Complex Number Review

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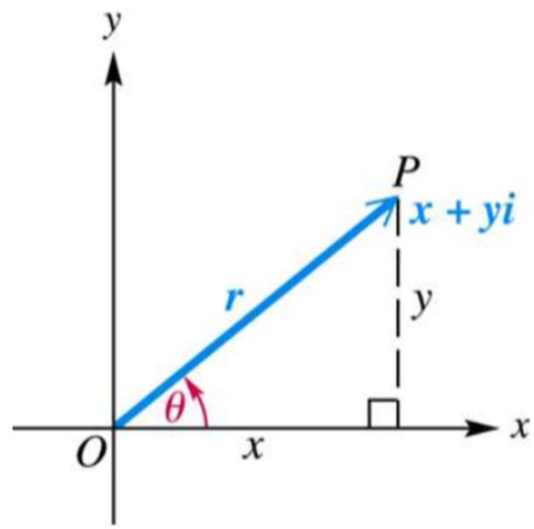
Converting complex numbers back and forth between rectangular and polar representation



Rectangular to Polar

$$|x + yi| = r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x},\ x \neq 0$$

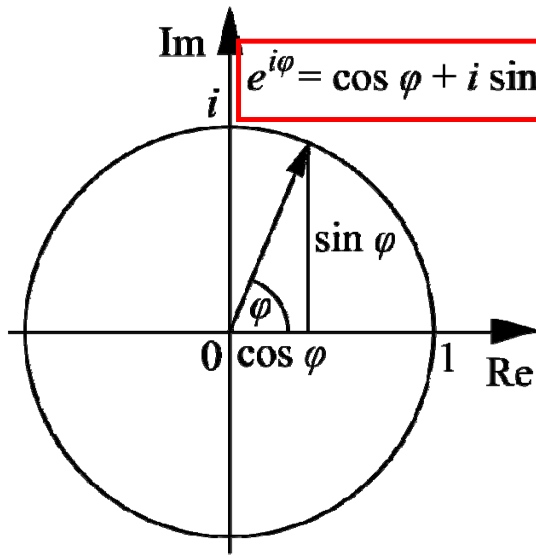
Polar to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

This is just trigonometry!

Euler's formula, named after Leonhard **Euler**, is a mathematical **formula** in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function.



using this definition, I can write:

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

$$= re^{i\vartheta}$$

Some Review of Complex Numbers

Equality:

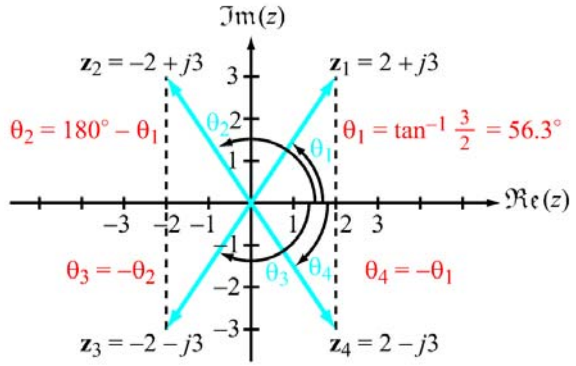
if two complex numbers, $\mathbf{z}_1$ and $\mathbf{z}_2$ are given by:

Do you see why I can write it this way???

$$\mathbf{z}_2 = x_2 + jy_2 = |\mathbf{z}_2| e^{j\vartheta_2}$$

$$\mathbf{z}_1 = x_1 + jy_1 = |\mathbf{z}_1| e^{j\vartheta_1}$$

then $\mathbf{z}_1 = \mathbf{z}_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$ or $|z_1| = |z_2|$ and $\theta_1 = \theta_2$



For complex numbers z_1 through z_4

Complex numbers z1 to z4 have the same magnitude but their polar angles depend on the polarities of their real and imaginary components.

Addition:

if two complex numbers, $\mathbf{z}_1$ and $\mathbf{z}_2$ are given by:

$$\mathbf{z}_2 = x_2 + jy_2 = |\mathbf{z}_2| e^{j\vartheta_2}$$

$$\mathbf{z}_1 = x_1 + jy_1 = |\mathbf{z}_1| e^{j\vartheta_1}$$

then $\mathbf{z}_1 + \mathbf{z}_2 = x_1 + x_2 + j(y_1 + y_2)$

Multiplication:

$\mathbf{z}_1\,\mathbf{z}_2 = (x_1 + jy_1)(x_2 + jy_2) = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$

or

$\mathbf{z}_1\,\mathbf{z}_2 =$ $|\mathbf{z}_1| e^{j\vartheta_1} \cdot |\mathbf{z}_2| e^{j\vartheta_2} = |\mathbf{z}_1||\mathbf{z}_2| e^{j(\vartheta_1 + \vartheta_2)}$

Division:

For $\mathbf{z}_2$ not equal to 0

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{|\mathbf{z}_1| e^{j\vartheta_1}}{|\mathbf{z}_2| e^{j\vartheta_2}} = \frac{|\mathbf{z}_1|}{|\mathbf{z}_2|} e^{j(\vartheta_1 - \vartheta_2)}$$

or

$$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

This is a useful trick for dividing two complex numbers ..multiply the numerator and denominator by the complex conjugate of the denominator

Exponentiation:

For any positive integer, n,

$$\mathbf{z}^n = (|\mathbf{z}|e^{j\vartheta})^n = |\mathbf{z}|^n e^{jn\vartheta} = |\mathbf{z}|^n(\cos n\theta + j\sin n\theta)$$

Multiplication and division of complex numbers are sometimes facilitated by converting to polar representation – but it's not absolutely necessary.

For example:

$$(8+j10)(5-j4)$$
$$=40-j32+j50+40$$
$$=80+j18$$

..... and after a little trig

$$=82e^{12.68^o}$$

do you see how I got this?

If we had converted the two complex numbers immediately to polar representation:

$$(8+j10)(5-j4)=(12.81\angle 51.34^o)(6.40\angle -38.66^o)$$
$$=82\angle 12.68^o$$
$$=80+j18$$

I get the product by multiplying the amplitudes and adding the polar angles.

When dividing complex numbers in the rectangular representation, the following trick is often useful as we saw previously:

If, for example, $$\frac{n_1}{n_2} = \frac{6+j3}{3-j1}$$

Multiply the numerator and denominator by the complex conjugate of the denominator ---- this gets all the imaginary stuff in the numerator:

$$\frac{n_1}{n_2} = \frac{6+j3}{3-j1} = \frac{6+j3(3+j1)}{3-j1(3+j1)} = \frac{18+j6+j9-3}{9+1}$$

$$= \frac{15+j15}{10} = 1.5+j1.5$$

You could have also done this by first converting to polar coordinates:

$$\frac{n_1}{n_2} = \frac{6 + j3}{3 - j1} = \frac{6.71\angle 26.57^o}{3.16\angle -18.43^o}$$

$$= 2.12\angle 45$$ The solution is obtained by dividing the amplitudes and subtracting the polar angles.

$$= 2.12\sin(45^o) + j2.12\cos(45^o) = 1.5 + j1.5$$ Converting back to Cartesian representation

Like we saw before

How you do it comes down to whether you'd rather do trigonometry or algebra!

..or maybe you just press REC→ POL on your calculator

Here are some exercises with complex numbers:

(a.) show $(-1+j)^7 = -8(1+j)$

(b.) show $(5j)/(2+j) = 1 + 2j$

(c.) show that $j(1-j\sqrt{3})(\sqrt{3}+j) = 2+2j\sqrt{3}$

What would be a good first step be in each of these exercises?

Review of Complex Numbers in a Single Chart

Table 7-2: Properties of complex numbers.

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$	
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy = \lvert\mathbf{z}\rvert e^{j\theta}$	$\mathbf{z}^* = x - jy = \lvert\mathbf{z}\rvert e^{-j\theta}$
$x = \mathfrak{Re}(\mathbf{z}) = \lvert\mathbf{z}\rvert\cos\theta$	$\lvert\mathbf{z}\rvert = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$
$y = \mathfrak{Im}(\mathbf{z}) = \lvert\mathbf{z}\rvert\sin\theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n = \lvert\mathbf{z}\rvert^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm\lvert\mathbf{z}\rvert^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1\mathbf{z}_2 = \lvert\mathbf{z}_1\rvert\lvert\mathbf{z}_2\rvert e^{j(\theta_1+\theta_2)}$	$\dfrac{\mathbf{z}_1}{\mathbf{z}_2} = \dfrac{\lvert\mathbf{z}_1\rvert}{\lvert\mathbf{z}_2\rvert} e^{j(\theta_1-\theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm180^\circ$	
$j = e^{j\pi/2} = 1\angle 90^\circ$	$-j = e^{-j\pi/2} = 1\angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm\dfrac{(1+j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm\dfrac{(1-j)}{\sqrt{2}}$