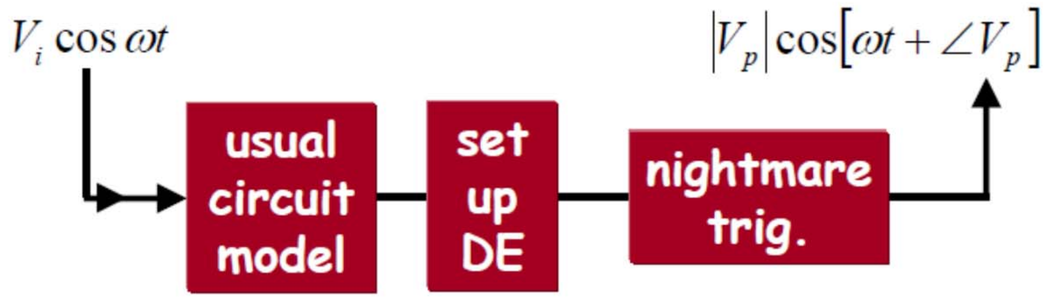
ELEN 50 Class 23 –Phasor Analysis Examples

S. Hudgens

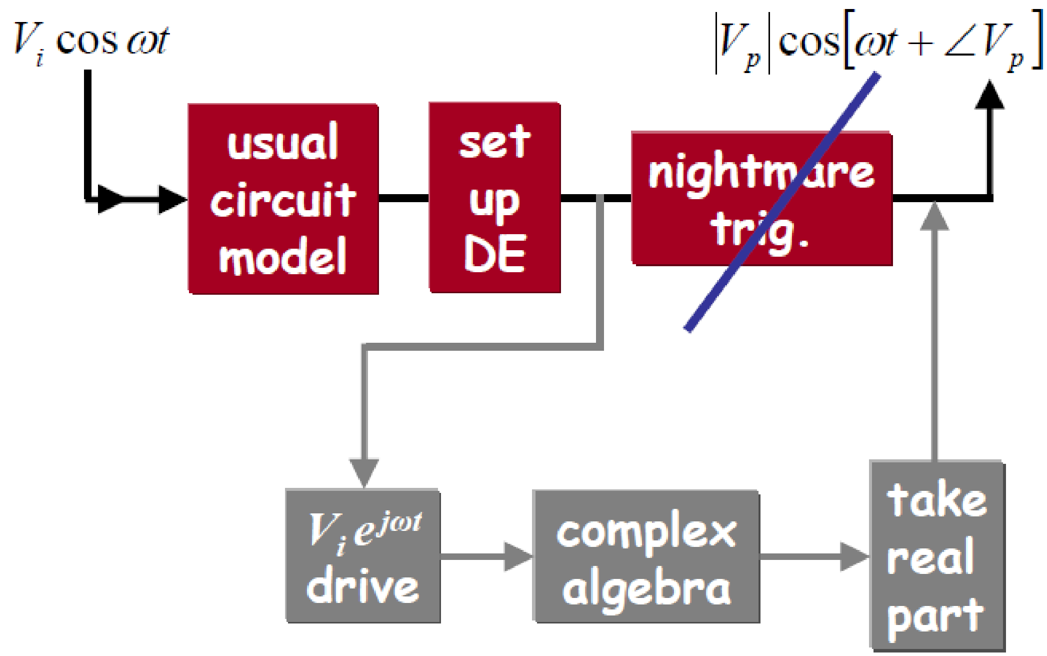
The Big Picture…

As we saw last time



The differential eq. approach

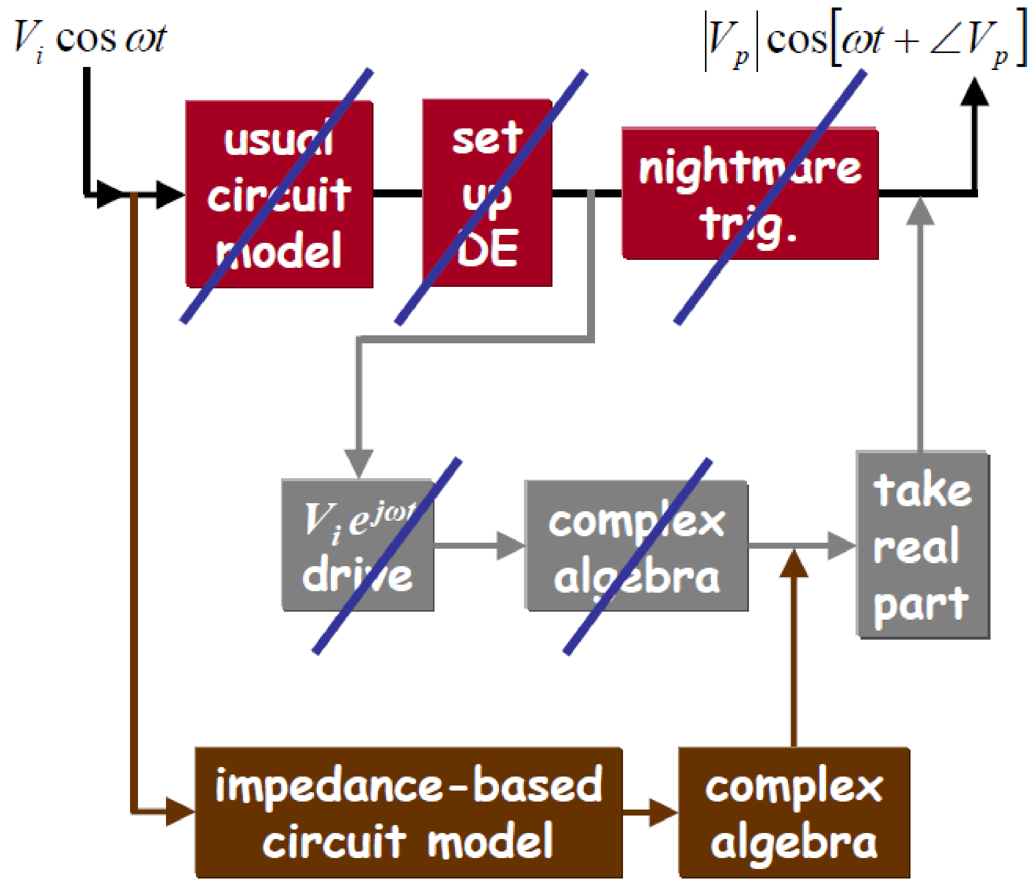
The Big Picture…



The phasor approach

The Big Picture…

The impedance model approach



No D.E.s, no trig!

Once Again – Important Relations Among Complex Numbers

(all derivable from Euler's Identity and some trig)

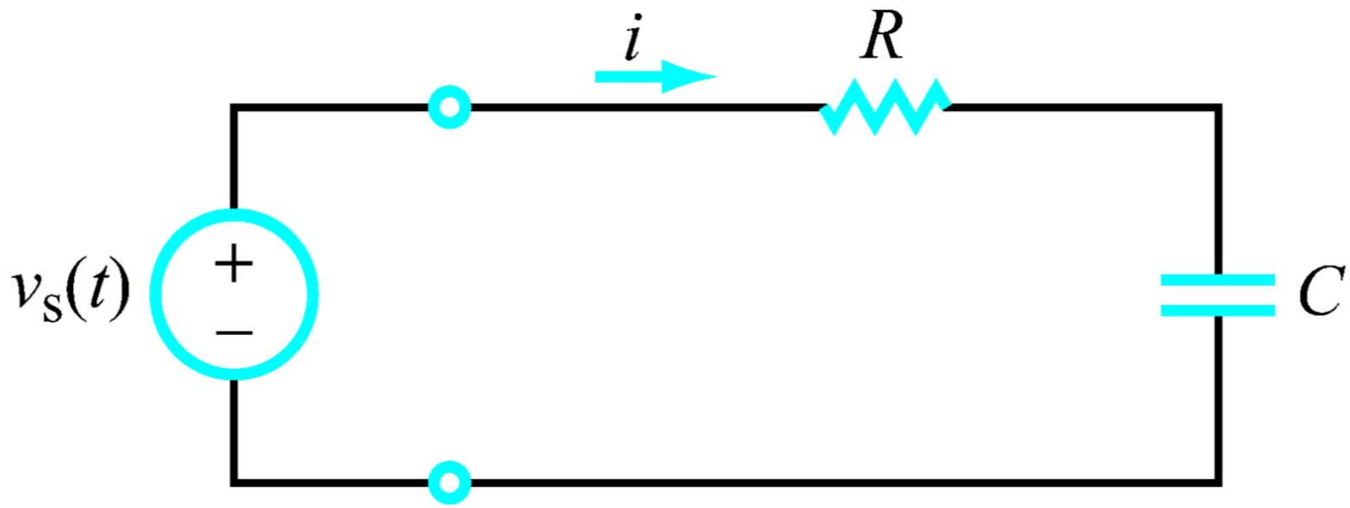
Table 7-2: Properties of complex numbers.

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$	
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \mathfrak{Re}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \mathfrak{Im}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm180^\circ$	
$j = e^{j\pi/2} = 1\angle 90^\circ$	$-j = e^{-j\pi/2} = 1\angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$

Let's formally analyze this simple RC circuit using phasors. The voltage source is sinusoidal (we just did a similar series connected resistor and inductor circuit – with a sinusoidal current source) :

$$v_s(t) = 12\sin(\omega t - 45^o)$$

and we want to calculate i(t), the steady state sinusoidal current. Let's first do a formal phasor analysis solution and then just use the simpler impedance model.



$$\omega = 10^3 rad / s$$

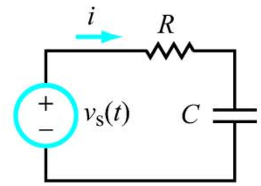
$$R = \sqrt{3}k\Omega$$

$$C = 1\mu F$$

Here's the **5 –step procedure** for the phasor analysis:

Just to be systematic

Step 1
Adopt Cosine Reference
(Time Domain)



$v_s(t) = 12\sin(\omega t - 45^\circ)$ (V)

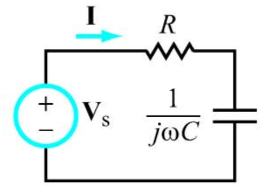
Using cosine reference this becomes

$v_0(t) = 12\cos(\omega t - 45^\circ - 90^\circ)$

Why is this?

Step 2
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135^\circ}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

This is just KVL in the phasor domain

Step 4
Solve for Unknown Variable
(Phasor Domain)

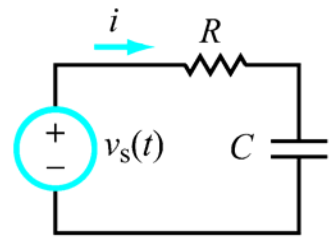
$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

Step 5
Transform Solution
Back to Time Domain

$$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}] = 6\cos(\omega t - 105^\circ) \text{ (mA)}$$

OK…let's go slowly …the first two steps …adopt a cosine reference for the source(s) and transform to the phasor domain

Step 1
Adopt Cosine Reference
(Time Domain)

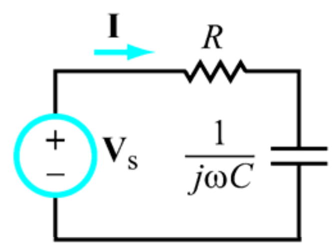


$v_s(t) = 12\sin(\omega t - 45°)$ (V)

$$v_s(t) = 12\sin(\omega t - 45^0)V$$
$$= 12\cos(\omega t - 45^0 - 90^0)V$$
$$= 12\cos(\omega t - 135^0)V$$

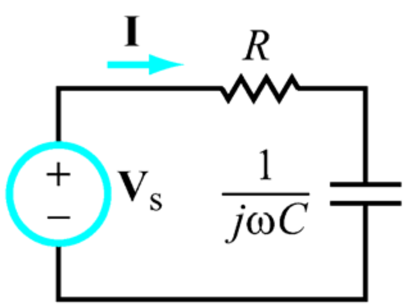
Step 2
Transfer to Phasor Domain

$i \rightarrow \mathbf{I}$
$v \rightarrow \mathbf{V}$
$R \rightarrow \mathbf{Z}_R = R$
$L \rightarrow \mathbf{Z}_L = j\omega L$
$C \rightarrow \mathbf{Z}_C = 1/j\omega C$



$\mathbf{V}_s = 12e^{-j135°}$ (V)

In phasor domain



$$\mathbf{V}_s = 12e^{-j135^\circ} \text{ (V)}$$

Step 3
Cast Equations in Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

Step 4
Solve for Unknown Variable (Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

We write the KVL (and/or the KCL) equation(s) in the phasor domain and solve for the unknown variable

Step 4
Solve for Unknown Variable
(Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

$$\mathbf{I} = \frac{12e^{-j135^0}}{R + \frac{1}{j\omega C}}$$

$$= \frac{j12\omega C e^{-j135^0}}{1 + j\omega RC}$$

since

$$\omega = 10^3 \, rad/s$$

$$R = \sqrt{3}k\Omega$$

$$C = 1\mu F$$

Notice – this is where the 10^{-3} went

$$\mathbf{I} = \frac{j12e^{-135^0}}{1 + j\sqrt{3}} mA$$

Step 5 Transform Solution Back to Time Domain	$i(t) = \Re e[\mathbf{I}e^{j\omega t}]$ $= 6\cos(\omega t - 105°)$ (mA)

$$\mathbf{I} = \frac{j12e^{-135^0}}{1+j\sqrt{3}}mA$$

We'll take our phasor domain solution for the current and put it <u>entirely in polar form</u>:

since

$$1+j\sqrt{3} = \sqrt{1+3}e^{j\phi}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^0$$

And since $e^{j90^0} = j$

I can write:

$$\mathbf{I} = \frac{12e^{-135^0}.e^{j90^0}}{2e^{j60^0}}mA$$

$$\mathbf{I} = \frac{12e^{-j135^0}.e^{j90^0}}{2e^{j60^0}} mA$$

$$\mathbf{I} = 6e^{j(-135^0+90^0-60^0)}$$

$$\mathbf{I} = 6e^{-j105^0} mA$$

This is the solution to the problem in the phasor domain

transforming back from the phasor domain to the time domain

Step 5 Transform Solution Back to Time Domain	$i(t) = \mathfrak{Re}[\mathbf{I}e^{j\omega t}]$ $= 6\cos(\omega t - 105^\circ)$ (mA)

and we're done: $i(t) = 6\cos(\omega t - 105^0) mA$

$$\omega = 10^3 rad/s$$

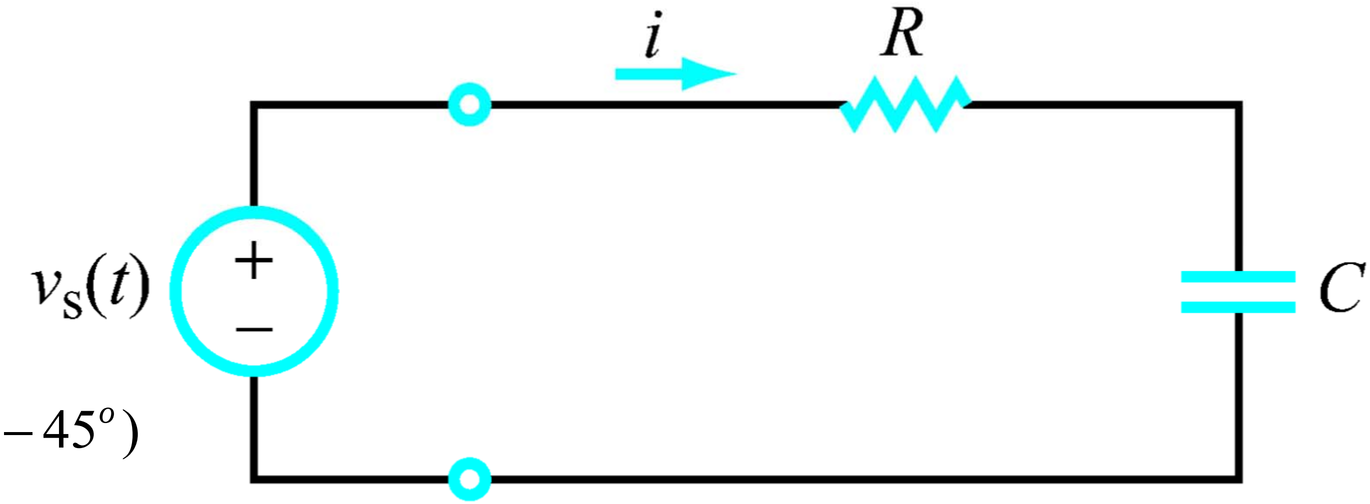
$$R = \sqrt{3} k\Omega$$

$$C = 1\mu F$$

$$\omega = 10^3 \, rad/s$$

$$R = \sqrt{3} k\Omega$$

$$C = 1 \mu F$$

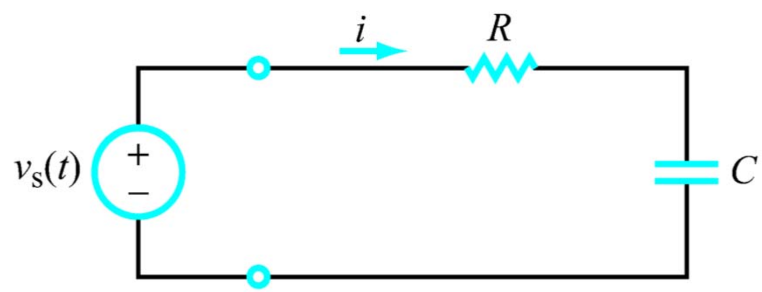


$$v_s(t) = 12 \sin(\omega t - 45^o)$$

Notice that in our solution for the current, the answer (as we expected) has the same frequency as the input voltage (of course)…but the phase angle has shifted. This is because of the capacitor!!!

$$i(t) = 6 \cos(\omega t - 105^0) mA$$

If we had done the same problem using the impedance model how would it have gone?



Well, we would just have to calculate the equivalent impedance of the circuit connected to the voltage source and then use the fact that:

$$\mathbf{I} = \frac{\mathbf{V}}{R}$$

We already know the phasor representation for the voltage and the equivalent impedance of the circuit is just the series combination of the impedance of the capacitor and the resistor ---which we have already calculated.

$v_S(t) = 12 \sin(\omega t - 45°)$ so

$V_S = 12 e^{-j135}$ as we saw earlier

the series combination of the resistor and capacitor impedance is:

$$Z_{eq} = \sqrt{3}\,k - j\,10^3 = 10^3\left(\sqrt{3} - j\right) = 2000\,e^{-j30°}$$

and, therefore, the current through the resistor is given by Ohm's Law: $I = \dfrac{V}{Z}$

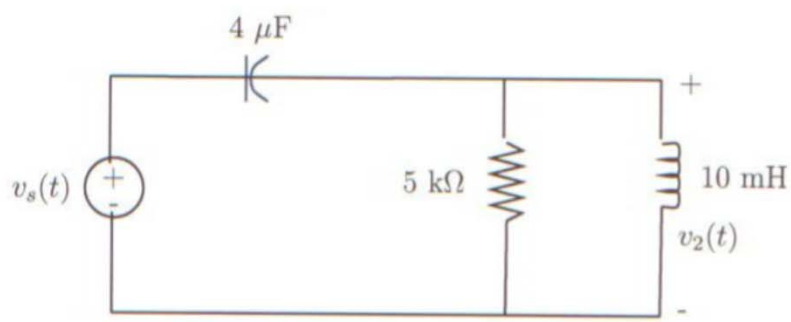
$$\text{so } I = \frac{12\,e^{-j135}}{2000\,e^{-j30}} = 6\,e^{-j105°} \text{ mA}$$

as we saw before

$$i(t) = 6\cos(\omega t - 105°) \text{ mA}$$

Here's a problem – can you solve it now using what you know about the complex impedance model approach?

4. Phasor Domain and Source Transformation:
Consider the RLC circuit below. The sinusoidal voltage source is given by $v_s(t) = 5\cos(5000t)$.



4(a). Convert the circuit to the phasor domain and draw it below.

4(b). What is the equivalent impedance seen by the voltage source. Is this impedance reactive?

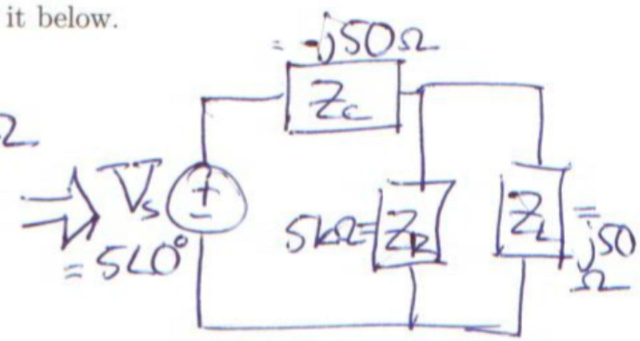
4(a). Convert the circuit to the phasor domain and draw it below.

$V_s(t) \longrightarrow \mathbf{V}_s = 5\angle 0^\circ$

$C \longrightarrow Z_C = -j/\omega C = \frac{-j}{5000(4\times10^{-6})} = -j50\ \Omega$

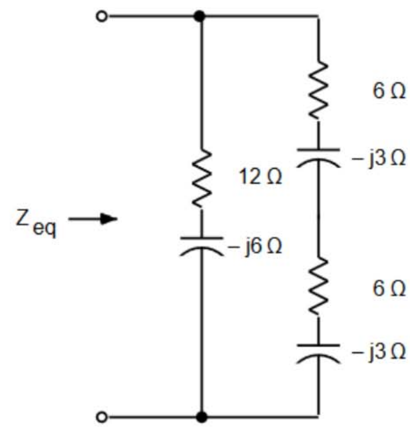
$R \longrightarrow Z_R = R = 5000\ \Omega$

$L \longrightarrow Z_L = j\omega L = j(5000)(.01) = j50\ \Omega$



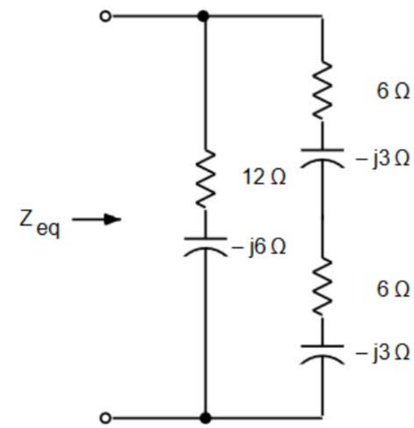
You can obtain the equivalent impedance of this circuit by series and parallel combinations of these impedances. The equivalent impedance is going to be entirely real because the inductive and capacitive impedances cancel.

Determine the equivalent impedance, Z_{eq}, of the circuit shown below.



Notice: the circuit elements have already been transformed into the phasor domain. To calculate the equivalent impedance, all you need to do is to combine the impedances.

8. Determine the equivalent impedance, Z_{eq}, of the circuit shown below.



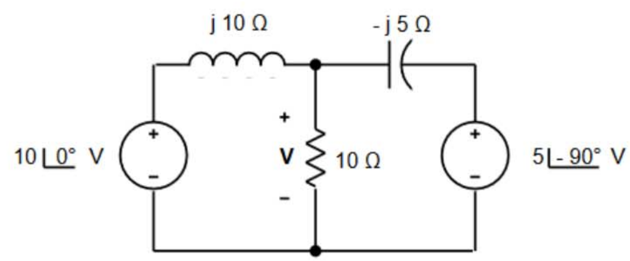
Solution: The equivalent impedance of this series-parallel circuit is therefore

$$Z_{eq} = (6 - j\,3 + 6 - j\,3) \,||\, (12 - j\,6) = (12 - j\,6) \,||\, (12 - j\,6)$$

thus,

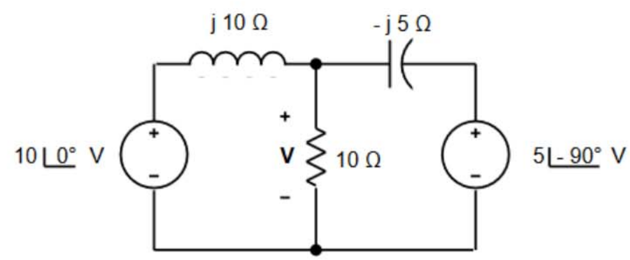
$$Z_{eq} = 6 - j\,3\ \Omega \quad \text{or} \quad 6.71\,\underline{/-26.57^\circ}\ \Omega .$$

Use the node-voltage method to find the voltage V across the 10-Ω resistor, expressed as a phasor.



Notice: the circuit has already been transformed into the phasor domain. What is the next step in a node voltage solution. What would you have done if this were a simple DC resistive circuit?

Use the node-voltage method to find the voltage V across the 10-Ω resistor, expressed as a phasor.



Writing KCL at the upper node

$$\frac{10\ \angle 0^\circ - \mathbf{V}}{j\,10} + \frac{5\ \angle -90^\circ - \mathbf{V}}{-j\,5} = \frac{\mathbf{V}}{10}$$

Which simplifies to:

$$\frac{10-\mathbf{V}}{j10} + \frac{-j5-\mathbf{V}}{-j5} = \frac{\mathbf{V}}{10}$$

Because 10 <0^{O} is 10 and 5 < -90^{o} is –j5

Putting everything over j10

$$10 - \mathbf{V} + j10 + 2\mathbf{V} = j\mathbf{V}$$

Solving for **V**

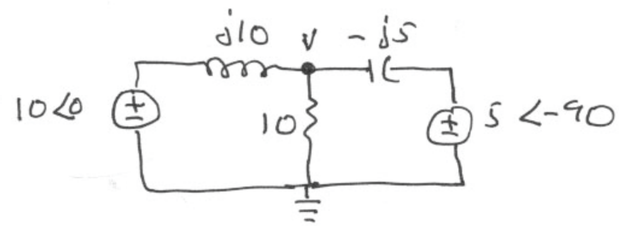
$$\mathbf{V} = -10\frac{(1+j)}{(1-j)}$$

Multiplying the fraction, top and bottom, by the complex conjugate of the denominator:

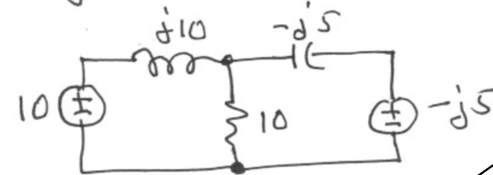
$$\mathbf{V} = \frac{-10}{2}2j = -j10$$

Could we have solved this circuit using source transforms?

source transform solution



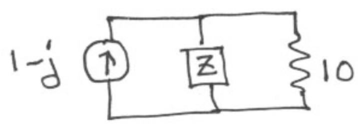
Solution by source transform



Source transform both voltage sources

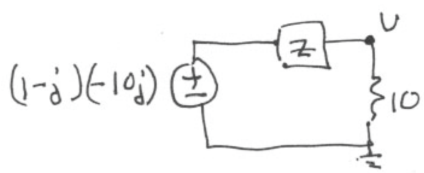
$$\frac{-j5}{-j5} = 1$$

combine current sources $\frac{10}{j10} + 1 = 1 - j$



$$Z = j10 \parallel -j5 = \frac{(j10)(-j5)}{j5}$$

$$= -10j$$



Source transform back to a voltage source ..and notice that v is part of a voltage divider

so $V = \dfrac{(1-j)(-10j)\,10}{10-10j} = \dfrac{(-10j)(10)(1-j)}{(10)(1-j)}$

$$V = -10j$$

Just to review

In the last two classes we discussed the technique for analyzing sinusoidal steady state circuits called **phasor analysis.**

Phasors are a way of representing the amplitude and phase of a sinusoidal signal (current or voltage) that permits solution of circuits that contain inductors and capacitors – while completely avoiding having to solve differential equations.

We define the phasor as a complex number containing amplitude and phase information:

$$\mathbf{V} = V_m e^{j\phi}$$

In other words, if a voltage in the time domain is :

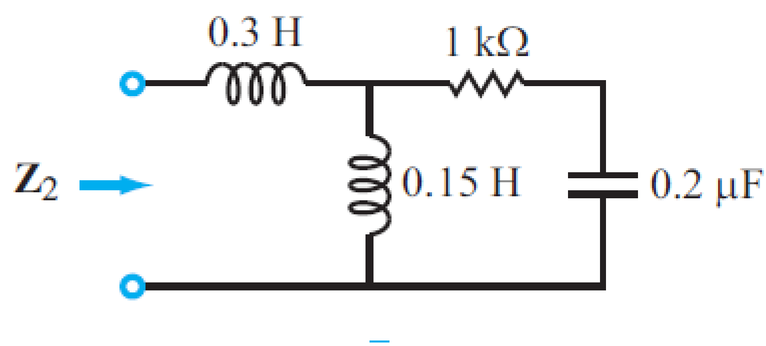
$$v = V_m \cos(\omega t + \phi)$$

transforming to the phasor domain gives us:

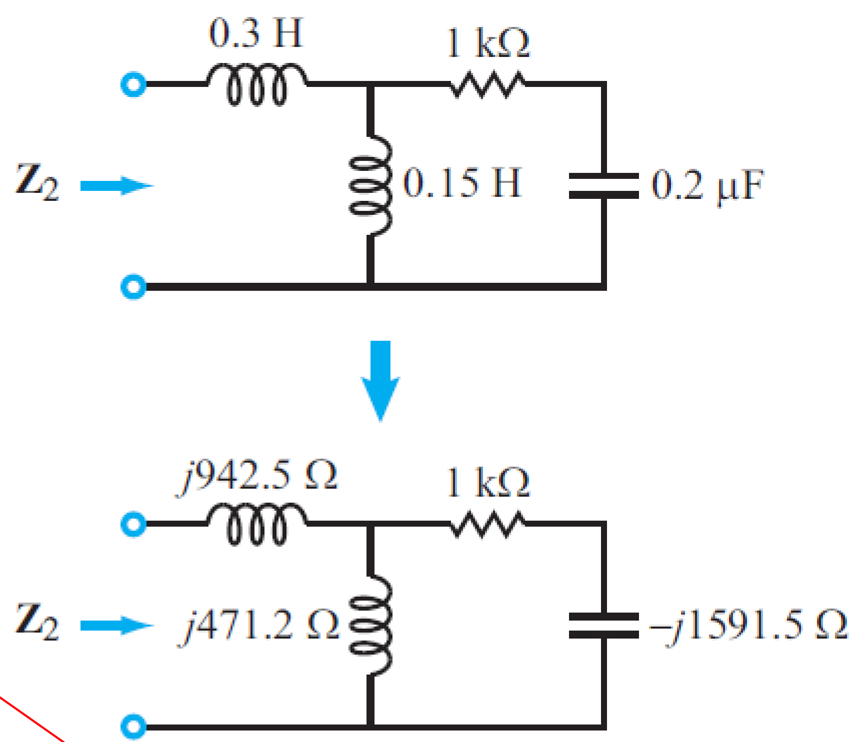
$$\mathbf{V} = V_m e^{j\phi}$$

Here are some problems just using the impedance model (so no KCL/KVL equations)

… determine Z_2 at 500 Hz for this circuit:



This is a much easier problem to solve than the previous one since it's only asking for the equivalent impedance. Here we need only to convert the circuit elements into their phasor domain equivalent impedances …..and then <u>combine them in series and parallel</u>.



Notice – **be careful**
---the frequency in this problem was specified in Hz. …….ω is in units or radians/s. ω (rad/s) = 2π f (Hz)

$$\mathbf{Z}_{L_1} = j\omega L_1 = j2\pi \times 500 \times 0.3 = j942.5\ \Omega$$

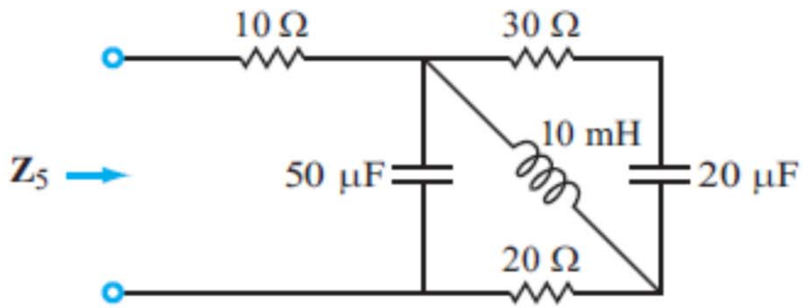
$$\mathbf{Z}_{L_2} = j\omega L_2 = j471.2\ \Omega$$

$$\mathbf{Z}_C = \frac{-j}{\omega C} = \frac{-j}{2\pi \times 500 \times 0.2 \times 10^{-6}} = -j1591.5\ \Omega$$

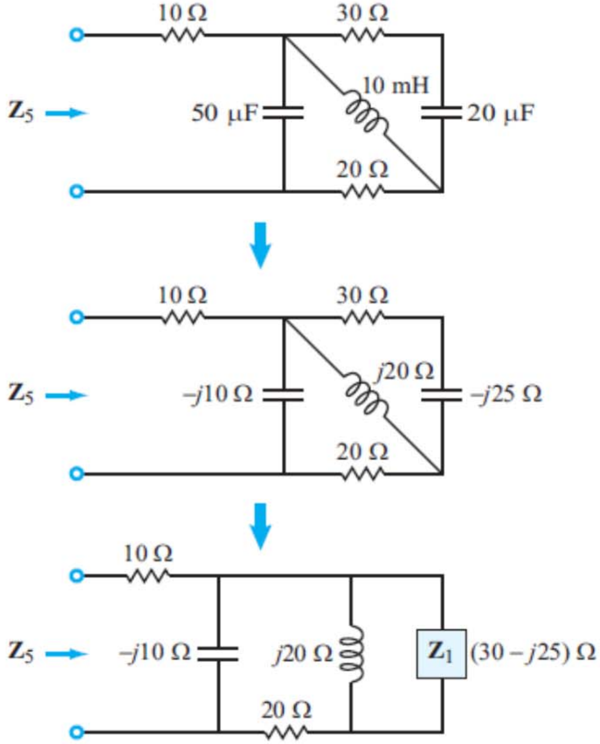
How did I get this?

so $$\mathbf{Z}_2 = j942.5 + j471.2 \parallel (1000 - j1591.5)$$

$$= j942.5 + \frac{j471.2(1000 - j1591.5)}{1000 - j1591.5 + j471.2} = (98.5 + j1524.0)\ \Omega$$



Calculate Z_5 at $\omega = 2000$ rad/s

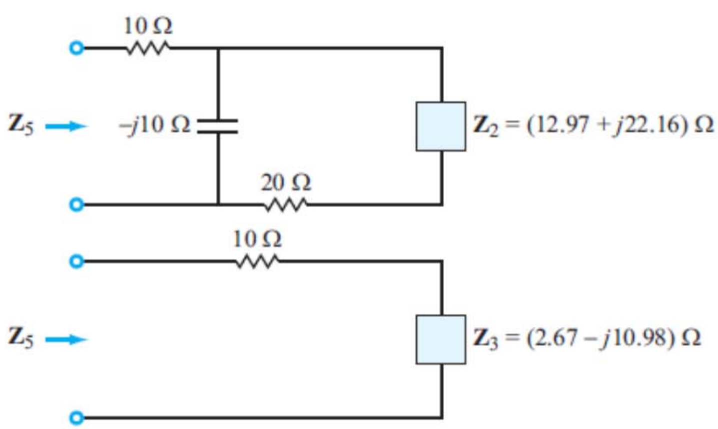


Transform to phasor domain

$$\mathbf{Z}_{C_1} = \frac{-j}{\omega C_1} = \frac{-j}{2000 \times 50 \times 10^{-6}} = -j10\ \Omega$$

$$\mathbf{Z}_{C_2} = \frac{-j}{\omega C_2} = \frac{-j}{2000 \times 20 \times 10^{-6}} = -j25\ \Omega$$

$$\mathbf{Z}_{L} = j\omega L = j2000 \times 10 \times 10^{-3} = j20\ \Omega$$



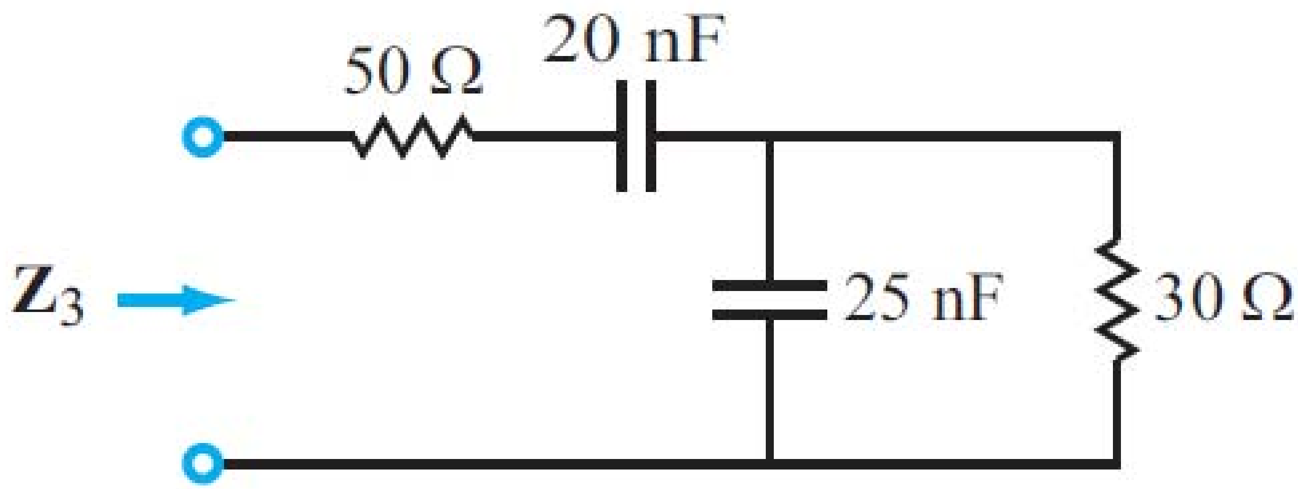
Simplify with series and parallel combinations

$$\begin{aligned}
\mathbf{Z}_2 &= \mathbf{Z}_1 \parallel (j20) \\
&= \frac{(30 - j25)(j20)}{30 - j25 + j20} \\
&= \frac{500 + j600}{30 - j5} \cdot \frac{(30 + j5)}{(30 + j5)} \\
&= \frac{12000 + j20500}{925} = (12.97 + j22.16)\ \Omega
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}_3 &= (20 + \mathbf{Z}_2) \parallel (-j10) \\
&= \frac{(32.97 + j22.16)(-j10)}{32.97 + j22.16 - j10} \\
&= \frac{221.6 - j329.7}{32.97 + j12.16} \cdot \frac{(32.97 - j12.16)}{(32.97 - j12.16)} \\
&= (2.67 - j10.98)\ \Omega
\end{aligned}$$

$$\mathbf{Z}_5 = 10 + \mathbf{Z}_3 = (12.67 - j10.98)\ \Omega.$$

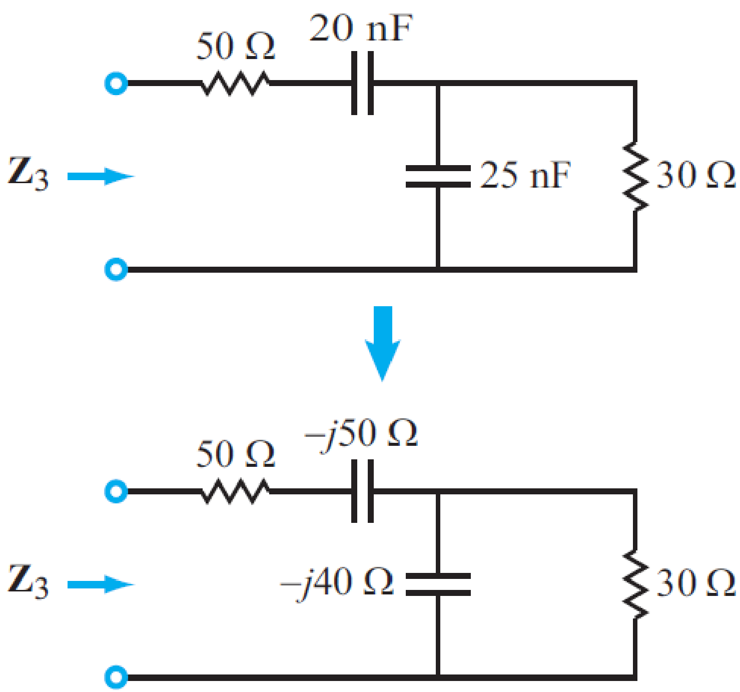
What is the value of Z_3 for this circuit



Is Z_3 going to be real or complex? Have you been given enough information to calculate it?

Actually Z_3 is going to be a complex value (why is this true?) …but to evaluate it you need to know the frequency because it will also be frequency dependent…so you haven't been given enough information.

OK…for practice, lets evaluate Z_3 at a frequency of 10^6 rad/s

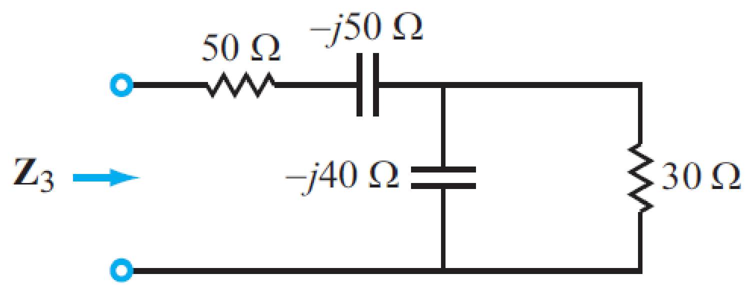


First we transfer to the phasor domain using:

$$R \rightarrow \mathbf{Z}_R = R$$
$$L \rightarrow \mathbf{Z}_L = j\omega L$$
$$C \rightarrow \mathbf{Z}_C = 1/j\omega C$$

All resistances, capacitances, and inductances are converted to complex impedances – at the specified frequency

Now we combine the impedances using well-known techniques



$$\mathbf{Z}_3 = 50 - j50 + \frac{30 \times (-j40)}{30 - j40} = (69.2 - j64.4)\ \Omega.$$

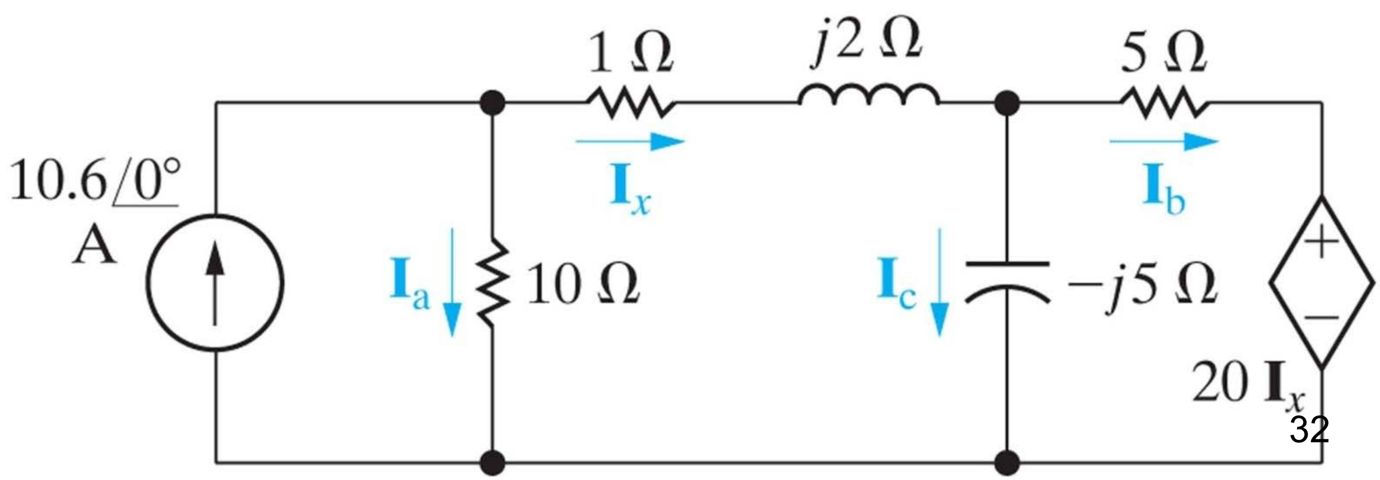
What would the impedance be at zero frequency (DC)
…or at very high frequency? Can you do this without
doing any calculations?

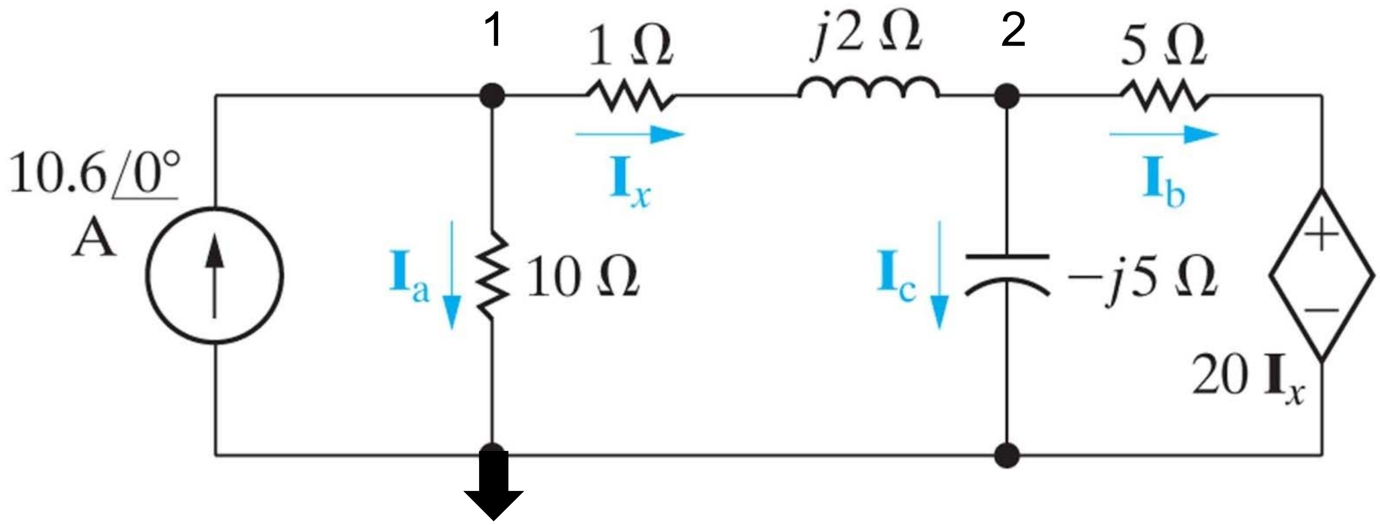
Obviously, as we just saw, if you can solve a sinusoidal steady state circuit problem by simply using the “impedance model” it is much easier than solving for phasor voltages or currents using a node voltage or mesh current approach.

As we discussed before, this is the equivalent of solving a DC circuit by series and parallel combinations of resistors and then using Ohms law to calculate the current or voltage as required.

It is possible, however, to use phasors to do a complete node voltage analysis in the frequency domain as we saw before.

Consider for example this circuit – where we’ve already transformed to the phasor domain:





We define voltages at the two extraordinary nodes and define a reference node as shown. Now, we write the node voltage equations at the two nodes using phasor voltages.

At node 1:

$$-10.6+\frac{\mathbf{V}_1}{10}+\frac{\mathbf{V}_1-\mathbf{V}_2}{1+j2}=0$$

This looks just like an ordinary node-voltage equation ….except the voltages are phasors ..and the equation has constants that are complex numbers.

$$-10.6+\frac{\mathbf{V}_1}{10}+\frac{\mathbf{V}_1-\mathbf{V}_2}{1+j2}=0$$

Multiplying by (1+j2) and collecting coefficients of the $\mathbf{V}_1$ and $\mathbf{V}_2$ phasors:

$$\mathbf{V}_1(1.1+j0.2)-\mathbf{V}_2=10.6+j21.2$$

Now we write the node voltage equation at the second node:

$$\frac{\mathbf{V}_2-\mathbf{V}_1}{1+j2}+\frac{\mathbf{V}_2}{-j5}+\frac{\mathbf{V}_2-20\mathbf{I}_x}{5}=0$$

$$and \quad \mathbf{I}_x=\frac{\mathbf{V}_1-\mathbf{V}_2}{1+j2}$$

Substituting the expression for $\mathbf{I}_x$ into the 2nd node equation, and multiplying through by (1+j2) …and collecting coefficients for the two phasor voltages:

$$-5\mathbf{V}_1 + (4.8 + j0.6)\mathbf{V}_2 = 0$$

So this equation along with the 1st equation:

$$\mathbf{V}_1(1.1 + j0.2) - \mathbf{V}_2 = 10.6 + j21.2$$

has the simultaneous solution for the phasor voltage: (by substitution)

$$\mathbf{V}_1 = 68.4 - j16.8V$$

$$\mathbf{V}_2 = 68 - j26V$$

from which the branch current phasors can be calculated:

In MATLAB ..the code looks like this:

```
>> A=[1.1+0.2*j, -1;-5,4.8+0.6*j]

A =

   1.1000 + 0.2000i  -1.0000
  -5.0000             4.8000 + 0.6000i

>> B=[10.6+21.2*j;0]

B =

  10.6000 +21.2000i
        0

>> x=A\B

x =

  68.4000 -16.8000i
  68.0000 -26.0000i
```

Notice: MATLAB <u>easily does matrix algebra with complex numbers</u>

You can also (obviously) do mesh current solutions using phasors as well as find Thevenin equivalent circuits for AC steady-state circuits. I'm not going to cover these explicitly here. If you would like to se examples, you should look at Chapter 9 in Nilsson. I won't ask you to solve any AC steady state circuits explicitly using mesh current solutions or Thevenin equivalent circuits on the final exam.

What you should remember, however, is that you can use KCL and KVL and any of the circuit solution techniques we've discussed for DC resistive circuits on AC steady state circuits using phasors.