ELEN 50 Class 12 – Maximum Power Transfer

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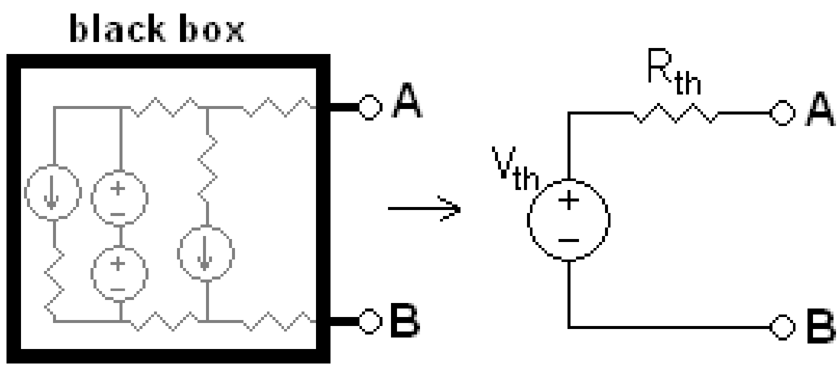
Last time we introduced the concept of the Thevenin Equivalent Circuit, where we can replace an arbitrary linear circuit (shown in the black box) with an equivalent circuit containing a voltage source in series with a resistor:



We also discussed how to determine V_{th} , and three different ways to determine R_{th}.

- **Obtain Vth** by calculating the voltage across the two specified terminals when no load is present (open circuit voltage) – sometimes you can do this by using source transforms.

- **Obtain Rth** by:

    1. Calculating the current that will flow between the specified terminals in a short circuit. Rth is obtained from Rth = Vth/Isc

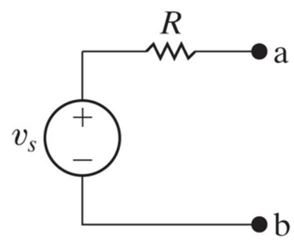
       …….or

    2. If the circuit doesn't contain dependent sources, you can calculate the equivalent resistance between the specified terminals after all independent voltage sources are replaced with short circuits and all independent current sources are replaced with open circuits. This in effect is "deactivating" the independent sources and and equivalent resistance in this case is Rth, the Thevenin resistance. ….or

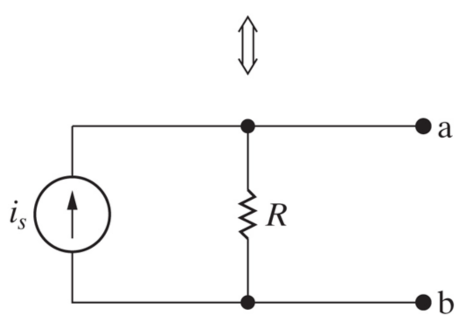
    3. If the circuit contains independent and dependent sources, Rth can be determined by deactivating independent sources, and adding an external source (vex)…then solve the circuit to determine the current iex supplied by the external source. Rth = vex/iex

Also…using a source transformation, we can convert the Thevenin equivalent circuit into a Norton equivalent circuit …composed of a current source in parallel with a resistance.

Remember, a source transformation allows a voltage source in series with a resistor to be transformed into a current source in parallel with a resistor.



(a)



(b)

If these two circuits are equivalent, then:

$$\boxed{i_s = v_s/R}$$

You can show this by attaching a load resistor to a and b and calculating the current flowing

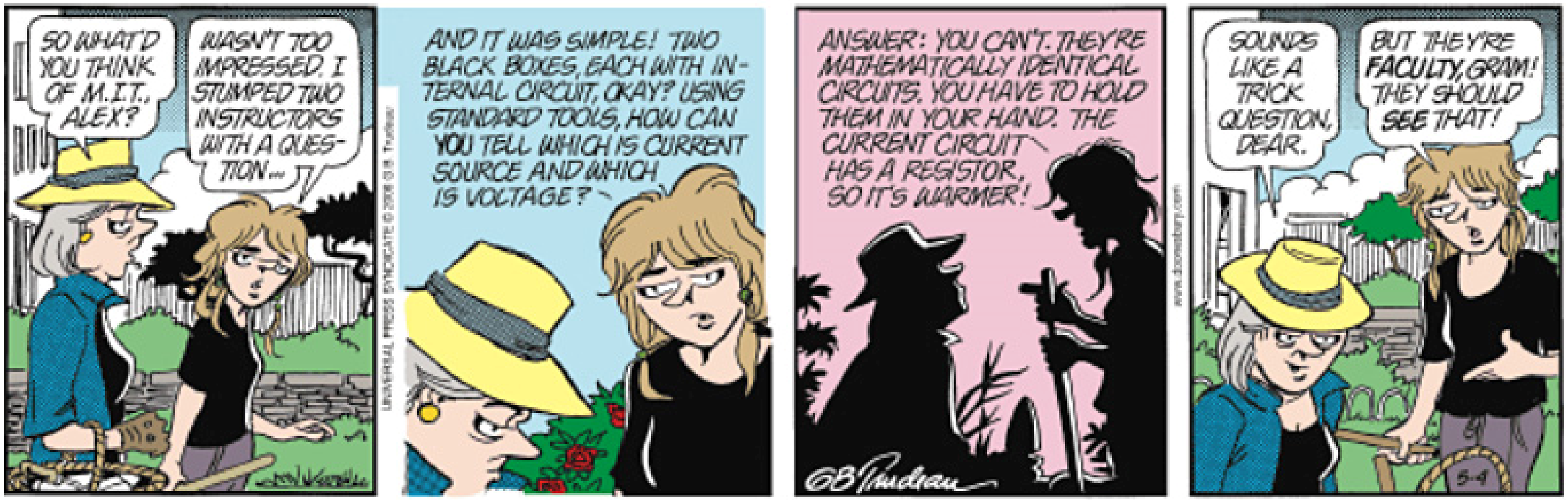
We also looked at some circuit examples where it was easy to use repetitive source transforms and series and parallel circuit combinations to convert a circuit into the form of a Thevenin equivalent circuit.

Obviously, this approach will also work …and, if it's simpler than the methods described in the previous chart, you should use it.

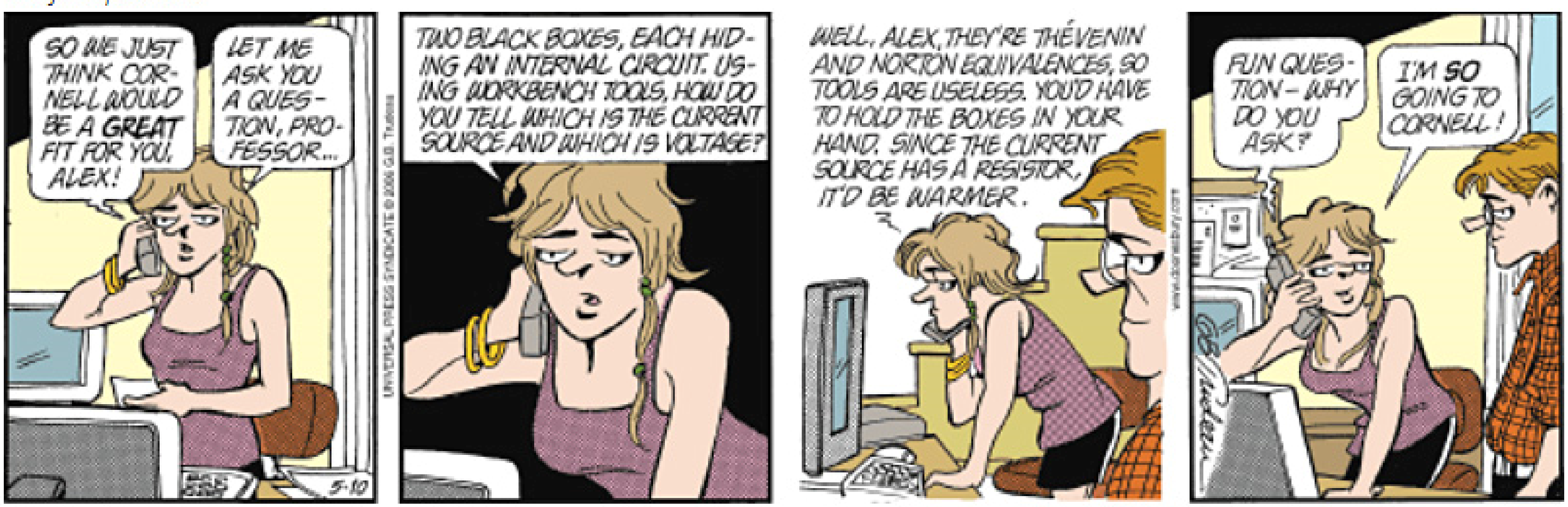
We also learned that if a circuit contains only dependent sources it has <u>Vth = 0</u>, and the value of Rth <u>has</u> to be calculated by "method 3" …the use of an external voltage or current source.

Thevenin and Norton Equivalents in Doonesbury!

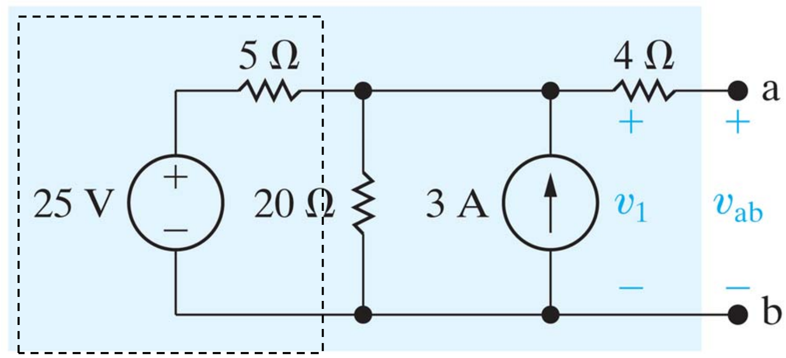
May 4, 2006:



May 10, 2006:

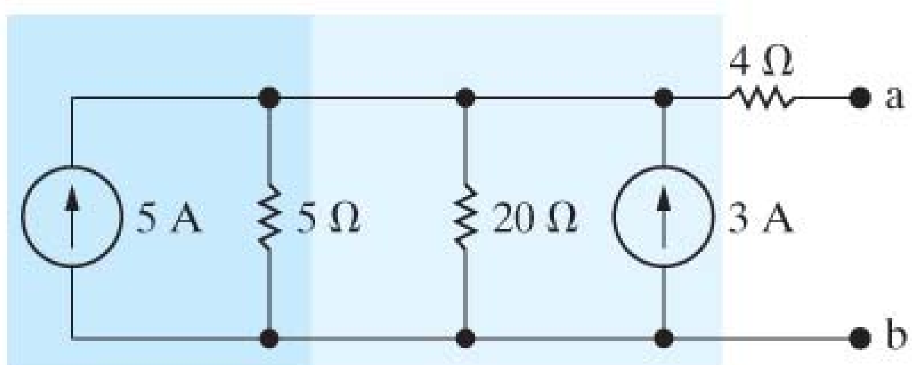


As we saw,sometimes source transforms are all you need to do to get a Thevenin equivalent circuit!

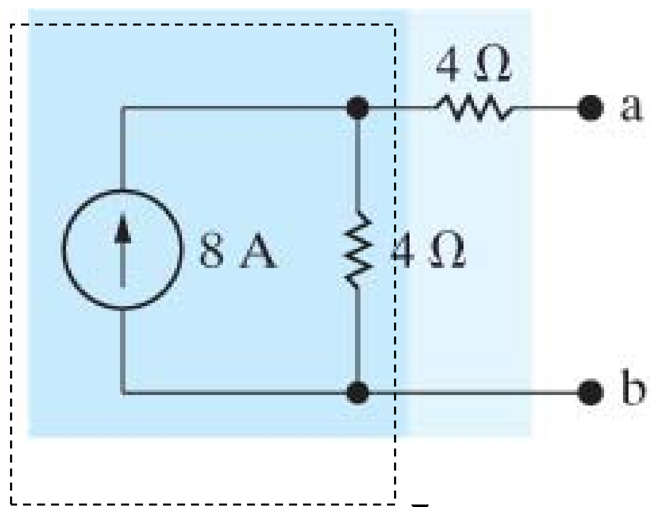


Calculate the Thevenin and Norton equivalent circuits for terminals a and b.

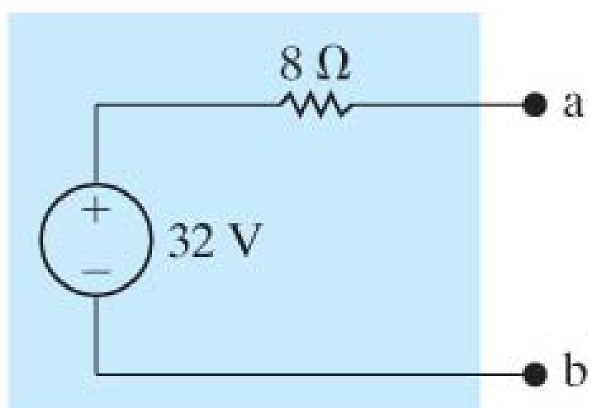
We can't eliminate any resistors – there are none in series with current sources or in parallel across voltage sources. However, we can transform this part to get:



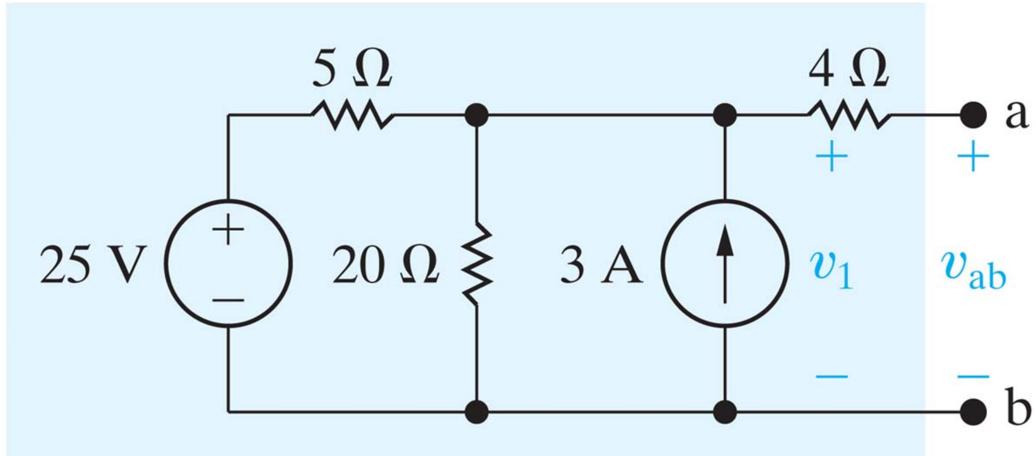
Then, combining the 2 parallel resistors and 2 current sources, we get:



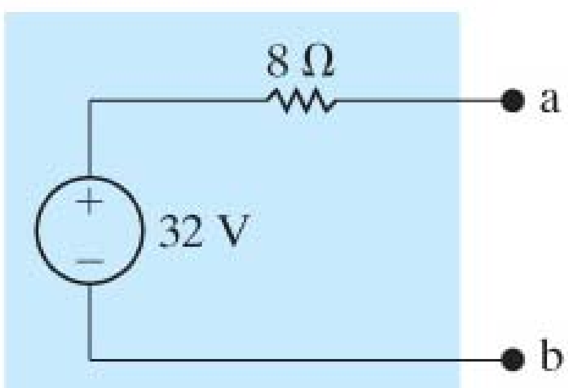
One more source transform of this part, and we get:

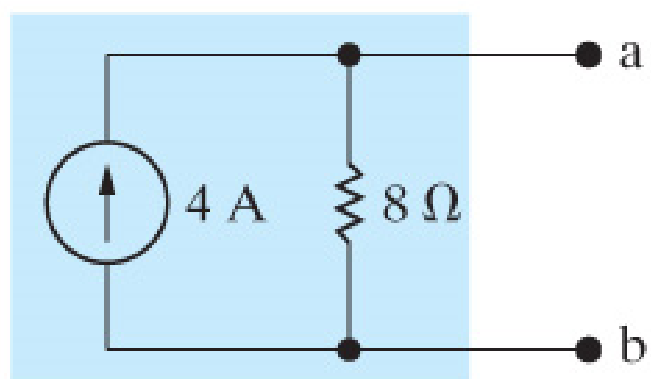


Original circuit

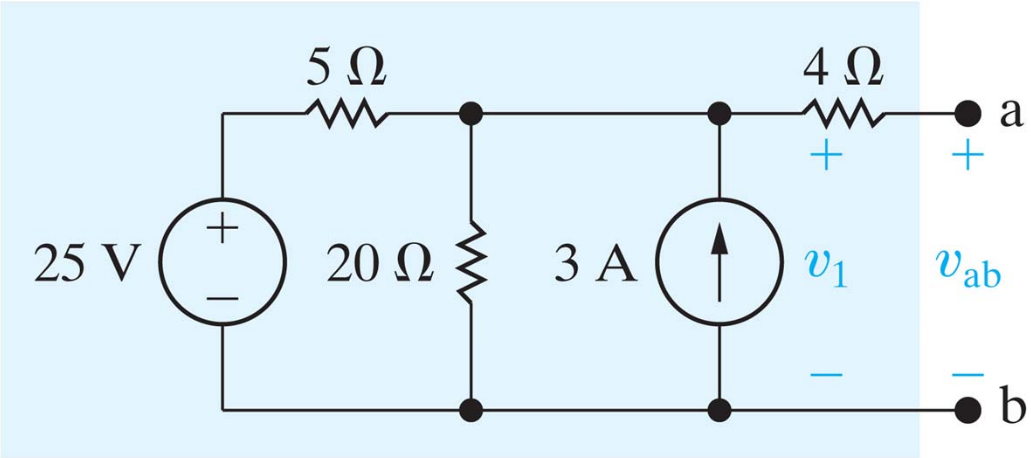


Thevenin Equivalent ← → Norton Equivalent





Is it clear how to get the Thevenin equivalent circuit using the other methods?



For example, <u>open circuit / short circuit</u> –
First we calculate Vth using node voltage:

There are only two nodes present …picking the bottom one as ground reference and labelling the top one as v1, we can write:

$$\frac{v_1 - 25}{5} + \frac{v_1}{20} - 3 = 0$$

so, v1 = Vth = 32 (as we saw before)

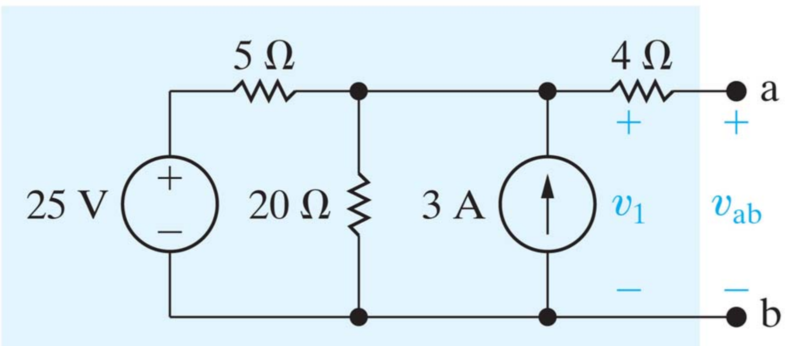
Now to get Rth using the short circuit method, we rewrite the node voltage equation with a short circuit between a and b:

$$\frac{v_1 - 25}{5} + \frac{v_1}{20} - 3 + \frac{v_1}{4} = 0$$

so, now v1 = 16 and isc = 4A
….therefore Rth = Vth/isc = 8 Ω (as before)

We could also have used “method three” the external source method – although we certainly wouldn’t have been required to do this since all of the sources present are independent sources

No doubt the easiest approach is the source transform we used originally…..the next easiest would be to get Vth by a node voltage calculation and then to get Rth using “method two” – deactivating independent sources and just calculating the equivalent resistance in the remaining circuit:



You can probably do this in your head – the remaining circuit (after source deactivation) is a 4Ω resistor in series with a 5Ω and a 20Ω in parallel – 8Ω   as we saw earlier

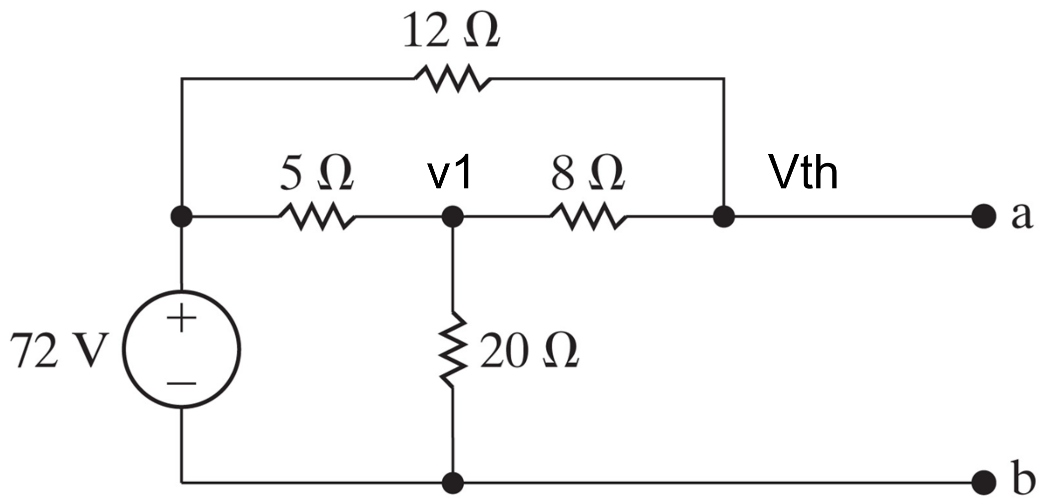
Once again -- a systematic way to obtain Vth and Rth for the Thevenin equivalent:

- **Obtain Vth** by calculating the voltage across the two specified terminals when no load is present (open circuit voltage) – sometimes you can do this by using source transforms.
- **Obtain Rth** by:
  1. Calculating the current that will flow between the specified terminals in a short circuit. Rth is obtained from Rth = Vth/Isc

     …….or

  2. If the circuit doesn't contain dependent sources, you can calculate the equivalent resistance between the specified terminals after all independent voltage sources are replaced with short circuits and all independent current sources are replaced with open circuits. This in effect is "deactivating" the independent sources and and equivalent resistance in this case is Rth, the Thevenin resistance. ….or
  3. If the circuit contains independent and dependent sources, Rth can be determined by deactivating independent sources, and adding an external source (vex)…then solve the circuit to determine the current iex supplied by the external source. Rth = vex/iex

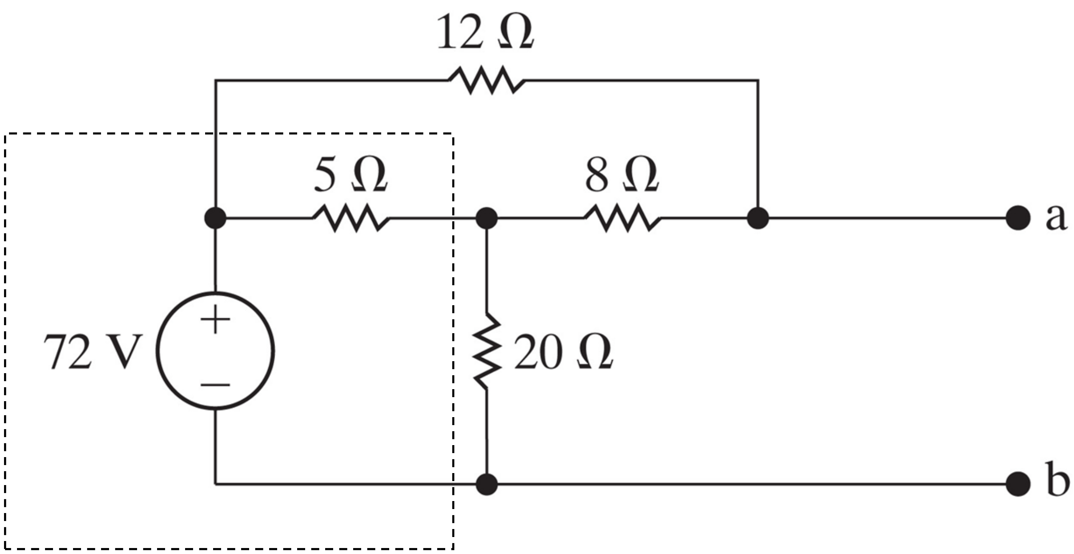
let’s consider this circuit:



We want to find is the Thevenin equivalent circuit as seen from terminals a and b. We can get Vth with a node voltage analysis for two nodes v1 and Vth (shown above) and we can get Rth by replacing the voltage source with a short circuit (deactivating it) and then solving for the series parallel resistor combinations (method #2).

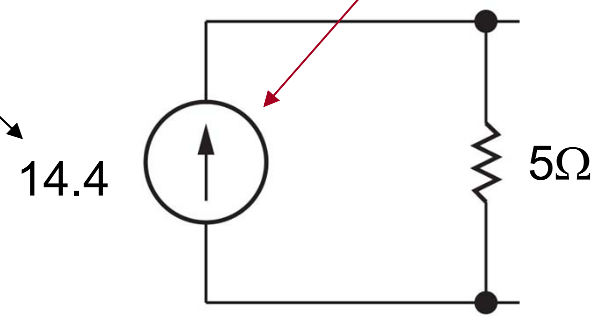
Could we simplify this circuit first with source transforms?

The simple answer is NO!



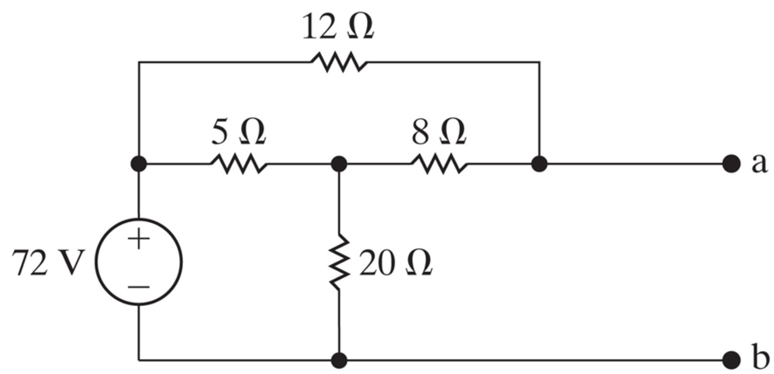
The current from this equivalent current source has to account for all of the current being supplied by the original voltage source – and this is only the current going through the 5Ω resistor – it doesn't account for current through the 12 Ω resistor

It's tempting to transform the part of the circuit contained in the box to:

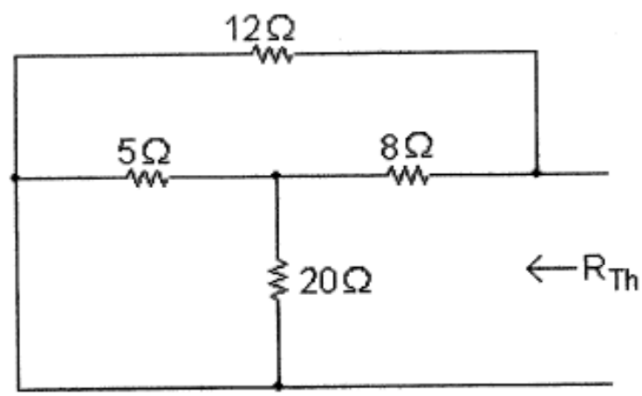


But This is Wrong!

Be careful with source transforms!

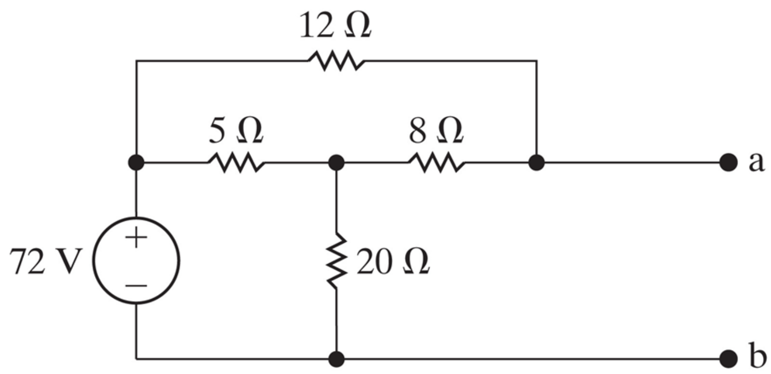


To find R_{Th}, replace the 72 V source with a short circuit:

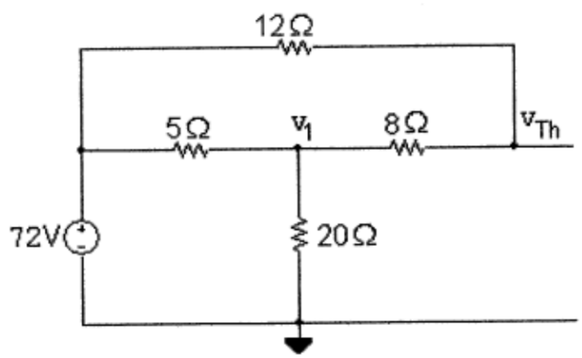


this is the method of 'deactivating the sources"

Note that the $5\,\Omega$ and $20\,\Omega$ resistors are in parallel, with an equivalent resistance of $5\|20 = 4\,\Omega$. The equivalent $4\,\Omega$ resistance is in series with the $8\,\Omega$ resistor for an equivalent resistance of $4 + 8 = 12\,\Omega$. Finally, the $12\,\Omega$ equivalent resistance is in parallel with the $12\,\Omega$ resistor, so $R_{\text{Th}} = 12\|12 = 6\,\Omega$.



Use node voltage analysis to find v_{Th}. Begin by redrawing the circuit and labeling the node voltages:



The node voltage equations are

$$\frac{v_1 - 72}{5} + \frac{v_1}{20} + \frac{v_1 - v_{\mathrm{Th}}}{8} = 0$$

$$\frac{v_{\mathrm{Th}} - v_1}{8} + \frac{v_{\mathrm{Th}} - 72}{12} = 0$$

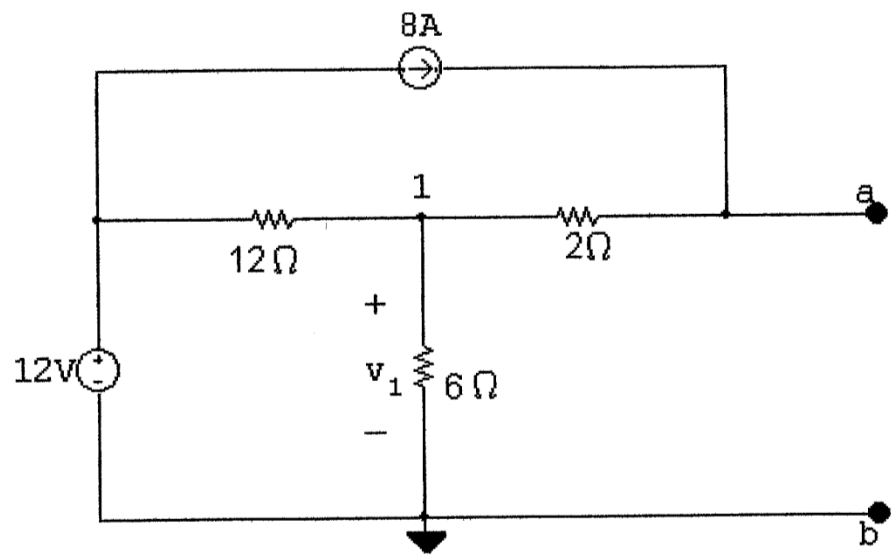
$$
\begin{aligned}
\frac{v_1 - 72}{5} + \frac{v_1}{20} + \frac{v_1 - v_{\mathrm{Th}}}{8} &= 0 \\
\frac{v_{\mathrm{Th}} - v_1}{8} + \frac{v_{\mathrm{Th}} - 72}{12} &= 0
\end{aligned}
$$

Place these equations in standard form:

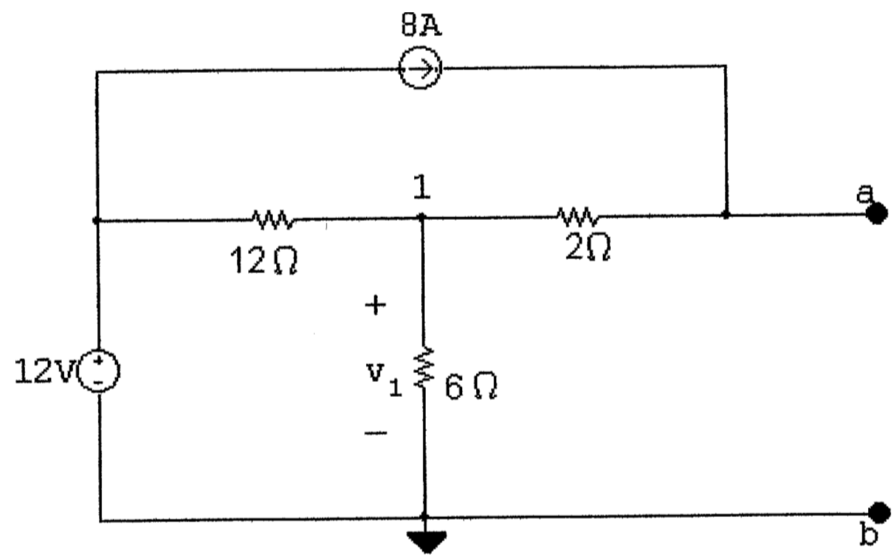
$$
\begin{aligned}
v_1\left(\frac{1}{5} + \frac{1}{20} + \frac{1}{8}\right) \quad + \quad v_{\mathrm{Th}}\left(-\frac{1}{8}\right) &= \frac{72}{5} \\
v_1\left(-\frac{1}{8}\right) \quad + \quad v_{\mathrm{Th}}\left(\frac{1}{8} + \frac{1}{12}\right) &= 6
\end{aligned}
$$

Solving, $v_1 = 60$ V and $v_{\mathrm{Th}} = 64.8$ V. Therefore, the Thévenin equivalent circuit is a 64.8 V source in series with a $6\,\Omega$ resistor.

Here's another circuit – find the Thevenin equivalent at terminals a and b. The circuit contains only dependent sources, so we can use method #1 or method #2.



Let's use method #2 (deactivating independent sources)

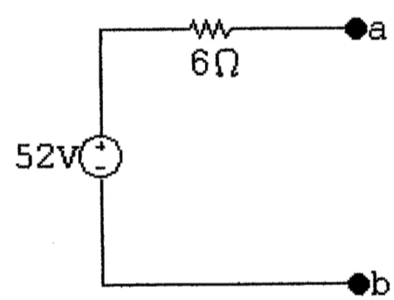


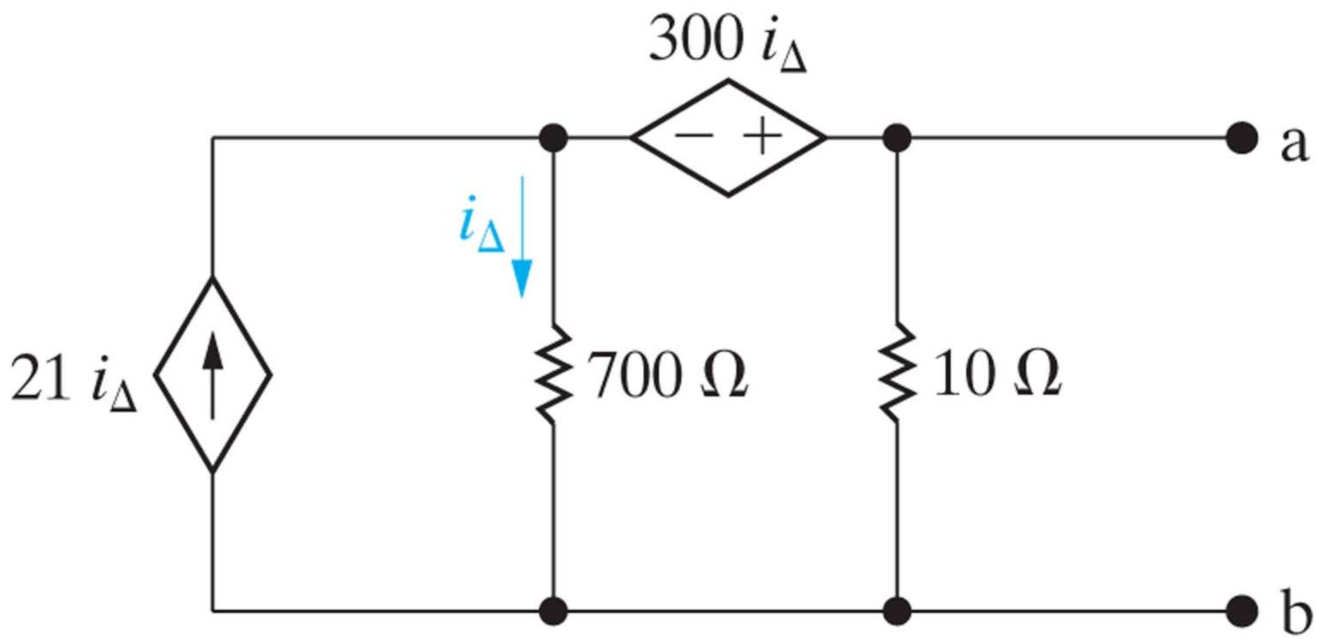
KCL at node 1 $\dfrac{v_1 - 12}{12} + \dfrac{v_1}{6} - 8 = 0$

$v_1 = 36\ \mathrm{V}$

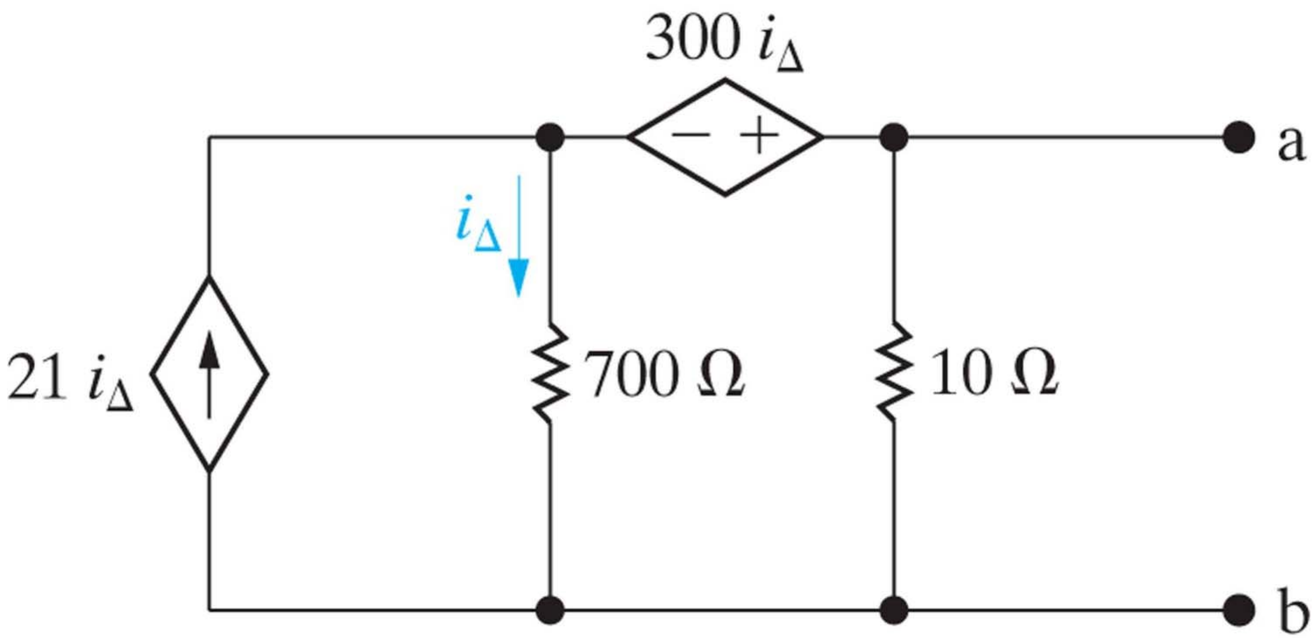
$v_{\mathrm{Th}} = v_1 + (2)(8) = 52\ \mathrm{V}$

$R_{\mathrm{Th}} = 2 + \dfrac{(12)(6)}{18} = 6\,\Omega$



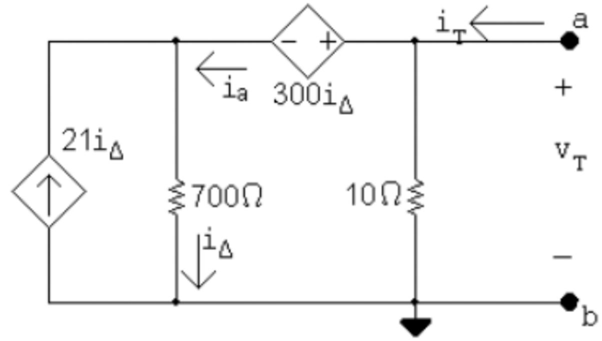


What is the Norton equivalent circuit with respect to terminals a and b? Notice that there are only dependent sources present. What does this immediately tell you about the Thevenin equivalent? What method should you use to determine Rth?



Vth = 0 …..can you see why? The current and voltage supplied by the two dependent sources can't "get started!" There are no independent sources in the circuit to cause any current to flow! As we saw earlier, this is a general rule – <u>if the circuit contains only dependent sources, Vth = 0!</u>

To get Rth, since there are dependent sources present, we'll have to use method #3 -- the external voltage or current method.



We apply an external current, i_T and calculate the resulting voltage, v_T

$$i_{\mathrm{T}} = \frac{v_{\mathrm{T}}}{10} + i_{\mathrm{a}}$$

$$i_{\mathrm{a}} = i_\Delta - 21 i_\Delta = -20 i_\Delta$$

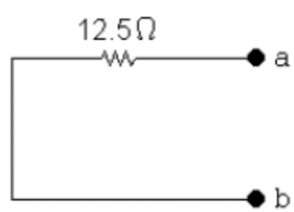
$$i_\Delta = \frac{v_{\mathrm{T}} - 300 i_\Delta}{700}, \qquad 1000 i_\Delta = v_{\mathrm{T}}$$

$$\therefore \quad i_{\mathrm{T}} = \frac{v_{\mathrm{T}}}{10} - 20\frac{v_{\mathrm{T}}}{1000} = 0.08 v_{\mathrm{T}}$$

$$\frac{v_{\mathrm{T}}}{i_{\mathrm{T}}} = 1/0.08 = 12.5\,\Omega$$

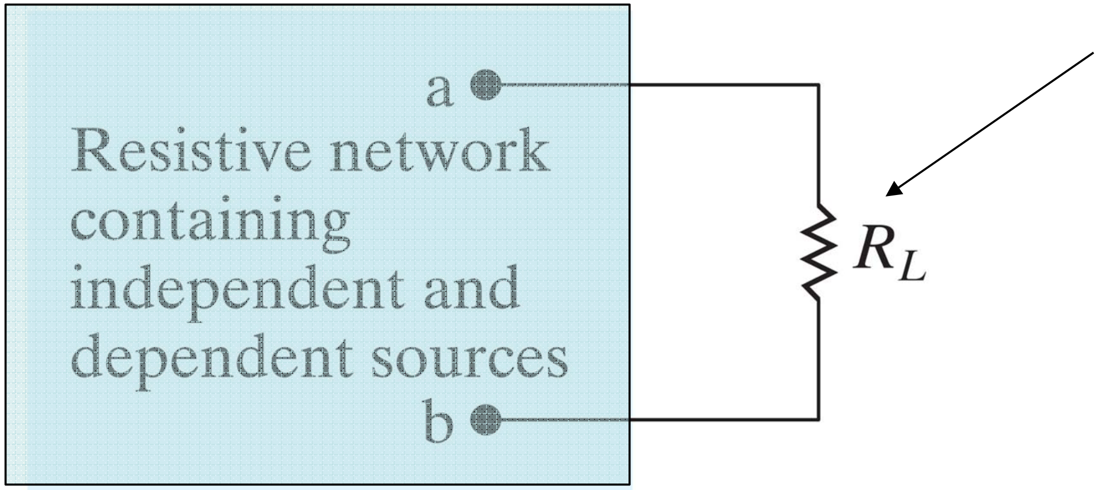
$$\therefore \quad R_{\mathrm{Th}} = 12.5\,\Omega$$

This is the Thevenin equivalent circuit



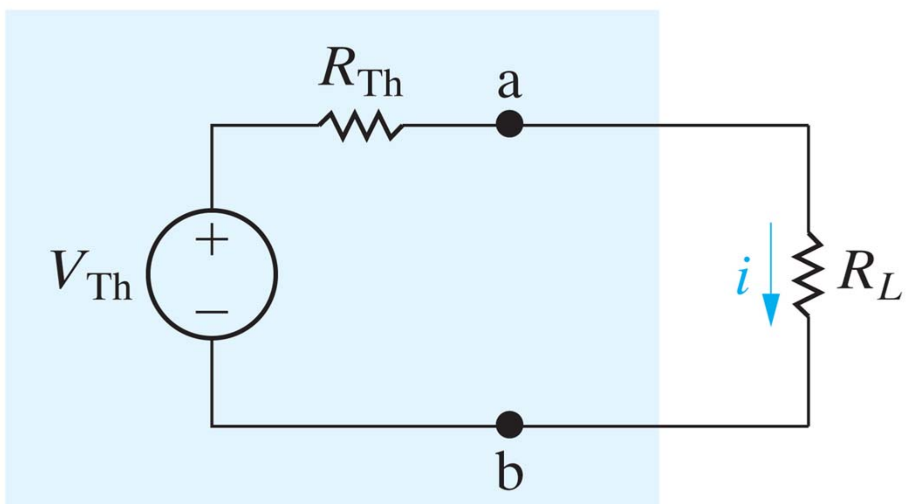
Maximum Power Transfer

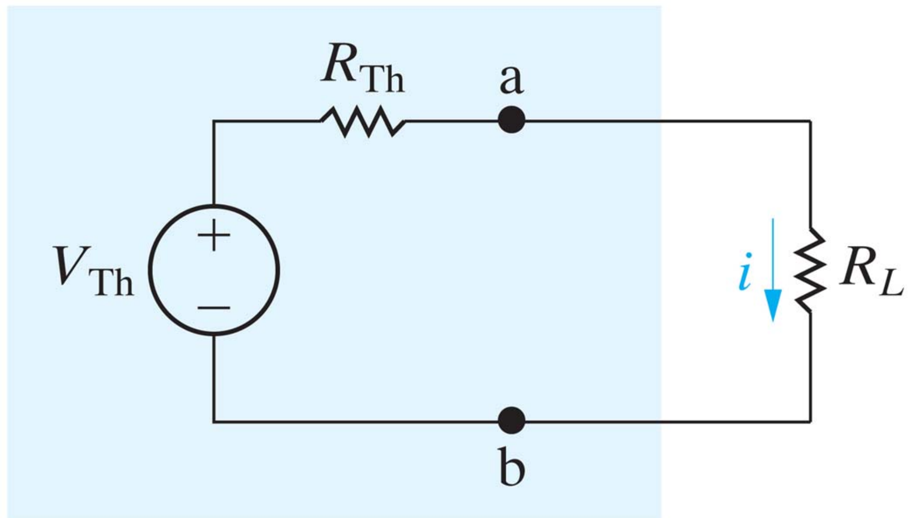
(the primary reason we want to determine the Thevenin equivalent)



We want to pick R_L so that maximum power will be transferred to it from the network

We know a resistive network can always be replaced by its Thevenin equivalent





The power dissipated in R_L : $p = i^2 R_L = (Vth / [Rth + R_L])^2 R_L$

To find out how the power delivered to R_L varies as we vary R_L, we can differentiate this expression with respect to R_L. The derivative will be zero at an extreme value of the function. We know that power is zero at both $R_L = 0$ and $R_L = \infty$, so, somewhere between 0 and ∞ is a value of R_L at which power dissipation is a maximum….and at this point the derivative will be zero.

$$p = i^2 R_L = \left(\frac{V_{th}}{R_{Th} + R_L} \right)^2 R_L$$

$$(1.) \quad p = \frac{V_{Th}^2}{\frac{R_{Th}^2}{R_L} + 2R_{Th} + R_L}$$

We could differentiate the expression in the parentheses with respect to R_L and set it equal to zero to get the value of RL for max power ..or just find the value of R_L that minimizes the denominator in eq. (1.)

To do this, we write:

$$\frac{d}{dR_L}\left(\frac{R_{Th}^2}{R_L} + 2R_{Th} + R_L \right) = -\frac{R_{Th}^2}{R_L^2} + 1 = 0$$

$$\text{and solving we get}: \quad \boxed{R_L = R_{Th}}$$

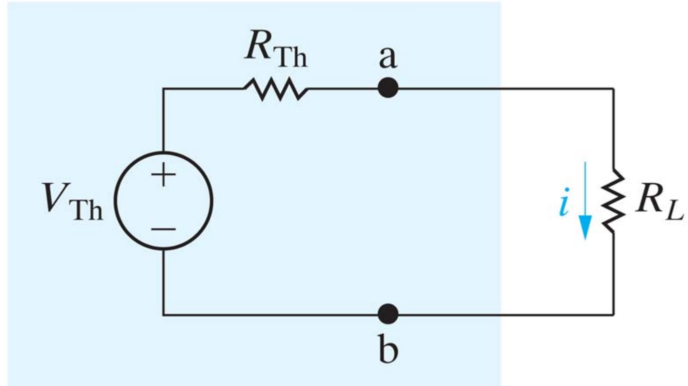
So…..power transfer is maximum when the load resistance, $R_L = R_{Th}$

and we can determine the value of the maximum power by substituting into:

$p = i2\ R_L = (Vth / [R_{Th} + R_L])^2\ R_L$

$\boxed{p_{max} = V_{Th}^2 / 4R_L}$

Does $R_L = R_{Thevenin}$ seem reasonable for max power transfer?

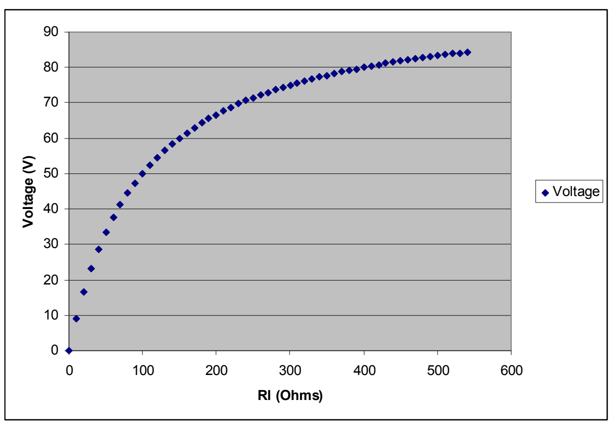


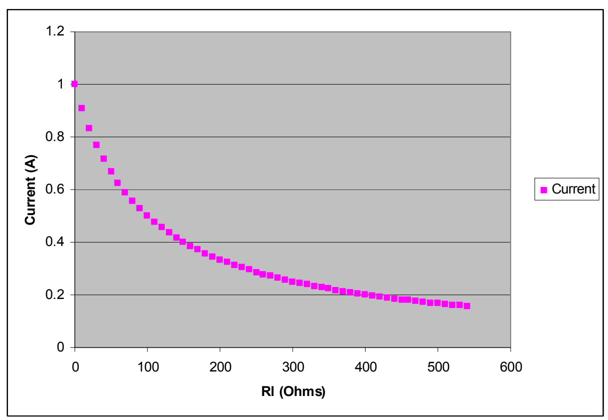
We can actually see that it is true graphically by plotting voltage, current, and power as we vary the load resistor. Let's consider, for example, Vth = 100V, and Rth = 100Ω

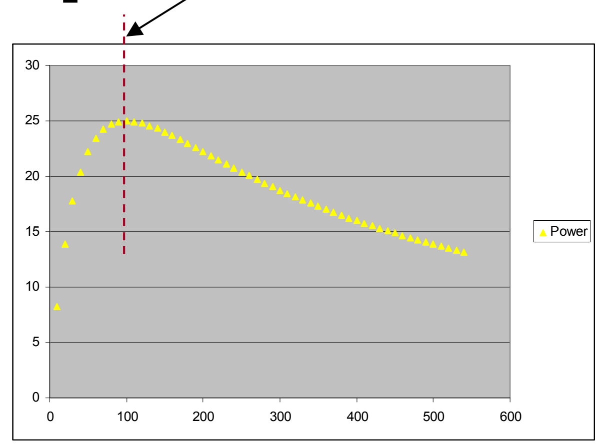
We calculate: $p_{max} = V_{Th}^2 / 4R_L = 25W$

Peak power transfer at 100 Ω

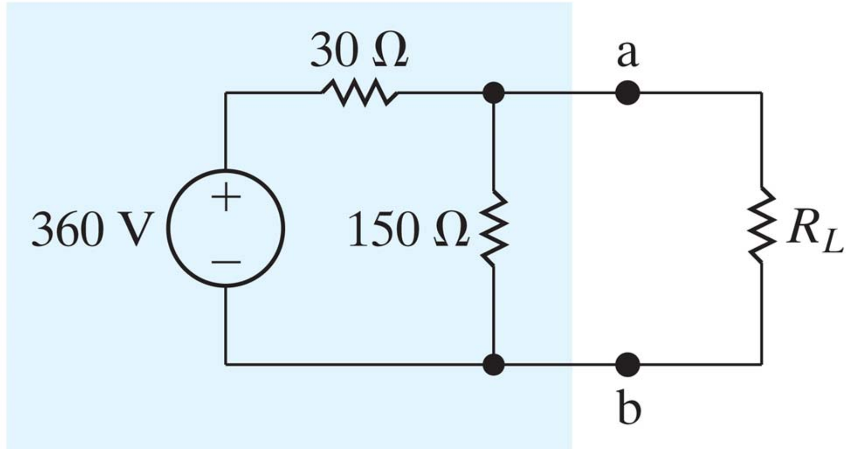
Now plotting current, voltage, and power as a function of R_L







Here's another example from the book



What is the value of R_L for maximum power transfer and what percentage of the power delivered by the voltage source is dissipated in R_L when it's chosen for max power transfer?

We can write down the Thevenin voltage at terminals a and b immediately. The 30Ω and 150Ω resistors make a voltage divider and the voltage across the 150Ω resistor is: 360 * (150/180) = 300V. We can get Rth using the trick of replacing the voltage source with a short circuit (method #2)…so Rth is 150 || 30 = 25Ω.

Maximum power transfer will occur with $R_L = R_{Th}$ =25Ω, and power delivered will be:

$P_{max} = Vth^2/4R_L = (300V)^2/(4 * 25) = 900W$ the total power delivered by the voltage source is:

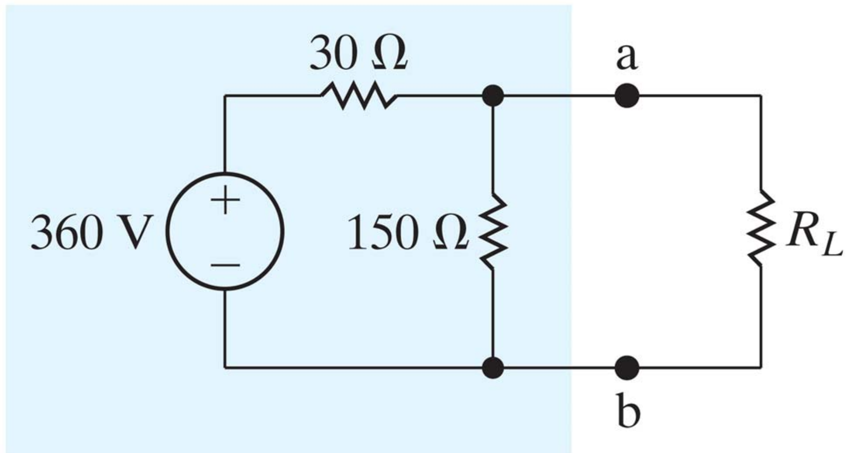
$(360)^2/[(25\Omega \parallel 150\Omega) + 30\Omega] = - 2520$ W …so the max power in R_L= 900/2520

= 35.7%

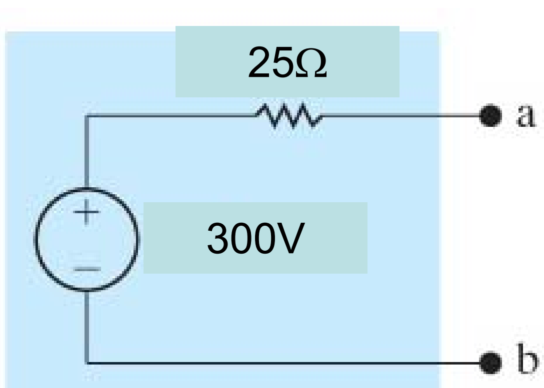
So the maximum power dissipated in the load resistor is only 35.7% of the power being delivered by the voltage source? How can this be?

If max power transfer occurs when Rl = Rth, how can the maximum power delivered ever be less than 50%?

The answer is that the maximum power delivered by a circuit will always be 50% of the power generated by the Thevenin equivalent voltage source! What we calculated was the power generated by the original voltage source …and some of its power is dissipated in the 150Ω resistor



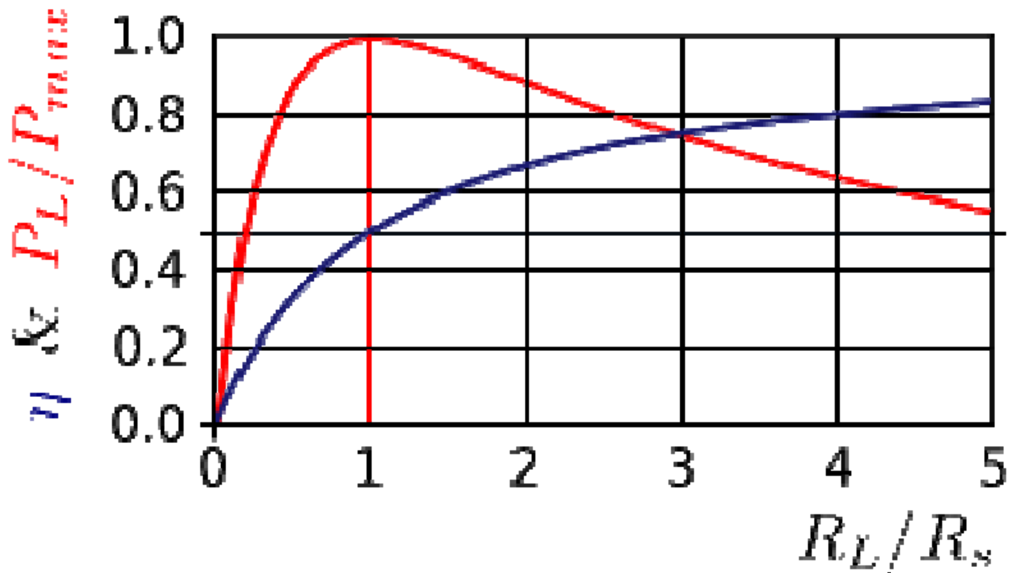
The Thevenin equivalent is:



We saw before that the maximum power delivered was 900W …but we can also see that the total power supplied by the Thevenin equivalent voltage source is $(300\text{ V})^2/50\Omega = 1800$ W

Notice: As we just saw……maximum power transmission happens when the load resistance matches the Thevenin resistance of a source …but this isn't the most efficient power transfer! Here is a plot of the power, P_L, dissipated in a load resistor, R_L, as a function of the ratio of R_L to the source (Thevenin) resistance, R_S. We have also plotted the power transfer efficiency. Maximum efficiency of power transfer, defined as power dissipated in the load over the power delivered by the source, is given by:

$$\eta = \frac{R_L}{R_L + R_S}$$

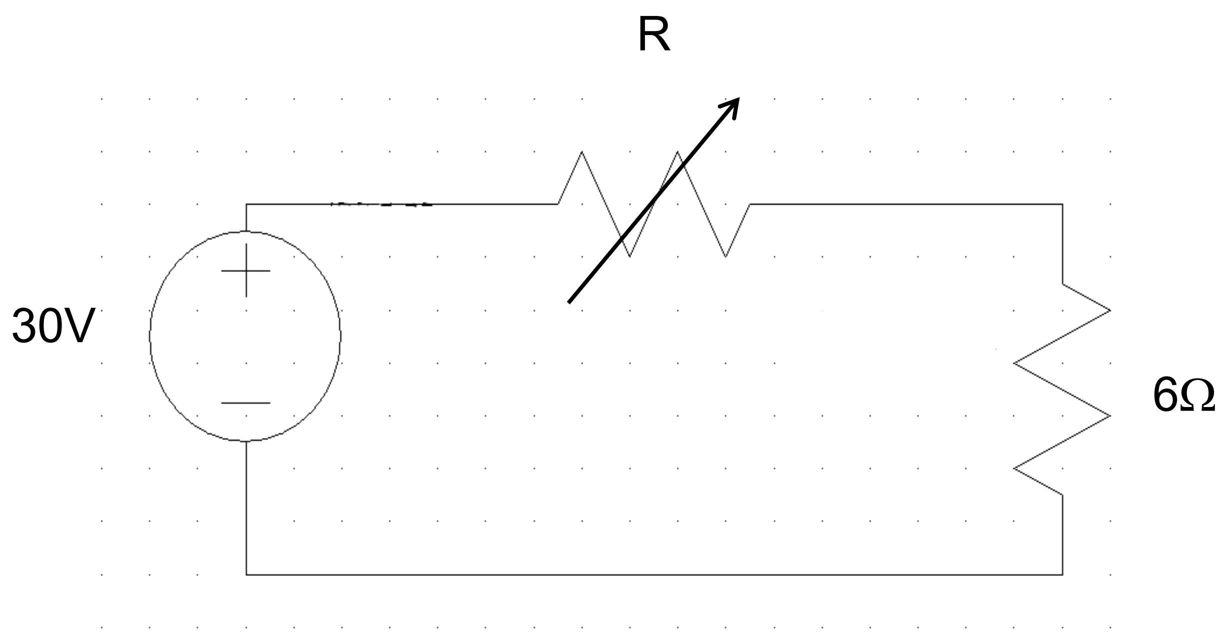


Most efficient power transfer occurs when the load resistance is very large or the source resistance is very small. Both of these conditions reduce the maximum power transferred …but they increase the efficiency of power transfer!

Which is more important? Maximum power transfer …or maximum power transfer efficiency?

It depends on what you are trying to do?

OK…here's one final (simple looking) maximum power transfer example:



What value of the variable resistor, R, will permit maximum power transfer to the 6Ω load resistor? What will the maximum power value be?

Be careful not to answer too quickly!

Well the maximum current from the voltage source will certainly flow when R = 0 …and since $p = I^2R_L$ causing maximum power to be dissipated in the $R_L = 6\ \Omega$ load.

This maximum power will be (30V)(30V/6Ω) = 150W. Any nonzero value of R will reduce the current delivered ….reducing the transferred power!

Did you think the best value for R would be 6Ω? Why did you think that? If the value of the series resistor were given as 6Ω, then this would be the optimum value for the load resistor …but that's not what is happening here! The series resistor (which is the Thevenin resistance) is the adjustable one …and the load resistance is fixed.

In the next class we'll have a review of everything we've covered so far in ELEN 50 .

Then there will be the first mid-term exam.