexpected changes in the force of mortality, implying under or over reserving.

The optimal investment problem of pension funds in the accumulation phase has been studied for instance by Battocchio and Menoncin (2004) and Delong et al. (2008), while Battocchio et al. (2007) propose a unified model for describing both the APh and the DPh. The role of longevity-linked assets in investor's optimal portfolio has been addressed first by Menoncin (2008). Maurer et al. (2013), solving a life-cycle portfolio investment problem with longevity risk, assess the importance of variable annuities to smooth consumption, while Horneff et al. (2010) analyse the role of deferred annuities. They find that these products should optimally account for 78% of the financial wealth of a retiree.

While insurance products are non-marketable, longevity assets on the market allow individuals to dynamically hedge against mortality fluctuations (we abstract from transaction costs). Cocco and Gomes (2012) analyse, in the context of a life-cycle model, the demand for a perfect hedge against shocks in the life expectancy of a CRRA agent. They study the optimal investment in a longevity bond, which is akin to our zero-coupon longevity asset. In their numerical simulations, they find that individuals—at old ages and especially approaching retirement—should invest a relevant fraction of their wealth in the longevity asset.

Here, we show how to obtain a closed form solution to the problem of a fund, under Hyperbolic Absolute Risk Aversion (HARA) preferences, when mortality intensity is stochastic. We also provide a calibrated application, which allows to appreciate the relevance of longevity products in the optimal portfolio.

1.1 The Structure of the Book

The aim of this work is to show how to solve the asset allocation problem of a pension fund which aims to maximise the expected utility of the wealth remaining at the death time of a representative pensioner. Of course, this framework is akin to that of a pension fund which works on a cohort of workers.

If we call τ the stochastic death time of a representative pensioner, w_t the vector containing the fund's portfolio and ρ_t the rate used by the fund for discounting future cash flows, the problem of the fund can be written as

$$\max_{w_{t\in[t_0,\tau]}} \mathbb{E}_{t_0}\left[U\left(R_\tau\right)e^{-\int_{t_0}^{\tau}\rho_s ds}\right],$$

where R_τ is the wealth of the fund at the death time, $U(\bullet)$ is the utility function and $\mathbb{E}_{t_0}[\bullet]$ is the expected value operator conditional to the information available at time t_0. The fund's wealth dynamics is driven by both the values of the assets held by the fund in its portfolio and the external cash flows given by the contributions and the pensions. Finally, the fund must face the risks modelled by many state variables such as: interest rate risk, mortality and longevity risk (stochastic force of mortality), inflation risk. In our general framework we will take into account any (finite) number of state variables, but we will also present some particular cases.

Each element of this optimisation problem is studied in this work according to the following schedule.

1. The utility function is useful for taking into account the risk aversion that the pension fund is assumed to inherit from its subscribers (the workers/pensioners). In the first chapter we show how to interpret the shape of the utility function for representing investor's preferences.
2. The second chapter presents the stochastic processes that will be used in the work in order to model the state variables of the optimisation problem. Some estimation techniques will be shown too.

3. The third chapter shows how a financial market can be modelled through the stochastic processes presented in the previous chapter. In particular, we show how to model a portfolio of risky and risk-less asset and how to price assets through the fundamental theorem of asset pricing in an arbitrage free financial market. The issue of an incomplete financial market will also be shown.
4. Chapter 4 will show how to use stochastic processes for modelling the actuarial variables, like the survival probability or the mortality intensity.
5. In Chap. 5 the fundamental theorem of asset pricing will be used in a version suitable for pricing actuarial assets thought of as derivatives written on the force of mortality which is a stochastic process itself (modelled in the previous chapter).
6. The Chap. 6 presents the theoretical computation of the optimal portfolio for a pension fund by using the so-called "martingale method". We demonstrate that this optimal portfolio is formed by a speculative and a hedging component and their roles and characteristics are described in full details.
7. In Chap. 7 we perform a full numerical example with US data and we show the values of the optimal portfolio containing a risk-less asset, a stock, a rolling zero coupon bond, and a rolling longevity zero coupon bond. The dynamics of the optimal asset shares are shown and commented.
8. The last chapter is devoted to the case of a pure accumulation fund that receives contributions and pays a final amount of money at retirement, without any mortality/longevity risk (i.e. all the wealth matured for the contributors is paid to the heirs in case of death). This case is show just as a benchmark for underlining in a deeper way how the longevity risk affects the optimal asset allocation of a pension fund.

1.2 The R Software

In this book we will use the R free software https://www.r-project.org/ and its free interface RStudio https://www.rstudio.com/. A full introduction to the R programming language is out of the scope of this book. Here, we just outline the main features of R. Other (and similar) software like Matlab (or its freeware clones like Scilab—https://www.scilab.org—or Octave—https://www.gnu.org/software/octave/), use mainly a vector/matrix approach to computations, while R also and mainly works with data frames and lists.

In this work we use the package Knitr for LaTeX (http://yihui.name/knitr/) which allows to code and execute R scripts directly on a LaTeX document without using externally the R software.

In the following code we show how to create three sets of data (with command `c`) and show the first one.

```
A = c(11, 12, 14)
B = c(19, 20, 21)
C = c(10, 9, 7)
A
## [1] 11 12 14
```

The number between brackets is the number of the first element of the row. In this example, the first element of the first row of A is the element number 1.

All the sets that have been created can be put together into a data frame through the following commands, where we also show the whole set and a subset of it. Finally, the mean of the second subset is computed. In order to recall a subset of a data frame, the dollar symbol is needed. In the following example, the data frame X contains the subsets A, B, and C which are identified as $X\$A$, $X\$B$, and $X\$C$ respectively.

```
X = data.frame(A, B, C)
X

##    A  B  C
## 1 11 19 10
## 2 12 20  9
## 3 14 21  7

X$A

## [1] 11 12 14

mean(X$B)

## [1] 20
```

A matrix can be created by concatenation of the sets A, B and C (row by row through the command `rbind`—or column by column through the command `cbind`). The new matrix (called M) can be used as argument of a matrix command like computing its determinant (`det`) or its transposition (`t`).

```
M = rbind(A, B, C)
det(M)

## [1] -21

t(M)

##       A  B  C
## [1,] 11 19 10
## [2,] 12 20  9
## [3,] 14 21  7
```

When two matrices are multiplied through the command `*`, the product is applied element wise. In order to apply the matrix product, we must use the command `%*%`, as we can see in the following example.

```
M * M
```

```
##   [,1] [,2] [,3]
## A  121  144  196
## B  361  400  441
## C  100   81   49
```

```
M %*% t(M)
```

```
##     A    B   C
## A 461  743 316
## B 743 1202 517
## C 316  517 230
```

If the command `rbind` is used on a matrix and a single number, a new row is appended to the matrix and all the elements of this row are equal to that single number.

```
rbind(M, 0)
```

```
##   [,1] [,2] [,3]
## A   11   12   14
## B   19   20   21
## C   10    9    7
##      0    0    0
```

The elements of a matrix or a set are identified by their coordinates inside brackets.

```
M[2, 1]
```

```
##  B
## 19
```

```
A[3]
```

```
## [1] 14
```

A subset of elements can be selected by using brackets, comas, and colon as follows.

```
M[1:2, 1]

##  A  B
## 11 19
```

The meaning of the above command is: “Take form matrix M, the row from 1 to 2, and the column 1”.

A sequence can be created through the command `seq` whose arguments are as follows:

```
seq(from = , to = , by = ,length.out = , along.with = )
```

where “`by`” contains the constant difference between two adjacent elements of the sequence, “`length.out`” is the number of elements in the sequence, and “`along.with`” is the variable whose dimension we want the sequence to replicate. Here are some examples.

```
seq(0, 2, by = 0.5)

## [1] 0.0 0.5 1.0 1.5 2.0

seq(0, 1, length.out = 10)

##  [1] 0.0000000 0.1111111 0.2222222 0.3333333 0.4444444 0.5555556 0.6666667
##  [8] 0.7777778 0.8888889 1.0000000

seq(0, 1, along.with = A)

## [1] 0.0 0.5 1.0
```

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Chapter 2
Decision Theory Under Uncertainty

2.1 Introduction

The choice of any agent on the financial market is guided by his/her preferences. Under some technical hypotheses, these preferences can be fully described through a function, the so-called "utility function". This function is a pure theoretical artifice but allows to use a powerful tool like the calculus for working with preferences and makes computations much easier.

In this chapter we show how to switch from preferences to a utility function and the characteristics of such a function. In particular, we show that it can suitably represent the risk aversion of an agent.

The knowledge of the utility function is fundamental for our approach, since we base our model on the assumption that the pension fund maximises an objective function which must consistently take into account its risk aversion on the financial markets.

2.2 Decision Theory (Without Risk)

Economic agents (households, firms, governments, institutional investors, and so on) are assumed to behave according to their "preferences". In Economics, preferences are always given and do not depend on any other variable (they are the so-called "**primitives**").

Preferences are usually defined on goods (and services). Nevertheless, in finance, preferences are rather defined on cash flows (which may be available either immediately or in the future). The main relationship used for describing preferences is the following one:

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- "**to be (weakly) preferred to**" ($\succeq$): when the cash flows in vector x are preferred to the cash flows in vector y, we write

$$x \succeq y,$$

where the adverb "weakly" means that for some elements of the vectors an agent might be indifferent between x and y.

The indifference relationship can be obtained from the previous one:

- "**to be indifferent to**" ($\sim$): when the cash flows in x are indifferent to the cash flows in y we write

$$x \sim y.$$

Given x and y, if both $x \succeq y$ and $y \succeq x$ hold, then the only conclusion is that the elements in x must be indifferent to the elements in y.

An economic agent is said to be "**rationale**" if his/her preferences (described by $\succeq$) are both

- **complete**: this means that an agent is always able to declare his/her preferences when facing a choice; in formal terms, we can write that, given x and y, either $x \succeq y$ or $y \succeq x$ or both (this last case coincides with the "indifference" case, i.e. $x \sim y$); this is a strong hypothesis since agents may not know the possible choices (what about choosing between a *gulasch* and a *moussaka* if you have never tasted those dishes?);
- **transitive**: this means that given three vectors x, y, and z, if an agent prefers x to y, and prefers y to z, then he/she must also prefer x to z; formally

$$x \succeq y, \quad y \succeq z \quad \Rightarrow \quad x \succeq z.$$

Rationality is the less restrictive requirement, but without any additional hypothesis we are not able to represent preferences through any function. The hypothesis we still need to this purpose is

- **continuity**: given three outcomes $x \succeq y \succeq z$, it is always possible to find a linear strictly convex combination of x and z which is indifferent to y; formally

$$\forall x \succeq y \succeq z, \quad \exists \alpha \in [0, 1] : y \sim \alpha x + (1 - \alpha) z.$$

Theorem 2.1 *If preferences are rationale (i.e. complete and transitive), and continuous, then there exists a continuous function $U(\bullet)$ (so-called "**utility function**") such that*

$$x \succeq y \Longleftrightarrow U(x) \geq U(y).$$

For a formal proof of Theorem 2.1 see Gollier (2001).

The utility function is, of course, not unique. Since the result of Theorem 2.1 just implies an inequality, any monotonically increasing transformation of the utility function $U(\bullet)$ does not alter such inequality. In other words, if $V(\bullet)$ is an increasing function, then

$$U(x) \geq U(y) \Longleftrightarrow V(U(x)) \geq V(U(y)).$$

This result allows us to conclude that a utility function cannot in any way be used for measuring the satisfaction of an economic agent. Instead, the utility function is just suitable for ordering the outcomes according to the preferences. In other words, the value $U(x)$ has no meaning by itself (and defining any unit of measure for it does not make any sense), and the utility function $U(\bullet)$ is relevant only for checking whether $U(y)$ is higher or lower than $U(x)$.

Rationality and continuity allow to switch from the field of preferences to the field of functions where all the powerful tools of the mathematical analysis can be applied.

2.3 Decision Theory (With Risk)

Theorem 2.1 is very useful for studying the behaviour of agents in a framework without risk, i.e. if the arguments of the utility function are deterministic. In case of risk, instead, the same framework does not allow to reach any significant result. If we are in a risky framework, x and y are stochastic variables and they cannot be compared through an inequality like that in Theorem 2.1.

A stochastic cash flow x can be represented as follows:

$$x = \begin{cases} x_1, & p_1 \geq 0 \\ x_2, & p_2 \geq 0 \\ \dots & \dots \\ x_k, & 1 - \sum_{i=1}^{k-1} p_i \end{cases}$$

where x_i ($i \in \{1, 2, \dots, k\}$) is the cash flow that occurs with probability p_i. If we think of x as the payoff of a risky asset, then each x_i is the payoff obtained in a particular state of the world and is not known ex ante. If the cash flows in x are continuously defined, then there exists a (non negative) density function $f(x)$ (whose integral on the whole domain is 1) through which we can measure the probability that a cash flow takes value in $[a, b]$:

$$\mathbb{P}\{a \leq x \leq b\} = \int_a^b f(x)\, dx.$$

In this framework, preferences defined on stochastic cash flows can be described through a utility function if and only if they satisfy another property:

- **independence (of irrelevant alternatives)**: if an agent prefers x to y, then between the two bundles (x, z) and (y, z) that contain z in the same proportion, he/she must prefer the bundle containing x. Formally:

$$x \succeq y \Longleftrightarrow \forall z, \alpha \in [0, 1] : \alpha x + (1 - \alpha) z \succeq \alpha y + (1 - \alpha) z.$$

Under this axiom, one of the most important result for the choice theory under uncertainty is reached (von Neumann and Morgenstern 1947).

Theorem 2.2 *If preferences are rationale (i.e. complete and transitive), continuous, and independent of irrelevant alternatives, then there exists a continuous function* $U(\bullet)$ *(so-called "**utility function**") such that*

$$x \succeq y \Longleftrightarrow \mathbb{E}[U(x)] \geq \mathbb{E}[U(y)].$$

Here $\mathbb{E}[\bullet]$ is the "expected value" operator, and a formal demonstration of Theorem 2.2 can be also found in Gollier (2001).

Also in this case the utility function is not unique, but this time the result of Theorem 2.2 is unaffected if and only if we apply an affine increasing transformation to the utility function (any other increasing transformation may alter the result). In fact, if we use $a + bU(x)$ (with b positive) , the previous relationship becomes

$$\mathbb{E}[a + bU(x)] \geq \mathbb{E}[a + bU(y)],$$

which can be simplified to

$$a + b\mathbb{E}[U(x)] \geq a + b\mathbb{E}[U(y)],$$

and

$$\mathbb{E}[U(x)] \geq \mathbb{E}[U(y)],$$

where we clearly see the need for b to be positive (if b is negative, the last passage does not preserve the sign of the inequality).

All the results we have exposed so far are summarised in Table 2.1.

Table 2.1 Relationship between axioms on preferences and utility theory

Preferences $(\succeq, \sim)$		Utility	Expected utility
Complete either $x \succeq y$ or $y \succeq x$ or both (i.e. $x \sim y$)			
Transitive $x \succeq y, y \succeq z$ $\Longrightarrow x \succeq z$	$\Rightarrow$ Rationale		
	Continuity $\forall x \succeq y \succeq z$ $\exists \alpha \in [0,1]:$ $y \sim \alpha x + (1-\alpha) z$	$\Rightarrow \exists U(\bullet): x \succeq y$ $\Longleftrightarrow$ $U(x) \geq U(y)$	
		Independence $\forall x, y, z, \alpha \in [0,1]:$ $x \succeq y$ $\Longleftrightarrow$ $\alpha x + (1-\alpha) z$ $\succeq \alpha y + (1-\alpha) z$	$\Rightarrow \exists U(\bullet): x \succeq y$ $\Longleftrightarrow$ $\mathbb{E}[U(x)] \geq \mathbb{E}[U(y)]$

Example 2.1 Let us assume that the cash flow x has the following distribution:

$$x = \begin{cases} a, & p \\ b, & 1-p \end{cases}$$

where we assume $b > a$ and p is the probability that the cash flow a happens. The cash flow y, instead, has the distribution

$$y = \begin{cases} a, & 2p \\ b, & 1-2p \end{cases}$$

where the probability of the event a has been doubled.

Any agent who prefers to have more than less (i.e. $b > a \Rightarrow U(b) > U(a)$) prefers the stochastic cash flow x. Actually, in x the worst case has a higher probability, and the better case has a lower probability. In fact, the two expected utilities are

$$\mathbb{E}[U(x)] = U(a)\,p + U(b)(1-p) = U(b) - p\,(U(b) - U(a)),$$

$$\mathbb{E}[U(y)] = U(a)\,2p + U(b)(1-2p) = U(b) - 2p\,(U(b) - U(a)),$$

(continued)

Example 2.1 (continued)

where, since $U(b) - U(a) > 0$, it is evident that

$$\mathbb{E}[U(x)] > \mathbb{E}[U(y)].$$

Table 2.2 Lotteries for the so-called Allais' paradox

	Probability		
Lottery	0.01	0.89	0.1
x	50	50	50
y	0	50	250
w	50	0	50
z	0	0	250

2.4 Critics to the Expected Utility

One of the most important critics to the expected utility framework came from Allais (1953) who proposed an experiment based on the following lotteries (stochastic cash flows):

$$x = \begin{cases} 50, & 1 \end{cases} \qquad w \begin{cases} 0, & 0.89 \\ 50, & 0.11 \end{cases}$$

$$y \begin{cases} 0, & 0.01 \\ 50, & 0.89 \\ 250, & 0.1 \end{cases} \qquad z \begin{cases} 0, & 0.9 \\ 250, & 0.1 \end{cases}$$

We see that lottery x is "degenerate" since it pays 50 with probability 1. These lotteries can also be represented as in Table 2.2.

We can set $U(0) = 0$ and $U(250) = 1$ (without any loss of generality)[1] and so the expected utilities from the four lotteries are:

$$\begin{aligned} \mathbb{E}[U(x)] &= U(50), \\ \mathbb{E}[U(y)] &= 0.10 + 0.89U(50), \\ \mathbb{E}[U(w)] &= 0.11U(50), \\ \mathbb{E}[U(z)] &= 0.10. \end{aligned}$$

[1] Note that give any utility function $U(x)$, the transformation $\frac{U(x)-U(0)}{U(250)-U(0)}$ (which implies $U(0) = 0$ and $U(250) = 1$) is an affine increasing transformation of $U(x)$ and, accordingly, it does not alter investor's preferences.

These lotteries were actually presented to some agents and Allais found that most of the people in the experiment preferred x to y, and z to w. According to the expected utility framework, if x is preferred to y then we must have

$$\mathbb{E}\left[U\left(x\right)\right] > \mathbb{E}\left[U\left(y\right)\right],$$

which means

$$U\left(50\right) > 0.10 + 0.89U\left(50\right),$$

or

$$0.11U\left(50\right) > 0.10.$$

It is evident that this result implies that w must be preferred to z (and not vice versa). Thus, if an agent prefers x to y and z to w, his/her preferences are violating the independence axiom.

In a financial framework, this violation is avoided through an argument known as "**Dutch book**" (a situation where the bookmaker proposes a set of trades which leave the other part strictly worse off).

Let us take four lotteries: a, b, c, and d where d is a linear combination of b and c (for instance $d = \frac{1}{2}b + \frac{1}{2}c$). This means that d can also be interpreted as a lottery whose outcomes are again lotteries: with probability $\frac{1}{2}$ the outcome of the lottery will be b, and with the same probability the outcome will be c.

If an agent has preferences such that

$$a \succeq b,$$
$$a \succeq c,$$

then he/she should also prefer a to d (since d is a combination of lotteries that he/she does not like).

Let us assume, instead, that $d \succeq a$ (i.e. preferences are not "independent of irrelevant alternatives"). In this case, when deciding whether to invest in a, b, or c the investor coherently chooses a. Nevertheless, a bank may offer him/her to exchange this asset with the combination d. The agent accepts (and he/she pays a fee for such a trade). When the outcome of the compound lottery d happens, the investor has either b or c. Nevertheless, now he/she is willing to give this asset back to the bank and receive the asset a that he/she prefers (and for this trade he pays other fees). The strategy of the bank can start again!

So, an agent whose preferences are not independent, can be Dutch-booked by any bank and will be eliminated from the financial market in no time.

2.5 Risk Aversion

We start by taking into account a lottery (R) whose outcomes can be either R_1 (with probability p) or R_2 (we assume $R_2 > R_1$):

$$R = \begin{cases} R_1, & p \\ R_2, & 1 - p \end{cases}$$

The expected value of R is, of course, between the two values R_1 and R_2 (Fig. 2.1). If an agent receives either R_1 or R_2, his/her utility is either $U(R_1)$ or $U(R_2)$, respectively. The expected utility lies between $U(R_1)$ and $U(R_2)$, on a straight line (Fig. 2.1) since the expected value is a linear operator.

Now, if the agent faces the choice either to play the lottery (and receive either R_1 or R_2 at the end) or to receive its expected value without any risk, he/she has the following alternatives:

$$\begin{aligned} \mathbb{E}[U(R)] &= \text{expected utility of the lottery}, \\ U(\mathbb{E}[R]) &= \text{utility of the expected value of the lottery}. \end{aligned}$$

We say that an agent is risk averse if his/her preferences are such that

$$U(\mathbb{E}[R]) > \mathbb{E}[U(R)]. \tag{2.5.1}$$

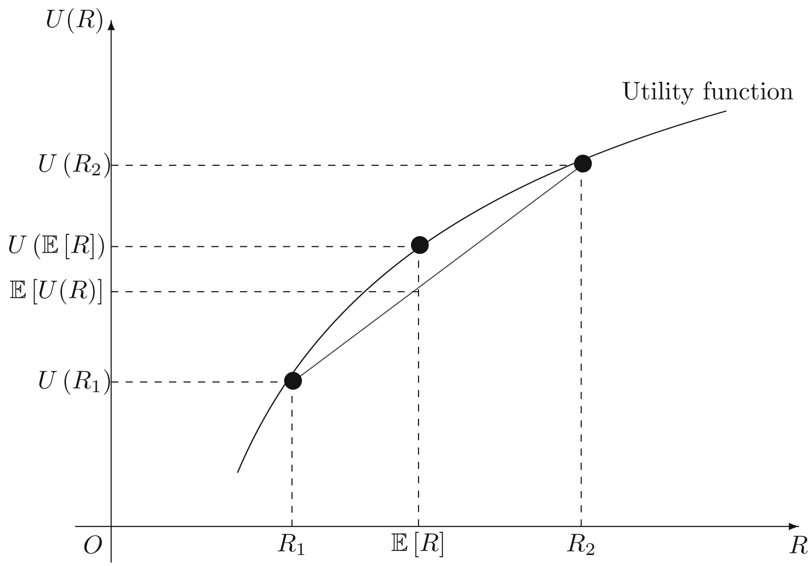


Fig. 2.1 Behaviour of a concave utility function

Definition 2.1 An agent is risk averse if and only if he/she prefers to receive the expected value of a lottery for sure, rather thank playing the lottery.

If the utility of the expected value is higher than the expected utility, then the utility function lies above the straight line drawn between two of its points: this is exactly the definition of a concave (utility) function. The inequality (2.5.1) is the so-called "**Jensen inequality**": Jensen stated that a concave transformation of an integral is greater than the integral of the same concave transformation.

Accordingly, we can conclude what follows.

Proposition 2.1 *An agent is risk averse if and only if his utility function is concave.*

Of course an agent whose utility function is convex is a risk lover, while an agent whose utility is linear is risk neutral. Thus, in order to measure risk aversion, we must compute the "degree of concavity" of a function (and the risk aversion is accordingly linked to the second derivative).

Arrow and Pratt (Arrow 1965; Pratt 1964) propose another approach which leads to the same result linking concavity and risk aversion. The idea is to start from Jensen inequality (2.5.1) and to make it an equality by reducing the argument of the utility in the left hand side. In other words, we assume that an agent is willing to pay an amount of money (χ) in order to avoid the risk; thus χ is like an insurance premium and it is called "**risk premium**". The maximum amount of money an agent is willing to pay for avoiding the risk solves the equation

$$U(\mathbb{E}[R] - \chi) = \mathbb{E}[U(R)]. \tag{2.5.2}$$

Remark 2.1 The argument of the utility in the left hand side ($\mathbb{E}[R] - \chi$) is called "**certainty equivalent**", and is the certain amount of money which is equivalent to an uncertain amount in terms of expected utility.

Now, both sides are developed in Taylor series under the hypothesis of having a small risk (i.e. either a wealth close to its expected value or $\chi = 0$). When the utility $U(R)$ is expanded in Taylor series (up to the second order) around $R = \mathbb{E}[R]$, we obtain

$$U(R) \simeq U(\mathbb{E}[R]) + U_R(\mathbb{E}[R])(R - \mathbb{E}[R]) + \frac{1}{2}U_{RR}(\mathbb{E}[R])(R - \mathbb{E}[R])^2,$$

where the subscripts on U indicate partial derivatives (a double subscript is the second derivative), and whose expected value is[2]

$$\mathbb{E}\left[U\left(R\right)\right] \simeq U\left(\mathbb{E}\left[R\right]\right) + \frac{1}{2}U_{RR}\left(\mathbb{E}\left[R\right]\right)\mathbb{V}\left[R\right], \tag{2.5.3}$$

where $\mathbb{V}\left[\bullet\right]$ is the variance operator.

The utility function $U\left(\mathbb{E}\left[R\right] - \chi\right)$ is expanded (up to the first order) around $\chi = 0$:

$$U\left(\mathbb{E}\left[R\right] - \chi\right) \simeq U\left(\mathbb{E}\left[R\right]\right) - U_{R}\left(\mathbb{E}\left[R\right]\right)\chi. \tag{2.5.4}$$

This time we stop to the first order since we do not want to have a second order polynomial in χ. By equating the two approximations (2.5.3) and (2.5.4), we have

$$U\left(\mathbb{E}\left[R\right]\right) + \frac{1}{2}U_{RR}\left(\mathbb{E}\left[R\right]\right)\mathbb{V}\left[R\right] = U\left(\mathbb{E}\left[R\right]\right) - U_{R}\left(\mathbb{E}\left[R\right]\right)\chi,$$

or

$$\chi = -\frac{1}{2}\frac{U_{RR}\left(\mathbb{E}\left[R\right]\right)}{U_{R}\left(\mathbb{E}\left[R\right]\right)}\mathbb{V}\left[R\right].$$

This result allows us to conclude that the amount of money an agent is willing to pay for avoiding the risk is proportional to both the variance of wealth and the opposite of the ratio between the second and the first derivative of the utility function. This last ratio, in particular, is called "**Arrow Pratt (absolute) risk aversion index**":

$$ARA = -\frac{U_{RR}\left(R\right)}{U_{R}\left(R\right)}.$$

Since the behaviour of an agent may change with the level of his/her wealth, then also the so-called "**Arrow Pratt relative risk aversion index**" is often used:

$$RRA = -\frac{U_{RR}\left(R\right)R}{U_{R}\left(R\right)}.$$

The inverse of the risk aversion index is called "**risk tolerance**" (RT) index and it can be computed either in absolute (ART) or in relative (RRT) terms too:

$$\begin{aligned} ART &= -\frac{U_{R}\left(R\right)}{U_{RR}\left(R\right)}, \\ RRT &= -\frac{U_{R}\left(R\right)}{U_{RR}\left(R\right)R}. \end{aligned}$$

[2]We recall that $\mathbb{E}\left[R - \mathbb{E}\left[R\right]\right] = 0$ while $\mathbb{E}\left[\left(R - \mathbb{E}\left[R\right]\right)^2\right]$ is the variance of R.

The utility functions are often classified according to the risk aversion RA index by acronyms formed by four letters:

- the first letter indicates whether the index is "Constant", "Increasing", "Decreasing" or "Hyperbolic" with respect to wealth;
- the second letter indicates whether the index is in the "Absolute" form or in the "Relative" form;
- the two last letters are "RA" (standing for "Risk Aversion").

Accordingly, we may have HARA, CARA, CRRA, DARA, IARA, DRRA, IRRA, and so on.

One of the most general forms of the utility function is the following one:

$$U(R) = \frac{(\alpha + \gamma R)^{1-\frac{\delta}{\gamma}} - 1}{\gamma - \delta}. \tag{2.5.5}$$

If the exponent $1 - \frac{\delta}{\gamma}$ is not integer, then the power is well defined only if its base is positive

$$\alpha + \gamma R > 0.$$

Thus, the domain of the function is

$$R \in \begin{cases} R > -\frac{\alpha}{\gamma}, & \gamma > 0, \\ R < -\frac{\alpha}{\gamma}, & \gamma < 0. \end{cases} \tag{2.5.6}$$

The derivatives of (2.5.5) are

$$U_R(R) = (\alpha + \gamma R)^{-\frac{\delta}{\gamma}}$$
$$U_{RR}(R) = -\delta (\alpha + \gamma R)^{-\frac{\delta}{\gamma}-1}.$$

Accordingly, this function is always increasing in wealth, and is concave if and only if δ is positive. The Arrow Pratt absolute risk aversion is

$$ARA = \frac{\delta}{\alpha + \gamma R},$$

which has a hyperbolic form (thus, this function is classified as a HARA utility).

If $\gamma > 0$, the extreme of the domain (2.5.6), i.e. $-\frac{\alpha}{\gamma}$, can be easily interpreted as a "subsistence level", that is an amount of wealth below which agents do not want to go. In what follows, we will discuss in details the signs of the parameters α and γ.

Let us now look at some important sub-cases of the general HARA utility (2.5.5):

1. with $\gamma = 0$ and $\alpha = 1$ we have a CARA case:

$$\begin{aligned} U(R) &= \lim_{\gamma \to 0} \frac{(1+\gamma R)^{1-\frac{\delta}{\gamma}} - 1}{\gamma - \delta} \\ &= \lim_{\gamma \to 0} -\frac{1}{\delta}(1+\gamma R)^{1-\frac{\delta}{\gamma}} + \frac{1}{\delta} \\ &= -\frac{1}{\delta} e^{-\delta R} + \frac{1}{\delta}, \end{aligned}$$

   and the utility function has an exponential form;
2. with $\alpha = 0$ (and $\gamma = 1$ for simplifying) we have a CRRA case:

$$U(R) = \frac{R^{1-\delta} - 1}{1 - \delta},$$

   i.e. a power utility function; this is one of the most commonly used utility function in economics;
3. with $\alpha = 0$, $\gamma = 1$, and $\delta = 1$ we can use De l'Hôpital theorem on the previous result and have

$$U(R) = \lim_{\delta \to 1} \frac{R^{1-\delta} - 1}{1 - \delta} = \lim_{\delta \to 1} \frac{-R^{1-\delta} \ln R}{-1} = \ln R,$$

   which is a log function;
4. with $\delta = -\gamma$ and $\gamma < 0$ we have

$$U(R) = \frac{(\alpha + \gamma R)^2 - 1}{2\gamma},$$

   which is a quadratic utility function. The main disadvantage of this function (which allows for some closed form solutions in economics) is that it has a maximum point and after that point an increase in wealth implies a reduction in utility (i.e. the first derivative is not always positive). In this case, the domain of wealth is the whole real line, because the exponent of the power is integer. Nevertheless, the marginal utility is positive (as we want it) only in the domain $\left]-\infty, -\frac{\alpha}{\gamma}\right]$. This means that a quadratic function does not prevent the wealth of an agent from being negative. Furthermore, the optimal wealth will never be higher than $-\alpha/\gamma$, which is a kind of target for the optimisation problem. In fact, in some papers, the approach which uses this quadratic utility function is also called "target approach".

The above mentioned results are summarised in Table 2.3.

Inada (1963) proposed a list of "good properties" that a utility function must satisfy in order to have a consistent model:

Table 2.3 Particular cases of utility (2.5.5)

Parameters	Utility	ARA	RRA
$\gamma = 0,\ \alpha = 1$	$\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta R}$ exponential	δ CARA	δR IRRA
$\alpha = 0,\ \gamma = 1$	$\frac{R^{1-\delta}-1}{1-\delta}$ power	$\frac{\delta}{R}$ DARA	$\frac{1}{\delta}$ CRRA
$\alpha = 0,\ \gamma = 1,\ \delta = 1$	$\ln R$ logarithm	$\frac{1}{R}$ DARA	1 CRRA
$\gamma = -\delta$	$\frac{1-(\alpha-\delta R)^2}{2\delta}$ quadratic	$\frac{\delta}{\alpha-\delta R}$ IARA	$\frac{\delta R}{\alpha-\delta R}$ IRRA

1. $U(0) = 0$; this condition is not restrictive since any utility function can be modified for satisfying it; in the case of HARA function (2.5.5), for instance, we can rewrite it as

$$\begin{aligned} V(R) &= \frac{(\alpha+\gamma R)^{1-\frac{\delta}{\gamma}} - 1}{\gamma-\delta} - U(0) \\ &= \frac{(\alpha+\gamma R)^{1-\frac{\delta}{\gamma}} - 1}{\gamma-\delta} - \frac{\alpha^{1-\frac{\delta}{\gamma}} - 1}{\gamma-\delta} \\ &= \frac{(\alpha+\gamma R)^{1-\frac{\delta}{\gamma}} - \alpha^{1-\frac{\delta}{\gamma}}}{\gamma-\delta}, \end{aligned}$$

   and now $V(R)$ satisfies the first Inada's condition;
2. $U(R) \in \mathcal{C}^1$, i.e. $U(R)$ is continuously differentiable (which also implies that it is continuous);
3. $U_R(R) > 0$ (increasing function); the function (2.5.5) has a derivative which is always positive in the domain; this hypothesis means that any agent always prefer to have more than less;
4. $U_{RR}(R) < 0$ (concave function); the function (2.5.5) is concave for $\delta > 0$; this hypothesis means that the agent is risk averse;
5. $\lim_{R\to 0} U_R(R) = +\infty$: this means that when wealth reaches zero, any (even infinitesimal) increase in wealth gives an infinite increase in utility and, accordingly, the optimal wealth can never be zero; the function (2.5.5) does not satisfy this condition for $R = 0$ but it satisfies it for R equal to the left bound of the domain $(-\frac{\alpha}{\gamma})$, in fact for $\gamma > 0$ and $\delta > 0$,

$$\lim_{R\to-\frac{\alpha}{\gamma}} (\alpha+\gamma R)^{-\frac{\delta}{\gamma}} = +\infty;$$

   in this case we can conclude that it will never be optimal to have a wealth equal to the "subsistence" level $-\frac{\alpha}{\gamma}$;
6. $\lim_{R\to+\infty} U_R(R) = 0$: this means that when wealth tends towards infinity, any increase in wealth does not change the utility level (if an agent is infinitely reach,

the utility to have 1 more cent should definitely be zero!); the function (2.5.5) satisfies such a condition for $\gamma > 0$ and $\delta > 0$.

We have seen that the utility function (2.5.5) satisfies a modified version of the Inada's conditions if $\gamma > 0$. Accordingly, the only parameter which can be either positive or negative is α. In particular:

- if $\alpha < 0$, then there exists a positive amount of wealth, that we can call a "subsistence wealth", below which agents cannot survive;
- if $\alpha > 0$, then the wealth is allowed to be lower than zero, which means that an agent is allowed to borrow money and be indebted during a period of time.

2.6 The Stone–Geary Utility Function

What we have defined as a HARA utility function is also known as the Stone–Geary utility function (Geary 1950) and can be written in the following form:

$$U(R) = \frac{(R-\alpha)^{1-\delta} - 1}{1-\delta},$$

where α can be interpreted as a subsistence level of wealth. In fact, the domain of this function is $[\alpha, +\infty[$, and it is never optimal, for wealth, to go below the value α, whose unit of measure is the same as wealth. The Arrow–Pratt risk aversion index is

$$ARA = \frac{\delta}{R-\alpha},$$

and so we can conclude that:

- the higher δ the higher the risk aversion; thus δ is a measure of risk aversion;
- the higher α the higher the risk aversion: if an agent has a high subsistence level of wealth, he can reach such a wealth at the end of a management period only if he increases the amount of riskless asset held in his portfolio;
- the higher the wealth, the lower the risk aversion: this is a behaviour that is often observed on financial market, where wealthier agents invest in portfolios that contain higher percentages of risky assets;
- the risk aversion depends on the distance between agent's wealth and the subsistence level: in this way a wealthier agent may behave in a more risk averse way than a poorer agent if his subsistence level is higher.

When α is set to zero, the so-called Cobb–Douglas utility function is obtained. The main weak point of the Cobb–Douglas is that it allows for a very low level of wealth that may not be compatible with the preferences shown by many agents on financial markets. In fact, the investment strategies of the agents are often defined in terms of a minimum level of wealth that is required, and this level coincides with α.

Both the Stone–Geary and the Cobb–Douglas utility functions show an absolute risk aversion that decreases with respect to wealth. Nevertheless, the Cobb–Douglas ARA has an absolute elasticity with respect to wealth equal to 1, while the Stone–Geary ARA has a higher absolute elasticity (given by $R/(R-\alpha)$).

In the case of a pension fund, the parameter α plays an important role, since it can be interpreted as a minimum wealth (or return) that is guaranteed to the contributors at the retirement date.

2.7 Certainty Equivalent on Financial Markets

In R there exist two useful packages which allow to download (financial) data directly from internet: “quantmod” and “downloader”.[3] For installing these packages, the following commands are needed:

```
install.packages('quantmod')
install.packages('downloader')
```

Once these packages are installed on R, they must be uploaded into R memory through the following commands.

```
library(quantmod)
library(downloader)
```

If we want to download the value of the S&P500 from Yahoo starting from January 1950, we must use the following command (the symbol “^GSPC” stands for the S&P500 index on Yahoo):

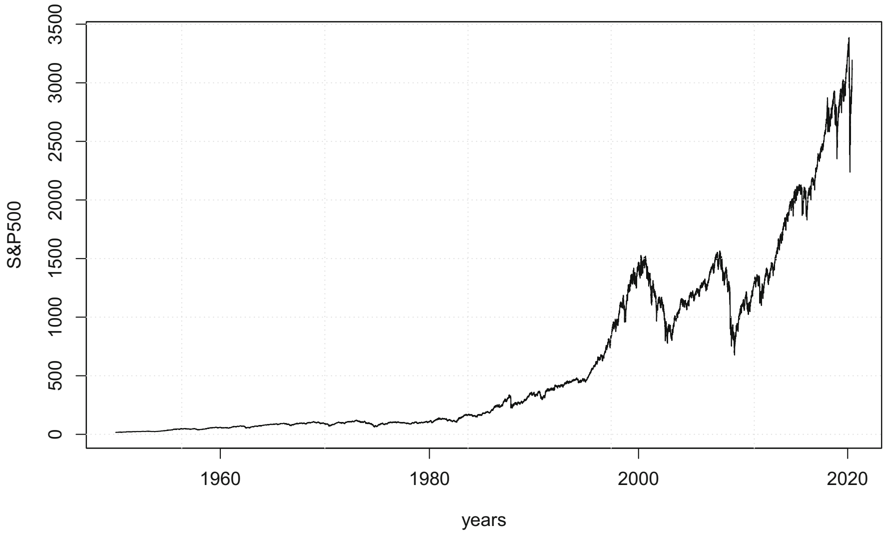
```
getSymbols("^GSPC", src = "yahoo", from = "1950-01-01",
    return.class = "zoo")

## [1] "^GSPC"
```

Here, we have used the “`return.class`” zoo, since it is one of the best class for managing time series, like the S&P500 prices over time.

Now, in the new time series GSPC there are as many elements as the daily values of the index S&P500 from January 1950 to the date when the command has been given. This time series is formed by seven sub-sets (the date, the open price, the high price, the low price, the close price, the volume and the adjusted close). The

[3] On internet there are detailed manuals on how these packages work.




```
SP = GSPC$GSPC.Close
plot(SP, xlab = "years", ylab = "S&P500")
grid()
```

Fig. 2.2 Behaviour of the US index S&P500 starting from January 1959 [Source: Yahoo Finance]

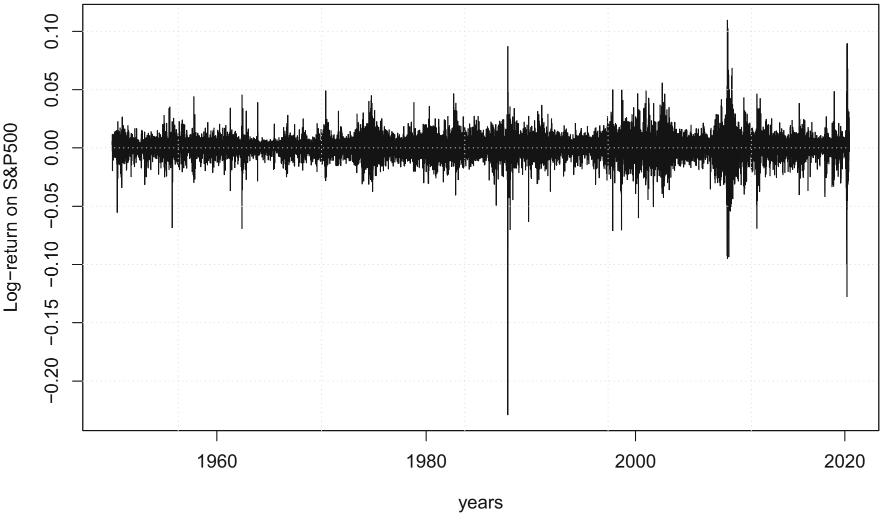
structure of this data frame can be seen through the command (the first letter must be capital)

```
View(GSPC)
```

Since we just need the closing prices, we can isolate and plot them with the commands shown in Fig. 2.2.

The commands `xlab` and `ylab` allow to write labels on the x axis and y axis, respectively. The command `grid()` creates a grid on the plot for making it easier to read.

The plot of the log-return can be obtained by computing the differences of the log of the prices as follows (we also compute the mean daily return), as shown in Fig. 2.3. The return can be transformed into a daily basis by multiplying it by 250 (the working days per year).




```
plot(return, xlab = "years", ylab = "Log-return on S&P500")
grid()
```

Fig. 2.3 Daily log returns on the US index S&P500 starting from January 1959 [Source: yahoo finance]

```
return = diff(log(SP))
mean(return)

## [1] 0.0002966138

mean(return) * 250

## [1] 0.07415345
```

Now, we can create a function in R which allows to compute the certainty equivalent for the stochastic return of the S&P500. A function has always the following syntax:

```
name=function(input1,input2,...){
commands
}
```

The last line of the commands is always the output of the function, while other software declare the output at the beginning when defining the function. In R, if more than one output are needed, all the output must be gathered in a list or a frame.

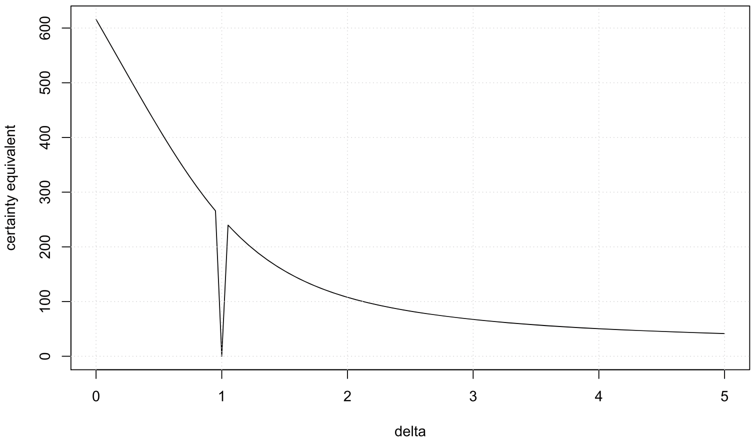
If we call S the value of the S&P500 index, the certainty equivalent ($\hat{S}$) for an agent whose preferences are represented by a CRRA utility function solves the following equation

$$\mathbb{E}\left[\frac{S^{1-\delta}-1}{1-\delta}\right] = \frac{\hat{S}^{1-\delta}-1}{1-\delta},$$

which gives

$$\hat{S} = \mathbb{E}\left[S^{1-\delta}\right]^{\frac{1}{1-\delta}},$$

which is of course a function of the risk aversion δ. The function for computing the certainty equivalent ("CE") is defined in the following code. Then, the function is applied (element-wise through command "`sapply`") to a set of $\delta \in [0, 5]$ (going from 0 to 5 with step 0.05). The figure is finally plotted (and a grid is added to the plot).




```
CE = function(delta) {
    mean(SP^(1 - delta))^(1/(1 - delta)) - 1
}
delta = seq(0, 5, by = 0.05)
plot(delta, sapply(delta, CE), type = "l", ylab = "certainty equivalent")
grid()
```

The command "`type='l'`" is needed in order to represent the figure as a continuous line and not as a sequence of dots like in the default setting of the command `plot`.

While the risk aversion increases, the certainty which makes an agent indifferent between it and the average S&P value becomes smaller and smaller. Such a relationship is concave as we can see in the figure (where also the discontinuity for $\delta = 1$ is present).

2.8 Utility and Time

The utility of some monetary unit available at time t may be different from the utility of the same amount of money available at time $t + 1$. Actually, any agent who is impatient must prefer money closer to the present time. If the utility function is allowed to depend on time $U(t, x)$, then the following inequality must hold:

$$U(t, R) > U(t + 1, R).$$

Another way to take into account this impatience is to apply a discount factor to the utility function, assuming that it is separable in time (t) and money (R) according to the following form:

$$U(t, R) = v_t u(R).$$

Here, $u(R)$ is a timeless utility function satisfying all the properties listed in the previous sections, while v_t is a discount factor whose value decreases while time goes on, i.e.

$$\frac{\partial v_t}{\partial t} < 0,$$

so that amounts of money available farther and farther are twined with lower and lower weight.

Discounting is needed to make comparable the utility of monetary amounts available at different time points. Indeed, the additivity of say two utilities $u(R_t)$ and $u(R_T)$ in the following form

$$v_t u(R_t) + v_T u(R_T),$$

requires time equivalence:

$$v_t \left(u(R_t) + \frac{v_T}{v_t} u(R_T) \right),$$

where v_t is the current value of a unit value at t. Thus, the utility obtained from R_T can be compared with that obtained from R_t if it is multiplied by the ratio $\frac{v_T}{v_t}$. Such a ratio is a kind of discount factor itself which allows to discount money from T to t.

The discount factor may take as many functional forms as agent preferences (a complete review of these forms and their implications is presented in Shane et al. 2002). In discrete time the most common discount factor has the following form:

$$v_t = (1+\rho)^{-t}\,,$$

where $\rho > 0$ is a "subjective" discount rate measuring agent's impatience. The main advantage of this formulation is that the ratio between two discount factors is a discount factor itself, in fact:

$$\frac{v_T}{v_t} = \frac{(1+\rho)^{-T}}{(1+\rho)^{-t}} = (1+\rho)^{-(T-t)} = v_{T-t}.$$

When each period is divided into n sub-periods, the discount factor can be written as

$$v_t = \left(1+\frac{\rho}{n}\right)^{-nt}\,,$$

and when the number of sub-periods tends towards infinity, the continuous time discount factor is obtained

$$\lim_{n\to\infty}\left(1+\frac{\rho}{n}\right)^{-nt} = e^{-\rho t}.$$

Also in this case the ratio between two discount factors is a discount factor itself:

$$\frac{v_T}{v_t} = \frac{e^{-\rho T}}{e^{-\rho t}} = e^{-\rho(T-t)} = v_{T-t}.$$

Definition 2.2 A discount factor v_t is said to be "**separable**" if $\frac{v_T}{v_t} = v_{T-t}$ for any $T \geq t$.

A discount factor is not separable if discounting from T to $t < T$ is different than discounting at first from T to $s < T$ and then from s to $t < s$. Algebraically, we can say that v_t is not separable if

$$\frac{v_T}{v_t} \neq v_{T-t}.$$

When the discount factor is not separable, a problem of inter-temporal consistency arises. This is the case of the so-called hyperbolic discount factor:

$$v_t = \frac{1}{1+\rho t},$$

which is not separable, in fact:

$$\frac{v_T}{v_t} = \frac{\frac{1}{1+\rho T}}{\frac{1}{1+\rho t}} = \frac{1+\rho t}{1+\rho T} \neq v_{T-t}.$$

Another way to have preferences affected by time is to take into account a utility which depends not only on present wealth, but also on past wealth. If, for instance, utility depends on a weighted sum of past wealth, we can write

$$v_t u \left(R_t, \sum_{i=1}^{t} \alpha_i R_{t-i} \right),$$

where α_i are weights. In this case, the preferences take into account the so-called "**habit formation**", i.e. an investor's utility increases only if his/her wealth is higher than the past average wealth. For instance, the following utility function may be considered

$$u \left(R_t, \sum_{i=1}^{t} \alpha_i R_{t-i} \right) = \frac{\left(R_t - \sum_{i=1}^{t} \alpha_i R_{t-i} \right)^{1-\delta}}{1-\delta}.$$

2.9 A First Pension Model

With the tools that we have developed in the previous sections, we can now face a first simple pension problem. Let us assume an agent works for one period, then retires and becomes a pensioner for another period. In both periods the agent takes utility from his/her consumption. In the first period the agent receives a wage (s) and a percentage of this wage (κ) is paid into a fund (or saved), while the remaining wage is consumed (c_t). During the second period (in $t+1$) the agent can consume the pension which is equal to the amount of money saved during the first period (κs) compounded at a rate (μ). This rate can be viewed as the average return on the assets in which the saving has been invested.

Finally, we assume that the utility is separable in consumption and time according to the following law:

$$U(c_t, t) = v_t u(c_t) = \frac{1}{(1+\rho)^t} u(c_t),$$

where ρ is the subjective discount rate.

If the agent wants to maximise the utility of his/her consumption during both periods by choosing the percentage of wage to save, he/she must solve the following problem

$$\max_{\kappa} \mathbb{E}_t \left[\frac{1}{(1+\rho)^t} u(c_t) + \frac{1}{(1+\rho)^{t+1}} u(c_{t+1}) \right] \tag{2.9.1}$$

s.t.

$$c_t = s(1-\kappa),$$

$$c_{t+1} = s\kappa(1+\mu),$$

where we have used the notation $\mathbb{E}_t[\bullet]$ for indicating the expected value computed under the information set available at time t. In particular, we are assuming that the values of all the variable are known at time t, while some variables (like μ in this framework) are not known at time t and, accordingly, are stochastic variables.

In such a framework, the only stochastic variable may be μ, which, in $t+1$, may have a higher or a lower value according to the return obtained on the financial market.

The problem can also be written as

$$\max_{\kappa} \mathbb{E}_t \left[\frac{1}{(1+\rho)^t} u(s(1-\kappa)) + \frac{1}{(1+\rho)^{t+1}} u(s\kappa(1+\mu)) \right],$$

whose First Order Condition (FOC) on κ is[4]

$$\mathbb{E}_t \left[-s\frac{1}{(1+\rho)^t} u'(s(1-\kappa)) + s(1+\mu)\frac{1}{(1+\rho)^{t+1}} u'(s\kappa(1+\mu)) \right] = 0,$$

where the primes on the utility functions are the first derivatives (with respect to the whole argument).

The FOC can be simplified as follows (recall that s is not stochastic):

$$-u'(s(1-\kappa)) + \mathbb{E}_t \left[\frac{1+\mu}{1+\rho} u'(s\kappa(1+\mu)) \right] = 0,$$

and

$$\mathbb{E}_t \left[(1+\mu)\frac{u'(c_{t+1})}{u'(c_t)} \right] = 1+\rho. \tag{2.9.2}$$

The ratio between the marginal utilities inside the expected value is called "**Intertemporal Marginal Rate of Substitution**" (IMRS). It measures how much of future consumption an agent is willing to give in exchange of present consumption.

[4]Here, we have differentiated the objective function with respect to κ by using the property that the derivative of an expected value is equal to the expected value of the derivative.

Equation (2.9.2) allows us to conclude that if all the agents on the financial market optimally behave by the same rule, then the expected compounding factor $(1 + \mu)$ on the pension system, weighted by the IMRS, must equate the subjective compounding rate $(1 + \rho)$.

Example 2.2 If we take into account an agent whose preferences belong to the CRRA family, then the utility function of consumption is

$$u(c) = \frac{c^{1-\delta}}{1-\delta}.$$

In this case the FOC (2.9.2) becomes

$$\mathbb{E}_t\left[(1+\mu)\left(\frac{c_{t+1}}{c_t}\right)^{-\delta}\right] = 1+\rho,$$

and if we substitute c_t and c_{t+1} from the constraints of Problem (2.9.1), we have

$$\mathbb{E}_t\left[(1+\mu)\left(\frac{s\kappa(1+\mu)}{s(1-\kappa)}\right)^{-\delta}\right] = 1+\rho,$$

whose only solution for κ is

$$\kappa^* = \frac{1}{1+\left(\frac{1+\rho}{\mathbb{E}_t\left[(1+\mu)^{1-\delta}\right]}\right)^{\frac{1}{\delta}}}.$$

This quasi-explicit solution allows us to conclude that, with these preferences, the optimal percentage of salary invested in t for having a pension in $t+1$ (with return μ), is always greater than 0 and lower than 1.

Even in this very simple framework, the behaviour of the optimal investment in a pension scheme does not unequivocally depend on δ and μ, while we are only able to conclude that when ρ increases, i.e. the agent becomes more impatient, the optimal value κ^* decreases. In fact, when the agent's impatience increases, he/she will prefer to consume more at time t and less at time $t+1$, because the future consumption will have a less weight in the total discounted utility.

Please note that the optimal value κ^* does not depend on the salary level s. This result is due to the particular shape of preferences. In fact, the CRRA utility function implies that the (relative) risk aversion does not depend on the level of wealth (and, thus, on the level of salary).

If we use as μ the returns obtained on the S&P500 in the previous sections, we can check the shape of κ^* as a function of ρ and δ. The first step is to define the function κ^* and obtain its values for a sequence of ρ and for a set of values of δ.

In the following code we use the command "`which`" that is the analogous of "find" in other software and that gives the coordinate of any element satisfying a given condition; in this case there is a double condition (defined with "and" written as "&") that μ must be different from (the symbol $\neq$ is rendered as "!=") zero and greater than 1.

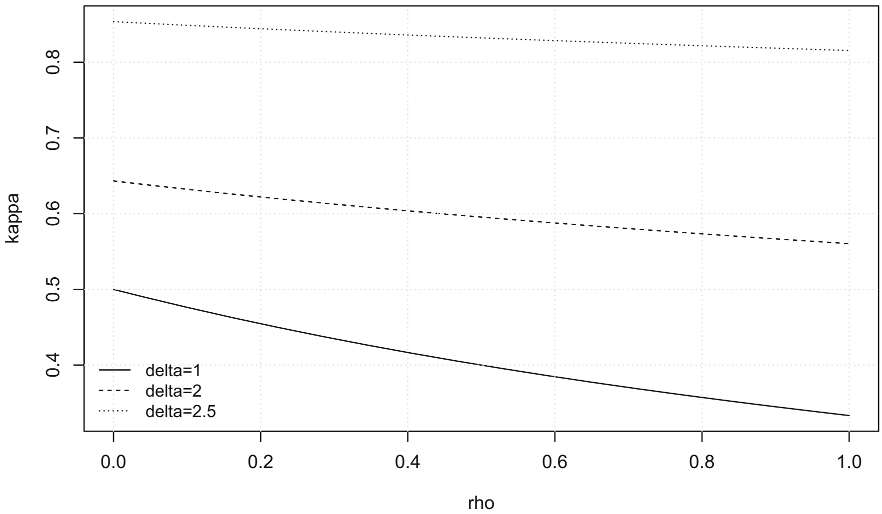
```
kap = function(rho, delta) {
    mu = return * 250
    mu = return[which(mu != 0 & mu > -1)]
    1/(1 + ((1 + rho)/mean((1 + mu * 250)^(1 - delta)))^(1/delta))
}
rho = seq(0, 1, 0.01)
kap1 = kap(rho, 1)
kap2 = kap(rho, 2)
kap3 = kap(rho, 2.5)
```

Then we can plot the figure with the following commands. The command "`matplot`" allows us to plot a matrix, in which any column is taken as the series that must be plot. Accordingly, if we have a matrix with C columns and R rows, this command will draw C curves, each made of R points.

In Fig. 2.4 we see that κ^* is a negative function of ρ and a positive function of δ. Furthermore, we see that the higher δ the less reactive κ^* to ρ.

In the code we added a legend, with the corresponding command, which takes some arguments:

- The first argument (between marks) is the string indicating the position of the legend: it is possible to chose 'top' (the legend is put on top and in the middle of the figure), 'bottom', 'right', 'left', or a combination of two of these options (like in the example).
- Another argument (called `legend`) contains the strings to appear in the descriptions of the curves.
- In the legend we use the same options that are used in the plot command for setting the curves (e.g. line type, width, colour). The option lty takes the following values: 0—blank, 1—solid, 2—dashed, 3—dotted, 4—dotdash, 5—longdash, 6—twodash.
- The command "`bty='n'`" produces a legend without the black border around it. This command stands for "box type = none".




```
matplot(rho, cbind(kap1, kap2, kap3), type = "l", lty = seq(1,
    3), xlab = "rho", ylab = "kappa", col = 1)
grid()
legend("bottomleft", legend = c("delta=1", "delta=2",
    "delta=2.5"), lty = seq(1, 3), col = 1, bty = "n")
```

Fig. 2.4 The optimal percentage of wage (κ) that must be optimally invested on financial market as a function of the subjective discount rate (ρ)

Example 2.3 Now, let us assume that the utility function belong to the HARA family and has the following form:

$$u\left(c\right) = \frac{\left(c - \alpha\right)^{1-\delta}}{1 - \delta}.$$

The FOC (2.9.2) now becomes

$$\mathbb{E}_t\left[\left(1 + \mu\right)\left(\frac{s\kappa\left(1 + \mu\right) - \alpha}{s\left(1 - \kappa\right) - \alpha}\right)^{-\delta}\right] = \left(1 + \rho\right),$$

which cannot be simplified as in the previous case. Nevertheless, we can immediately check that, this time, the salary s does matter. Actually, in

(continued)

Example 2.3 (continued)

a HARA framework, the distance between the wealth (salary) and the subsistence level α, crucially affects the risk aversion.

Unfortunately, in this case we are not able to find a closed form solution for the optimal percentage κ and we could only proceed with numerical methods, after estimating both the values of all the parameters and the distribution of μ.

In the easy framework we have studied in this section there are many missing characteristics of a more realistic framework.

1. A mortality risk: in Problem (2.9.1) the agent is sure to live two periods; instead, in a more realistic framework, a stochastic death time must be taken into account.
2. A longevity risk: even if a death risk is considered, it is unlikely that the parameters of the force of mortality determistically behave over time; instead, a more realistic model, would take into account a force of mortality which is stochastic itself.
3. The presence of a financial market: in Problem (2.9.1) the agent just chooses either to consumer or to save for his/her pension. In the real world, there is also a financial investing decision about how much to invest in risk-less and risky assets. In this case, the decision how much to invest in a pension scheme, would become part of a portfolio allocation problem, where the pension is just a new class of risky asset.

All these aspects will be introduced in the following chapters and the optimal portfolio for a pension fund will be computed in a quasi-explicit form.

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Chapter 3
Stochastic Processes

3.1 Introduction

We have already argued that the analysis of a consistent framework for a pension fund must take into account stochastic variables such as the death time, the force of mortality, the asset prices/returns, the interest rate, the cash flows of contributions and pensions. These stochastic variables follow different paths over time and so they can be modelled through different stochastic processes.

In this chapter we show the properties of a class of stochastic differential equations that are used in finance for describing the dynamics of some variables. In particular, we stress that some variables like the wages should growth over time (on average), while other variables, like the interest rate, should move around a kind of "equilibrium" value. Both these behaviours can be modelled by using the same family of stochastic differential equations. Thus, this chapter will give us the tools for effectively describing the dynamics of all the main variables that describe the framework that a pension fund must face in its management.

3.2 Deterministic Linear Differential Equation

The dynamics of most of the economic variables can be described by a linear differential equation like the following one:

$$dx_t = (a_t x_t + b_t)\, dt, \tag{3.2.1}$$

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_3

for $t \in [t_0, T]$, where a_t and b_t are deterministic functions. If a boundary condition is provided, a unique solution to this equation exists. In particular, if an initial value x_{t_0} is known, the solution to (3.2.1) is

$$x_T = x_{t_0} e^{\int_{t_0}^T a_u du} + \int_{t_0}^T b_s e^{\int_s^T a_u du} ds, \tag{3.2.2}$$

while, if the final value x_T is known, the solution to (3.2.1) is

$$x_{t_0} = x_T e^{-\int_{t_0}^T a_u du} - \int_{t_0}^T b_s e^{-\int_{t_0}^s a_u du} ds. \tag{3.2.3}$$

Both cases are useful in finance, and their relevance depends on the sign of the functions a_t and b_t. The most important cases take into account a_t and b_t with different signs.

- $a_t < 0$ and $b_t > 0$: in this case (3.2.2) represents the behaviour of a variable which tends to revert towards an equilibrium value. In particular, if a is constant, it measures the strength of the reversion towards the equilibrium and if b is constant, $-\frac{b}{a}$ is the equilibrium value. When $a_t = -\alpha$ and $b_t = b$ are both constant, then (3.2.2) simplifies to

$$x_T = x_{t_0} e^{-\alpha(T-t_0)} + b\frac{1 - e^{-\alpha(T-t_0)}}{\alpha} = \left(x_{t_0} - \frac{b}{\alpha}\right) e^{-\alpha(T-t_0)} + \frac{b}{\alpha},$$

  where we see that while T increases, the value of x_T becomes closer and closer to its equilibrium value $\frac{b}{\alpha}$. In particular, if the initial value x_{t_0} is higher (lower) than $\frac{b}{\alpha}$, x_T decreases (increases) over time.
- $a_t > 0$ and $b_t < 0$: in this case Eq. (3.2.3) describes the price of an asset at time t_0 whose future cash flows are given by $-b_t$, x_T is the final value of the asset (for bonds, it mainly coincides with the face value), and a_t is a discount factor.

The main characteristic of a financial framework is that the future cash flows and the discount rate are not deterministic. This is the reason why, in modern finance, Eq. (3.2.1) is used in a modified (stochastic) version that we are about to present.

3.3 Stochastic Linear Differential Equation

In order to take into account the risk, Eq. (3.2.1) is modified by adding an error term as follows:

$$dx_t = (a_t x_t + b_t)\, dt + \phi(t, x_t)\, dW_t, \tag{3.3.1}$$

where $\phi(t, x_t)$ is a deterministic function of both time and state x_t, and W_t is a Wiener process (normally distributed with zero mean and variance t).

A complete analysis and presentation of Wiener processes and stochastic calculus applied to finance can be found in many dedicated manuals (Karatzas and Shreve 1991; Øksendal 1998; Karatzas and Shreve 1998; Duffie 2001; Björk 2009).

If Itô's lemma is applied to the variable $x_t e^{-\int_{t_0}^{t} a_u du}$, we obtain the following differential:

$$d\left(x_t e^{-\int_{t_0}^{t} a_u du}\right) = b_t e^{-\int_{t_0}^{t} a_u du} dt + e^{-\int_{t_0}^{t} a_u du} \phi(t, x_t)\, dW_t,$$

whose expected value at time t_0 is

$$\mathbb{E}_{t_0}\left[d\left(x_t e^{-\int_{t_0}^{t} a_u du}\right)\right] = \mathbb{E}_{t_0}\left[b_t e^{-\int_{t_0}^{t} a_u du} dt\right].$$

We use the notation $\mathbb{E}_{t_0}[\bullet]$ in order to represent the expected value computed under the filtration at time t_0 or, in other words, given all the information available at time t_0. This expected value is computed under the so-called historical probability $\mathbb{P}$, i.e. the probability which can be calibrated on the historical data on financial market.

Now, if both sides of the previous equation are integrated between t_0 and T, the following result is obtained

$$\mathbb{E}_{t_0}\left[x_T e^{-\int_{t_0}^{T} a_u du} - x_{t_0}\right] = \mathbb{E}_{t_0}\left[\int_{t_0}^{T} b_s e^{-\int_{t_0}^{s} a_u du} ds\right],$$

and

$$x_{t_0} = \mathbb{E}_{t_0}\left[x_T e^{-\int_{t_0}^{T} a_u du} - \int_{t_0}^{T} b_s e^{-\int_{t_0}^{s} a_u du} ds\right]. \tag{3.3.2}$$

The value of the variable x_t at time t_0 is the same we have obtained in Eq. (3.2.3) but, this time, the future cash flows (x_T and b_t) are computed under the expected value.

We can obtain a very important result when both a_t and b_t are deterministic and there exists a function β_t such that

$$b_s = \frac{\partial \beta_s}{\partial s} - a_s \beta_s.$$

In this case Eq. (3.3.2) becomes

$$\begin{aligned} x_{t_0} &= \mathbb{E}_{t_0}\left[x_T\right] e^{-\int_{t_0}^T a_u du} - \int_{t_0}^T \left(\frac{\partial \beta_s}{\partial s} - a_s \beta_s\right) e^{-\int_{t_0}^s a_u du} ds \\ &= \mathbb{E}_{t_0}\left[x_T\right] e^{-\int_{t_0}^T a_u du} - \int_{t_0}^T \frac{\partial \left(\beta_s e^{-\int_{t_0}^s a_u du}\right)}{\partial s} ds \\ &= \mathbb{E}_{t_0}\left[x_T\right] e^{-\int_{t_0}^T a_u du} - \beta_T e^{-\int_{t_0}^T a_u du} + \beta_{t_0}, \end{aligned}$$

and, finally

$$\mathbb{E}_{t_0}\left[x_T\right] = \beta_T + \left(x_{t_0} - \beta_{t_0}\right) e^{\int_{t_0}^T a_u du}.$$

This means that if the process starts at $x_{t_0} = \beta_{t_0}$, then its expected value is equal to β_t for any $t > t_0$. Furthermore, if the function a_t is negative, then the limit of this process is

$$\lim_{T \to \infty} \mathbb{E}_{t_0}\left[x_T\right] = \lim_{T \to \infty} \beta_T,$$

and the expected value of x_t tends to converge towards β_t while time goes on. Such a behaviour is called "**mean reversion**" and x_t is said to follow a "**mean reverting process**".

Either the differential equation (3.3.1) together with its final value x_T, or the expected value (3.3.2) are two equivalent ways to represent the same quantity. Equation (3.3.2) is a kind of solution to the differential equation (3.3.1), but a strong form solution of it can be found only under particular conditions that must hold on the diffusion term $\phi\left(t, x_t\right)$. For instance, if $\phi\left(t, x_t\right)$ is constant, then a strong closed form solution to (3.3.1) exists. We refer the reader to the previously mentioned literature for details about these conditions. Here, we just show that case of a linear diffusion term

$$\phi\left(t, x_t\right) = \phi_{0,t} + \phi_{1,t} x_t,$$

where $\phi_{0,t}$ and $\phi_{1,t}$ are two deterministic functions of time. The solution to the process dx_t in this case is

$$\begin{aligned} x_t &= x_{t_0} e^{\int_{t_0}^t \left(a_u - \frac{1}{2}\phi_{1,u}^2\right) ds + \int_{t_0}^t \phi_{1,u} dW_u} \\ &+ \int_{t_0}^t e^{\int_s^t \left(a_u - \frac{1}{2}\phi_{1,u}^2\right) ds + \int_s^t \phi_{1,u} dW_u} \left(b_s - \phi_{0,s}\phi_{1,s}\right) ds \\ &+ \int_{t_0}^t e^{\int_s^t \left(a_u - \frac{1}{2}\phi_{1,u}^2\right) ds + \int_s^t \phi_{1,u} dW_u} \phi_{1,s} dW_s. \end{aligned} \tag{3.3.3}$$

This solution and the closed form solutions of other stochastic differential equations are presented in Kloeden and Platen (1992).

The Wiener process is not the only way for describing the stochastic behaviour of a state variable. Some recent developments in the literature have concentrated on the Lévy processes which, contrary to the Wiener processes, are not driven by a Gaussian distribution. For a presentation of the Lévy processes used in finance the reader is referred to Cont and Tankov (2004); Øksendal and Sulem (2010); Le Courtois and Walter (2014). For an application of them to the pension scheme of a representative consumer, instead, the reference is Le Courtois and Menoncin (2015).

3.4 Stochastic Models Used in Finance

Most of the stochastic models used in finance are particular cases of the following stochastic differential equation:

$$dx_t = (ax_t + b)\,dt + \sigma x_t^\gamma \,dW_t, \tag{3.4.1}$$

which is a heteroscedastic process whose variance

$$\mathbb{V}_t\left[dx_t\right] = \sigma^2 x_t^{2\gamma} dt,$$

is proportional to x_t itself through the parameter γ. The model (3.4.1) is usually identified as CKLS because of the initials of the authors who first estimated its parameters (a, b, σ and γ) for modelling the short-term interest rate in USA (Chan et al. 1992).

Three very common financial models are particular cases of (3.4.1).

- **Geometric Brownian Motion** GBM—($\gamma = 1$ and $b = 0$): this model is often used for describing the behaviour of stock indexes on financial markets. In this case (3.4.1) has a closed form solution (which can be obtained by applying Itô's lemma to the logarithm of x_t):

$$x_t = x_{t_0} e^{\left(a-\frac{1}{2}\sigma^2\right)(t-t_0)+\sigma\left(W_t - W_{t_0}\right)},$$

  which allows to conclude that x_t is log-normally distributed. In this case σ is the standard deviation of the log-return of the index, while the expected log-return is $a - \frac{1}{2}\sigma^2$.
- **Mean Reverting** Vasiček (1977) model ($\gamma = 0$ and $a < 0$): in this case x_t is normally distributed, homoscedastic, and can accordingly take negative values; the solution to (3.4.1) is

$$x_t = x_{t_0} e^{-|a|(t-t_0)} + \frac{b}{|a|}\left(1 - e^{-|a|(t-t_0)}\right) + \int_{t_0}^{t} e^{-|a|(t-s)} \sigma \,dW_s.$$

For instance, the GDP growth rate or the inflation rate can take negative values and they actually tends to revert towards a kind of equilibrium value. Unfortunately, this model does not allow to suitably describe heteroscedastic phenomena, and most of the economic variables are actually heteroscedastic.

- **Mean Reverting** Cox et al. (1985) model—CIR—($\gamma = \frac{1}{2}$, and $a < 0$): the expected value of this variable behaves exactly like the previous one, but the variance is not constant and it is instead proportional to the variable itself (thus allowing for heteroscedasticity). Furthermore, if $2b \geq \sigma^2$ (known as "Feller condition"), the variable x_t never takes negative values and never reaches zero. In other words, zero is a so-called reflecting barrier. There exists no close form solution for x_t in this case, but this model has a closed form solution for the expected value of the discount factor $e^{-\int_{t_0}^{t} x_s ds}$, which plays a crucial role in financial models since it coincides with the price of a zero-coupon bond when x_t is the (instantaneously) riskless interest rate. This model is often used for describing interest rates (or other kinds of discounting rates, like the force of mortality).

When the process represents a mean reverting variable, the interpretation of the equilibrium value $\frac{b}{|a|}$ is straightforward, while the interpretation of the force of the mean reversion effect ($|a|$) is less clear. Hopefully, the interpretation of this parameter as a measure of the mean reversion strength, has a counterpart as a time measure. Let us assume that we want to find the time t^* at which the difference between the expected value of x_t and its initial value x_{t_0} is half the difference between the equilibrium value $\frac{b}{|a|}$ and the initial value. Algebraically, we can write

$$\mathbb{E}_{t_0}\left[x_{t^*}\right] - x_{t_0} = \frac{1}{2}\left(\frac{b}{|a|} - x_{t_0}\right),$$

and, if we substitute the expected value

$$\mathbb{E}_{t_0}\left[x_{t^*}\right] = x_{t_0} e^{-|a|(t-t_0)} + \frac{b}{|a|}\left(1 - e^{-|a|(t-t_0)}\right),$$

we obtain

$$t^* = t_0 + \frac{\ln 2}{|a|}.$$

The value t^* is called **half-life** and its unit of measure is in years. Thus, while the value of ($|a|$) may not have a precise meaning, the value of t^* is the number of years that the process takes to cover half the distance to the equilibrium value.

3.5 Parameter Estimation

A stochastic differential equation like (3.4.1) is usually estimated by taking into account its finite difference version, where $dx_t = x_{t+dt} - x_t$ is replaced by $x_{i+1} - x_i$ as follows

$$x_{i+1} = b \cdot dt + (1 + a \cdot dt)\, x_i + \sigma x_i^{\gamma} \sqrt{dt} \varepsilon_i, \tag{3.5.1}$$

and where ε_i is a standard normal variable. Now, dt is the ratio between the time unit of measure used for the variable x_i and the number of observations in each unit of measure. Thus, if we take into account daily data and we want to estimate year rate of return, then $dt = \frac{1}{250}$ (where 250 are the working days in a year). Note that the parameter γ is not affected by the time unit of measure, while σ is proportional to the square root of dt.

The estimation of the parameters can be done through two main methods: (1) the method of moments, based on the idea to match the theoretical and empirical moments of a stochastic process,[1] and (2) the method of maximum likelihood, which maximises the probability to extract the empirical data from a given distribution.

If the value of γ is known (and must not be estimated), Eq. (3.2.1) can be transformed into a homoscedastic process through a suitable transformation. For instance, we can define the new variable

$$y_t := \frac{x_t^{1-\gamma} - 1}{1 - \gamma}, \tag{3.5.2}$$

whose dynamics can be found by using Itô's lemma

$$dy_t = \left(a\,(1 + (1-\gamma)\, y_t) - \frac{1}{2} \frac{\gamma \sigma^2}{1 + (1-\gamma)\, y_t} + b\,(1 + (1-\gamma)\, y_t)^{-\frac{\gamma}{1-\gamma}} \right) dt + \sigma dW_t. \tag{3.5.3}$$

This process is homoscedastic since its instantaneous variance is constant:

$$\mathbb{V}_t\,[dy_t] = \sigma^2 dt.$$

[1] We recall that the Ordinary Least Square (OLS) method can be traced back to the method of moments.

An immediate application of the method of moments gives the estimated volatility:

$$\hat{\sigma} = \sqrt{\frac{\mathbb{V}_t\left[dy_t\right]}{dt}}. \tag{3.5.4}$$

Remark 3.1 The validity of the model for describing the data can be checked through the homoscedastic transformation (3.5.2). If the model (3.4.1) is correct, then after applying the transformation (3.5.2) to the data, we should obtain a homoscedastic process.

Now, the equation for dy_t can be estimated through Ordinary Least Square, since it is now a homoscedastic process. The estimation is performed on the following discretised version of (3.5.3):

$$\begin{aligned} y_{i+1} =& \underbrace{a \cdot dt}_{\beta_0} + \underbrace{\left(1 + a \cdot dt\left(1-\gamma\right)\right)}_{\beta_1} y_i + \underbrace{\left(-\frac{1}{2}\gamma\sigma^2 \cdot dt\right)}_{\beta_2} \frac{1}{1+\left(1-\gamma\right)y_i} \\ &+ \underbrace{b \cdot dt}_{\beta_3} \left(1 + \left(1-\gamma\right)y_i\right)^{-\frac{\gamma}{1-\gamma}} + \sigma\sqrt{dt}\varepsilon_i. \end{aligned} \tag{3.5.5}$$

The case of a GBM is even simpler. In fact, if $b = 0$ and $\gamma = 1$, Eq. (3.5.3) becomes[2]

$$d\ln x_t = \left(a - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t.$$

In order to apply the method of moments, we must compute the first and the second moment of $d\ln x_t$:

$$\begin{cases} \mathbb{E}_t\left[d\ln x_t\right] = \left(a - \frac{1}{2}\sigma^2\right)dt, \\ \mathbb{V}_t\left[d\ln x_t\right] = \sigma^2 dt. \end{cases}$$

[2]We recall that $\lim_{\gamma\to 1}\frac{x_t^{1-\gamma}-1}{1-\gamma} = \ln x_t$, and $\lim_{\gamma\to 1}\left(1+\left(1-\gamma\right)y_t\right)^{-\frac{\gamma}{1-\gamma}} = e^{-y}$.

If we solve this system for a and σ, we obtain the following solutions/estimations:

$$\hat{\sigma} = \sqrt{\frac{\mathbb{V}_t\left[d \ln x_t\right]}{dt}},$$

$$\hat{a} = \frac{\mathbb{E}_t\left[d \ln x_t\right] + \frac{1}{2}\mathbb{V}_t\left[d \ln x_t\right]}{dt}.$$

If we use the log-returns already computed on S&P500 (gathered in the variable `return`), we have the following estimations (with $dt = 1/250$). Here, we use the function `sd` which computes the standard deviation of data.

```
dt = 1/250
sigma_hat = sd(return)/sqrt(dt)
sigma_hat

## [1] 0.1565085

a_hat = (mean(return) + 0.5 * sd(return)^2)/dt
a_hat

## [1] 0.08640092
```

When also the parameter γ must be estimated, the methods we have just presented cannot be applied any longer, and other methods must be used. In R there exists a package called `Sim.DiffProc` that allows to estimate parameters for any form of a stochastic process.

3.6 The Interest Rate

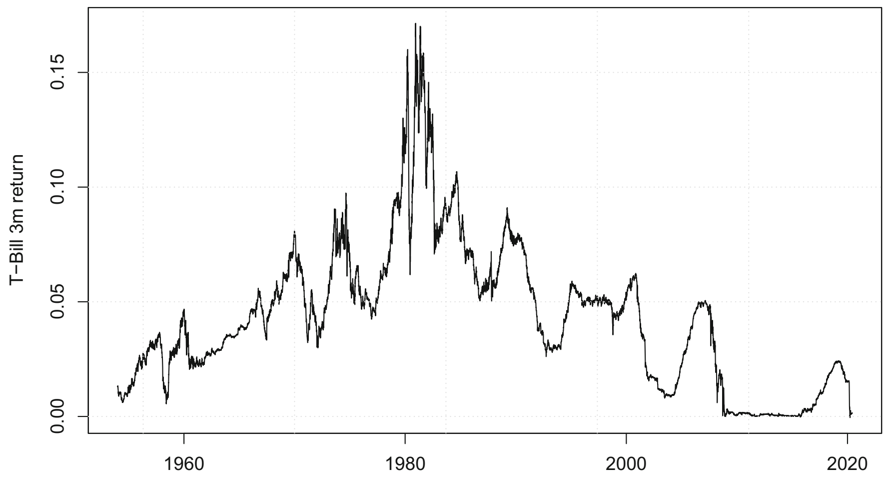
Now we can download the time series of the returns on the secondary market of the 3-month US Treasury Bill (from FRED database https://research.stlouisfed.org/fred2/). In what follows we use the command "`na.omit`" in order to eliminate from the database the values which are not numbers.

```
getSymbols("DTB3", src = "FRED", return.class = "zoo")

## [1] "DTB3"

DTB3 = na.omit(DTB3)/100
```

The graphical result is shown in Fig. 3.1.

The behaviour of the interest rate on T-Bills can be clearly divided into two main periods: from 1954 to the first and second oil shocks in the mid Seventies and first Eighties, and from this time to the date of the sub-prime crisis (2007/2008). During the first period, interest rates are increasing and then they start decreasing. Many sub-periods can then be identified.

One of the most striking effects of the sub-prime crisis, is that some short term interest rates have become negative (like the 3 month T-Bills in US and the 3 month Euribor in EU). It is possible to find the negative (or zero) values in the set DTB3 through the following commands, which also show the dates when these negative rates happened.




```
plot(DTB3, xlab = "", ylab = "T-Bill 3m return")
grid()
```

Fig. 3.1 Returns on the secondary market of the 3-month US Treasury Bill [Source: https://research.stlouisfed.org/fred2/]

```
which(DTB3 <= 0)

##  [1] 13725 13731 13735 14423 14432 14480 14481 14926 15421 15423 15426 15429
## [13] 15430 15431 15433 15434 15435 15438 15444 16548 16549

DTB3[which(DTB3 <= 0)]

## 2008-12-10 2008-12-18 2008-12-24 2011-09-22 2011-10-05 2011-12-15 2011-12-16
##      0e+00      0e+00      0e+00      0e+00      0e+00      0e+00      0e+00
## 2013-09-26 2015-09-18 2015-09-22 2015-09-25 2015-09-30 2015-10-01 2015-10-02
##      0e+00     -1e-04     -1e-04     -1e-04     -1e-04     -2e-04      0e+00
## 2015-10-06 2015-10-07 2015-10-08 2015-10-14 2015-10-22 2020-03-25 2020-03-26
##      0e+00      0e+00     -1e-04      0e+00      0e+00     -4e-04     -5e-04
```

We can see that the first zero interest rate on T-Bills appears on 10 December 2008 (the year of the sub-prime crisis), while the first negative interest rate appears on 18 September 2015. The presence of negative interest rates makes the model (3.4.1) inappropriate for any rational value of γ.

The Vasiček (1977) model (with $\gamma = 0$) allows $x(t)$ to be negative and its parameters can be estimated with the procedure shown in the previous section. If $\gamma = 0$, Eq. (3.5.5) becomes

$$y_{i+1} = (a+b) \cdot dt + (a \cdot dt + 1)\, y_i + \sigma \sqrt{dt} \varepsilon_i,$$

and, since in this case $y_i = x_i - 1$, we finally have

$$x_{i+1} = \underbrace{b \cdot dt}_{\beta_0} + \underbrace{(a \cdot dt + 1)}_{\beta_1} x_i + \sigma \sqrt{dt} \varepsilon_i.$$

After estimating β_0 and β_1, the parameters of the model are obtained as follows:

$$\begin{cases} \beta_0 = b \cdot dt, \\ \beta_1 = a \cdot dt + 1, \end{cases} \iff \begin{cases} b = \frac{\beta_0}{dt}, \\ a = \frac{\beta_1 - 1}{dt}. \end{cases}$$

The estimation of σ can be easily obtained from (3.5.4) where, since $\gamma = 0$, $dx_t = dy_t$. Here, $dt = 1/250$ since we use again daily values. In R we write the following code (we call the estimation `sigma_V` since it refers to Vasiček 1977).

```
sigma_V = sd(diff(DTB3))/sqrt(dt)
sigma_V

## [1] 0.01375297
```

The commands for the regression of x_{i+1} on x_i are as follows, where also the estimations of a and b are obtained. We recall that the equilibrium value towards which the process reverts, is given by $\frac{b}{|a|}$.

In the following code we use the double square brackets for telling R to take just the numerical value of the command (without any label). The command `lm` (linear model) estimates an OLS regression, where the dependent variable is written on the left hand side of the tilde, while the independent variables are on the right. The command "summary" shows a kind of table gathering the results of the regression: residuals, estimation of the coefficients, their standard error, t-values, and p-values, and other statistical indexes.

```
y = DTB3[2:length(DTB3)]
x1 = DTB3[1:(length(DTB3) - 1)]
vasicek = lm(y ~ x1)
summary(vasicek)

##
## Call:
## lm(formula = y ~ x1)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0126555 -0.0002021 -0.0000067  0.0002022  0.0134389
##
## Coefficients:
##              Estimate Std. Error  t value Pr(>|t|)
## (Intercept) 1.512e-05  1.156e-05    1.309    0.191
## x1          9.996e-01  2.180e-04 4584.626   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0008698 on 16594 degrees of freedom
## Multiple R-squared:  0.9992,Adjusted R-squared:  0.9992
## F-statistic: 2.102e+07 on 1 and 16594 DF,  p-value: < 2.2e-16

b_V = vasicek$coefficients[[1]]/dt
a_V = (vasicek$coefficients[[2]] - 1)/dt
# equilibrium value
b_V/abs(a_V)

## [1] 0.04108868
```

Cox et al. (1985) model (with $\gamma = 0.5$) can be estimated through model (3.5.5) written as follows

$$y_{i+1} = a \cdot dt + \left(1 + \frac{1}{2} a \cdot dt\right) y_i + \left(b - \frac{1}{4}\sigma^2\right) dt \frac{1}{1 + \frac{1}{2} y_i} + \sigma \sqrt{dt}\varepsilon_i,$$

and, since $y_i = 2\left(x_i^{\frac{1}{2}} - 1\right)$,

$$x_{i+1}^{\frac{1}{2}} = \underbrace{\left(1 + \frac{1}{2}a \cdot dt\right)}_{\beta_1} x_i^{\frac{1}{2}} + \underbrace{\frac{1}{2}\left(b - \frac{1}{4}\sigma^2\right)dt}_{\beta_2}\frac{1}{x_i^{\frac{1}{2}}} + \frac{1}{2}\sigma\sqrt{dt}\varepsilon_i.$$

After estimating β_1 and β_2, the parameters of the model are obtained as follows:

$$\begin{cases} \beta_1 = 1 + \frac{1}{2}a \cdot dt, \\ \beta_2 = \frac{1}{2}\left(b - \frac{1}{4}\sigma^2\right)dt, \end{cases} \iff \begin{cases} a = \frac{2(\beta_1 - 1)}{dt}, \\ b = \frac{2\beta_2}{dt} + \frac{1}{4}\sigma^2. \end{cases}$$

Before estimating the parameters, we must take the negative and zero interest rate out of the sample. In fact, with $\gamma = \frac{1}{2}$, the square root is not defined for the negative values of the interest rates, and the ratio $x_i^{-\frac{1}{2}}$ is not defined for $x_i = 0$. Thus, we now write the code for taking the time series of the interest rate just before the first date when the interest rate becomes zero or is negative.

```
DTB3_0 = DTB3[1:which(DTB3 <= 0)[1] - 1]
```

The estimation of σ is obtained again from (3.5.4) where $y_t := 2\left(x_t^{\frac{1}{2}} - 1\right)$ and $dy_t = d\left(2\sqrt{x_t}\right)$. We call the estimation sigma_C since it refers to Cox et al. (1985).

```
sigma_C = sd(diff(2 * sqrt(DTB3_0)))/sqrt(dt)
sigma_C
```

```
## [1] 0.06623315
```

Finally, we can estimate the other parameters a and b. In this particular case, the command for the OLS regression "`lm`" contains one more option "-1" which orders R to perform the regression without the constant.

```
y = DTB3_0[2:length(DTB3_0)]^0.5
x1 = DTB3_0[1:(length(DTB3_0) - 1)]^0.5
x2 = DTB3_0[1:(length(DTB3_0) - 1)]^(-0.5)
CIR = lm(y ~ x1 + x2 - 1)
summary(CIR)
```

```
##
## Call:
## lm(formula = y ~ x1 + x2 - 1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.074519 -0.000654 -0.000023  0.000634  0.050112
##
## Coefficients:
##     Estimate Std. Error  t value Pr(>|t|)
## x1 9.995e-01  1.256e-04 7957.113  < 2e-16 ***
## x2 2.116e-05  4.965e-06    4.262 2.04e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.002093 on 13721 degrees of freedom
## Multiple R-squared:  0.9999,Adjusted R-squared:  0.9999
## F-statistic: 7.973e+07 on 2 and 13721 DF,  p-value: < 2.2e-16
```

```
a_C = 2 * (CIR$coefficients[[1]] - 1)/dt
b_C = 2 * CIR$coefficients[[2]]/dt + sigma_C^2/4
# equilibrium value
b_C/abs(a_C)
```

```
## [1] 0.04990227
```

The values that we have obtained for the parameters of both model are gathered in Table 3.1.

Table 3.1 Estimation of the parameters of models Vasiček (1977) and Cox et al. (1985) on the 3-month US Treasury Bill (from FRED database https://research.stlouisfed.org/fred2/)

Parameters	Vasiček (1977)	Cox et al. (1985)
a	−0.0920262	−0.2339697
b	0.0037812	0.0116756
σ	0.013753	0.0662331
Mean reversion ($-a$)	0.0920262	0.2339697
Equilibrium ($\frac{b}{-a}$)	0.0410887	0.0499023
Half-life ($\frac{\ln 2}{\lvert a\rvert}$)	7.5320663	2.9625506

3.7 Simulation

When a stochastic differential equation is simulated, it is discretised as in (3.5.1). The path of the variable x_t can be obtained by starting from its initial value x_{t_0} and then computing x_{t_i} for any i by generating a random (normal) variable at each passage.

Here, we can create a function for simulating the CKLS model (3.4.1) whose input are: the parameters a, b, σ and γ, the initial value x_{t_0}, the time interval dt, the time when the simulation must end ($t_T = T$), and the number of simulations to generate (N).

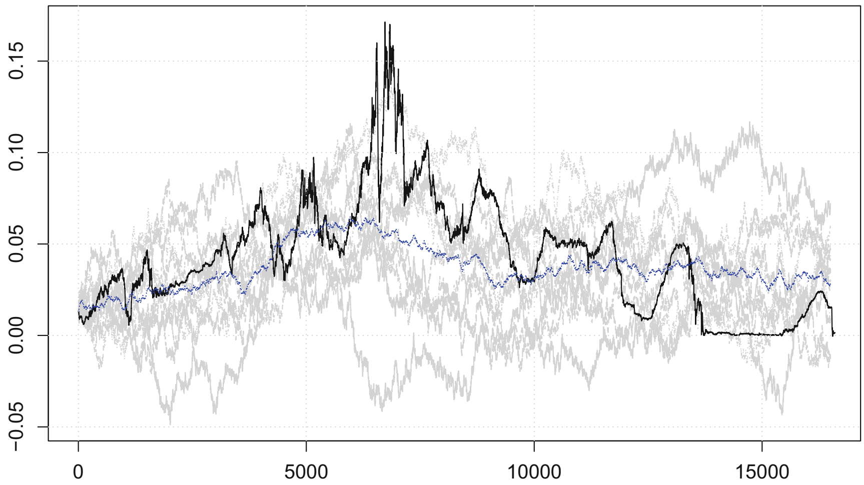
The output is a matrix containing, in each row $i \in \{1, 2, \ldots, T\}$, the set of N possible values of the variable x_{t_i}, and, in each column $j \in \{1, 2, \ldots, N\}$, the simulation of the whole path for a given state of the world. Thus, we initially define a matrix of zeros, whose dimension is $T \times N$. Actually, since each unit of time (1 year) is divided into sub-periods of length dt, the number of rows is $\frac{T}{dt}$.

In the code below we use a `for` cycle which allows to produce any new simulation starting from the previous one. We also use the command "`rnorm(x,mean,sd)`" in which the first argument is mandatory and indicated the quantity of numbers that must be drawn from the normal distribution. The other two optional arguments indicate the mean and the standard deviation, respectively. If these two options are not specified, a standardised normal variable is used.

```
CKLS = function(a = 0, b = 0, sigma = 1, g = 0, x0 = 1,
    dt = 1/250, T = 1, N = 1) {
    x = array(0, dim = c(T/dt, N))
    x[1, ] = rep(x0, N)
    for (i in 2:(T/dt)) {
        dx = (a * x[i - 1, ] + b) * dt + sigma * x[i -
            1, ]^g * rnorm(N) * sqrt(dt)
        x[i, ] = x[i - 1, ] + dx
    }
    x
}
```

Now, we can use such a function for generating a fake series of the 3-month T-Bill rate, with the following characteristics:

- we set the parameters at the values estimated in the previous section;
- the initial value coincides with the first element of the dataset DTB3;
- we generate the same numbers of days as the length of the dataset DTB3, expressed in terms of years; since we have daily data, the number of years is given by the length of DTB3 divided by 250 (the working day per year);
- we create 10 simulations of the same path. The 10 paths are drawn in light grey and the true time series of the 3-month T-Bill rate is added to the plot;
- finally, the mean of the 10 simulations is shown (dotted line).



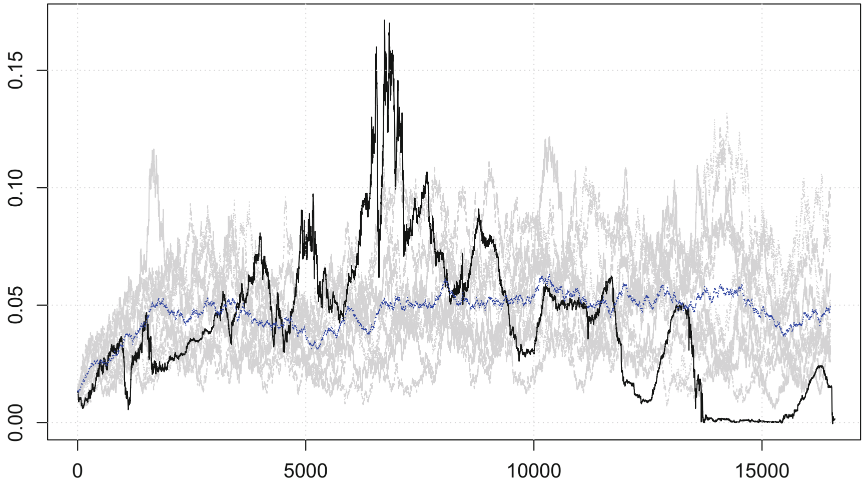

```
sym_V = CKLS(a = a_V, b = b_V, sigma = sigma_V, g = 0,
    x0 = DTB3[1], dt = 1/250, T = round(length(DTB3) *
        dt), N = 10)
matplot(sym_V, type = "l", col = "lightgray", ylab = "",
    ylim = c(min(sym_V, DTB3), max(sym_V, DTB3)))
lines(as.numeric(DTB3))
lines(rowMeans(sym_V), type = "l", lty = 3, col = "blue")
grid()
```

Fig. 3.2 Simulation of 10 trajectories (in light grey) of the Vasiček (1977) process estimated on the historical data of 3-month Treasury Bill

The commands and the results are shown in Fig. 3.2 for the model Vasiček (1977), and in Fig. 3.3 for the model Cox et al. (1985). In particular, the command "`lines`" overlay a new curve on the current plot (thus it cannot be used alone without a plot command set before). Furthermore, the option "`ylim`" inside the command `plot` contains the boundaries of the vertical axes, needed for avoiding that the simulations belong to a set that is too different with respect to the domain of the empirical data.

We can check that the both processes are actually mean reverting and the interest rate moves around an equilibrium value with two relevant exceptions:

- the high inflation period during both oil crises (mid Seventies and initial Eighties);
- the 2007/2008 crisis when the interest rate is set to zero by the Federal Reserve System.



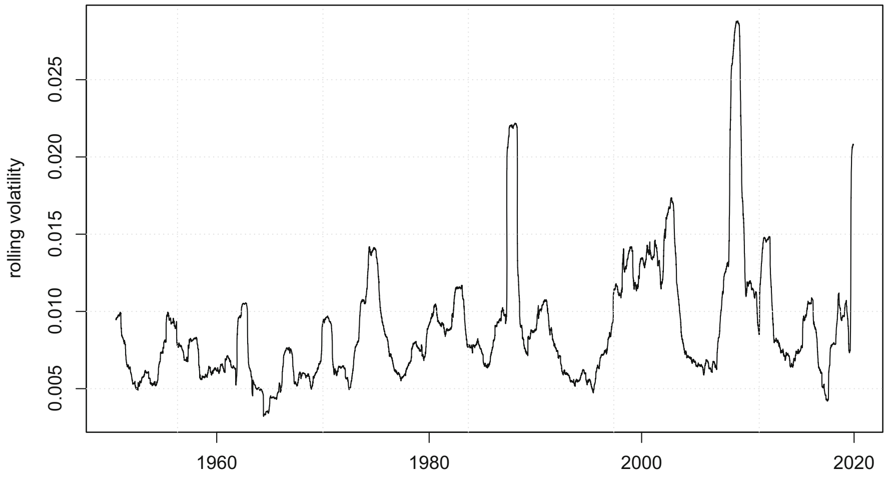

```
sym_C = CKLS(a = a_C, b = b_C, sigma = sigma_C, g = 0.5,
    x0 = DTB3[1], dt = 1/250, T = round(length(DTB3) *
        dt), N = 10)
matplot(sym_C, type = "l", col = "lightgray", ylab = "",
    ylim = c(min(sym_C, DTB3), max(sym_C, DTB3)))
lines(as.numeric(DTB3))
lines(rowMeans(sym_C), type = "l", lty = 3, col = "blue")
grid()
```

Fig. 3.3 Simulation of 10 trajectories (in light grey) of the Cox et al. (1985) process estimated on the historical data of 3-month Treasury Bill

3.8 The State Variables

We assume that our economic framework can be fully described by a set of (s) state variables solving stochastic differential equations. In a dynamic programming problem such variables are called "state variables" and if they are gathered in the vector $z_t \in \mathbb{R}^s$, we now assume that z_t solves the following matrix stochastic differential equation

$$\underset{s\times 1}{dz_t} = \underset{s\times 1}{\mu_z}(t, z_t)dt + \underset{s\times k}{\Omega}(t, z_t)^{\top} \underset{k\times 1}{dW_t}, \tag{3.8.1}$$




```
roll.vol = na.omit(rollapply(return, 250, sd))
plot(roll.vol, xlab = "", ylab = "rolling volatility")
grid()
```

Fig. 3.4 Rolling volatility of the S&P500 returns (daily computed on the basis of 250 days)

where the k Wiener processes driving z_t are assumed to be independent,[3] and $\top$ indicates transposition.

The functional forms of both the drift μ_z and the diffusion Ω can be specified in any way which guarantees the existence of a solution to (3.8.1).

Typical state variables are:

- the riskless interest rate, like the 3-month T-Bill rate we have already seen in the previous sections;
- the market prices of risks as we are about to define in the next chapters;
- the general level of prices, for taking into account the inflation risk;
- the volatility of asset returns, which is stochastic itself as one can easily perceive from Fig. 3.4. In this figure we use the command `rollapply` (available inside both the package `zoo` and the package `quantmod`) that allow to apply a function to a window of data whose width is specified as an argument of the function.

[3]The independence hypothesis is not restrictive since we can always switch from a vector of dependent Wiener processes to a vector of independent Wiener processes (and vice versa) through the Cholesky decomposition of the variance and covariance matrix.

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Chapter 4
The Financial Market

4.1 Introduction

The description of a financial market through stochastic processes is fundamental for solving an dynamic optimisation problem. In particular, in this chapter we show how to check whether a market is arbitrage free and complete. We will always assume that the financial market is arbitrage free, since this is an obvious assumption that no model can disregard. Instead, the completeness hypothesis will be technically necessary for solving the optimal portfolio in semi-closed form and, in particular, for using the so-called "martingale method".

The absence of arbitrage is the hypothesis that we rely on for showing one of the most powerful result in finance: the fundamental theorem of asset pricing. Under this hypothesis there exists a (risk neutral) probability under which the expected return on any asset on the financial market is equal to the risk-less interest rate. We will show how to price any risky asset by switching between the historical and the risk neutral probability.

All these tools are necessary for developing the analysis in the following chapters.

4.2 Financial Assets

Let us assume that on financial market there are n risky assets whose prices $S_t \in \mathbb{R}^n_+$ follow the matrix stochastic differential equation

$$\begin{aligned} \underset{n\times n}{I_S^{-1}}\underset{n\times 1}{dS_t} &= \underset{n\times 1}{\mu}(t, S_t)dt + \underset{n\times k}{\Sigma}(t, S_t)'\underset{k\times 1}{dW_t}, \\ S(t_0) &= S_0, \end{aligned} \tag{4.2.1}$$

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where I_S is the diagonal matrix containing the asset values $(S_1, S_2, \ldots S_n)$ and the prime denotes transposition. The functional forms of the drift μ and the diffusion Σ can be specified in any way which guarantees the existence of a solution to (4.2.1).

The expected (instantaneous) returns on these assets are

$$\mathbb{E}_t \left[I_S^{-1} dS_t \right] = \mu(t, S_t)\, dt,$$

while their (instantaneous) variance and covariance $(n \times n)$ matrix is

$$\mathbb{V}_t \left[I_S^{-1} dS_t \right] = \Sigma(t, S_t)^{\mathsf{T}} \Sigma(t, S_t)\, dt.$$

Hereafter, we will neglect the functional dependencies of both μ and Σ^{T} with respect to time and space in order to keep the notation as simple as possible.

Furthermore, on financial market there exists a riskless asset (issued by the Government) whose price G_t solves

$$\frac{dG_t}{G_t} = r_t dt. \tag{4.2.2}$$

Remark 4.1 If we know the value in t_0 of the asset G then the (unique) solution of differential equation (4.2.2) is

$$G_t = G_{t_0} e^{\int_{t_0}^{t} r_u du},$$

and the ratio

$$\frac{G_t}{G_T} = \frac{G_{t_0} e^{\int_{t_0}^{t} r_u du}}{G_{t_0} e^{\int_{t_0}^{T} r_u du}} = e^{-\int_t^T r_u du},$$

for any $T > t$ is the **discount factor** between t and T.

4.3 Portfolio and Wealth

A portfolio is a linear combination of assets S_t and G_t whose value R_t is given by

$$R_t = \underset{1\times n}{w_t^{\mathsf{T}}} \underset{n\times 1}{S_t} + w_{G,t} G_t, \tag{4.3.1}$$

where w_t and $w_{G,t}$ are the number of risky and riskless assets held in the portfolio, respectively (if the element $w_{i,t}$ is negative then corresponding asset $S_{i,t}$ is short sold).

Since the portfolio allocation w_t and $w_{G,t}$ may change over time according to the changes in the asset values then both w_t and $w_{G,t}$ must be considered as stochastic variables. This means that the differential of R_t must be computed as

$$\begin{aligned} dR_t = & \underbrace{dw_t^{\mathsf{T}} \times S_t + w_t^{\mathsf{T}} \times dS_t + dw_t^{\mathsf{T}} \times dS_t}_{d(w_t^{\mathsf{T}} S_t)} \\ & + \underbrace{dw_{G,t} \times G_t + w_{G,t} \times dG_t}_{d(w_t^{\mathsf{T}} G_t)}, \end{aligned}$$

where the term $dw_{G,t}^{\mathsf{T}} dG_t$ lacks because G_t is deterministic.[1] After rearranging the terms we have

$$\begin{aligned} dR_t = & \underbrace{w_t^{\mathsf{T}} dS_t + w_{G,t} dG_t}_{dR_{1,t}} \\ & + \underbrace{dw_t^{\mathsf{T}} (S_t + dS_t) + dw_{G,t} \times G_t}_{dR_{2,t}}, \end{aligned}$$

where $dR_{1,t}$ are changes in wealth due to the changes in asset prices (dS_t and dG_t) while $dR_{2,t}$ are changes in wealth due to changes in portfolio allocation (dw_t and $dw_{G,t}$).

The changes in the portfolio composition are subject to a constraint, since we cannot invest more than the available wealth. Accordingly, we can distinguish three cases.

1. **Strict self-financing condition**: the agent has no more wealth than his/her portfolio value and he/she does not want to withdraw any money from it. Accordingly, in each period, he/she can invest more in one asset only if he/she suitably decreases the amount of money invested in the other assets. This condition can be written as

$$dw_t^{\mathsf{T}} (S_t + dS_t) + dw_{G,t} \times G_t = 0,$$

   where we see that $S_t + dS_t$ is the new price of asset S_t, after the period dt.
2. **Outflows**: at each period, the agent wants to withdraw some money from his/her portfolio in order, for instance, to finance consumption. If we call $c_t dt$ the amount

[1] We recall that if both X_t and Y_t are stochastic variables, then the differential of their product is $d(X_t Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t$.

of consumption in the instant dt, then

$$dw_t^\intercal (S_t + dS_t) + dw_{G,t} \times G_t = -c_t dt,$$

since the total change in wealth due to agent's allocation decision (i.e. $dR_{2,t}$) must coincide with the amount of money that is withdrawn.

3. **Inflows**: at each instant in time, the agent receives some yield $y_t dt$ and a percentage α_t of it is invested in the portfolio; this means that when deciding the asset allocation, the agent allows to invest more than his/her portfolio value as follows:

$$dw_t^\intercal (S_t + dS_t) + dw_{G,t} \times G_t = \alpha_t y_t dt.$$

Case 2 is typical of a pension fund when it starts paying pensions during the so-called "distribution phase": at each time the amount of pensions are deducted from its wealth. Instead, case 3 is typical of a pension fund when it receives contributions from its sponsors during the so-called "accumulation phase".

In order to accommodate all the mentioned cases, we take into account the following equation:

$$dw_t^\intercal (S_t + dS_t) + dw_{G,t} \times G_t = k_t dt,$$

where k_t is a variable which may take either positive, or negative values over time.

Accordingly, the wealth differential can be written as

$$dR_t = w_t^\intercal dS_t + w_{G,t} dG_t + k_t dt, \tag{4.3.2}$$

which is the dynamic version of the constraint (4.3.1). Since both constraints must be verified at any time, we can merge them by taking $w_{G,t}$ from (4.3.1) and plugging it into (4.3.2). Accordingly, we have just one (dynamic) constraint:

$$dR_t = w_t^\intercal dS_t + \underbrace{\frac{R_t - w_t^\intercal S_t}{G_t}}_{w_{G,t}} dG_t + k_t dt,$$

and after substituting for both dS_t and dG_t from (4.2.1) and (4.2.2) respectively, we have

$$dR_t = \left(R_t r_t + \underset{1\times n}{w_t^\intercal} \underset{n\times n}{I_S} \left(\underset{n\times 1}{\mu_t} - r_t \underset{n\times 1}{\mathbf{1}} \right) + k_t \right) dt + \underset{1\times n}{w_t^\intercal} \underset{n\times n}{I_S} \underset{n\times k}{\Sigma_t^\intercal} \underset{k\times 1}{dW(t)}, \tag{4.3.3}$$

where $\mathbf{1}$ is a vector containing just 1's.

The differences

$$\mu_t - r_t \mathbf{1},$$

between the expected returns on the risky asset and the riskless interest rate, are called **risk premia**. In fact, they measure the excess return (with respect to r_t) that agents need in order to bear a given risk. Thus, there is a risk premium for any asset on the financial market.

4.4 External Cash Flows and Modified Wealth

In many financial textbooks the stochastic equation describing wealth does not contain any external cash flow (i.e. $k_t = 0$ in (4.3.3)). Here, we can formally trace back our analysis to that case by choosing a suitable function K_t which solves the following differential equation:

$$dK_t = (K_t r_t + k_t)\, dt. \tag{4.4.1}$$

Accordingly, the difference $dR_t - dK_t = d\,(R_t - K_t)$ is given by (recall (4.3.3))

$$d\,(R_t - K_t) = \big((R_t - K_t)\, r_t + w_t^{\intercal} I_S\, (\mu_t - r_t \mathbf{1})\big)\, dt + w_t^{\intercal} I_S \Sigma_t^{\intercal} dW_t,$$

and, if the modified wealth is defined as

$$\widetilde{R}_t := R_t - K_t,$$

we can write

$$d\widetilde{R}_t = \big(\widetilde{R}_t r_t + w_t^{\intercal} I_S\, (\mu_t - r_t \mathbf{1})\big)\, dt + w_t^{\intercal} I_S \Sigma_t^{\intercal} dW_t. \tag{4.4.2}$$

Accordingly, all the definition of arbitrage and market completeness that we are about to present apply to our framework in the same way they do in the usual financial approach without any external cash flow. In the following section we will present such ideas.

Now, we concentrate on Eq. (4.4.1) which is linear like (3.2.1) and whose solution, if k_{t_0} is known, is given by:

$$k_t = k_{t_0} e^{\int_{t_0}^{t} r_u du} + \int_{t_0}^{t} k_s e^{\int_s^t r_u du} ds.$$

Thus, at each time t, the value of k_t is given by the sum of its past values compounded by the interest rate r_t. In the case of a pension fund, the variable k_t plays a relevant role which will be investigated in details in the future chapters.

In (4.4.2), $w_t^\intercal I_S$ is the amount of wealth invested in each asset. Sometimes, the wealth differential equation is written as a function of the percentage of (modified) wealth invested in each asset:

$$\theta_t := \frac{1}{\widetilde{R}_t} w_t^\intercal I_S.$$

In this case, the stochastic equation for wealth becomes

$$\frac{d\widetilde{R}_t}{\widetilde{R}_t} = \left(r_t + \theta_t^\intercal (\mu_t - r_t \mathbf{1})\right) dt + \theta_t^\intercal \Sigma_t^\intercal dW_t. \tag{4.4.3}$$

4.5 Arbitrage

A financial market is well defined if there is no arbitrage (gain without risk). We define an arbitrage as a portfolio without risk and whose return is different from r_t. Accordingly, θ_t in (4.4.3) is an arbitrage if the two following conditions hold

$$\begin{cases} \underset{1\times n}{\theta_t^\intercal} \; \underset{n\times k}{\Sigma_t^\intercal} = \underset{1\times k}{\mathbf{0}}, \\ \underset{1\times n}{\theta_t^\intercal} \left(\underset{n\times 1}{\mu_t} - r_t \underset{n\times 1}{\mathbf{1}} \right) \neq 0. \end{cases}$$

For checking whether there is an arbitrage on financial market, we can use the following result.

Lemma 4.1 (Fredholm) *One and only one of the two following cases is true:*

$$\exists x \in \mathbb{R}^k : \underset{n\times k}{A^\intercal} \; \underset{k\times 1}{x} = \underset{n\times 1}{b},$$

$$\exists y \in \mathbb{R}^n : \begin{cases} \underset{1\times n}{y^\intercal} \; \underset{n\times k}{A^\intercal} = \mathbf{0}, \\ \underset{1\times n}{y^\intercal} \; \underset{n\times 1}{b} \neq 0. \end{cases}$$

Fredholm (1903)'s lemma allows us to conclude what follows.

Proposition 4.1 *On the financial market (4.2.1)–(4.2.2) there is no arbitrage if and only if there exists a vector $\xi_t \in \mathbb{R}^k$ such that*

$$\underset{n\times k}{\Sigma_t^\top}\ \underset{k\times 1}{\xi_t} = \underset{n\times 1}{\mu_t} - r_t \underset{n\times 1}{\mathbf{1}}. \tag{4.5.1}$$

The matrix equation (4.5.1) is a linear system of n equations in k unknowns which can have:

1. only one solution (the market is arbitrage free);
2. infinite solutions (the market is arbitrage free);
3. no solution (the market is not arbitrage free).

Since in the real world the financial markets are actually arbitrage free, then we never take into account the third case.

We highlight that Eq. (4.5.1) has (at least) a solution if there exists the so-called **left inverse** of matrix $\Sigma^\top$. In particular, the matrix $\Sigma_l^\top$ is said to be the left inverse of $\Sigma^\top$ if

$$\Sigma_l^\top \Sigma^\top = I,$$

where I is the identity matrix.

Example 4.1 A financial market with one risky asset driven by one risk source:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t,$$
$$\frac{dG_t}{G_t} = r_t dt,$$

is always arbitrage free since there exists the scalar ξ_t which solves

$$\sigma_t \xi_t = \mu_t - r_t.$$

Furthermore, ξ_t coincides with the **Sharpe ratio**. Let us stress that if $\sigma_t = 0$ (i.e. both assets are riskless), then the market is arbitrage free if and only if $\mu_t = r_t$ (on the financial market there cannot be more than one risk free return).

The vector ξ_t has a nice economic interpretation. If σ_t measures the risk and $\mu_t - r_t$ is the risk premium, then the ratio between $\mu_t - r_t$ and σ_t is the risk premium for any unit of risk: actually, this is the "**market price of risk**". If there are k risk sources on the financial market, then there must be k prices of risk. Thus, a financial market works well (is arbitrage free) if and only if it is able to provide a price for any risk source.

Example 4.2 Let us take into account the following market:

$$\frac{dS_{1,t}}{S_{1,t}} = \mu_{1,t}dt + \sigma_{1,t}dW_{1,t},$$
$$\frac{dS_{2,t}}{S_{2,t}} = \mu_{2,t}dt + \sigma_{2,t}dW_{1,t},$$
$$\frac{dG_t}{G_t} = r_t dt,$$

where there are two risky assets driven by just one risk source $dW_{1,t}$. In this case we can check the existence of arbitrage by solving the linear system

$$\underbrace{\begin{bmatrix} \sigma_{1,t} \\ \sigma_{2,t} \end{bmatrix}}_{\Sigma_t^\intercal} \xi_t = \underbrace{\begin{bmatrix} \mu_{1,t} - r_t \\ \mu_{2,t} - r_t \end{bmatrix}}_{\mu_t - r_t \mathbf{1}}.$$

This system has a solution if and only if

$$\frac{\mu_{1,t} - r_t}{\sigma_{1,t}} = \frac{\mu_{2,t} - r_t}{\sigma_{2,t}},$$

i.e. if the Sharpe ratios of the two risky assets coincide. If this is not the case then the financial market is not arbitrage free.

The interpretation of this result is easy: if two assets depend on the same risk source, then their market price of risk must be the very same. There exists an actual financial market which works in this way: the bond market. If we just neglect the credit risk (i.e. we consider bonds issued by the most reliable Government), then the bonds are affected just by the interest rate risk. Accordingly, on this market, there are many assets, but there is only one source of risk. This means that we expect the Sharpe ratios of these bonds to be all the same.

Example 4.3 Let us take into account the following market:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_{1,t} dW_{1,t} + \sigma_{2,t} dW_{2,t},$$
$$\frac{dG_t}{G_t} = r_t dt,$$

where there is just one risky asset whose price depends on two risk sources. In this case there is no arbitrage if we are able to solve

$$\underbrace{\begin{bmatrix} \sigma_{1,t} & \sigma_{2,t} \end{bmatrix}}_{\Sigma_t^\top} \begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix} = \underbrace{\mu_t - r_t}_{\mu_t - r_t \mathbf{1}}.$$

Since this equation has infinite solutions, then the financial market is arbitrage free.

Example 4.4 Let us take into account the following market

$$\frac{dS_{1,t}}{S_{1,t}} = \mu_{1,t} dt + \sigma_{1,1,t} dW_{1,t} + \sigma_{1,2,t} dW_{2,t},$$
$$\frac{dS_{2,t}}{S_{2,t}} = \mu_{2,t} dt + \sigma_{2,1,t} dW_{1,t} + \sigma_{2,2,t} dW_{2,t},$$
$$\frac{dG_t}{G_t} = r_t dt,$$

where there are two risky assets driven by two risk sources. There is no arbitrage if and only if we are able to solve

$$\underbrace{\begin{bmatrix} \sigma_{1,1,t} & \sigma_{1,2,t} \\ \sigma_{2,1,t} & \sigma_{2,2,t} \end{bmatrix}}_{\Sigma_t{}^\top} \begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu_{1,t} - r_t \\ \mu_{2,t} - r_t \end{bmatrix}}_{\mu_t - r_t \mathbf{1}}.$$

There exists only one solution to this system if

$$\sigma_{1,1,t}\sigma_{2,2,t} - \sigma_{1,2,t}\sigma_{2,1,t} \neq 0.$$

In this case we have

(continued)

Example 4.4 (continued)

$$\begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix} = \begin{bmatrix} \sigma_{1,1,t} & \sigma_{1,2,t} \\ \sigma_{2,1,t} & \sigma_{2,2,t} \end{bmatrix}^{-1} \begin{bmatrix} \mu_{1,t} - r_t \\ \mu_{2,t} - r_t \end{bmatrix}$$

$$= \frac{1}{\sigma_{1,1,t}\sigma_{2,2,t} - \sigma_{1,2,t}\sigma_{2,1,t}} \begin{bmatrix} \sigma_{2,2,t}\sigma_{1,2,t} \left(\frac{\mu_{1,t}-r_t}{\sigma_{1,2,t}} - \frac{\mu_{2,t}-r_t}{\sigma_{2,2,t}} \right) \\ \sigma_{2,1,t}\sigma_{1,1,t} \left(\frac{\mu_{2,t}-r_t}{\sigma_{2,1,t}} - \frac{\mu_{1,t}-r_t}{\sigma_{1,1,t}} \right) \end{bmatrix}.$$

Remark 4.2 Hereafter, we will always work with an arbitrage free financial market.

4.6 Completeness (and Asset Pricing)

Definition 4.1 A financial market is said to be **complete** if and only if any asset can be replicated by a suitable portfolio.

Now, let us assume there is an asset whose price follows

$$\frac{dF_t}{F_t} = \mu_{F,t} dt + \underset{1\times k}{\sigma_{F,t}^{\top}} \underset{k\times 1}{dW_t}.$$

Since the market is arbitrage free, then the drift and diffusion terms of this asset must verify the no arbitrage condition (like it happens for any other asset on the market):

$$\sigma_{F,t}^{\top} \xi_t = \mu_{F,t} - r_t.$$

This means that the previous differential equation can be written as

$$\frac{dF_t}{F_t} = \left(r_t + \sigma_{F,t}^{\top} \xi_t \right) dt + \underset{1\times k}{\sigma_{F,t}^{\top}} \underset{k\times 1}{dW_t}. \tag{4.6.1}$$

We can conclude that on an arbitrage free financial market, the expected return on any asset is given by the riskless interest rate augmented by the product between the diffusion term of the asset and the market price of risk.

Proposition 4.2 *On an arbitrage free financial market (where* $\exists \xi_t : \Sigma_t^{\mathsf{T}} \xi_t = \mu_t - r_t \mathbf{1}$*), the drift of any asset having diffusion* $\sigma_{F,t}^{\mathsf{T}}$ *must be* $r_t + \sigma_{F,t}^{\mathsf{T}} \xi_t$*.*

In order to replicate asset F_t, we must look for a portfolio θ_t such that the investor's wealth

$$\frac{d\widetilde{R}_t}{\widetilde{R}_t} = \left(r_t + \theta_t^{\mathsf{T}} (\mu_t - r_t \mathbf{1})\right) dt + \theta_t^{\mathsf{T}} \Sigma_t^{\mathsf{T}} dW_t, \tag{4.6.2}$$

coincides with $\frac{dF_t}{F_t}$. The two stochastic processes (4.6.1) and (4.6.2) are equal if both their drift and diffusion terms are equal. Nevertheless, the absence of arbitrage allows us to ask just for the diffusion terms to be equal, in fact, if we are able to find a portfolio such that

$$\underset{1\times n}{\theta_t^{\mathsf{T}}} \; \underset{n\times k}{\Sigma_t^{\mathsf{T}}} = \underset{1\times k}{\sigma_{F,t}^{\mathsf{T}}}, \tag{4.6.3}$$

then Eq. (4.6.1) becomes

$$\begin{aligned} \frac{dF_t}{F_t} &= \left(r_t + \underbrace{\sigma_{F,t}^{\mathsf{T}}}_{\theta_t{}^{\mathsf{T}} \Sigma_t{}^{\mathsf{T}}} \xi_t \right) dt + \underbrace{\sigma_{F,t}^{\mathsf{T}}}_{\theta_t^{\mathsf{T}} \Sigma_t^{\mathsf{T}}} dW_t \\ &= \left(r_t + \theta_t^{\mathsf{T}} \Sigma_t^{\mathsf{T}} \xi_t\right) dt + \theta_t^{\mathsf{T}} \Sigma_t^{\mathsf{T}} dW_t, \end{aligned}$$

and, because of the no arbitrage condition (i.e. $\Sigma_t^{\mathsf{T}} \xi_t = \mu_t - r_t \mathbf{1}$), we finally have

$$\frac{dF_t}{F_t} = \left(r_t + \theta_t^{\mathsf{T}} (\mu_t - r_t \mathbf{1})\right) dt + \theta_t^{\mathsf{T}} \Sigma_t^{\mathsf{T}} dW_t,$$

which is exactly the differential equation of wealth $\widetilde{R}_t$. Accordingly, condition (4.6.3) is necessary and sufficient for replicating the asset F_t.

Remark 4.3 Condition (4.6.3) implies that the asset F_t and the wealth $\widetilde{R}_t$ behave in the very same way. Nevertheless, it does not guarantee that they have the same value. This last condition (i.e. $F_t = \widetilde{R}_t$) is satisfied by finding the amount of riskless asset $w_{G,t}$ which solves

$$\widetilde{R}_t = w_{G,t} G_t + w_t^{\mathsf{T}} S_t = F_t,$$

(continued)

Remark 4.3 (continued)
i.e.

$$w_{G,t} = \frac{F_t - w_t^{\mathsf{T}} S_t}{G_t},$$

or, which is the same,

$$\frac{1}{\widetilde{R}_t} w_{G,t} G_t + \frac{1}{\widetilde{R}_t} w_t^{\mathsf{T}} S_t = \frac{F_t}{\widetilde{R}_t},$$

$$\theta_{G,t} + \theta_t^{\mathsf{T}} \mathbf{1} = \frac{F_t}{\widetilde{R}_t},$$

i.e.

$$\theta_{G,t} = \frac{F_t}{\widetilde{R}_t} - \theta_t^{\mathsf{T}} \mathbf{1}.$$

System (4.6.3) has a solution if the matrix Σ_t^{T} has a so-called **right inverse**. In particular, we say that $\Sigma_{r,t}^{\mathsf{T}}$ is the right inverse of matrix Σ_t^{T} if

$$\Sigma_t^{\mathsf{T}} \Sigma_{r,t}^{\mathsf{T}} = I.$$

Nevertheless, we recall that we have already assumed that Σ_t^{T} has a left inverse for the market to be arbitrage free. Accordingly, if we now assume that Σ_t^{T} has also the right inverse, then we are assuming that it is invertible (in fact a matrix which has both the left and the right inverse is invertible). If Σ_t^{T} is invertible then Eq. (4.5.1) has a unique solution:

$$\xi_t = \left(\Sigma_t^{\mathsf{T}}\right)^{-1} (\mu_t - r_t \mathbf{1}).$$

Proposition 4.3 *The financial market is complete if and only if there exists only one vector of market price of risk solving Eq. (4.5.1).*

No arbitrage	Completeness
$\exists \xi_t : \Sigma_t^{\mathsf{T}} \xi_t = \mu_t - r_t \mathbf{1}$	$\exists! \xi_t : \Sigma_t^{\mathsf{T}} \xi_t = \mu_t - r_t \mathbf{1}$

4.7 Change of Probability and Asset Pricing

We take back the stochastic differential equation for a generic asset F_t:

$$\frac{dF_t}{F_t} = \left(r_t + \sigma_{F,t}^{\mathsf{T}}\xi_t\right)dt + \sigma_{F,t}^{\mathsf{T}}dW_t,$$

which can be written as

$$\frac{dF_t}{F_t} = r_t dt + \sigma_{F,t}^{\mathsf{T}}\left(\xi_t dt + dW_t\right).$$

Girsanov has demonstrated that the term $\xi_t dt + dW_t$ is equal to another Wiener process under a new probability measure. Thus, under this new measure, the expected return on any asset must be equal to the riskless interest rate (this is the reason why the new probability is often called “risk neutral probability”).

Theorem 4.1 (Girsanov) *Given the market (4.2.1)–(4.2.2), if there exists a vector ξ_t such that*

$$\Sigma_t^{\mathsf{T}}\xi_t = \mu_t - r_t\mathbf{1}$$

then there exists a probability measure $\mathbb{Q}_t$ such that

$$dW_t^{\mathbb{Q}} = \xi_t dt + dW_t, \tag{4.7.1}$$

*provided that the so-called **Radon–Nikodym derivative***

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{-\frac{1}{2}\int_{t_0}^{t}\xi_s^{\mathsf{T}}\xi_s ds - \int_{t_0}^{t}\xi_s^{\mathsf{T}}dW_s}, \tag{4.7.2}$$

is a martingale (i.e. $\mathbb{E}_{t_0}\left[\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right] = 1$).

A sufficient (but not necessary) condition for (4.7.2) to be a martingale on a given horizon T, is the so-called **Novikov's condition**

$$\mathbb{E}_{t_0}\left[e^{\frac{1}{2}\int_{t_0}^{t}\xi_s^{\mathsf{T}}\xi_s ds}\right] < \infty.$$

A demonstration of Theorem 4.1 can be found in Karatzas and Shreve (1991). Theorem 4.1 allows us to conclude that if the financial market is arbitrage free (and ξ_t is such that the Radon–Nikodym is a martingale), then we can switch from the historical probability $\mathbb{P}_t$ to another probability $\mathbb{Q}_t$ under which the expected return on any financial asset coincides with the riskless interest rate. This is the reason why the new probability $\mathbb{Q}_t$ is called **risk neutral probability**.

Theorem 4.1 is not based on the hypothesis of completeness. In fact, it is sufficient that the financial market is arbitrage free.

Now we take Eq. (4.2.1) and we rewrite it under the new probability by using (4.7.1):

$$\begin{aligned} I_S^{-1} dS_t &= \mu_t dt + \Sigma_t^{\mathsf{T}} dW_t \\ &= \mu_t dt + \Sigma_t^{\mathsf{T}} \left(dW_t^{\mathbb{Q}} - \xi_t dt \right) \\ &= \left(\mu_t - \Sigma_t^{\mathsf{T}} \xi_t \right) dt + \Sigma_t^{\mathsf{T}} dW_t^{\mathbb{Q}}. \end{aligned}$$

Since the market is arbitrage free, then $\mu_t - \Sigma_t^{\mathsf{T}} \xi_t = r_t \mathbf{1}$ (from Eq. (4.5.1)), and so

$$I_S^{-1} dS_t = r_t \mathbf{1} dt + \Sigma_t^{\mathsf{T}} dW_t^{\mathbb{Q}}.$$

Remark 4.4 Girsanov's theorem allows us to change the drift of any stochastic process, while the diffusion cannot be changed.

The new probability measure $\mathbb{Q}_t$ allows us to obtain the following result:[2]

$$\begin{aligned} d\left(\frac{1}{G_t} S_t\right) &= d\left(\frac{1}{G_t}\right) S_t + \frac{1}{G_t} dS_t + \underbrace{d\left(\frac{1}{G_t}\right) dS_t}_{0} \\ &= -\frac{1}{G_t} r_t S_t dt + \frac{1}{G_t} I_S \left(r_t \mathbf{1} dt + \Sigma_t^{\mathsf{T}} dW_t^{\mathbb{Q}} \right) \\ &= \frac{1}{G_t} I_S \Sigma_t^{\mathsf{T}} dW_t^{\mathbb{Q}}. \end{aligned}$$

If we compute the integral from t to T of both sides we have

$$\int_t^T d\left(\frac{1}{G_s} S_s\right) = \int_t^T \frac{1}{G_s} I_S \Sigma_s^{\mathsf{T}} dW_s^{\mathbb{Q}},$$

$$\frac{S_T}{G_T} - \frac{S_t}{G_t} = \int_t^T \frac{1}{G_s} I_S \Sigma_s^{\mathsf{T}} dW_s^{\mathbb{Q}},$$

[2]We recall that given $dG_t = G_t r_t dt$, we have $d\left(\frac{1}{G_t}\right) = -\frac{1}{G_t} r_t dt$.

whose expected value is

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_T}{G_T}\right] = \frac{S_t}{G_t}.$$

This is the formula of a martingale. This is why $\mathbb{Q}_t$ is also known as **martingale equivalent measure** (under it, the asset prices are martingales if measured in term of the riskless asset, which is accordingly the *numéraire* of the economy).

Furthermore, since G_t belongs to the information set in t, we can rewrite the previous equation as

$$S_t = \mathbb{E}_t^{\mathbb{Q}}\left[S_T \underbrace{\frac{G_t}{G_T}}_{\text{discount factor}}\right],$$

where we have already underlined that $\frac{G_t}{G_T}$ is the discount factor between t and T. This is a fundamental result in asset pricing.

Theorem 4.2 (Fundamental Theorem of Asset Pricing (I)) *On an arbitrage free financial market, the price of any asset is given by the expected value, under the risk neutral probability, of its future value discounted by the riskless interest rate.*

According to Theorem 4.2, the value in t of an asset which pays 1 Euro in T (for sure), which is called a zero coupon bond (ZCB), is given by

$$B(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[1 \times \frac{G_t}{G_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]. \tag{4.7.3}$$

This means that the value of a ZCB is given by the expected value of the discount factor (under the risk neutral probability).

If an asset pays some cash flows δ_t at any instant t, and it is sold in T at the price S_T (which is unknown in t), its value can be computed as the sum of each cash flow which is interpreted as a single asset (the strategy of trading asset cash flows independently of the main asset is called **"stripping"**) as shown in Fig. 4.1.

Accordingly, the value of a cash flow paying asset is

$$\begin{aligned} S_t &= \int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[\delta_s \frac{G_t}{G_s}\right] ds + \mathbb{E}_t^{\mathbb{Q}}\left[S_T \frac{G_t}{G_T}\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta_s \frac{G_t}{G_s} ds + S_T \frac{G_t}{G_T}\right]. \end{aligned}$$

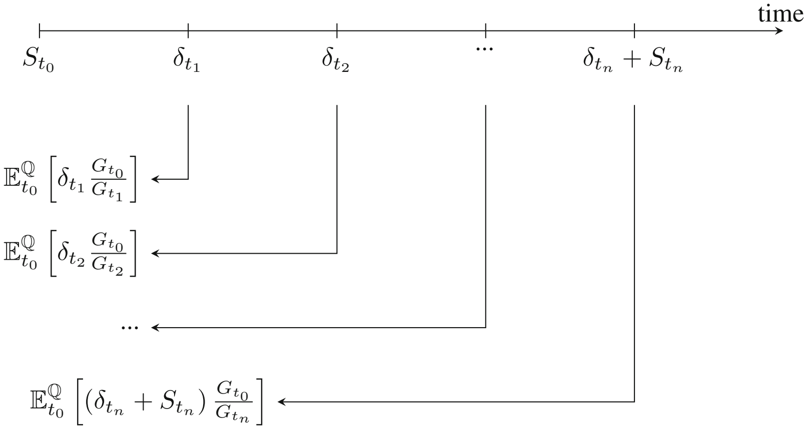

Fig. 4.1 Expected present value of dividends

Theorem 4.3 (Fundamental Theorem of Asset Pricing (II)) *On an arbitrage free financial market, the value of any asset is given by the expected value, under the risk neutral probability, of its future cash flows discounted by the riskless interest rate.*

4.8 Bond Pricing: Closed Form and Simulations

Given the value of a ZCB as in (4.7.3), if we assume that the interest rate r_t solves a stochastic differential equation like (3.4.1), the expected value in (4.7.3) cannot be found in closed form. Nevertheless, it is possible to price a ZCB by simulations (as we are about to do in this section). Instead, if r_t follows a process like (3.4.1) with either $\gamma = 0$ or $\gamma = \frac{1}{2}$, then a closed form for (4.7.3) exists. More generally, let us take the following differential equation

$$dr_t = (ar_t + b_t)\,dt + \sqrt{f_t + gr_t}\,dW_t,$$

with the market price of risk that takes the form

$$\xi_t = \phi\sqrt{f_t + gr_t},$$

so that the differential dr_t does not change its statistical properties under the risk neutral probability $\mathbb{Q}$:

$$\begin{aligned} dr_t &= (ar_t + b_t)\,dt + \sqrt{f_t + gr_t}\left(dW_t^{\mathbb{Q}} - \xi_t dt\right) \\ &= \left(\underbrace{(a-\phi g)}_{a^{\mathbb{Q}}} r_t + \underbrace{b_t - \phi f_t}_{b_t^{\mathbb{Q}}}\right) dt + \sqrt{f_t + gr_t}\,dW_t^{\mathbb{Q}}. \end{aligned}$$

Remark 4.5 This model coincides with Vasiček (1977) if $g = 0$ and with Cox et al. (1985) if $f = 0$.

This particular setting allows a closed form solution for the value of a ZCB.

Proposition 4.4 *Let us assume that, under the risk neutral probability $\mathbb{Q}$, the interest rate r_t follows the process*

$$dr_t = \left(a^{\mathbb{Q}} r_t + b_t^{\mathbb{Q}}\right) dt + \sqrt{f_t + gr_t}\,dW_t^{\mathbb{Q}},$$

then the price of a ZCB is given by

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right] = e^{-A(t,T) - C(t,T) r_t},$$

where

$$k \equiv \sqrt{\left(a^{\mathbb{Q}}\right)^2 + 2g},$$

$$C(t,T) = 2\frac{1 - e^{-k(T-t)}}{k - a^{\mathbb{Q}} + \left(k + a^{\mathbb{Q}}\right) e^{-k(T-t)}},$$

$$A(t,T) = \int_t^T \left(C(s,T)\, b_s^{\mathbb{Q}} - \frac{1}{2} C(s,T)^2 f_s\right) ds.$$

Proof. Let us assume that $B(t,T)$ can actually be written in the form

$$B(t,T) = e^{-A(t,T)-C(t,T)r_t},$$

where the functions $A(t,T)$ and $C(t,T)$ must satisfy

$$A(T,T) = C(T,T) = 0,$$

since the value of a ZCB at the expiration is always equal to its face value 1 (without any credit risk).

Under the probability $\mathbb{Q}$, we know that the expected return on $B(t,T)$ must be r_t and, accordingly, through Itô's lemma, we can write

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{dB(t,T)}{B(t,T)}\right] = \begin{pmatrix} -\frac{\partial A(t,T)}{\partial t} - \frac{\partial C(t,T)}{\partial t} r_t - C(t,T)\left(a^{\mathbb{Q}} r_t + b_t^{\mathbb{Q}}\right) \\ +\frac{1}{2} C(t,T)^2 (f_t + g r_t) \end{pmatrix} dt = r_t dt,$$

which can be split into two ordinary differential equations for the functions $A(t,T)$ and $C(t,T)$as follows:

$$\begin{cases} 0 = \frac{\partial A(t,T)}{\partial t} + C(t,T)\, b_t^{\mathbb{Q}} - \frac{1}{2} C(t,T)^2 f_t, \\ 0 = r_t \left(\frac{\partial C(t,T)}{\partial t} + 1 + a^{\mathbb{Q}} C(t,T) - \frac{1}{2} g C(t,T)^2\right). \end{cases}$$

The value of $C(t,T)$ is obtained from the second differential equation, which is a Riccati differential equation with constant coefficients (the case with constant coefficients is the only one which allows for a closed form solution). The unique solution of this equation, with boundary condition $C(T,T) = 0$, is

$$C(t,T) = 2\frac{1 - e^{-k(T-t)}}{k - a^{\mathbb{Q}} + \left(k + a^{\mathbb{Q}}\right) e^{-k(T-t)}},$$

where $k \equiv \sqrt{\left(a^{\mathbb{Q}}\right)^2 + 2g}$. After computing $C(t,T)$, the value of $A(t,T)$ is obtained by integrating the first equation of the previous system (and recalling the boundary condition $A(T,T) = 0$):

$$A(t,T) = \int_t^T \left(C(s,T)\, b_s^{\mathbb{Q}} - \frac{1}{2} C(s,T)^2 f_s\right) ds.$$

□

The R code which allows to compute this value of a ZCB can be written as follows, where we create some functions nested inside another function.

```
ZCB.affine = function(a, b, f, g, r0, T, t) {
    C = function(s) {
        k = sqrt(a^2 + 2 * g)
        2 * (1 - exp(-k * (T - s)))/(k - a + (k + a) *
            exp(-k * (T - s)))
    }
    integrand = function(s) {
        C(s) * b - 0.5 * C(s)^2 * f
    }
    exp(-integrate(integrand, lower = t, upper = T)$value -
        C(t) * r0)
}
```

The command "`integrate`" contains three mandatory options: the integrand function, the lower boundary, and the upper boundary of integration. The output of this command contains five elements, but here we are just interested in the "`value`" output (i.e. the numerical value of the integral).

We can use this function for computing the value of a ZCB in a numerical example with values of parameters taken from an estimation of the CIR case (i.e. $f = 0$). If we assume that the initial value of the interest rate is equal to its equilibrium value (i.e. $\frac{b}{|a|}$), then we can check that the ZCB is not much different from that obtained by using a constant interest rate.

```
ZCB.affine(a = a_C, b = b_C, f = 0, g = sigma_C^2,
    r0 = b_C/abs(a_C), T = 5, t = 0)

## [1] 0.7807778
exp(-b_C/abs(a_C) * 5)

## [1] 0.7791814
```

When a closed form solution is not available, we can price this asset through simulations. Thus, we can use the R function CKLS already created in the previous chapter and simulate a sufficiently high number of paths for the interest rate. Finally, the mean of the ZCB computed for each path will approximate the true value of the ZCB.

In the following code we replace the exponential of the integral of the interest rates with the sum of the interest rates weighted by dt. The command "`apply`" contains three arguments: (1) the array which is the base of the computations, (2) the dimension of the array that we want to work on (1 stands for rows, and 2 stands for columns), and (3) the function that we want to apply to the array. In this particular case we want to compute the sum through the columns on the array `x`.

```
CKLS.ZCB = function(a = 0, b = 0, sigma = 1, g = 0,
    x0 = 1, dt = 1/250, T = 1, N = 1) {
    x = array(0, dim = c(T/dt, N))
    x[1, ] = rep(x0, N)
    for (i in 2:(T/dt)) {
        dx = (a * x[i - 1, ] + b) * dt + sigma * x[i -
            1, ]^g * rnorm(N) * sqrt(dt)
        x[i, ] = x[i - 1, ] + dx
    }
    mean(exp(-apply(x, 2, sum) * dt))
}
CKLS.ZCB(a = a_C, b = b_C, sigma = sigma_C^2, g = 0.5,
    x0 = b_C/abs(a_C), T = 5, N = 10000)
```

```
## [1] 0.7791784
```

Performing the same procedure many times will always give (not too) different values, which are all also different from the true value. While the number of simulations increases, the precision of the simulated result increases too. Unfortunately, the rate of this increment is quite slow because of the high variance of the results. There are some so-called "variance reduction techniques" that allow to reduce this variance (the interested reader is referred to Iacus 2009, Section 1.4).

4.9 The Switch Between Probabilities

Since $\mathbb{P}_t$ and $\mathbb{Q}_t$ are probabilities, then their differentials ($d\mathbb{P}_t$ and $d\mathbb{Q}_t$) coincide with density functions. Thus, it should be clear that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{t_0}[X_s] &= \int_{\Omega_0} X_s d\mathbb{Q}_s = \int_{\Omega_0} X_s \frac{d\mathbb{Q}_s}{d\mathbb{P}_s} d\mathbb{P}_s \\ &= \mathbb{E}_{t_0}\left[X_s \frac{d\mathbb{Q}_s}{d\mathbb{P}_s}\right],\end{aligned}$$

where $\frac{d\mathbb{Q}_s}{d\mathbb{P}_s}$ is the Radon–Nikodym derivative (4.7.2), and Ω_0 is the domain of the stochastic variable X at time t_0.

Thus, we can always switch from an expected value under $\mathbb{Q}_t$ to an expected value under the historical probability $\mathbb{P}_t$ and vice versa.

From Girsanov theorem we know that

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{-\frac{1}{2}\int_{t_0}^t \xi_s^\intercal \xi_s ds - \int_t^T \xi_s^\intercal dW_s} := m_{t_0,t}, \tag{4.9.1}$$

and by using Itô's lemma we can write

$$\frac{dm_{t_0,t}}{m_{t_0,t}} = -\xi_t^{\intercal} dW_t. \tag{4.9.2}$$

Accordingly, the value of any asset can be written as

$$S_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[S_T e^{-\int_{t_0}^T r_s ds}\right],$$

or, alternatively,

$$\begin{aligned} S_{t_0} &= \mathbb{E}_{t_0}\left[S_T m_{t_0,T} e^{-\int_{t_0}^T r_s ds}\right] \\ &= \mathbb{E}_{t_0}\left[S_T e^{-\frac{1}{2}\int_{t_0}^T \xi_s{}^{\intercal}\xi_s ds - \int_{t_0}^T \xi_s^{\intercal} dW_s} e^{-\int_{t_0}^T r_s ds}\right]. \end{aligned}$$

Because of the role played by the martingale $m_{t_0,T}$ in asset pricing, it is sometimes called "**price kernel**".

Thus, we can conclude that the value of an asset can also be computed under the historical probability but the discount factor must be stochastic:

$$S_{t_0} = \mathbb{E}_{t_0}\left[S_T \underbrace{e^{-\int_{t_0}^T \left(r_s + \frac{1}{2}\xi_s^{\intercal}\xi_s\right)ds - \int_{t_0}^T \xi_s^{\intercal} dW_s}}_{\text{Stochastic Discount Factor (SDF)}}\right].$$

Since the SDF between time t_0 and time t_0 is of course equal to 1, then we can write

$$S_{t_0} SDF_{t_0,t_0} = \mathbb{E}_{t_0}\left[S_T SDF_{t_0,T}\right],$$

i.e. under the historical probability the asset prices are martingales if they are discounted by the stochastic discount factor.

Another probability that is very useful in the financial applications allows to split the expected value of a product into the product of two expected values. Let us take again the result of the fundamental theorem of asset pricing:

$$S_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[S_T e^{-\int_{t_0}^T r_u du}\right].$$

We can write this equation under the form of the integral of the function weighted by the density $d\mathbb{Q}_t$:

$$S_{t_0} = \int_{\Omega_0} S_T e^{-\int_{t_0}^T r_u du} d\mathbb{Q}_T.$$

Inside the integral we can multiply and divide by the same quantity, equal to the price of a ZCB:

$$S_{t_0} = \int_{\Omega_0} S_T e^{-\int_{t_0}^T r_u du} \frac{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]} d\mathbb{Q}_T.$$

Sine the price of a ZCB is known at time t_0, it can be collected outside the integral:

$$S_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right] \int_{\Omega_0} S_T \underbrace{\frac{e^{-\int_{t_0}^T r_u du}}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]} d\mathbb{Q}_T}_{d\mathbb{F}_T}.$$

Now, we can check that the term we have called $d\mathbb{F}_T$ is a positive martingale. The exponential function is always positive and so its expected value. Furthermore, since $d\mathbb{Q}_T$ is a density function, it is always positive too. Finally, the expected value of $d\mathbb{F}_T$ under $\mathbb{Q}$ is

$$\int_{\Omega_0} \frac{e^{-\int_{t_0}^T r_u du}}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]} d\mathbb{Q}_T = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\frac{e^{-\int_{t_0}^T r_u du}}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]}\right] = 1.$$

Any stochastic variable which is always positive and whose integral on the domain is 1, can be interpreted as a density function. Furthermore, since $d\mathbb{F}_T$ is a density function, then $\mathbb{F}_T$ is a probability:

$$S_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right] \int_{\Omega_0} S_T d\mathbb{F}_T = B(t_0, T)\, \mathbb{E}_{t_0}^{\mathbb{F}}[S_T].$$

Finally, since the value of a ZCB at expiration is always 1 (if we neglect the credit risk), then we can write

$$\frac{S_{t_0}}{B(t_0, T)} = \mathbb{E}_{t_0}^{\mathbb{F}}\left[\frac{S_T}{B(T, T)}\right].$$

This result allows us to conclude that the asset prices are martingales under the new $\mathbb{F}_T$ probability if the *numéraire* of the economy is the ZCB.

4.10 Change of *Numéraire*

Let us assume on the market there are two assets:

$$\frac{dS_t}{S_t} = rdt + \sigma^\intercal dW_t^{\mathbb{Q}},$$

$$\frac{dN_t}{N_t} = rdt + \sigma_N^\intercal dW_t^{\mathbb{Q}},$$

and the second one is used like a *numéraire* on the market. Now, we want to check whether there exists another probability (that we call $\mathbb{N}$) under which the ration S_t/N_t is a martingale. In other words, we want to check the existence of $\mathbb{N}$ such that

$$\frac{S_t}{N_t} = \mathbb{E}_t^{\mathbb{N}}\left[\frac{S_T}{N_T}\right].$$

If we use Itô's lemma on the ratio between S_t and N_t, we have

$$\frac{d\left(\frac{S_t}{N_t}\right)}{\frac{S_t}{N_t}} = (\sigma_N - \sigma)^\intercal \sigma_N dt - (\sigma_N - \sigma)^\intercal dW_t^{\mathbb{Q}}.$$

Now, through Girsanov Theorem 4.1, we can change the probability $\mathbb{Q}$ to the probability $\mathbb{N}$ as follows

$$dW_t^{\mathbb{Q}} = \nu dt + dW_t^{\mathbb{N}},$$

and, thus

$$\frac{d\left(\frac{S_t}{N_t}\right)}{\frac{S_t}{N_t}} = (\sigma_N - \sigma)^\intercal (\sigma_N - \nu)\, dt - (\sigma_N - \sigma)^\intercal dW_t^{\mathbb{N}}.$$

We know that any martingale can be written as a stochastic process with zero drift. Thus, we can conclude that the new probability $\mathbb{N}$ exists if

$$\nu = \sigma_N,$$

ad if ν satisfies the Novikov's condition.

We can conclude that the price of any asset can be written as a martingale under a new probability if it is divided by the price of another asset which is used as *numéraire*. The difference between the Wiener process under the risk neutral probability and the new probability is given by the volatility of the *numéraire*.

4.11 Assets with Coupons/Dividends

If a financial asset instantaneously pays coupons/dividends δ_t (in monetary units), then the fundamental asset pricing theorem must be interpreted in the following way: under the probability $\mathbb{Q}_t$, the "total" expected return on the asset must equate the riskless interest rate r_t. In this case, the total return is the sum of the capital return ($\frac{dS_t}{S_t}$) and the coupon/dividend return ($\frac{\delta_t}{S_t}dt$). Thus, we can write

$$\frac{dS_t}{S_t} + \frac{\delta_t}{S_t}dt = r_t dt + \sigma_t^\top dW_t^{\mathbb{Q}},$$

which becomes

$$\frac{dS_t}{S_t} = \left(r_t - \frac{\delta_t}{S_t}\right) dt + \sigma_t^\top dW_t^{\mathbb{Q}}.$$

This means that an asset which pays coupons/dividends has a growth rate lower than that of an asset which does not pay any cash flow. In fact, on financial markets we notice that the prices of stocks suddenly fall when dividends are paid.

Example 4.5 Let us take into account the simplest case: everything is deterministic (coupons/dividends and interest rate). The value of an asset paying cash flows δ_t from t to T is given by

$$V_t = \int_t^T \delta_s e^{-\int_t^s r_u du} ds.$$

Now, if we differentiate with respect to time we have

$$dV_t = \left(-\delta_t + r_t \int_t^T \delta_s e^{-\int_t^s r_u du} ds\right) dt,$$

which can be written as

(continued)

Example 4.5 (continued)

$$\frac{dV_t}{V_t} = \left(r_t - \frac{\delta_t}{V_t}\right) dt,$$

which is, of course, the result we were looking for.

According to the result of the example, if the value of a bond is given by

$$V_{t,T} = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T \delta_s \frac{G_t}{G_s} ds + \frac{G_t}{G_T} \right],$$

then the expected value of its differential is

$$\mathbb{E}_t^{\mathbb{Q}} \left[dV_{t,T} \right] = \left(V_{t,T} r_t - \delta_t \right) dt.$$

If the coupons are deterministic then the value of $V_{t,T}$ can be simplified as follows

$$\begin{aligned} V_{t,T} &= \int_t^T \delta_s \mathbb{E}_t^{\mathbb{Q}} \left[\frac{G_t}{G_s} \right] ds + \mathbb{E}_t^{\mathbb{Q}} \left[\frac{G_t}{G_T} \right] \\ &= \int_t^T \delta_s B\left(t, s\right) ds + B\left(t, T\right). \end{aligned}$$

Once the values of all the ZCB's are known, the asset price $V_{t,T}$ can be easily computed.

Example 4.6 An interesting case is that of a perfectly indexed bond whose coupon is equal to the riskless interest rate:

$$V_{t,T} = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T r_s \frac{G_t}{G_s} ds + \frac{G_t}{G_T} \right].$$

Now we recall that

$$\frac{dG_s}{G_s} = r_s ds,$$

and so we can substitute $r_s ds$ in the integral with its corresponding value:

(continued)

Example 4.6 (continued)

$$
\begin{aligned}
V_{t,T} &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{G_t}{G_s}\underbrace{\frac{dG_s}{G_s}}_{r_s ds} + \frac{G_t}{G_T}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{G_t}{G_s^2} dG_s + \frac{G_t}{G_T}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\left[-\frac{G_t}{G_s}\right]_{s=t}^{s=T} + \frac{G_t}{G_T}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[-\frac{G_t}{G_T} + \frac{G_t}{G_t} + \frac{G_t}{G_T}\right] = 1.
\end{aligned}
$$

The result of this example shows that a perfectly indexed bond must always be listed at par. We will use this result later for showing some interesting case about the pricing of assets whose cash flows are subject to credit risk.

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Chapter 5
The Actuarial Framework

5.1 Introduction

In this chapter we show how to use the stochastic tools developed in the previous chapters for measuring the actuarial risk. The main measure that we deal with is the force of mortality. There exist a lot of deterministic models for this particular variable, and after summarising that models, in this chapter we show how to create a stochastic version of them. For this purpose, we use a particular version of the stochastic variables described in the previous chapter. In fact, the force of mortality must diverge over time, since the survival probability must converge towards zero while the agent grows older and older. Nevertheless, we use a mean reverting process since we assume that the force of mortality will stay close to one of its deterministic models. To this purpose, we show a mean reverting process that has a time varying (divergent) equilibrium value and we will show how to calibrate this model to the US actuarial data.

5.2 Actuarial Measures

Let us call π_τ the density function of the death time τ (which can also be interpreted as the "default time" for a firm) whose domain is assumed to be $[0, \omega]$. The results obtained in this framework are still valid if $\omega \to \infty$. The density function is non negative, and its integral on the whole domain must be

$$\int_0^\omega \pi_s ds = 1.$$

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_5

Remark 5.1 We can define a similar and equivalent framework where, instead of measuring the death time τ, we measure the death age of the agent. So, if ω is the maximum length of human life, and ι is the current age of the agent, then the previous integral should be written as

$$\int_{\iota}^{\iota+\omega} \pi_s ds = 1,$$

and if we define the new variable $t = s - \iota$, the integral becomes

$$\int_0^{\omega} \pi_{\iota+t} dt = 1.$$

Thus, the density function π_t will also depend on the age ι of the agent.

The probability to die (or to go bankrupt) between time 0 and time t is given by

$$({}_t q_0) = \int_0^t \pi_s ds,$$

while the probability to survive from time 0 to time t is of course given by

$$({}_t p_0) = 1 - \int_0^t \pi_s ds.$$

If we differentiate this equation we have

$$d\,({}_t p_0) = -\pi_t dt,$$

with the natural boundary condition $({}_0 p_0) = 1$, i.e. the probability to be alive at time 0 given that we are alive at the same time is, of course, 1.

The previous differential equation can be written as

$$\frac{d\,({}_t p_0)}{({}_t p_0)} = -\frac{\pi_t}{({}_t p_0)} dt, \tag{5.2.1}$$

where

$$\frac{\pi_t}{({}_t p_0)} = \frac{\pi_t}{1 - \int_0^t \pi_s ds} := \lambda_t, \tag{5.2.2}$$

is called **hazard rate**, but it takes many other names according to the field where it is applied.

If π_t is the density of mortality, then λ_t is called **force of mortality**, while in the literature about the credit risk, this same measure is called **default intensity**. Since both the numerator and the denominator of this ratio are positive then λ_t is positive for any t.

The survival probability between time 0 and time t can be obtained as a function of λ_t by solving (5.2.1) with its boundary condition:

$$\frac{d\,({}_t p_0)}{({}_t p_0)} = -\lambda_t dt, \qquad ({}_0 p_0) = 1,$$

which gives

$$({}_t p_0) = e^{-\int_0^t \lambda_s ds}.$$

Bayes formula for computing conditional probability

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)},$$

allows us to write

$$\mathbb{P}(\tau > T|\tau > t) = \frac{\mathbb{P}(\tau > t|\tau > T)\,\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)},$$

where $\mathbb{P}(\tau > T|\tau > t)$ is the probability to be alive at time T given that one is alive at time t (with $T > t$). We can also write it as $({}_{T-t} p_t)$. The probability of being alive at time t given that we are alive at time T is, of course 1 (i.e. $\mathbb{P}(\tau > t|\tau > T) = 1$). Then, since we have

$$\mathbb{P}(\tau > T) = ({}_T p_0)\,,$$

we can finally write

$$\mathbb{P}(\tau > T|\tau > t) = \frac{({}_T p_0)}{({}_t p_0)},$$

and

$$({}_{T-t} p_t) = \frac{({}_T p_0)}{({}_t p_0)} = \frac{e^{-\int_0^T \lambda_s ds}}{e^{-\int_0^t \lambda_s ds}} = e^{-\int_t^T \lambda_s ds}. \tag{5.2.3}$$

It is worth noting the analogy between r_t and λ_t and between the vale of a zero coupon $B(t, T)$ and the value of the survival probability $({}_{T-t} p_t)$. This analogy is confirmed even in a stochastic environment as we are about to show in the next section.

5.3 Double Stochastic Force of Mortality and Asset Pricing

The death time τ is a stochastic variable. Nevertheless, in the previous section we have assumed that both λ_t and π_t are deterministic. A more realistic assumption is that λ_t (or π_t) is stochastic itself (this is the so-called **double stochastic model**). In this case, Eq. (5.2.3) is valid only as an expected value:

$$({}_{T-t}p_t) = \mathbb{E}_t\left[e^{-\int_t^T \lambda_s ds}\right], \tag{5.3.1}$$

where the expected value is computed under the historical probability.

Remark 5.2 When λ_t is stochastic, the so-called **longevity risk** can be taken into account. In fact, the longevity risk can be defined as the unforeseen change in the force of mortality. Accordingly, if λ_t is deterministic, there is no longevity risk. In fact, in this case, even after many years, we are assuming that the mortality law remains unchanged.

The analogy between the survival probability in (5.3.1) and the value of a zero coupon bond in (4.7.3) is apparent and it is summarised in Table 5.1.

Since λ_t plays exactly the same role as an interest rate, it is often modelled by using the same stochastic processes used for r_t like (3.4.1).

Remark 5.3 The advantage of modelling the force of mortality instead of the time τ, is that λ_t is defined by itself and is no more linked with τ.

Let us assume that we have to evaluate an insurance contract (on the financial market) which pays a given amount of money δ_t at any instant in time until the death age τ (i.e. an annuity). Such a pricing exercise is used, for instance, when we want to trade the insurance contract on the financial market. Under the hypothesis

Table 5.1 Comparison between the value of a zero-coupon bond and the survival probability

Financial framework	Actuarial framework
$\frac{dG_t}{G_t} = r_t dt$	$\frac{d({}_tp_0)}{({}_tp_0)} = -\lambda_t dt$
$G_t = G_0 e^{\int_0^t r_u du}$	$({}_tp_0) = e^{-\int_0^t \lambda_u du}$
$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]$	$({}_{T-t}p_t) = \mathbb{E}_t\left[e^{-\int_t^T \lambda_u du}\right]$

that the agent is still alive, the value of the annuity at time t is given by

$$A_{1,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^\tau \delta_s e^{-\int_t^s r_u du} ds\right],$$

where the expected value must be jointly computed with respect to both the financial risk (represented by the probability $\mathbb{Q}$) and the actuarial risk (represented by the death time τ). These two risks generate two different filtrations ($\sigma-$algebras): the financial $\sigma-$algebra ($\mathcal{F}_t$) is generated by the Wiener processes (which describe the risk sources on the financial market), while the actuarial $\sigma-$algebra ($\mathcal{G}_t$) is generated by the stochastic variable τ. Then, the conditional expected value could be written as

$$\mathbb{E}_{t,t}^{\mathbb{Q},\tau}[\bullet] \equiv \mathbb{E}^{\mathbb{Q},\tau}[\bullet|\, \mathcal{F}_t \wedge \mathcal{G}_t].$$

The two subscripts in the expected value on the left hand side indicate the information sets.

The integral in $A_{1,t}$ can be simplified by using an **indicator function** $\mathbb{I}_\epsilon$ whose value is 1 if the event ϵ happens and 0 otherwise:

$$\mathbb{I}_\epsilon = \begin{cases} 1 & \text{if } \epsilon \text{ happens,} \\ 0 & \text{otherwise.} \end{cases}$$

The value of $A_{1,t}$ can accordingly be written as

$$A_{1,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^\omega \mathbb{I}_{s<\tau}\delta_s e^{-\int_t^s r_u du} ds\right],$$

where we recall that ω is the right bound of human life domain (many models assume $\omega \to \infty$).

In order to simplify the computations we now use the rule of **iterated expected values** (the so-called **tower rule for expected values**)

$$\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_T^{\mathbb{Q}}[\bullet]\right] = \mathbb{E}_t^{\mathbb{Q}}[\bullet], \ \forall T \geq t$$

and we write (this trick has been firstly introduced by Lando 1998)

$$A_{1,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega,t}^{\mathbb{Q},\tau}\left[\int_t^\omega \mathbb{I}_{s<\tau}\delta_s e^{-\int_t^s r_u du} ds\right]\right],$$

where the inner expected value is computed with respect to the financial information available at time ω (i.e. under the filtration $\mathcal{F}_\omega$). Now, since all the financial variables between t and ω are perfectly known in ω, then we can collect all the financial variables outside the inner expected value

$$A_{1,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^{\omega} \mathbb{E}_{\omega,t}^{\mathbb{Q},\tau}\left[\mathbb{I}_{s<\tau}\right]\delta_s e^{-\int_t^s r_u du} ds\right],$$

and since the expected value of the indicator function of an event is the probability of the event, we can write

$$A_{1,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^{\omega} \mathbb{E}_{\omega}^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right]\delta_s e^{-\int_t^s r_u du} ds\right].$$

Finally, we use again the iterated expected value (but the other way round) for eliminating the inner expected value:

$$A_{1,t} = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \delta_s e^{-\int_t^s (\lambda_u + r_u) du} ds\right], \tag{5.3.2}$$

where we can forget about the actuarial information set (generated by τ) since λ_t does not explicitly depend on τ.

The value of $A_{1,t}$ can then be interpreted as the value of a bond whose coupons are discounted by the riskless interest rate augmented by the force of mortality.

Remark 5.4 For the sake of simplicity, it is quite common to assume that λ_t is independent of any other stochastic variable. Under this assumption, Eq. (5.3.2) can be significantly simplified as follows:

$$\begin{aligned} A_{1,t} &= \int_t^{\omega} \mathbb{E}_t^{\mathbb{Q}}\left[\delta_s e^{-\int_t^s r_u du}\right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds \\ &= \int_t^{\omega} \mathbb{E}_t^{\mathbb{Q}}\left[\delta_s e^{-\int_t^s r_u du}\right]\left({}_{s-t}p_t^{\mathbb{Q}}\right) ds, \end{aligned}$$

where we see that the value of the asset $A_{1,t}$ is the sum (the integral) of all the expected discounted future cash flows, weighted by the probability that the reference agent is still alive at any coupon maturity date.

Now, we take into account the case of an insurance contract paying 1 monetary unit at the death time of an agent. Its value can be written as

$$A_{2,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[e^{-\int_t^{\tau} r_u du}\right].$$

Here, we can use the same trick previously presented and write

$$A_{2,t} = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega,t}^{\mathbb{Q},\tau}\left[e^{-\int_t^\tau r_u du}\right]\right].$$

Now, we can compute the inner expected value just with respect to the stochastic variable τ (which is the only stochastic variable in the inner expected value since the whole path of r_t is known at time ω):

$$\begin{aligned} A_{2,t} &= \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega}^{\mathbb{Q}}\left[\int_t^\omega \pi_s e^{-\int_t^s r_u du} ds\right]\right] \\ &= \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^\omega \pi_s e^{-\int_t^s r_u du} ds\right], \end{aligned}$$

and by using (5.2.2)

$$A_{2,t} = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega \lambda_s e^{-\int_t^s (r_u + \lambda_u) du} ds\right]. \tag{5.3.3}$$

In this case, we see that an asset paying only one cash flow at a stochastic time (like the death time of an agent) is priced like an asset paying coupons given by the original cash flow weighted by the death intensity λ_t.

5.4 Annuities in the Gompertz Framework

In the actuarial literature one of the most general function form for the force of mortality is shown in Perks (1932):

$$\lambda_t = \frac{\varepsilon_0 + \varepsilon_1 e^{\varepsilon_2(\iota+t)}}{1 + \varepsilon_3 e^{\varepsilon_2(\iota+t)}}, \tag{5.4.1}$$

for some constant parameters ε_i, $i \in \{0, 1, 2, 3\}$, and where ι is the age of the agent.

The probability to survive from time t (i.e. from age $\iota + t$) for $T - t$ periods is

$$({}_{T-t}p_t) = e^{-\varepsilon_0(T-t)}\left(\frac{1 + \varepsilon_3 e^{\varepsilon_2(\iota+T)}}{1 + \varepsilon_3 e^{\varepsilon_2(\iota+t)}}\right)^{\frac{\varepsilon_0\varepsilon_3 - \varepsilon_1}{\varepsilon_2\varepsilon_3}},$$

where we see that the parameter ε_0 measures the age independent component of the survival probability, for instance the mortality due to accidents.

Most of the functions used in the actuarial literature are just particular cases of (5.4.1) as summarised in Table 5.2.

Table 5.2 Some models for the force of mortality used in the actuarial literature, as particular cases of (5.4.1)

Author (model)	Force of mortality λ_t	Parameters in (5.4.1)
Exponential	ε_0	$\varepsilon_1 = \varepsilon_3 = 0$
Gompertz (1825)	$\varepsilon_1 e^{\varepsilon_2(\iota+t)}$	$\varepsilon_0 = \varepsilon_3 = 0$
Makeham (1890)	$\varepsilon_0 + \varepsilon_1 e^{\varepsilon_2(\iota+t)}$	$\varepsilon_3 = 0$
Kannisto	$\frac{\varepsilon_1 e^{\varepsilon_2(\iota+t)}}{1+\varepsilon_1 e^{\varepsilon_2(\iota+t)}}$	$\varepsilon_0 = 0$, $\varepsilon_1 = \varepsilon_3$
Makeham-Perks	$\frac{\varepsilon_0+\varepsilon_1 e^{\varepsilon_2(\iota+t)}}{1+\varepsilon_1 e^{\varepsilon_2(\iota+t)}}$	$\varepsilon_1 = \varepsilon_3$
Beard (1959)	$\frac{\varepsilon_1 e^{\varepsilon_2(\iota+t)}}{1+\rho\varepsilon_1 e^{\varepsilon_2(\iota+t)}}$	$\varepsilon_0 = 0$, $\varepsilon_3 = \rho\varepsilon_1$
Makehma-Beard	$\frac{\varepsilon_0+\varepsilon_1 e^{\varepsilon_2(\iota+t)}}{1+\rho\varepsilon_1 e^{\varepsilon_2(\iota+t)}}$	$\varepsilon_3 = \rho\varepsilon_1$

The Gompertz-Makeham (Gompertz 1825) mortality is able to describe the human mortality in a sufficiently accurate way and, furthermore, it allows for some interesting closed form solutions of many actuarial measures. Thus, we have chosen to use it in our work. Of course all the main results remain valid for any other functional form chosen for the force of mortality.

The Gompertz-Makeham mortality that we use has the following form:

$$\lambda_t = \phi + \frac{1}{b} e^{\frac{\iota+t-m}{b}}, \tag{5.4.2}$$

where ϕ is a positive constant measuring accidental deaths linked to non-age factors, while m and b are modal and scaling parameters of the distribution, respectively. Function (5.4.2) is defined for $\omega \to \infty$. When either b or m tend to infinity we obtain a constant force of mortality (given by ϕ) and, accordingly, an exponential distribution function.

Function (5.4.2) is able to describe the actual survival probability in a quite precise way (except for child mortality). In particular, in the so-called "pure Gompertz" case (with $\phi = 0$) some calibrations lead to the following values for the parameters: m is 88.18 for male and 92.63 for female, and b is 10.5 for male and 8.78 for females (see, for instance, Milevsky (2006)). In the following sections we will present how to estimate the parameters of (5.4.2) from an online human mortality database.

The exponential of the force of mortality, in this case, is given by

$$\begin{aligned} e^{-\int_t^T \lambda_u du} &= e^{-\phi(T-t) - e^{\frac{\iota-m+T}{b}}\left(1-e^{-\frac{T-t}{b}}\right)} \\ &= e^{-\phi(T-t) - e^{\frac{\iota-m+t}{b}}\left(e^{\frac{T-t}{b}}-1\right)}, \end{aligned} \tag{5.4.3}$$

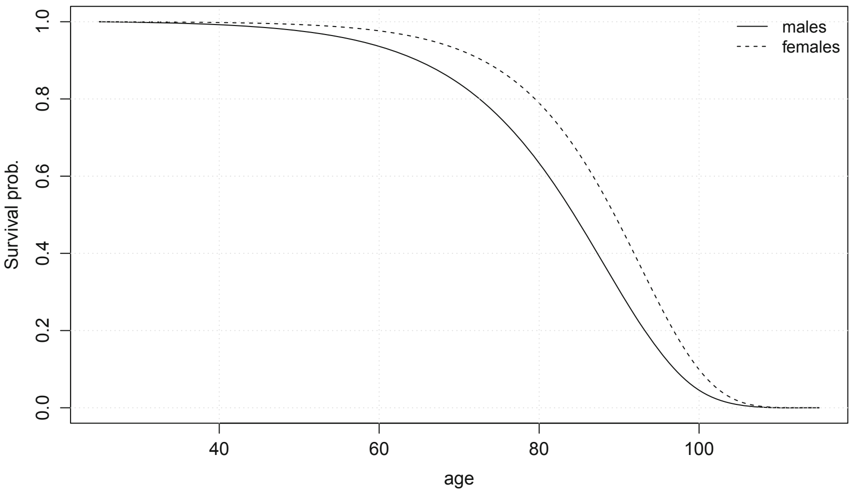
which coincides with the survival probability for $T - t$ periods starting at the age of $\iota + t$. A function which computes the survival probability in this setting can be written in R as follows.

```
GMprob = function(age, t, T, phi, m, b) {
    exp(-phi * (T - t) - exp((age - m + T)/b) * (1 -
        exp(-(T - t)/b)))
}
```

We use this function to plot the survival probability for different initial ages as in Fig. 5.1 for both males (continuous line) and female (dashed line).

Let us assume we have to evaluate an annuity which pays $1dt$ monetary units at each instant in time until the death of an agent under the hypotheses that: (1) the riskless interest rate is constant, and (2) the force of mortality follows (5.4.2). Thus, we have to solve the following integral (recall (5.3.2)):

$$\int_t^\infty e^{-\phi(s-t)-e^{\frac{\iota-m+t}{b}}\left(e^{\frac{s-t}{b}}-1\right)} e^{-r(s-t)} ds.$$




```
years = seq(0, 90, by = 0.5)
plot(25 + years, GMprob(age = 25, t = 0, T = years,
    phi = 0, m = 88.18, b = 10.5), type = "l", ylab = "Survival prob.",
    xlab = "age")
lines(25 + years, GMprob(age = 25, t = 0, T = years,
    phi = 0, m = 92.63, b = 8.78), type = "l", lty = 2)
grid()
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 5.1 Survival probability for males (continuous line) and females (dashed line) aged 25 (according to the pure Gompertz survival function)

Before solving it algebraically, we can use R to compute it numerically through the code that follows, where the command “integrate” is used. This command takes at least three input: (1) the integrand function, defined as an R function, (2) the lower limit, and (3) the upper limit of the integration. The output of the “integrate” command also shows the “absolute error”, which is a measure of the goodness of the result. Here, in order to have the numerical value of the integral as the only output, we use the code “$value”.

```
GMannuity = function(age, t, phi, m, b, r) {
    integrand = function(s) {
        exp(-phi * (s - t) - exp((age - m + t)/b) *
            (exp((s - t)/b) - 1)) * exp(-r * (s - t))
    }
    integrate(integrand, t, Inf)$value
}
GMannuity(age = 25, t = 0, phi = 0, m = 88.18, b = 10.5,
    r = 0.05)
```

```
## [1] 18.51519
```

```
GMannuity(age = 25, t = 0, phi = 0, m = 92.63, b = 8.78,
    r = 0.05)
```

```
## [1] 18.93728
```

The previous integral can also be solved algebraically as shown in the following proposition.

Proposition 5.1 *If the force of mortality follows Eq. (5.4.2) and the riskless interest rate is constant, then the value of a life annuity is*

$$\int_t^{\infty} e^{-\phi(s-t)-e^{\frac{\iota-m+t}{b}}\left(e^{\frac{s-t}{b}}-1\right)} e^{-r(s-t)} ds$$
$$= be^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right),$$

where

$$\Gamma(y_1, y_2) = \int_{y_2}^{\infty} e^{-t} t^{y_1-1} dt,$$

is the so-called incomplete gamma function.

Proof. Let us make the following variable substitution:

$$z = e^{\frac{\iota - m + s}{b}} \Longleftrightarrow s = m - \iota + b \ln z,$$

$$dz = \frac{z}{b} ds \Longleftrightarrow ds = \frac{b}{z} dz,$$

and so the integral can be written as

$$b e^{-(\phi + r)(m - \iota - t) + e^{\frac{\iota - m + t}{b}}} \int_{e^{\frac{\iota - m + t}{b}}}^{\infty} e^{-z} z^{-(\phi + r)b - 1} dz,$$

where the incomplete Gamma function can be immediately seen. □

Now, we can use R to compute the value of the same annuity as before, but this time we use the incomplete gamma function which can be found in the “`gls`” package.

install.packages('gsl')

Remark 5.5 For Linux users, the installation of this package on R needs the package libgsl-dev to be installed on Linux.

Finally, we compare this value with the one obtained through the “integrate” command (in order to do so we just take the “value” of the “integrate” output). The difference between the two values is small indeed.

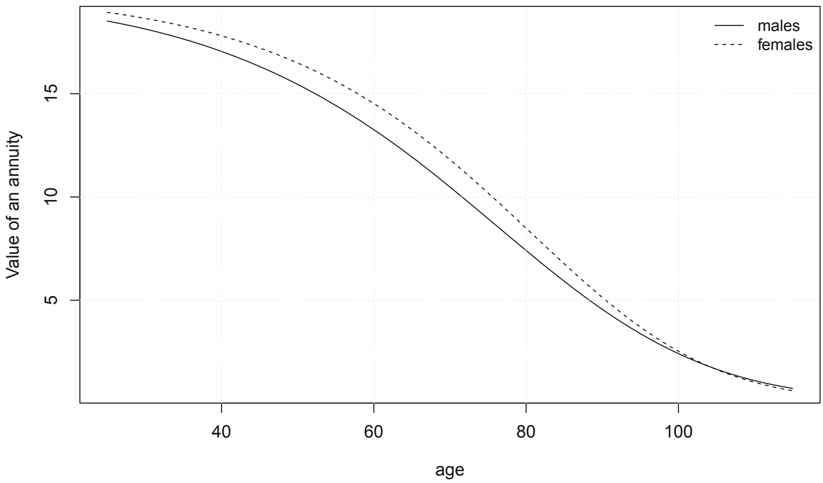
```
library(gsl)
age = 25
t = 0
phi = 0
m = 88.18
b = 10.5
r = 0.05
annuity = b * exp(-(phi + r) * (m - age - t) + exp((age +
    t - m)/b)) * gamma_inc(-(phi + r) * b, exp((age +
    t - m)/b))
annuity - GMannuity(age = 25, t = 0, phi = 0, m = 88.18,
    b = 10.5, r = 0.05)

## [1] -8.668621e-13
```

The value of the annuity is decreasing over time, since the older the agent the lower the expected number of instalments he/she is about to receive. Figure 5.2 shows how the value of the annuity depends on the initial age of the agent (t_0) and on sex. Since females live longer than males, the value of their annuities is of course higher. In the code, note that the arguments of the function "GMannuity" different from the initial age are listed outside the function itself (as arguments of the command "sapply").

5.5 The Human Mortality Database

An important database for human mortality in many countries (HMD) can be found at www.mortality.org where a free registration is needed for downloading the life tables. A direct link between R and this database is provided by a package called




```
years = seq(0, 90, by = 0.5)
plot(25 + years, sapply(years, GMannuity, age = 25,
    phi = 0, m = 88.18, b = 10.5, r = 0.05), type = "l",
    xlab = "age", ylab = "Value of an annuity")
lines(25 + years, sapply(years, GMannuity, age = 25,
    phi = 0, m = 92.63, b = 8.78, r = 0.05), type = "l",
    lty = 2)
grid()
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 5.2 Value of an annuity instantaneously paying $1dt$ Euros for males (continuous line) and females (dashed line) at different ages (according to the pure Gompertz survival function) "demography", that can be installed through the following command:

```
install.packages('demography')
```

> *Remark 5.6* Linux users who have troubles in installing this package, can try to install on their system the package "libcurl4-gnutls-dev" (through the command "sudo apt-get install").

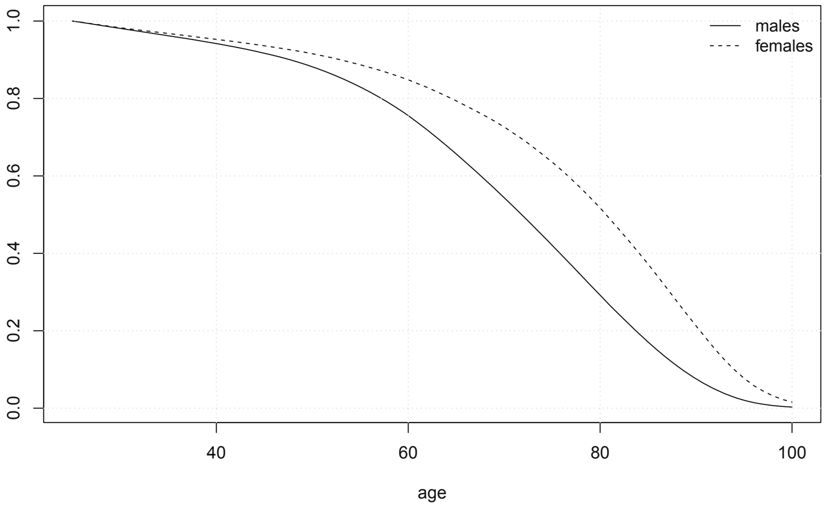
The life table for any available country can be obtained by the command "hmd.mx" whose arguments are: (1) the code of the country, (2) the user name, and (3) the password chosen for signing up. The following commands allow to upload the library in the package "demography" for the USA data on males and females for the cohort of those who are aged 25.

```
library(demography)
tableUSA = hmd.mx("USA", "francesco.menoncin@unibs.it",
    "12345678")
age = 25
USAlife_M = lifetable(tableUSA, type = "cohort", series = "male",
    age = age)
USAlife_F = lifetable(tableUSA, type = "cohort", series = "female",
    age = age)
N = length(USAlife_M$mx[, 1])
```

Furthermore, we can use and plot the number of males and females survived at any age (whose initial value is normalised to 1) as in Fig. 5.3. It is evident that females live longer, since they clearly have a higher survival probability for any age.

The force of mortality for US males and females aged 25 in 1933 can be shown as in Fig. 5.4, where we see that between 25 and 28 the force of mortality is increasing, then it decreases until 30 and, finally, it starts increasing again. This behaviour is due to the high mortality of young males because of accidents and, in general, non-life related events.

Again the difference between the sexes is apparent: the force of mortality of females (dashed line) is much lower than that of males. This difference were already implied in the previous figure, and this is just a confirmation of the same phenomenon.




```
matplot(seq(age, age + N - 1), cbind(USAlife_M$lx[,
    1], USAlife_F$lx[, 1]), type = "l", xlab = "age",
    ylab = "", lty = c(1, 2), col = c(1, 1))
grid()
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 5.3 Number of US males and females (dashed), aged 25 in 1933, who survived at any age [Source: www.mortality.org]

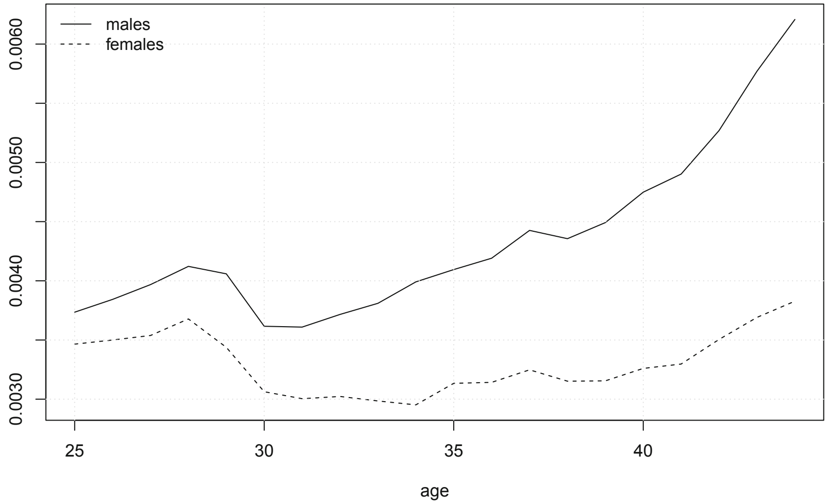
5.6 Estimation of the Gompertz Deterministic Model

We can use the non-linear least square estimator for fitting the parameters of the Gompertz-Makeham mortality (5.4.2) on the data of the HMD. The equation to estimate is

$$\lambda_t = a_1 + a_2 e^{a_3 t} + \varepsilon_t,$$

where λ_t is the time series of the force of mortality and ε_t is the error term (white noise). Once a_1, a_2 and a_3 are estimated, the values of the parameters in (5.4.2) are obtained as

$$\phi = a_1, \qquad b = \frac{1}{a_3}, \qquad m = \iota - b \ln (b a_2)\,.$$




```
matplot(seq(age, age + 20 - 1), cbind(USAlife_M$mx[1:20,
    1], USAlife_F$mx[1:20, 1]), type = "l", ylab = "",
    xlab = "age", lty = c(1, 2), col = c(1, 1))
grid()
legend("topleft", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 5.4 Force of mortality of US males and females (dashed), aged 25 in 1933 [Source: www.mortality.org]

The same procedure must be performed for both males and females. Since the data that have been downloaded correspond to a cohort of agents whose initial age is 25, then in the following command we set $\iota = 25$. The command in R for the non-linear least square is "`nls`", that can be used as follows.

```
phi_GM = c(NA, NA)
b_GM = c(NA, NA)
m_GM = c(NA, NA)
```

```
lambda_M = USAlife_M$mx[, 1]
lambda_F = USAlife_F$mx[, 1]
fit_M = nls(lambda_M ~ a1 + a2 * exp(a3 * seq(0, N -
    1)), start = list(a1 = 0.1, a2 = 0.1, a3 = 0.1))
summary(fit_M)

##
## Formula: lambda_M ~ a1 + a2 * exp(a3 * seq(0, N - 1))
##
## Parameters:
##     Estimate Std. Error t value Pr(>|t|)
## a1 4.196e-03  9.556e-04   4.391 3.75e-05 ***
## a2 7.681e-04  5.336e-05  14.395  < 2e-16 ***
## a3 8.634e-02  9.599e-04  89.945  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.005288 on 73 degrees of freedom
##
## Number of iterations to convergence: 6
## Achieved convergence tolerance: 4.288e-07

phi_GM[1] = coef(fit_M)[1]
b_GM[1] = 1/coef(fit_M)[3]
m_GM[1] = age - b_GM[1] * log(b_GM[1] * coef(fit_M)[2])
c(phi_GM[1], b_GM[1], m_GM[1])

## [1]  0.004195854 11.581891066 79.692121111

fit_F = nls(lambda_F ~ a1 + a2 * exp(a3 * seq(0, N -
    1)), start = list(a1 = 0.1, a2 = 0.1, a3 = 0.1))
summary(fit_F)

##
## Formula: lambda_F ~ a1 + a2 * exp(a3 * seq(0, N - 1))
##
## Parameters:
##     Estimate Std. Error t value Pr(>|t|)
## a1 2.790e-03  9.231e-04   3.022  0.00346 **
## a2 1.818e-04  1.907e-05   9.531  1.9e-14 ***
## a3 1.036e-01  1.449e-03  71.527  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.005662 on 73 degrees of freedom
##
```

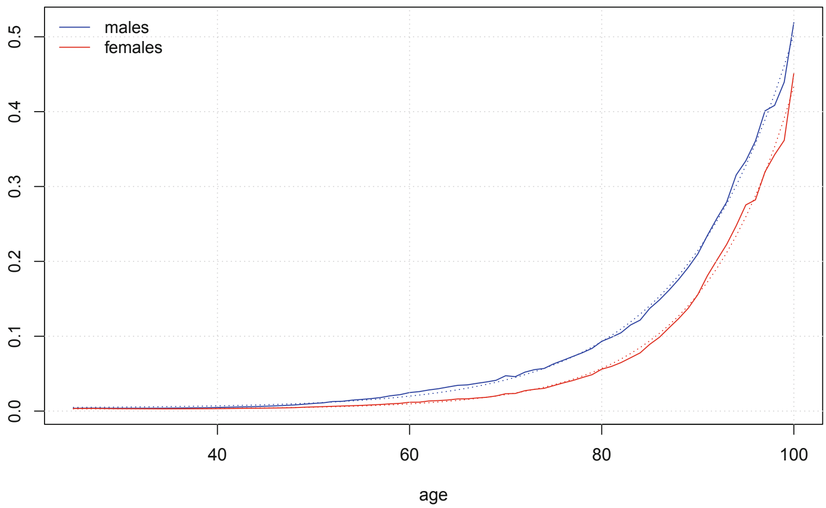
```
## Number of iterations to convergence: 5
## Achieved convergence tolerance: 9.516e-07
```

```
phi_GM[2] = coef(fit_F)[1]
b_GM[2] = 1/coef(fit_F)[3]
m_GM[2] = age - b_GM[2] * log(b_GM[2] * coef(fit_F)[2])
c(phi_GM[2], b_GM[2], m_GM[2])
```

```
## [1]  0.002790043  9.651150629 86.243866299
```

The interpolation looks very good as witnessed by the p-values of the estimated parameters. Furthermore, the values estimated for the parameters are in line with the main findings in this field (see, for instance, Milevsky 2006).

In Fig. 5.5 the comparison between the data and the fitted curve is presented.




```
matplot(seq(age, age + N - 1), cbind(lambda_M, lambda_F),
    type = "l", ylab = "", xlab = "age", col = c("blue",
        "red"), lty = 1)
grid()
matlines(seq(age, age + N - 1), cbind(predict(fit_M,
    list(x = seq(age, age + N - 1))), predict(fit_F,
    list(x = seq(age, age + N - 1)))), col = c("blue",
    "red"), lty = 3)
legend("topleft", legend = c("males", "females"), col = c("blue",
    "red"), lty = 1, bty = "n")
```

Fig. 5.5 Comparison between the force of mortality of US males (blue) and females (red), aged 25 in 1933, and the interpolated Gompertz-Makeham model (dotted) [Source: www.mortality.org]

5.7 A Stochastic Model for the Force of Mortality

The (instantaneous) force of mortality λ_t is assumed to solve a stochastic differential equation of the form we have used for all the other state variables. Nevertheless, in this case, a process which reverts to a constant mean is not suitable. For showing that, let us take into account a constant force of mortality λ. In this case the probability to survive for t periods, from t_0 to $t - t_0$ is

$$\left({}_t p_{t_0}\right) = e^{-\int_{t_0}^{t_0+t} \lambda ds} = e^{-\lambda t},$$

that is the probability to survive for a given length of time is independent of the present time and, thus, of the agent's age. This is not reasonable for agents whose survival probability for the same length should decrease with their age. In other words, the force of mortality should be increasing over time (on average) and diverge for $t \to \infty$. This is, for instance, the case of the Gompertz force of mortality.

Here, we show a stochastic model that actually reverts towards a mean $\gamma(t)$ which is a function of time. To this purpose, we propose the following model

$$d\lambda_t = \alpha \left(\underbrace{\frac{1}{\alpha} \frac{\partial \gamma_t}{\partial t} + \gamma_t}_{\beta_t} - \lambda_t \right) dt + \phi(t, \lambda) dW_t. \tag{5.7.1}$$

If Itô's lemma is applied to the process $Y_t := \lambda_t e^{\alpha(t-t_0)}$, we obtain

$$\begin{aligned} dY_t &= \left(\alpha \lambda_t e^{\alpha(t-t_0)} + \alpha e^{\alpha(t-t_0)} (\beta_t - \lambda_t)\right) dt + e^{\alpha(t-t_0)} \phi(t, \lambda) dW_t \\ &= \alpha e^{\alpha(t-t_0)} \beta_t dt + e^{\alpha t} \phi(t, \lambda) dW_t, \end{aligned}$$

whose expected value is

$$\mathbb{E}_{t_0}[dY_t] = \alpha e^{\alpha(t-t_0)} \beta_t dt,$$

and, after integrating,

$$\mathbb{E}_{t_0}[Y_t] = Y_{t_0} + \int_{t_0}^{t} \alpha e^{\alpha(s-t_0)} \beta_s ds,$$

or,

$$\mathbb{E}_{t_0}\left[\lambda_t e^{\alpha(t-t_0)}\right] = \lambda_{t_0} + \int_{t_0}^{t} \alpha e^{\alpha(s-t_0)} \beta_s ds.$$

Now, because of the form of the function β_t, the integral in the right had side can be simplified as follows:

$$\begin{aligned}\mathbb{E}_{t_0}\left[\lambda_t e^{\alpha(t-t_0)}\right] &= \lambda_{t_0} + \int_{t_0}^{t} e^{\alpha(s-t_0)}\left(\frac{\partial \gamma_s}{\partial s} + \alpha\gamma_s\right) ds \\ &= \lambda_{t_0} + \int_{t_0}^{t} e^{\alpha(s-t_0)}\frac{\partial \gamma_s}{\partial s} ds + \int_{t_0}^{t} \gamma_s \alpha e^{\alpha(s-t_0)} ds.\end{aligned}$$

The first integral can be solved by parts:

$$\begin{aligned}\mathbb{E}_{t_0}\left[\lambda_t e^{\alpha(t-t_0)}\right] &= \lambda_{t_0} + \left[\gamma_s e^{\alpha(s-t_0)}\right]_{s=t_0}^{s=t} - \int_{t_0}^{t} \gamma_s \alpha e^{\alpha(s-t_0)} ds + \int_{t_0}^{t} \gamma_s \alpha e^{\alpha(s-t_0)} ds \\ &= \lambda_{t_0} + \gamma_t e^{\alpha(t-t_0)} - \gamma_{t_0},\end{aligned}$$

and finally

$$\mathbb{E}_{t_0}\left[\lambda_t\right] = \gamma_t + \left(\lambda_{t_0} - \gamma_{t_0}\right) e^{-\alpha(t-t_0)}.$$

If the initial value of the function γ_t coincides with the initial value of λ_t (i.e. $\lambda_{t_0} = \gamma_{t_0}$), then we can conclude that the expected value of λ_t coincides with the value of γ_t for any $t \geq t_0$:

$$\mathbb{E}_{t_0}\left[\lambda_t\right] = \gamma_t.$$

If we want the expected force of mortality to coincide with the Gompertz-Makeham model (5.4.2), then we must set

$$\gamma_t = \phi + \frac{1}{b} e^{\frac{\iota+t-m}{b}}.$$

Furthermore, in order to exploit the properties of the CIR process, we can set

$$\phi\left(t, \lambda_t\right) = \sigma_\lambda \sqrt{\lambda_t},$$

and, finally, the stochastic process (5.7.1) becomes

$$d\lambda_t = \alpha_\lambda \left(\underbrace{\phi + \left(\frac{1}{\alpha_\lambda}\frac{1}{b} + 1\right)\frac{1}{b} e^{\frac{\iota+t-m}{b}}}_{\beta_{\lambda,t}} - \lambda_t \right) dt + \sigma_\lambda \sqrt{\lambda_t} dW_t. \tag{5.7.2}$$

Given the data we have already obtained from the HMD (in the previous section), we can estimate the parameters of this stochastic process through the procedures shown in Sects. 3.5 and 3.6.

In particular, we use the homoscedastic transformation (3.5.2) with $\gamma = \frac{1}{2}$ (i.e. $y_t := 2\left(\sqrt{\lambda_t} - 1\right)$. In this case, the dynamics of variable y_t is

$$dy_t = \left(-\frac{\alpha}{2}y_t + \left(\alpha\beta_t - \frac{1}{4}\sigma^2\right)\frac{2}{y_t}\right)dt + \sigma dW_t.$$

From this equation we obtain the estimation of σ from

$$\mathbb{V}_t\left[dy_t\right] = \sigma^2 dt \Longleftrightarrow \sigma = \sqrt{\frac{\mathbb{V}_t\left[dy_t\right]}{dt}}.$$

The dynamics dy_t can be discretised as follows

$$y_{i+1} = \left(1 - \frac{\alpha dt}{2}\right)y_i + 2dt\left(\alpha\phi - \frac{1}{4}\sigma^2\right)\frac{1}{y_i} + 2dt\left(\frac{1}{b} + \alpha\right)\frac{1}{b}e^{\frac{\iota - m}{b}}\frac{e^{\frac{t}{b}}}{y_i} + \sigma dW_t.$$

The equation whose parameters we are about the estimate can be written as

$$y_{i+1} = \beta_1 y_i + \beta_2\frac{1}{y_i} + \beta_3\frac{e^{\beta_4 t}}{y_i} + \varepsilon_i,$$

where ε_i is the error (homoscedastic) term and

$$\begin{cases} \beta_1 = 1 - \frac{\alpha dt}{2}, \\ \beta_2 = 2dt\left(\alpha\phi - \frac{1}{4}\sigma^2\right), \\ \beta_3 = 2dt\left(\frac{1}{b} + \alpha\right)\frac{1}{b}e^{\frac{\iota - m}{b}}, \\ \beta_4 = \frac{1}{b}, \end{cases} \Longleftrightarrow \begin{cases} \alpha = 2\frac{1-\beta_1}{dt}, \\ \phi = \frac{1}{\alpha}\left(\frac{\beta_2}{2dt} + \frac{1}{4}\sigma^2\right), \\ b = \frac{1}{\beta_4}, \\ m = \iota - b\ln\frac{\beta_3 b}{2dt\left(\frac{1}{b} + \alpha\right)}. \end{cases}$$

For the estimation, we will use the `nls` (non-linear least square) since the variable y_{i+1} is linked with $\frac{1}{y_i}$ and with e^t through non linear functions. This function asks for the initial values of the parameters. For ϕ, b, and m we will use the values estimated from the Gompertz model in the previous section. The value of dt is set to 1 since we have annual data, and the age ι is set to 25.

```
dt = 1
age = 25
sigma_l = c(sd(diff(2 * sqrt(lambda_M)))/sqrt(dt),
    sd(diff(2 * sqrt(lambda_F)))/sqrt(dt))
alpha_l = c(0.1, 0.1)
phi_l = phi_GM
m_l = m_GM
b_l = b_GM
```

The estimation is performed in R through the following commands for both males and females.

```
N = length(lambda_M)
y = 2 * sqrt(lambda_M[2:N])
x1 = 2 * sqrt(lambda_M[1:(N - 1)])
x2 = (2 * sqrt(lambda_M[1:(N - 1)]))^(-1)
x3 = seq(1, N - 1) * dt
longevity = nls(y ~ beta1 * x1 + beta2 * x2 + beta3 *
    exp(beta4 * x3) * x2, start = list(beta1 = 1 -
    alpha_l[1] * dt/2, beta2 = 2 * dt * (alpha_l[1] *
    phi_l[1] - 0.25 * sigma_l[1]^2), beta3 = 2 * dt *
    (1/b_l[1] + alpha_l[1])/b_l[1] * exp((age - m_l[1])/b_l[1]),
    beta4 = 1/b_l[1]))
summary(longevity)

##
## Formula: y ~ beta1 * x1 + beta2 * x2 + beta3 * exp(beta4 * x3) * x2
##
## Parameters:
##       Estimate Std. Error t value Pr(>|t|)
## beta1 0.9267334  0.0392287  23.624   <2e-16 ***
## beta2 0.0008027  0.0004780   1.679   0.0975 .
## beta3 0.0003605  0.0001559   2.312   0.0237 *
## beta4 0.0853385  0.0025569  33.376   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0102 on 71 degrees of freedom
##
## Number of iterations to convergence: 4
## Achieved convergence tolerance: 2.014e-06

alpha_l[1] = 2 * (1 - coef(longevity)[1])/dt
phi_l[1] = (coef(longevity)[2]/(2 * dt) + 0.25 * sigma_l[1]^2)/alpha_l[1]
b_l[1] = 1/coef(longevity)[4]
m_l[1] = age - b_l[1] * log(coef(longevity)[3] * b_l[1]/(2 *
    dt * (1/b_l[1] + alpha_l[1])))
c(alpha_l[1], phi_l[1], b_l[1], m_l[1], sigma_l[1])

## [1]  0.146533298  0.003408856 11.718041787 80.055483426  0.019817450

# Estimation for females
N = length(lambda_F)
y = 2 * sqrt(lambda_F[2:N])
x1 = 2 * sqrt(lambda_F[1:(N - 1)])
x2 = (2 * sqrt(lambda_F[1:(N - 1)]))^(-1)
x3 = seq(1, N - 1) * dt
longevity = nls(y ~ beta1 * x1 + beta2 * x2 + beta3 *
```

```
    exp(beta4 * x3) * x2, start = list(beta1 = 1 -
    alpha_l[2] * dt/2, beta2 = 2 * dt * (alpha_l[2] *
    phi_l[2] - 0.25 * sigma_l[2]^2), beta3 = 2 * dt *
    (1/b_l[2] + alpha_l[2])/b_l[2] * exp((age - m_l[2])/b_l[2]),
    beta4 = 1/b_l[2]))
summary(longevity)

##
## Formula: y ~ beta1 * x1 + beta2 * x2 + beta3 * exp(beta4 * x3) * x2
##
## Parameters:
##        Estimate Std. Error t value Pr(>|t|)
## beta1 8.303e-01  4.921e-02  16.871  < 2e-16 ***
## beta2 2.042e-03  6.238e-04   3.273 0.001643 **
## beta3 1.325e-04  3.618e-05   3.663 0.000477 ***
## beta4 1.051e-01  1.772e-03  59.307  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01087 on 71 degrees of freedom
##
## Number of iterations to convergence: 4
## Achieved convergence tolerance: 1.258e-06

alpha_l[2] = 2 * (1 - coef(longevity)[1])/dt
phi_l[2] = (coef(longevity)[2]/(2 * dt) + 0.25 * sigma_l[2]^2)/alpha_l[2]
b_l[2] = 1/coef(longevity)[4]
m_l[2] = age - b_l[2] * log(coef(longevity)[3] * b_l[2]/(2 *
    dt * (1/b_l[2] + alpha_l[2])))
c(alpha_l[2], phi_l[2], b_l[2], m_l[2], sigma_l[2])

## [1]  0.339406305  0.003380312  9.513472650 87.393601332  0.022483728
```

All the parameters that have been estimated are gathered in Table 5.3.

Finally, we can simulate such a process with a "`for`" cycle analogous to that already used for the CKLS function. In the case of males, if we want to create 10 possible paths of the λ_t that solves (5.7.2), we can give the following commands.

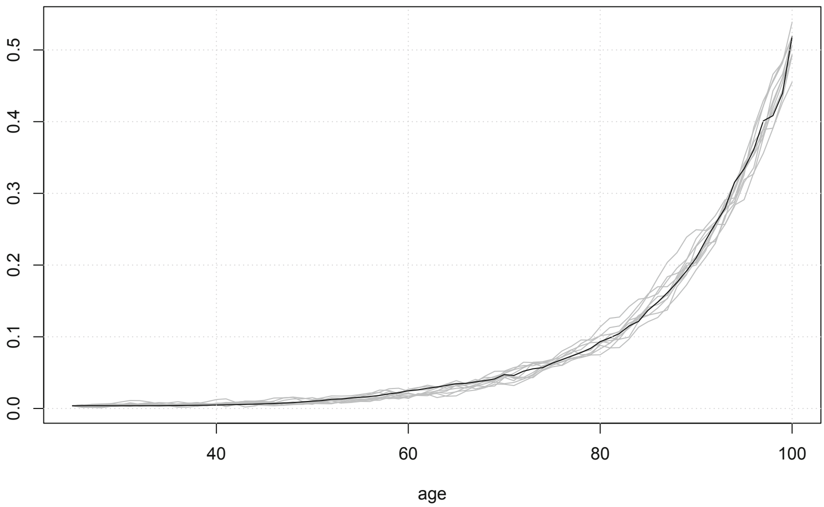
Table 5.3 Parameter estimations of both the Gompertz (5.4.2) and its stochastic version (5.7.2) on the HMD

Parameters	Models			
	Gompertz (5.4.2)		Stochastic Gompertz (5.7.2)	
	Males	Females	Males	Females
ϕ	0.0041959	0.0027900	0.0034089	0.0033803
b	11.5818911	9.6511506	11.7180419	9.5134727
m	79.6921211	86.2438664	80.0554833	87.3936013
α_λ	–	–	0.1465334	0.3394063
σ_λ	–	–	0.0198175	0.0224837

```
M = 10
x = array(0, dim = c(N, M))
x[1, ] = rep(lambda_M[1], M)
dt = 1
for (i in 2:N) {
    dx = alpha_l[1] * (phi_l[1] + (1/alpha_l[1]/b_l[1] +
        1)/b_l[1] * exp((age + i - 1 - m_l[1])/b_l[1]) -
        x[i - 1, ]) * dt + sigma_l[1] * sqrt(x[i -


        1, ]) * rnorm(M) * sqrt(dt)
    # dx[which(x[i-1,]+dx<0)]=0
    x[i, ] = x[i - 1, ] + dx
}
```

Then, the simulated paths can be compared with the actual behaviour of the force of mortality, like in Fig. 5.6.



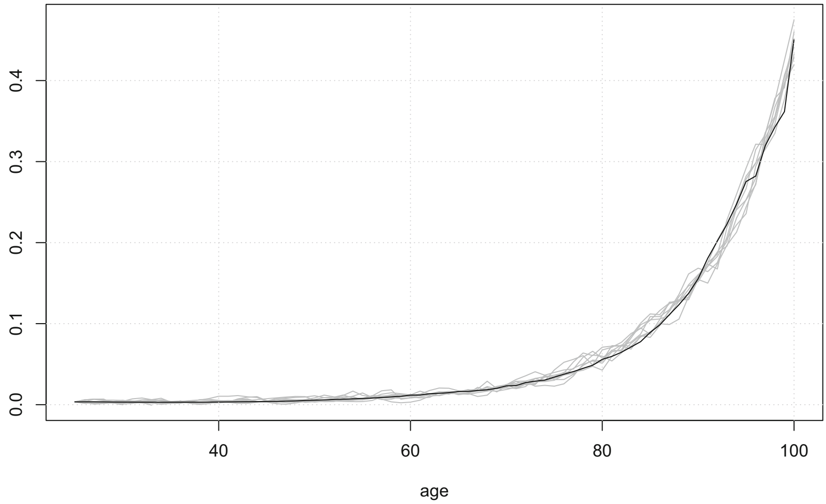

```
matplot(seq(age, age + N - 1), x, type = "l", col = "gray",
    lty = 1, xlab = "age", ylab = "")
lines(seq(age, age + N - 1), lambda_M, type = "l")
grid()
```

Fig. 5.6 Simulation of 10 trajectories of the stochastic process (5.7.2)—in grey—compared with the actual behaviour of the force of mortality for US males aged 25 in 1933—in black

The same commands can be written for the case of females.

```
M = 10
x = array(0, dim = c(N, M))
x[1, ] = rep(lambda_F[1], M)
dt = 1
for (i in 2:N) {
    dx = alpha_l[2] * (phi_l[2] + (1/alpha_l[2]/b_l[2] +
        1)/b_l[2] * exp((age + i - 1 - m_l[2])/b_l[2]) -
        x[i - 1, ]) * dt + sigma_l[2] * sqrt(x[i -
        1, ]) * rnorm(M) * sqrt(dt)
    # dx[which(x[i-1,]+dx<0)]=0
    x[i, ] = x[i - 1, ] + dx
}
```

Finally, the simulations for the females is shown in Fig. 5.7.




```
matplot(seq(age, age + N - 1), x, type = "l", col = "gray",
    lty = 1, xlab = "age", ylab = "")
lines(seq(age, age + N - 1), lambda_F, type = "l")
grid()
```

Fig. 5.7 Simulation of 10 trajectories of the stochastic process (5.7.2)—in grey—compared with the actual behaviour of the force of mortality for US females aged 25 in 1933—in black

5.8 A Stochastic Model for the Survival Probability

Given the stochastic process for λ_t in (5.7.2), we can compute the survival probability in closed form as we have already done for the value of a ZCB. In particular, we obtain

$$\left({}_{T-t}p_t\right) = \mathbb{E}_t^{\tau}\left[\mathbb{I}_{\tau<T}\right] = \mathbb{E}_t\left[e^{-\int_t^T \lambda_u du}\right] = e^{-A(t,T)-C(t,T)\lambda_t}, \tag{5.8.1}$$

where, after Proposition 4.4,

$$k := \sqrt{\alpha_\lambda^2 + 2\sigma_\lambda^2},$$

$$C\left(t,T\right) = 2\frac{1-e^{-k(T-t)}}{k+\alpha_\lambda+\left(k-\alpha_\lambda\right)e^{-k(T-t)}},$$

$$A\left(t,T\right) = \alpha_\lambda \int_t^T \beta_{\lambda,s} C\left(s,T\right) ds.$$

Now, if we apply Itô's lemma to (5.8.1), we obtain

$$\frac{d\left({}_{T-t}p_t\right)}{\left({}_{T-t}p_t\right)} = \lambda_t dt - C\left(t,T\right)\sigma_\lambda\sqrt{\lambda_t}dW_t.$$

Given the process (5.7.1), the integral $\int_t^T \beta_{\lambda,s} C\left(s,T\right) ds$ does not have a closed form solution, but it can be solved numerically.

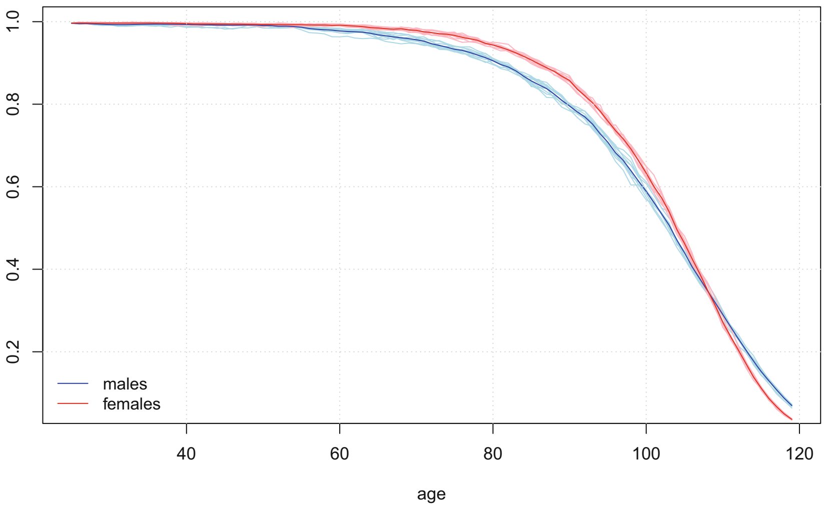
The following code allows us to compute the survival probability, given a value of $\lambda\left(t\right)$.

```
Prob = function(t, T, l, age, sex) {
    C = function(t, T) {
        k = sqrt(alpha_l[sex]^2 + 2 * sigma_l[sex]^2)
        2 * (1 - exp(-(T - t) * k))/(k + alpha_l[sex] +
            (k - alpha_l[sex]) * exp(-(T - t) * k))
    }
    beta = function(s) {
        phi_l[sex] + (1/(alpha_l[sex] * b_l[sex]) +
            1)/b_l[sex] * exp((age + s - m_l[sex])/b_l[sex])
    }
    INT = function(s) {
        beta(s) * C(t, s)
    }
    exp(-alpha_l[sex] * integrate(INT, t, T)$value -
        C(t, T) * l)
}
```

With this function we can now simulate the value of λ_t and compute the corresponding survival probability. In particular, we want to show the behaviour of the survival probability for 1 year, when the agent becomes older and older. Algebraically, in the previous function we set $t = 0$, $T = 1$, and the age will go from 25 to 120. The commands for simulating these values for males and females are as follows.

```
age = 25
M = 10
N = 120 - age
x_m = array(0, dim = c(N, M))
x_f = array(0, dim = c(N, M))
P_m = array(0, dim = c(N, M))
P_f = array(0, dim = c(N, M))
x_m[1, ] = rep(lambda_M[1], M)
x_f[1, ] = rep(lambda_F[1], M)
P_m[1, ] = rep(Prob(t = 0, T = 1, l = lambda_M[1],
    age = age, sex = 1), M)
P_f[1, ] = rep(Prob(t = 0, T = 1, l = lambda_F[1],
    age = age, sex = 2), M)
dt = 1
for (i in 2:N) {
    dx_m = alpha_l[1] * (phi_l[1] + (1/alpha_l[1]/b_l[1] +
        1)/b_l[1] * exp((age + i - 1 - m_l[1])/b_l[1]) -
        x_m[i - 1, ]) * dt + sigma_l[1] * sqrt(x_m[i -
        1, ]) * rnorm(M) * sqrt(dt)
    dx_f = alpha_l[2] * (phi_l[2] + (1/alpha_l[2]/b_l[2] +
        1)/b_l[2] * exp((age + i - 1 - m_l[2])/b_l[2]) -
        x_f[i - 1, ]) * dt + sigma_l[2] * sqrt(x_f[i -
        1, ]) * rnorm(M) * sqrt(dt)
    x_m[i, ] = x_m[i - 1, ] + dx_m
    x_f[i, ] = x_f[i - 1, ] + dx_f
    P_m[i, ] = Prob(t = 0, T = 1, l = x_m[i, ], age = 24 +
        i, sex = 1)
    P_f[i, ] = Prob(t = 0, T = 1, l = x_f[i, ], age = 24 +
        i, sex = 2)
}
```

Finally, we can show the results of this simulations with some graphs. In Fig. 5.8 we simulate 10 paths of the probability that we have just calculated, for both males and females. Inside the command "`rowMeans`" we specify the option "`na.rm=T`" so that R will neglect any `NA` values if present.




```
matplot(seq(age, age + N - 1), P_m, type = "l", col = "lightblue",
    lty = 1, xlab = "age", ylab = "")
matlines(seq(age, age + N - 1), P_f, type = "l", col = "lightpink",
    lty = 1, xlab = "age", ylab = "")
lines(seq(age, age + N - 1), rowMeans(P_m, na.rm = T),
    type = "l", col = "blue")
lines(seq(age, age + N - 1), rowMeans(P_f, na.rm = T),
    type = "l", col = "red")
legend("bottomleft", legend = c("males", "females"),
    lty = 1, col = c("blue", "red"), bty = "n")
grid()
```

Fig. 5.8 Simulations of 10 paths of the survival probability for 1 year ahead, starting from the age of 25, for both males (in blue) and females (in red). The process for λ_t is (5.7.1) where the parameters have been estimated from the MHD on US population aged 25

5.9 The Evolution of Wealth Subject to Actuarial Risk

Let us assume that an economic agent, like a pension fund, invests an initial wealth R_{t_0} and, because of this investment, receives some periodic cash flows k_t until his death time τ, when it receives back a percentage (q_τ) of the final wealth R_τ. Since the price of any asset on the financial market can be computed as presented in Theorem 4.3, also in this case we can write

$$R_{t_0} = \mathbb{E}^{\mathbb{Q},\tau}_{t_0,t_0}\left[\int_{t_0}^{\omega} \mathbb{I}_{s<\tau} k_s e^{-\int_{t_0}^{s} r_u du} ds + q_\tau R_\tau e^{-\int_{t_0}^{\tau} r_u du}\right], \tag{5.9.1}$$

which is simplified as follows

$$R_{t_0} = \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\int_{t_0}^{\omega} k_s e^{-\int_{t_0}^{s}(r_u+\lambda_u)du} ds + \int_{t_0}^{\omega} \lambda_s q_s R_s e^{-\int_{t_0}^{s}(r_u+\lambda_u)du} ds\right]$$
$$= \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\int_{t_0}^{\omega} (k_s + \lambda_s q_s R_s)\, e^{-\int_{t_0}^{s}(r_u+\lambda_u)du} ds\right].$$

Given the relationship between (3.3.1) and (3.3.2) already presented in Sect. 3.3, this value of R_{t_0} can be associated with the following differential equation

$$\begin{aligned} dR_t &= (R_t\,(r_t+\lambda_t) - (k_t + \lambda_t q_t R_t))\,dt + \sigma^{\intercal}_{R,t} dW^{\mathbb{Q}}_t \\ &= (R_t\,(r_t + (1-q_t)\,\lambda_t) - k_t)\,dt + \sigma^{\intercal}_{R,t} dW^{\mathbb{Q}}_t, \end{aligned} \tag{5.9.2}$$

where we have indicated with $\sigma_{R,t}$ a generic diffusion term for wealth. Now, we can see that, according to the value of q_t, two interesting cases can be discussed. In what follows we show the cases with $q_t = 1$ and $q_t = 0$ even if, of course, all the intermediate cases $q_t \in [0, 1]$ are possible.

- With $q_t = 1$, the agent (or his/her heirs) receives the whole amount of remaining wealth at the death time τ and, accordingly, the force of mortality λ_t does not affect the wealth dynamics dR_t. In fact, independently of the death time, the whole wealth is paid back to the investor.
- With $q_t = 0$, the agent does not receive any amount of wealth at the death time τ and, accordingly, the force of mortality λ_t affects the wealth dynamics dR_t. In fact, at any instant in time, the entire wealth could be "erased" by death.

Girsanov's theorem allows us to rewrite the dynamics of wealth under the historical probability:

$$\begin{aligned} dR_t &= (R_t\,(r_t + (1-q_t)\,\lambda_t) - k_t)\,dt + \sigma^{\intercal}_{R,t}\,(dW_t + \xi_t dt) \\ &= \left(R_t\,(r_t + (1-q_t)\,\lambda_t) + \sigma^{\intercal}_{R,t}\xi_t - k_t\right) dt + \sigma^{\intercal}_{R,t} dW_t. \end{aligned}$$

In the framework of a pension fund, the parameter q_t may measure the percentage of the final wealth that is paid back to the heirs of the pensioner.

Note that if we invert the previous process, and we go from the differential equation to the expected value representation of its solution (the so-called Feynman-Kač theorem—Øksendal 1998; Duffie 2001), we can write the initial wealth R_{t_0} also in the following way

$$R_{t_0} = \mathbb{E}^{\mathbb{Q}}_{t_0}\left[\int_{t_0}^{\omega} k_s e^{-\int_{t_0}^{s}(r_u+(1-q_u)\lambda_u)du} ds\right],$$

where we see that the mortality risk, represented by λ_t, is reduced by the percentage q_t. In fact, if the agent's heir is reimbursed by this percentage, the agent's wealth is not fully lost and it suffers a smaller level of risk.

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Chapter 6
Financial-Actuarial Assets

6.1 Introduction

In this chapter we show how to price a derivative on human life by using the tools already developed in the previous chapter. In particular, we show three cases: (1) the longevity bond, (2) the out of date Tontine, and (3) the death bond. These particular assets are just examples of a design that could be used to create many other actuarial derivatives whose underlying is the force of mortality. From a theoretical point of view, such assets are very useful for completing the financial market. In fact, it is very difficult to find pure financial assets that have a correlation with the force of mortality which is sufficiently high to make these assets suitable for hedging purposes.

In the following chapters we will use actuarial assets for completing the financial market on which a pension fund can invest its wealth, and we will show numerical simulations for understanding the dynamics of the optimal share of these assets in the fund's portfolio.

6.2 Derivatives on Human Life

Derivatives on human life must be interpreted as bonds whose coupons (or final payments) are linked to some demographic measure. Accordingly, the fundamental theorem of asset pricing allows us to write the value of a demographic asset D_t paying coupons $\delta(t, \lambda_t)$ as

$$D(t, r_t, \lambda_t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \delta(s, \lambda_s)\, e^{-\int_t^s r_u du} ds\right]. \tag{6.2.1}$$

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_6

If both r_t and λ_t are stochastic, then on this asset there are two main risk sources: the stochastic changes in the interest rate and in the force of mortality. Let us assume that r_t and λ_t solve

$$dr_t = \mu_{r,t} dt + \sigma_{r,t}^{\mathsf{T}} dW_t,$$
$$d\lambda_t = \mu_{\lambda,t} dt + \sigma_{\lambda,t}^{\mathsf{T}} dW_t,$$

with k risk sources. The dynamic behaviour of $D(t, r_t, \lambda_t)$ can be obtained through Itô's lemma on (6.2.1) as follows

$$dD(t, r_t, \lambda_t) = (D(t, r_t, \lambda_t) r_t - \delta(t, \lambda_t)) dt + \frac{\partial D(t, r_t, \lambda_t)}{\partial r_t} \sigma_{r,t}^{\mathsf{T}} dW_t^{\mathbb{Q}} + \frac{\partial D(t, r_t, \lambda_t)}{\partial \lambda_t} \sigma_{\lambda,t}^{\mathsf{T}} dW_t^{\mathbb{Q}},$$

or, alternatively

$$\begin{aligned}\frac{dD(t, r_t, \lambda_t)}{D(t, r_t, \lambda_t)} &= \left(r_t - \frac{\delta(t, \lambda_t)}{D(t, r_t, \lambda_t)} \right) dt \\ &\quad + \frac{\partial D(t, r_t, \lambda_t)}{\partial r_t} \frac{1}{D(t, r_t, \lambda_t)} \sigma_{r,t}^{\mathsf{T}} dW_t^{\mathbb{Q}} \\ &\quad + \frac{\partial D(t, r_t, \lambda_t)}{\partial \lambda_t} \frac{1}{D(t, r_t, \lambda_t)} \sigma_{\lambda,t}^{\mathsf{T}} dW_t^{\mathbb{Q}},\end{aligned}$$

where we have omitted the drift terms since they are not relevant to our purposes.

If we call

$$\nabla_{Y,X} := \frac{\partial Y}{\partial X} \frac{1}{Y},$$

the **semi-elasticity** of Y with respect to X,[1] then we can write dD_t as

$$\begin{aligned}\frac{dD(t, r_t, \lambda_t)}{D(t, r_t, \lambda_t)} &= \left(r_t - \frac{\delta(t, \lambda_t)}{D(t, r_t, \lambda_t)} \right) dt \\ &\quad + \nabla_{D,r} \sigma_{r,t}^{\mathsf{T}} dW_t^{\mathbb{Q}} + \nabla_{D,\lambda} \sigma_{\lambda,t}^{\mathsf{T}} dW_t^{\mathbb{Q}},\end{aligned}$$

[1] We recall that, instead, the elasticity of Y with respect to X is given by $\eta_{Y,X} := \frac{\partial Y}{\partial X} \frac{X}{Y}$.

where $\nabla_{D,r}$ is a measure of the reaction of the price $D(t, r_t, \lambda_t)$ to the changes in the interest rate, and $\nabla_{D,\lambda}$ measures the reaction of $D(t, r_t, \lambda_t)$ to the changes in the force of mortality. In the financial literature, the opposite of the variable $\nabla_{D,r}$ is called **duration**.[2]

Now, thanks to Proposition 4.2, we can specify the drift as

$$\frac{dD(t, r_t, \lambda_t)}{D(t, r_t, \lambda_t)} = \left(r_t - \frac{\delta_t}{D(t, r_t, \lambda_t)} + \nabla_{D,r}\sigma_{r,t}^{\mathsf{T}}\xi_t + \nabla_{D,\lambda}\sigma_{\lambda,t}^{\mathsf{T}}\xi_t\right) dt$$
$$+ \nabla_{D,r}\sigma_{r,t}^{\mathsf{T}} dW_t + D_{D,\lambda}\sigma_{\lambda,t}^{\mathsf{T}} dW_t,$$

where ξ_t is the vector of the market prices for risk.

If the force of mortality is not stochastic (i.e. $\sigma_{\lambda,t} = \mathbf{0}$), then the demographic asset becomes exactly as any other bond on the financial market:

$$\left.\frac{dD(t, r_t, \lambda_t)}{D(t, r_t, \lambda_t)}\right|_{\sigma_{\lambda,t}=\mathbf{0}} = \left(r_t - \frac{\delta_t}{D(t, r_t, \lambda_t)} + \nabla_{D,r}\sigma_{r,t}^{\mathsf{T}}\xi_t\right) dt + \nabla_{D,r}\sigma_{r,t}^{\mathsf{T}} dW_t.$$

As it is easy to check, this differential equation is not different with respect to that of a coupon bond (whose coupons may depend on λ_t which is no more a stochastic variable).

We have already argued that the longevity risk is measured by the stochasticity of λ_t. Thus, a model where λ_t is deterministic implies the absence of any longevity risk. In other words, the presence of the longevity risk implies the need for hedging against one more risk source (that which drives λ_t).

[2] The reason for this name is clear if we take into account a deterministic case. The value of a bond is

$$V_t = \int_t^T \delta_s e^{-r(s-t)} ds,$$

and its semi-elasticity with respect to r is

$$\frac{dV_t}{dr}\frac{1}{V_t} = -\frac{\int_t^T \delta_s (t-s) e^{-r(s-t)} ds}{\int_t^T \delta_s e^{-r(s-t)} ds},$$

whose opposite is the weighted mean of the time to maturity $(t-s)$ of the coupons, where the weights are the discounted values of the coupons. Accordingly, the opposite of the semi-elasticity is a "weighted average duration" of the bond.

6.3 Longevity Bond

A longevity bond is a bond whose coupons are strictly proportional to the survival rate of a given population.

$$L_B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{({}_s p_0)}{({}_t p_0)} e^{-\int_t^s r_u du} ds\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r_u+\lambda_u) du} ds\right], \tag{6.3.1}$$

where we note that the ratio $({}_s p_0) / ({}_t p_0)$ is the number of agents who survive from t to s, i.e. $({}_s p_t)$.

If we apply the method shown in Sect. 5.3, we can rewrite the price $L_B(t,T)$ in the following way

$$L_B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] e^{-\int_t^s r_u du} ds\right]$$
$$= \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[\int_t^T \mathbb{I}_{s<\tau} e^{-\int_t^s r_u du} ds\right].$$

From this version of $L_B(t,T)$, we understand that it can be interpreted like an insurance contract that pays a coupon of 1 Euro until the maturity T if a reference agent is still alive. This reference agent may also be the representative member of a reference population.

In November 2004 BNP-Paribas made an attempt to issue such a bond but it was not appreciated by the market. One of the reasons was that it was written on the survival rate of a population of Welsh males aged 60. The European pension funds, which were the main potential buyers of such an asset, were not interested in the Welsh population because they would have been exposed to a too high basis risk.

If λ_t and r_t are independent then we can write

$$L_B(t,T) = \int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds$$
$$= \int_t^T B(t,s)\,({}_s p_t)^{\mathbb{Q}}\, ds,$$

where $({}_s p_t)^{\mathbb{Q}}$ stands for the survival probability from t to s, computed under the probability $\mathbb{Q}$.

From Eq. (6.3.1) we can see that a longevity bond can be interpreted as the sum of many zero-coupon longevity bonds which pay 1 at any instant in time:

$$L_B(t,T) = \int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s (r_u+\lambda_u)du}\right] ds := \int_t^T L(t,s)\,ds.$$

The zero-coupon longevity bond $L(t,T)$ has the following differential

$$\frac{dL(t,T)}{L(t,T)} = \left(\underbrace{r_t+\lambda_t}_{\text{discount rate}} + \nabla_{L,r}\sigma_{r,t}^{\intercal}\xi_t + \nabla_{L,\lambda}\sigma_{\lambda,t}^{\intercal}\xi_t\right)dt$$
$$+ \nabla_{L,r}\sigma_{r,t}^{\intercal}dW_t + \nabla_{L,\lambda}\sigma_{\lambda,t}^{\intercal}dW_t.$$

6.4 The Tontine

A Tontine[3] is a financial-actuarial asset named after the Neapolitan banker Lorenzo de Tonti, who invented it in France in 1653 for the prime minister Cardinal Mazarino. The financial structure of the Tontine is as follows: the Government pays each period a given amount of money which is divided among all the holders. If some holders die, their shares are redistributed to the survivors. This process continues until only one investor survives (or the maturity is reached). While once very popular in France, Britain, and the United States, Tontines have been banned in Britain and the United States due to the incentive for subscribers to kill one another.

The value of a Tontine for the holder (agent i) is given by

$$\Theta_i(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{1}{({}_s p_0)} e^{-\int_t^s r_u du} ds\right],$$

and since there are $({}_t p_0)$ surviving agents on the market at time t, then the total value of the Tontine is

$$\Theta_B(t,T) = ({}_t p_0)\,\Theta_i(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{({}_t p_0)}{({}_s p_0)} e^{-\int_t^s r_u du} ds\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{\int_t^s \lambda_u du} e^{-\int_t^s r_u du} ds\right].$$

Note that the ratio $({}_t p_0)/({}_s p_0)$ measures the inverse of the number of surviving agents between t and s. With respect to the force of mortality, we can immediately see that the Tontine and the longevity bond behave in the opposite way. When the

[3] The role of such an asset in optimal portfolio is studied in Menoncin (2008).

force of mortality increases, the value of the Tontine increases while the value of the longevity bond decreases.

If we take into account the zero-coupon bond Tontine

$$\Theta(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_u-\lambda_u)du}\right],$$

then the dynamics of $\Theta(t,T)$ is

$$\begin{aligned}\frac{d\Theta(t,T)}{\Theta(t,T)} = &\left(r_t - \lambda_t + \nabla_{\Theta,\lambda}\sigma_{\lambda,t}^{\mathsf{T}}\xi_t + \nabla_{\Theta,r}\sigma_{r,t}^{\mathsf{T}}\xi_t\right)dt \\ &+ \nabla_{\Theta,\lambda}\sigma_{\lambda,t}^{\mathsf{T}}dW_t + \nabla_{\Theta,r}\sigma_{r,t}^{\mathsf{T}}dW_t,\end{aligned}$$

where λ_t in the drift has the opposite sign with respect to the longevity bond. Since the longevity bond and the Tontine react in the opposite way to the changes in the force of mortality, then if they have the same time to maturity $(T-t)$, it must be verified that

$$\nabla_{\Theta,\lambda} = -\nabla_{L,\lambda}.$$

It is easy to demonstrate that on a market where there are: (1) an ordinary bond, (2) a longevity bond, and (3) a Tontine, one of the three assets is redundant. For instance, if we hold θ_L longevity bonds and θ_B ordinary bonds, the return of such a portfolio is

$$\frac{d\tilde{R}_t}{\tilde{R}_t} = \theta_{L,t}\frac{dL(t,T)}{L(t,T)} + \theta_{B,t}\frac{dB(t,T)}{B(t,T)},$$

or

$$\begin{aligned}\frac{d\tilde{R}_t}{\tilde{R}_t} =&\theta_{L,t}\left(r_t + \lambda_t + \nabla_{L,\lambda}\sigma_{\lambda,t}^{\mathsf{T}}\xi_t + \nabla_{L,r}\sigma_{r,t}^{\mathsf{T}}\xi_t\right)dt \\ &+ \theta_{L,t}\nabla_{L,\lambda}\sigma_{\lambda,t}^{\mathsf{T}}dW_t + \theta_{L,t}\nabla_{L,r}\sigma_{r,t}^{\mathsf{T}}dW_t \\ &+ \theta_{B,t}\left(r_t + \nabla_{B,r}\sigma_{r,t}^{\mathsf{T}}\xi_t\right)dt + \theta_{B,t}\nabla_{B,r}\sigma_{r,t}^{\mathsf{T}}dW_t.\end{aligned}$$

In order to guarantee that this portfolio replicates the Tontine, we must choose $\theta_{L,t}$ and $\theta_{B,t}$ such that the diffusion terms of $\frac{d\tilde{R}_t}{\tilde{R}_t}$ and $\frac{d\Theta(t,T)}{\Theta(t,T)}$ are identical:

$$\theta_{L,t}\nabla_{L,\lambda}\sigma_{\lambda,t} + \theta_{L,t}\nabla_{L,r}\sigma_{r,t} + \theta_{B,t}\nabla_{B,r}\sigma_{r,t} = \nabla_{\Theta,\lambda}\sigma_{\lambda,t} + \nabla_{\Theta,r}\sigma_{r,t}.$$

If we collect the terms containing $\sigma_{r,t}$ and those containing $\sigma_{\lambda,t}$:

$$\left(\theta_{L,t}\nabla_{L,\lambda} - \nabla_{\Theta,\lambda}\right)\sigma_{\lambda,t} + \left(\theta_{L,t}\nabla_{L,r} + \theta_{B,t}\nabla_{B,r} - \nabla_{\Theta,r}\right)\sigma_{r,t} = 0,$$

which is zero if and only if

$$\begin{cases} \theta_{L,t} \nabla_{L,\lambda} - \nabla_{\Theta,\lambda} = 0, \\ \theta_{L,t} \nabla_{L,r} + \theta_{B,t} \nabla_{B,r} - \nabla_{\Theta,r} = 0, \end{cases}$$

whose solution is

$$\theta_{L,t} = \frac{\nabla_{\Theta,\lambda}}{\nabla_{L,\lambda}},$$
$$\theta_{B,t} = \frac{\nabla_{\Theta,r}}{\nabla_{B,r}} - \frac{\nabla_{\Theta,\lambda}}{\nabla_{L,\lambda}} \frac{\nabla_{L,r}}{\nabla_{B,r}}.$$

If the force of mortality and the interest rate are independent and all the bonds have the same time to maturity, then we know that $L(t,T)$, $B(t,T)$, and $\Theta(t,T)$ must all react in the same way to the changes in the interest rate, i.e.

$$\nabla_{\Theta,r} = \nabla_{B,r} = \nabla_{L,r},$$

and since we know that $\nabla_{\Theta,\lambda} = -\nabla_{L,\lambda}$, then the previous result becomes

$$\theta_{L,t} = -1,$$
$$\theta_{B,t} = 2,$$

i.e. in order to replicate a Tontine it is sufficient to buy two ordinary bonds and short sell one longevity bond (all having the same maturity).

It is straightforward to show that this portfolio composition also makes portfolio return equal to the return on the Tontine (and this is true because the drift terms has been obtained through a non arbitrage condition). Thus, if Tontines are put out of law, then also short selling longevity bonds should be illegal. Nevertheless, since the issuance of a longevity bond coincides with a short position on it, we can conclude that preventing the existence of a particular asset on a complete financial market is a quite challenging task.

6.5 Death Bond

A death bond is like any other *Asset Backed Security* (ABS). The process for changing a death insurance into a death bond is made by four steps (as in Fig. 6.1).

Let us see such steps in details.

1. The so-called seller is the subscriber of the death insurance. When the agent grows older (typically 70) and he does not have any further need for the insurance on his life, he may want to cash out his policy.

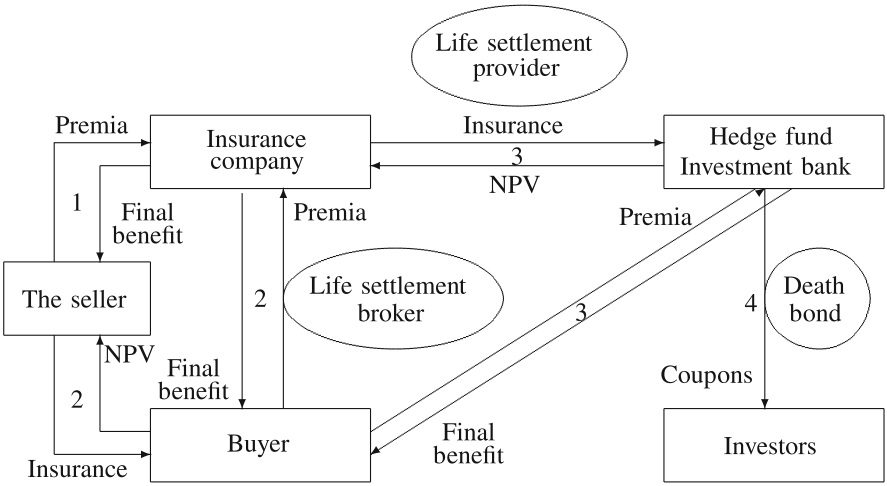

Fig. 6.1 How a death insurance becomes a death bond

2. The seller hires a *life settlement* broker who will find a buyer for his policy. The buyer pays the net present value of the policy (that we have called D_t in the previous section) and receives the insurance policy. Thus, the buyer will continue paying the settlements to the insurance company. The buyer will also receive the final benefit from the insurance company when the seller dies. The up-front payout to the seller varies widely, from 20% of the death benefit to 40%. The seller pays to the broker a commission ranged from 5 to 6%.
3. Another character in this game is the so-called *life settlement provider*. Through him, a hedge fund or an investment bank buys a pool of death insurances from insurance company (or insurance companies). Now, the hedge fund will receive the premia from the buyer and will pay the final benefit.
4. In the last step, after a sufficient number of policies has been collected, these policies can back the emission of a death bond. Accordingly, the policies play the same role as the assets in an ABS or the mortgages in a mortgage backed security. We say that the new death bond is a **pass through** asset when the premia received by the hedge fund are directly paid to the investors without any guarantee from the hedge fund (i.e. if the hedge fund does not receive any payment, then it does not pay any coupon). Instead, the death bond is **pay through** when the hedge fund guarantees the payments of the coupons.

The death bond is based upon death insurances. Thus, let us show how a death insurance works. With a death insurance contract, the subscriber agrees to pay settlements (δ) during his life time in order to receive, at his death time, a given amount of money (final benefit). For the sake of simplicity we will set such amount to 1.

Let us assume the death insurance is subscribed at time t_0, then the actuarial equilibrium asks for the expected present value of the settlements to be equal to the expected present value of the final benefit (available at the death time τ and equal to 1). If we assume that δ is constant and continuously paid, then the actuarial/financial equilibrium asks for the following equality to hold:

$$\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\tau}\delta\frac{G_{t_0}}{G_s}ds\right]=\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[1\times\frac{G_{t_0}}{G_\tau}\right].$$

In Sect. 5.3 we have already seen how to handle both sides, which can be simplified as follows:

$$\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega}\delta\mathbb{I}_{s<\tau}\frac{G_{t_0}}{G_s}ds\right]=\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega,t_0}^{\mathbb{Q},\tau}\left[\frac{G_{t_0}}{G_\tau}\right]\right],$$

$$\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega}\delta\mathbb{I}_{s<\tau}\frac{G_{t_0}}{G_s}ds\right]\right]=\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\mathbb{E}_{\omega}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\frac{G_{t_0}}{G_s}\pi_s ds\right]\right],$$

$$\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega}\delta\mathbb{E}_{\omega,t_0}^{\mathbb{Q},\tau}\left[\mathbb{I}_{s<\tau}\right]\frac{G_{t_0}}{G_s}ds\right]=\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\lambda_s e^{-\int_{t_0}^{s}r_u+\lambda_u du}ds\right],$$

$$\mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega}\delta\mathbb{E}_{\omega}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s}\lambda_u du}\right]\frac{G_{t_0}}{G_s}ds\right]=\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\lambda_s e^{-\int_{t_0}^{s}r_u+\lambda_u du}ds\right],$$

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\delta e^{-\int_{t_0}^{s}r_u+\lambda_u du}ds\right]=\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\lambda_s e^{-\int_{t_0}^{s}r_u+\lambda_u du}ds\right].$$

Accordingly, we are now able to compute the fair premium δ of this insurance contract:

$$\delta^*=\frac{\int_{t_0}^{\omega}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\lambda_s e^{-\int_{t_0}^{s}r_u+\lambda_u du}\right]ds}{\int_{t_0}^{\omega}\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{s}r_u+\lambda_u du}\right]ds},$$

i.e. the premium coincides with the weighted mean of the forces of mortality, and the weights are given by the values of longevity bonds. If the force of mortality λ is assumed to be constant, then

$$\delta^*=\lambda,$$

i.e. the premium that must be paid coincides with the force of mortality.

The numerator of δ^* can be further simplified through a change in probability. If we a longevity bond as the *numéraire* of the economy, and we call $\mathbb{L}$ the corresponding probability (see Sect. 4.10), then we can write

$$\delta^* = \frac{\int_{t_0}^{\omega} \mathbb{E}_{t_0}^{\mathbb{L}} [\lambda_s] \, L(t_0, s) \, ds}{\int_{t_0}^{\omega} L(t_0, s) \, ds}.$$

This new version of the previous formula shows in an even clearer way that δ^* is actually a weighted average of the expected value of the force of mortality under a new probability.

Reference

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Chapter 7
Pension Fund Management

7.1 Introduction

This chapter is the core of this work. By using all the tools that have been developed in the previous chapters, here we show how to compute the dynamic optimal asset allocation of a pension fund that can invest its wealth in four asset classes: (1) a risk-less asset, (2) a stock index, (3) a rolling zero coupon bond, which is a kind of derivative on the risk-less interest rate, and (4) a rolling longevity bond that can be thought as a derivative on the stochastic force of mortality.

We show that the optimal portfolio is formed by two parts: a speculative component and a hedging component. We show in details the form and the properties of these components.

7.2 Contributions and Pensions

The management period for a pension fund can be divided into two phases.

1. **Accumulation phase (A-Ph)**: this period lasts from the time when the sponsor (worker) signs his contract with the pension fund (t_0) and the retirement date (T). The retirement date may be either chosen by the worker or imposed by the law (this is the case in many European countries). During the accumulation phase the worker periodically pays contributions (c_t) into the fund.
2. **Distribution phase (D-Ph)**: after the retirement date (T), the pension fund starts paying pensions (p_t) to the worker-pensioner until his/her death time (τ). Of course death may occur even before time T and, in this case, the pension fund has just collected the contributions without having to pay any pension.

The worker chooses either the contribution or the pension stream, while the other stream of cash flows must satisfy a so-called **feasible condition** (or **fairness**

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_7

condition), i.e. the expected present value of both cash flow streams (contributions and pensions) must be equal at inception (in t_0). This condition can be written as

$$0 = \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\tau}\left(c_s\mathbb{I}_{s<T} - p_s\mathbb{I}_{s\geq T}\right)e^{-\int_{t_0}^{s} r_u du}ds\right].$$

Note that this formula may as well describe a particular form of swap derivative. In this kind of swap, the stream of contributions between t_0 and T is exchanged with the stream of pensions between T and the death time. Here, we have evaluated the streams of contributions and pensions under the risk neutral probability since we are assuming that it is possible to fully replicate them, on a complete financial market.

As we have already shown in previous chapters, this expected value can be written only in terms of $\mathbb{Q}$ and λ_t in the following form (it is like an annuity):

$$0 = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\left(c_s\mathbb{I}_{s<T} - p_s\mathbb{I}_{s\geq T}\right)e^{-\int_{t_0}^{s}(r_u+\lambda_u)du}ds\right].$$

Let us assume that both c_t and p_t are constant. In this case the feasible condition becomes

$$\begin{aligned}\frac{p^*}{c^*} &= \frac{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\mathbb{I}_{s<T}e^{-\int_{t_0}^{s}(\lambda_u+r_u)du}ds\right]}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega}\mathbb{I}_{s\geq T}e^{-\int_{t_0}^{s}(\lambda_u+r_u)du}ds\right]}\\ &= \frac{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T}e^{-\int_{t_0}^{s}(\lambda_u+r_u)du}ds\right]}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{T}^{\omega}e^{-\int_{t_0}^{s}(\lambda_u+r_u)du}ds\right]}.\end{aligned} \tag{7.2.1}$$

Finally, if we recall the value of a longevity bond, we can also write

$$\frac{p^*}{c^*} = \frac{\int_{t_0}^{T} L(t_0,s)\,ds}{\int_{T}^{\omega} L(t_0,s)\,ds}.$$

Example 7.1 If both r and λ are constant, and we assume $\omega\to\infty$, then the feasibility condition brings to the following result

$$\begin{aligned}\frac{p^*}{c^*} &= \frac{\int_{t_0}^{T}e^{-(r+\lambda)(s-t_0)}ds}{\int_{T}^{\infty}e^{-(r+\lambda)(s-t_0)}ds} = \frac{\int_{t_0}^{T}e^{-(r+\lambda)s}ds}{\int_{T}^{\infty}e^{-(r+\lambda)s}ds}\\ &= e^{(r+\lambda)(T-t_0)} - 1.\end{aligned}$$

(continued)

Example 7.1 (continued)

From this equation we immediately see that:

1. when r increases also $\frac{p^*}{c^*}$ increases: pensions are paid after receiving contributions, this means that their duration is longer than that of contributions and changes in the interest rate affect pensions in a heavier way; when the interest rate increases the present value of future cash flows reduces (and, of course, pensions reduce much more than contributions and, accordingly, they can be higher);
2. when λ increases also $\frac{p^*}{c^*}$ increases: if the force of mortality increases, then also the death probability is higher and, given the amount of contributions, pensions can be higher too (since the probability to pay pensions for a longer period of time becomes lower);
3. when $(T - t_0)$ increases also $\frac{p^*}{c^*}$ increases: if contributions are paid for a longer period of time, then pensions can be higher.

This result can be obtained in closed form when the force of mortality follows a Gompertz function and the interest rate is constant.

Proposition 7.1 *If the force of mortality follows Eq. (5.4.2) and the riskless interest rate is constant, then the feasible ratio (7.2.1) is given by*

$$\frac{p^*}{c^*} = \frac{\Gamma\left(-(\phi + r)\,b, e^{\frac{\iota + t_0 - m}{b}}\right)}{\Gamma\left(-(\phi + r)\,b, e^{\frac{\iota + T - m}{b}}\right)} - 1.$$

Proof. When λ follows (5.4.2) and the interest rate is constant, Eq. (7.2.1) can be written as

$$\frac{p^*}{c^*} = \frac{\int_{t_0}^{T} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds}{\int_{T}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds} = \frac{\int_{t_0}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds - \int_{T}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds}{\int_{T}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds}$$

$$= \frac{\int_{t_0}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds}{\int_{T}^{\infty} e^{-\int_{t_0}^{s}(r+\lambda_u)du} ds} - 1,$$

(continued)

where the numerator is a life annuity starting from t_0 as shown in Proposition 5.1. In order to trace back also the denominator the that result, we can write it as

$$\int_T^\infty e^{-\int_{t_0}^s (r+\lambda_u)du} ds = \int_T^\infty \frac{e^{-\int_T^s (r+\lambda_u)du}}{e^{-\int_T^s (r+\lambda_u)du}} e^{-\int_{t_0}^s (r+\lambda_u)du} ds$$
$$= e^{-\int_{t_0}^T (r+\lambda_u)du} \int_T^\infty e^{-\int_T^s (r+\lambda_u)du} ds,$$

and finally we have

$$\frac{p^*}{c^*} = \frac{\int_{t_0}^\infty e^{-\int_{t_0}^s (r+\lambda_u)du} ds}{e^{-\int_{t_0}^T (r+\lambda_u)du} \int_T^\infty e^{-\int_T^s (r+\lambda_u)du} ds} - 1.$$

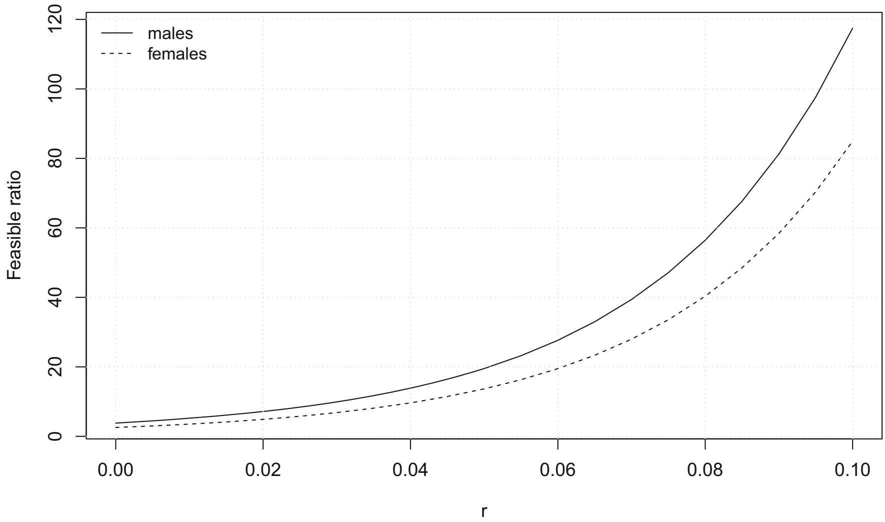
Now, in the denominator, we have terms whose values have already been computed in (5.4.3). If we substitute both (5.4.3) and the result of Proposition 5.1 we obtain

$$\frac{p^*}{c^*} = \frac{b e^{-(\phi+r)(m-\iota-t_0)+e^{\frac{\iota+t_0-m}{b}}}}{e^{-(\phi+r)(T-t_0)-e^{\frac{\iota-m+t_0}{b}}\left(e^{\frac{T-t_0}{b}}-1\right)} b e^{-(\phi+r)(m-\iota-T)+e^{\frac{\iota+T-m}{b}}}}$$
$$\times \frac{\Gamma\left(-(\phi+r)\,b, e^{\frac{\iota+t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b, e^{\frac{\iota+T-m}{b}}\right)} - 1.$$

which can be simplified as shown in the proposition. □

If we use the parameters estimated for US citizens who were 25 in 1933, we can show in Fig. 7.1 the feasible ratio $\frac{p^*}{c^*}$ for males (continuous line) and females (dashed line) for different values of the (constant) interest rate.

Since females live longer, then they receive a lower amount of pensions even if they pay the same contributions as males. From Fig. 7.1 we can see that changes




```
age = 25
t0 = 0
T = 40
r = seq(0, 0.1, by = 0.005)
ratio_male = gamma_inc(-(phi_GM[1] + r) * b_GM[1],
    exp((age + t0 - m_GM[1])/b_GM[1]))/gamma_inc(-(phi_GM[1] +
    r) * b_GM[1], exp((age + T - m_GM[1])/b_GM[1])) -
    1
ratio_female = gamma_inc(-(phi_GM[2] + r) * b_GM[2],
    exp((age + t0 - m_GM[2])/b_GM[2]))/gamma_inc(-(phi_GM[2] +
    r) * b_GM[2], exp((age + T - m_GM[2])/b_GM[2])) -
    1
plot(r, ratio_male, type = "l", ylab = "Feasible ratio",
    xlab = "r")
lines(r, ratio_female, type = "l", lty = 2)
grid()
legend("topleft", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 7.1 Feasible ratio $\frac{p^*}{c^*}$ for US males (continuous line) and females (dashed line) who enter pension fund at 25 (in 1993) and retire at 65, as a function of (constant) interest rate in the interest rate are less important when the interest rate is low while they may affect the feasible ratio in a relevant way when the interest rate level is high.

This result suggests that the best moment for a reform of a pension system is a period when interest rates are stable and low, i.e. during a period of economic growth.

In this section we have computed the ratio between the fair value of pensions and contributions. Actually, contributions and pensions cannot be separately computed. If the sponsor chooses a level of contribution, then the pension level must be set in order to meet the feasibility condition. In the same way, if the sponsor chooses the amount of pensions, then the contributions are suitably set in order to make the feasibility condition hold.

Accordingly, there are two main schemes for a pension fund management.

1. **Defined Contributions** (DC): contributions are defined in advance (in t_0), generally as a percentage of sponsor's wage, while pensions are paid in the D-Ph just according to the returns the pension fund has been able to obtain on the financial market (a minimum pension level could be guaranteed). This scheme makes the sponsor to bear the risk during the D-Ph, while the fund bears the risk during the A-Ph.
2. **Defined Benefits** (DB): the pension level is defined in advance (in t_0) and the pension fund commits itself to meet such an obligation. The contributions may be accordingly change in order to make the fund able to provide the defined pension stream. This scheme makes the sponsor bear the risk during the A-Ph, while the fund bears the risk during the D-Ph.

7.3 Reserves

We have already argued that the expected present values of contributions and pensions must be equal. In fact, if this were not the case, either the pension or the sponsor would not have any interest in subscribing the contract.

Nevertheless, after time t_0, the difference between expected values of contributions and pensions is no longer equal to zero.

In what follows we will call "**net outflow**" for the pension fund the expected present value of the future pensions diminished by the expected present value of the future contributions (the opposite could be called "net inflow").

In order to simplify the notation, we can define the following variable:

$$k_t \equiv p_t^* \mathbb{I}_{t \geq T} - c_t^* \mathbb{I}_{t < T}, \tag{7.3.1}$$

where k_t coincides with the (negative) contribution during the A-Ph and with the pension during the D-Ph.

In actuarial literature, the net outflow is called "**mathematical reserve**". Furthermore, a similar reserve can be computed by taking into account what have already been paid (and received).

1. **Prospective Mathematical Reserve** (PMR): at any instant t, it is the expected present value of all the future net outflows for the pension fund:

$$\Delta_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} k_s e^{-\int_t^s (r_u+\lambda_u)du} ds\right], \tag{7.3.2}$$

where, of course, we must have

$$\Delta_{t_0} = 0,$$

since the value of c^* and p^* are computed in order to meet such a condition. This reserve is mainly used for measuring how much capital the pension fund must keep in order to guarantee the payment of pensions. Accordingly, Δ_t is mainly informative in a DB scheme where a given level of pensions is guaranteed. In the next sections we will show the behaviour of such a reserve.

2. **Retrospective Mathematical Reserve** (RMR): at any instant t, it is the amount of contributions and pensions that have already been received and paid respectively,

$$K_t = -\int_{t_0}^{t} k_s e^{\int_s^t r_u du} ds. \tag{7.3.3}$$

This reserve is mainly computed by pension fund in a DC scheme and allows to measure the efficiency of pension fund management. In fact, if K_t is subtracted from the fund's wealth $(R_t - K_t)$, then we obtain the amount of wealth that the fund has been able to accumulate without taking into account neither contributions nor pensions.

We will concentrate on the PMR since it is the reserve which "naturally" arises when computing the optimal portfolio of a pension fund, as we will show in the following sections.

7.4 Prospective Mathematical Reserve

Let us start with the simplest case where all variables (r, λ, c, and p) are constant (and $\omega \to \infty$). The PMR can be simplified as follows:

$$\Delta_t = \int_t^{\infty} \left(p^* \mathbb{I}_{s\geq T} - c^* \mathbb{I}_{s<T}\right) e^{-(r+\lambda)(s-t)} ds,$$

and now some further simplifications can be done if we know whether we are in the A-Ph or in the D-Ph. In fact, during the first phase (when $t < T$):

$$\begin{aligned}\Delta_{t|t<T} &= p^* \int_T^\infty e^{-(r+\lambda)(s-t)} ds - c^* \int_t^T e^{-(r+\lambda)(s-t)} ds \\ &= p^* \frac{1}{r+\lambda} e^{-(r+\lambda)(T-t)} - c^* \frac{1-e^{-(r+\lambda)(T-t)}}{r+\lambda} \\ &= p^* \frac{1}{r+\lambda} - \left(c^* + p^*\right) \frac{1-e^{-(r+\lambda)(T-t)}}{r+\lambda}.\end{aligned}$$

Instead, during the distribution phase (when $t \geq T$), we have

$$\begin{aligned}\Delta_{t|t\geq T} &= p^* \int_t^\infty e^{-(r+\lambda)(s-t)} ds \\ &= p^* \frac{1}{r+\lambda}.\end{aligned}$$

We can write Δ_t as a unique function as follows

$$\Delta_t = p^* \frac{1}{r+\lambda} - \left(c^* + p^*\right) \frac{1-e^{-(r+\lambda)\max(T-t,0)}}{r+\lambda},$$

and, if we substitute the feasible ratio

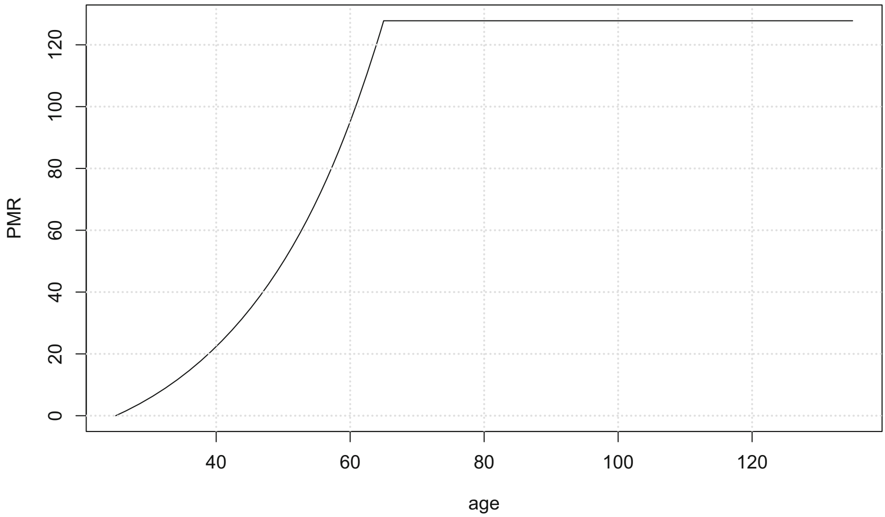
$$\frac{p^*}{c^*} = e^{(r+\lambda)(T-t_0)} - 1,$$

we finally obtain

$$\Delta_t = \frac{c^*}{r+\lambda} \left(e^{(r+\lambda)(T-t_0-\max(T-t,0))} - 1 \right).$$

With $r + \lambda = 0.05$, $t_0 = 0$, $T = 40$, i.e. the length of the contribution period is 40 years, and $c^* = 1$ we obtain the Fig. 7.2.

The particular behaviour of the prospective reserve is due to the constancy of λ. In this case, the death probability between t and $t + s$ does not depend on the age of the agent but just on s. This means that the probability to die between 65 and 70 is exactly the same as that to die between 100 and 105, and the pension fund cannot reduce the amount of capital to keep in order to guarantee the payment of pensions. This situation is very unlikely because in the real world the probability to die constantly increases over time (and mainly after T).




```
r = 0.025
l = 0.025
t0 = 0
T = 40
c = 1
t = seq(t0, 110, by = 0.5)
plot(25 + t, c/(r + l) * (exp((r + l) * (T - t0 - pmax(T -
    t, 0))) - 1), type = "l", ylab = "PMR", xlab = "age")
grid()
```

Fig. 7.2 Prospective mathematical reserve with $r + \lambda = 0.05$, $t_0 = 0$, $T = 40$, and $c^* = 1$

If we use, instead, the Gompertz function we have

$$\Delta_{t|t<T} = p^* \int_T^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds - c^* \int_t^T e^{-r(s-t)-\int_t^s \lambda_u du} ds.$$

Since Proposition (5.1) allows us to compute the value of the integral when it is semi-definite, then we must write the integral between t and T as the difference between two semi-definite integrals in the following way

$$\int_t^T e^{-r(s-t)-\int_t^s \lambda_u du} ds = \int_t^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds - \int_T^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds,$$

and so we can finally write

$$\Delta_{t|t<T} = \left(c^* + p^*\right) e^{-r(T-t)-\int_t^T \lambda_u du} \int_T^\infty e^{-r(s-T)-\int_T^s \lambda_u du} ds$$
$$- c^* \int_t^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds.$$

Now, we can use the results shown in Proposition (5.1) and in Eq. (5.4.3) in order to write

$$\begin{aligned}\Delta_{t|t<T} =& c^* b e^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \\ &\times \left(\left(1 + \frac{p^*}{c^*}\right) \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+T-m}{b}}\right) - \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right)\right),\end{aligned}$$

and, after substituting for the feasible ratio in Proposition (7.1), we obtain

$$\begin{aligned}\Delta_{t|t<T} =& c^* b e^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right) \\ &\times \left(\frac{\Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right)} - 1 \right).\end{aligned}$$

When we are in the distribution phase, the prospective reserve is

$$\Delta_{t|t\geq T} = p^* \int_t^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds,$$

and, after the suitable substitutions, we have

$$\begin{aligned}\Delta_{t|t\geq T} &= p^* b e^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right) \\ &= c^* b e^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}}\right) \\ &\times \left(\frac{\Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+T-m}{b}}\right)} - 1 \right).\end{aligned}$$

Both reserves can be written with a unique notation in the following way:

$$\Delta_t = c^* b e^{-(\phi+r)(m-\iota-t)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\,b, e^{\frac{\iota+t-m}{b}}\right)$$
$$\times \left(\frac{\Gamma\left(-(\phi+r)\,b, e^{\frac{\iota+t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)\,b, e^{\frac{\iota+\min(t,T)-m}{b}}\right)} - 1 \right). \quad (7.4.1)$$

With the parameters estimated on US males and females, the behaviour of the prospective mathematical reserve is shown in Fig. 7.3.

This behaviour of the prospective mathematical reserve has an immediate interpretation. The value in t_0 is zero because of the feasibility condition. Then, while time passes, there are less and less contributions and, accordingly, until time T, the relative weight of the pensions increases. In T all the contributions has been received and all the pensions still have to be paid. Thus, the maximum amount reached in T coincides with the expected present value of all the future pensions.

After T, the probability that the agent dies becomes higher and higher over time, and the value of Δ_t asymptotically goes back towards zero.

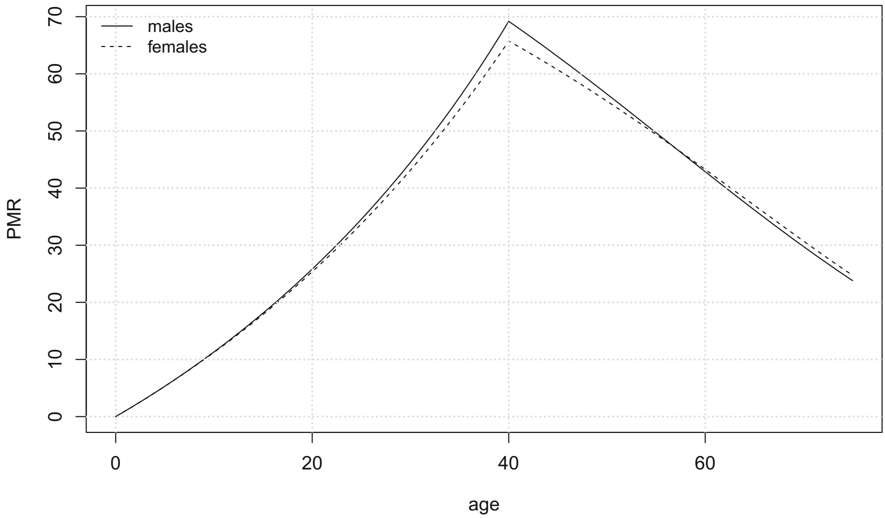
7.5 Fund's Budget Constraint

The budget constraint of the fund asks for the total expected positive cash flows to finance the total expected negative cash flows. The inflows to the fund's portfolio are the initial wealth R_{t_0} and the contributions, while the outflows are the pensions and the final wealth R_τ that is withdrawn from the fund. In particular, if we assume that the fund can retain only a percentage q_τ of its final wealth, the budget constraint of the fund can be written as

$$R_{t_0} + \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{T} c_t \mathbb{I}_{t<\tau} e^{-\int_{t_0}^{t} r_u du} dt\right]$$
$$= \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{T}^{\omega} p_t \mathbb{I}_{t<\tau} e^{-\int_{t_0}^{t} r_u du} dt\right] + \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[q_\tau R_\tau e^{-\int_{t_0}^{\tau} r_u du}\right].$$

This equation can also be written as

$$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{T}^{\omega} p_t e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right] - \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T} c_t e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right]$$
$$+ \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega} \lambda_t q_t R_t e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right]$$




```
r = 0.02
age = 25
T = 40
c = 1
t0 = 0
t = seq(0, 75, by = 0.5)
PMR = function(c, phi, b, m, r, t0, t) {
    c * b * exp((phi + r) * (t - m) + exp((t - m)/b)) *
        gamma_inc(-(phi + r) * b, exp((t - m)/b)) *
        (gamma_inc(-(phi + r) * b, exp((t0 - m)/b)) *
            gamma_inc(-(phi + r) * b, exp((pmax(t,
                T) - m)/b))/gamma_inc(-(phi + r) *
            b, exp((t - m)/b))/gamma_inc(-(phi + r) *
            b, exp((T - m)/b)) - 1)
}
plot(t, PMR(c, phi_GM[1], b_GM[1], m_GM[1], r, t0,
    t), type = "l", ylab = "PMR", xlab = "age")
lines(t, PMR(c, phi_GM[2], b_GM[2], m_GM[2], r, t0,
    t), type = "l", lty = 2)
grid()
legend("topleft", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 7.3 Behaviour of the prospective mathematical reserve (dashed line for females) with a Gompertz mortality low ($\iota = 25$, $T = 40$, and $c^* = 1$) estimated on US citizens aged 25 in 1933

$$
\begin{aligned}
=&\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} p_t \mathbb{I}_{t\geq T} e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right] - \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} c_t \mathbb{I}_{t<T} e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right] \\
&+ \mathbb{E}_{t_0,t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\omega} \lambda_t q_t R_t e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right] \\
=&\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \left(p_t \mathbb{I}_{t\geq T} - c_t \mathbb{I}_{t<T} + \lambda_t R_t\right) e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right] \\
=&\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \left(k_t + \lambda_t q_t R_t\right) e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right].
\end{aligned}
$$

If we recall the definition of PMR (7.3.2), we can finally write the constraint as

$$
R_{t_0} = \Delta_{t_0} + \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_t q_t R_t e^{-\int_{t_0}^{t} r_u+\lambda_u du} dt\right]. \tag{7.5.1}
$$

Remark 7.1 If q_t is constant over time, then this constraint can be alternatively written as if $q = 1$ and rescaling the contributions, the pensions, and the initial wealth by dividing them by q.

The differential of this wealth, as already shown in Sect. 5.9, is computed as follows

$$
dR_t = \left(R_t \left(r_t + \lambda_t\right) - k_t - \lambda_t q_t R_t\right) dt + \left(\ldots\right) dW_t^{\mathbb{Q}},
$$

or

$$
dR_t = \left(R_t \left(r_t + \left(1 - q_t\right) \lambda_t\right) - k_t\right) dt + \left(\ldots\right) dW_t^{\mathbb{Q}},
$$

where, now, we just have to define the diffusion term.

When differentiating the wealth as in Sect. 4.3, we can write the wealth dynamics as

$$
\begin{aligned}
dR_t =& w_t^{\intercal} dS_t + w_{G,t} dG_t \\
&+ dw_t^{\intercal} \left(dS_t + S_t\right) + G_t dw_{G,t},
\end{aligned}
$$

where the "self-financing" condition implies that the second part of the portfolio must be

$$
dw_t^{\intercal} \left(dS_t + S_t\right) + G_t dw_{G,t} = -k_t dt,
$$

since the change in fund's portfolio must finance the pensions and must be financed by contributions.

Thus, after the substitutions shown in Sect. 4.3, we can write

$$\begin{aligned} dR_t &= \big(R_t\,(r_t + (1-q_t)\,\lambda_t) + w_t^\intercal I_S\,(\mu_t - r_t\mathbf{1}) - k_t\big)\,dt \\ &\quad + w_t^\intercal I_S \Sigma_t^\intercal dW_t, \end{aligned} \tag{7.5.2}$$

which, under $\mathbb{Q}$, is

$$dR_t = (R_t\,(r_t + (1-q_t)\,\lambda_t) - k_t)\,dt + w_t^\intercal I_S \Sigma_t^\intercal dW_t^{\mathbb{Q}}.$$

Remark 7.2 We note that when the pension fund keeps all the residual wealth ($q_t = 1$) at the death time of the pensioner, i.e. no fund is paid back to the pensioner's heirs, the dynamics of wealth dR_t does not depend on λ_t.

7.6 Pension Fund's Ratios

In the previous sections we presented the "reserves" and, in particular, the PMR Δ_t. The goodness of a pension fund management is often measured with other ratios. A well known ratio is the "funding ratio" which is computed as the value of a pension assets, i.e. the fund's investments in equity and bonds, divided by the value of its liabilities, i.e. the current and future pension benefits to be paid out.

With our notation, this funding ratio (FR_t) can be computed as

$$FR_t := \frac{R_t}{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega p_s^* \mathbb{I}_{s\geq T} e^{-\int_t^s r_u + \lambda_u du} ds\right]}.$$

In the previous sections we have already defined the fund's wealth at time t_0 in (7.5.1). If we write it at any date $t > t_0$, we have

$$\begin{aligned} R_t &= \Delta_t + \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega \lambda_s q_s R_s e^{-\int_t^s r_u + \lambda_u du} ds\right] \\ &= \Delta_t + \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[q_\tau R_\tau e^{-\int_t^\tau r_u du}\right]. \end{aligned}$$

Thus, we can rewrite the funding ratio as follows:

$$FR_t := 1 + \frac{\mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[q_\tau R_\tau e^{-\int_t^\tau r_u du}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega c_s^* \mathbb{I}_{s<T} e^{-\int_t^s r_u + \lambda_u du} ds\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega p_s^* \mathbb{I}_{s\geq T} e^{-\int_t^s r_u + \lambda_u du} ds\right]},$$

and since we require that FR_t is greater than 1, this condition can be also restated as

$$FR_t > 1 \iff \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[q_\tau R_\tau e^{-\int_t^\tau r_u du}\right] > \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\omega c_s^* \mathbb{I}_{s<T} e^{-\int_t^s r_u + \lambda_u du} ds\right],$$

i.e. if the expected wealth at the death time is greater than the contributions that still have to be received.

Another ratio that is often used in the pension fund industry is the active to passive members ratios. In our framework, such an index is not suitable since we are dealing with the management of a homogeneous cohort and we do not compare cohorts of workers/pensioners belonging to different ages.

In our framework, we manage fund's wealth in order to be able to face the payment of benefits (pensions) by using the compounded value of the contributions that they paid themselves during their working life. Instead, the ratio between active (workers) and passive (pensioners) members is suitable when a fund pays pensions to a cohort by using the contributions paid by another (younger) cohort. This is the so-called Pay-As-You-Go (PAYG) scheme whose equilibrium requires a balance between the workers and the pensioners.

If we call m_t the number of workers subscribing to a pension scheme and paying contributions, and n_t the number of retirees receiving pensions, the ratio active/passive is just $\frac{m_t}{n_t}$. If we want to measure this ratio in monetary terms, then we must multiply m_t by the average contribution and n_t by the average pension.

In our framework, the solvency of the fund is measured by the comparison between its wealth R_t and the PMR Δ_t. We recall that Δ_t measures the benefits that the fund still has to pay, net of the contributions that still have to be received. Thus, if, at any time, R_t is greater than Δ_t, the fund is able (on average) to face its futures duties.

We will show that the optimal portfolio strategy allows to have an optimal wealth that is sufficiently high to fully finance the value of the PMR. In other words, we could argue that our fund always satisfy the following condition

$$\frac{R_t}{\Delta_t} > 1,$$

where, on the left hand side, we have the wealth to PMR ratio.

7.7 Fund's Optimisation Problem

A pension fund may have different objectives and in the literature many approaches have been considered. Here, we take into account one of the most commonly used approach which is to maximise the expected utility of the fund's final wealth, i.e. the wealth at time τ when the (representative) subscriber dies. The argument of the utility is the wealth the fund has been able to obtain after paying all the pensions (at time τ):

$$\max_{\{w_t\}_{t\in[t_0,\tau]}} \mathbb{E}_{t_0}^{\tau}\left[U\left(R_\tau\right)e^{-\int_{t_0}^{\tau}\rho_u du}\right], \tag{7.7.1}$$

where the function ρ_t is a subjective discount factor which accounts for the fund's time preferences. The higher ρ_t the more myopic is the fund.

The objective function can be modified as follows:

$$\max_{\{w_t\}_{t\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega}\lambda_t e^{-\int_{t_0}^{t}\rho_u+\lambda_u du}U\left(R_t\right)dt\right], \tag{7.7.2}$$

and we will use a HARA utility function

$$U\left(R_t\right) = \frac{\left(q_t R_t - \alpha\right)^{1-\delta}}{1-\delta},$$

in which $\delta > 0$ and α can be interpreted as a minimum level of wealth that must be reached by the fund. Here, of course, we assume that the fund takes utility only from the percentage (q_t) of wealth that it can keep.

Remark 7.3 The utility function can alternatively be written by collecting the term q_t as follows:

$$U\left(R_t\right) = \frac{\left(q_t R_t - \alpha\right)^{1-\delta}}{1-\delta} = q_t^{1-\delta}\frac{\left(R_t - \frac{\alpha}{q_t}\right)^{1-\delta}}{1-\delta},$$

where we see that the "subsistence" level of wealth to be reached is given by α/q_t exactly because the fund will be able to retain only a percentage q_t of the final wealth.

The constraint of the optimisation problem can be written in terms of either a stochastic differential equation as in (7.5.2) or a static constraint as in (7.5.1).

7.8 Dynamic Optimisation (the Martingale Approach)

According to the martingale approach, the dynamic optimisation problem can be solved as a static problem where the control variable is the wealth. Instead of looking for the optimal portfolio, we then look for the optimal wealth and, after finding it, we compute the portfolio which replicates the optimal wealth (of course we need the market completeness assumption).

In the dynamic programming technique, instead, the solution to the optimisation problem is approached through a backward induction technique due to Bellman, which leads to a partial differential equation called Hamilton-Jacobi-Bellman equation.

Dynamic Programming

$$\max_{\{w_t\}_{t\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s \frac{(q_s R_s - \alpha)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^{s} \rho_u + \lambda_u du} ds\right],$$

$$\begin{aligned} dR_t =& \big(R_t\,(r_t + (1-q_t)\,\lambda_t) + w_t^{\mathsf{T}} I_S\,(\mu_t - r_t\mathbf{1}) - k_t\big)\,dt \\ &+ w_t^{\mathsf{T}} I_S \Sigma_t^{\mathsf{T}} dW_t, \\ dz_t =& \mu_z dt + \Omega_t^{\mathsf{T}} dW_t. \end{aligned}$$

The so-called value function is defined as

$$J\,(t, R_t, z_t) := \max_{\{w_t\}_{t\in[t_0,\omega]}} \mathbb{E}_t\left[\int_t^{\omega} \lambda_s \frac{(q_s R_s - \alpha)^{1-\delta}}{1-\delta} e^{-\int_t^s \rho_u + \lambda_u du} ds\right],$$

and the expected value of its differential dJ is computed (through Ito's lemma) and maximised with respect to w_t.

The optimal value w_t^* is obtained by solving the partial differential equation in J (which contains its derivatives with respect to t, R_t, and z_t and the second derivatives with respect to R_t and z_t).

Martingale Approach

$$\max_{\{R_s\}_{s\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s \frac{(q_s R_s - \alpha)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^{s} \rho_u + \lambda_u du} ds\right],$$

$$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_s q_s R_s e^{-\int_{t_0}^{s} (r_u + \lambda_u) du} ds\right] + \Delta_{t_0}.$$

The solution is the optimal wealth R_s^*. Then the wealth at any time t is computed

(continued)

$$R_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s q_s R_s^* e^{-\int_t^s (r_u+\lambda_u)du} ds\right] + \Delta_t.$$

After differentiating R_t through Itô's lemma by recalling that

$$dz_t = \mu_{z,t}dt + \Omega_t^{\top} dW_t,$$

the optimal portfolio w_t^* is finally obtained in order to make dR_t replicate the differential equation:

$$\begin{aligned} dR_t =& \big(R_t\,(r_t + (1-q_t)\,\lambda_t) + w_t{}^{\top} I_S\,(\mu_t - r_t\mathbf{1}) - k_t\big)\,dt \\ &+ w_t^{\top} I_S \Sigma_t^{\top} dW_t. \end{aligned}$$

The maximisation problem at timet_0 is

$$\max_{\{R_s\}_{s\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s \frac{(q_s R_s - \alpha)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^s \rho_u+\lambda_u du} ds\right],$$

under the constraint

$$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_s q_s R_s e^{-\int_{t_0}^s r_u+\lambda_u du} ds\right] + \Delta_{t_0}.$$

In order to solve this problem we use the Lagrangian method and, after defining the (constant) Lagrangian multiplier l, we can write the Lagrangian function as

$$\begin{aligned} \mathcal{L} =& \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s e^{-\int_{t_0}^s \rho_u+\lambda_u du} \frac{(q_s R_s - \alpha)^{1-\delta}}{1-\delta} ds\right] \\ &- l\left(\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_s q_s R_s e^{-\int_{t_0}^s r_u+\lambda_u du} ds\right] + \Delta_{t_0} - R_{t_0}\right), \end{aligned}$$

where now the control variable is R_s.

Actually, with the martingale approach the original problem whose choice variables are the portfolio weights, is transformed into a maximisation problem whose choice variable it the wealth. Finally, this optimal wealth is replicated by using the market completeness hypothesis.

Thus, now we have to compute the first order condition with respect to wealth:[1]

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial R_s} =& \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s q_s e^{-\int_{t_0}^{s} \rho_u+\lambda_u du} (q_s R_s - \alpha)^{-\delta} ds\right] \\ &- l\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_s q_s e^{-\int_{t_0}^{s} r_u+\lambda_u du} ds\right].\end{aligned}$$

The two expected values are computed under two different risk measures. In order to simplify them we can write the expected value under $\mathbb{Q}$ as the equivalent expected value under the historical probability by using the Radon-Nydomin derivative (recall (4.9.2)):

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\omega} \lambda_s q_s e^{-\int_{t_0}^{s} r_u+\lambda_u du} ds\right] = \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s q_s m_{t_0,s} e^{-\int_{t_0}^{s} r_u+\lambda_u du} ds\right].$$

The first order condition can accordingly be written as

$$\frac{\partial \mathcal{L}}{\partial R_s} = \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s q_s \left(e^{-\int_{t_0}^{s} \rho_u du} (q_s R_s - \alpha)^{-\delta} - l m_{t_0,s} e^{-\int_{t_0}^{s} r_u du}\right) e^{-\int_{t_0}^{s} \lambda_u du} ds\right].$$

The first order condition asks for this expected integral to be equal to zero. Nevertheless, we want this first order condition to be valid for each and any state of the world. This means that we can set to zero the integrand function[2] :

$$e^{-\int_{t_0}^{s} \rho_u du} (q_s R_s - \alpha)^{-\delta} - l m_{t_0,s} e^{-\int_{t_0}^{s} r_u du} = 0,$$

or

$$R_s^* = \frac{\alpha}{q_s} + \frac{1}{q_s}\left(l m_{t_0,s} \frac{e^{-\int_{t_0}^{s} r_u du}}{e^{-\int_{t_0}^{s} \rho_u du}}\right)^{-\frac{1}{\delta}}, \tag{7.8.1}$$

and the value of the multiplier l will be found through the constraint on the initial wealth R_{t_0}. We see that the optimal wealth is always greater than α/q_s as the form of the utility function implies. In particular, we stress that the optimal wealth is simply rescaled by the parameter q_s.

[1] We can avoid to check for the second order condition since we already know that the utility function has suitable properties for obtaining a well defined optimum.

[2] If we want $\mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} f_s ds\right] = 0$ for any state of the world, then we need to have $\int_{t_0}^{\omega} f_s ds = 0$ but if we want this condition to hold for any time s (and not for the overall period) then we need to have $f_s = 0$ for any $s \in [t_0, \omega]$.

Now, we are interested in replicating the wealth R_t at any instant in time t. In order to compute the differential dR_t, we apply Itô's lemma to the constraint (7.5.1)

$$R_t^* = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s q_s R_s^* e^{-\int_t^s r_u+\lambda_u du} ds\right] + \Delta_t,$$

where we can substitute the optimal value R_s^*:

$$\begin{aligned} R_t^* &= \alpha \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s e^{-\int_t^s r_u+\lambda_u du} ds\right]}_{H_t} \\ &\quad + \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s l^{-\frac{1}{\delta}} m_{t_0,s}^{-\frac{1}{\delta}} \frac{e^{-\int_t^s r_u+\lambda_u du}}{e^{\frac{1}{\delta}\int_{t_0}^s \rho_u - r_u du}} ds\right] + \Delta_t. \end{aligned}$$

Remark 7.4 We can see that the dynamics of the optimal wealth is not affected by the percentage q_t. In fact, the optimal portfolio will be independent of q_t because the optimal strategy aimed at maximising the final wealth is the same as the optimal strategy that maximises an exogenous percentage of the final wealth.

Remark 7.5 We stress that the function H_t coincides with the value of an insurance contract as in (5.3.3). Thus, we can write

$$H_t := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s e^{-\int_t^s r_u+\lambda_u du} ds\right] = \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[e^{-\int_t^{\tau} r_u du}\right]. \tag{7.8.2}$$

Since the martingale $m_{t_0,s}$ is such that

$$m_{t_0,s} = m_{t_0,t} m_{t,s},$$

we have

$$\begin{aligned} & l^{-\frac{1}{\delta}} \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s \left(m_{t_0,t} m_{t,s}\right)^{-\frac{1}{\delta}} e^{-\int_t^s r_u+\lambda_u du} ds\right] \\ &= \frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u - r_u du}} \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s m_{t,s}^{-\frac{1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r_u + \lambda_u + \frac{1}{\delta}\rho_u\right) du} ds\right]}_{F_t}. \end{aligned}$$

Remark 7.6 The function F_t is the expected present value of a transformation of the price kernel (4.9.1):

$$\begin{aligned} F_t &:= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s m_{t,s}^{-\frac{1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r_u + \lambda_u + \frac{1}{\delta}\rho_u\right) du} ds\right] \\ &= \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[m_{t,\tau}^{-\frac{1}{\delta}} e^{-\int_t^{\tau}\left(\frac{\delta-1}{\delta} r_u + \frac{1}{\delta}\rho_u\right) du}\right]. \end{aligned} \tag{7.8.3}$$

Finally, the wealth of the fund can be written in the following way

$$R_t(z_t) = \alpha H_t(z_t) + \frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u - r_u du}} F_t(z_t) + \Delta_t(z_t), \tag{7.8.4}$$

where we have indicated the functional dependencies of R_t, H_t, and F_t on the state variables z_t.

Accordingly, the differential dR_t can be computed by using Itô's lemma (by recalling that both functions F_t and Δ_t do depend on the state variables z_t):

$$\begin{aligned} dR_t &= (\ldots) dt - \frac{\partial R_t}{\partial m_{t_0,t}} m_{t_0,t} \xi_t^{\mathsf{T}} dW_t + \left(\frac{\partial R_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t \\ &= (\ldots) dt - \frac{-\frac{1}{\delta} l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}-1}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u - r_u du}} F_t m_{t_0,t} \xi_t^{\mathsf{T}} dW_t \\ &\quad + \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u - r_u du}} \frac{\partial F_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t \\ &= (\ldots) dt + \frac{1}{\delta}(R_t - \alpha H_t - \Delta_t) \xi_t^{\mathsf{T}} dW_t \\ &\quad + \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{R_t - \alpha H_t - \Delta_t}{F_t} \frac{\partial F_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t. \end{aligned}$$

Since the wealth evolves according to (7.5.2), then replication asks for the diffusion terms of the wealth differentials (the optimal and the non-optimal) to be equal:

$$\begin{aligned} w_t^{\mathsf{T}} I_S \Sigma_t^{\mathsf{T}} &= \frac{1}{\delta}(R_t - \alpha H_t - \Delta_t) \xi_t^{\mathsf{T}} \\ &\quad + \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{R_t - \alpha H_t - \Delta_t}{F_t} \frac{\partial F_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}}. \end{aligned} \tag{7.8.5}$$

Remark 7.7 Now, we can see that the market completeness plays a crucial role in this setting. In fact, if the market is complete, then the matrix Σ_t is invertible (as shown in Sect. 4.6) and there exists a unique solution to the previous matrix equation.

If the market is complete, the optimal portfolio is

$$I_S w_t^* = \underbrace{\frac{R_t - \alpha H_t - \Delta_t}{\delta}\Sigma_t^{-1}\xi_t}_{\text{speculative portfolio}} \\ \underbrace{+\alpha\Sigma_t^{-1}\Omega_t\frac{\partial H_t}{\partial z_t} + \frac{R_t - \alpha H_t - \Delta_t}{F_t}\Sigma_t^{-1}\Omega_t\frac{\partial F_t}{\partial z_t} + \Sigma_t^{-1}\Omega_t\frac{\partial \Delta_t}{\partial z_t}}_{\text{hedging portfolio}}. \tag{7.8.6}$$

The first portfolio component, the so-called "speculative component" coincides with the Merton's portfolio. The second portfolio component, the so-called "hedging portfolio", arises because of the need to hedge against the stochastic changes in the value of the state variables.

The amount of money invested in the riskfree asset is, of course, obtained from the static budget constraint

$$R_t = \left(w_t^*\right)^{\top} S_t + w_{G,t}G_t.$$

Thus, we have

$$G_t w_{G,t} = R_t - \left(w_t^*\right)^{\top} S_t = R_t - \left(w_t^*\right)^{\top} I_S \mathbf{1},$$

where $\mathbf{1}$ is a vector of 1s.

In the following sections we are about to describe in details the roles of each portfolio component shown in (7.8.6).

7.9 The Optimal Wealth

From (7.8.4) and from the optimal portfolio (7.8.6) we can easily see that the relevant wealth for the pension fund is not just R_t. Instead, the fund's decisions are based on a kind of corrected wealth given by

$$\hat{R}_t := R_t - \alpha H_t - \Delta_t, \tag{7.9.1}$$

which is given by the nominal wealth reduced by the minimum amount that the fund wants to have at the end of the management period (αH_t), and net of the prospective mathematical reserve PMR. We stress that both αH_t and Δ_t are amounts of money that the fund needs to "save" in order to be able to face its engagements. Thus, we can argue that once the fund has saved in each period the amount $\alpha H_t + \Delta_t$, the remaining wealth $\hat{R}_t$ can be freely invested in the financial market.

Since at time t_0 we know that, by construction, $\Delta_{t_0} = 0$, the cash flows of contributions and pensions perfectly finance each other and so at inception the fund does not need to saving more than αH_t.

Given the value of the function H_t in (7.8.2), we can write its dynamics as follows

$$dH_t = (H_t (r_t + \lambda_t) - \lambda_t) \, dt + \left(\frac{\partial H_t}{\partial z_t} \right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t^{\mathbb{Q}}, \tag{7.9.2}$$

while the dynamics of Δ_t, given (7.3.2), is

$$d\Delta_t = (\Delta_t (r_t + \lambda_t) - k_t) \, dt + \left(\frac{\partial \Delta_t}{\partial z_t} \right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t^{\mathbb{Q}}. \tag{7.9.3}$$

If we combine (7.9.2), and (7.3.2) with (7.5.2), we can write

$$\begin{aligned} d\hat{R}_t =& dR_t - \alpha dH_t - d\Delta_t \\ =& \left(\hat{R}_t (r_t + \lambda_t) - \lambda_t (q_t R_t - \alpha) \right) dt \\ &+ w_t^{\mathsf{T}} I_S \Sigma_t^{\mathsf{T}} dW_t^{\mathbb{Q}} - \alpha \left(\frac{\partial H_t}{\partial z_t} \right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t^{\mathbb{Q}} - \left(\frac{\partial \Delta_t}{\partial z_t} \right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t^{\mathbb{Q}}, \end{aligned}$$

and, after substituting the optimal portfolio, we get

$$\begin{aligned} d\hat{R}_t^* =& \left(\hat{R}_t^* (r_t + \lambda_t) - \lambda_t \left(q_t R_t^* - \alpha \right) \right) dt \\ &+ \hat{R}_t^* \left(\frac{1}{\delta} \xi_t^{\mathsf{T}} + \frac{1}{F_t} \left(\frac{\partial F_t}{\partial z_t} \right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} \right) dW_t^{\mathbb{Q}}. \end{aligned}$$

Such dynamics allow us to conclude that the modified wealth $\hat{R}_t^*$ grows at the rate $r_t + \lambda_t$ because of both the financial and the actuarial risk, but at any moment in time it could happen (with intensity λ_t) that the management period suddenly ends (because of the worker/pensioner's death) and the fund receives utility from the amount $q_t R_t^* - \alpha$.

An alternative way for writing this differential equation is

$$d\hat{R}_t^* = \left(\hat{R}_t^* \left(r_t + (1-q_t)\lambda_t\right) - \lambda_t \left(q_t \left(\alpha H_t + \Delta_t\right) - \alpha\right)\right) dt$$
$$+ \hat{R}_t^* \left(\frac{1}{\delta}\xi_t^\top + \frac{1}{F_t}\left(\frac{\partial F_t}{\partial z_t}\right)^\top \Omega_t^\top\right) dW_t^{\mathbb{Q}},$$

which is a function of just the modified wealth $\hat{R}_t^*$.

We stress that without any actuarial risk (i.e. $\lambda_t = 0$), the dynamics of wealth is much simpler since under $\mathbb{Q}$ it simply grows at the rate r_t.

Finally, given (7.8.4), we know that

$$\frac{R_t - \alpha H_t - \Delta_t}{F_t} = \frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u - r_u du}},$$

and since the right hand side is always positive and so is F_t, we can conclude that the optimal wealth is always able to finance both the minimum wealth α, and the PMR Δ_t. This means that, in such a framework, with a complete market where any risk can be fully hedged, the fund is always able to face its engagement.

The condition that $R_t - \alpha H_t - \Delta_t > 0$, is the kind of "ratio" that our fund always satisfies through the optimal investment on the financial market.

7.10 The Speculative Portfolio Component

In (7.8.6), if all the state variables are deterministic (i.e. $\Omega_t = \mathbf{0}$), the only remaining portfolio component is the first one

$$I_S w_t^* \big|_{\Omega_t = \mathbf{0}} = \frac{R_t - \alpha H_t - \Delta_t}{\delta} \Sigma_t^{-1} \xi_t := I_S w_{M,t},$$

and this is the reason why it is interpreted as a "speculative" investment. In fact, if the state variables are deterministic, then there is no need to hedge against them and the only reason for investing in risky assets is to obtain a suitable return for financing the payment of pensions.

The hypothesis that $\Omega_t = \mathbf{0}$ is very strong. In fact, on the actual financial market:

- the riskless interest rate is stochastic as shown in Fig. 3.1;
- the risky asset returns are stochastic: on average, the returns on risky assets are quite stable over time and, nevertheless, they present a time varying volatility, as shown in Fig. 2.3 for the S&P500 index;
- the force of mortality is stochastic if we want to take into account the longevity risk.

The portfolio component $I_S w_{M,t}$ is usually identified with the so-called Merton's portfolio, since it coincides with the optimal portfolio originally obtained by Merton (1969) who further assumed that r_t, μ_t, and Σ_t are all constant. In Merton (1969), of course, the portfolio is strictly self-financing and, thus, there are neither inflows nor outflows to the wealth.

We see that the amount of each risky asset held in portfolio is:

- proportional to the market price of risk ξ_t;
- proportional to the fund's wealth net of both the minimum wealth to be reached and the PMR;
- inversely proportional to the risky asset diffusion Σ_t;
- proportional to the fund's risk tolerance (i.e. the inverse of the risk aversion); an infinitely risk averse agent ($\delta \to \infty$) would invest nothing in risky assets and, accordingly, all the wealth would be invested in the riskfree asset.

Given the equation for the market price of risk (4.5.1), we recall that the ratio between the market price of risk and the diffusion term of risky assets Σ_t can also be written as

$$\Sigma_t^{-1}\xi_t = \left(\Sigma_t^{\intercal}\Sigma_t\right)^{-1}(\mu_t - r_t\mathbf{1}).$$

If we assume that: (1) ξ_t and Σ_t are constant, (2) there are no contributions/pensions, and (3) the fund's preferences belong to the CRRA family (i.e. $\alpha = 0$), then the portfolio simplifies to

$$\frac{1}{R_t}\, I_S w_{M,t}\big|_{\Delta_t=0,\alpha=0} = \frac{1}{\delta}\Sigma^{-1}\xi,$$

which means that the percentage of wealth invested in each asset is constant over time. With HARA preference and a PMR, the percentage of wealth invested in risky asset is given by

$$\frac{1}{R_t} I_S w_{M,t} = \left(1 - \frac{\alpha H_t + \Delta_t}{R_t}\right)\frac{1}{\delta}\Sigma_t^{-1}\xi_t.$$

In this case, even if ξ and Σ are constant, the percentage of wealth invested in each asset is not constant and its behaviour depends on the behaviour of both functions H_t and Δ_t. The dynamics of Δ_t has already be shown in Fig. 7.3: it is increasing over time during the accumulation period and then it starts decreasing. The dynamics of H_t depends on the behaviour of both λ_t and r_t, as it is apparent from (7.8.2).

7.11 The Speculative Portfolio Component: A Numerical Example

In this section we present the behaviour of the speculative portfolio component in a model where it is the only component of the optimal portfolio. In order to obtain such a result, we need the hypothesis that there are no stochastic state variables (i.e. $\Omega_t = \mathbf{0}$). This implies that:

1. the riskless interest rate is deterministic (we further assume that it is constant);
2. the force of mortality is deterministic and follows a Gompertz-Makeham model like (5.4.2);
3. contributions and pensions are deterministic and, furthermore, we assume that they are constant;
4. the return on the only risky asset listed on the market has constant average (μ) and variance (σ^2). Thus, the market price of risk is constant $\xi = \frac{\mu - r}{\sigma}$.

Under these assumptions, the function H_t in (7.8.2) takes the following value[3]

$$\begin{aligned} H_t &= \int_t^{\infty} \lambda_s e^{-\int_t^s (r+\lambda_u) du} ds \\ &= 1 - rbe^{(\phi+r)(\iota+t-m)+e^{\frac{\iota+t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{\iota+t-m}{b}} \right), \end{aligned}$$

which can also be written, in differential form, as follows:

$$dH_t = (H_t (r + \lambda_t) - \lambda_t)\, dt.$$

Since r, c, and p are all constant, then the PMR is given by (7.4.1). Furthermore, the differential form of the PMR is

$$d\Delta_t = (\Delta_t (r + \lambda_t) - k_t)\, dt.$$

If we finally call S_t the price of the risky asset, the optimal portfolio in this case is given by

$$S_t w_t^* = \frac{R_t - \alpha H_t - \Delta_t}{\delta} \frac{\mu - r}{\sigma^2},$$

[3]This result can be easily obtained by using (5.4.3) and Proposition 5.1. Furthermore, the following property has been used:

$$\Gamma(s+1, x) = s\Gamma(s, x) + x^s e^{-x}.$$

and the optimal wealth follows the differential equation

$$dR_t = \left(R_t r + \frac{R_t - \alpha H_t - \Delta_t}{\delta} \xi^2 - k_t \right) dt + \frac{R_t - \alpha H_t - \Delta_t}{\delta} \xi dW_t,$$

which is a linear differential equation in R_t whose solution, given the initial wealth R_{t_0}, can be found in closed form as shown in (3.3.3):

$$\begin{aligned} R_t = {} & R_{t_0} e^{\left(r+\frac{1}{\delta}\left(1-\frac{1}{2}\frac{1}{\delta}\right)\xi^2\right)(t-t_0)+\frac{1}{\delta}\xi\left(W_t - W_{t_0}\right)} \\ & + \int_{t_0}^{t} \left(\frac{1-\delta}{\delta} \xi^2 \frac{\alpha H_s + \Delta_s}{\delta} - k_s \right) e^{\left(r+\frac{1}{\delta}\left(1-\frac{1}{2}\frac{1}{\delta}\right)\xi^2\right)(t-s)+\frac{1}{\delta}\xi(W_t - W_s)} ds \\ & + \frac{1}{\delta}\xi \int_{t_0}^{t} e^{\left(r+\frac{1}{\delta}\left(1-\frac{1}{2}\frac{1}{\delta}\right)\xi^2\right)(t-s)+\frac{1}{\delta}\xi(W_t - W_s)} dW_s. \end{aligned}$$

The expected value of R_t at time t_0 is

$$\begin{aligned} \mathbb{E}_{t_0}\left[R_t\right] = {} & R_{t_0} e^{\left(r+\frac{1}{\delta}\xi^2\right)(t-t_0)} \\ & + \int_{t_0}^{t} \left(\frac{1-\delta}{\delta} \xi^2 \frac{\alpha H_s + \Delta_s}{\delta} - k_s \right) e^{\left(r+\frac{1}{\delta}\xi^2\right)(t-s)} ds, \end{aligned}$$

and, accordingly, we can conclude that this expected value is positive if the initial wealth is sufficiently high, i.e.

$$\mathbb{E}_{t_0}\left[R_t\right] > 0 \Longleftrightarrow R_{t_0} > \int_{t_0}^{t} e^{-\left(r+\frac{1}{\delta}\xi^2\right)(s-t_0)} \left(k_s + \frac{\delta - 1}{\delta} \xi^2 \frac{\alpha H_s + \Delta_s}{\delta} \right) ds.$$

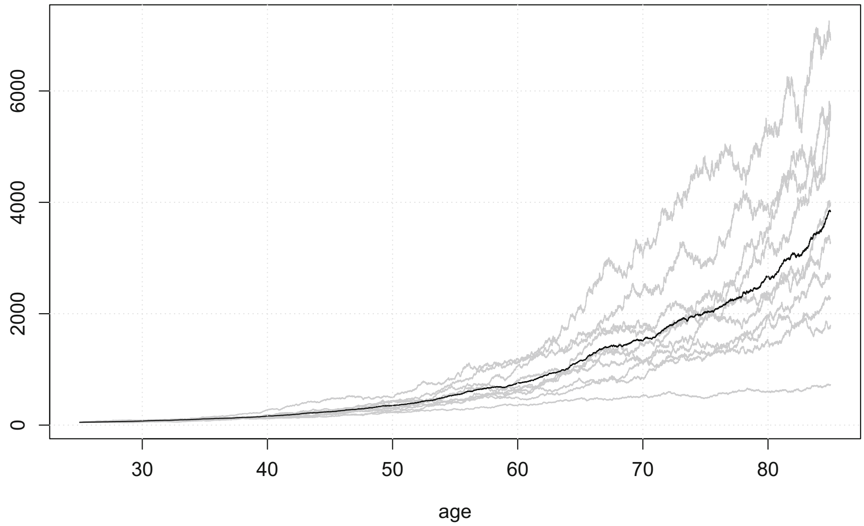
This means that the fund's wealth may become negative at a given point in time if the initial wealth is not sufficiently high. Since $\delta > 1$, we note that the higher α, the higher the initial wealth for preventing the fund's wealth from becoming negative. This result is in line with the budget constraint on the fund's wealth: the initial wealth must be enough for financing the minimum wealth α and, of course, all the other outflows.

The simulation of the optimal portfolio can be easily performed in R through the following commands where we use `as.numeric(`*`condition`*`)` as the indicator function. In fact, it takes value 1 if the "condition" is true and 0 otherwise.

```
M = 10  #number of simulations
N = 60  #number of years
R0 = 50
age = 25
t0 = 0
T = 40  #time to retirement
dt = 1/250
r = 0.05
delta = 2.5
alphaR = R0  #we want to preserve the initial wealth
muS = 0.08
sigmaS = 0.15
xi = (muS - r)/sigmaS
c = 1
p = c * (gamma_inc(-(phi_GM[1] + r) * b_GM[1], exp((age +
    t0 - m_GM[1])/b_GM[1]))/gamma_inc(-(phi_GM[1] +
    r) * b_GM[1], exp((age + T - m_GM[1])/b_GM[1])) -
    1)
H = function(t, age, phi, b, m, r) {
    1 - r * b * exp((phi + r) * (age + t - m) + exp((age +
        t - m)/b)) * gamma_inc(-(phi + r) * b, exp((age +
        t - m)/b))
}
R = array(0, dim = c(N/dt, M))
R[1, ] = rep(R0, M)
ptf = array(NA, dim = c(N/dt, M))
for (i in 2:(N/dt)) {
    ptf[i, ] = (R[i - 1, ] - alphaR * H(t0 + (i - 1) *
        dt, age = age, phi_GM[1], b_GM[1], m_GM[1],
        r) - PMR(c, phi_GM[1], b_GM[1], m_GM[1], r,
        age, t0 + (i - 1) * dt))/(R[i - 1, ] * delta) *
        (muS - r)/sigmaS^2
    dR = (R[i - 1, ] * r + R[i - 1, ] * ptf[i, ] *
        (muS - r) - (p * as.numeric(t0 + (i - 1) *
        dt > T) - c * as.numeric(t0 + (i - 1) * dt <=
        T))) * dt + R[i - 1, ] * ptf[i, ] * sigmaS *
        rnorm(M) * sqrt(dt)
    R[i, ] = R[i - 1, ] + dR
}
```

Through the previous commands, 10 simulations of both the optimal wealth and the optimal portfolio have been created. Now, we can plot them as shown in Figs. 7.4 and 7.5.

In Fig. 7.5, we see that, on average, the optimal speculative portfolio is decreasing over time during the A-Ph and then the riskiness increases slightly during the D-Ph. The reason for this behaviour is very intuitive: during the A-Ph the PMR is increasing since the fund can count on a stream of future contributions which




```
matplot(seq(age, age + N - dt, dt), R, type = "l",
    col = "gray", lty = 1, xlab = "age", ylab = "")
lines(seq(age, age + N - dt, dt), rowMeans(R), type = "l")
grid()
```

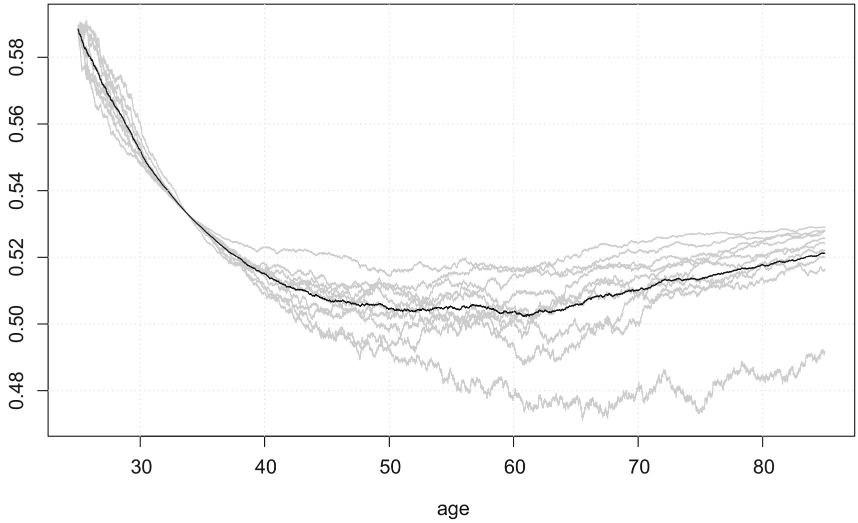
Fig. 7.4 In grey we show 10 simulations of the optimal wealth for a pension fund that does not face any risk on the state variables (i.e. $\Omega_t = \mathbf{0}$). In black we show the mean of the simulations

becomes smaller and smaller, while the total amount of pensions that must be paid keeps unchanged. Thus, the fund must reduce the speculation and reach the retirement date with the lowest share of risky asset. Instead, during the D-Ph the probability that another pension instalment has to be paid in the period ahead becomes lower and lower and, accordingly, the fund can increase the riskiness of its portfolio. In our simulation, the initial wealth is invested in the risky asset for 58.8444157115043% on average. At the age of retirement the share of risky asset is 49.8908064057496% (on average) and at the age of 85 the share is 51.3833872097995% (on average).

The pure Merton's speculative component, with neither minimum wealth nor PMR (i.e. $\alpha = c = p = 0$), would have a share of risky asset equal to

$$\frac{w_t^* S_t}{R_t} = \frac{1}{\delta} \frac{\mu - r}{\sigma^2},$$

whose value is 53.3333333333333%, which is higher than the portfolio shown in Fig. 7.5 and is close, instead, to the optimal portfolio corresponding to a very old




```
matplot(seq(age, age + N - dt, dt), ptf, type = "l",
    col = "gray", lty = 1, xlab = "age", ylab = "")
lines(seq(age, age + N - dt, dt), rowMeans(ptf), type = "l")
grid()
```

Fig. 7.5 In grey we show 10 simulations of the optimal portfolio for a pension fund that does not face any risk on the state variables (i.e. $\Omega_t = \mathbf{0}$). In black we show the mean of the simulations

age. In fact, for a very long financial horizon the PMR becomes closer and closer to zero and the relevance of the minimum wealth is fading away since the discount factor H_t is very small for high values of t.

7.12 Hedging Portfolio Component for Minimum Wealth

In (7.8.6), among the portfolio components aimed at hedging against the stochastic state variables, there is one term that depends on the minimum wealth that the fund wants to reach at the financial horizon. This term can be written as

$$I_S w_{H,t} := \alpha \Sigma_t^{-1} \Omega_t \frac{\partial H_t}{\partial z},$$

and it is the product of two terms:

1. the derivative of the discounted value of α with respect to z_t; from (7.8.2), we know that αH_t is the expected present value of the amount α available at the death time τ;
2. the covariance between the risky assets and the state variables. In fact, since Σ_t is invertible, the term $\Sigma_t^{-1}\Omega_t$ can be written as

$$\Sigma_t^{-1}\Omega_t = \left(\Sigma_t^{\mathsf{T}}\Sigma_t\right)^{-1}\Sigma_t^{\mathsf{T}}\Omega_t,$$

where $\Sigma_t^{\mathsf{T}}\Sigma_t$ is the variance-covariance matrix of asset prices and $\Sigma_t^{\mathsf{T}}\Omega_t$ is the matrix measuring the correlation between the asset returns and the state variables; it is in fact true that

$$\mathbb{C}_t\left[I_S^{-1}dS_t, dz_t\right] = \Sigma_t^{\mathsf{T}}\Omega_t dt.$$

Accordingly, $\Sigma_t^{-1}\Omega_t$ can be seen as the "beta" coefficient (ratio between correlation and variance) between asset returns and state variables.

This portfolio component exists only because the fund wants to keep at least a minimum level of wealth at the financial horizon τ. In fact, if we trace back our model to the CRRA case (with $\alpha = 0$), this portfolio component disappears.

The presence of α deeply affects the optimal portfolio because of the stochastic behaviour of the state variables. At each instant in time, both r_t and λ_t stochastically change and the term αH_t changes accordingly.

This portfolio term also vanishes if the covariance between the asset returns and the state variables is zero. Nevertheless, in this case, the absence of a hedging term does not depend on the fact that we do not need to hedge. Instead, its absence is due to the impossibility to hedge the risk by using the assets. In fact, the assets are not correlated with the risk that we want to hedge against.

7.13 Hedging Portfolio Component for Prospective Mathematical Reserve

The value of the PMR stochastically evolves over time and, accordingly, it is necessary to hedge the portfolio against the changes of Δ_t at each instant in time.

Together with the minimum wealth α, the amount of the PMR is what the fund must save in order to be solvable. In this framework, "to save" means "not to invest in risky assets". The total reserve the fund must save at each time t is given by

$$\alpha H_t + \Delta_t.$$

The stochastic behaviour of this "total reserve" can be computed through Itô's lemma:

$$d\left(\alpha H_t + \Delta_t\right) = (\ldots)\, dt + \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}} dW_t,$$

where the drift term has been omitted since it is immaterial to our purposes. Given the dynamics of fund's wealth (7.5.2), we can find the portfolio that replicates, at any instant, such a reserve. As shown in Sect. 4.6, the replicating portfolio $w_{\Delta,t}$ is such that the diffusion term of $d\left(\alpha H_t + \Delta_t\right)$ and the diffusion term of fund's wealth are equal:

$$w_{\Delta,t}^{\mathsf{T}} I_S \Sigma_t^{\mathsf{T}} = \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right)^{\mathsf{T}} \Omega_t^{\mathsf{T}}.$$

Since the market is complete (i.e. Σ_t is invertible), the only solution to this equation is

$$I_S w_{\Delta,t} = \Sigma_t^{-1} \Omega_t \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t}\right),$$

which is a component of the optimal portfolio (7.8.6). Thus, we can conclude that this portfolio component replicates, at each time, the total wealth that must be put in a reserve for being able both to have a sufficient level of wealth (α) at the financial horizon (τ) and to finance the net outflows.

Also in this case the coefficient of this portfolio is $\Sigma_t^{-1}\Omega_t$ which measures the correlation between the asset returns and the state variables.

7.14 Hedging Portfolio Component for Discount Factor

The last portfolio component we have to analyse is the one which hedges the portfolio against the stochastic fluctuations of the function F_t as defined in (7.8.3). From Sect. 4.6, we know that the price kernel $m_{t_0,t}$ is a martingale, but any power of it is not. In fact, if we apply Itô's lemma to $m_{t_0,t}^{\kappa}$ for any real number κ, we obtain

$$\frac{d\left(m_{t_0,t}^{\kappa}\right)}{m_{t_0,t}^{\kappa}} = \frac{1}{2}\kappa\left(\kappa - 1\right) \xi_t{}^{\mathsf{T}} \xi_t dt - \kappa \xi_t^{\mathsf{T}} dW_t,$$

which is not a martingale since its drift is not zero.

We can check whether there exists a function of time f_t that allows to transform $m^{\kappa}_{t_0,t}$ into a martingale in the following form:

$$\frac{d\left(m^{\kappa}_{t_0,t} f_t\right)}{m^{\kappa}_{t_0,t} f_t} = \left(\frac{\partial f_t}{\partial t}\frac{1}{f_t} + \frac{1}{2}\kappa\left(\kappa - 1\right)\xi_t^{\mathsf{T}}\xi_t\right) dt - \kappa \xi_t^{\mathsf{T}} dW_t.$$

Thus, the function f_t must solve

$$\frac{\partial f_t}{\partial t}\frac{1}{f_t} + \frac{1}{2}\kappa\left(\kappa - 1\right)\xi_t^{\mathsf{T}}\xi_t = 0,$$

with the boundary condition that $f_{t_0} = 1$ in order to preserve the martingale property. This differential equation has a unique solution

$$f_t = e^{-\frac{1}{2}\kappa(\kappa-1)\int_{t_0}^{t}\xi_s^{\mathsf{T}}\xi_s ds}.$$

We can accordingly conclude that if $m_{t_0,t}$ is a martingale, then

$$m^{\kappa}_{t_0,t} e^{-\frac{1}{2}\kappa(\kappa-1)\int_{t_0}^{t}\xi_s^{\mathsf{T}}\xi_s ds}, \tag{7.14.1}$$

is a martingale too.

Remark 7.8 The martingale $m_{t_0,t}$ allows to switch from the historical to the risk neutral probability through Girsanov's theorem:

$$dW_t^{\mathbb{Q}} = \xi_t dt + dW_t.$$

The new martingale (7.14.1) allows to switch from the historical to a new probability that we could call $\mathbb{Q}_{\kappa}$ (since it depends on the parameter κ) and the corresponding Girsanov's theorem is

$$dW_t^{\mathbb{Q}_{\kappa}} = \kappa\xi_t dt + dW_t.$$

The function F_t can be written under a new probability measure with three passages:

1. we switch initially from $\mathbb{Q}$ to $\mathbb{P}$:

$$\begin{aligned} F_t &\equiv \mathbb{E}^{\mathbb{Q},\tau}_{t,t}\left[m_{t,\tau}^{-\frac{1}{\delta}} e^{-\int_t^{\tau}\frac{\delta-1}{\delta}r_u + \frac{1}{\delta}\rho_u du}\right] \\ &= \mathbb{E}^{\tau}_{t}\left[m_{t,\tau}^{1-\frac{1}{\delta}} e^{-\int_t^{\tau}\frac{\delta-1}{\delta}r_u + \frac{1}{\delta}\rho_u du}\right], \end{aligned}$$

2. then we use the previously defined function f_t (with $\kappa = 1 - \frac{1}{\delta}$) to write

$$F_t = \mathbb{E}_t^\tau \left[m_{t,\tau}^{1-\frac{1}{\delta}} \frac{e^{\frac{1}{2}\left(1-\frac{1}{\delta}\right)\frac{1}{\delta}\int_t^\tau \xi_s^\top \xi_s ds}}{e^{\frac{1}{2}\left(1-\frac{1}{\delta}\right)\frac{1}{\delta}\int_t^\tau \xi_s^\top \xi_s ds}} e^{-\int_t^\tau \frac{\delta-1}{\delta} r_u + \frac{1}{\delta}\rho_u du} \right],$$

3. and, finally, if we define the new probability by

$$dW_t^{\mathbb{Q}_\delta} = \frac{\delta - 1}{\delta} \xi_t dt + dW_t, \tag{7.14.2}$$

we obtain

$$F_t = \mathbb{E}_{t,t}^{\mathbb{Q}_\delta,\tau} \left[e^{-\int_t^\tau \left(\frac{\delta-1}{\delta} r_u + \frac{1}{\delta}\rho_u + \frac{1}{2}\frac{\delta-1}{\delta}\frac{1}{\delta}\xi_u^\top \xi_u \right) du} \right], \tag{7.14.3}$$

where we see that the function F_t is the expected value, under a new "subjective" probability measure, of a discount factor which is a combination of: (1) the riskless interest rate, (2) the subjective discount rate, (3) the square of the market price of risk, and (4) the force of mortality (implicit in the stochastic death time τ).

Remark 7.9 If an agent has log preferences (i.e. $\delta = 1$) then, the function F_t is computed under the historical probability, in fact:

$$dW_t^{\mathbb{Q}_\delta}\Big|_{\delta=1} = dW_t.$$

Instead, if an agent is infinitely risk averse (i.e. $\delta \to \infty$) then he/she computes F_t under the risk neutral probability, in fact:

$$dW_t^{\mathbb{Q}_\delta}\Big|_{\delta\to\infty} = \xi_t dt + dW_t = dW_t^{\mathbb{Q}}.$$

The new "subjective" probability measure $\mathbb{Q}_\delta$ is a weighted mean of the historical and the risk neutral probabilities. It is easy to show that by using twice Girsanov's theorem:

$$\begin{cases} dW_t^{\mathbb{Q}_\delta} = \left(1 - \frac{1}{\delta}\right) \xi_t dt + dW_t, \\ dW_t^{\mathbb{Q}} = \xi_t dt + dW_t, \end{cases}$$

the following result is obtained

$$dW_t^{\mathbb{Q}_\delta} = \left(1 - \frac{1}{\delta}\right) dW_t^{\mathbb{Q}} + \frac{1}{\delta} dW_t. \tag{7.14.4}$$

Since δ is a positive parameter, the terms $\frac{1}{\delta}$ and $1 - \frac{1}{\delta}$ are positive and sum up to 1, that is they are the weights of a weighted mean. The higher the risk aversion δ, the closer the subjective probability ($\mathbb{Q}_\kappa$) to the risk neutral probability ($\mathbb{Q}$).

The same weights are used for the discounting rate in defining the function F_t as in (7.14.3). The term $\frac{1}{\delta}$ is the weight of the subjective discount factor ρ_t, while $1 - \frac{1}{\delta}$ is the weight of the riskless interest rate, and this result is consistent with (7.14.4).

Remark 7.10 When the risk aversion (δ) tends towards infinity, the subjective probability $\mathbb{Q}_\delta$ coincides with the risk neutral probability and the discount rate in F_t coincides with the riskless interest rate:

$$\lim_{\delta \to \infty} \left(1 - \frac{1}{\delta}\right) r_t + \frac{1}{\delta}\rho_t + \frac{1}{2}\frac{\delta - 1}{\delta}\frac{1}{\delta}\xi_t^\mathsf{T}\xi_t = r_t.$$

Thus, we can conclude

$$\lim_{\delta \to \infty} F_t = H_t.$$

The portfolio component that hedges against the stochastic changes in the function F_t is

$$I_S w_{F,t} := \frac{R_t - \alpha H_t - \Delta_t}{F_t} \Sigma_t^{-1} \Omega_t \frac{\partial F_t}{\partial z_t},$$

which contains the semi-elasticity of F_t with respect to the state variables z_t.[4]

The presence of this component in the optimal portfolio comes from the need to hedge against the stochastic behaviour of a discount factor that accounts for all the risk sources on the financial market and takes into account also the risk aversion (δ).

The results shown in the last sections can be summarised as in Table 7.1 where we split the portfolio into three components: (1) the speculative component, (2) the component which replicates the total reserve $\alpha H_t + \Delta_t$, and (3) the component which hedges against the stochastic shocks on the discount factor F_t.

[4]Recall that $\frac{1}{F_t}\frac{\partial F_t}{\partial z_t} = \frac{\partial \ln F_t}{\partial z_t}$.

Table 7.1 Components of the optimal portfolio for a pension fund, as shown in (7.8.6)

Type	Algebra
Speculative	$I_S w_{M,t} = \frac{R_t - \alpha H_t - \Delta_t}{\delta} \Sigma_t^{-1} \xi_t$
Replicating	$I_S w_{\Delta,t} = \Sigma_t^{-1} \Omega_t \left(\alpha \frac{\partial H_t}{\partial z_t} + \frac{\partial \Delta_t}{\partial z_t} \right)$
Hedging	$I_S w_{F,t} = \frac{R_t - \alpha H_t - \Delta_t}{F_t} \Sigma_t^{-1} \Omega_t \frac{\partial F_t}{\partial z_t}$

7.15 The Case of an Incomplete Market

When the market is incomplete (i.e. the matrix $\Sigma^\top$ is not invertible in (4.5.1)) it is not possible to replicate the optimal wealth, since the matrix equation (7.8.5) has no solution. This means that we cannot find any portfolio that is able to behave exactly like the optimal wealth at any instant in time.

Let us call $\Pi_t^\top$ the right hand side of (7.8.5), then the market incompleteness implies that the following equation has no solution:

$$w_t^\top I_S \Sigma_t^\top = \Pi_t^\top .$$

One alternative strategy, in this case, would be to find the portfolio that allows to stay “as close as possible” to Π_t. The distance to be minimised can be computed in many different ways and one of the most common is the square distance. Thus, we can look for the minimum-least-square-portfolio (MLS)

$$w_{MLS,t} = \arg\min_{w_t} \left(w_t^\top I_S \Sigma_t^\top - \Pi_t^\top \right) \left(\Sigma_t I_S w_t - \Pi_t \right) .$$

The first order condition (FOC) on this problem is

$$2 I_S \Sigma_t^\top \Sigma_t I_S w_{MLS,t} - 2 I_S \Sigma_t^\top \Pi_t = 0,$$

while the second order condition asks for the matrix $\Sigma_t^\top \Sigma_t$ to be positive definite, which is always true for a variance-covariance matrix. The only solution to the FOC is

$$I_S w_{MLS,t} = \left(\Sigma_t^\top \Sigma_t \right)^{-1} \Sigma_t^\top \Pi_t , \tag{7.15.1}$$

where the variance-covariance matrix is invertible because of the non arbitrage condition. In fact, the market is arbitrage free if, after eliminating all the redundant assets, the number of assets is not greater than the number of risk sources (i.e. Wiener drivers). This implies that the matrix $\Sigma_t^\top$ has full row rank and, accordingly, $\Sigma_t^\top \Sigma_t$ is invertible.

If we compare the optimal portfolio in (7.8.6) and the MLS portfolio in (7.15.1), we see that the matrix Σ_t^{-1} in the complete market is substituted by the matrix $\left(\Sigma_t^\top \Sigma_t \right)^{-1} \Sigma_t^\top$ in the incomplete market.

The (square) minimum error is

$$\min_{w_t} \left(w_t^\top I_S \Sigma_t^\top - \Pi_t^\top \right) \left(\Sigma_t I_S w_t - \Pi_t \right)$$
$$= \Pi_t^\top \left(I - \Sigma_t \left(\Sigma_t^\top \Sigma_t \right)^{-1} \Sigma_t^\top \right) \Pi_t,$$

where the matrix $\left(I - \Sigma_t \left(\Sigma_t^\top \Sigma_t \right)^{-1} \Sigma_t^\top \right)$ is idempotent and it measures, in some way, the degree of market incompleteness. Of course, if the market is complete, i.e. the matrix $\Sigma_t^\top$ is invertible, then this idempotent matrix is zero, and the MLS portfolio coincides with the optimal portfolio:

$$\exists \left(\Sigma_t^\top \right)^{-1} \Rightarrow w_{MLS,t} = w_t^*,$$

and, accordingly, the MLS portfolio perfectly replicates the optimal wealth.

In an incomplete market there exist infinite vectors ξ_t, i.e. market prices of risk, that solve (4.5.1) and, accordingly, we have to choose one of them which is consistent with the minimum least square approach. Thus, we take the market price of risk ξ_t that minimises the square $\xi_t^\top \xi_t$, under the condition that there is no arbitrage on the financial market. In algebraic terms:

$$\xi_{MLS,t} = \arg\min_{\xi_t} \xi_t^\top \xi_t$$
$$\text{s.t.}$$
$$\Sigma_t^\top \xi_t = \mu_t - r_t \mathbf{1}.$$

The Lagrangian of this problem is

$$\mathcal{L} = \xi_t^\top \xi_t - l^\top \left(\Sigma_t^\top \xi_t - \mu_t + r_t \mathbf{1} \right),$$

where l is the vector of the Lagrange multipliers. The first order condition is

$$\frac{\partial \mathcal{L}}{\partial \xi_t} = 2\xi_{MLS,t} - \Sigma_t l = 0,$$

and the second order condition is trivially satisfied for a minimum. The candidate for the minimum is

$$\xi_{MLS,t} = \frac{1}{2} \Sigma_t l,$$

which can be substituted into the constraint in order to obtain the value of the multiplier

$$\Sigma_t^\top \xi_{MLS,t} = \mu_t - r_t \mathbf{1} \Rightarrow l = 2 \left(\Sigma_t^\top \Sigma_t \right)^{-1} \left(\mu_t - r_t \mathbf{1} \right).$$

Thus, we can finally write

$$\xi_{MLS,t} = \Sigma_t \left(\Sigma_t^\intercal \Sigma_t\right)^{-1} (\mu_t - r_t \mathbf{1}),$$

which is consistent with the whole framework. In fact, also in this case, the matrix $\left(\Sigma_t^\intercal\right)^{-1}$ is substituted by $\Sigma_t \left(\Sigma_t^\intercal \Sigma_t\right)^{-1}$ and they coincide when the market is complete.

7.16 The Role of Longevity Bonds and Ordinary Bonds

In this section we present the case of a pension fund that faces two risks: the interest rate risk and the longevity risk. Accordingly, we model r_t and λ_t as the solutions to the following stochastic differential equations:

$$\begin{aligned} dr_t &= \mu_r(t, r_t)\, dt + \sigma_r(t, r_t)\, dW_{r,t}, \\ d\lambda_t &= \mu_\lambda(t, \lambda_t)\, dt + \sigma_\lambda(t, \lambda_t)\, dW_{\lambda,t}, \end{aligned}$$

where we have assumed that these two processes are instantaneously independent, i.e. $\mathbb{C}_t[dr_t, d\lambda_t] = 0$.

Remark 7.11 It is very important to stress the difference between independence and instantaneous independence of two stochastic processes. Given the two processes

$$\begin{aligned} dx_t &= \mu_x(t, x_t, y_t)\, dt + \sigma_{x,t}^\intercal dW_t, \\ dy_t &= \mu_y(t, x_t, y_t)\, dt + \sigma_{y,t}^\intercal dW_t, \end{aligned}$$

they are instantaneously independent if $\sigma_{x,t}^\intercal \sigma_{y,t} = 0$, even if they are not independent since their drift terms depend on one another. Thus, we can conclude that the instantaneous changes in x_t and y_t are independent, but the processes x_t and y_t do depend on each other.

On the financial market three risky assets are listed.

1. A zero-coupon bond whose value is:

$$\frac{dB(t,T)}{B(t,T)} = \left(r_t + \nabla_{B,r}\sigma_r\xi_r\right) dt + \nabla_{B,r}\sigma_r dW_{r,t},$$

and which can be seen as a derivative on the interest rate r_t, where the term $\nabla_{B,r}$ is the semi-elasticity of the bond with respect to the interest rate:

$$\nabla_{B,r} := \frac{\partial B_t}{\partial r_t}\frac{1}{B_t}.$$

This semi-elasticity is often approximated through the duration.

2. A stock:

$$\frac{dS_t}{S_t} = (r_t + \sigma_S \xi_S + \sigma_{Sr}\xi_r)\,dt + \sigma_S dW_{S,t} + \sigma_{Sr} dW_{r,t},$$

which is correlated with the bond but which is also driven by a risk source of its own. Note that the instantaneous correlation of the two first risky assets is

$$\mathbb{C}_t\left[\frac{dB(t,T)}{B(t,T)}, \frac{dS_t}{S_t}\right] = \nabla_{B,r}\sigma_r\sigma_{Sr}dt.$$

3. One risky asset which is a derivative on the human life (actuarial bond) whose price solves:

$$\frac{dL(t,T)}{L(t,T)} = \left(r_t + \nabla_{L,r}\sigma_r\xi_r + \nabla_{L,\lambda}\sigma_\lambda\xi_\lambda\right)dt$$
$$+ \nabla_{L,r}\sigma_r dW_{r,t} + \nabla_{L,\lambda}\sigma_\lambda dW_{\lambda,t},$$

where $\nabla_{L,r}$ and $\nabla_{L,\lambda}$ are the semi-elasticities of this bond with respect to the interest rate and the force of mortality respectively:

$$\nabla_{L,r} := \frac{\partial L(t,T)}{\partial r_t}\frac{1}{L(t,T)}, \quad \nabla_{L,\lambda} := \frac{\partial L(t,T)}{\partial \lambda_t}\frac{1}{L(t,T)}.$$

Here, we assume that the actuarial asset is correlated with both the interest rate and the force of mortality, but it is not correlated with the stock.

If we define dW_t as the vector which contains all the Wiener processes

$$dW_t = \begin{bmatrix} dW_{r,t} \\ dW_{\lambda,t} \\ dW_{S,t} \end{bmatrix},$$

then the dynamics of the vector of the state variables $z_t = \begin{bmatrix} r_t \\ \lambda_t \end{bmatrix}$ can be written as

$$\begin{bmatrix} dr_t \\ d\lambda_t \end{bmatrix} = \begin{bmatrix} \mu_r \\ \mu_\lambda \end{bmatrix} dt + \underbrace{\begin{bmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\lambda & 0 \end{bmatrix}}_{\Omega^\top} \begin{bmatrix} dW_{r,t} \\ dW_{\lambda,t} \\ dW_{S,t} \end{bmatrix},$$

while the financial market can be summarised through the following matrix differential equation:

$$\begin{bmatrix} \frac{dB(tT)}{B(t,T)} \\ \frac{dS_t}{S_t} \\ \frac{dL(t,T)}{L(t,T)} \end{bmatrix} = \begin{bmatrix} r_t + \nabla_{B,r}\sigma_r\xi_r \\ r_t + \sigma_S\xi_S + \sigma_{Sr}\xi_r \\ r_t + \nabla_{L,r}\sigma_r\xi_r + \nabla_{L,\lambda}\sigma_\lambda\xi_\lambda \end{bmatrix} dt + \underbrace{\begin{bmatrix} \nabla_{B,r}\sigma_r & 0 & 0 \\ \sigma_{Sr} & 0 & \sigma_S \\ \nabla_{L,r}\sigma_r & \nabla_{L,\lambda}\sigma_\lambda & 0 \end{bmatrix}}_{\Sigma^\intercal} \begin{bmatrix} dW_{r,t} \\ dW_{\lambda,t} \\ dW_{S,t} \end{bmatrix}.$$

This market is complete since the matrix $\Sigma_t^\intercal$ is invertible (in fact, its determinant is $-\nabla_{L,\lambda}\sigma_\lambda\sigma_S\nabla_{B,r}\sigma_r \neq 0$) and its inverse is

$$\Sigma_t^{-1} = \begin{bmatrix} \frac{1}{\nabla_{B,r}\sigma_r} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}\sigma_\lambda} & -\frac{\sigma_{Sr}}{\nabla_{B,r}\sigma_r\sigma_S} \\ 0 & 0 & \frac{1}{\sigma_S} \\ 0 & \frac{1}{\nabla_{L,\lambda}\sigma_\lambda} & 0 \end{bmatrix}.$$

The matrix products necessary for computing the optimal portfolio are as follows:

$$\Sigma_t^{-1}\Omega_t = \begin{bmatrix} \frac{1}{\nabla_{B,r}\sigma_r} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}\sigma_\lambda} & -\frac{\sigma_{Sr}}{\nabla_{B,r}\sigma_r\sigma_S} \\ 0 & 0 & \frac{1}{\sigma_S} \\ 0 & \frac{1}{\nabla_{L,\lambda}\sigma_\lambda} & 0 \end{bmatrix} \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_\lambda \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\nabla_{B,r}} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}} \\ 0 & 0 \\ 0 & \frac{1}{\nabla_{L,\lambda}} \end{bmatrix},$$

$$\Sigma_t^{-1}\xi_t = \begin{bmatrix} \frac{1}{\nabla_{B,r}\sigma_r} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}\sigma_\lambda} & -\frac{\sigma_{Sr}}{\nabla_{B,r}\sigma_r\sigma_S} \\ 0 & 0 & \frac{1}{\sigma_S} \\ 0 & \frac{1}{\nabla_{L,\lambda}\sigma_\lambda} & 0 \end{bmatrix} \begin{bmatrix} \xi_r \\ \xi_\lambda \\ \xi_S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\xi_r}{\nabla_{B,r}\sigma_r} - \frac{\nabla_{L,r}\xi_\lambda}{\nabla_{B,r}\nabla_{L,\lambda}\sigma_\lambda} - \frac{\sigma_{Sr}\xi_S}{\nabla_{B,r}\sigma_r\sigma_S} \\ \frac{\xi_S}{\sigma_S} \\ \frac{\xi_\lambda}{\nabla_{L,\lambda}\sigma_\lambda} \end{bmatrix}.$$

Since there are two state variables on this market, then the vectors $\frac{\partial F_t}{\partial z_t}$ and $\frac{\partial \Delta_t}{\partial z_t}$ contain two terms each and can be written as

$$\frac{\partial F_t}{\partial z_t} = \begin{bmatrix} \frac{\partial F_t}{\partial r_t} \\ \frac{\partial F_t}{\partial \lambda_t} \end{bmatrix}, \quad \frac{\partial \Delta_t}{\partial z_t} = \begin{bmatrix} \frac{\partial \Delta_t}{\partial r_t} \\ \frac{\partial \Delta_t}{\partial \lambda_t} \end{bmatrix}.$$

Finally, the following products are computed:

$$\Sigma_t^{-1}\Omega_t\frac{\partial F_t}{\partial z_t}=\begin{bmatrix}\frac{1}{\nabla_{B,r}} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\\ 0 & 0\\ 0 & \frac{1}{\nabla_{L,\lambda}}\end{bmatrix}\begin{bmatrix}\frac{\partial F_t}{\partial r_t}\\ \frac{\partial F_t}{\partial \lambda_t}\end{bmatrix}=\begin{bmatrix}\frac{1}{\nabla_{B,r}}\frac{\partial F_t}{\partial r_t}-\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\frac{\partial F_t}{\partial \lambda_t}\\ 0\\ \frac{1}{\nabla_{L,\lambda}}\frac{\partial F_t}{\partial \lambda_t}\end{bmatrix},$$

$$\Sigma_t^{-1}\Omega_t\frac{\partial \Delta_t}{\partial z_t}=\begin{bmatrix}\frac{1}{\nabla_{B,r}} & -\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\\ 0 & 0\\ 0 & \frac{1}{\nabla_{L,\lambda}}\end{bmatrix}\begin{bmatrix}\frac{\partial \Delta_t}{\partial r_t}\\ \frac{\partial \Delta_t}{\partial \lambda_t}\end{bmatrix}=\begin{bmatrix}\frac{1}{\nabla_{B,r}}\frac{\partial \Delta_t}{\partial r_t}-\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\frac{\partial \Delta_t}{\partial \lambda_t}\\ 0\\ \frac{1}{\nabla_{L,\lambda}}\frac{\partial \Delta_t}{\partial \lambda_t}\end{bmatrix},$$

and the optimal portfolio is

$$\begin{aligned}\begin{bmatrix}B(t,T)\,w_{B,t}\\ S_t w_{S,t}\\ L(t,T)\,w_{L,t}\end{bmatrix}=&\frac{R_t-\Delta_t}{\delta}\begin{bmatrix}\frac{\xi_r}{\nabla_{B,r}\sigma_r}-\frac{\nabla_{L,r}\xi_\lambda}{\nabla_{B,r}\nabla_{L,\lambda}\sigma_\lambda}-\frac{\sigma_{Sr}\xi_S}{\nabla_{B,r}\sigma_r\sigma_S}\\ \frac{\xi_S}{\sigma_S}\\ \frac{\xi_\lambda}{\nabla_{L,\lambda}\sigma_\lambda}\end{bmatrix}\\ &+\frac{R_t-\Delta_t}{F_t}\begin{bmatrix}\frac{1}{\nabla_{B,r}}\frac{\partial F_t}{\partial r_t}-\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\frac{\partial F_t}{\partial \lambda_t}\\ 0\\ \frac{1}{\nabla_{L,\lambda}}\frac{\partial F_t}{\partial \lambda_t}\end{bmatrix}\\ &+\begin{bmatrix}\frac{1}{\nabla_{B,r}}\frac{\partial \Delta_t}{\partial r_t}-\frac{\nabla_{L,r}}{\nabla_{B,r}\nabla_{L,\lambda}}\frac{\partial \Delta_t}{\partial \lambda_t}\\ 0\\ \frac{1}{\nabla_{L,\lambda}}\frac{\partial \Delta_t}{\partial \lambda_t}\end{bmatrix},\end{aligned}$$

or, in other terms:

$$\begin{aligned}B(t,T)\,w_{B,t}^*=&\underbrace{\frac{R_t-\Delta_t}{\delta}\left(\frac{\xi_r}{\nabla_{B,r}\sigma_r}-\frac{\sigma_{Sr}\xi_S}{\nabla_{B,r}\sigma_r\sigma_S}\right)+\frac{R_t-\Delta_t}{F_t}\frac{1}{\nabla_{B,r}}\frac{\partial F_t}{\partial r_t}+\frac{1}{\nabla_{B,r}}\frac{\partial \Delta_t}{\partial r_t}}_{B(t,T)w_{B,0}^*}\\ &-\frac{\nabla_{L,r}}{\nabla_{B,r}}L(t,T)\,w_{L,t}^*,\end{aligned}\tag{7.16.1}$$

$$S_t w_{S,t}^*=\frac{R_t-\Delta_t}{\delta}\frac{\xi_S}{\sigma_S},\tag{7.16.2}$$

$$L(t,T)\,w_{L,t}^*=\frac{R_t-\Delta_t}{\delta}\frac{\xi_\lambda}{\nabla_{L,\lambda}\sigma_\lambda}+\frac{R_t-\Delta_t}{F_t}\frac{1}{\nabla_{L,\lambda}}\frac{\partial F_t}{\partial \lambda_t}+\frac{1}{\nabla_{L,\lambda}}\frac{\partial \Delta_t}{\partial \lambda_t}.\tag{7.16.3}$$

From this result we can draw some relevant conclusions.

1. The stock does not play any hedging role and it is just used for speculative purposes. All the hedging portfolio components, in this case, are optimally set to zero and only the Merton's speculative component remains. Even if the stock

is correlated with the bond, we see that the stock is not used for hedging against the interest rate risk. This means that, in this optimal portfolio, only the risky asset with the highest correlation with the interest rate risk is used for hedging against that risk.

2. The demographic asset is used just for hedging against the demographic risk, even if this asset is correlated with the interest rate. In fact, in the optimal demographic asset allocation there are no derivatives with respect to the interest rate. This result is consistent with what we have already argued in the previous point: only the risky asset with the highest correlation with the longevity risk is used for hedging against that risk.
3. The money that must be invested in the demographic asset is partially taken from the money invested in the bond and partially form the money invested in the riskless asset. The proportion taken form the bond is $\frac{\nabla_{L,r}}{\nabla_{B,r}}$, which is the ratio between the semi-elasticities of the actuarial and the ordinary bond with respect to the interest rate. If we assume that these two bonds have the same semi-elasticity (duration), then the amount of money to be invested in demographic asset is fully taken from the money invested in ordinary bond.
4. The ordinary bond is used for hedging against both the demographic risk and the interest rate risk. In fact, it contains both derivatives with respect to r_t and λ_t.

7.17 The Role of Longevity Bonds and Ordinary Bonds in an Incomplete Market

In this section we take into account the same framework as in the previous section, but without the possibility to invest in a demographic asset. In other words, we study the case of an incomplete market. In this case the market can be summarised through the following matrix differential equation:

$$\begin{bmatrix} \frac{dB(t,T)}{B(t,T)} \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r_t + \nabla_{B,r}\sigma_r\xi_r \\ r_t + \sigma_S\xi_S + \sigma_{Sr}\xi_r \end{bmatrix} dt + \underbrace{\begin{bmatrix} \nabla_{B,r}\sigma_r & 0 & 0 \\ \sigma_{Sr} & 0 & \sigma_S \end{bmatrix}}_{\Sigma^\top} \begin{bmatrix} dW_{r,t} \\ dW_{\lambda,t} \\ dW_{S,t} \end{bmatrix},$$

where we see that there is no asset which depends on the risk source given by $dW_{\lambda,t}$. Here, the market prices of risk ξ_r and ξ_S must be interpreted in the “least square” sense we have exposed in the previous sections.

In this framework, one relevant matrix is the following one:

$$
\begin{aligned}
&\left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \Sigma_t^\intercal \\
&= \left(\begin{bmatrix} \nabla_{B,r}\sigma_r & 0 & 0 \\ \sigma_{Sr} & 0 & \sigma_S \end{bmatrix} \begin{bmatrix} \nabla_{B,r}\sigma_r & \sigma_{Sr} \\ 0 & 0 \\ 0 & \sigma_S \end{bmatrix} \right)^{-1} \begin{bmatrix} \nabla_{B,r}\sigma_r & 0 & 0 \\ \sigma_{Sr} & 0 & \sigma_S \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\nabla_{B,r}\sigma_r} & 0 & -\frac{\sigma_{Sr}}{\nabla_{B,r}\sigma_r\sigma_S} \\ 0 & 0 & \frac{1}{\sigma_S} \end{bmatrix},
\end{aligned}
$$

and, accordingly, we obtain

$$
\begin{aligned}
&\left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \Sigma_t^\intercal \Omega_t \\
&= \begin{bmatrix} \frac{1}{\nabla_{B,r}\sigma_r} & 0 & -\frac{\sigma_{Sr}}{\nabla_{B,r}\sigma_r\sigma_S} \\ 0 & 0 & \frac{1}{\sigma_S} \end{bmatrix} \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_\lambda \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\nabla_{B,r}} & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \Sigma_t^\intercal \xi_{MLS,t} = \left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \left(\mu_t - r_t \mathbf{1}\right) \\
&= \frac{1}{\nabla_{B,r}^2 \sigma_r^2 \sigma_S^2} \begin{bmatrix} \sigma_{Sr}^2 + \sigma_S^2 & -\nabla_{B,r}\sigma_r\sigma_{Sr} \\ -\nabla_{B,r}\sigma_r\sigma_{Sr} & \nabla_{B,r}^2\sigma_r^2 \end{bmatrix} \begin{bmatrix} \nabla_{B,r}\sigma_r\xi_r \\ \sigma_S\xi_S + \sigma_{Sr}\xi_r \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma_S\xi_r - \sigma_{Sr}\xi_S}{\nabla_{B,r}\sigma_r\sigma_S} \\ \frac{\xi_S}{\sigma_S} \end{bmatrix}.
\end{aligned}
$$

Finally, the portfolio hedging components are obtained as follows

$$
\left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \Sigma_t^\intercal \Omega_t \frac{\partial F_t}{\partial z_t} = \begin{bmatrix} \frac{1}{\nabla_{B,r}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F_t}{\partial r_t} \\ \frac{\partial F_t}{\partial \lambda_t} \end{bmatrix} = \begin{bmatrix} \frac{1}{\nabla_{B,r}} \frac{\partial F_t}{\partial r_t} \\ 0 \end{bmatrix},
$$

$$
\left(\Sigma_t^\intercal \Sigma_t\right)^{-1} \Sigma_t^\intercal \Omega_t \frac{\partial \Delta_t}{\partial z_t} = \begin{bmatrix} \frac{1}{\nabla_{B,r}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta_t}{\partial r_t} \\ \frac{\partial \Delta_t}{\partial \lambda_t} \end{bmatrix} = \begin{bmatrix} \frac{1}{\nabla_{B,r}} \frac{\partial \Delta_t}{\partial r_t} \\ 0 \end{bmatrix},
$$

and the optimal portfolio is

$$
\begin{aligned}
\begin{bmatrix} B(t,T)\, w_{MLS,B}^* \\ S_t w_{MLS,S}^* \end{bmatrix} =& \frac{R_t - \Delta_t}{\delta} \begin{bmatrix} \frac{\xi_r}{\nabla_{B,r}\sigma_r} - \frac{\sigma_{Sr}\xi_S}{\nabla_{B,r}\sigma_r\sigma_S} \\ \frac{\xi_S}{\sigma_S} \end{bmatrix} \\
&+ \frac{R_t - \Delta_t}{F_t} \begin{bmatrix} \frac{1}{\nabla_{B,r}} \frac{\partial F_t}{\partial r_t} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\nabla_{B,r}} \frac{\partial \Delta_t}{\partial r_t} \\ 0 \end{bmatrix},
\end{aligned}
$$

Table 7.2 The comparison between optimal portfolios in three models: (I) a complete market whose only state variable is the riskless interest rate, (II) an incomplete market where there exists a longevity risk which cannot be hedged because a correlated asset is missing, (III) a complete market where a longevity risk can be hedged because of the presence of an actuarial asset

	I	II	III
Longevity risk	No	Yes	Yes
Longevity asset	No	No	Yes
Completeness	Yes	No	Yes
Optimal ptf. (bond)	$B(t,T)\,w^*_{B,0}$		$B(t,T)\,w^*_{B,0} - \frac{\nabla_{L,r}}{\nabla_{B,r}} L(t,T)\,w^*_{L,t}$
Optimal ptf. (stock)	$S_t w^*_{S,t}$		$S_t w^*_{S,t}$
Optimal ptf. (longevity)	–		$L(t,T)\,w^*_{L,t}$
MLS ptf. (bond)		$B(t,T)\,w^*_{B,0}$	
MLS ptf. (stock)		$Sw^*_{S,t}$	
MLS ptf. (longevity)		–	

or, in other terms, recalling (7.16.1) and (7.16.2):

$$B(t,T)\,w^*_{MLS,B} = B(t,T)\,w^*_{B,0},$$

$$S_t w^*_{MLS,S} = S_t w^*_S.$$

This MLS portfolio coincides with that obtained in a complete market without any longevity risk. These results are summarised in Table 7.2, from which we can draw the following conclusions.

1. In an incomplete market where the longevity risk is present but there are no assets which may allow to hedge against that risk, the MLS strategy is to behave as if the longevity risk did not exist at all.
2. When a new derivative written on the force of mortality (i.e. a demographic asset) is introduced in such a market, it becomes complete. Thus, the optimal portfolio exists and is unique. More precisely, it is given by the previous MLS portfolio where the money to be invested in the demographic asset is partially taken from the investment in ordinary bonds ($\frac{\nabla_{L,r}}{\nabla_{B,r}}$) and partially from the investment in the riskless asset ($1 - \frac{\nabla_{L,r}}{\nabla_{B,r}}$). The proportions are chosen based on the ratio between the duration of the demographic bond and the ordinary bond.

7.18 The Inflation Risk

A pension fund manages its wealth for a period of time whose length can be even longer than 50 years. For instance, let us think of a young worker who starts contributing when he/she is 20, retires when he/she is 70, and dies when he/she is 85. During such a long period of time, the inflation risk may strongly affect the pension fund. In this section we show how to take into account such a risk in the optimisation problem.

The first step is to model the Consumer Price Index (CPI) P_t as a stochastic process:

$$\frac{dP_t}{P_t} = \pi_t dt + \sigma_{P,t}^{\top} dW_t,$$

where π_t is the instantaneous inflation rate, i.e. the relative change of the CPI. Of course the inflation rate may be stochastic itself and so also π_t may follow a stochastic process or be function of the stochastic state variables z_t.

If we assume that the CPI is affected by a risk source of its own, then the market remains complete if and only if there exists a derivative written either on the CPI or on the inflation rate. The Italian Government, for instance, issues two kinds of bond called "Buoni del Tesoro Poliennali indicizzati—BTPi" (Multiyear indexed Treasury Bills) and "Buoni del Tesoro Poliennali Italia—BTP-Italia" (Multiyear Treasury Bills on Italy) that are indexed to the European and Italian CPI respectively.

Now, we can assume that the objective of the fund is to maximise the expected utility of the real wealth, which is defined as the ratio between the nominal wealth and the CPI ($\frac{R_t}{P_t}$).

The variable α in the utility function plays the role of a minimum wealth. When the wealth is measured in nominal terms, also α mast have the same magnitude. Thus, we normalise by the price level also the level of α, and the instantaneous utility function is

$$\frac{\left(\frac{R_t}{P_t} - \frac{\alpha}{P_t}\right)^{1-\delta}}{1-\delta} = P_t^{\delta-1}\frac{(R_t-\alpha)^{1-\delta}}{1-\delta}.$$

The objective function can accordingly be written as follows:

$$\max_{\{R_s\}_{s\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\int_{t_0}^{\omega} \lambda_s P_s^{\delta-1}\frac{(R_s-\alpha)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^{s}\rho_u+\lambda_u du} ds\right].$$

By following the same procedure presented in the previous sections, we find the very same function H_t, but a different discount factor

$$\begin{aligned} F_{P,t} &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega} \lambda_s P_s^{1-\frac{1}{\delta}} m_{t,s}^{-\frac{1}{\delta}} e^{-\int_t^s\left(\frac{\delta-1}{\delta}r_u+\lambda_u+\frac{1}{\delta}\rho_u\right)du} ds\right] \\ &= \mathbb{E}_{t,t}^{\mathbb{Q}_\delta,\tau}\left[P_\tau^{1-\frac{1}{\delta}} e^{-\int_t^{\tau}\left(\frac{\delta-1}{\delta}r_u+\frac{1}{\delta}\rho_u+\frac{1}{2}\frac{\delta-1}{\delta}\frac{1}{\delta}\xi_u^{\top}\xi_u\right)du}\right]. \end{aligned}$$

Accordingly, the optimal portfolio has the same form as that obtained without the inflation risk, with the exception of the function F_t that must become $F_{P,t}$.

We can see that the price level P_τ affects the function F_t (and then optimal portfolio) through the agent's risk aversion. In particular we can see two particular cases.

- For a log investor (with $\delta = 1$) the inflation risk does not affect the portfolio. In fact, in this case, the exponent of the CPI is zero.
- For an infinitely risk averse investor (with $\delta \to \infty$), it is the CPI itself that affect the portfolio since the exponent of the CPI is 1.

Reference

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics, 51*, 247–257.

Chapter 8
A Workable Framework

8.1 Introduction

The model we have solved in the previous chapter is very general and is able to accommodate many frameworks with a lot of different dynamics for both the state variables and the asset prices.

The values of the functions H_t, F_t, and Δ_t in (7.8.2), (7.8.3), and (7.3.2) respectively, can be obtained through numerical simulations, but the computation of their derivatives with respect to all the state variables is more difficult and requires a much stronger effort in terms of simulations. This is the reason why we often prefer to define the stochastic processes of the state variables in such a way that H_t, F_t, and Δ_t have a closed form representation. This is the case, for instance, when the state variables follow square root mean reverting processes.

In this chapter we present a framework that can be solved with numerical simulations and we show the empirical dynamics of the optimal portfolio by calibrating the state variables and the asset prices on the US data.

8.2 The State Variables

Here, we assume that the force of mortality follows the same process already defined and estimated in (5.7.2):

$$d\lambda_t = \alpha_\lambda \left(\underbrace{\phi + \left(\frac{1}{\alpha_\lambda b} + 1 \right) \frac{1}{b} e^{\frac{\iota + t - m}{b}}}_{\beta_{\lambda,t}} - \lambda_t \right) dt + \sigma_\lambda \sqrt{\lambda_t} dW_{\lambda,t}, \tag{8.2.1}$$

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_8

and, accordingly, the upper bound of the domain of the death time τ is infinite (i.e. $\omega \to \infty$).

The interest rate follows a CIR process as defined in Sect. 3.4:

$$dr_t = \alpha_r \left(\beta_r - r_t\right) dt + \sigma_r \sqrt{r_t} dW_{r,t}. \tag{8.2.2}$$

Finally, we assume that the worker pays a constant percentage of his salary into the fund. Furthermore, we assume that the contribution follows a geometric Brownian motion:

$$\frac{dc_t}{c_t} = \mu_c dt + \sigma_{c,r} \sqrt{r_t} dW_{r,t} + \sigma_{c,A} dW_{A,t}, \tag{8.2.3}$$

which is affected by two risk sources: that of the riskless interest rate ($W_{r,t}$) and that of the stock on the financial market ($W_{A,t}$). The instantaneous correlation between the changes in c_t and r_t is set proportional to the level of the interest rate itself

$$\mathbb{C}_t \left[\frac{dc_t}{c_t}, dr_t\right] = \sigma_{c,r} \sigma_r r_t dt,$$

and this assumption is needed for being able to find a closed form solution to the PMR.

In matrix notation, the state variables can be written as

$$\begin{bmatrix} d\lambda_t \\ dr_t \\ dc_t \end{bmatrix} = \begin{bmatrix} \alpha_\lambda \left(\beta_{\lambda,t} - \lambda_t\right) \\ \alpha_r \left(\beta_r - r_t\right) \\ c_t \mu_c \end{bmatrix} dt + \underbrace{\begin{bmatrix} 0 & 0 & \sigma_\lambda \sqrt{\lambda_t} \\ 0 & \sigma_r \sqrt{r_t} & 0 \\ c_t \sigma_{c,A} & c_t \sigma_{c,r} \sqrt{r_t} & 0 \end{bmatrix}}_{\Omega_t^\intercal} \underbrace{\begin{bmatrix} dW_{A,t} \\ dW_{r,t} \\ dW_{\lambda,t} \end{bmatrix}}_{dW_t}.$$

Remark 8.1 We assume that the worker wants to obtain a fixed pension p during his/her retirement and the fair value of this pension (p^*) is obtained by setting the PMR to zero at the inception.

The assumption that the level of contribution is affected by the risk source of the financial market is justified because contributions are proportional to wages and they depend on the performances of the hiring firms whose future expected profits are embedded in their price on financial market.

Remark 8.2 The three risk sources introduced in this section are of course independent on each other. This means that we are assuming that the interest rate r_t and the force of mortality λ_t are instantaneously independent. Not only does this hypothesis allow to significantly simplify the computations, but it is also quite weak.

Since in our model we have introduced three risk sources, we assume that four assets (one riskless and three risky) are listed on the financial market. The riskless asset has already been defined in (4.2.2). In the following sections we define the other risky assets.

Now, if we want to keep unchanged the statistical properties of these processes even under the new probabilities $\mathbb{Q}$ and $\mathbb{Q}_\delta$, we must assume that the market prices of both the interest rate risk and the longevity risk have the following forms

$$\xi_r = \kappa_r \sqrt{r_t}, \qquad \xi_\lambda = \kappa_\lambda \sqrt{\lambda_t},$$

while the market price of risk for the stock market (ξ_A) is assumed to be constant. Thus, under these assumptions, we can write (8.2.1)–(8.2.3) as follows (recall (4.7.1)):

$$d\lambda_t = \underbrace{(\alpha_\lambda + \kappa_\lambda \sigma_\lambda)}_{\alpha_\lambda^{\mathbb{Q}}} \left(\underbrace{\frac{\alpha_\lambda \beta_{\lambda,t}}{\alpha_\lambda + \kappa_\lambda \sigma_\lambda}}_{\beta_{\lambda,t}^{\mathbb{Q}}} - \lambda_t \right) dt + \sigma_\lambda \sqrt{\lambda_t} dW_{\lambda,t}^{\mathbb{Q}}, \tag{8.2.4}$$

$$dr_t = \underbrace{(\alpha_r + \kappa_r \sigma_r)}_{\alpha_r^{\mathbb{Q}}} \left(\underbrace{\frac{\alpha_r \beta_r}{\alpha_r + \kappa_r \sigma_r}}_{\beta_r^{\mathbb{Q}}} - r_t \right) dt + \sigma_r \sqrt{r_t} dW_{r,t}^{\mathbb{Q}}, \tag{8.2.5}$$

$$\frac{dc_t}{c_t} = \left(\mu_c - \sigma_{c,A}\xi_A - \kappa_r \sigma_{c,r} r_t\right) dt + \sigma_{c,r}\sqrt{r_t} dW_{r,t}^{\mathbb{Q}} + \sigma_{c,A} dW_{A,t}^{\mathbb{Q}}. \tag{8.2.6}$$

The behaviour of these state variables under $\mathbb{Q}_\delta$ is definitely similar and, recalling (7.14.2), it can be obtained just by substituting ξ_t by $\left(1 - \frac{1}{\delta}\right) \xi_t$.

There is another rate which is relevant in the optimisation problem: the subjective discount rate ρ_t. Here, we assume that it is constant, so some computations are significantly simplified, without any detriment to the quality of the results.

8.3 The Auxiliary Functions

The function H_t in (7.8.2), because of the independence between r_t and λ_t can be simplified as

$$H_t = \int_t^{\infty} \mathbb{E}_t^{\mathbb{Q}}\left[\lambda_s e^{-\int_t^s \lambda_u du}\right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right] ds.$$

The function F_t in (7.8.3) and (7.14.3) contains the square of the market price of risk ξ_t. Under the hypotheses made in the previous section, this square is

$$\xi_t^{\intercal}\xi_t = \kappa_r^2 r_t + \kappa_\lambda^2 \lambda_t + \xi_A^2,$$

and so F_t can be written as

$$F_t = \int_t^{\infty} \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\lambda_s e^{-\left(1+\frac{1}{2}\frac{\delta-1}{\delta}\frac{1}{\delta}\kappa_\lambda^2\right)\int_t^s \lambda_u du}\right]$$
$$\times\, \mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1-\frac{1}{\delta}+\frac{1}{2}\frac{\delta-1}{\delta}\frac{1}{\delta}\kappa_r^2\right)\int_t^s r_u du}\right] e^{-\left(\frac{1}{\delta}\rho+\frac{1}{2}\frac{\delta-1}{\delta}\frac{1}{\delta}\xi_A^2\right)(s-t)} ds.$$

The value of the PMR under the probability $\mathbb{Q}$ is given by

$$\begin{aligned}
\Delta_t =& \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty}\left(p^*\mathbb{I}_{s\geq T} - c_s\mathbb{I}_{s<T}\right) e^{-\int_t^s r_u+\lambda_u du} ds\right] \\
=& p^*\int_t^{\infty} \mathbb{I}_{s\geq T}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds \\
& - \int_t^{\infty} \mathbb{I}_{s<T}\mathbb{E}_t^{\mathbb{Q}}\left[c_s e^{-\int_t^s r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds,
\end{aligned}$$

and it can be written as a piecewise function for $t < T$ and $t \geq T$:

$$\Delta_t = \begin{cases} p^*\int_T^{\infty}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds & \\ \quad -\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[c_s e^{-\int_t^s r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds, & t < T \\ p^*\int_t^{\infty}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\right] ds. & t \geq T \end{cases}$$

Remark 8.3 The equilibrium value of the constant pension p^* is such that $\Delta_{t_0} = 0$, i.e.

$$p^* = \frac{\int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[c_s e^{-\int_{t_0}^{s} r_u du} \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} \lambda_u du} \right] ds}{\int_{T}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} r_u du} \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} \lambda_u du} \right] ds}.$$

Thus, we see that there is a common term that can be computed in order to find the values of all these functions. For this purpose, we rely on the result of the following proposition.

Proposition 8.1 *Given the following stochastic differential equation*

$$dx_t = \alpha \left(\beta_t - x_t \right) dt + \sigma \sqrt{x_t} dW_t,$$

for any real constant u, v, and q, we can write

$$\mathbb{E}_t \left[\left(u + v x_T \right) e^{-q \int_t^T x_s ds} \right]$$
$$= \frac{u + \alpha v \int_t^T \beta_s e^{-\alpha (s-t) - \sigma^2 \int_t^s C(i) di} ds + v e^{-\alpha (T-t) - \sigma^2 \int_t^T C(i) di} x_t}{e^{\alpha \int_t^T C(s) \beta_s ds + C(t) x_t}},$$

where

$$k := \sqrt{\alpha^2 + 2\sigma^2 q},$$

$$C \left(t; T, \alpha, \sigma, q \right) = 2q \frac{1 - e^{-k(T-t)}}{k + \alpha + (k - \alpha) e^{-k(T-t)}}, \tag{8.3.1}$$

and

$$\int_t^T C(i) \, di = \frac{2}{\sigma^2} \ln \left(\frac{k + \alpha + (k - \alpha) e^{-k(T-t)}}{2k} \right) + \frac{2q}{k + \alpha} (T - t).$$

Proof. By using Itô's lemma, we know that the function

$$Y\left(t,x_t\right)=\mathbb{E}_t\left[\left(u+vx_T\right)e^{-q\int_t^T x_s ds}\right],$$

must satisfy

$$\frac{\partial Y}{\partial t}+\frac{\partial Y}{\partial x_t}\alpha\left(\beta_t-x_t\right)+\frac{1}{2}\frac{\partial^2 Y}{\partial x_t^2}\sigma^2 x_t=qx_tY,$$

with the boundary condition

$$Y\left(T,x_T\right)=u+vx_T.$$

Now, we use the guess function

$$Y\left(t,x_t\right)=\left(E\left(t\right)+D\left(t\right)x_t\right)e^{-A(t)-C(t)x_t},$$

where the function A, C, E, and D must be computed in order to solve the previous differential equation. The boundary condition translates into the following conditions:

$$E\left(T\right)=u,$$
$$D\left(T\right)=v,$$
$$A\left(T\right)=0,$$
$$C\left(T\right)=0.$$

Once the partial derivatives of Y are substituted into the differential equation we obtain[1]

$$\begin{aligned}0=&\frac{\partial E}{\partial t}+\frac{\partial D}{\partial t}x_t+\left(E+Dx_t\right)\left(-\frac{\partial A}{\partial t}-\frac{\partial C}{\partial t}x_t\right)\\&+\left(D-\left(E+Dx_t\right)C\right)\alpha\left(\beta_t-x_t\right)\\&+\frac{1}{2}\left(-2CD+\left(E+Dx_t\right)C^2\right)\sigma^2x_t-qx_t\left(E+Dx_t\right),\end{aligned}$$

[1]For the sake of simplicity, we have omitted the functional dependencies (except for the function β_t).

which is an ordinary differential equation in A, C, E, and D. Since this equation must hold for any value of x_t then we can split it into three ordinary differential equations as follows

$$\begin{cases} 0 = \frac{\partial E}{\partial t} + D\alpha\beta_t - E\left(\frac{\partial A}{\partial t} + C\alpha\beta_t\right), \\ 0 = \frac{\partial D}{\partial t} - D\alpha - CD\sigma^2 - D\left(\frac{\partial A}{\partial t} + C\alpha\beta_t\right), \\ 0 = \frac{\partial C}{\partial t} + q - \alpha C - \frac{1}{2}C^2\sigma^2. \end{cases} \tag{8.3.2}$$

We immediately see that the value of function $C(t)$ can be computed from the third equation. With the suitable boundary condition the only solution of the differential equation for $C(t)$ is given by

$$C(t) = 2q\frac{1 - e^{-(T-t)\sqrt{\alpha^2+2\sigma^2 q}}}{\sqrt{\alpha^2 + 2\sigma^2 q} + \alpha + \left(\sqrt{\alpha^2 + 2\sigma^2 q} - \alpha\right)e^{-(T-t)\sqrt{\alpha^2+2\sigma^2 q}}}.$$

The values of all the other functions can be written in terms of $C(t)$. From the second equation in System (8.3.2) we see that $D(t) = 0$ is actually a solution to the equation. This solution also satisfies the boundary condition $D(T) = 0$, which is compatible with $E(T) = 1$. If we set also $E(t)$ as a constant (i.e. $\frac{\partial E(t)}{\partial t} = 0$), then the first equation in System (8.3.2) implies that

$$\frac{\partial A(t)}{\partial t} = -C(t)\alpha\beta_t,$$

with the boundary condition $A(T) = 0$. The only solution of this equation is

$$A(t) = \int_t^T C(s)\alpha\beta_s ds.$$

Given this value for $A(t)$, the two first equations of system (8.3.2) become

$$\frac{\partial E(t)}{\partial t} = -D(t)\alpha\beta_t,$$
$$\frac{\partial D(t)}{\partial t} = D(t)\left(\alpha + C(t)\sigma^2\right).$$

We now compute the value of $D(t)$ from the second equation by obtaining

$$D(t) = ve^{-\int_t^T\left(\alpha + C(u)\sigma^2\right)du},$$

(continued)

and the value of $E(t)$ can then be computed from the first equation

$$E(t) = u + \alpha \int_t^T D(s)\,\beta_s ds,$$

and the result of the proposition follows. □

Finally, the last term we need is the discounted value of the contributions that can be computed in closed form as shown in the following result.

Proposition 8.2 *Given the stochastic differential equations (8.2.5) and (8.2.6), we can write*

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du}\right] &= c_t e^{(\mu_c - \sigma_{c,A}\xi_A)(T-t)} \\ &\times e^{-\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}\int_t^T C\left(s;T,\alpha_r^{\mathbb{Q}}-\sigma_r\sigma_{c,r},\sigma_r,1+\kappa_r\sigma_{c,r}\right)ds} \\ &\times e^{-C\left(t;T,\alpha_r^{\mathbb{Q}}-\sigma_r\sigma_{c,r},\sigma_r,1+\kappa_r\sigma_{c,r}\right)r_t},\end{aligned}$$

where the function $C(t;\ldots)$ is defined in (8.3.1).

Proof. If we call Y_t the expected value that must be computed, Itô's lemma can be applied to it for having

$$\begin{aligned}r_t Y_t = {} & \frac{\partial Y_t}{\partial t} + \frac{\partial Y_t}{\partial c_t} c_t\left(\mu_c - \sigma_{c,A}\xi_A - \kappa_r\sigma_{c,r}r_t\right) + \frac{\partial Y_t}{\partial r_t}\alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}} - r_t\right) \\ & + \frac{1}{2}\frac{\partial^2 Y_t}{\partial c_t^2}c_t^2\left(\sigma_{c,r}^2 r_t + \sigma_{c,A}^2\right) + \frac{1}{2}\frac{\partial^2 Y_t}{\partial r_t^2}\sigma_r^2 r_t + \frac{\partial^2 Y_t}{\partial c_t \partial r_t}c_t\sigma_r\sigma_{c,r}r_t.\end{aligned}$$

The guess function for solving this differential equation is

$$Y_t = c_t e^{-A(t)-C(t)r_t},$$

whose final condition is

$$Y_T = c_T,$$

(continued)

which implies the final conditions

$$A(T) = C(T) = 0.$$

Once the guess function is substituted in the differential equation, it gives

$$r_t Y_t = Y_t \left(-\frac{\partial A(t)}{\partial t} - \frac{\partial C(t)}{\partial t} r_t \right) + Y_t \left(\mu_c - \sigma_{c,A}\xi_A - \kappa_r \sigma_{c,r} r_t \right)$$
$$- C(t) Y_t \alpha_r^{\mathbb{Q}} \left(\beta_r^{\mathbb{Q}} - r_t \right) + C(t)^2 \frac{1}{2} Y_t \sigma_r^2 r_t - C(t) Y_t \sigma_r \sigma_{c,r} r_t,$$

that can be split into two differential equations, one that contains the term that are multiplied by r_t and one that contains the terms that are not:

$$0 = \frac{\partial A(t)}{\partial t} - \mu_c + \sigma_{c,A}\xi_A + C(t)\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}$$
$$0 = \left(\frac{\partial C(t)}{\partial t} + 1 + \kappa_r \sigma_{c,r} - C(t)\left(\alpha_r^{\mathbb{Q}} - \sigma_r \sigma_{c,r} \right) - C(t)^2 \frac{1}{2}\sigma_r^2 \right) r_t,$$

Thus, the function $C(t)$ that solves the second equation has the same form as the function $C(t)$ in (8.3.1), where $q = 1 + \kappa_r \sigma_{c,r}$ and $\alpha = \alpha_r^{\mathbb{Q}} - \sigma_r \sigma_{c,r}$, and the function $A(t)$ has the form

$$A(t) = -\left(\mu_c - \sigma_{c,A}\xi_A \right)(T - t) + \int_t^T C(s)\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}} ds.$$

□

Given the result of Proposition 8.1, we can immediately check that

$$\frac{\partial}{\partial c_t}\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du} \right] = \frac{1}{c_t}\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du} \right],$$

and, accordingly, the value of the derivative of the PMR with respect to the contribution is:

$$\frac{\partial \Delta_t}{\partial c_t} = \begin{cases} -\frac{1}{c_t}\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[c_s e^{-\int_t^s r_u du} \right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du} \right] ds, & t < T \\ 0. & t \geq T \end{cases} \tag{8.3.3}$$

In fact, after the retirement date, we are taking into account the case of constant pensions (equal to p^*) and there is no more need for hedging against the stochastic behaviour of the cash flows.

8.4 The Financial Market

Since we have assumed that there are three risk sources on the financial market, the market completeness can be achieved only if there are also three (non perfectly correlated) risky assets. To this purpose we introduce the following risky assets.

- A stock: whose price A_t solves the stochastic differential equation

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dW_{A,t} + \sigma_{A,r}\sqrt{r_t}dW_{r,t}, \tag{8.4.1}$$

  where all the parameters are constant.
- A rolling ZCB with constant time to maturity (h). The price of such a bond can be written as (see Proposition 8.1)

$$\begin{aligned} B(t,t+h) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+h} r_u du}\right] \\ &= e^{-\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}\int_t^{t+h} C\left(s;t+h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)ds - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)r_t}, \end{aligned}$$

  that can also be written in differential form as

$$\begin{aligned} \frac{dB(t,t+h)}{B(t,t+h)} =& r_t dt - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}dW_{r,t}^{\mathbb{Q}} \\ =& r_t dt - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}\left(dW_{r,t}+\xi_r dt\right) \\ =& \left(1 - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\kappa_r\right)r_t dt \\ & - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}dW_{r,t}. \end{aligned} \tag{8.4.2}$$

  This result allows us to see that the diffusion term of the rolling bond does not depend on the financial horizon T. Thus, we can conclude that, in this framework,

$$\nabla_{B,r} = -C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}.$$

- A rolling zero-coupon longevity bond. This bond pays one monetary unit at a given horizon if a reference agent is still alive. For the sake of simplicity we use for this bond the same maturity as the previously defined rolling ZCB. Thus, its value can be written as

$$\begin{aligned} L(t,t+h) &= \mathbb{E}_{t,t}^{\mathbb{Q},\tau}\left[e^{-\int_t^{t+h} r_u du}\mathbb{I}_{t+h<\tau}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+h} r_u + \lambda_u du}\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+h} r_u du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+h} \lambda_u du}\right] \\ &= B(t,t+h)\, e^{-\alpha_\lambda^{\mathbb{Q}}\int_t^{t+h} \beta_{\lambda,s}^{\mathbb{Q}} C\left(s;t+h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)ds - C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\lambda_t}. \end{aligned}$$

In differential form, we have

$$\begin{aligned}\frac{dL\left(t,t+h\right)}{L\left(t,t+h\right)} &= \left(r_t+\lambda_t\right)dt - C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}dW_{\lambda,t}^{\mathbb{Q}} \\ &\quad - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}dW_{r,t}^{\mathbb{Q}} \\ &= \left(\left(1-C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\kappa_r\right)r_t\right. \\ &\quad \left.+\left(1-C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\kappa_\lambda\right)\lambda_t\right)dt \\ &\quad - C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}dW_{\lambda,t} \\ &\quad - C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}dW_{r,t},\end{aligned} \tag{8.4.3}$$

and the semi-elasticities of this longevity bond with respect to both the interest rate and the longevity are

$$\nabla_{L,r} = -C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t} = \nabla_{B,r},$$

$$\nabla_{L,\lambda} = -C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}.$$

The relationship $\nabla_{L,r} = \nabla_{B,r}$ is valid because we have assumed that the variables r_t and λ_t are independent and, furthermore, that the two bonds have the same constant maturity h. This second hypothesis is not too strong in a complete market, since it is always possible to replicate any zero-coupon of any maturity.

This financial market can be written in matrix form as

$$\begin{bmatrix}\frac{dA_t}{A_t}\\ \frac{dB(t,t+h)}{B(t,t+h)}\\ \frac{dL(t,t+h)}{L(t,t+h)}\end{bmatrix} = \left(r_t+\Sigma_t^{\intercal}\xi_t\right)dt$$

$$+\underbrace{\begin{bmatrix}\sigma_A & \sigma_{A,r}\sqrt{r_t} & 0\\ 0 & -C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t} & 0\\ 0 & -C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t} & -C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}\end{bmatrix}}_{\Sigma_t^{\intercal}}$$

$$\times\underbrace{\begin{bmatrix}dW_{A,t}\\ dW_{r,t}\\ dW_{\lambda,t}\end{bmatrix}}_{dW_t},$$

where, of course $\xi_t = \left[\xi_A \;\; \kappa_r\sqrt{r_t} \;\; \kappa_\lambda\sqrt{\lambda_t}\right]^{\intercal}$.

For computing the optimal portfolio we need to invert the (transposed) diffusion matrix

$$\Sigma_t^{-1} = \begin{bmatrix} \frac{1}{\sigma_A} & 0 & 0 \\ \frac{\sigma_{A,r}}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sigma_A} & -\frac{1}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sqrt{r_t}} & \frac{1}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}} \\ 0 & 0 & -\frac{1}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda\sqrt{\lambda_t}} \end{bmatrix},$$

and compute the following products:

$$\Sigma_t^{-1}\xi_t = \begin{bmatrix} \frac{\xi_A}{\sigma_A} \\ \frac{\sigma_{A,r}\xi_A}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r\sigma_A} - \frac{\kappa_r}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)\sigma_r} + \frac{\kappa_\lambda}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda} \\ -\frac{\kappa_\lambda}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)\sigma_\lambda} \end{bmatrix},$$

$$\Sigma_t^{-1}\Omega_t = \begin{bmatrix} 0 & 0 & c_t\frac{\sigma_{c,A}}{\sigma_A} \\ \frac{1}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)} & -\frac{1}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)} & \frac{c_t}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)}\left(\frac{\sigma_{A,r}\sigma_{c,A}}{\sigma_r\sigma_A} - \frac{\sigma_{c,r}}{\sigma_r}\right) \\ -\frac{1}{C\left(0;h,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,1\right)} & 0 & 0 \end{bmatrix}.$$

In the product $\Sigma_t^{-1}\xi_t$ everything is constant, while in the product $\Sigma_t^{-1}\Omega_t$ the only variable which depends on time is c_t.

8.5 The Data

In this section we collect all the data needed for calibrating the stochastic processes shown in the previous sections. In collecting these data we have two aims: (1) dealing with a sufficiently long time series, and (2) avoiding the problems linked to the 2007/2008 financial crisis. To these purposes, we chose the following time interval

$$01/01/1970 \quad - \quad 01/01/2007,$$

in fact, we recall that the first problem on the interbank market raised on 9/8/2007 because of the default of some funds owned by the French investment bank BNP-Paribas.

Since we have assumed that there is no correlation between the force of mortality λ_t and all the other stochastic variables, then for the stochastic process of λ_t we keep the values already calibrated in Sect. 5.7 on the US population.

For what concerns the other variables, we refer to the following data sets.

- The riskless interest rate r_t is calibrated on the 3-month Treasury Bill whose returns are gathered by FRED in the series code “DTB3”.

```
getSymbols("DTB3", src = "FRED", return.class = "zoo")

## [1] "DTB3"

r = na.omit(DTB3)/100
r = window(r, start = "1970-01-01", end = "2007-01-01")
```

- The return on the ZCB $B(t, t+h)$ is obtained, for $h = 10$, from the returns on 10-year Treasury Bill that are gathered by FRED in the series code "DGS10".

```
getSymbols("DGS10", src = "FRED", return.class = "zoo")

## [1] "DGS10"

r10Y = na.omit(DGS10)/100
r10Y = window(r10Y, start = "1970-01-01", end = "2007-08-09")
```

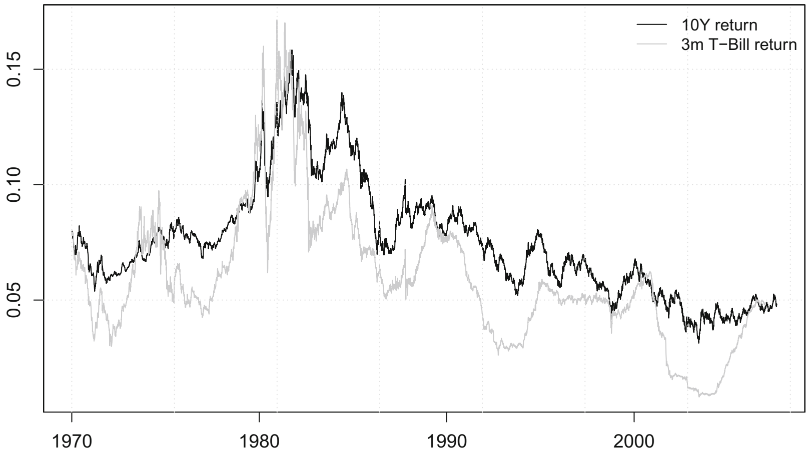
If we want to compare on a graph the return on the 10 year bond and the riskless interest rate, we can give the commands in Fig. 8.1. From this figure we immediately see that the recessions usually coincide with a difference between the long and short term interest rate that is close to zero or even negative.

- The risky asset A_t is assumed to be the S&P500 index on the US financial market. In this case we can define the starting and ending date directly into the “`getSymbols`” command since yahoo accepts these parameters.

```
getSymbols("^GSPC", src = "yahoo", return.class = "zoo",
    from = "1970-01-01", to = "2007-01-01")

## [1] "^GSPC"

A = na.omit(GSPC$GSPC.Adjusted)
```

In Fig. 8.2, we can see the behaviour of both the stock index and its daily returns. From this figure we can appreciate that the average return is stable over time, while the variance changes a lot among different periods. For instance, the variance increases during both the accumulation and the burst of the dot-com bubble.

For drawing the figure we use the following commands: (1) “`par(new=T)`” that creates a new graph in the same framework, (2) “`xaxt='n'`” and “`xaxt='n'`” that prevent the “`plot`” command from drawing the x and y axis, (3) “`axis(side=4)`” that creates an axes on position “4” (we recall that position




```
plot(merge(r10Y, r), screens = 1, xlab = "", ylab = "",
    col = c("black", "gray"), lty = 1)
grid()
legend("topright", legend = c("10Y return", "3m T-Bill return"),
    col = c("black", "gray"), lty = 1, bty = "n")
```

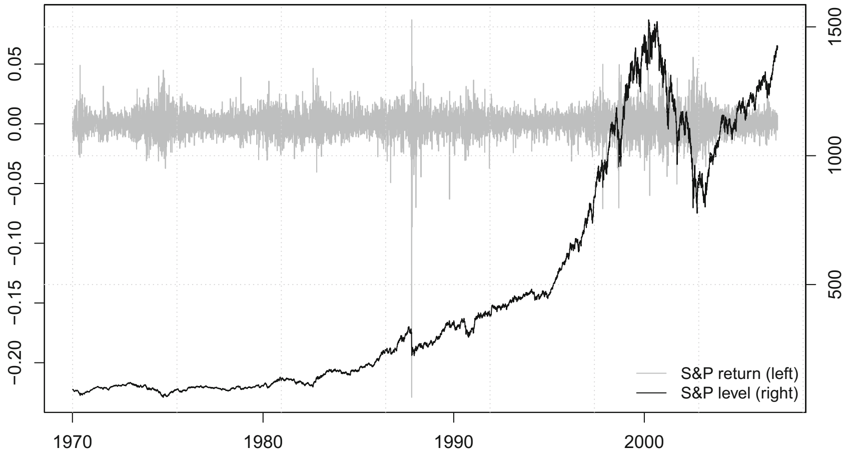
Fig. 8.1 Comparison between the time series of the return on 10 year US bonds and 3 month T-Bill

"1" is the bottom axes, "2" is the left axes, "3" is the upper axes, and "4" is the right axes).

- We have assumed that the contributions are proportional to the wages. Thus, we take the historical series of wages paid in US (the FRED code is "A576RC1A027NBEA"). We highlight that the data downloaded from FRED have an annual frequency and, accordingly, we will have to take that into account when computing the correlation between the wages and the other variables (that have daily frequency).

```
getSymbols("A576RC1A027NBEA", src = "FRED", return.class = "zoo")

## [1] "A576RC1A027NBEA"

w = na.omit(A576RC1A027NBEA)
w = window(w, start = "1970-01-01", end = "2007-01-01")
```

In Fig. 8.3 we show the log-return on the total amount of wages paid in US. It is clear that the average growth rate of wages has been quite higher till 1990 and then




```
plot(diff(log(A)), type = "l", col = "gray", xlab = "",
    ylab = "")
par(new = T)
plot(A, xaxt = "n", yaxt = "n", ylab = "", xlab = "")
axis(side = 4)
grid()
legend("bottomright", legend = c("S&P return (left)",
    "S&P level (right)"), lty = 1, col = c("gray",
    "black"), bty = "n")
```

Fig. 8.2 Comparison between the daily level of the S&P500 and its daily return

it has reduced. We can check the significant difference between the average growth rate before and after 1990 through the following commands.

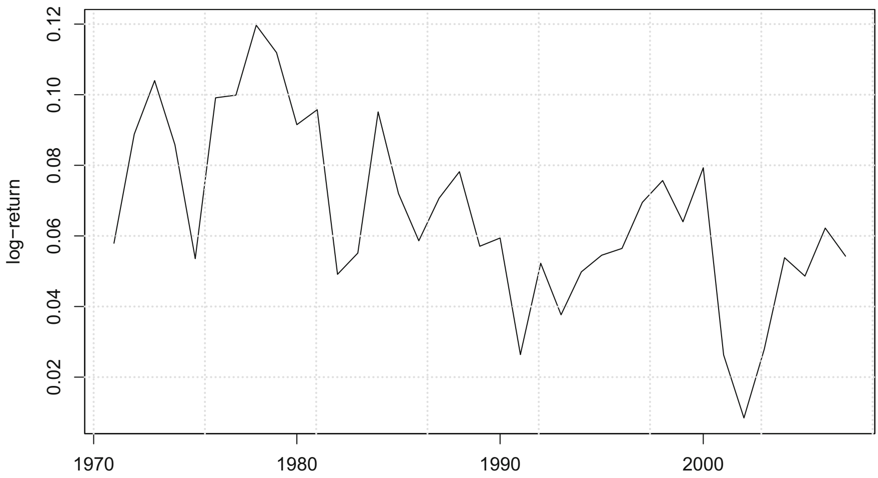
```
mean(window(diff(log(w)), end = "1990-01-01"))

## [1] 0.0801664

mean(window(diff(log(w)), start = "1990-01-01"))

## [1] 0.0503643
```

8.6 Calibration of the Riskless Interest Rate

Given the process

$$dr_t = \alpha_r (\beta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_{r,t},$$

the estimation of the parameters α_r, β_r, and σ_r can be performed through the same procedure seen in the previous chapters. Since we have daily data we take $dt = 1/250$.




```
plot(diff(log(w)), xlab = "", ylab = "log-return")
grid()
```

Fig. 8.3 Log-difference of annual wages in US (FRED: A576RC1A027NBEA)

```
dt = 1/250
sigma_r = sd(diff(2 * sqrt(r)))/sqrt(dt)
y = r[2:length(r)]^0.5
x1 = r[1:(length(r) - 1)]^0.5
x2 = r[1:(length(r) - 1)]^(-0.5)
CIR = lm(y ~ x1 + x2 - 1)
summary(CIR)
```

```
##
## Call:
## lm(formula = y ~ x1 + x2 - 1)
##
## Residuals:
##        Min          1Q     Median          3Q         Max
## -0.0160398 -0.0006411  0.0000028  0.0006564  0.0171645
##
## Coefficients:
##     Estimate Std. Error  t value Pr(>|t|)
## x1 9.998e-01  1.473e-04 6785.801   <2e-16 ***
## x2 8.513e-06  7.437e-06    1.145    0.252
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.00184 on 9234 degrees of freedom
## Multiple R-squared:  0.9999,Adjusted R-squared:  0.9999
## F-statistic: 8.13e+07 on 2 and 9234 DF,  p-value: < 2.2e-16
```

```
a_C = as.numeric(2 * (CIR$coefficients[1] - 1)/dt)
b_C = as.numeric(2 * CIR$coefficients[2]/dt) + sigma_r^2/4
alpha_r = -a_C
beta_r = b_C/abs(a_C)
c(alpha_r, beta_r, sigma_r)
```

```
## [1] 0.09934780 0.05136836 0.05819260
```

8.7 Calibration of the ZCB

Once the parameters of the r_t process have been obtained, we can switch to the rolling bond $B(t, t+h)$. If we take into account a 10-year US Government bond ($h = 10$), we know that its expected return is

$$\mathbb{E}_t\left[\frac{dB(t, t+h)}{B(t, t+h)}\right] = \left(1 - C\left(0; h, \alpha_r^{\mathbb{Q}}, \sigma_r, 1\right)\sigma_r\kappa_r\right) r_t dt$$
$$= \frac{k + \alpha_r - \sigma_r\kappa_r + (k - \alpha_r + \sigma_r\kappa_r)\, e^{-h\kappa}}{k + \alpha_r + \sigma_r\kappa_r + (k - \alpha_r - \sigma_r\kappa_r)\, e^{-h\kappa}} r_t dt,$$

where $k := \sqrt{(\alpha_r + \sigma_r \kappa_r)^2 + 2\sigma_r^2}$. In this equation:

- the expected return can be estimated on the market data over a sufficiently long time period
- the value r_t, in equilibrium, can be replaced by its long term mean β_r
- all the parameters but κ_r have already been estimated. Thus, κ_r is the only unknown of the equation and its value can be found numerically.

Of course, the 10 year return is usually higher than the return on the T-Bill. A difference between these two returns which is close to zero or even negative, usually announces a period of crisis. Actually, before a crisis there has always been a zero or negative difference between these two returns, even if the contrary is not always true.

Now, for computing κ_r we use the following steps.

- We use the command `uniroot(function,interval)` for finding the solution of an equation (`function`) in one unknown, by trying values of the unknown in the defined `interval`. The solution is the value of the unknown that sets the value of the function to zero.
- The interval for computing the κ_r that solves the equation is negative since we know that the market price of interest rate risk must be lower than zero. In fact, the semi-elasticity of any bond with respect to the interest rate is negative (a positive shock on the interest rate implies a negative shock on the bond prices).

```
compute_k_r = function(k_r) {
    k = sqrt((alpha_r + sigma_r * k_r)^2 + 2 * sigma_r^2)
    h = 10
    (k + alpha_r - sigma_r * k_r + (k - alpha_r + sigma_r *
        k_r) * exp(-h * k))/(k + alpha_r + sigma_r *
        k_r + (k - alpha_r - sigma_r * k_r) * exp(-h *
        k)) * beta_r - mean(r10Y)
}
k_r = uniroot(compute_k_r, c(-2, 0))$root
k_r
```

```
## [1] -1.018731
```

8.8 Calibration of the Risky Asset

Given Eq. (8.4.1), the log-return on the asset prices is given by

$$d \ln A_t = \left(\mu_A - \frac{1}{2}\left(\sigma_A^2 + \sigma_{A,r}^2 r_t\right)\right) dt + \sigma_A dW_{A,t} + \sigma_{A,r}\sqrt{r_t} dW_{r,t},$$

whose first and second moments are

$$\mathbb{E}_t\left[d\ln A_t\right] = \left(\mu_A - \frac{1}{2}\left(\sigma_A^2 + \sigma_{A,r}^2 r_t\right)\right) dt,$$

$$\mathbb{V}_t\left[d\ln A_t\right] = \left(\sigma_A^2 + \sigma_{A,r}^2 r_t\right) dt.$$

Then, the estimation of μ_A can be obtained (via the method of moments) as

$$\mu_A = \frac{\mathbb{E}_t\left[d\ln A_t\right] + \frac{1}{2}\mathbb{V}_t\left[d\ln A_t\right]}{dt}.$$

```
dt = 1/250
dlnA = diff(log(A))
mu_A = (mean(dlnA) + 0.5 * var(dlnA))/dt
mu_A
```

```
## [1] 0.08516942
```

For estimating also the two components of the S&P 500 volatility (i.e. σ_A and $\sigma_{A,r}$) we must take into account the correlation between the stock index and the interest rate. Given (8.2.2) and (8.4.1), the instantaneous correlation between the two variables is

$$\mathbb{C}_t\left[d\ln A_t, dr_t\right] = \sigma_r \sigma_{A,r} r_t dt.$$

We can assume that, in equilibrium, the value of the interest rate coincides with its long term value β_r. Accordingly, the values of the stock diffusion parameters can be found by solving the following system:

$$\begin{cases} \frac{1}{dt}\mathbb{V}_t\left[d\ln A_t\right] = \sigma_A^2 + \sigma_{A,r}^2 \beta_r, \\ \frac{1}{dt}\mathbb{C}_t\left[d\ln A_t, dr_t\right] = \sigma_r \sigma_{A,r} \beta_r. \end{cases}$$

This system has two solutions, for a positive and a negative value of σ_A, respectively. Here, we take only the positive solution:

$$\sigma_{A,r} = \frac{\mathbb{C}_t\left[d\ln A_t, dr_t\right]}{\sigma_r \beta_r dt}, \quad \sigma_A = \sqrt{\frac{1}{dt}\mathbb{V}_t\left[d\ln A_t\right] - \sigma_{A,r}^2 \beta_r}.$$

The command for obtaining these parameters are as follows, where we use:

- The function `merge` for merging two data sets. This function allows us to correctly take into account the corresponding dates on the two time series. Inside this function we use the option "`all=FALSE`" so that only the data with the same dates are included in the final set.

- The function `na.omit` for omitting the values NA. In fact, when the function `merge` is used, the data whose dates do not match are replenished by a NA.

```
dr = diff(r)
dlnA_dr = merge(dlnA, dr, all = FALSE)
sigma_Ar = (cov(dlnA_dr[, 1], dlnA_dr[, 2]))/(sigma_r *
    beta_r * dt)
sigma_A = sqrt((var(dlnA))/dt - sigma_Ar^2 * beta_r)
c(sigma_Ar, sigma_A)
```

```
## [1] -0.0757339  0.1555213
```

We can see that the value of $\sigma_{A,r}$ is negative, which means that the interest rate is negatively correlated with the S&P 500 index. Nevertheless, this negative correlation plays a marginal role in defining the volatility of the index, since σ_A is much higher (in absolute value) than $\sigma_{A,r}\sqrt{\beta_r}$.

Because of the Girsanov's theorem, we know that, under the probability $\mathbb{Q}$, the expected return on any asset must equate the riskless interest rate. Thus, for A_t, (8.4.1) can be written as

$$\begin{aligned}\frac{dA_t}{A_t} &= \mu_A dt + \sigma_A \left(dW^{\mathbb{Q}}_{A,t} - \xi_A dt\right) + \sigma_{A,r}\sqrt{r_t}\left(dW^{\mathbb{Q}}_{r,t} - \kappa_r\sqrt{r_t}dt\right)\\ &= \left(\mu_A - \xi_A\sigma_A - \kappa_r\sigma_{A,r}r_t\right)dt + \sigma_A dW^{\mathbb{Q}}_{A,t} + \sigma_{A,r}\sqrt{r_t}dW^{\mathbb{Q}}_{r,t},\end{aligned}$$

which implies

$$\mu_A - \xi_A\sigma_A - \kappa_r\sigma_{A,r}r_t = r_t.$$

Accordingly, since all the other parameters have already been estimated, we obtain

$$\xi_A = \frac{\mu_A - \kappa_r\sigma_{A,r}r_t - r_t}{\sigma_A}.$$

Again, we substitute the riskless interest rate with its equilibrium value β_r and we obtain an estimated market price of stock's own risk.

```
xi_A = (mu_A - k_r * sigma_Ar * beta_r - beta_r)/sigma_A
xi_A
```

```
## [1] 0.1918571
```

8.9 Calibration of the Contributions

We now have to estimate the parameters of the contribution process

$$\frac{dc_t}{c_t} = \mu_c dt + \sigma_{c,r}\sqrt{r_t}dW_{r,t} + \sigma_{c,A}dW_{A,t}.$$

Again, in order to obtain the values of three parameters (μ_c, $\sigma_{c,r}$, and $\sigma_{c,A}$), we need three conditions to match. In particular, we compute the first and second moment on $d\ln c_t$, and we also check for the correlation between $d\ln c_t$ and $d\ln A_t$:

$$\begin{cases} \mathbb{E}_t\left[d\ln c_t\right] = \left(\mu_c - \frac{1}{2}\left(\sigma_{c,r}^2 r_t + \sigma_{c,A}^2\right)\right)dt, \\ \mathbb{V}_t\left[d\ln c_t\right] = \left(\sigma_{c,r}^2 r\left(t\right) + \sigma_{c,A}^2\right)dt, \\ \mathbb{C}_t\left[d\ln c_t, d\ln A_t\right] = \sigma_{c,A}\sigma_A dt. \end{cases}$$

If we again substitute r_t buy its long term equilibrium value β_r, and we solve for the unknowns, we obtain

$$\begin{cases} \mu_c = \frac{\mathbb{E}_t[d\ln c_t]+\frac{1}{2}\mathbb{V}_t[d\ln c_t]}{dt}, \\ \sigma_{c,A} = \frac{\mathbb{C}_t[d\ln c_t, d\ln A_t]}{\sigma_A dt}, \\ \sigma_{c,r} = \sqrt{\frac{\frac{1}{dt}\mathbb{V}_t[d\ln c_t]-\sigma_{c,A}^2}{\beta_r}}. \end{cases}$$

The return μ_c can be directly obtained after computing the mean and the variance of the process $d\ln c$. This time we take $dt = 1$ since the data have an annual frequency.

```
dt = 1
dlnw = diff(log(w))
mu_c = as.numeric(mean(dlnw) + 0.5 * var(dlnw))/dt
mu_c
```

```
## [1] 0.06655217
```

From this estimation we can check that the average growth rate of wages in USA is lower than the growth rate of the S&P 500.

Now, the two series that we are working on (S&P 500 and wages) have a different frequency. Thus, in order to compute the covariance, we have to trace back both series to the same frequency. This result can be achieved in two equivalent ways by using the following R commands (both commands belong to the `quantmod` package):

- `yearlyReturn` that can be applied to a time series of any frequency for obtaining a corresponding yearly series, where each element is the average of the corresponding data. Since this command applies to time series, we must suitable transform its argument with the command "`as.xts`".

- `apply.yearly` that allows to apply a given function with an annual frequency (in our case the suitable function is `mean`).

Once the frequency is the same, we can take subsets of the two series where the dates are the same in order to compute the covariance.

```
# annual return on SP500
dlnAY = apply.yearly(dlnA, mean) * 250
# dlnAY=yearlyReturn(as.xts(A),type='log')
sigma_cA = as.numeric(cov(dlnw, dlnAY)/(sigma_A * dt))
sigma_cr = as.numeric(sqrt((var(dlnw)/dt - sigma_cA^2)/beta_r))
c(sigma_cA, sigma_cr)
```

```
## [1] 0.007844577 0.106664952
```

The last parameter we need is κ_λ. This is the most difficult parameter to estimate since we would need a liquid market for actuarial assets (either death bonds, or longevity bonds, or any other similar asset). Many papers that deal with the problem set the market price of actuarial risk to zero. If $\kappa_\lambda = 0$, then the expected return on the actuarial asset coincides with the riskless rate under both the historical and the risk neutral probabilities. Nevertheless, this hypothesis does not seem to be reasonable since we instead expect any actuarial asset to have an average return higher than $r(t)$.

In our framework, the actuarial asset has the form of a ZCB and, accordingly, its value must react to the changes in $\lambda(t)$ in the same way the ordinary zero-coupon react to the changes in the interest rate. Thus, we expect the market price of actuarial risk to be negative and its magnitude should be similar to that of the market price of interest rate risk (κ_r). For this reason in this work we assume $\kappa_\lambda = \kappa_r$. The values of the parameters are gathered in Table 8.1.

```
# market price of actuarial risk
k_l = k_r
```

Table 8.1 Parameters in the basic scenario

			Mortality λ_t	
Rate r_t	Stock A_t	Wage c_t	Males	Females
$\alpha_r = 0.09935$	$\mu_A = 0.08517$	$\mu_c = 0.06655$	$\alpha_\lambda = 0.1465$	$\alpha_\lambda = 0.3394$
$\beta_r = 0.05137$	$\sigma_A = 0.1555$	$\sigma_{c,A} = 0.007845$	$\phi = 0.003409$	$\phi = 0.00338$
$\sigma_r = 0.05819$	$\sigma_{A,r} = -0.07573$	$\sigma_{c,r} = 0.1067$	$b = 11.72$	$b = 9.513$
$\kappa_r = -1.019$	$\xi_A = 0.1919$		$m = 80.06$	$m = 87.39$
			$\kappa_\lambda = \kappa_r$	

8.10 The Behaviour of the Auxiliary Functions

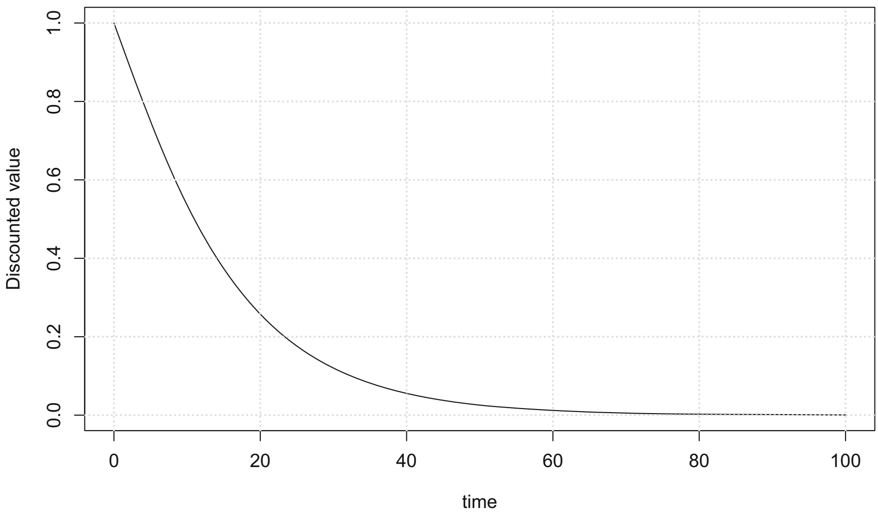
The first function we study here is the expected value of the discounted factor, under both probabilities $\mathbb{Q}$ and $\mathbb{Q}_\beta$:

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right].$$

The value of such a function can be computed through the following R code, where the choice between the two probabilities can be performed via a string whose value is either "Q" or "Qb".

```
C = function(t, T, alpha, sigma, q) {
    k = sqrt(alpha^2 + 2 * sigma^2 * q)
    2 * q * (1 - exp(-(T - t) * k))/(k + alpha + (k -
        alpha) * exp(-(T - t) * k))
}
int_C = function(t, T, alpha, sigma, q) {
    k = sqrt(alpha^2 + 2 * sigma^2 * q)
    2/sigma^2 * log((k + alpha + (k - alpha) * exp(-k *
        (T - t)))/(2 * k)) + 2 * q * (T - t)/(k + alpha)
}
EQ_r = function(t, T, q, r, prob) {
    if (prob == "Q") {
        alpha = alpha_r + sigma_r * k_r
        beta = alpha_r * beta_r/(alpha_r + sigma_r *
            k_r)
    }
    if (prob == "Qd") {
        alpha = alpha_r + sigma_r * k_r * (1 - 1/delta)
        beta = alpha_r * beta_r/(alpha_r + sigma_r *
            k_r * (1 - 1/delta))
    }
    exp(-alpha * beta * int_C(t, T, alpha, sigma_r,
        q) - C(t, T, alpha, sigma_r, q) * r)
}
```

If we take into account a risk aversion $\delta = 2.5$ (that is nevertheless useless for the value of the function under $\mathbb{Q}$), the graphical behaviour of the expected value of the discount factor can be drawn as in Fig. 8.4, where we see that the function has value 1 when $t = T$ and tends towards zero for higher and higher values of T.




```
delta = 2.5
plot(seq(0, 100), lapply(seq(0, 100), EQ_r, q = 1,
    prob = "Q", t = 0, r = beta_r), type = "l", xlab = "time",
    ylab = "Discounted value")
grid()
```

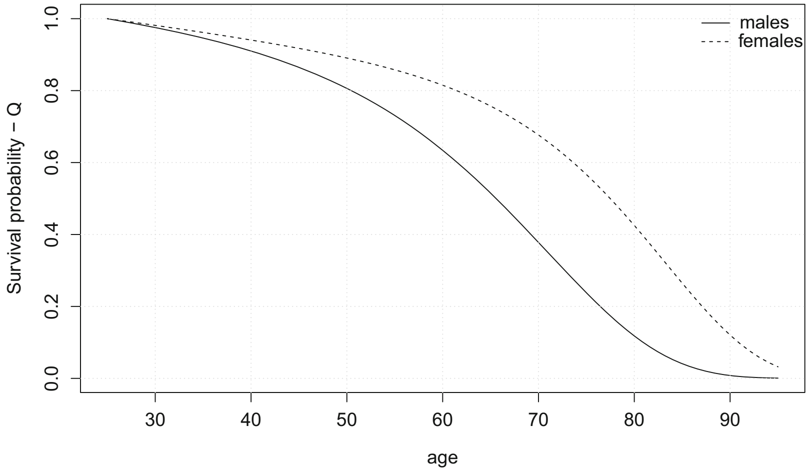
Fig. 8.4 Value of the function $\mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\right]$ for $T \in [0, 100]$, given the values of the parameters calibrated in the previous sections

The second function that we write in this section is useful for computing the expected value of the discount factor where the discount rate is the force of mortality:

$$\mathbb{E}_t^{\mathbb{Q}}\left[(u + v\lambda_T)\, e^{-\int_t^T \lambda_u du}\right].$$

```
EQ_l = function(t, T, q, u, v, l, prob, sex) {
    alpha = alpha_l[sex]
    phi = phi_l[sex]
    m = m_l[sex]
    b = b_l[sex]
    sigma = sigma_l[sex]
    if (prob == "Q") {
        alpha1 = alpha + sigma * k_l
        beta = function(s) {
            alpha1 * (phi + (1/(alpha1 * b) + 1)/b *
                exp((age + s - m)/b))/(alpha1 + sigma *
                k_l)
        }
    }
    if (prob == "Qd") {
        alpha1 = alpha + sigma * k_l * (1 - 1/delta)
        beta = function(s) {
            alpha * (phi + (1/(alpha1 * b) + 1)/b *
                exp((age + s - m)/b))/(alpha1 + sigma *
                k_l * (1 - 1/delta))
        }
    }
    INT_A = function(s) {
        beta(s) * exp(-alpha1 * (s - t) - sigma^2 *
            int_C(t, s, alpha1, sigma, q))
    }
    INT_B = function(s) {
        beta(s) * C(t, s, alpha1, sigma, q)
    }
    (u + alpha1 * v * integrate(INT_A, t, T)$value +
        v * exp(-alpha1 * (T - t) - sigma^2 * int_C(t,
            T, alpha1, sigma, q)) * l)/exp(alpha1 *
        integrate(INT_B, t, T)$value + C(t, T, alpha1,
        sigma, q) * l)
}
# Initial value of force of mortality for males and
# females
l0 = (phi_l + (1/(alpha_l * b_l) + 1)/b_l * exp((age +
    0 - m_l)/b_l))
```

In Figs. 8.5 and 8.6 the values of the functions $\mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T \lambda_u du}\right]$ and $\mathbb{E}_0^{\mathbb{Q}}\left[\lambda_T e^{-\int_0^T \lambda_u du}\right]$ are drawn, respectively. The value of the first function coincides with the survival probability (under $\mathbb{Q}$) from the initial age (set at 25) to the time $25 + T$. Such a probability is, of course, decreasing over time, starts with a value of 1 and tends towards zero while the time horizon increases. Instead, the value of the second function is first increasing, when the growth of λ_t prevails on the reduction



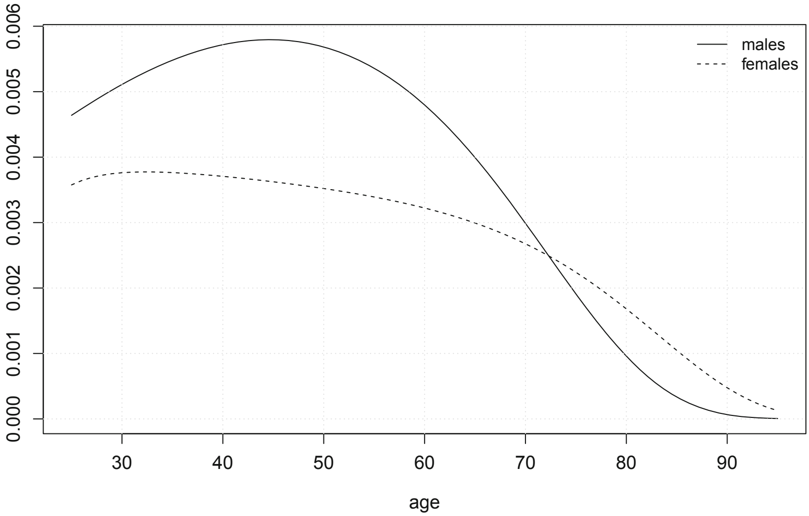

```
age = 25
plot(age + seq(0, 70), lapply(seq(0, 70), EQ_l, t = 0,
    q = 1, u = 1, v = 0, l = l0[1], prob = "Q", sex = 1),
    type = "l", xlab = "age", ylab = "Survival probability - Q")
lines(age + seq(0, 70), lapply(seq(0, 70), EQ_l, t = 0,
    q = 1, u = 1, v = 0, l = l0[2], prob = "Q", sex = 2),
    lty = 2)
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
grid()
```

Fig. 8.5 Value of the function $\mathbb{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T \lambda_u du}\right]$ (i.e. the survival probability under $\mathbb{Q}$) for $T \in [0, 70]$ and an age of 25, given the values of the parameters calibrated in the previous sections

of the discount factor, and then it is decreasing since the second effect prevails on the first one.

Now, we write a function for computing the expected present value of contributions:

$$\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du}\right].$$




```
plot(age + seq(0, 70), lapply(seq(0, 70), EQ_l, t = 0,
    q = 1, u = 0, v = 1, l = l0[1], prob = "Q", sex = 1),
    type = "l", xlab = "age", ylab = "")
lines(age + seq(0, 70), lapply(seq(0, 70), EQ_l, t = 0,
    q = 1, u = 0, v = 1, l = l0[2], prob = "Q", sex = 2),
    lty = 2)
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
grid()
```

Fig. 8.6 Value of the function $\mathbb{E}_0^{\mathbb{Q}}\left[\lambda_T e^{-\int_0^T \lambda_u du}\right]$ for $T \in [0, 70]$ and an age of 25, given the values of the parameters calibrated in the previous sections

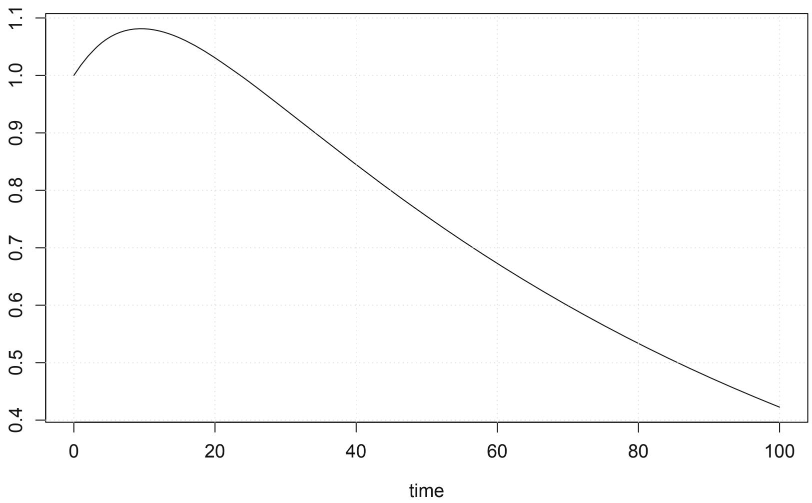
```
EQ_c_r = function(t, T, c, r) {
    alpha = alpha_r + sigma_r * k_r
    beta = alpha_r * beta_r/(alpha_r + sigma_r * k_r)
    c * exp((mu_c - sigma_cA * xi_A) * (T - t)) * exp(-alpha *
        beta * int_C(t, T, alpha - sigma_r * sigma_cr,
        sigma_r, 1 + k_r * sigma_cr)) * exp(-C(t, T,
        alpha - sigma_r * sigma_cr, sigma_r, 1 + k_r *
            sigma_cr) * r)
}
```

If we use the values calibrated on the US data in the previous subsections, we can obtain for $T \in [0, 100]$ the curve drawn in Fig. 8.7.

The expected discounted value of a future contribution is increasing for the first years while the growth of the discount factor prevails over the increment in the contribution, and then it is decreasing afterwards.

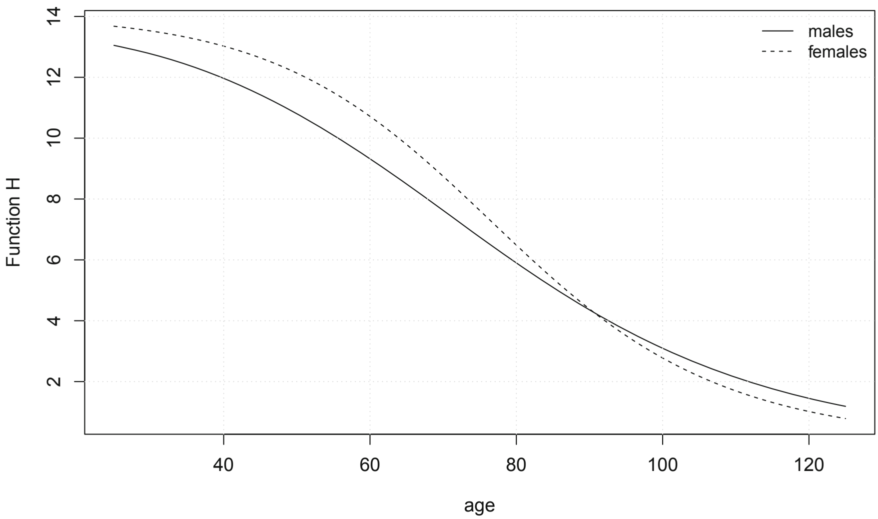
The numerical integration of the function $H(t)$ creates some problems if the infinity is suitably taken into account. Nevertheless, we can take a "sufficiently" long financial horizon and the value of the function $H(t)$ always converges.

The value of the function $H(t)$ can be coded as follows, where we use the user defined functions presented above.




```
plot(seq(0, 100), lapply(seq(0, 100), EQ_c_r, t = 0,
    c = 1, r = beta_r), type = "l", xlab = "time",
    ylab = "")
grid()
```

Fig. 8.7 Value of the function $\mathbb{E}_0^{\mathbb{Q}}\left[c_T e^{-\int_0^T r_u du}\right]$ for $T \in [0, 70]$ given the values of the parameters calibrated in the previous sections




```
matplot(age + seq(0, 100), cbind(lapply(seq(0, 100),
    H, r = beta_r, l = l0[1], sex = 1), lapply(seq(0,
    100), H, r = beta_r, l = l0[2], sex = 2)), type = "l",
    xlab = "age", ylab = "Function H", lty = c(1, 2),
    col = 1)
grid()
legend("topright", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 8.8 Behaviour of the "auxiliary" function $H(t)$ by changing t and starting from the age of 25, with the value of the parameters calibrated in the previous section

```
H = function(t, r, l, sex) {
    INT = function(s) {
        func = function(s) {
            EQ_l(t = t, T = s, q = 1, u = 1, v = 0,
                l = l, prob = "Q", sex = sex) * EQ_r(t = t,
                T = s, q = 1, r = r, prob = "Q")
        }
        sapply(s, func)
    }
    integrate(INT, t, 200)$value
}
```

The behaviour of the function $H(t)$ with respect to time t, given an initial age of 25 and the values of the parameters as in Table 8.1, can be shown as in Fig. 8.8.

The function $F(t)$ can be computed through a code akin to that just presented for the function $H(t)$. Here, we define the subjective discount rate (ρ) equal to the equilibrium value of the interest rate (β_r).

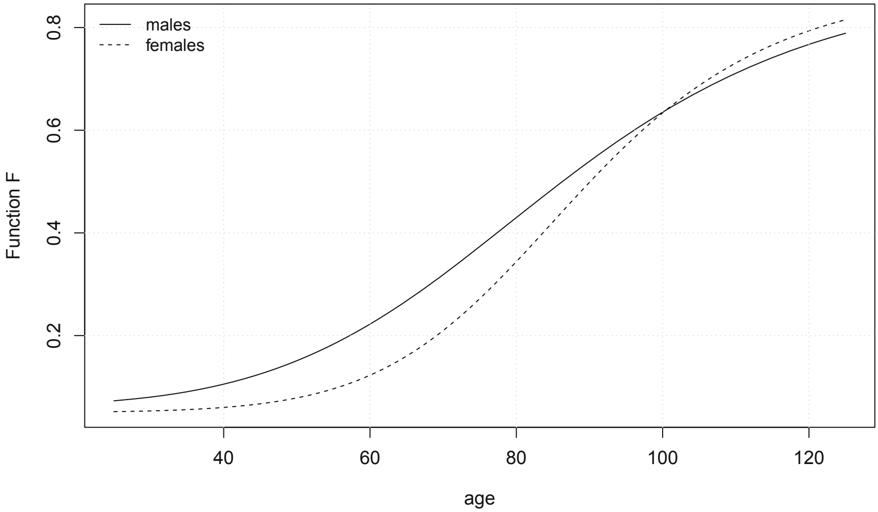
```
rho = beta_r
F = function(t, r, l, sex) {
    INT = function(s) {
        func = function(s) {
            q_l = 1 + 0.5 * (delta - 1)/delta^2 * k_l^2
            q_r = 1 - 1/delta + 0.5 * (delta - 1)/delta^2 *
                k_r^2
            EQ_l(t = t, T = s, q = q_l, u = 0, v = 1,
                l = l, prob = "Qd", sex = sex) * EQ_r(t = t,
                T = s, q = q_r, r = r, prob = "Qd") *
                exp(-(rho/delta + 0.5 * (delta - 1)/delta^2 *
                  xi_A^2) * (s - t))
        }
        sapply(s, func)
    }
    integrate(INT, t, 200)$value
}
```

The function $F(t)$ is drawn in Fig. 8.9 where we see that its value is a bit lower than that of the function $H(t)$.

Now, we are ready to write the code for computing the equilibrium pension p^* that makes $\Delta_{t_0} = 0$. Such a value is given by

$$p^* = \frac{\int_{t_0}^{T} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[c_s e^{-\int_{t_0}^{s} r_u du} \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} \lambda_u du} \right] ds}{\int_{T}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} \lambda_u du} \right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} r_u du} \right] ds},$$

and the code is as follows.




```
matplot(age + seq(0, 100), cbind(lapply(seq(0, 100),
    F, r = beta_r, l = l0[1], sex = 1), lapply(seq(0,
    100), F, r = beta_r, l = l0[2], sex = 2)), type = "l",
    xlab = "age", ylab = "Function F", lty = c(1, 2),
    col = 1)
grid()
legend("topleft", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 8.9 Behaviour of the "auxiliary" function $F(t)$ by changing t and starting from the age of 25, with the value of the parameters calibrated in the previous section

```r
pension = function(T, r0, l0, c0, sex) {
    INT_Num = function(s) {
        func = function(s) {
            EQ_c_r(t = 0, s, c = c0, r = r0) * EQ_l(t = 0,
                s, q = 1, u = 1, v = 0, l = l0, prob = "Q",
                sex = sex)
        }
        sapply(s, func)
    }
    INT_Den = function(s) {
        func = function(s) {
            EQ_r(t = 0, s, q = 1, r = r0, prob = "Q") *
                EQ_l(t = 0, s, q = 1, u = 1, v = 0,
                  l = l0, prob = "Q", sex = sex)
        }
        sapply(s, func)
    }
    integrate(INT_Num, 0, T)$value/integrate(INT_Den,
        T, 200)$value
}
p = c(NA, NA)
p[1] = pension(T = 65 - age, r0 = beta_r, l0 = l0[1],
    c0 = 1, sex = 1)
p[2] = pension(T = 65 - age, r0 = beta_r, l0 = l0[2],
    c0 = 1, sex = 2)
p

## [1] 187.6411 102.4462
```

We see that the pension for females is much lower than that for males. This is due to the longest lifetime for females. In fact, if a pension level must be guaranteed for a longer period of time, the pension level itself cannot be too high. Once p^* is known, we are able to compute the value of the PMR.

```
PMR = function(t, T, c, r, l, sex) {
    INT_A = function(s) {
        func = function(s) {
            EQ_c_r(t, s, c, r) * EQ_l(t, s, q = 1,
                u = 1, v = 0, l, prob = "Q", sex)
        }
        sapply(s, func)
    }
    INT_D = function(s) {
        func = function(s) {
            EQ_r(t, s, q = 1, r, prob = "Q") * EQ_l(t,
                s, q = 1, u = 1, v = 0, l, prob = "Q",
                sex)
        }
        sapply(s, func)
    }
    p[sex] * integrate(INT_D, max(t, T), 200)$value -
        integrate(INT_A, min(t, T), T)$value
}
```

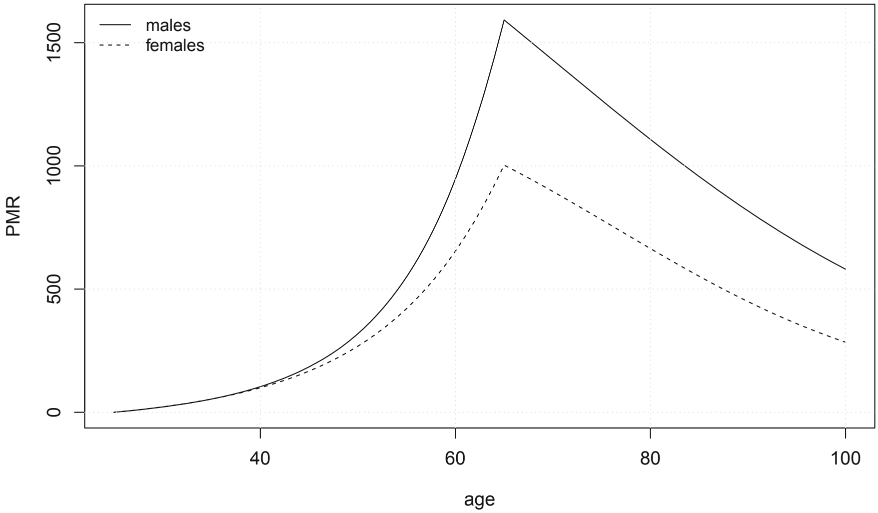
Finally, we can draw the PMR as in Fig. 8.10 where we see that the "triangular" behaviour already shown in the previous chapters is confirmed. The PMR starting value is zero (because it has been built in order to satisfy such a condition), then it is increasing until the retirement age (65) and, finally, it is decreasing.

Remark 8.4 In Fig. 8.10 we have assumed that the value of the force of mortality λ_t does not change over time and is always equal to the initial value. This is done just for the sake of simplicity. Instead, a full path of λ_t should be simulated and increasing values of the force of mortality should be used an input in the PMR function. Thus, the reduction in the value of PMR after the retirement age would be much stronger.

8.11 The Derivatives of the Auxiliary Functions

The last components we need in order to compute the optimal portfolio are the derivatives of the functions H_t, F_t, and Δ_t with respect to the state variables r_t, λ_t, and c_t. In particular, we have

$$\frac{\partial H_t}{\partial z_t} = \begin{bmatrix} \frac{\partial H_t}{\partial \lambda_t} \\ \frac{\partial Ht}{\partial r_t} \\ \frac{\partial H_t}{\partial c_t} = 0 \end{bmatrix}, \quad \frac{\partial F_t}{\partial z_t} = \begin{bmatrix} \frac{\partial F_t}{\partial \lambda_t} \\ \frac{\partial F_t}{\partial r_t} \\ \frac{\partial F_t}{\partial c_t} = 0 \end{bmatrix}, \quad \frac{\partial \Delta_t}{\partial z_t} = \begin{bmatrix} \frac{\partial \Delta_t}{\partial \lambda_t} \\ \frac{\partial \Delta_t}{\partial r_t} \\ \frac{\partial \Delta_t}{\partial c_t} \end{bmatrix}.$$




```
plot(age + seq(0, 75), lapply(seq(0, 75), PMR, c = 1,
    r = beta_r, l = l0[1], sex = 1, T = 65 - age),
    type = "l", xlab = "age", ylab = "PMR")
lines(age + seq(0, 75), lapply(seq(0, 75), PMR, c = 1,
    r = beta_r, l = l0[2], sex = 2, T = 65 - age),
    type = "l", lty = 2)
grid()
legend("topleft", c("males", "females"), lty = c(1,
    2), bty = "n")
```

Fig. 8.10 Behaviour of the PMR under the risk neutral probability by changing t and starting from the age of 25, with the value of the parameters calibrated in the previous section. The retirement age is set to 65 (i.e. 65–25 years after the initial time)

The derivative of the PMR Δ_t computed with respect to the contribution c_t takes an easy form already computed in (8.3.3) as follows

$$\frac{\partial \Delta_t}{\partial c_t} = \begin{cases} -\frac{1}{c_t} \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left[c_s e^{-\int_t^s r_u du} \right] \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^s \lambda_u du} \right] ds, & t < T \\ 0. & t \geq T \end{cases}$$

and it can be computed in R by using the functions that have already been created for the previous computations.

```
dDelta_dc = function(t, T, r, l, c, sex) {
    INT = function(s) {
        func = function(s) {
            EQ_c_r(t, s, c, r) * EQ_l(t, s, q = 1,
                u = 1, v = 0, l, prob = "Q", sex)
        }
        sapply(s, func)
    }
    -integrate(INT, t, T)$value/c * as.numeric(t <
        T)
}
```

If this derivative is drawn with respect to time, by taking the "initial" interest rate, contribution, and force of mortality constant, then we obtain the result shown in Fig. 8.11.

The behaviour of this derivative, which appears both in the stock and in the bond weights, means that there is a deep need for hedging against the stochastic changes in contributions c_t at the beginning of the management period. Instead, while time goes on, this need becomes less and less important. Finally, at the retirement age, there is no more need to hedge, since there are no more contributions left. Instead, the pensions start being paid and they are assumed to be constant.

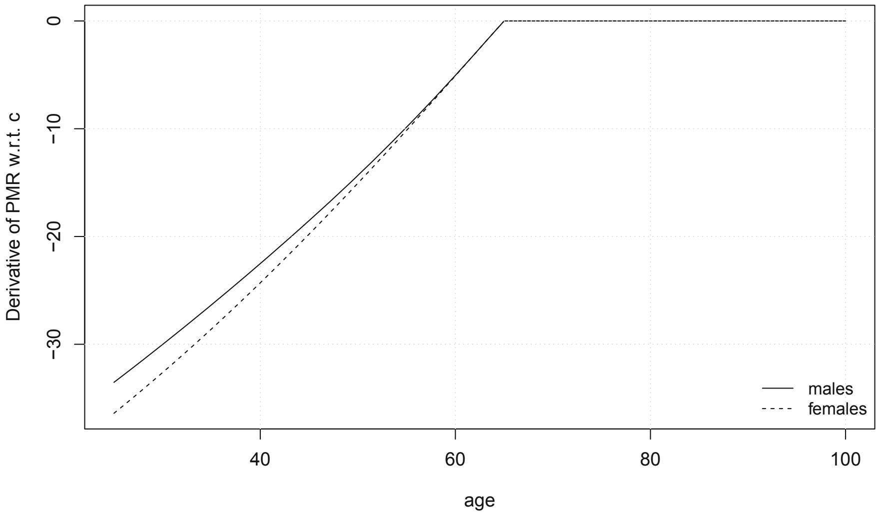
For computing the other derivatives it is sufficient to differentiate the result obtained in Proposition 8.1, from which we obtain

$$\begin{aligned}&\frac{\partial}{\partial x_t}\mathbb{E}_t\left[\left(u+vx_T\right)e^{-q\int_t^T x_s ds}\right]\\ =&ve^{-\int_t^T\left(\alpha+\left(\sigma^2+\alpha\beta_s\right)C(s)\right)ds-C(t)x_t}\\ &-C\left(t\right)\mathbb{E}_t\left[\left(u+vx_T\right)e^{-q\int_t^T x_s ds}\right].\end{aligned}$$

In the case of the interest rate, the result simplifies to

$$\frac{\partial}{\partial r_t}\mathbb{E}_t\left[e^{-q\int_t^T r_s ds}\right]=-C\left(t\right)\mathbb{E}_t\left[e^{-q\int_t^T r_s ds}\right].$$

Accordingly, we can use again the same function previously defined for computing the expected value $\mathbb{E}_t\left[e^{-q\int_t^T r_s ds}\right]$ where we just multiply by $-C\left(t\right)$ the final computation.




```
matplot(age + seq(0, 75), cbind(lapply(seq(0, 75),
    dDelta_dc, c = 1, r = beta_r, l = l0[1], sex = 1,
    T = 65 - age), lapply(seq(0, 75), dDelta_dc, c = 1,
    r = beta_r, l = l0[2], sex = 2, T = 65 - age)),
    type = "l", xlab = "age", ylab = "Derivative of PMR w.r.t. c",
    lty = c(1, 2), col = 1)
grid()
legend("bottomright", c("males", "females"), lty = c(1,
    2), bty = "n")
```

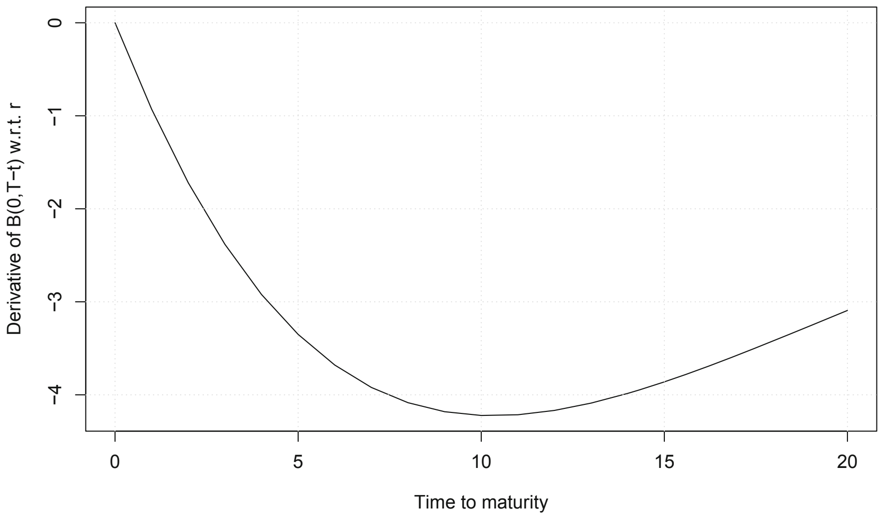
Fig. 8.11 Behaviour of the derivative of the PMR with respect to the contribution c_t, by changing t and starting from the age of 25, with the value of the parameters calibrated in the previous section. The retirement age is set to 65 (i.e. 65–25 years after the initial time)

```
dEQ_r_dr = function(q, prob, t, T, r) {
    if (prob == "Q") {
        alpha = alpha_r + sigma_r * k_r
        beta = alpha_r * beta_r/(alpha_r + sigma_r *
            k_r)
    }
```

```
    if (prob == "Qd") {
        alpha = alpha_r + sigma_r * k_r * (1 - 1/delta)
        beta = alpha_r * beta_r/(alpha_r + sigma_r *
            k_r * (1 - 1/delta))
    }
    -C(t, T, alpha, sigma_r, q) * exp(-alpha * beta *
        int_C(t, T, alpha, sigma_r, q) - C(t, T, alpha,
        sigma_r, q) * r)
}
```

If we take the case with the risk neutral probability $\mathbb{Q}$ and $q = 1$, then we can draw the derivative of a ZCB $B(0, T - t)$ with respect to the interest rate r_0 and for many different time to maturities as in Fig. 8.12.

When computing the derivative with respect to the value of λ_t, we can use the function that was already written for computing the value of the expected value in Proposition 8.1.




```
plot(20 - seq(0, 20), dEQ_r_dr(q = 1, prob = "Q", t = seq(0,
    20), T = 20, r = beta_r), type = "l", xlab = "Time to maturity",
    ylab = "Derivative of B(0,T-t) w.r.t. r")
grid()
```

Fig. 8.12 Derivative of the ZCB $B(0, T - t)$ computed with respect to r_0 for some time to maturities. In the figure we assume that $T = 20$ and so $t \in [0, 20]$

```
dEQ_l_dl = function(q, u, v, prob, t, T, l, sex) {
    alpha_l = alpha_l[sex]
    sigma_l = sigma_l[sex]
    phi = phi_l[sex]
    b = b_l[sex]
    m = m_l[sex]
    if (prob == "Q") {
        alpha = alpha_l + sigma_l * k_l
        beta = function(s) {
            alpha_l * (phi + (1/(alpha_l * b) + 1)/b *
                exp((age + s - m)/b))/(alpha_l + sigma_l *
                k_l)
        }
    }
    if (prob == "Qd") {
        alpha = alpha_l + sigma_l * k_l * (1 - 1/delta)
        beta = function(s) {
            alpha_l * (phi + (1/(alpha_l * b) + 1)/b *
                exp((age + s - m)/b))/(alpha_l + sigma_l *
                k_l * (1 - 1/delta))
        }
    }
    INT_A = function(s) {
        alpha + (sigma_l^2 + alpha * beta(s)) * C(t,
            s, alpha, sigma_l, q)
    }
    v * exp(-integrate(INT_A, t, T)$value - C(t, T,
        alpha, sigma_l, q) * l) - C(t, T, alpha, sigma_l,
        q) * EQ_l(t, T, q, u, v, l, prob, sex)
}
```

Finally, the derivative of the expected discounted contributions with respect to the interest rate is given by:

$$\frac{\partial}{\partial r_t}\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du}\right] = -C\left(t; T, \alpha_r^{\mathbb{Q}} - \sigma_r\sigma_{c,r}, \sigma_r, 1+\kappa_r\sigma_{c,r}\right)\mathbb{E}_t^{\mathbb{Q}}\left[c_T e^{-\int_t^T r_u du}\right].$$

```
dEQ_c_r_dr = function(t, T, c, r) {
    alpha = alpha_r + sigma_r * k_r
    beta = alpha_r * beta_r/(alpha_r + sigma_r * k_r)
    c * exp((mu_c - sigma_cA * xi_A) * (T - t)) * exp(-alpha *
        beta * int_C(t, T, alpha - sigma_r * sigma_cr,
        sigma_r, 1 + k_r * sigma_cr)) * exp(-C(t, T,
        alpha - sigma_r * sigma_cr, sigma_r, 1 + k_r *
            sigma_cr) * r) * (-C(t, T, alpha - sigma_r *
        sigma_cr, sigma_r, 1 + k_r * sigma_cr))
}
```

All these components can now be put together in order to compute the derivatives of the function $H(t)$, $F(t)$, and $\Delta^{\mathbb{Q}}(t)$ with respect to the state variables. In fact, in the codes used for computing these functions it will be sufficient to substitute the original expected values, with their derivatives as coded in the previous boxes.

```
dH_dr = function(t, r, l, sex) {
    INT = function(s) {
        func = function(s) {
            EQ_l(t, s, q = 1, u = 0, v = 1, l, prob = "Q",
                sex) * dEQ_r_dr(q = 1, prob = "Q",
                t, s, r)
        }
        sapply(s, func)
    }
    integrate(INT, t, 200)$value
}
dH_dl = function(t, r, l, sex) {
    INT = function(s) {
        func = function(s) {
            dEQ_l_dl(q = 1, u = 0, v = 1, prob = "Q",
                t, s, l, sex) * EQ_r(t, s, q = 1, r,
                prob = "Q")
        }
        sapply(s, func)
    }
    integrate(INT, t, 200)$value
}
dF_dr = function(t, r, l, sex) {
    q_l = 1 + 0.5 * (delta - 1)/delta^2 * k_l^2
    q_r = 1 - 1/delta + 0.5 * (delta - 1)/delta^2 *
        k_r^2
    INT = function(s) {
        func = function(s) {
            EQ_l(t, s, q = q_l, u = 0, v = 1, l, prob = "Qd",
                sex) * dEQ_r_dr(q = q_r, prob = "Qd",
                t, s, r) * exp(-(rho/delta + 0.5 *
                (delta - 1)/delta^2 * xi_A^2) * (s -
                t))
        }
        sapply(s, func)
```

```
    }
    integrate(INT, t, 200)$value
}
dF_dl = function(t, r, l, sex) {
    q_l = 1 + 0.5 * (delta - 1)/delta^2 * k_l^2
    q_r = 1 - 1/delta + 0.5 * (delta - 1)/delta^2 *
        k_r^2
    INT = function(s) {
        func = function(s) {
            dEQ_l_dl(q = q_l, u = 0, v = 1, prob = "Qd",
                t, s, l, sex) * EQ_r(t, s, q = q_r,
                r, prob = "Qd") * exp(-(rho/delta +
                0.5 * (delta - 1)/delta^2 * xi_A^2) *
                (s - t))
        }
        sapply(s, func)
    }
    integrate(INT, t, 200)$value
}
dDelta_dr = function(t, T, c, r, l, sex) {
    INT_A = function(s) {
        func = function(s) {
            dEQ_c_r_dr(t, s, c, r) * EQ_l(t, s, q = 1,
                u = 1, v = 0, l, "Q", sex)
        }
        sapply(s, func)
    }
    INT_D = function(s) {
        func = function(s) {
            dEQ_r_dr(q = 1, "Q", t, s, r) * EQ_l(t,
                s, q = 1, u = 1, v = 0, l, "Q", sex)
        }
        sapply(s, func)
    }
    p[sex] * integrate(INT_D, max(t, T), 200)$value -
        integrate(INT_A, min(t, T), T)$value
}
dDelta_dl = function(t, T, c, r, l, sex) {
    INT_A = function(s) {
        func = function(s) {
            EQ_c_r(t, s, c, r) * dEQ_l_dl(q = 1, u = 1,
                v = 0, "Q", t, s, l, sex)
        }
        sapply(s, func)
    }
```

```
    INT_D = function(s) {
        func = function(s) {
            EQ_r(t, s, q = 1, r, "Q") * dEQ_l_dl(q = 1,
                u = 1, v = 0, "Q", t, s, l, sex)
        }
        sapply(s, func)
    }
    p[sex] * integrate(INT_D, max(t, T), 200)$value -
        integrate(INT_A, min(t, T), T)$value
}
```

We finally have all the components necessary to compute the optimal portfolio as presented in the next section.

8.12 The Simulations

The first step for computing the optimal portfolio is to define the values of the preference parameters in the utility function. In particular, we set:

- the initial value of all the variables: stock, bond, longevity asset, wage, interest rate, force of mortality, wealth; in particular we set:
  - $R_0 = 100$, which is just an arbitrary value;
  - $r_0 = \beta_r$: the initial interest rate is assumed to be at its long term equilibrium value;
  - $\lambda_0 = \beta_{\lambda,0} (\iota = 25)$: the force of mortality is assumed to start from its equilibrium value (for both males and females) when the agents are aged 25;
  - $A_0 = B(0, h) = L(0, h) = c_0 = 1$, where we recall that $h = 10$;
  - $t_0 = 25$: this is the age of the agent at the beginning of the management period;
  - $T = 65 - t_0$: the age of retirement;
- the preference parameters α, δ, and ρ; in particular we set:
  - $\alpha = R_0$: we assume that the investor wants to guarantee, at the end of the management period, a wealth which is at least equal the starting value;
  - $\delta = 2.5$: this is a standard value for the Arrow–Pratt risk aversion;
  - $\rho = \beta_r$: we assume that the agent discount the future at the long term equilibrium value of the interest rate (Table 8.2).

Table 8.2 Parameters of the investor's preferences in the basic scenario

Initial values	Subjective values
$R_0 = 100$	$\alpha = 0$
$r_0 = 0.0513683572758613$	$\delta = 2.5$
$\lambda_0 = 0.00463909139909301$ (males)	$\rho = 0.0513683572758613$
$\lambda_0 = 0.00357553703354542$ (females)	$t_0 = 0$
$A_0 = B(0, 10) = L(0, 10) = c_0 = 1$	$T = 40$

```
# Initial values
R0 = 100
r0 = beta_r
A0 = B0 = L0 = c0 = 1
h = 10
age = 25
T = 65 - age
alpha = 0
delta = 2.5
rho = beta_r
```

In this framework, the optimal portfolio is given by

$$w_{A,t}^{*} A_t = \frac{R_t - \alpha H_t - \Delta_t}{\delta} \frac{\xi_A}{\sigma_A} + c_t \frac{\sigma_{c,A}}{\sigma_A} \frac{\partial \Delta_t}{\partial c_t}, \tag{8.12.1}$$

$$\begin{aligned} w_{B,t}^{*} B(t, t+h) = & \frac{R_t - \alpha H_t - \Delta_t}{C\left(0; h, \alpha_r^{\mathbb{Q}}, \sigma_r, 1\right)} \left(\frac{1}{\delta}\left(\frac{\sigma_{A,r}\xi_A}{\sigma_r \sigma_A} - \frac{\kappa_r}{\sigma_r}\right) - \frac{1}{F_t}\frac{\partial F_t}{\partial r_t}\right) \\ & - \frac{1}{C\left(0; h, \alpha_r^{\mathbb{Q}}, \sigma_r, 1\right)} \left(\alpha \frac{\partial H_t}{\partial r_t} + \frac{\partial \Delta_t}{\partial r_t} - c_t \left(\frac{\sigma_{A,r}\sigma_{c,A}}{\sigma_r \sigma_A} - \frac{\sigma_{c,r}}{\sigma_r}\right)\frac{\partial \Delta_t}{\partial c_t}\right) \\ & - w_{L,t}^{*} L(t, t+h), \end{aligned} \tag{8.12.2}$$

$$\begin{aligned} w_{L,t}^{*} L(t, t+h) = & - \frac{R_t - \alpha H_t - \Delta_t}{C\left(0; h, \alpha_\lambda^{\mathbb{Q}}, \sigma_\lambda, 1\right)} \left(\frac{\kappa_\lambda}{\delta \sigma_\lambda} + \frac{1}{F_t}\frac{\partial F_t}{\partial \lambda_t}\right) \\ & - \frac{1}{C\left(0; h, \alpha_\lambda^{\mathbb{Q}}, \sigma_\lambda, 1\right)} \left(\alpha \frac{\partial H_t}{\partial \lambda_t} + \frac{\partial \Delta_t}{\partial \lambda_t}\right). \end{aligned} \tag{8.12.3}$$

The simulations are based on the following procedure:

- the initials values of the state variables have already been set
- the optimal portfolio is computed by using the previous values
- the optimal wealth is computed by using the previously obtained optimal portfolio
- the new values of the state variables are simulated
- the process goes back to the second line

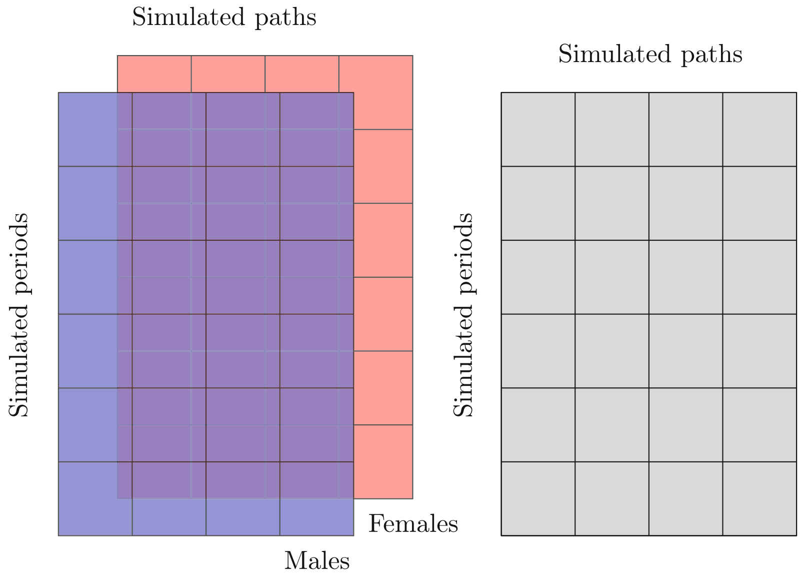


Fig. 8.13 Array for the simulations of stochastic variables that either depend (on the left) or do not depend (on the right) on the sex of the agents

We simulate N paths of the variables at each iteration and we computed two portfolios and state variables at each passage, one per sex (males and females). This means that most of the variables will be gathered in an array which has: (1) as may rows as simulated periods, (2) as many columns are simulated paths, and (3) as many "slices" are simulated sex. This scheme can be represented as in Fig. 8.13.

Since the code that we are about to write is quite complex and slow, it will take many minutes to conclude the simulations. This is the reason why we use a limited number of simulations $N = 100$ and quarterly (instead of daily) data ($dt = 1/4$).

```
# Simluations setting
N = 100  #number of paths
horizon = 60  #number of periods (years)
dt = 1/4  #subperiods (quarters)
```

Finally, we can write the code corresponding to the presented approach as follows.

```
# Setting of Wiener processes
dW_A = dW_r = dW_l = array(rnorm(horizon/dt * N, mean = 0,
    sd = sqrt(dt)), dim = c(horizon/dt, N))
# Setting of initial state variables
R = array(NA, dim = c(horizon/dt, N, 2))
R[1, , ] = R0
l = array(NA, dim = c(horizon/dt, N, 2))
l[1, , 1] = l0[1]
l[1, , 2] = l0[2]
r = array(NA, dim = c(horizon/dt, N))
r[1, ] = r0
co = array(NA, dim = c(horizon/dt, N))
co[1, ] = c0
wA = wB = wL = array(NA, dim = c(horizon/dt - 1, N,
    2))
RN = DN = HN = array(NA, dim = c(horizon/dt - 1, N,
    2))
C_r = C(0, h, alpha_r + k_r * sigma_r, sigma_r, 1)
C_l = C(0, h, alpha_l + k_l * sigma_l, sigma_l, 1)
# Cycle of simulations
for (i in 1:(horizon/dt - 1)) {
    t = i * dt  #transform the counter (days) in years
    for (sex in 1:2) {
        HN[i, , sex] = mapply(H, t, r[i, ], l[i, ,
            sex], sex)
        DN[i, , sex] = mapply(PMR, t, T, co[i, ], r[i,
            ], l[i, , sex], sex)
        RN[i, , sex] = R[i, , sex] - alpha * HN[i,
            , sex] - DN[i, , sex]
        wA[i, , sex] = RN[i, , sex] * xi_A/(delta *
            sigma_A) + co[i, ] * sigma_cA/sigma_A *
            mapply(dDelta_dc, t, T, r[i, ], l[i, ,
                sex], co[i, ], sex)
        wL[i, , sex] = -RN[i, , sex]/C_l[sex] * (k_l/(delta *
            sigma_l[sex]) + mapply(dF_dl, t, r[i, ],
            l[i, , sex], sex)/mapply(F, t, r[i, ],
            l[i, , sex], sex)) - (alpha * mapply(dH_dl,
            t, r[i, ], l[i, , sex], sex) + mapply(dDelta_dl,
            t, T, co[i, ], r[i, ], l[i, , sex], sex))/C_l[sex]
        wB[i, , sex] = RN[i, , sex]/C_r * ((sigma_Ar *
            xi_A/sigma_r/sigma_A - k_r/sigma_r)/delta -
            mapply(dF_dr, t, r[i, ], l[i, , sex], sex)/mapply(F,
                t, r[i, ], l[i, , sex], sex)) - (alpha *
```

```
            mapply(dH_dr, t, r[i, ], l[i, , sex], sex) +
            mapply(dDelta_dr, t, T, co[i, ], r[i, ],
                l[i, , sex], sex) - co[i, ] * (sigma_Ar *
            sigma_cA/sigma_r/sigma_A - sigma_cr/sigma_r) *
            mapply(dDelta_dc, t, T, r[i, ], l[i, ,
                sex], co[i, ], sex))/C_r - wL[i, ,
            sex]
        l[i + 1, , sex] = l[i, , sex] + alpha_l[sex] *
            (phi_l[sex] + (1/alpha_l[sex]/b_l[sex] +
                1)/b_l[sex] * exp((age + t - m_l[sex])/b_l[sex]) -
                l[i, , sex]) * dt + sigma_l[sex] *
            sqrt(l[i, , sex]) * dW_l[i, ]
        r[i + 1, ] = r[i, ] + alpha_r * (beta_r - r[i,
            ]) * dt + sigma_r * sqrt(r[i, ]) * dW_r[i,
            ]
        co[i + 1, ] = co[i, ] + co[i, ] * (mu_c * dt +
            sigma_cr * sqrt(r[i, ]) * dW_r[i, ] + sigma_cA *
            dW_A[i, ])
        R[i + 1, , sex] = R[i, , sex] + (R[i, , sex] *
            r[i, ] + wA[i, , sex] * (mu_A - r[i, ]) +
            wB[i, , sex] * (-C_r * sigma_r * k_r *
                r[i, ]) + wL[i, , sex] * (-C_r * sigma_r *
            k_r * r[i, ] + (1 - C_l[sex] * sigma_l[sex] *
            k_l) * l[i, , sex]) - (p[sex] * as.numeric(t >=
            T) - co[i, ] * as.numeric(t < T))) * dt +
            wA[i, , sex] * sigma_A * dW_A[i, ] + (wA[i,
            , sex] * sigma_Ar * sqrt(r[i, ]) + wB[i,
            , sex] * (-C_r * sigma_r * sqrt(r[i, ])) +
            wL[i, , sex] * (-C_r * sigma_r * sqrt(r[i,
                ]))) * dW_r[i, ] + wL[i, , sex] * (-C_l[sex] *
            sigma_l[sex] * sqrt(l[i, , sex])) * dW_l[i,
            ]
    }
    print(i)
}
```

We stress that the last command "`print(i)`" is just used for showing on the screen how the process is proceeding.

Remark 8.5 The simulation of the state variables r_t, c_t, and λ_t can be performed outside the for cycle for the portfolio. Actually, the for cycle that we have written in the previous code can be split into two codes: one for simulating the stochastic state variables, and one for simulating the portfolio. Of course these two cycles must use the very same simulated Wiener processes.

At first we check the behaviour of the state variables. In Figs. 8.14, 8.15, and 8.16 we show the results of the simulated force of mortality λ_t (for both males and females), interest rate r_t, and contribution c_t, respectively. In each figure we create a matrix containing the worst case (0 quantile), the median (50% quantile) , and the best case (100% quantile of 100 simulations.

```
time = seq(age, age + horizon - 2 * dt, dt)
l_Q = cbind(t(apply(l[1:(horizon/dt - 1), , 1], 1,
    quantile, probs = c(0, 0.5, 1))), t(apply(l[1:(horizon/dt -
    1), , 2], 1, quantile, probs = c(0, 0.5, 1))))
r_Q = cbind(t(apply(r[1:(horizon/dt - 1), ], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(r[1:(horizon/dt -
    1), ], 1, quantile, probs = c(0, 0.5, 1))))
co_Q = cbind(t(apply(co[1:(horizon/dt - 1), ], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(co[1:(horizon/dt -
    1), ], 1, quantile, probs = c(0, 0.5, 1))))
```

In Fig. 8.14 we can clearly check that females live longer on average and with a shorter variation of the extreme cases.

Figure 8.15 shows the mean reverting property of the interest rate, which is always positive. Finally, in Fig. 8.16 we see the exponential growth of contributions.

Another variable of interest for the optimal portfolio is the PMR, whose value for both males and females is shown in Fig. 8.17.

```
DN_Q = cbind(t(apply(DN[, , 1], 1, quantile, probs = c(0,
    0.5, 1))), t(apply(DN[, , 2], 1, quantile, probs = c(0,
    0.5, 1))))
```

We see that the PMR is increasing up the its maximum value at retirement, and then it start decreasing, as the theory indicates. The behaviour of PMR for males and females is definitely different. In particular, males are able to collect more contributions for their pensions, because they will receive those pensions for a shorter period of time on average (since they will live shorter). In other words, males discount future cash flows at a higher rate ($r_t + \lambda_t$) than females and, accordingly, their future pensions have a lower weight.

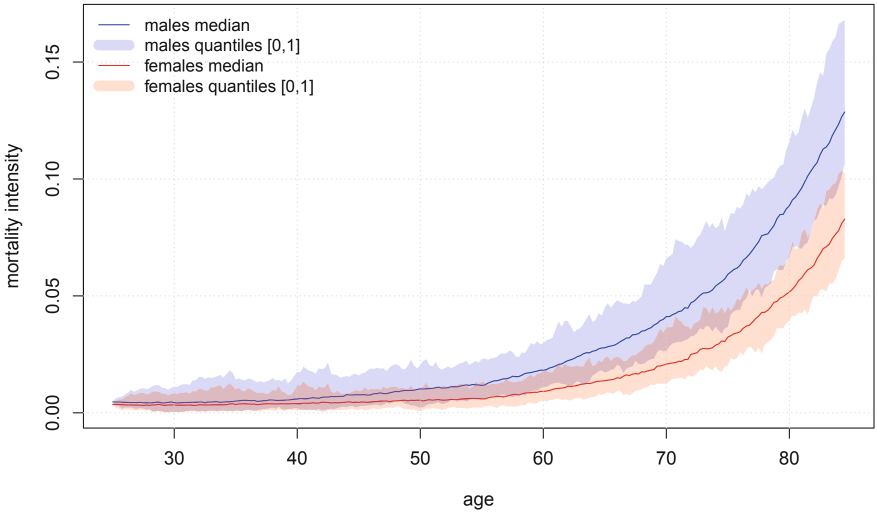
Finally, we can check the behaviour of the optimal wealth. Since this wealth is exponentially growing, we take into account the log-return. The code for computing the log-returns for both males and females is as follows.

```
dlnR = array(NA, dim = c(dim(R)[1] - 1, dim(R)[2],
    2))
dlnR[, , 1] = apply(log(R[, , 1]), 2, diff)
dlnR[, , 2] = apply(log(R[, , 2]), 2, diff)
```

Then, we can compute the median and the best and worst cases for these returns.

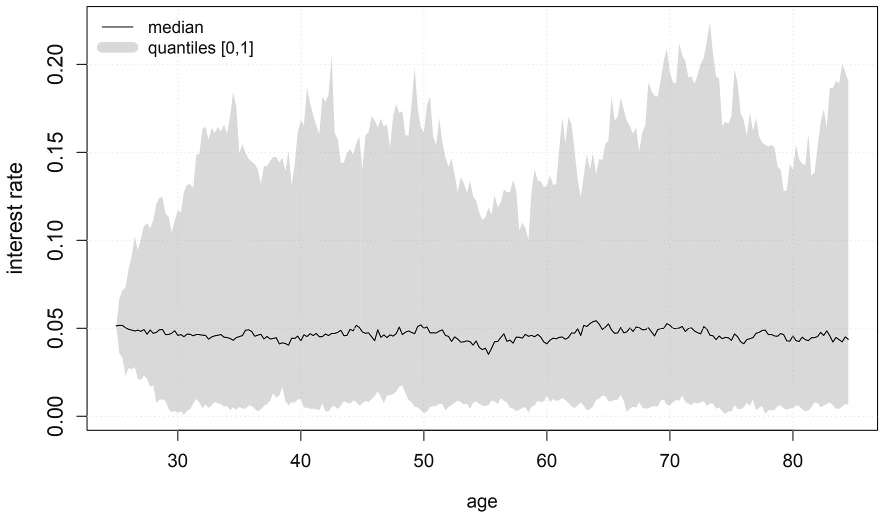
```
dlnR_Q = cbind(t(apply(dlnR[, , 1], 1, quantile, probs = c(0,
    0.5, 1))), t(apply(dlnR[, , 2], 1, quantile, probs = c(0,
    0.5, 1))))
time2 = head(time, -1)
```

The behaviour over time of the log-returns on wealth is shown in Fig. 8.18, where we see that the difference between sexes is negligible. Furthermore, the return on the optimal wealth is stable before retirement, while it starts increasing after retirement. This result is definitely due to the hedging strategy. In fact, before retirement, the




```
matplot(time, l_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "mortality intensity")
polygon(c(time, rev(time)), c(l_Q[, 3], rev(l_Q[, 1])),
    col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(l_Q[, 6], rev(l_Q[, 4])),
    col = rgb(1, 0, 0, alpha = 0.2), border = NA)
grid()
legend("topleft", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

Fig. 8.14 Result of 100 simulations of the force of mortality λ_t (for both males and females) with the values of parameter gathered in Table 8.1




```
matplot(time, r_Q, type = "l", lty = 1, col = c(NA,
    "black", NA), xlab = "age", ylab = "interest rate")
grid()
polygon(c(time, rev(time)), c(r_Q[, 3], rev(r_Q[, 1])),
    col = rgb(0, 0, 0, alpha = 0.2), border = NA)
legend("topleft", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c("black", rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

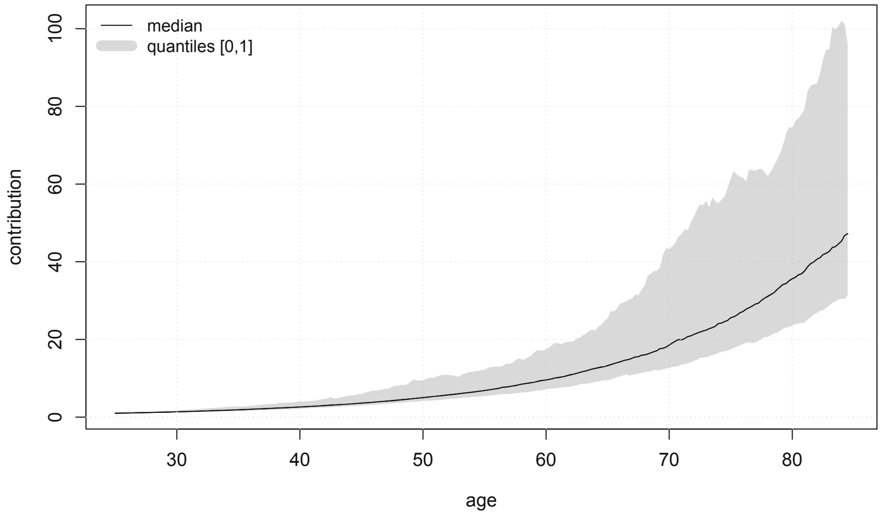
Fig. 8.15 Result of 100 simulations of the interest rate r_t with the values of parameter gathered in Table 8.1

need for hedging is higher and, accordingly, the fund must be renounce to a part of its return.

8.13 The Optimal Portfolio

Now, we can check the shape over time of the optimal portfolio allocation for both males and females. We start from the sum of all the risky assets as a percentage of wealth R_t:

$$\frac{w_{A,t}^* A_t + w_{B,t}^* B(t, t+h) + w_{L,t}^* L(t, t+h)}{R_t}.$$



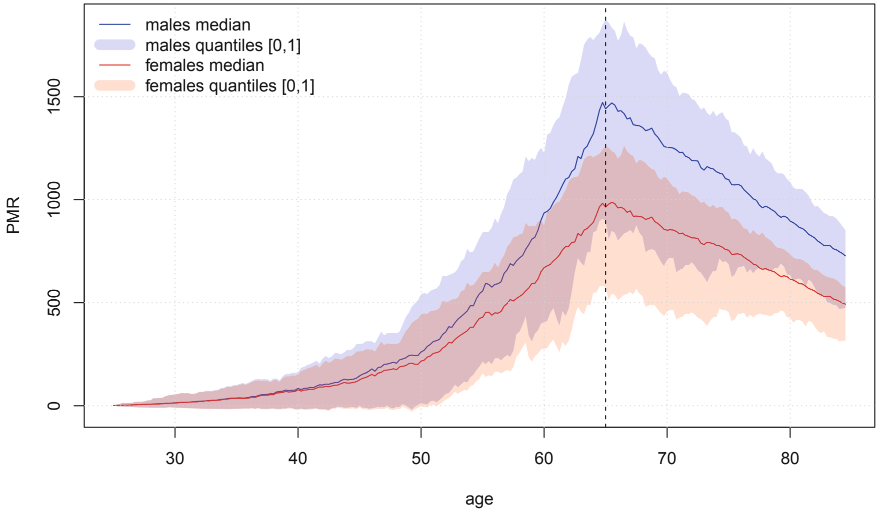

```
matplot(time, co_Q, type = "l", lty = 1, col = c(NA,
    "black", NA), xlab = "age", ylab = "contribution")
grid()
polygon(c(time, rev(time)), c(co_Q[, 3], rev(co_Q[,
    1])), col = rgb(0, 0, 0, alpha = 0.2), border = NA)
legend("topleft", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c("black", rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

Fig. 8.16 Result of 100 simulations of the contribution c_t with the values of parameter gathered in Table 8.1

This ratio helps us in understanding whether the whole portfolio is more or less risky over time. This percentage is shown, for both males and females, in Fig. 8.19, where we show three quantiles: 0% for presenting the so-called "worst case" scenario, 50% for showing the median, and 100% for presenting the best case scenario. These quantiles are obtained with the following commands.

```
wAwLwBR_Q = cbind(t(apply((wA[, , 1] + wL[, , 1] +
    wB[, , 1])/R[, , 1], 1, quantile, probs = c(0,
    0.5, 1))), t(apply((wA[, , 2] + wL[, , 2] + wB[,
    , 2])/R[, , 2], 1, quantile, probs = c(0, 0.5,
    1))))
```

As a percentage of R_t, we see that the behaviour of risky asset share is almost the same for males and females and, in particular, the share reduces over time till



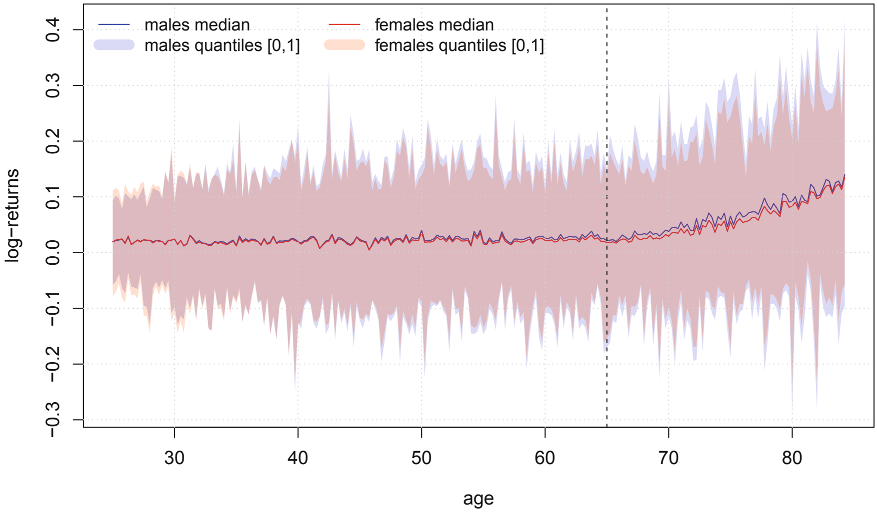

```
matplot(time, DN_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "PMR")
polygon(c(time, rev(time)), c(DN_Q[, 3], rev(DN_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(DN_Q[, 6], rev(DN_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
grid()
legend("topleft", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
abline(v = T + age, lty = 2)
```

Fig. 8.17 Result of 100 simulations of the PMR Δ_t for both males and females with the values of parameter gathered in Table 8.1 (the vertical lines coincides with retirement)

the retirement age. After this time, the shape increases and then after some years it reaches a kind "equilibrium" value.

We stress that this behaviour is clearly in contrast with what is usually argued by the financial advisers who suggest to invest lower and lower percentages of wealth while the agent becomes older and older. Actually, this seems to be true only till retirement, when, instead, the percentage of risky asset increases again.

This particular behaviour is due to the risk faced by the agents. At the beginning of the accumulation phase, the agent still has to receive all his/her future wages and, thus, the risk is relatively low. Nevertheless, while the time goes on, the agent



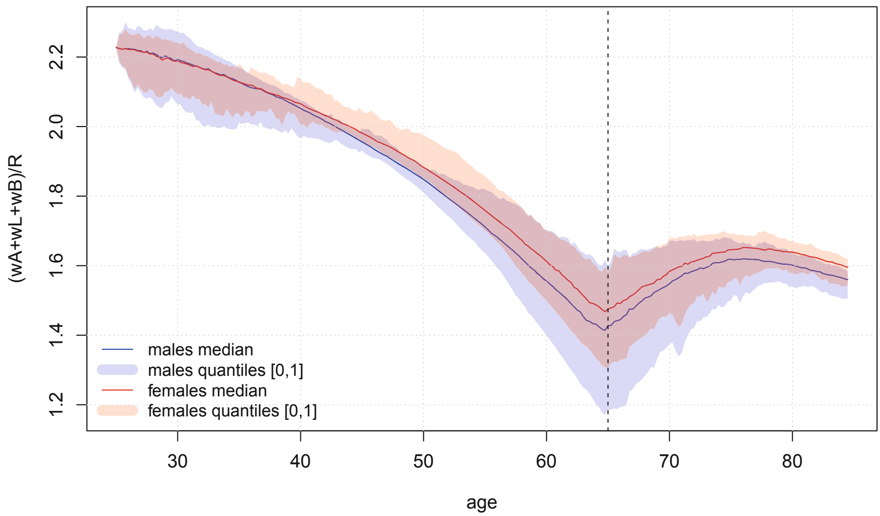

```
matplot(time2, dlnR_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "log-returns")
polygon(c(time2, rev(time2)), c(dlnR_Q[, 3], rev(dlnR_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time2, rev(time2)), c(dlnR_Q[, 6], rev(dlnR_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, lty = 2)
grid()
legend("topleft", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10), ncol = 2)
```

Fig. 8.18 Median and extreme scenarios of the optimal fund's wealth log-returns over time (100 simulations) for both males and females

has less and less future wages, while the pensions that must be financed remain the same. This means that the risk represented by the future pensions is less and less hedged by the wages and, accordingly, the amount of stock in the portfolio must be reduced.

We check that this behaviour is different if we compute the share of the optimal risky assets as a percentage of the corrected wealth as defined in (7.9.1):

$$\frac{w^*_{A,t} A_t + w^*_{B,t} B\,(t, t+h) + w^*_{L,t} L\,(t, t+h)}{\hat{R}_t},$$




```
matplot(time, wAwLwBR_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "(wA+wL+wB)/R")
polygon(c(time, rev(time)), c(wAwLwBR_Q[, 3], rev(wAwLwBR_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wAwLwBR_Q[, 6], rev(wAwLwBR_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, lty = 2)
grid()
legend("bottomleft", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

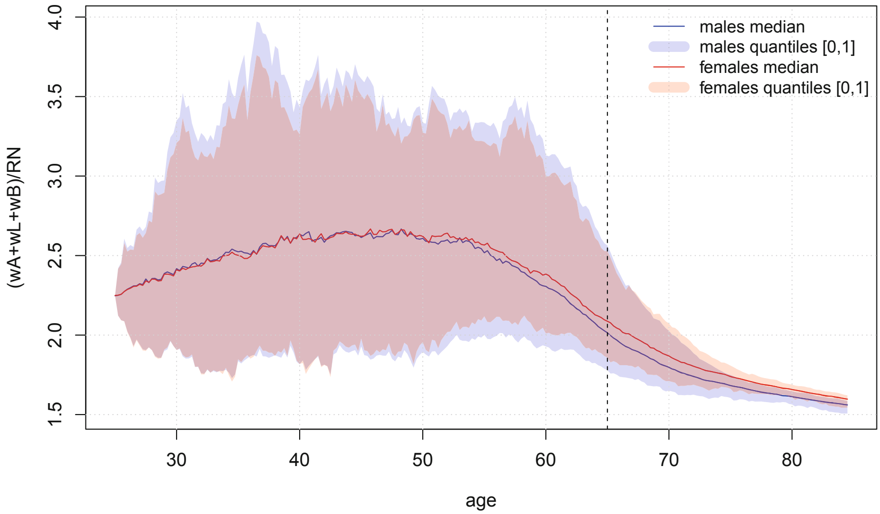
Fig. 8.19 Median and extreme scenarios of the risky assets share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{A,t}A_t+w^*_{B,t}B(t,t+h)+w^*_{L,t}L(t,t+h)}{R_t}$ for both males and females

and the shape is shown in Fig. 8.20. Again, we show here the commands for computing the quantiles.

```
wAwLwBRN_Q = cbind(t(apply((wA[, , 1] + wL[, , 1] +
    wB[, , 1])/RN[, , 1], 1, quantile, probs = c(0,
    0.5, 1))), t(apply((wA[, , 2] + wL[, , 2] + wB[,
    , 2])/RN[, , 2], 1, quantile, probs = c(0, 0.5,
    1))))
```

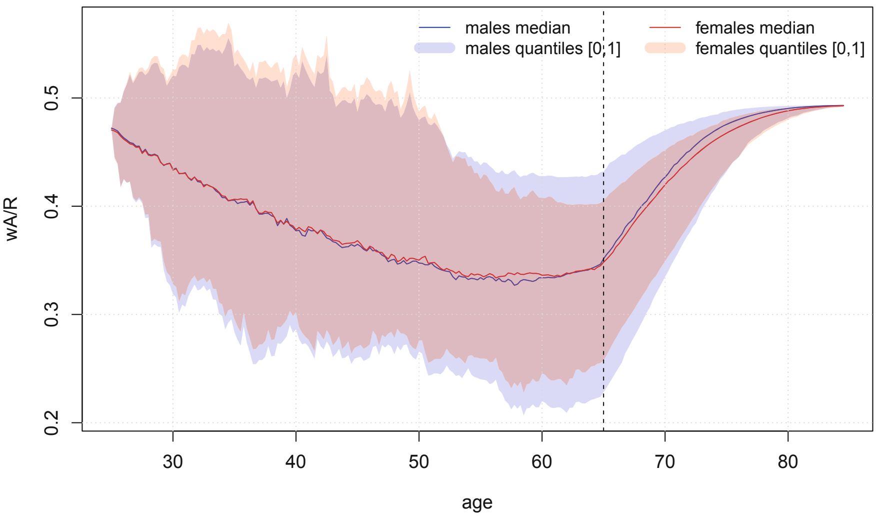
This time the changes over time are less strong and the portfolio is more stable during the accumulation phase. Instead, while the retirement approaches, the total risky asset over the corrected wealth decreases over time, and continue decreasing.

Now, we can disentangle the three asset classes and show their particular shape. Figures 8.21 and 8.22, show the behaviour of the stock share for both males and females with respect to the wealth R_t and the corrected wealth $\hat{R}_t$, respectively.




```
matplot(time, wAwLwBRN_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "(wA+wL+wB)/RN")
polygon(c(time, rev(time)), c(wAwLwBRN_Q[, 3], rev(wAwLwBRN_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wAwLwBRN_Q[, 6], rev(wAwLwBRN_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, lty = 2)
grid()
legend("topright", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

Fig. 8.20 Median and extreme scenarios of the risky assets share in the optimal portfolio over time (100 simulations) as a percentage of corrected wealth $\frac{w^*_{A,t}A_t+w^*_{B,t}B(t,t+h)+w^*_{L,t}L(t,t+h)}{\hat{R}_t}$ for both males and females

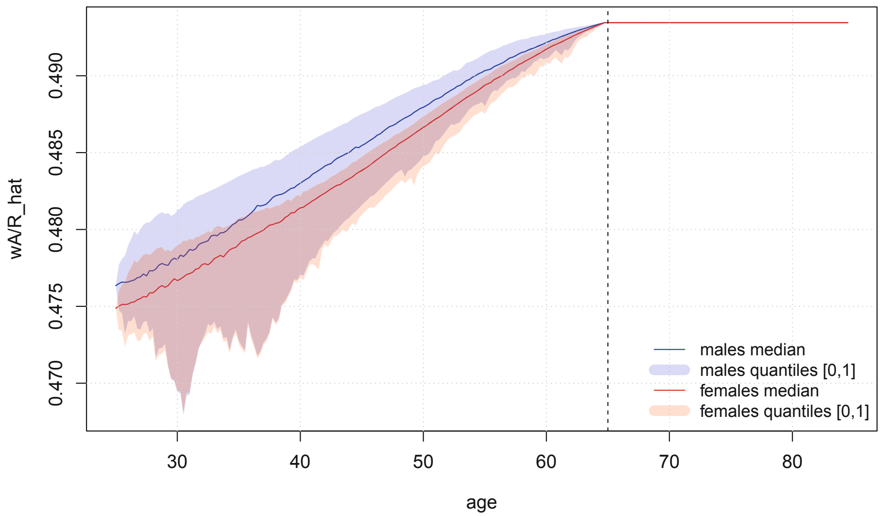



```
matplot(time, wAR_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "wA/R")
polygon(c(time, rev(time)), c(wAR_Q[, 3], rev(wAR_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wAR_Q[, 6], rev(wAR_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, lty = 2)
grid()
legend("topright", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10), ncol = 2)
```

Fig. 8.21 Median and extreme scenarios of the stock share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w_{A,t}A_t}{R_t}$ for both males and females

```
wAR_Q = cbind(t(apply(wA[, , 1]/R[, , 1], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(wA[, , 2]/R[, ,
    2], 1, quantile, probs = c(0, 0.5, 1))))
wARN_Q = cbind(t(apply(wA[, , 1]/RN[, , 1], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(wA[, , 2]/RN[,
    , 2], 1, quantile, probs = c(0, 0.5, 1))))
```

The shape of this percentage with respect to R_t is akin to what we have already seen for the whole portfolio, nevertheless, an important difference is that after the retirement the stock share reverts to an equilibrium and constant value. This share



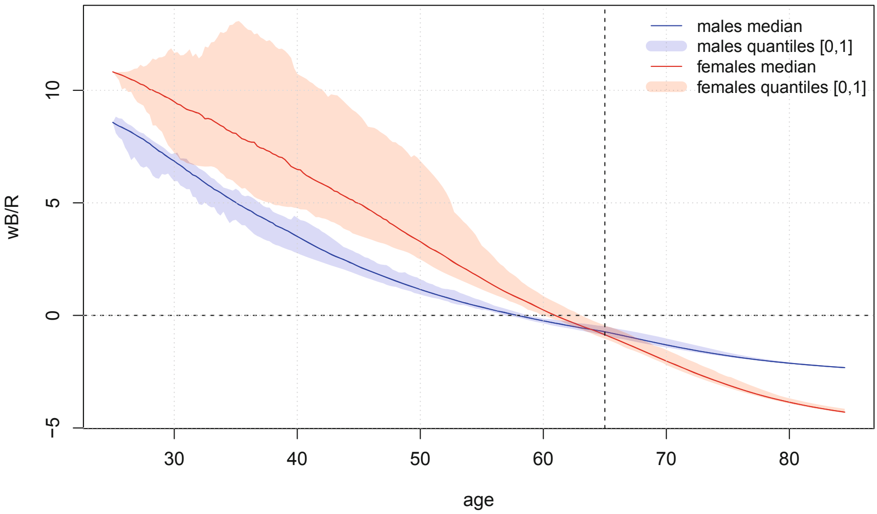

```
matplot(time, wARN_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "wA/R_hat")
polygon(c(time, rev(time)), c(wARN_Q[, 3], rev(wARN_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wARN_Q[, 6], rev(wARN_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, lty = 2)
grid()
legend("bottomright", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

Fig. 8.22 Median and extreme scenarios of the stock share in the optimal portfolio over time (100 simulations) as a percentage of modified wealth $\frac{w_{A,t}A_t}{\hat{R}_t}$ for both males and females

becomes constant because after retirement there is no more need to hedge against the wage risk (while the interest rate and the mortality risks are managed via the other two asset classes).

The share of stock with respect to the corrected wealth $\hat{R}_t$ is more stable over time (the field of variation is very small) and, again, becomes constant after retirement.

The behaviour of the investment in bonds $w^*_{B,t}B\,(t, t+h)$ is presented just with respect to wealth R_t, because its shape with respect to the modified wealth $\hat{R}_t$ is very similar.




```
matplot(time, wBR_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "wB/R")
polygon(c(time, rev(time)), c(wBR_Q[, 3], rev(wBR_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wBR_Q[, 6], rev(wBR_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, h = 0, lty = 2)
grid()
legend("topright", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

Fig. 8.23 Median and extreme scenarios of the bond share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{B,t} B(t,t+h)}{R_t}$ for both males and females

```
wBR_Q = cbind(t(apply(wB[, , 1]/R[, , 1], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(wB[, , 2]/R[, ,
    2], 1, quantile, probs = c(0, 0.5, 1))))
```

From Fig. 8.23, we can highlight some interesting points:

- Before retirement, females must optimally buy a higher percentage of bonds.

- The optimal share of bonds for males is very stable over time (the shaded are between the best and worst cases is very small), while the initial share for females is very volatile.
- Before retirement, both males and females must have long positions on bonds.
- Few years before retirement, the bond share becomes negative.
- At retirement the optimal bond shares for males and females are the same.

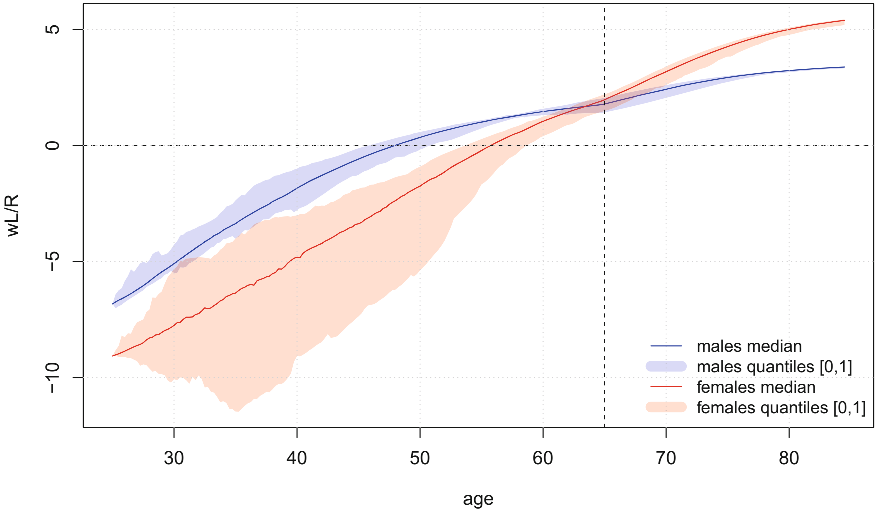
Finally, we can check the optimal shape of the investment in the longevity asset $w^*_{L,t}L\,(t, t+h)$ as a percentage of wealth R_t (again, the shape with respect to the modified wealth $\hat{R}_t$ is definitely similar).

```
wLR_Q = cbind(t(apply(wL[, , 1]/R[, , 1], 1, quantile,
    probs = c(0, 0.5, 1))), t(apply(wL[, , 2]/R[, ,
    2], 1, quantile, probs = c(0, 0.5, 1))))
```

From Fig. 8.24 we can highlight some interesting features:

- The initial share of longevity assets for both males and females are negative.
- The optimal share of longevity assets becomes positive few years before retirement.
- At retirement the optimal share is the same for males and females.
- The initial volatility of the females' optimal strategy is very high, while males have a very stable strategy over time.

An important conclusion is that there is for sure room for a market of longevity assets between generations. In particular, it is apparent that young people (both males and females) optimally want to sell longevity assets, while they are interested in a long position when older. An insurance company or any other institutional investor that would be willing to issue longevity assets, could then find both demand and supply by matching agents belonging to different cohorts.




```
matplot(time, wLR_Q, type = "l", lty = 1, col = c(NA,
    "blue", NA, NA, "red", NA), xlab = "age", ylab = "wL/R")
polygon(c(time, rev(time)), c(wLR_Q[, 3], rev(wLR_Q[,
    1])), col = rgb(0, 0, 1, alpha = 0.2), border = NA)
polygon(c(time, rev(time)), c(wLR_Q[, 6], rev(wLR_Q[,
    4])), col = rgb(1, 0, 0, alpha = 0.2), border = NA)
abline(v = age + T, h = 0, lty = 2)
grid()
legend("bottomright", legend = c("males median", "males quantiles [0,1]",
    "females median", "females quantiles [0,1]"), lty = 1,
    col = c("blue", rgb(0, 0, 1, alpha = 0.2), "red",
        rgb(1, 0, 0, alpha = 0.2)), bty = "n", lwd = c(1,
        10, 1, 10))
```

Fig. 8.24 Median and extreme scenarios of the longevity asset share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{L,t} L(t,t+h)}{R_t}$ for both males and females

Chapter 9
A Pure Accumulation Fund

9.1 Introduction

In this final chapter we compute the optimal portfolio for another institutional investor which deals with neither the mortality nor the longevity risk. Instead, this fund just manages the contributions of a worker and provides him/her with an amount of money at the end of the management period. We assume that the same amount of money is due independently of the agent's survival or, in other words, the heirs have the right to receive the whole amount due to the original subscriber.

9.2 The Optimisation Problem

If we call T the moment when the management period ends, then the objective function for the fund can be written as a modified version of (7.7.1) as follows:

$$\max_{R_T} \mathbb{E}_{t_0}\left[\frac{(q_T R_T - \alpha)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^T \rho_u du}\right]. \tag{9.2.1}$$

The absence of mortality and longevity risk transforms the inter-temporal problem, as written in (7.7.2), in a final horizon problem.

In a pure accumulation case, the fund receives again the contribution until time T, but pays only one amount of money at time T. Thus, the prospective mathematical reserve at any time t is

$$\hat{\Delta}_t := \mathbb{E}_t^{\mathbb{Q}}\left[p_T^* e^{-\int_t^T r_u du} - \int_t^T c_s^* e^{-\int_t^s r_u du} ds\right],$$

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3_9

Table 9.1 Summary of the objective function and constraints of two institutional investors: a pension fund which deals with mortality/longevity risk, and an accumulation fund which does not

Pension fund	
Objective function	$\max_{\{R_\tau\}_{\tau\in[t_0,\omega]}} \mathbb{E}_{t_0}\left[\frac{(q_\tau R_\tau-\alpha)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{\tau}\rho_u du}\right]$
Budget constraint	$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[q_\tau R_\tau e^{-\int_{t_0}^{\tau} r_u du}\right] + \Delta_{t_0}$
PMR	$\Delta_t := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\omega}\left(p_s^*\mathbb{I}_{s\geq T} - c_s^*\mathbb{I}_{s<T}\right)e^{-\int_t^s (r_u+\lambda_u)du}ds\right]$
Accumulation fund	
Objective function	$\max_{R_T} \mathbb{E}_{t_0}\left[\frac{(q_T R_T-\alpha)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{T}\rho_u du}\right]$
Budget constraint	$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[q_T R_T e^{-\int_{t_0}^{T} r_u du}\right] + \hat{\Delta}_{t_0}$
PMR	$\hat{\Delta}_t := \mathbb{E}_t^{\mathbb{Q}}\left[p_T^* e^{-\int_t^T r_u du} - \int_t^T c_s^* e^{-\int_t^s r_u du}ds\right]$

in which p_T^* and $\left\{c_t^*\right\}_{t\in[t_0,T]}$ are the cash flows that solve the fairness condition

$$\hat{\Delta}_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[p_T^* e^{-\int_{t_0}^T r_u du} - \int_{t_0}^T c_s^* e^{-\int_{t_0}^s r_u du}ds\right] = 0.$$

If we set p_T^* constant, then its value is given by

$$p_T^* = \frac{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^T c_s^* e^{-\int_{t_0}^s r_u du}ds\right]}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[e^{-\int_{t_0}^T r_u du}\right]}.$$

In this approach we have substituted the stochastic horizon τ with a deterministic horizon T and, in what follows, we will check the implications of this change for the optimal portfolio.

In Table 9.1 we summarise the two optimisation problems for a pension fund and for a "pure" accumulation fund.

9.3 The Optimal Portfolio

As we have already shown for the case of the pension fund, we write the Lagrangian function of Problem (9.2.1) as

$$\begin{aligned}\mathcal{L} :=& \mathbb{E}_{t_0}\left[\frac{(q_T R_T - \alpha)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^T \rho_u du}\right] \\ & - l\left(\mathbb{E}_{t_0}^{\mathbb{Q}}\left[q_T R_T e^{-\int_{t_0}^T r_u du}\right] + \Delta_{t_0} - R_{t_0}\right),\end{aligned}$$

in which l is the Lagrangian multiplier and whose FOC is

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial R_T} =&\mathbb{E}_{t_0}\left[q_T\left(q_T R_T-\alpha\right)^{-\delta} e^{-\int_{t_0}^T \rho_u du}\right] \\ &-l \mathbb{E}_{t_0}^{\mathbb{Q}}\left[q_T e^{-\int_{t_0}^T r_u du}\right]=0.\end{aligned}$$

After switching from the probability $\mathbb{Q}$ to the probability $\mathbb{P}$ through the martingale $m_{t_0,T}$ (defined in (4.9.2)), we can compute the FOC with respect to each period and each state of the world as

$$\left(q_T R_T-\alpha\right)^{-\delta} e^{-\int_{t_0}^T \rho_u du}-l m_{t_0,T} e^{-\int_{t_0}^T r_u du}=0,$$

from which the optimal wealth is

$$R_T^*=\frac{\alpha}{q_T}+\frac{1}{q_T}\left(l m_{t_0,T} \frac{e^{-\int_{t_0}^T r_u du}}{e^{-\int_{t_0}^T \rho_u du}}\right)^{-\frac{1}{\delta}}.$$

After rewriting the budget constraint at any time t and substituting for the optimal wealth, we get

$$R_t^*=\alpha \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]+\frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^t \rho_u-r_u du}} \mathbb{E}_t^{\mathbb{Q}}\left[m_{t,T}^{-\frac{1}{\delta}} e^{-\int_t^T\left(\frac{\delta-1}{\delta} r_u+\frac{1}{\delta} \rho_u\right) du}\right]+\hat{\Delta}_t.$$

This formula is definitely analogue to that use for the pension fund. The only difference can be found in defining the auxiliary functions H_t and F_t that, instead of having a stochastic horizon τ, have a deterministic horizon T. Thus, in line with Equations (7.8.2) and (7.14.3), these two functions are defined as

$$H_t:=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right],$$

$$F_t:=\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\int_t^T\left(\frac{\delta-1}{\delta} r_u+\frac{1}{\delta} \rho_u+\frac{1}{2} \frac{\delta-1}{\delta} \frac{1}{\delta} \xi_u^{\top} \xi_u\right) du}\right].$$

Here, we do not use a different notation for the auxiliary functions H_t and F_t because this form is actually a particular form of the original Equations (7.8.2) and (7.14.3), in which τ has been substituted with T.

Now, since the optimal wealth can be written as

$$R_t^* = \alpha H_t + \frac{l^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}}{e^{\frac{1}{\delta}\int_{t_0}^{t} \rho_u - r_u du}} F_t + \hat{\Delta}_t,$$

once it is differentiated with respect to the stochastic variables $m_{t_0,t}$ and z_t, we obtain the same functional form for the optimal portfolio that has already shown in (7.8.6) for a pension fund:

$$I_S \hat{w}_t^* = \underbrace{\frac{R_t - \alpha H_t - \hat{\Delta}_t}{\delta} \Sigma_t^{-1} \xi_t}_{\text{speculative portfolio}} \tag{9.3.1}$$

$$+ \underbrace{\alpha \Sigma_t^{-1} \Omega_t \frac{\partial H_t}{\partial z_t} + \frac{R_t - \alpha H_t - \hat{\Delta}_t}{F_t} \Sigma_t^{-1} \Omega_t \frac{\partial F_t}{\partial z_t} + \Sigma_t^{-1} \Omega_t \frac{\partial \hat{\Delta}_t}{\partial z_t}}_{\text{hedging portfolio}}.$$

9.4 A Workable Framework

We now take into account the same framework that we have already presented for the case of a pension fund. Nevertheless, here the only state variables are the interest rate and the contributions as already described in (8.2.2) and (8.2.3), respectively. Furthermore, we do not need any longer the longevity asset, and so we restrict our analysis to the case of a market formed by a riskless asset, a stock as in (8.4.1), and a bond as in (8.4.2).

In this particular case the two main matrix products useful for computing the optimal portfolio are

$$\Sigma_t^{-1} \xi_t = \begin{bmatrix} \frac{\xi_A}{\sigma_A} \\ \frac{1}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)} \left(\frac{\sigma_{A,r}\xi_A}{\sigma_r \sigma_A} - \frac{\kappa_r}{\sigma_r} \right) \end{bmatrix},$$

$$\Sigma_t^{-1} \Omega_t = \begin{bmatrix} 0 & c_t \frac{\sigma_{c,A}}{\sigma_A} \\ -\frac{1}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)} & \frac{c_t}{C\left(0;h,\alpha_r^{\mathbb{Q}},\sigma_r,1\right)} \left(\frac{\sigma_{A,r}\sigma_{c,A}}{\sigma_r \sigma_A} - \frac{\sigma_{c,r}}{\sigma_r} \right) \end{bmatrix},$$

and the optimal portfolio itself is

$$
\begin{aligned}
w_{A,t}^{*} A_t =& \frac{R_t - \alpha H_t - \hat{\Delta}_t}{\delta} \frac{\xi_A}{\sigma_A} + c_t \frac{\sigma_{c,A}}{\sigma_A} \frac{\partial \hat{\Delta}_t}{\partial c_t}, \\
w_{B,t}^{*} B\left(t, t+h\right) =& \frac{R_t - \alpha H_t - \hat{\Delta}_t}{C\left(0; h, \alpha_r^{\mathbb{Q}}, \sigma_r, 1\right)} \left(\frac{1}{\delta} \left(\frac{\sigma_{A,r} \xi_A}{\sigma_r \sigma_A} - \frac{\kappa_r}{\sigma_r} \right) - \frac{1}{F_t} \frac{\partial F_t}{\partial r_t} \right) \\
& - \frac{1}{C\left(0; h, \alpha_r^{\mathbb{Q}}, \sigma_r, 1\right)} \left(\alpha \frac{\partial H_t}{\partial r_t} + \frac{\partial \hat{\Delta}_t}{\partial r_t} - c_t \left(\frac{\sigma_{A,r} \sigma_{c,A}}{\sigma_r \sigma_A} - \frac{\sigma_{c,r}}{\sigma_r} \right) \frac{\partial \hat{\Delta}_t}{\partial c_t} \right),
\end{aligned}
$$

which coincides with the optimal portfolio already computed in the previous chapter, where $w_{L,t}^{*} = 0$.

In this particular case we can see that the function H_t coincides with the function "`EQ_r`" that we have already written for the pension fund case. Accordingly, also the derivative $\frac{\partial H_t}{\partial r_t}$ coincides with the already defined function "`dEQ_r_dr`".

The semi-elasticity of F_t with respect to the interest rate r_t is

$$
\begin{aligned}
\frac{1}{F_t} \frac{\partial F_t}{\partial r_t} &= \frac{1}{\mathbb{E}_t^{\mathbb{Q}_\delta} \left[e^{-\left(1 - \frac{1}{\delta} + \frac{1}{2} \frac{\delta - 1}{\delta} \frac{1}{\delta} \kappa_r^2 \right) \int_t^T r_u du} \right]} \frac{\partial \mathbb{E}_t^{\mathbb{Q}_\delta} \left[e^{-\left(1 - \frac{1}{\delta} + \frac{1}{2} \frac{\delta - 1}{\delta} \frac{1}{\delta} \kappa_r^2 \right) \int_t^T r_u du} \right]}{\partial r_t}. \\
&= -C\left(t; T, \alpha_r^{\mathbb{Q}_\delta}, \sigma_r, 1 - \frac{1}{\delta} + \frac{1}{2} \frac{\delta - 1}{\delta} \frac{1}{\delta} \kappa_r^2 \right).
\end{aligned}
$$

Finally, we just have to define the value of the new PMR and its derivatives. The value of the reserve at any time t is

$$
\hat{\Delta}_t = p_T^{*} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_u du} \right] - \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left[c_s^{*} e^{-\int_t^s r_u du} \right] ds,
$$

thus, again, we have already coded all the necessary values. The fair value of p_T^{*} can be computed as follows.

```
p2 = integrate(EQ_c_r, 0, T, t = 0, c = c0, r = r0)$value/EQ_r(t = 0,
    T = T, q = 1, r = r0, prob = "Q")
p2
```

```
## [1] 719.9645
```

While the new value of the PMR is coded in the following commands.

```
PMR2 = function(t, T, c, r) {
    p2 * EQ_r(t, T, q = 1, r, prob = "Q") - integrate(EQ_c_r,
        t, T, t = t, c = c, r = r)$value
}
```

Its derivatives are

$$\frac{\partial \Delta_t}{\partial c_t} = -\frac{1}{c_t}\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[c_s e^{-\int_t^s r_u du}\right] ds,$$

and

$$\frac{\partial \Delta_t}{\partial r_t} = p^* \frac{\partial \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]}{\partial r_t} - \int_t^T \frac{\partial \mathbb{E}_t^{\mathbb{Q}}\left[c_s e^{-\int_t^s r_u du}\right]}{\partial r_t} ds,$$

that can be computed in R by using the functions that have already been created for the previous computations.

```
dDelta2_dc = function(t, T, r, c) {
    -integrate(EQ_c_r, t, T, t = t, c = c, r = r)$value/c
}
dDelta2_dr = function(t, T, c, r) {
    p2 * dEQ_r_dr(q = 1, "Q", t, T, r) - integrate(dEQ_c_r_dr,
        t, T, c = c, r = r)$value
}
```

9.5 The Optimal Portfolio: Numerical Results

The simulations that we need for computing the optimal portfolio in this case are based on the very same figures that we have already shown for the case of the pension fund. The main difference is that now the horizon for the simulation coincides with the retirement date T. In the present case we do not need any more to define a three dimension array since the "sex dimension" is not relevant any more (in fact sex only affects the morality/longevity risk).

```
N = 100
horizon = T
dt = 1/4
dW_A = dW_r = array(rnorm(horizon/dt * N, mean = 0,
    sd = sqrt(dt)), dim = c(horizon/dt, N))
R = array(NA, dim = c(horizon/dt, N))
R[1, ] = R0
r = array(NA, dim = c(horizon/dt, N))
r[1, ] = r0
co = array(NA, dim = c(horizon/dt, N))
co[1, ] = c0
wA = wB = array(NA, dim = c(horizon/dt - 1, N))
RN = array(NA, dim = c(horizon/dt - 1, N))
C_r = C(0, h, alpha_r + k_r * sigma_r, sigma_r, 1)
for (i in 1:(horizon/dt - 1)) {
    t = i * dt
    RN[i, ] = R[i, ] - alpha * EQ_r(t, T, q = 1, r = r[i,
        ], prob = "Q") - mapply(PMR2, t, T, co[i, ],
        r[i, ])
    wA[i, ] = RN[i, ] * xi_A/(delta * sigma_A) + co[i,
        ] * sigma_cA/sigma_A * mapply(dDelta2_dc, t,
        T, co[i, ], r[i, ])
    wB[i, ] = RN[i, ]/C_r * ((sigma_Ar * xi_A/sigma_r/sigma_A -
        k_r/sigma_r)/delta + C(t, T, alpha_r + k_r *
        sigma_r * (delta - 1)/delta, sigma_r, 1 - 1/delta +
        0.5 * (delta - 1)/delta^2 * k_r^2)) - (alpha *
        dEQ_r_dr(q = 1, prob = "Q", t, T, r[i, ]) +
        mapply(dDelta2_dr, t, T, co[i, ], r[i, ]) -
        co[i, ] * (sigma_Ar * sigma_cA/sigma_r/sigma_A -
            sigma_cr/sigma_r) * mapply(dDelta2_dc,
            t, T, co[i, ], r[i, ]))/C_r
    r[i + 1, ] = r[i, ] + alpha_r * (beta_r - r[i,
        ]) * dt + sigma_r * sqrt(r[i, ]) * dW_r[i,
        ]
    co[i + 1, ] = co[i, ] + co[i, ] * (mu_c * dt +
        sigma_cr * sqrt(r[i, ]) * dW_r[i, ] + sigma_cA *
        dW_A[i, ])

    R[i + 1, ] = R[i, ] + (R[i, ] * r[i, ] + wA[i,
        ] * (mu_A - r[i, ]) + wB[i, ] * (-C_r * sigma_r *
        k_r * r[i, ]) + co[i, ]) * dt + wA[i, ] * sigma_A *
        dW_A[i, ] + (wA[i, ] * sigma_Ar * sqrt(r[i,
        ]) + wB[i, ] * (-C_r * sigma_r * sqrt(r[i,
        ]))) * dW_r[i, ]
    print(i)
}
```

After simulating the portfolio, we can draw the optimal asset allocation. In order to do so we define the "time" variable and the quantiles for the optimal asset share both in terms of both wealth R_t and modified wealth $\hat{R}_t$.

We stress that in this case the time is measured in "years" and not in "age", since in this framework the optimal portfolio does not depend any longer on the age of the worker/pensioner. In fact, λ_t was the only state variable depending on the age (through the equilibrium value $\beta_{\lambda,t}$).

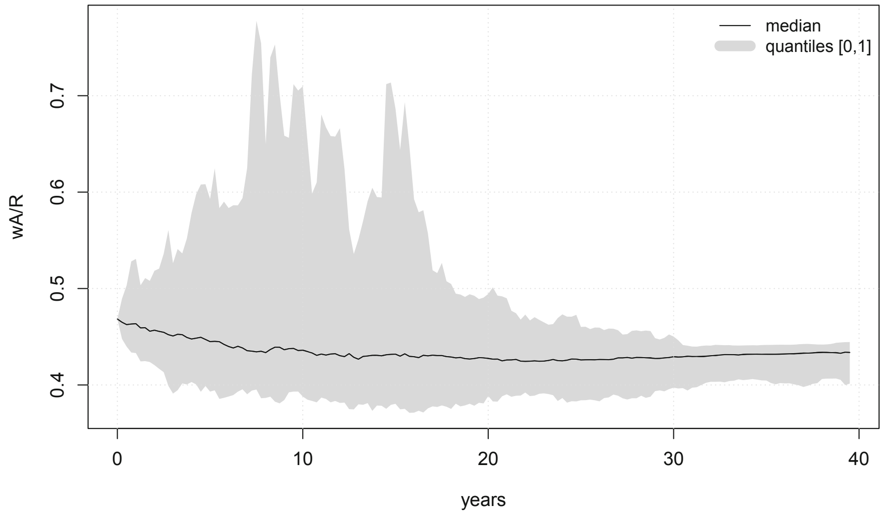
```
time = seq(0, horizon - 2 * dt, dt)
wAR_Q = t(apply(wA/head(R, -1), 1, quantile, probs = c(0,
    0.5, 1)))
wARN_Q = t(apply(wA/RN, 1, quantile, probs = c(0, 0.5,
    1)))
wBR_Q = t(apply(wB/head(R, -1), 1, quantile, probs = c(0,
    0.5, 1)))
wBRN_Q = t(apply(wB/RN, 1, quantile, probs = c(0, 0.5,
    1)))
```

Figures 9.1 and 9.2 present the dynamics of the optimal stock share as a percentage of wealth R_t and modified wealth $\hat{R}_t$, respectively. We highlight some results.

- The stock share in percentage of wealth R_t is slightly decreasing over time with a quantile range that is very wide at the beginning of the management period and shrinks over time. The median of the optimal stock share is very stable.
- The stock share as a percentage of the modified wealth $\hat{R}_t$ is slightly increasing. Its median is very stable, and varies over a small range. Again, the range of the quantiles is wider at the beginning of the management period and narrower at the end.
- The optimal share of stocks in both cases is similar.
- With respect to the case of a pension fund, we see that, here, the stock share is more stable. This means that the presence of the mortality/longevity risk does affect the stock share, even if the stock $w^*_{A,t}$ is not directly affected by the derivatives of H_t, F_t, and Δ_t with respect to λ_t. Actually, the stock share depends on the values of these functions, and these values are largely affected by the force of mortality.

Figures 9.3 and 9.4 present the dynamics of the optimal bond share as a percentage of wealth R_t and modified wealth $\hat{R}_t$, respectively. We highlight some results.

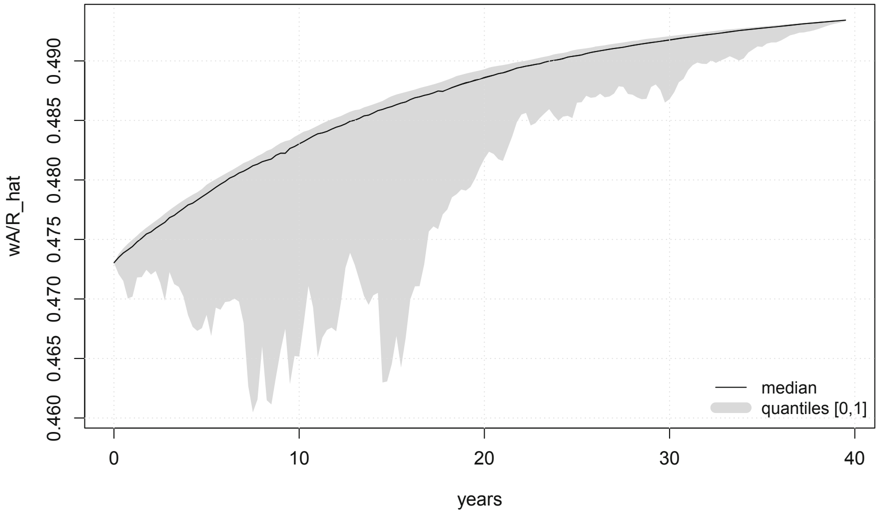
- The optimal bond share, in terms of both wealth measures, never becomes negative.
- The bond share is decreasing over time with a range of quantiles that is wider in the beginning and narrower in the end of the management period.




```r
matplot(time, wAR_Q, type = "l", lty = 1, col = c(NA,
    1, NA), xlab = "years", ylab = "wA/R")
polygon(c(time, rev(time)), c(wAR_Q[, 3], rev(wAR_Q[,
    1])), col = rgb(0, 0, 0, alpha = 0.2), border = NA)
grid()
legend("topright", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c(1, rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

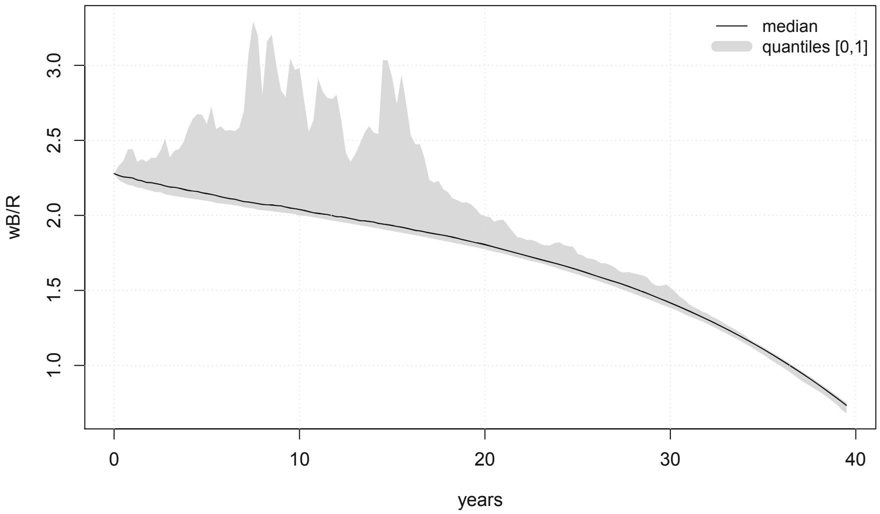
Fig. 9.1 Median and extreme scenarios of the stock share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{A,t} A_t}{R_t}$

- The bond share as a percentage of R_t, at the beginning of the period, may reach very high values, but with a low probability. In fact we see that the median is very close to the lowest quantile.
- We can compare this result with the optimal bond share for a pension fund. The dynamics is decreasing over time for both cases, but we see that, here, the bond share is mach smaller and never becomes negative. This behaviour is due to the result which shows that the optimal amount of longevity asset $w^*_{L,t}$ is taken from the amount invested in the bond as in (8.12.2).



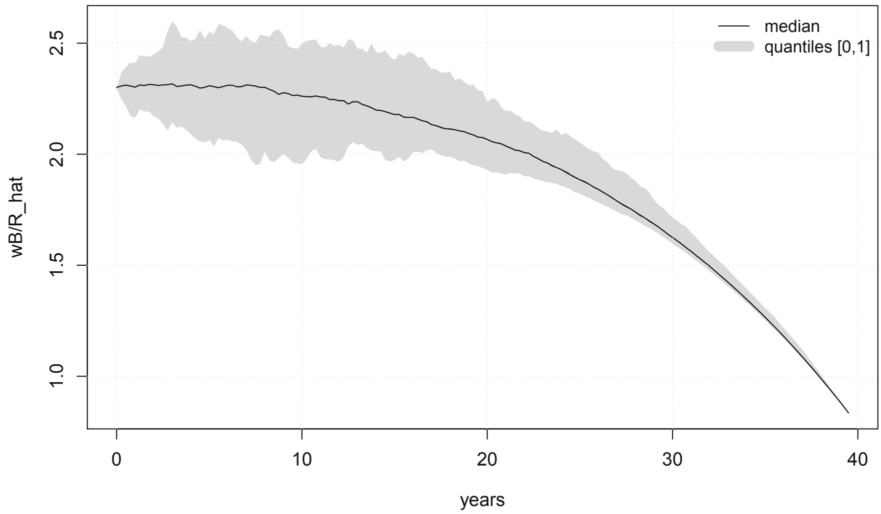

```r
matplot(time, wARN_Q, type = "l", lty = 1, col = c(NA,
    1, NA), xlab = "years", ylab = "wA/R_hat")
polygon(c(time, rev(time)), c(wARN_Q[, 3], rev(wARN_Q[,
    1])), col = rgb(0, 0, 0, alpha = 0.2), border = NA)
grid()
legend("bottomright", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c(1, rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

Fig. 9.2 Median and extreme scenarios of the stock share in the optimal portfolio over time (100 simulations) as a percentage of modified wealth $\frac{w^*_{A,t} A_t}{\hat{R}_t}$




```
matplot(time, wBR_Q, type = "l", lty = 1, col = c(NA,
    1, NA), xlab = "years", ylab = "wB/R")
polygon(c(time, rev(time)), c(wBR_Q[, 3], rev(wBR_Q[,
    1])), col = rgb(0, 0, 0, alpha = 0.2), border = NA)
grid()
legend("topright", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c(1, rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

Fig. 9.3 Median and extreme scenarios of the bond share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{B,t}B(t,t+h)}{R_t}$




```
matplot(time, wBRN_Q, type = "l", lty = 1, col = c(NA,
    1, NA), xlab = "years", ylab = "wB/R_hat")
polygon(c(time, rev(time)), c(wBRN_Q[, 3], rev(wBRN_Q[,
    1])), col = rgb(0, 0, 0, alpha = 0.2), border = NA)
grid()
legend("topright", legend = c("median", "quantiles [0,1]"),
    lty = 1, col = c(1, rgb(0, 0, 0, alpha = 0.2)),
    bty = "n", lwd = c(1, 10))
```

Fig. 9.4 Median and extreme scenarios of the bond share in the optimal portfolio over time (100 simulations) as a percentage of wealth $\frac{w^*_{B,t} B(t,t+h)}{\hat{R}_t}$

Conclusions

In this work we have approached the wealth management problem of a pension fund which aims at maximising the (utility of) wealth remaining at the death time of a representative agent.

We have shown how to model a risky framework that takes into account a set of stochastic variables that describe the risks faced by a pension fund, i.e. both financial and actuarial risks. We have presented how to model and calibrate (on US data) the main stochastic variables involved in the analysis: (1) the instantaneously risk-less interest rate, (2) the force of mortality for both males and females, which is stochastic in order to take into account not only the mortality risk, but also the longevity risk, and (3) the wages of workers.

In the final chapters we show how to compute the optimal portfolio in a closed-form by using the so-called martingale approach in a complete market, and we apply the empirical framework developed in the previous chapters for computing the numerical values of the optimal portfolio shares of stocks, bonds, and longevity assets.

The main result is that the longevity assets play a crucial role in the fund's optimal portfolio with a significantly different magnitude for males (smaller percentage of wealth in absolute value) and females. Both sexes go strongly short on longevity asset at the beginning of their working life, while they start going long when they are close to the retirement age. The optimal share for both sexes is the same at retirement and becomes strongly positive after that time. This result suggests that there is room for creating a market of longevity assets between agents of different ages: young agents would be on the supply side of the market, while old agents would be on the demand side.

Finally, we compare the optimal asset shares for a pension fund with the same optimal strategy for a pure accumulation fund that does not deal with the longevity risk. We are able to see that this last risk actually plays a very relevant role in affecting the optimal strategy for a pension fund and deeply affect the amount of money that must be invested in the bond asset class.

F. Menoncin, *Risk Management for Pension Funds*, EURO Advanced Tutorials on Operational Research, https://doi.org/10.1007/978-3-030-55528-3