ELEN 50 Class 22 – Sinusoidal Steady State and Phasors

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Up to now, we've considered time-independent DC circuits and the time response of circuits containing reactive elements (capacitors and inductors) when subjected to transient changes (step response or natural decay).

We've also seen that the transient response of circuits containing inductors and capacitors can be quite complicated – requiring the solution of differential equations….sometimes second order equations.

The remainder of ELEN 50 will focus on an important special class of time dependent circuits – <u>circuits containing steady state current and voltage sources which vary sinusoidally with time</u>. These "AC Steady State" circuits can be solved using a powerful technique called phasor analysis – and we'll introduce this approach today.

AC Steady State might seem like a special case …but it's not!

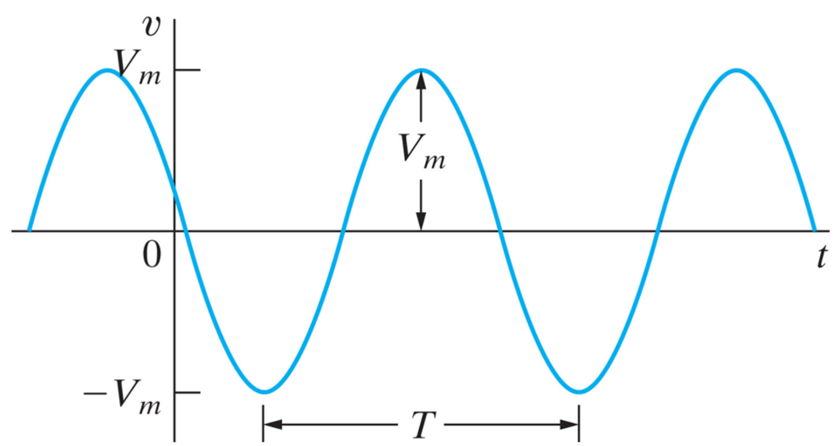
We are interested in the response of circuits to sinusoidal sources for several reasons:

- Generation, transmission, distribution, and consumption of electrical power around the world occurs under essentially sinusoidal steady-state conditions.
- The Fourier theorem permits the description of any time varying signal to be decomposed into a distribution of sinusoidal components …so understanding a system's response to a sinusoidal signal can permit the description of the system's response to any time varying signal.

We can write a general sinusoidal voltage signal as:

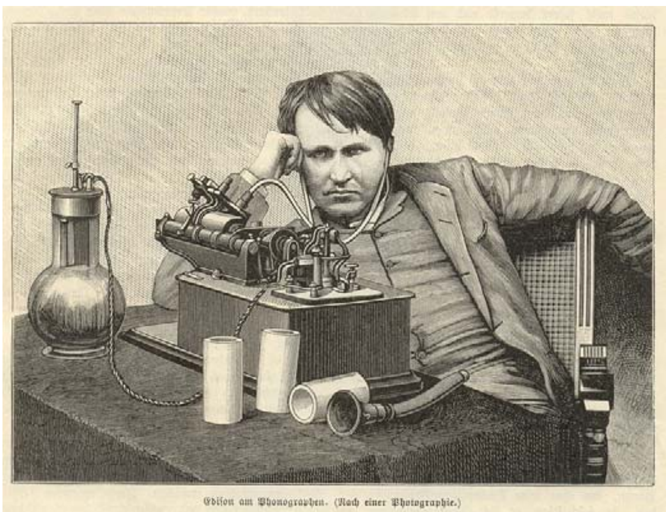
$$v = V_m \cos(\omega t + \phi)$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

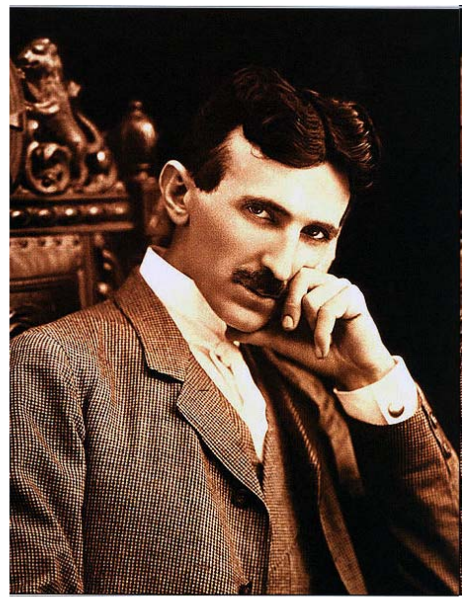


AC versus DC – just for fun

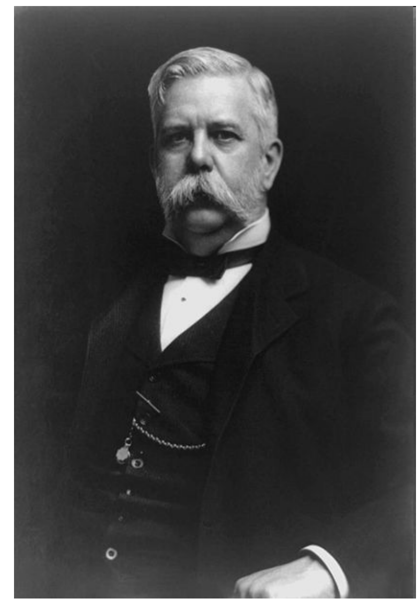
Electric power wasn't always distributed using AC. In the 1880's as the country's electric distribution system was just being developed there was a fierce battle between proponents of DC power and AC power. By 1896, with the construction of the Niagara Falls power plant, AC had won.



Thomas A. Edison



Nicola Tesla



George Westinghouse

DC power

AC power

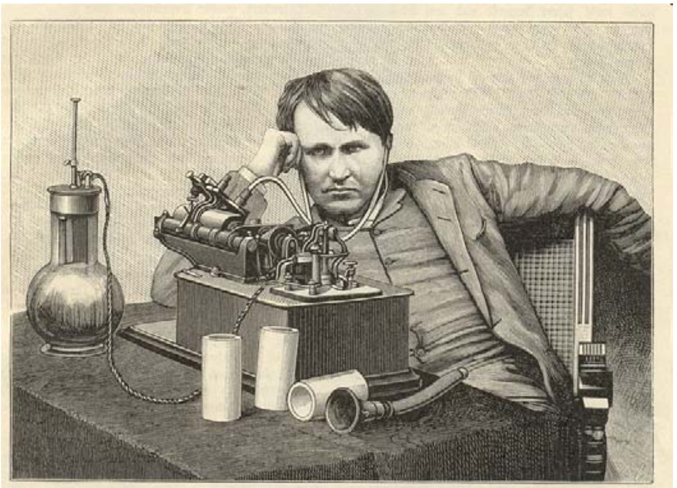
Courtesy of WIkipedia

"There is no plea which will justify the use of high-tension and alternating currents, either in a scientific or a commercial sense. They are employed solely to reduce investment in copper wire and real estate."

"...My personal desire would be to prohibit entirely the use of alternating currents. They are unnecessary as they are dangerous...I can therefore see no justification for the introduction of a system which has no element of permanency and every elements of danger to life and property."

"...I have always consistently opposed high-tension and alternating systems of electric lighting...not only on account of danger, but because of their general unreliability and unsuitability for any general system of distribution."

Source: Edison, Thomas A. The Dangers of Electric Lighting, North American Review, November, 1889. pp.630, 632, 633.

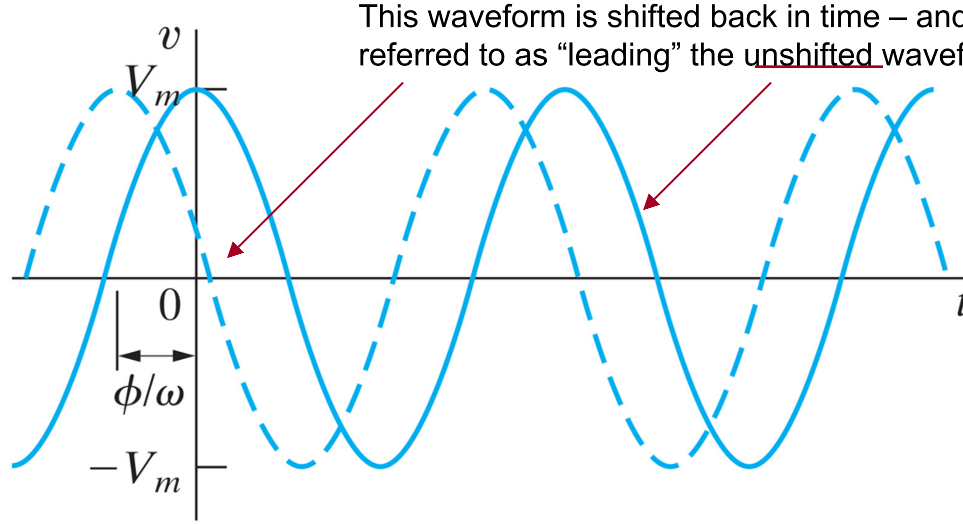


Edison am Phonographen. (Nach einer Photographie.)

Edison's company owned the patents for the Ni Fe batteries and DC motors that would be used in a DC power system!

This waveform is shifted back in time – and is referred to as "leading" the unshifted waveform

$$v = V_m \cos(\omega t + \phi)$$



The root mean square (RMS) value of a sinusoidal current or voltage is defined:

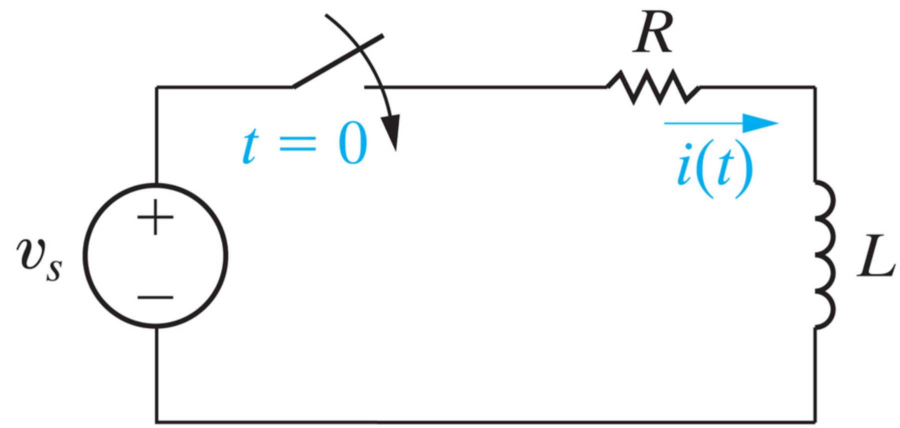
$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)dt}$$

$$\boxed{V_{RMS} = \frac{V_m}{\sqrt{2}}}$$

The RMS voltage (or current) is the <u>effective voltage or current in an AC system</u> – it is the value of a DC current or voltage that will deliver the same power to a load as the AC voltage or current.

The sinusoidal response of a circuit

$$v_s = V_m \cos(\omega t + \phi)$$



This is similar to the step-response solution for an RL circuit (which we discussed last time), however, here, the voltage source is sinusoidal. We want to obtain i(t) for time equal to or greater than t=0. We start, as usual, by writing KVL for the circuit:

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

This is similar to the differential equation for the step response of the RL circuit --- except for the $\cos(\omega t+\phi)$ term – it's a first order equation.

$$L\frac{di}{dt}+Ri=V_m\cos(\omega t+\phi)$$

Although this is a first order equation, it is more complicated than the step-response circuits. To solve it, we can't just separate variables and integrate. Instead, we propose the following as a solution: (although this is fairly non-obvious). You can convince yourself that this solution works by differentiating it but, as I said, it's non-obvious!

$$i=\frac{-V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\phi+\theta)e^{-(R/L)t}+\frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\phi-\theta)$$

transient component

steady-state component

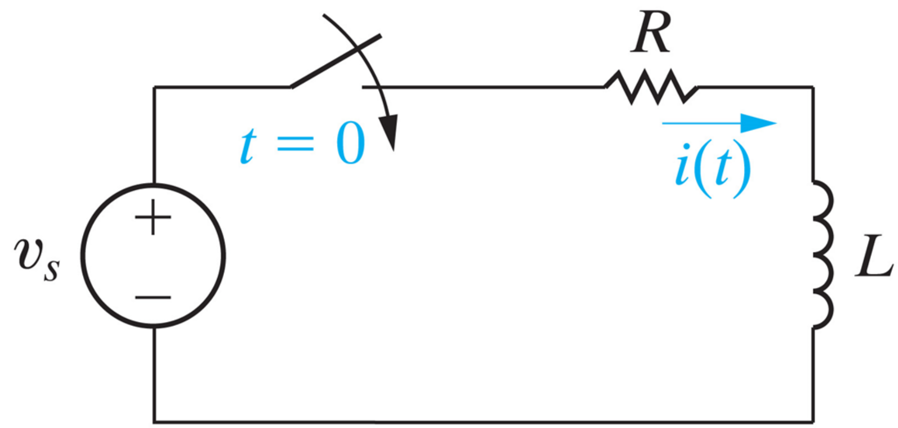
Here, $\theta=\arctan\frac{\omega L}{R}$

Notice that the sine wave has been shifted in time – a phase shift

The transient part of the solution dies out as exp(-t/τ) and the steady-state term persists as long as the switch remains closed. Also notice that the steady-state term has a magnitude:

$$\frac{V_m}{\sqrt{R^2+\omega^2L^2}}$$

$$v_s = V_m \cos(\omega t + \phi)$$



If this were a DC voltage source, the steady state current (after some large number of time constants) would be:

$$i = \frac{v_s}{R}$$

(since the inductor is a short circuit for DC):

But, for an AC voltage source -- after a long time: $$i = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

Also notice, the phase angle of the steady-state response signal is different from the source:

steady-state current response: $$\frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\omega t+\phi-\theta)$$

input voltage: $$v_s = V_m \cos(\omega t+\phi)$$

The phase of the current response has been shifted by a phase angle, θ, from the input. According to our convention, the current "lags" the voltage.

$$\theta = \arctan\frac{\omega L}{R}$$

Is there an easier way to solve AC circuits like this?

Yes there is …if we restrict the solution to the steady state part …i.e. don't consider the transient response that dies off exponentially.

The technique is called 'phasor analysis"

Phasors

The phasor is a complex number that carries the amplitude and phase angle information of a sinusoidal function. It is a convenient way of representing these properties and, as we'll see, it's possible to develop simple, powerful circuit analysis techniques for the response of circuits to steady-state sinusoidal inputs using phasors.

The phasor concept is based on the Euler identity which lets us express sine and cosine functions as an exponential function.

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

The cosine function is the real part of the complex exponential function and the sine function is the imaginary part. Notice we are using the symbol, "j" to indicate imaginary numbers – i.e.

$$\boxed{j^2 = -1}$$

We do that to avoid confusion with the symbol, i, for current.

So if: $$v = V_m \cos(\omega t + \phi)$$

The real part

We can write it as: $$= V_m \Re\left\{ e^{j\omega t} e^{j\phi} \right\}$$

$$= \Re\left\{ V_m e^{j\phi} e^{j\omega t} \right\}$$

We can define the phasor transform as:

symbol for phasor transform

$$\mathbf{V} = V_m e^{j\phi} = \boldsymbol{P}\left\{ V_m \cos(\omega t + \phi) \right\}$$

This is the representation for a phasor – it's written in <u>boldface</u> and it's a complex number containing <u>amplitude and phase</u> information.

Notice: <u>the phasor doesn't have any time dependence</u> – it is a transform from the time domain to the frequency or complex number domain.

Phasors

So, if:

$$v = V_m \cos(\omega t + \phi)$$

The phasor transform of v gives:

$$\mathbf{V} = V_m e^{j\phi}$$

The phasor has no explicit time dependence. It is a complex number containing only amplitude and phase information – and these are the two variables that change from node to node in a circuit attached to AC steady state sources. We'll see how to use phasors to solve AC steady state circuits in a while.

we can also write the phasor using the abbreviation:

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$$

For example if we have the phasor:

$$\mathbf{V} = 100\angle -26°$$

$$\mathbf{V} = 100 e^{-j26°}$$

$$v = 100\cos(\omega t - 26°)$$

Notice, as we just saw, the phasor doesn't contain any explicit information about the frequency either …just the amplitude and the phase.

Obviously, we can also do an <u>inverse phasor transfrom</u> – taking us from the complex number domain back to the time domain.

$$\boldsymbol{P}^{-1}\left\{V_m e^{j\phi}\right\} = \Re\left\{V_m e^{j\phi} e^{j\omega t}\right\}$$

Phasors can be used to add sinusoidal functions (without using trigonometric identities):

For example: if y_1=20 cos(ωt-30^o) and y_2=40 cos(ωt+60^o) it's easy to add:

y=y_1 + y_2. Using phasors, we can write:

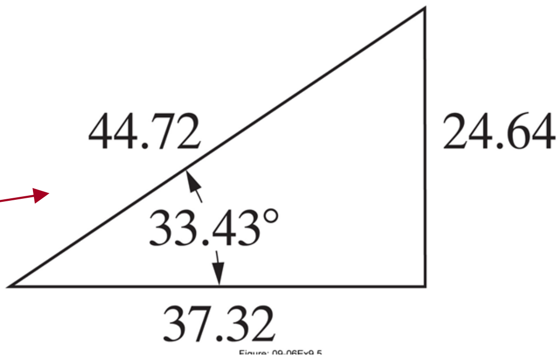
$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$
$$= 20\angle -30° + 40\angle 60°$$

To perform the addition we must add the real and imaginary parts of the phasors separately – of course….because it's a complex number.

$$20\angle -30° = A\cos\phi + jA\sin\phi = 20(0.866) + j\,20(-0.5)$$
$$= 17.32 - j10$$
$$40\angle 60° = 40(.5) + j40(0.866)$$
$$= 20 + j34.64$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$
$$= 17.32 - j10 + 20 + j34.64$$
$$= 37.32 + j24.64$$

And to put this back into phasor notation, we have to do a little trig to calculate the phase angle:



$$\phi = \arctan\frac{24.64}{37.32} = 33.43° \quad and \;\; A = \frac{24.64}{\sin\phi} = 44.72$$

$$so \;\; \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$
$$= 44.72\angle 33.43°$$

Writing this in the time domain: $y=44.72 \cos(\omega t + 33.43^{o})$

So …phasors are a convenient way to add sinusoidal functions without resorting to trigonometric identities …but what good are they for dealing with AC steady-state circuit analysis?

Passive Circuit Elements in the Frequency Domain

Resistors

If the current in a resistor varies sinusoidally with time:

$$i = I_m \cos(\omega t + \theta_i) \;\; then \;\; from\ Ohms\ law:$$

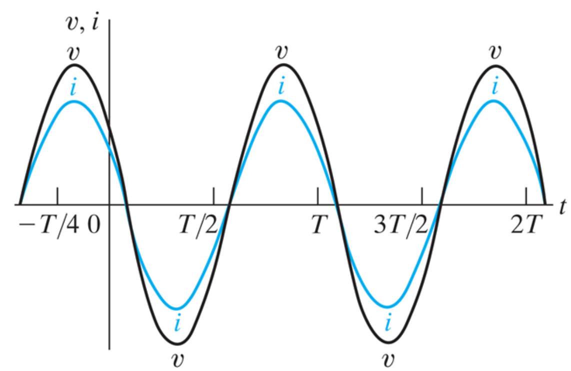
$$v = R\left[I_m \cos(\omega t + \theta_i)\right]$$

$$= RI_m\left[\cos(\omega t + \theta_i)\right]$$

The phasor transform of this voltage: $\mathbf{V} = V_m e^{j\phi} = \mathcal{P}\left\{V_m \cos(\omega t + \phi)\right\}$

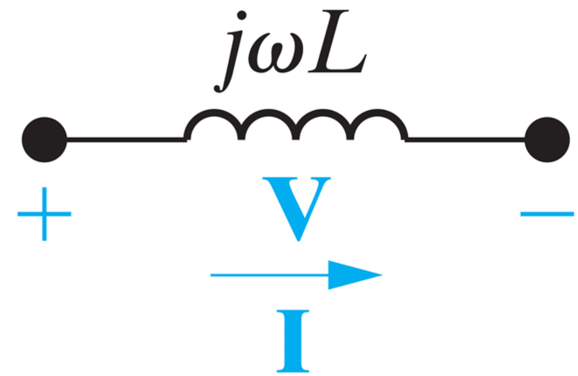
$$\mathbf{V} = RI_m e^{j\theta_i} = RI_m \angle\theta_i$$

So: $\boxed{\mathbf{V} = R\mathbf{I}}$ Ohm's law for phasors



Inductors

The voltage drop across and inductor which is passing a current:



$$i = I_m \cos(\omega t + \theta_i)$$

is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$

which we can rewrite using the cosine function as

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90°)$$

The phasor representation of:

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90°)$$

is

$$\mathbf{V} = -\omega L I_m e^{j(\theta_i - 90°)}$$

$$= -\omega L I_m e^{j\theta_i} e^{-j90°}$$

and , from Euler's identity,

$$e^{-j90°} = \cos(-90°) + j\sin(-90°) = -j \quad so$$

$$\mathbf{V} = j\omega L I_m e^{j\theta_i}$$

$$\boxed{\mathbf{V} = j\omega L\mathbf{I}}$$

where the phasor representaion of the current is :

$$\mathbf{I} = I_m e^{j\theta_i}$$

So by using phasor notation, we have been able to write the fundamental voltage-current relation for an inductor:

$$v = L\frac{di}{dt}$$

As:

$$\mathbf{V} = j\omega L\mathbf{I}$$

We've replaced the time derivative of the current with: $j\omega\mathbf{I}$

This is really important! Phasor transforms allow the use of all of the DC analysis tools we've developed for AC circuits …which contain inductors and capacitors …and with **no differential equations**! This is why we use phasors for AC steady-state circuits!

Going back to the exponential notation again using Euler's identity:

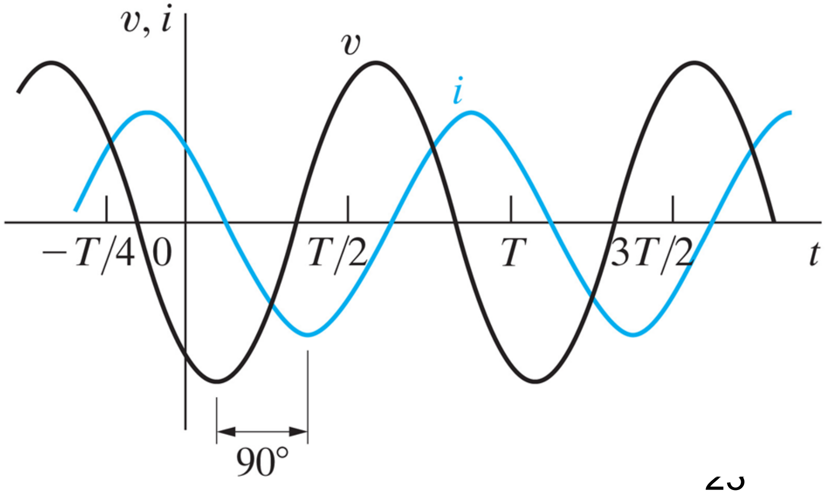
$$j\omega L \textit{ can be rewritten as}$$

$$\omega L\left(\cos(90°) + j\sin(90°)\right) = \omega L e^{j90°}$$

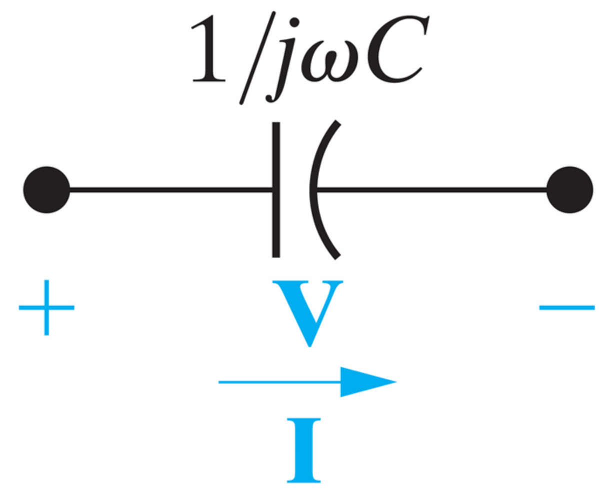
so

$$\mathbf{V} = j\omega L\mathbf{I} = \left(\omega L e^{j90°}\right)\left(I_m e^{j\theta_i}\right)$$
$$= \omega L I_m e^{j(\theta_i + 90°)} = \omega L I_m \angle(\theta_i + 90°)$$

The voltage across the inductor is shifted +90° in phase with respect to the current. We say the <u>voltage across an inductor leads the current by 90°</u>



Capacitors



If a sinusoidal voltage is applied to a capacitor:

$$v = V_m \cos(\omega t + \theta_i)$$

Then, since for a capacitor: $$i = C\frac{dv}{dt}$$

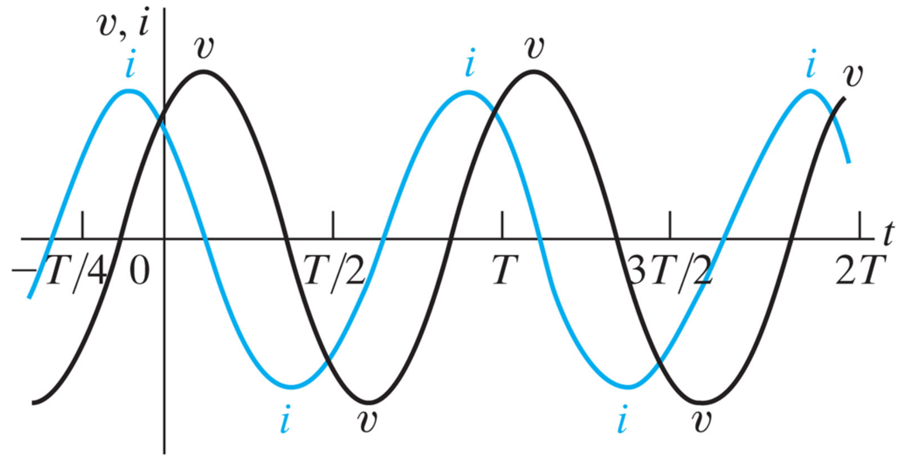
In phasor notation (using the trick we learned from the inductor example) .. we replace the time derivative of the voltage with $j\omega \mathbf{V}$

$$\textit{and we can write}:$$

$$\mathbf{I} = j\omega C\mathbf{V}$$

$$so \quad \boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$

Just like we did with the inductor, we can show that the voltage across the capacitor:



$$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$= \left(\frac{1}{\omega C}\angle -90^\circ\right)(I_m \angle \theta_i)$$

$$= \frac{I_m}{\omega C}\angle\left(\theta_i - 90^\circ\right)$$

So the <u>voltage across the capacitor lags the current by 90°.</u>
Notice that, just like the inductor, the phase shift across these elements doesn't depend on the value of the capacitor or inductor…it's just plus or minus 90°.

We can summarize all of the stuff about phasor representations of passive circuit elements by writing a general phasor equation:

$$\boxed{\mathbf{V} = Z\mathbf{I}}$$

Where Z is the impedance of the circuit element

Z can be real, complex, or purely imaginary – it has units of Ohms ….and it is not a phasor…it multiplies the current phasor to give the voltage phasor. The imaginary part of the impedance (if it exists) is called the reactance.

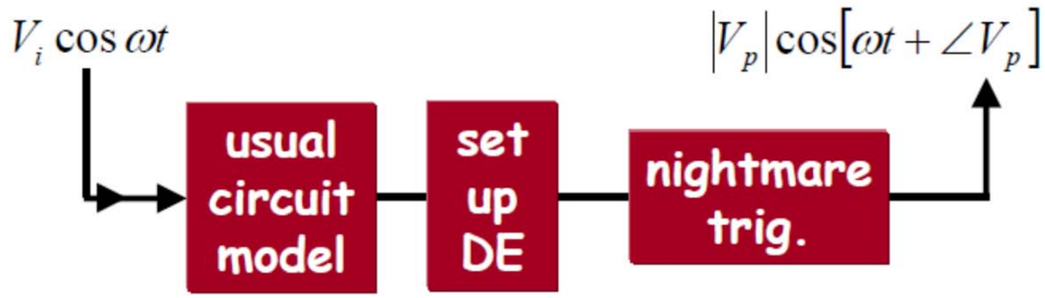
Circuit Element	**Impedance**	**Reactance**
Resistor	R	none
Inductor	$j\omega L$	ωL
Capacitor	$j(-1/\omega C)$	$-1/\omega C$

Impedances add in series and parallel just like resistances – and the delta to wye transforms for complex impedances also behave just like resistances

In fact, every one of the circuit analysis techniques you've learned in ELEN 50: KCL, KVL, node voltage analysis, mesh current analysis, source transformations, and Thevenin and Norton equivalents can be done for steady-state sinusoidal circuits in a straightforward way using the phasor convention – without differential equations.

So what has been accomplished?

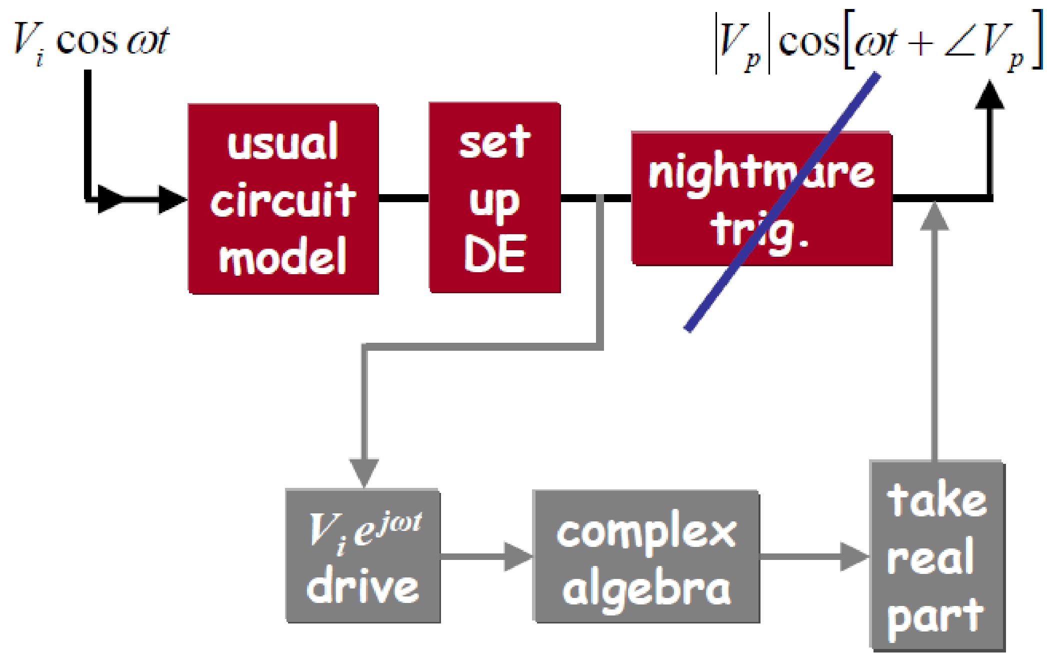
The Big Picture...



The differential eq. approach

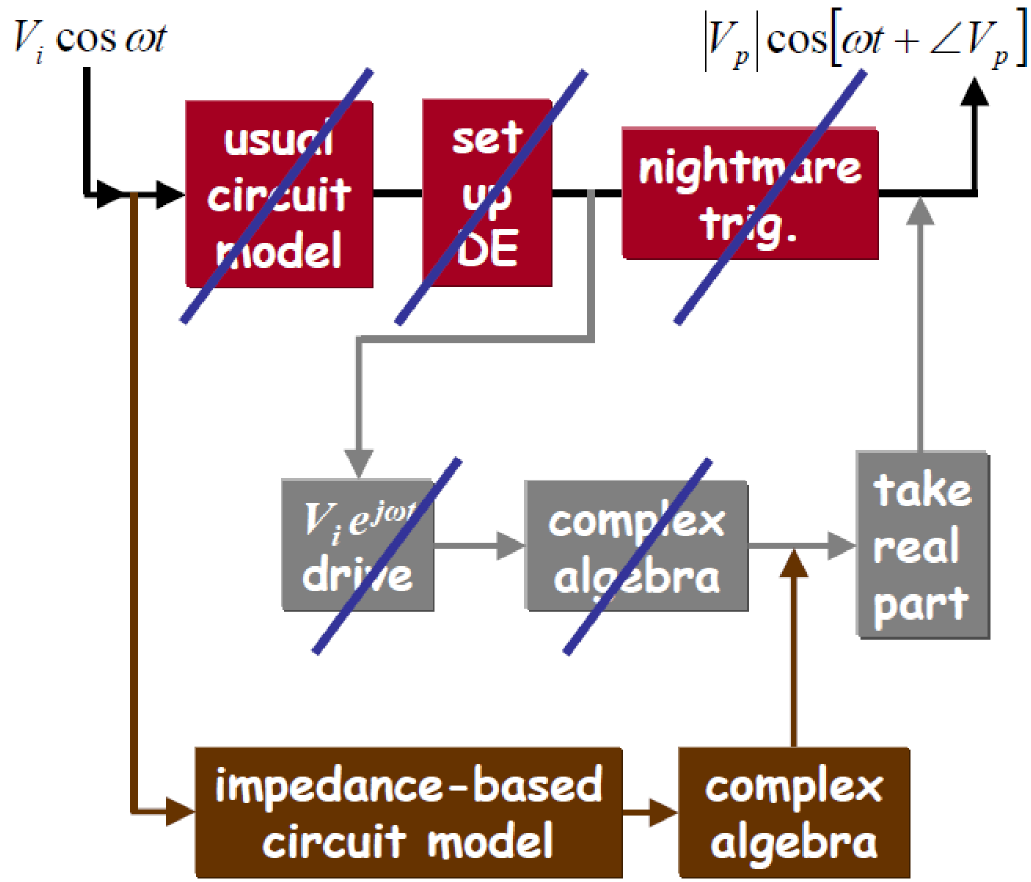
The Big Picture…

The phasor approach



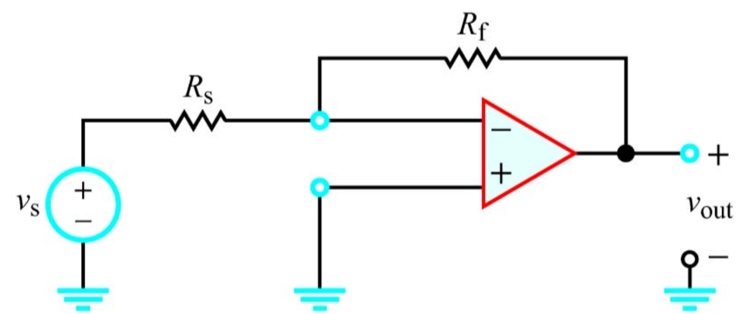
The impedance model approach

The Big Picture...

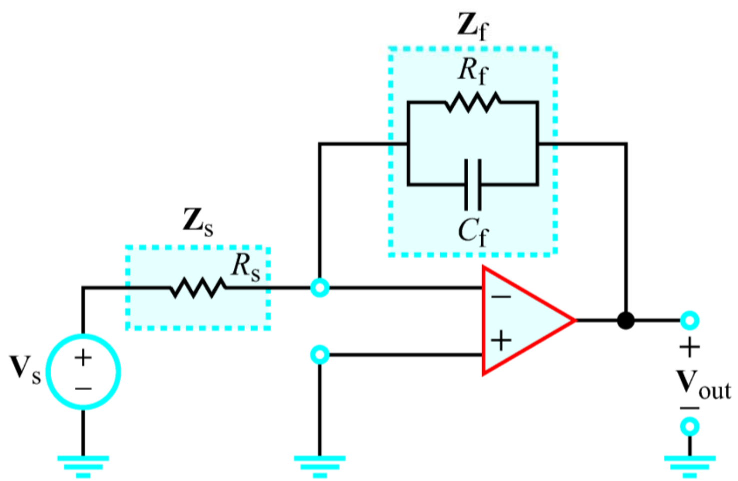


No D.E.s, no trig!

Here's an example from the textbook – it's an inverting op amp configuration with both a capacitor and resistor in the feedback loop.



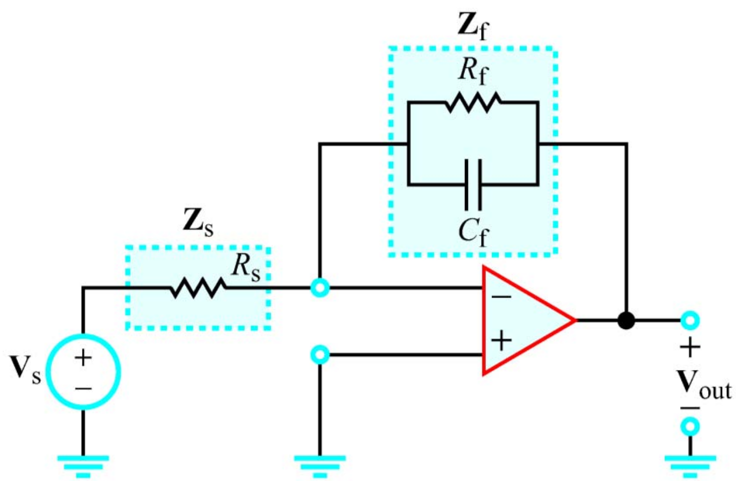
(a) Inverting amplifier



(b) Phasor domain with impedances

In the Phasor domain, the resistor and capacitor can be combined to form a complex impedance.

This should look familiar to you, it's Fig. 1 in your second lab project!



(b) Phasor domain with impedances

For the simple inverting amp configuration, we know: $v_{out} = -\frac{R_f}{R_s} v_s$

In the phasor domain we can write: $\mathbf{V}_{out} = -\frac{\mathbf{Z_f}}{\mathbf{Z_s}} \mathbf{V}_s$

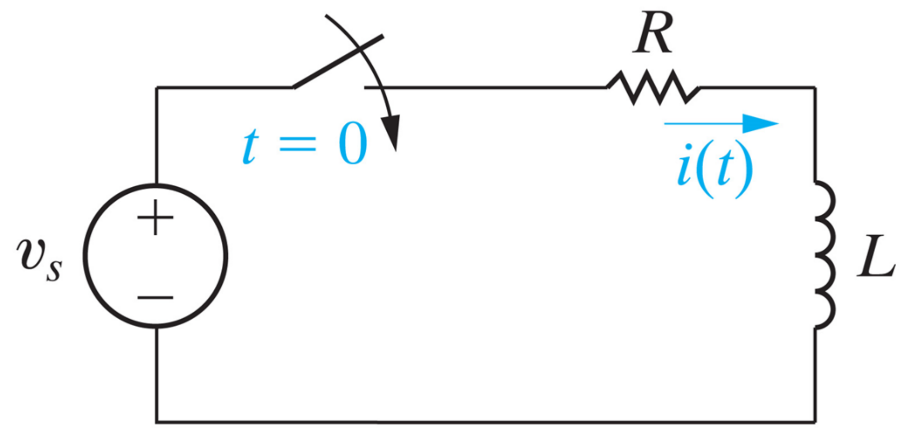
where: $\mathbf{Z}_s = R_s$

$$\mathbf{Z}_f = R_f \text{ in parallel with} \left(\frac{1}{j\omega C_f} \right)$$

$$\mathbf{Z}_f = \frac{R_f}{1 + j\omega R_f C_f}$$

In the beginning of this class, we analyzed this circuit when v_s was a sinusoidal source:

$$v_s = V_m \cos(\omega t + \phi)$$



We found the solution for i(t) by solving the differential equation and got:

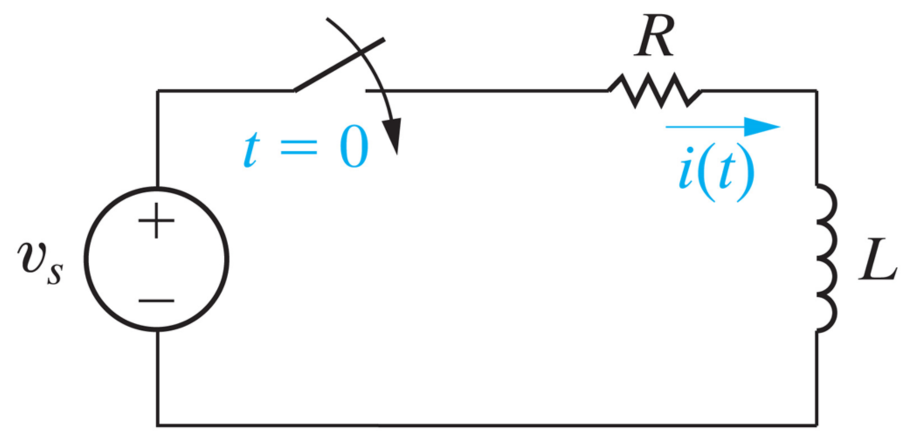
$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi + \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

transient component

steady-state component

$$\theta = \arctan \frac{\omega L}{R}$$

Phasors can not be used for the transient response part of the solution (they are for AC steady state only), but we could have gotten the steady state part easily, without solving any differential equations by using phasors.



This circuit is essentially a series combination of two impedances, a resistive impedance and an inductive impedance.

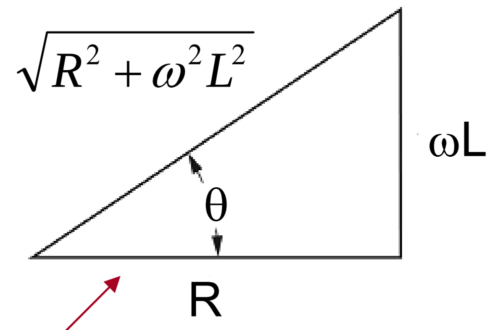
So the phasor equation for the steady state is simply: $\mathbf{I} = \dfrac{\mathbf{V}}{Z}$

Where: $Z = R + j\omega L$

Since:

$$v_s = V_m \cos(\omega t + \phi)$$

The voltage phasor is $\mathbf{V} = V_m e^{j\phi} = V_m \angle \phi$



To express Z in the same, polar form, we need to do some trig: the real part of Z is R and the imaginary part is jωL, so the modulus of the complex number is SQRT (R^2+$\omega^2$$L^2$) and the phase angle is arctan ωL/R =θ

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{V_m \angle \phi}{\sqrt{R^2 + \omega^2 L^2} \angle \theta} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

.. …our solution for the steady-state AC response using phasors was:

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{V_m \angle \phi}{\sqrt{R^2 + \omega^2 L^2} \angle \theta} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

And when we analyzed this circuit using a differential equation we got:

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi + \theta) e^{-\left(R/L\right)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

So the phasor solution is the same as the “long time” solution that we got by solving the messy differential equation….after the initial transient has died away.

We’ll do some more circuit analysis using phasors in the next class, plus do a little review on complex numbers.