ELEN 50 Class 10 – Solving Matrix Equations and Node Voltage
Method: More Examples

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First, a little discussion on methods of solving simple matrix equations……………

The fundamental laws used for circuit analysis (KVL and KCL) result in systems of linear equations that we have to solve. Matrix algebra is a powerful method for solving systems of simultaneous linear equations …as you probably learned in high school.

When you first started solving simultaneous linear equations you probably used the method of substitution:

For example -

$$2x - 3y = -2$$
$$4x + y = 24$$

You approach this by first solving for y in terms of x in the second equation: $y = 24 - 4x$

And then substitute into the first equation:

$$2x - 3(24 - 4x) = -2$$
$$2x - 72 + 12x + 2 = 0$$
$$14x = 70$$
$$x = 5$$
$$y = 24 - 20 = 4$$

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To solve this problem with matrix techniques we can write the two algebraic equations:

$$2x - 3y = -2$$

$$4x + y = 24$$

as a single matrix equation – of course using the rules for multiplying a matrix and a vector:

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 24 \end{pmatrix} \qquad \mathbf{Ax = B}$$

Here, **A** is a 2x2 matrix and **x** and **B** are column vectors. We can solve the matrix equation using any one of a variety of techniques. The technique you probably learned in high school was Cramer's Method. Cramer's Method says that for a system of *n* linear equations for *n* unknowns, represented in matrix multiplication form as follows:

$$\mathbf{Ax = B}$$

where the *n* by *n* matrix, **A** has a nonzero determinant, and the vector **x** is the column vector of the variables.

The theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

(do you remember from high school how to take determinants?)

Where **A**$_i$ is the matrix formed by replacing the *i*th column of **A** by the column vector, **b**.

In the problem we have been considering:

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 24 \end{pmatrix}$$

$$\det A_1 = \begin{vmatrix} -2 & -3 \\ 24 & 1 \end{vmatrix} = (-2) - (-72) = 70$$

$$\det A_2 = \begin{vmatrix} 2 & -2 \\ 4 & 24 \end{vmatrix} = (48) - (-8) = 56$$

$$\det A = \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = (2) - (-12) = 14$$

So, Cramer's Method gives:
x= 70/14=5 and y=56/14 = 4 as we saw before. The matrix solution doesn't require as much algebraic manipulation as the substitution method.

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If we have a system of three equations in three unknowns the matrix method is clearly easier than the substitution method. Cramer's Method for solving a 3x3 matrix is still workable….but this method becomes rapidly unworkable for bigger matrices.

For a system of three equations in three unknowns, for example:

Solve the following system of equations using Cramer's rule:

$$x_1 - x_2 + x_3 = 3$$

$$2x_1 + x_2 - x_3 = 0$$

$$3x_1 + 2x_2 + 2x_3 = 15$$

Convert the system of equations into matrix form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \\ 15 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{bmatrix}, \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \{b\} = \begin{Bmatrix} 3 \\ 0 \\ 15 \end{Bmatrix}$$

Define matrices $[A_1]$, $[A_2]$, and $[A_3]$ as : like we did for the 2X2 matrix

$$[A_1] = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & -1 \\ 15 & 2 & 2 \end{bmatrix}, [A_2] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 3 & 15 & 2 \end{bmatrix},$$

$$[A_3] = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 15 \end{bmatrix}$$

Remember – you can calculate the determinant of a 3x3 matrix using:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

so for:

$$[A] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{bmatrix}, \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \{b\} = \begin{Bmatrix} 3 \\ 0 \\ 15 \end{Bmatrix}$$

determinants of $[A]$, $[A_1]$, $[A_2]$, and $[A_3]$:

$$||[A]|| = 12$$
$$||[A_1]|| = 12$$
$$||[A_2]|| = 24$$
$$||[A_3]|| = 48$$

Unknowns x_1, x_2, and x_3 are then calculated as

$$x_1 = \frac{|[A_1]|}{|[A]|} = \frac{12}{12} = 1, \quad x_2 = \frac{|[A_2]|}{|[A]|} = \frac{24}{12} = 2, \quad x_3 = \frac{|[A_3]|}{|[A]|} = \frac{48}{12} = 4$$

For the matrix equation:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \\ 15 \end{Bmatrix}$$

How about using Cramer's Method for larger matrices? This is really <u>not a good idea!</u> This approach quickly becomes computationally unworkable!

As we saw – the Cramer's Method is straightforward:

- For a system of n equations, Cramer's rule requires that you calculate $n + 1$ determinants of $n \times n$ matrices.
- In the general case for a system of equations $[A]\{x\} = \{b\}$, the matrix $[A_i]$ is obtained by replacing the ith column of the original $[A]$ matrix with the contents of the $\{b\}$ vector.
- Each unknown variable x_i is found by dividing the determinant $|[A_i]|$ by the determinant of the original coefficient matrix $|[A]|$.

Cramer's Method – Advantages and Disadvantages

Advantages

- Easy to remember steps

Disadvantages

- Computationally intensive compared to other methods: the most efficient ways of calculating the determinant of an $n \times n$ matrix require $(n - 1)(n!)$ operations. So Cramer's rule would require $(n - 1)((n + 1)!)$ total operations. For 8 equations, that works out to $7(9!) = 2540160$ operations, or around 700 hours if you can perform one operation per second.
- Roundoff error may become significant on large problems with non-integer coefficients.

It's easy to imagine having to solve a network with 8 nodes or loops…which would result in an 8x8 matrix equation!

A plot of (n-1)*[(n+1)!] versus n

Plot with y-axis labeled "Log Number of Operations to Use Cramer's Rule" (0 to 9) and x-axis labeled "Number of Rows and Collumns" (0 to 10).
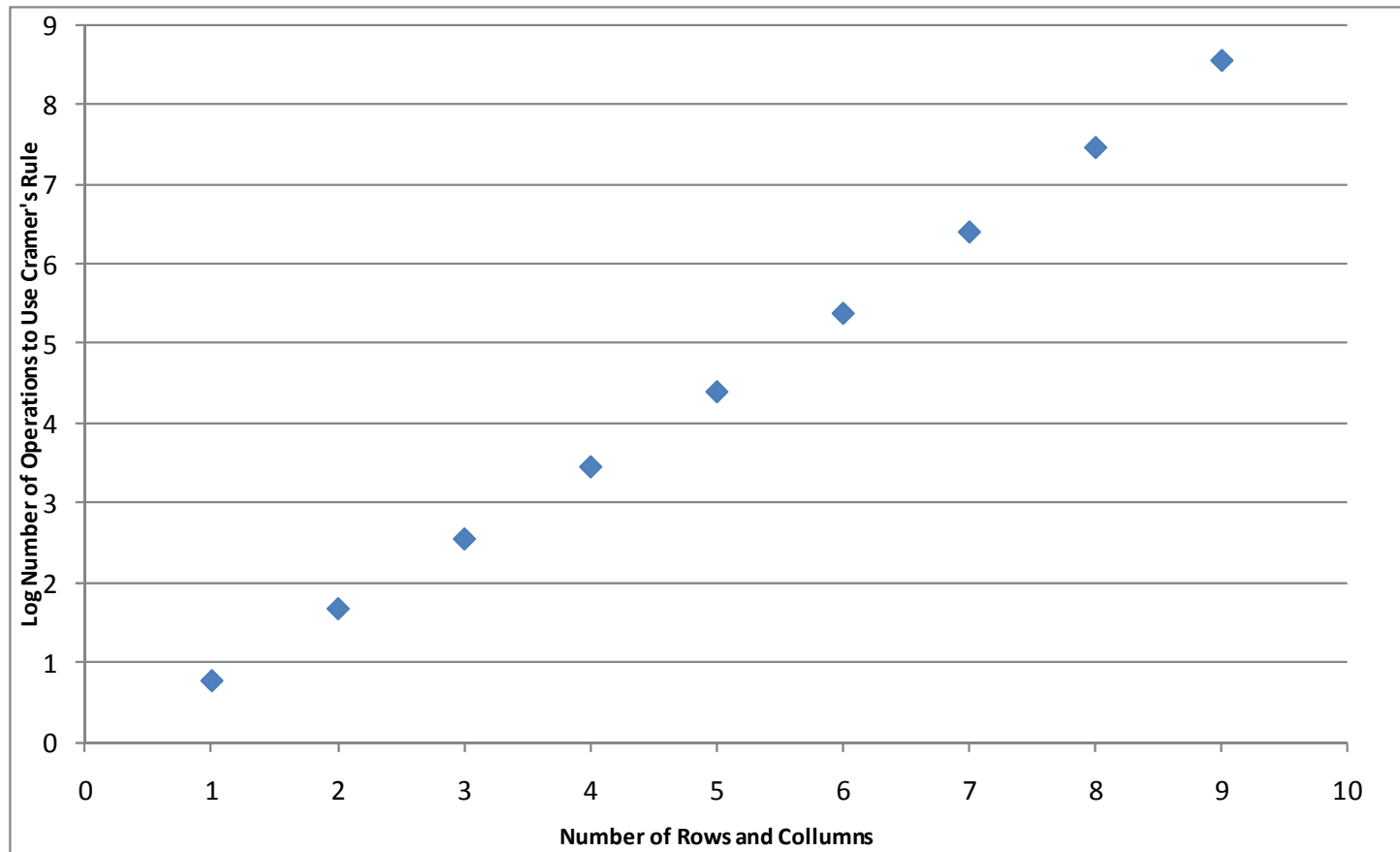
A much more computationally efficient method for (manually) solving matrix equations is the process of Gaussian elimination (another technique, known as Gauss-Jordan elimination is similar). The Gaussian elimination technique is also known simply as "row reduction" and it was named after the mathematician, Carl Friedrich Gauss – although the technique had actually been known prior to Gauss's using it.

The goal of Gaussian elimination is to perform elementary row operations on an augmented matrix so as to transform it into "right echelon form" or "upper triangular matrix." These elementary row operations don't change the value of the matrix equations represented by the augmented matrix, and once the right echelon form has been achieved, the system of equations represented by the matrix is easily solved by back substitution.

$$\left[\mathbf{C}^5\right] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Augmented Matrix in right echelon form

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Elementary Row Operations

- Type 1: Divide a row by a number
- Type 2: Multiply a row by a number and add it to another row
- Type 3: Interchange two rows

(why do these operations preserve the linear system ?)

The process of Gaussian elimination can always be represented as a sequence of these elementary row operations. In some cases, only Type 2 operations need to be performed.

Solving **Ax=B** by Gaussian Elimination

The goal of Gaussian Elimination is to transform a matrix into <u>right echelon form</u> through the use of the elementary row operations described before. At that point, the variables in the x vector may be obtained by back substitution.

Restating the elementary row operations …we can do the following to any row (equation) in the matrix:

- Any equation may be multiplied by a nonzero scalar.
- Any equation may be added to (or subtracted from) another equation.
- The positions of any two equations in the system may be swapped.

Here's an example:

An example:

Solve the following system of equations with Gauss elimination:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 6 \\ 15 \end{Bmatrix}$$

We first write the <u>augmented matrix, C_0</u> – obtained by adding a new column to the original matrix, **A**, composed of the constant vector, **B**.

The <u>augmented matrix</u> is, therefore:

$$[C_0] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 6 \\ 3 & 2 & 2 & 15 \end{bmatrix}$$

Two augmented matrices are <u>row equivalent</u> if they are related by a sequence of elementary row operations.

Let's eliminate the "4" in the beginning of row 2.

We can do this by multiplying all the elements of row 1 by 2 and subtracting this from row 2.

$$[\mathbf{C}_0] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 6 \\ 3 & 2 & 2 & 15 \end{bmatrix}$$

Row 2 will then be: $\begin{bmatrix} 4-(2)(2) & 3-(2)(-1) & -1-(2)(1) & 6-(2)(4) \end{bmatrix}$

$$= \begin{bmatrix} 0 & 5 & -3 & -2 \end{bmatrix}$$

This augmented matrix is equivalent to the original one.

So: $[\mathbf{C}] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 3 & 2 & 2 & 15 \end{bmatrix}$

Now we can eliminate the 3 in the beginning of row 3

…..by multiplying all the elements of row 1 by 1.5 and subtracting this from the elements of row 3.

$$[\mathbf{C}] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 3 & 2 & 2 & 15 \end{bmatrix}$$

Row 3 will then be:

$$[3-(1.5)(2) \quad 2-(1.5)(-1) \quad 2-(1.5)(1) \quad 15-(1.5)(4)]$$

$$= [0 \quad 3.5 \quad 0.5 \quad 9]$$

So: $[\mathbf{C}^1] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 0 & 3.5 & 0.5 & 9 \end{bmatrix}$

Now we can eliminate the 3.5 in column 2 of the third row

We can do this by multiplying all the elements of row 2 by 0.7 and subtracting this from the elements of row 3.

$$[\mathbf{C}^1] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 0 & 3.5 & 0.5 & 9 \end{bmatrix}$$

Row 3 will then be:

$$[0-(0.7)(0) \quad 3.5-(0.7)(5) \quad 0.5-(0.7)(-3) \quad 9-(0.7)(-2)]$$

$$= [0 \quad 0 \quad 2.6 \quad 10.4]$$

So: $[\mathbf{C}^2] = \begin{bmatrix} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 0 & 0 & 2.6 & 10.4 \end{bmatrix}$

And we're done. We've converted the original matrix into an upper right echelon matrix

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The original matrix equation:

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 6 \\ 15 \end{Bmatrix}$$

has become

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 2.6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ -2 \\ 10.4 \end{Bmatrix}$$

…and because we got this matrix equation by elementary row operations, it has the same roots as the original.

And we can now get the vector, **x,** by back substitution. Directly, $x_3 = 10.4/2.6 = 4$. Then $5x_2 - 3x_3 = -2$ from the second row requires that $x_2 = 2$. Finally, the first row equation, $2x_1 - x_2 + x_3 = 4$ requires $x_1 = 1$.

So:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 4 \end{Bmatrix}$$

We could have actually solved this system with comparable ease using Cramer's rule, of course…but the Gaussian elimination would have been far easier with a 4x4 matrix or larger.

Of course, far and away, the easiest way to solve this 3x3 matrix equation would have been with MATLAB using the \ operator! If you have access to MATLAB.

Here's the MATLAB solution: for **AX=B**

A=[2,-1,1;4,3,-1;3,2,2]

A =

$$\begin{array}{rrr} 2 & -1 & 1 \\ 4 & 3 & -1 \\ 3 & 2 & 2 \end{array}$$

>> B=[4;6;15]

B =

$$\begin{array}{r} 4 \\ 6 \\ 15 \end{array}$$

>> X=A\B

X =

$$\begin{array}{r} 1 \\ 2 \\ 4 \end{array}$$

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More Review of Node Voltage Analysis Method plus topics from last time:

- Node Voltage Method with dependent and independent sources

- Supernodes

Node Voltage Method :

- Identify all extraordinary nodes, set one as the reference (ground) node, and assign node voltages (v1, v2, v3, etc.) to the $n_{ex} - 1$ remaining nodes.

- At each of the n_{ex}-1 nodes, write the KCL equation requiring the sum of all currents <u>leaving</u> the node to be zero.

- Solve the n_{ex}-1 independent simultaneous equations to determine the unknown node voltages.

In the last class we introduced the concepts of supernodes and quasi-supernodes
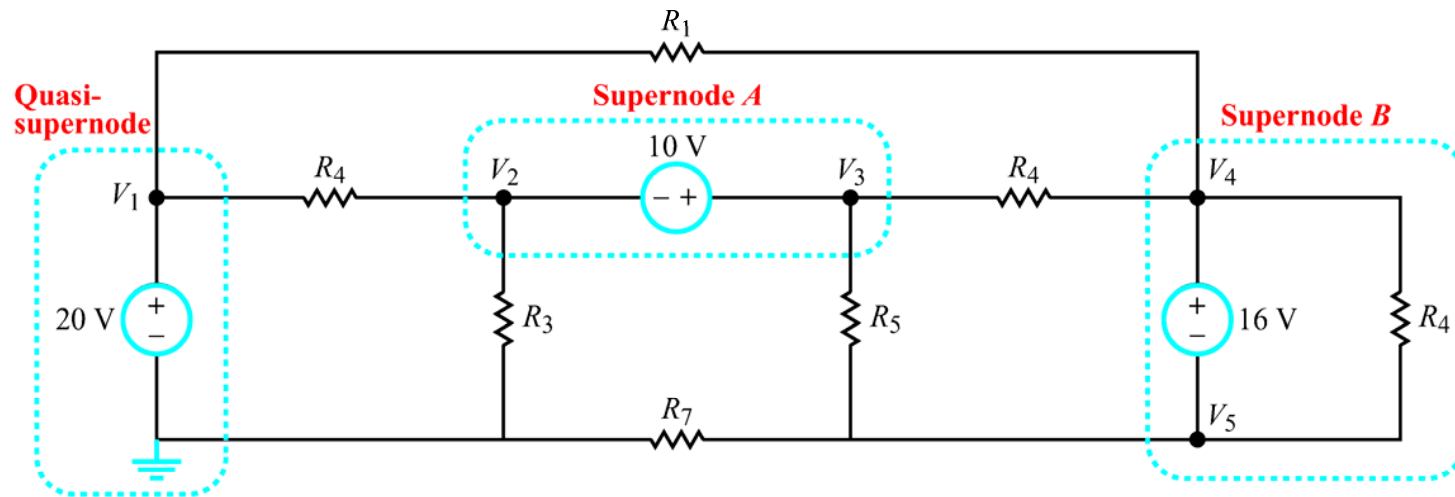
Independent Sources: -- we want to know v_0
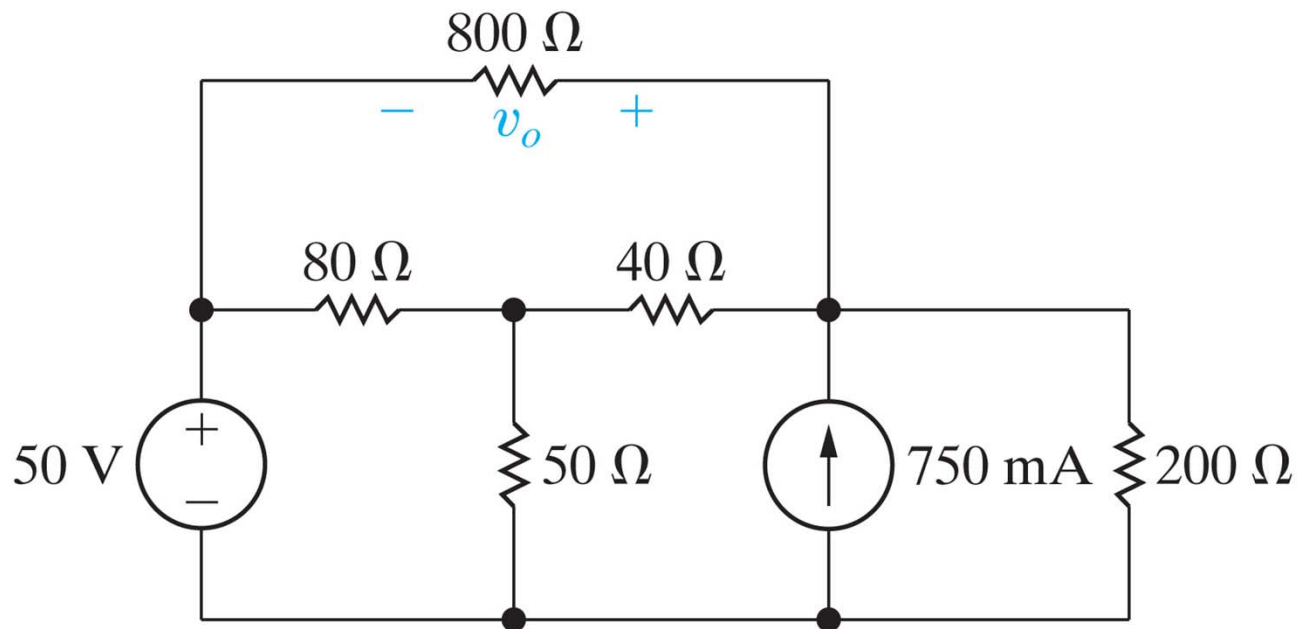
How many essential nodes are there? Which node should we choose for the reference node? How many KCL equations will we need to write?
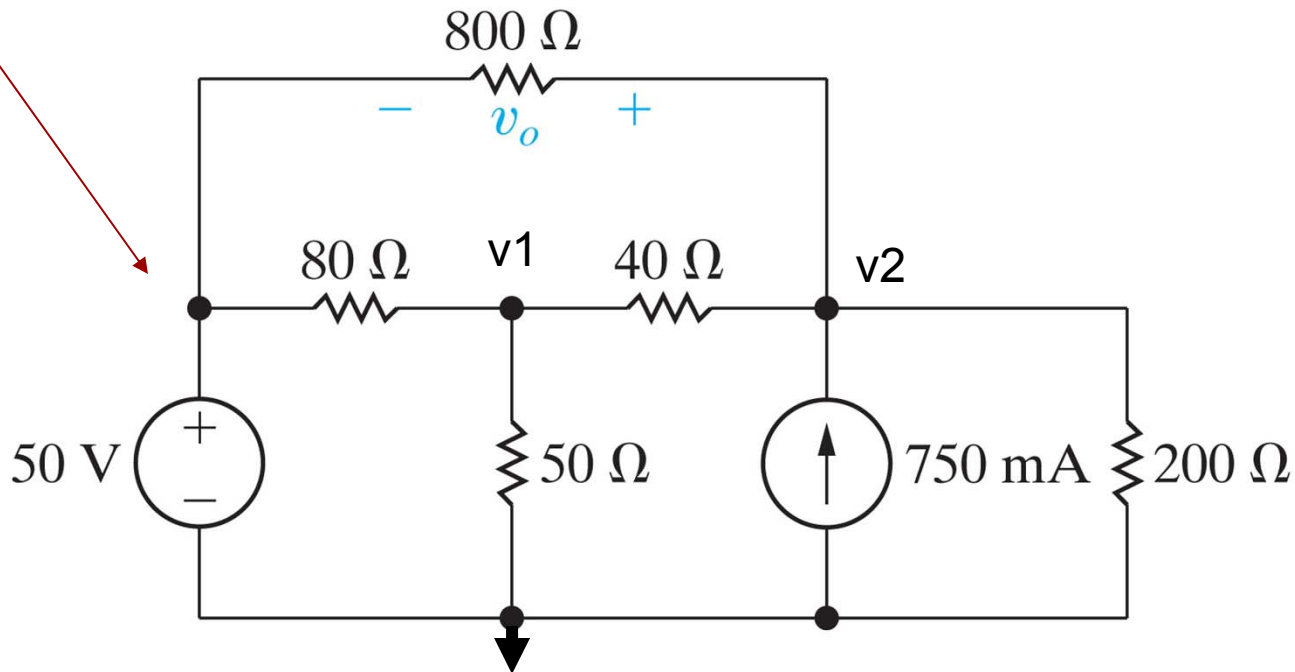
Yes…but, actually …it's a quasi-supernode because one of the two nodes is the reference node. The voltage at this node is, therefore, already known, and we don't need to write a node voltage equation for it.
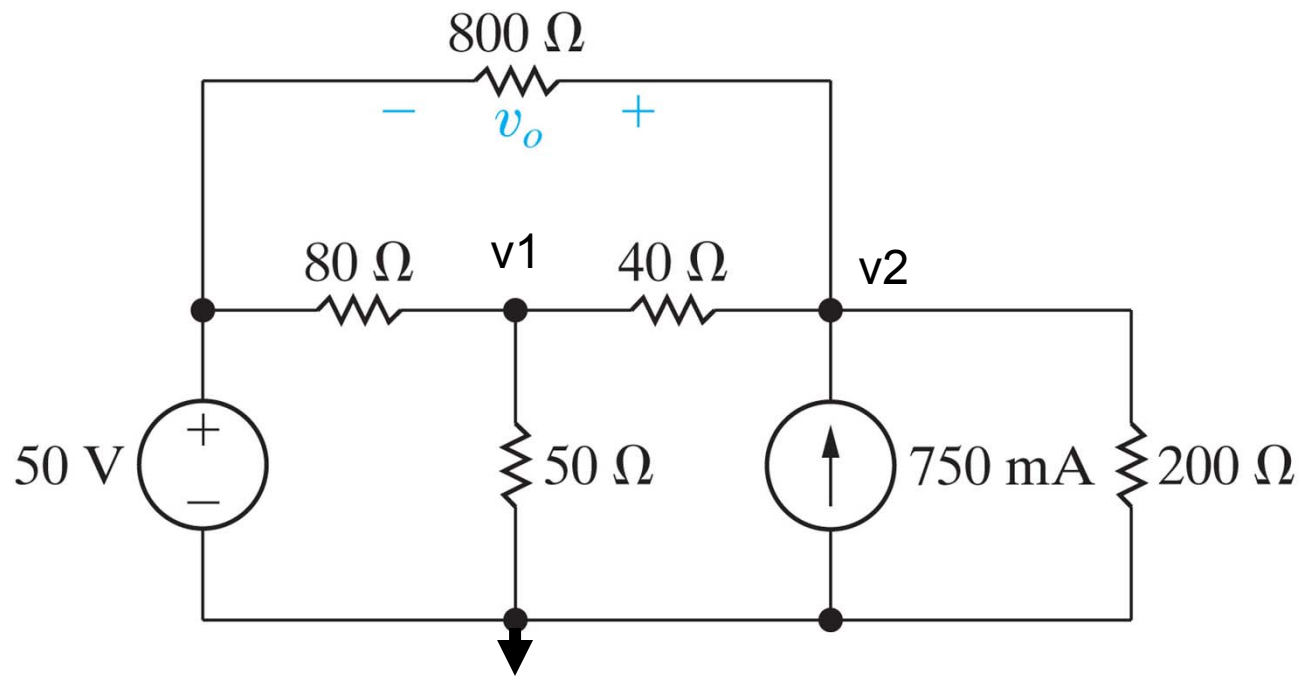
Write the node voltage equation for v1 and v2
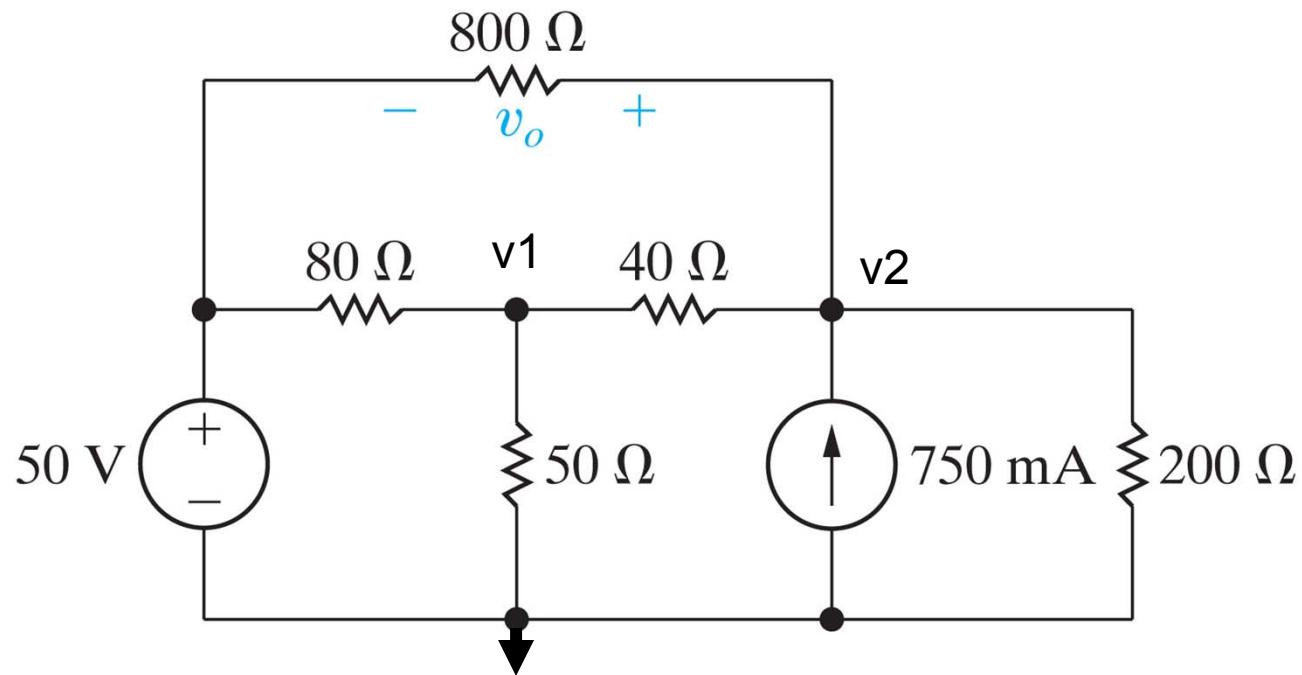
$(v1 -50)/80 + v1/50 + (v1-v2)/40 = 0$ at node v1

$v2/200 - 0.75 + (v2-v1)/40 + (v2-50)/800 = 0$ at node v2

Putting these two equations in standard form:

$$v_1 \left(\frac{1}{80} + \frac{1}{50} + \frac{1}{40} \right) \quad + \quad v_2 \left(-\frac{1}{40} \right) \quad = \quad \frac{50}{80}$$

$$v_1 \left(-\frac{1}{40} \right) \quad + \quad v_2 \left(\frac{1}{40} + \frac{1}{200} + \frac{1}{800} \right) \quad = \quad 0.75 + \frac{50}{800}$$

$$\begin{bmatrix} \left(\dfrac{1}{80} + \dfrac{1}{50} + \dfrac{1}{40} \right) & \left(-\dfrac{1}{40} \right) \\ \left(-\dfrac{1}{40} \right) & \left(\dfrac{1}{40} + \dfrac{1}{200} + \dfrac{1}{800} \right) \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{50}{80} \right) \\ \left(0.75 + \dfrac{50}{800} \right) \end{bmatrix}$$

We can solve this directly in MATLAB

\>\> C=[(1/80+1/50+1/40), (-1/40);(-1/40), (1/40+1/200+1/800)]

C =

  0.0575 -0.0250
 -0.0250   0.0313

\>\> S=[(50/80); (0.75+50/800)]

S =

  0.6250
  0.8125

Using the MATLAB "backslash" operation:

V = C\S

\>\> V=C\S

V =

  34.0000
  53.2000

So:

v1 = 34V;

v2 = 53.2V

So …… v0 = v2 – 50 = 3.2V
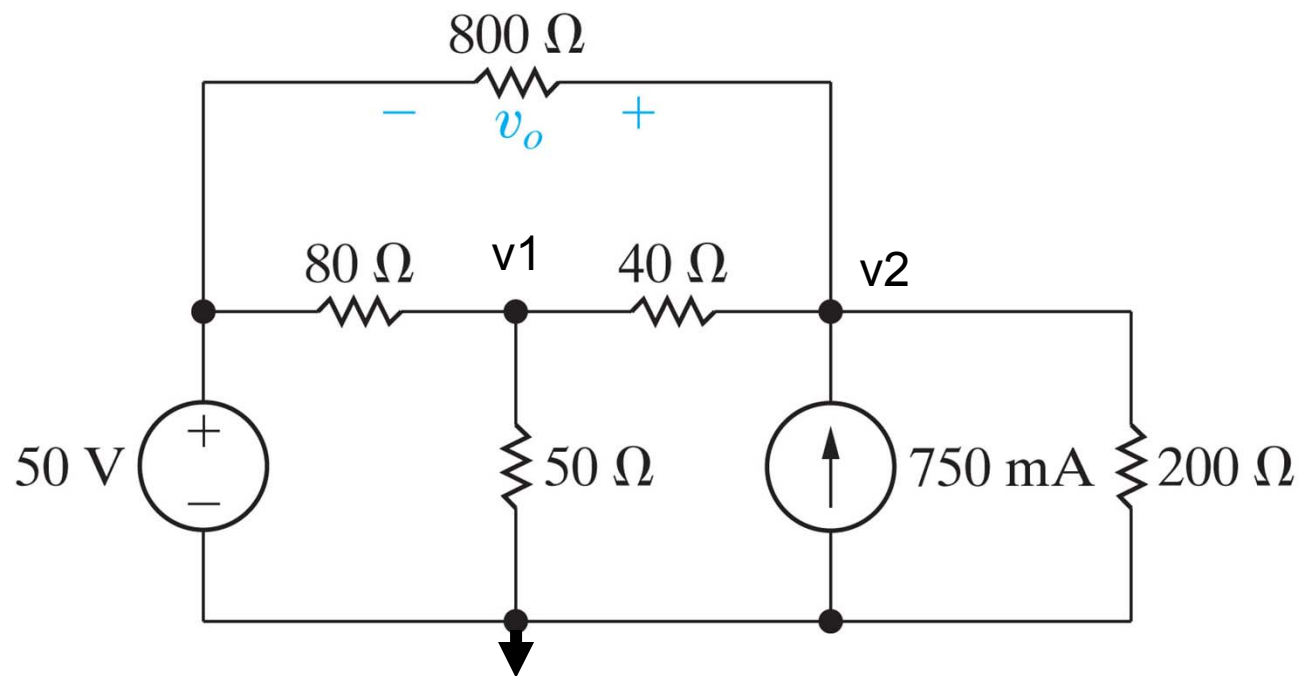
Dependent Sources: -- we want to know v_0
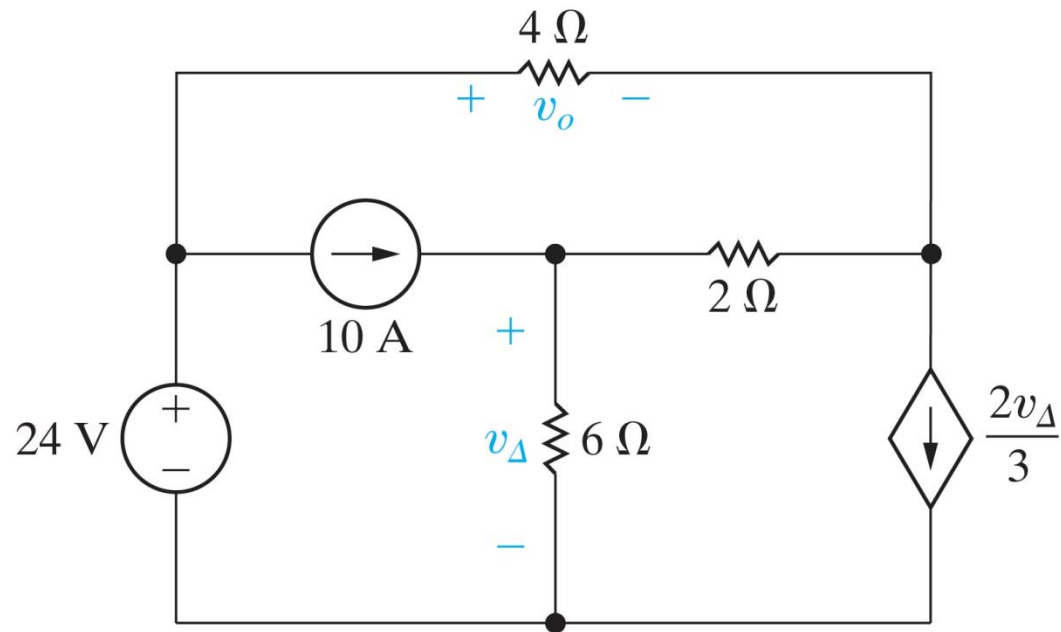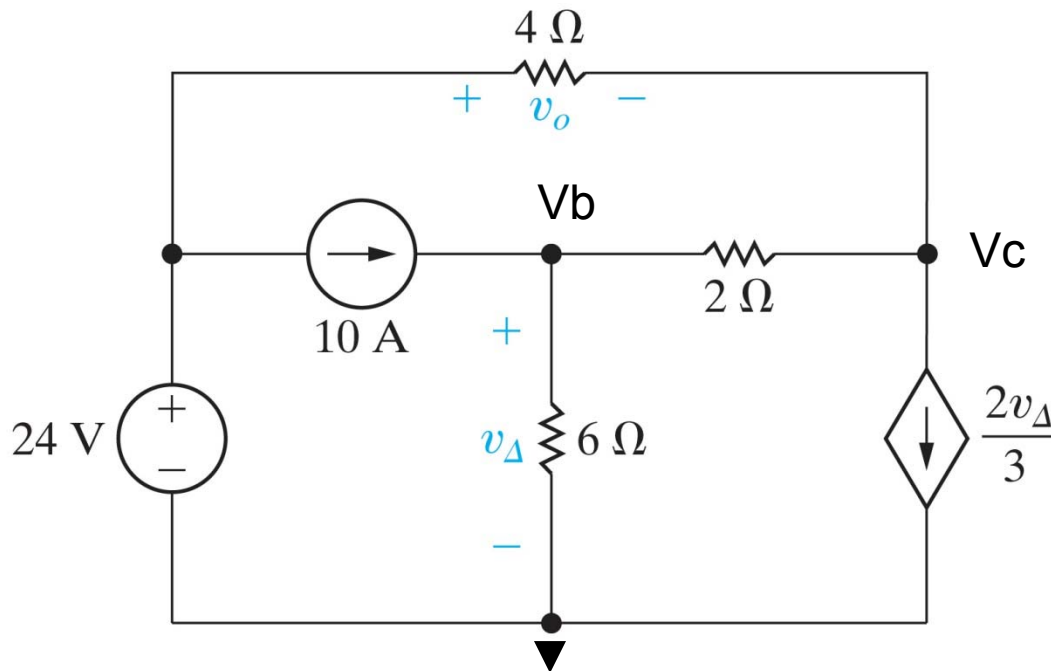
We label the essential nodes and chose a reference node

We know
the voltage
at this
quasi-
supernode
by
inspection
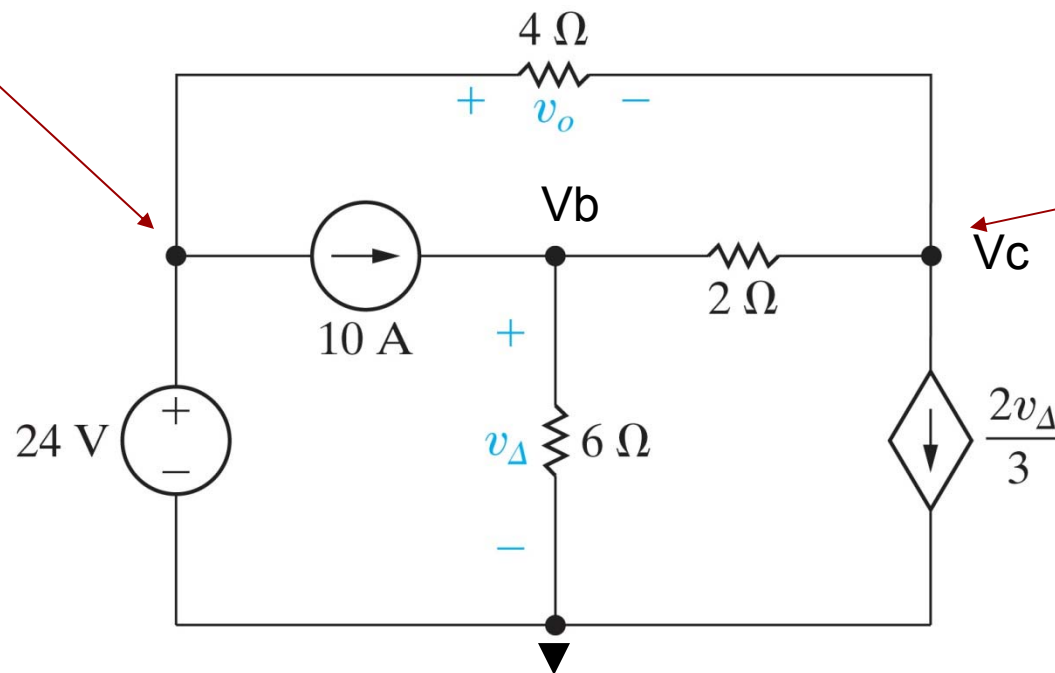
Is this a
quasi-
supernode
also?

$4\,\Omega$

$+$ v_o $-$

Vb

$10\,A$

$+$

$2\,\Omega$

Vc

$24\,V$

$+$
$-$

v_Δ $6\,\Omega$

$-$

$\dfrac{2v_\Delta}{3}$

It's not a quasi-supernode because it's a current source!
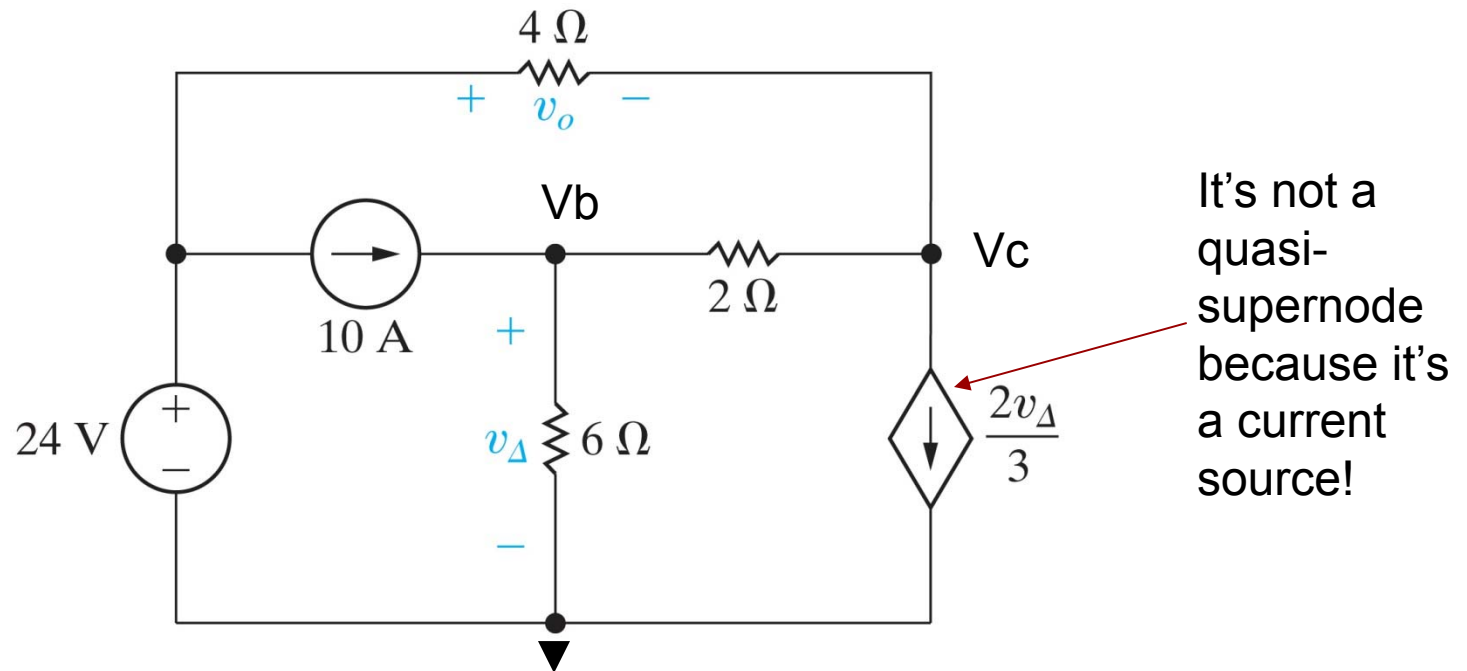
Write the node voltage equations at the two remaining essential nodes -- node b and node c

$$-10 + \frac{v_b}{6} + \frac{v_b - v_c}{2} = 0 \qquad \text{at node b}$$

$$\frac{2v_\Delta}{3} + \frac{v_c - v_b}{2} + \frac{v_c - 24}{4} = 0 \qquad \text{at node c}$$

We need another equation involving v_Δ because we've got two equations in three unknowns – <u>we always need constraint equations when using the node voltage method with dependent sources</u> -- what is the third equation involving v_Δ, v_c and v_b?

Well …the constraint equation here is trivial: $v_\Delta = V_b$
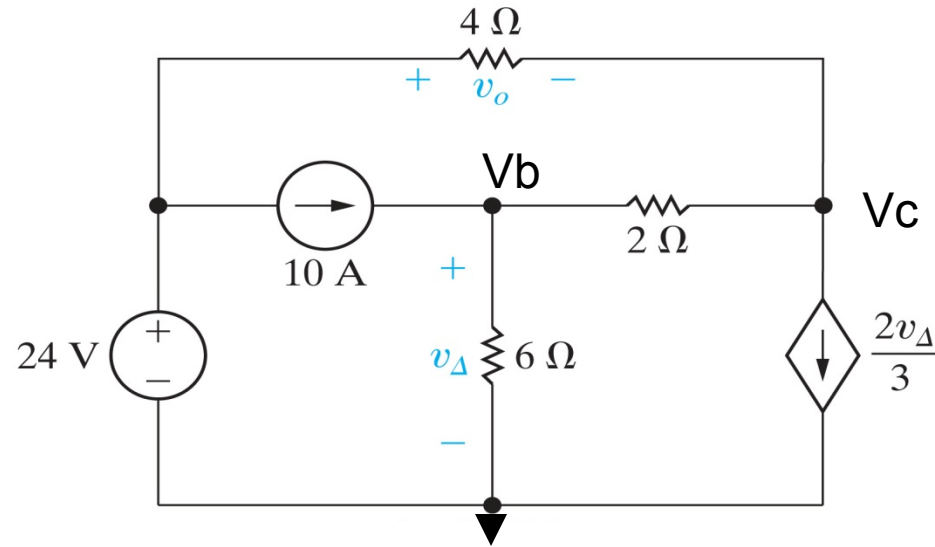
Putting the equations in standard form we could write:

$$v_b\left(\frac{1}{6}+\frac{1}{2}\right) \quad + \quad v_c\left(-\frac{1}{2}\right) \quad + \quad v_\Delta(0) \quad = \quad 10$$

$$v_b\left(-\frac{1}{2}\right) \quad + \quad v_c\left(\frac{1}{2}+\frac{1}{4}\right) \quad + \quad v_\Delta\left(\frac{2}{3}\right) \quad = \quad \frac{24}{4}$$

$$v_b(1) \quad + \quad v_c(0) \quad + \quad v_\Delta(-1) \quad = \quad 0$$

and solve a 3 X 3 matrix using MATLAB :

or ……..we can use the constraint equation to eliminate v_Δ immediately (giving us two KCL equations at two nodes)
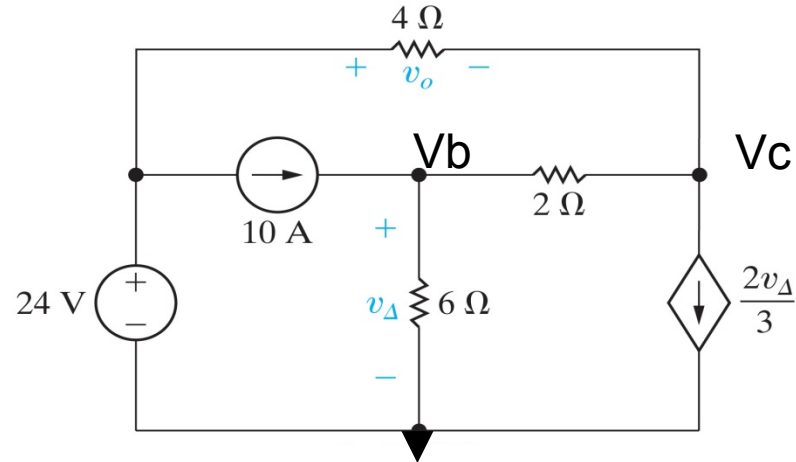
$$-10+\frac{V_b}{6}+\frac{V_b-V_c}{2}=0$$

$$\frac{2V_b}{3}+\frac{V_c-V_b}{2}+\frac{V_c-24}{4}=0$$

In standard form:

$$\begin{bmatrix} \left(\dfrac{1}{6}+\dfrac{1}{2}\right) & \left(-\dfrac{1}{2}\right) \\ \left(\dfrac{2}{3}-\dfrac{1}{2}\right) & \left(\dfrac{1}{2}+\dfrac{1}{4}\right) \end{bmatrix} \bullet \begin{bmatrix} v_b \\ v_c \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$$

And we can use MATLAB to solve this ……or we can just write:

$$4V_b - 3V_c = 60$$

$$2V_b + 9V_c = 72$$

$$\text{multiplying the second eq. by 2 and subtracting}$$

$$-21V_c = -84 \quad so \ V_c = 4, V_b = 18$$

We can solve:

Vb = 18V; Vc = 4V; v_Δ = 18, so v0 = 24 – Vc = 20V and we're done!
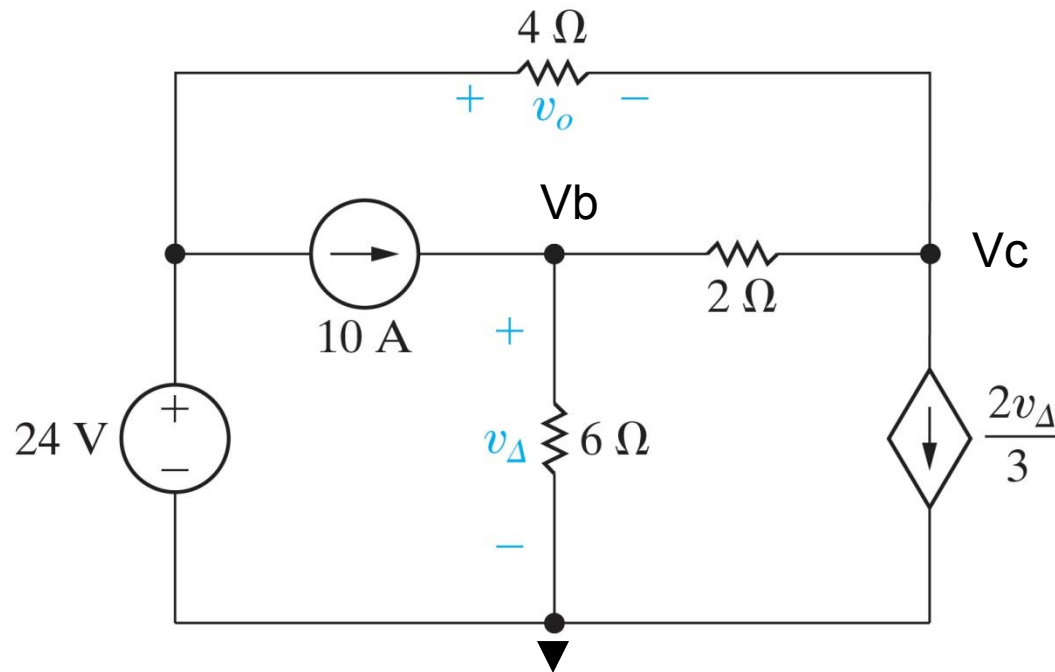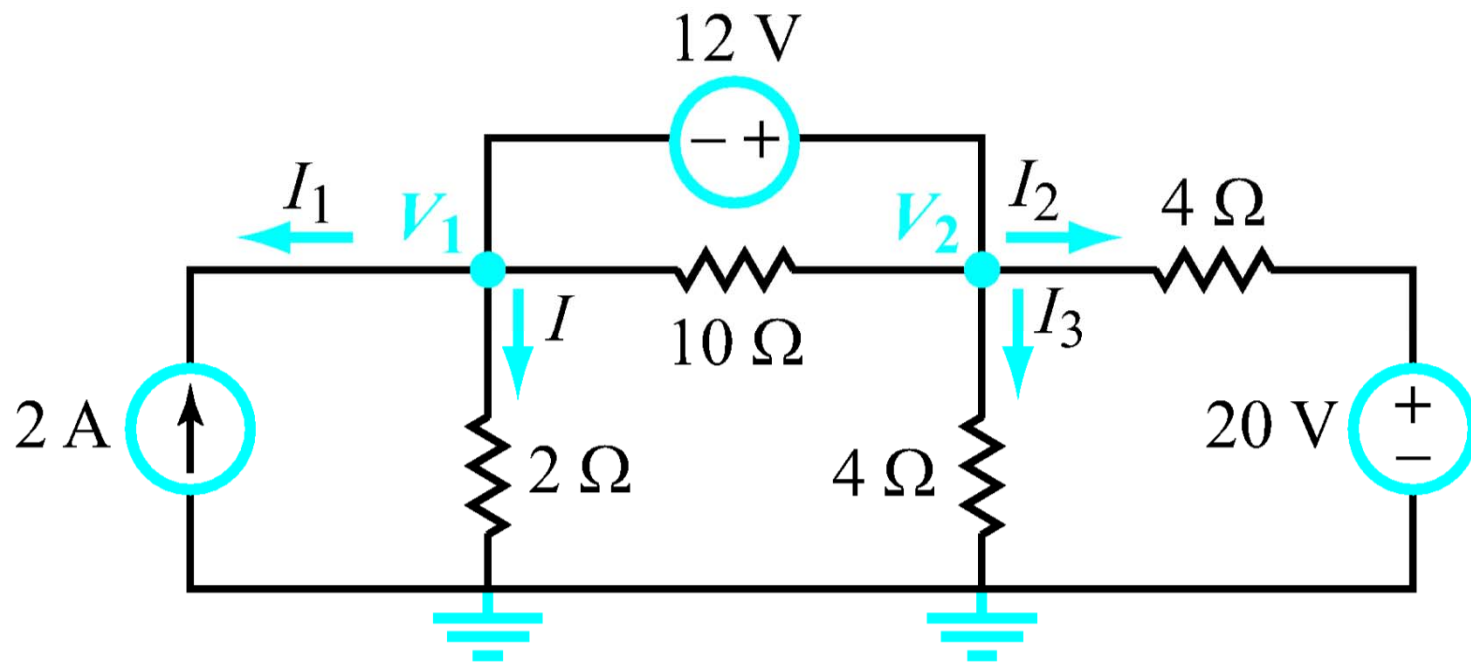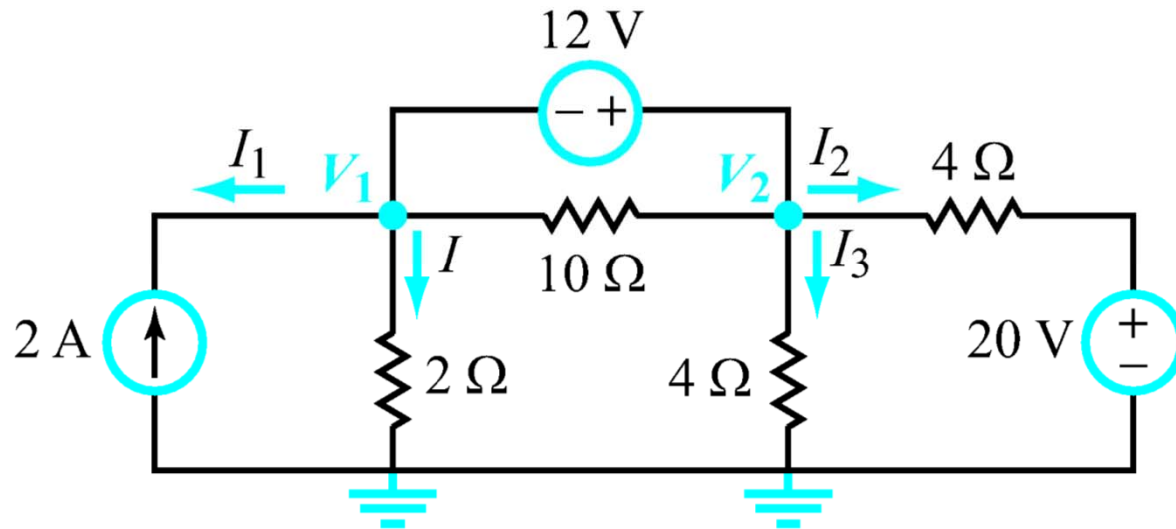
Nodes, Supernodes, and Quasi-Supernodes
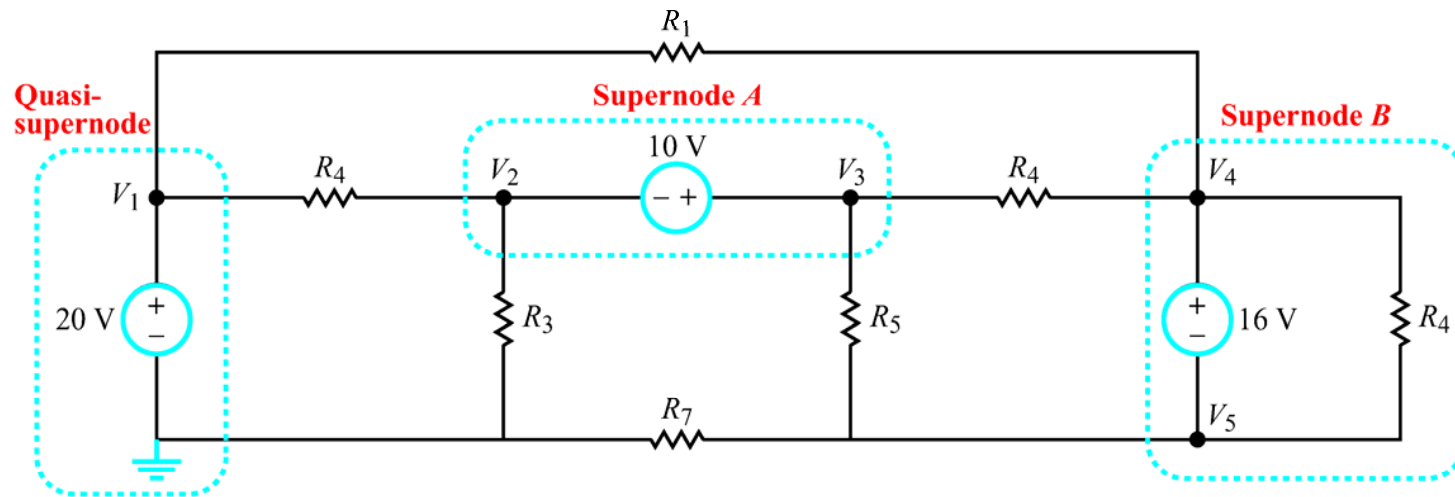
Here's an exercise from another textbook …apply the supernode concept to determine I in the circuit.
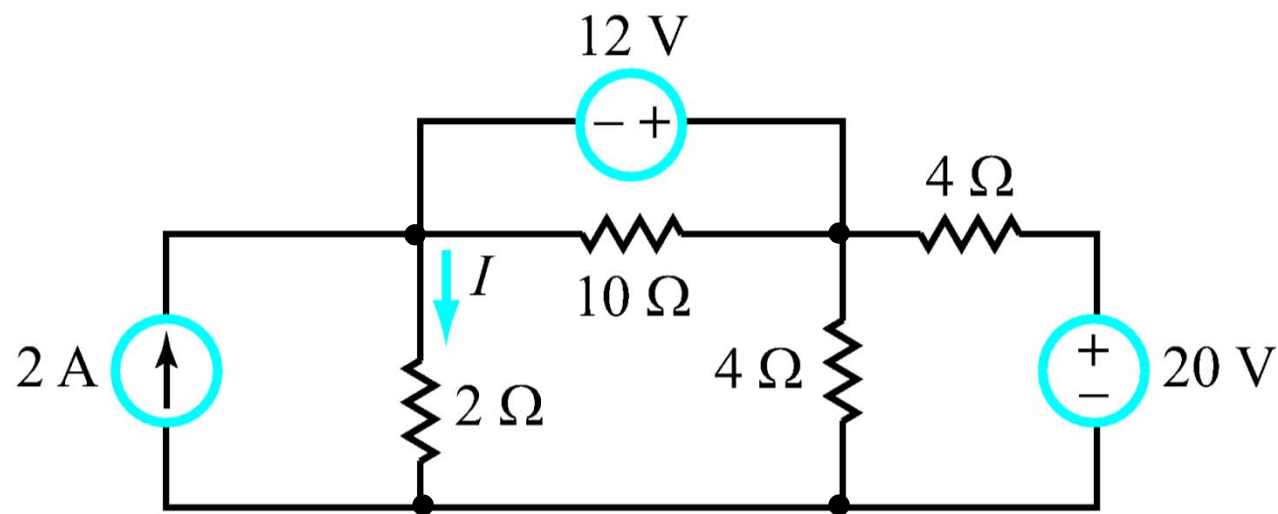
Figure E3.3

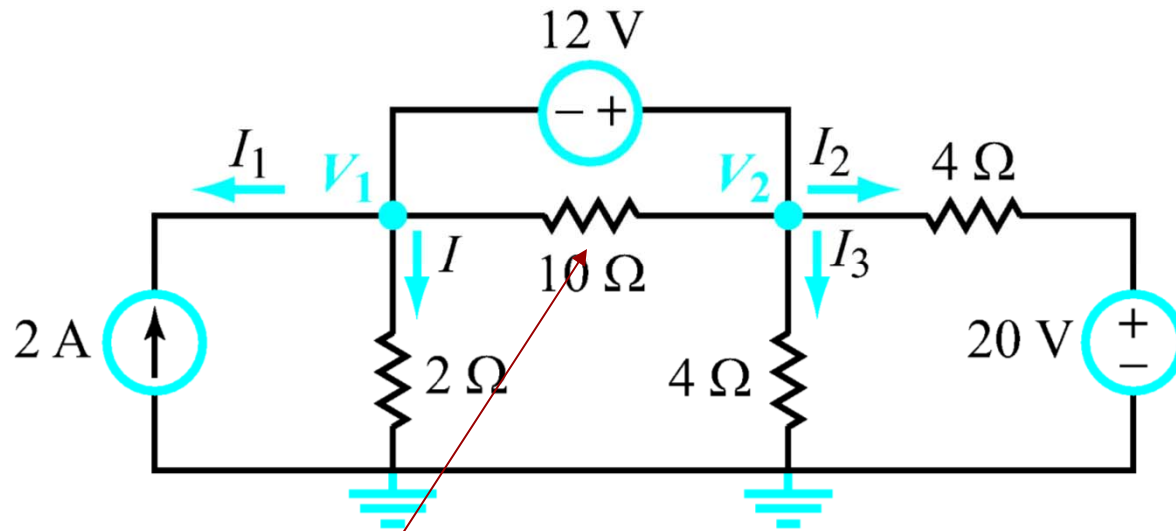Number essential nodes and chose a reference node. Is there a supernode in this circuit?

V1 and V2 form a supernode so we can write a single KCL equation for it:
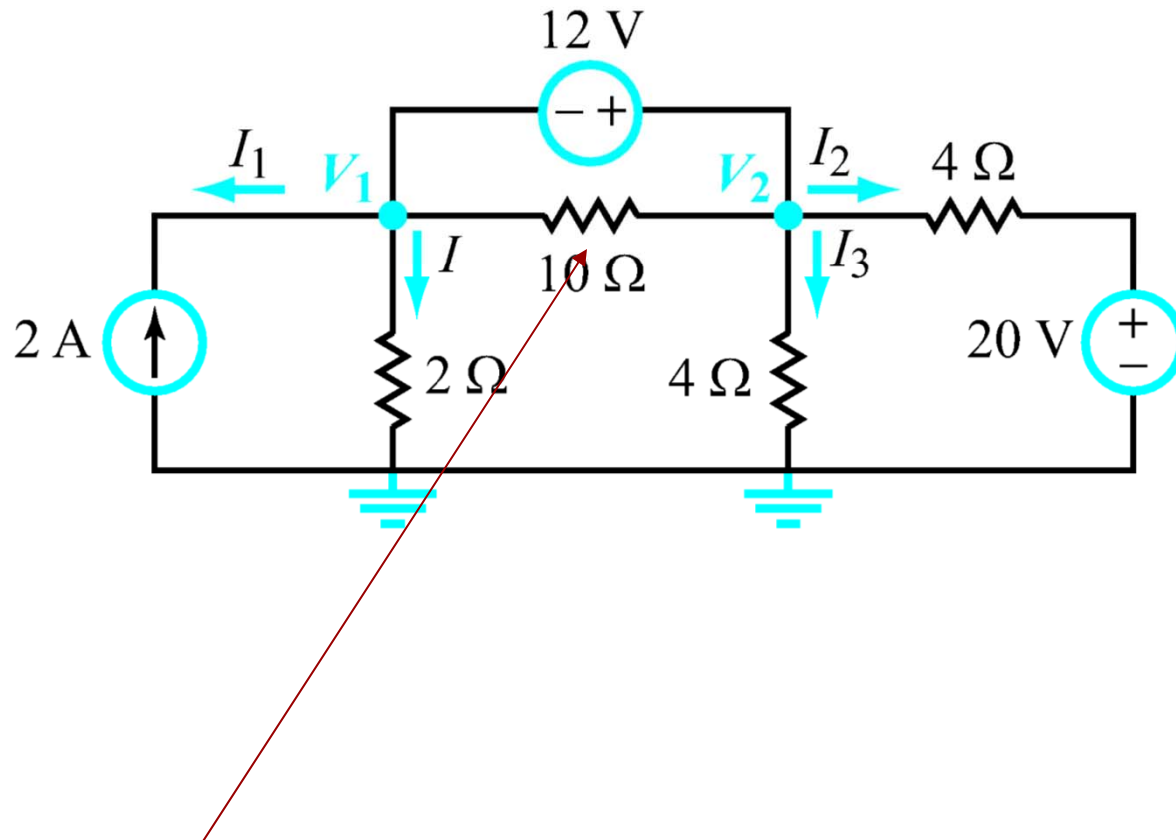
$I_1 + I + I_2 + I_3 = 0$

$I_1 = -2$, $I = V_1/2$, $I_3 = V_2/4$, $I_2 = (V_2-20)/4$ and KVL gives us $V_2 - V_1 = 12$

So, solving for V_1 and substituting:

$I = 0.5A$, $V_1 = 1$, $V_2 = 13$

How about the current through this resistor – we didn't account for it in any of the calculations? Does the behavior of this circuit depend on the value of this resistor?

This is one of these "superfluous resistors" we've talked about before… it's a resistor in parallel with a voltage source that can be removed from a circuit without affecting anything. We didn't explicitly remove it while solving the problem …we just ignored it when writing the KCL equations for the supernode.

Next time ….Thevenin and Norton Equivalent Circuits