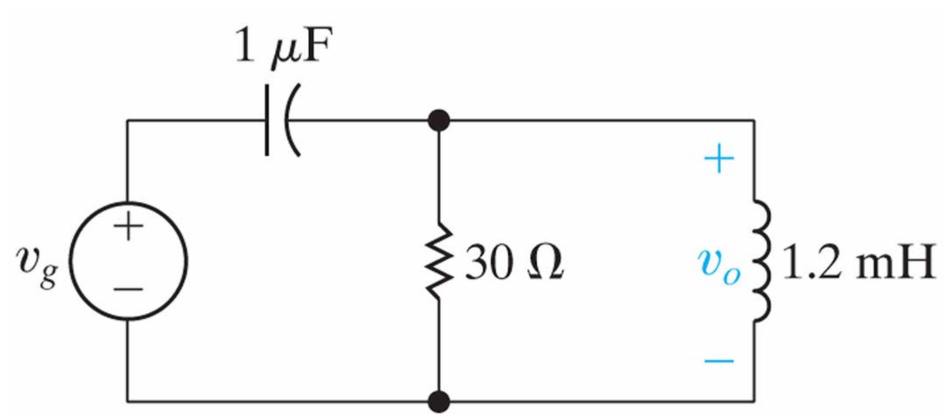
ELEN 50 Class 25 – RMS Values/ Complex Power

S. Hudgens

One more AC steady-state circuit to solve with phasors:

This circuit is operating in the sinusoidal steady state. Find the steady-state expression for $v_0(t)$ if v_g = 40 cos 50,000t V.

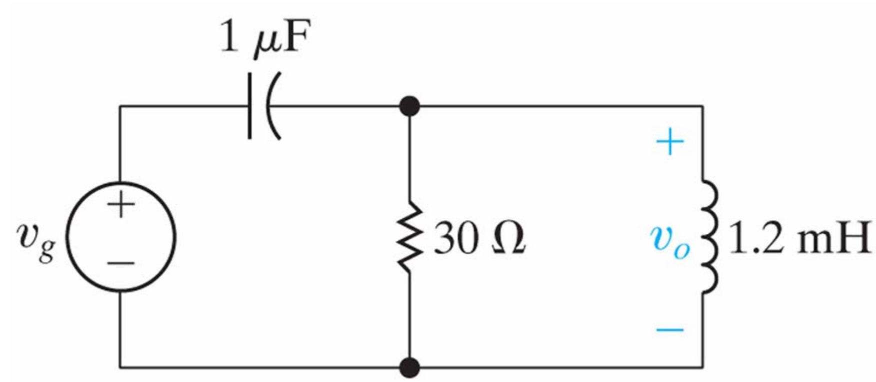


Sources in the time domain are already in the “cosine reference” form …so all we can go to step #2 and convert everything to the phasor domain….frequency is in radians/sec…..so:

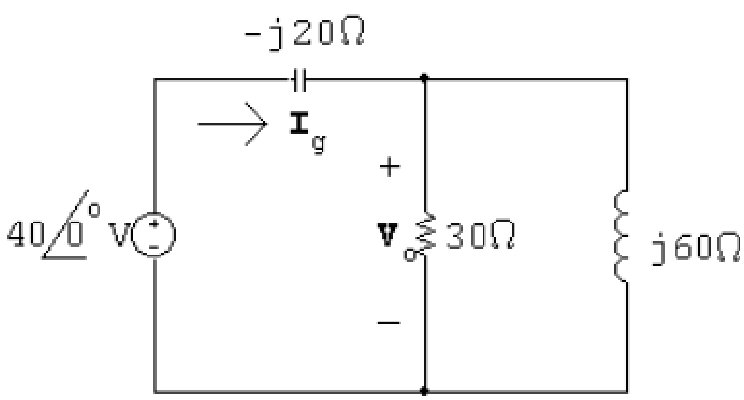
$$\frac{1}{j\omega C} = \frac{1}{(1\times10^{-6})(50\times10^{3})} = -j20\,\Omega$$

$$j\omega L = j50\times10^{3}(1.2\times10^{-3}) = j60\,\Omega$$

$$\mathbf{V}_g = 40\underline{/0^\circ}\ \mathrm{V}$$

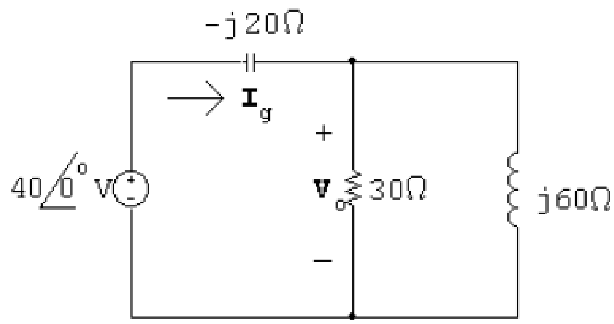


$v_g = 40 \cos 50{,}000t$ V



Do we have to use the node voltage or mesh current method?

No! We can solve this circuit easily just using the impedance model -- series and parallel combinations of impedances and Ohms Law….so no need for node voltage or mesh current solutions.



$$Z_e = -j20 + 30 || j60 = 24 - j8\,\Omega$$

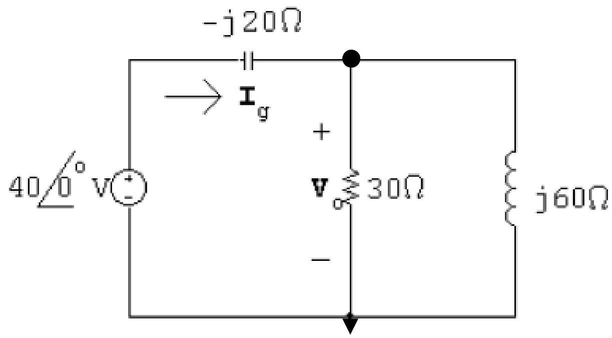
$$\mathbf{I}_g = \frac{40\angle 0^\circ}{24 - j8} = 1.5 + j0.5\,\text{mA}$$

$$\mathbf{V}_o = (30||j60)\mathbf{I}_g = \frac{30(j60)}{30 + j60}(1.5 + j0.5) = 30 + j30 = 42.43\angle 45^\circ\,\text{V}$$

$$v_o = 42.43\cos(50{,}000t + 45^\circ)\,\text{V}$$

Converting back to the time domain

What if we had done a Node-Voltage solution, how would it have looked?



$$\frac{\mathbf{V}_0 - 40}{j20\Omega} - \frac{\mathbf{V}_0}{30\Omega} - \frac{\mathbf{V}_0}{j60\Omega} = 0$$

multiplying through by j60 we get: $3V_0 - 120 - 2jV_0 - V_0 = 0$

collecting terms $V_0(2-2j) = 120$

so $V_0 = \dfrac{120}{2-2j} = 30 + 30j = 42.43\angle 45^o$ The result we got earlier

We can use phasors to deal with AC power also – but be careful!

From Class 24 – we saw that:

Average and Reactive Power

$$p = P + P\cos 2\omega t - Q\sin 2\omega t$$

$$where:$$

(5.)

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$

Average Power or Real Power (measured in Watts)

(6.)

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

Reactive Power (measured in VAR – volt amps reactive)

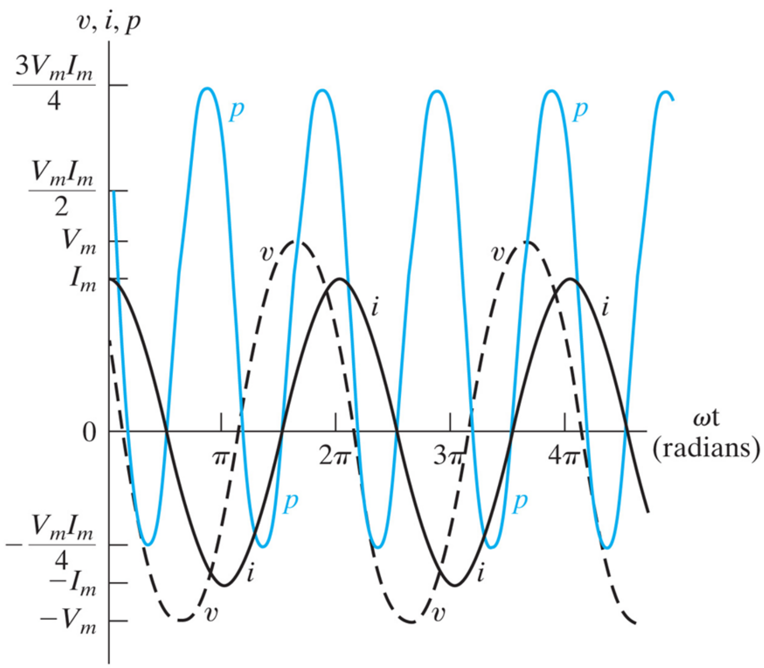
The expression in (5.) is called the "Real Power" because it is the power transformed from electrical energy to nonelectrical energy (heat, motion, light, etc.) …we might have called it useful power. It is clearly the average power because:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\, dt \quad .and \qquad \int_{t_0}^{t_0+T} \cos 2\omega t\, dt = \int_{t_0}^{t_0+T} \sin 2\omega t\, dt = 0$$

From Class 24:

Power Factor

The angle, $\theta_v - \theta_i$, the relative phase shift between the voltage and current, plays an important role in computation of the average and reactive power. The cosine of this angle is called the power factor and the sine of the angle is the reactive factor. Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, to uniquely define the phase shift angle in terms of power factor, we have to specify a lagging power factor (current lags voltage) or a leading power factor (current leads voltage).



In the example we considered earlier, where $\theta_v=60^o$ and $\theta_i=0^o$, the power factor = $\cos(\theta_v - \theta_i) = \cos(60^o) = 0.5$ ..and the current lags the voltage so this load has a lagging power factor of 0.5.

The average power for this load is:

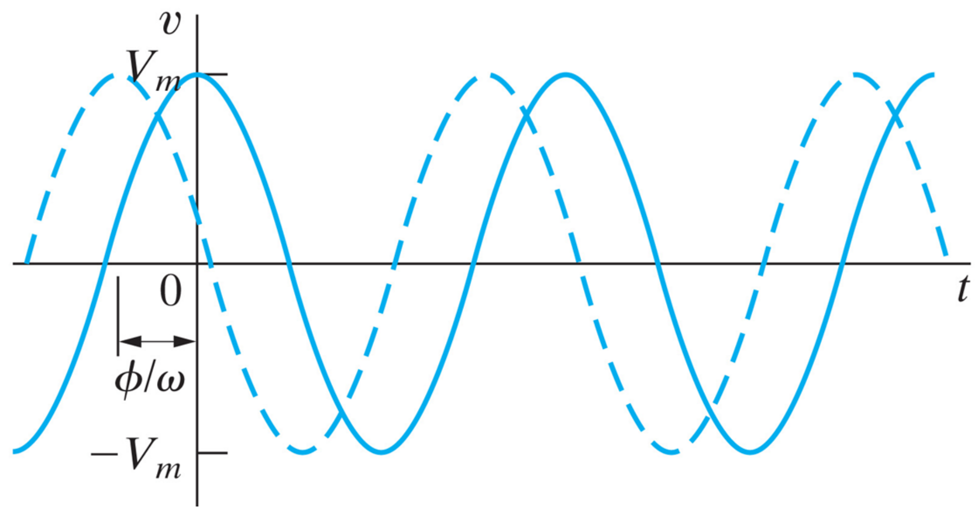
$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m I_m}{4}$$

And the peak reactive power, Q (VAR) is:

$$3\frac{V_m I_m}{4}$$

In Class 24, we introduced the concept of RMS voltage.

$$v = V_m \cos(\omega t + \phi)$$



The root mean square (RMS) value of a sinusoidal current or voltage is defined:

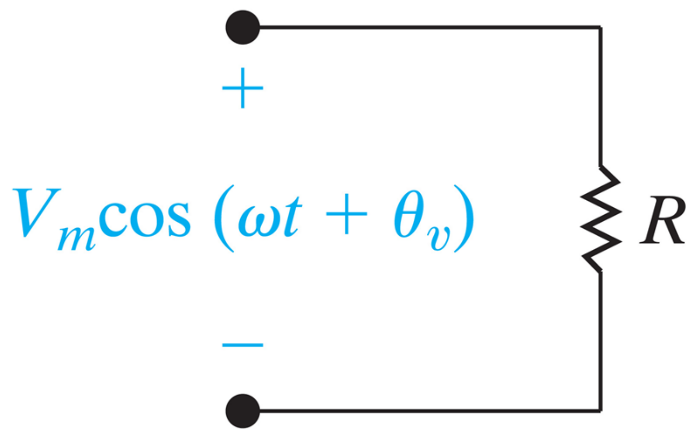
$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)dt}$$

$$\boxed{V_{RMS} = \frac{V_m}{\sqrt{2}}}$$

The RMS voltage (or current) is the square root of the mean of the square of the voltage or current in an AC system – it is the value of a DC current or voltage that will deliver the same power to a load as the AC voltage or current.

We can also define RMS power in the same way.
Assume that a sinusoidal voltage is applied to a resistor – we can write for the average power:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_m^2 \cos^2(\omega t + \phi_v)}{R} dt$$
$$= \frac{1}{R}\left[\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi_v) dt\right]$$



Notice that since (as we saw on the previous slide):

$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi) dt}$$

$$\boxed{P = \frac{V_{RMS}^2}{R} \qquad also, \quad P = I_{RMS}^2 R}$$

The RMS value of a sinusoidal source delivers the same power to a resistive load as a DC source of the same value – that's why the RMS value is also called the effective value.

The average and reactive power that we discussed earlier can also be expressed in terms of RMS (effective) values:

$$p = P + P\cos 2\omega t - Q\sin 2\omega t$$

$$where:$$

Average Power

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$

$$\boxed{= V_{eff} I_{eff} \cos(\theta_v - \theta_i)}$$

Reactive Power

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$\boxed{= V_{eff} I_{eff} \sin(\theta_v - \theta_i)}$$

Maybe you wondered why we didn't use phasors when we were deriving the expressions for average and reactive power earlier. Instead, we did all of the calculations in the time domain ….and this involved using a couple of obscure trig identities!

The short answer is that multiplying current and voltage phasors the way you would multiply ordinary complex numbers doesn't work!!!! It works OK to add phasors or to multiply a phasor by a constant – because the general assumption behind the use of phasors in circuit analysis holds…i.e. all of the values of interest in circuit analysis (voltages and currents) are varying in time with the same frequency – only their magnitudes and phase factors change as one moves around the circuit.

Because of this, we can represent a voltage, v(t) (for example) as the phasor:

$\mathbf{V} = Ve^{j\phi}$ when, in the time domain $v(t) = V\cos(\omega t + \phi)$

Where we don't have to take explicit account of the frequency ω

However, if you calculate power using phasors you presumably would have to multiply a voltage phasor with a current phasor …and the product phasor would have to have the same frequency as the voltage and the current….to be consistent with the fundamental assumption of the phasor representation.

As we saw …the product, the instantaneous AC steady-state power is not varying at a frequency of ω ..it's actually varying with time at a frequency of 2ω !!

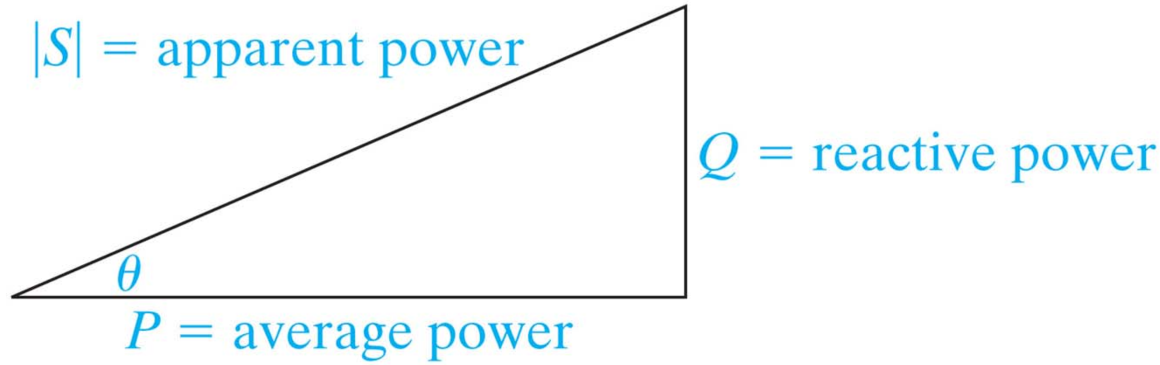
$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\boxed{\cos 2\omega t}$$
$$-\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\boxed{\sin 2\omega t}$$

However, we can combine real and reactive effective power by now defining a new phasor quantity -- complex power

$$\mathbf{S} = P + jQ$$

This is a complex number whose real part is the real power and whose imaginary part is the reactive power. The unit for complex power is VA (volt amps) and it is dimensionally the same as the unit for real power (Watts) i.e. energy/time.

Expressed geometrically:



Here, θ is the power factor angle i.e. cos (θ) = PF. This makes sense because a purely resistive load has zero reactive power so θ = 0 and cos (0) = unity ….the correct power factor for a resistive load.

Notice: the complex power is a phasor quantity defined in terms of the voltage and current phasors, but

$$\mathbf{S} \neq \mathbf{VI}$$

where

$$\mathbf{V} = V_m e^{j\phi_v}$$

$$\mathbf{I} = I_m e^{j\phi_i}$$

In fact, as we will see:

$$\mathbf{S} = \frac{1}{2}\mathbf{VI}^*$$

It is defined this way so that it causes the real part of **S** to be equal to P_{av}

The magnitude of the complex power is referred to as the apparent power:

$$|S| = \sqrt{P^2 + Q^2}$$

Apparent power is an important concept (especially to power companies!)

It is just the modulus of the complex power. Although the average power, P, is the useful output of an energy converting device (motor, heater, etc.), as we saw earlier in our discussion of power factor, the apparent power represents the volt-amp capacity required by a source to supply the average power.

Power Calculations

We can, of course, write the complex power in both rectangular and polar form in terms of maximum voltage and current values as:

$$S = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)]$$

$$= \frac{V_m I_m}{2}e^{j(\theta_v - \theta_i)} = \frac{V_m I_m}{2}\angle(\theta_v - \theta_i)$$

All of these expressions are in terms of the maximum amplitude of the voltage or current waveform … V_m or I_m

We can also write the complex power in terms of the RMS or effective values;

$$\mathbf{S} = \frac{V_m I_m}{2} \angle(\theta_v - \theta_i)$$
$$= V_{eff} I_{eff} \angle(\theta_v - \theta_i)$$
$$= V_{eff} e^{j\theta_v} I_{eff} e^{-j\theta_i}$$
$$= \mathbf{V}_{eff} \mathbf{I}^*_{eff}$$

So we can write the complex power as the product of the voltage phasor and the <u>complex conjugate of the current phasor</u>

Notice – we can write $I_{eff} e^{-j\theta_i} = \mathbf{I}^*_{eff}$ because

$$I_{eff} e^{-j\theta_i} = I_{eff} \cos(-\theta_i) + jI_{eff} \sin(-\theta_i)$$
$$= I_{eff} \cos(\theta_i) - jI_{eff} \sin(\theta_i)$$
$$= \mathbf{I}^*_{eff}$$

since

$$\cos(-\theta) = \cos(\theta) \quad and \; \sin(-\theta) = -\sin(\theta)$$

We can also show:

$$S = \mathbf{V}_{eff}\,\mathbf{I}^*{}_{eff} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$

Alternatively, we can write complex power in terms of impedance:

$$S = Z\,\mathbf{I}_{eff}\,\mathbf{I}^*{}_{eff} = \left|\mathbf{I}_{eff}\right|^2 Z$$

since $\mathbf{V}_{eff} = Z\,\mathbf{I}_{eff}$

Or we can write:

$$S = \mathbf{V}_{eff}\left(\frac{\mathbf{V}_{eff}}{Z}\right)^* = \frac{\left|\mathbf{V}_{eff}\right|^2}{Z^*}$$

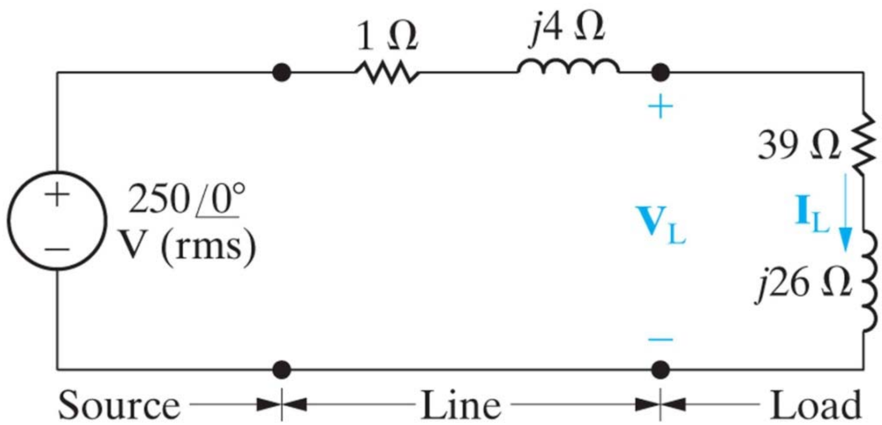
These are very useful relationships

$$S = \frac{\left|\mathbf{V}_{eff}\right|^2}{Z^*}$$

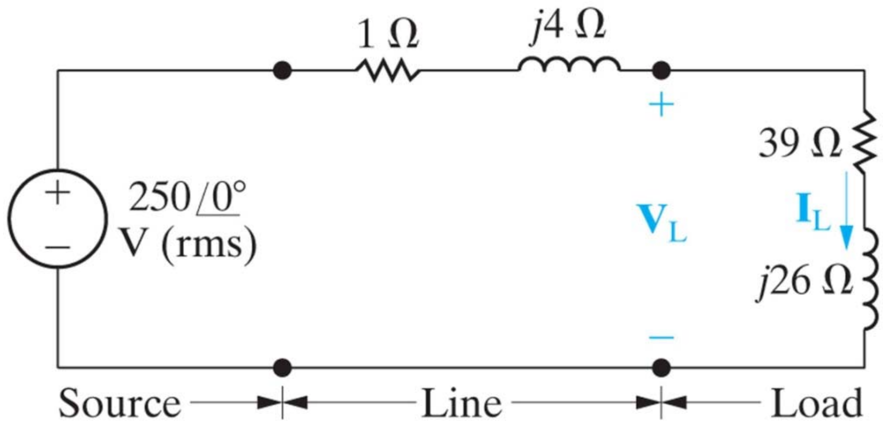
$$S = \left|\mathbf{I}_{eff}\right|^2 Z$$

We can use these relationships to calculate complex power in AC circuits using ordinary phasor analysis

Here's an example from Nilsson – a load of 39Ω +j26Ω is fed from a voltage source through a line having an impedance of 1Ω +j4Ω. The effective (RMS) value of the voltage source is 250V, as shown.



What is the load current phasor $\mathbf{I}_L$ and voltage phasor $\mathbf{V}_L$? Calculate the average and reactive power delivered to the load.



The line and load impedances are in series, so the load current is the voltage source divided by the total impedance.

$$\mathbf{I}_L = \frac{250\angle 0^o}{40 + j30} = \frac{250\angle 0^0}{50\angle 36.87^o} = 5\angle -36.87^o\, A(rms)$$

Since the load voltage is the load impedance times $\mathbf{I}_L$

After converting $\mathbf{I}_L$ into polar coordinates for convenience in multiplying

$$\mathbf{V}_L = (39 + j26)\mathbf{I}_L = (46.87\angle 33.69^o)(5\angle -36.87^o) = 234.36\angle -3.18^o V(rms)$$

So….since:

$$\mathbf{I}_L = 5\angle -36.87^o\, A(rms)$$

$$\mathbf{V}_L = 234.36\angle -3.18^o V(rms)$$

Notice ---complex conjugate

$$S = \mathbf{V}_L \mathbf{I}^*{}_L = (5\angle +36.87^o)(234.36\angle -3.18^o)$$

$$= 1171.8\angle 33.69^o$$

$$= 975 + j650\, VA$$

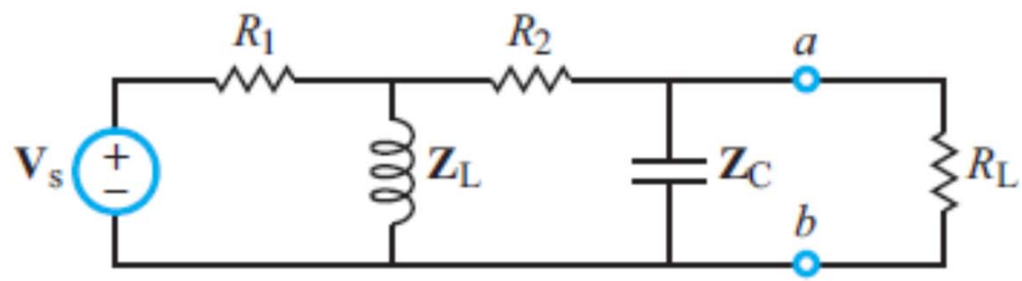
975 Watts and 650 VARs

This is the average and reactive power delivered to the load

Notice also that the <u>apparent power</u> delivered to the load, |S| = 1171 VA

Here's a more complicated problem:

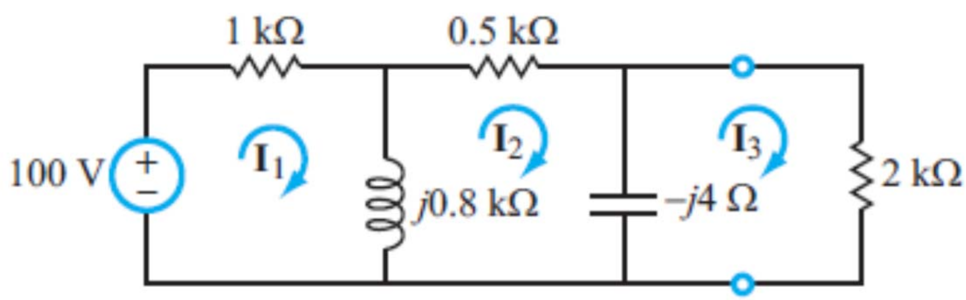
Determine the average power dissipated in the load resistor, RL, when:
$\mathbf{V}_s$=100V, R1=1 kΩ , R2=0.5 kΩ, R_L = 2 kΩ, Z_L=j 0.8 kΩ, Z_c= -j 4 kΩ



What would be a good solution strategy? We will use phasors, of course because it's sinusoidal steady-state.

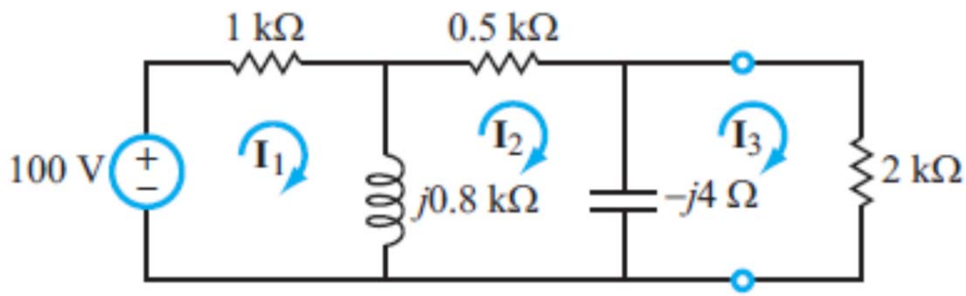
How about mesh current?

We can actually solve this circuit quickly using the mesh current by inspection technique



$$\begin{bmatrix} (1+j0.8) & -j0.8 & 0 \\ -j0.8 & (0.5-j3.2) & j4 \\ 0 & j4 & (2-j4) \end{bmatrix} \times 10^3 \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}.$$

Do you agree with this matrix equation?



MATLAB returns this for the current vector

$$\mathbf{I}_1 = 65.36\angle{-31.55^\circ} \quad \text{(mA)}$$

$$\mathbf{I}_2 = 24.87\angle{58.48^\circ} \quad \text{(mA)}$$

$$\mathbf{I}_3 = 22.28\angle{31.98^\circ} \quad \text{(mA)}$$

so $$P_{av} = \frac{1}{2}|\mathbf{I}_3|^2 R_L = 496.4 \text{ mW}.$$

Because, as we saw, $$\mathbf{I}_{eff} = \frac{\mathbf{I}}{\sqrt{2}}$$

and $$S = Z\mathbf{I}_{eff}\mathbf{I}^*_{eff} = \left|\mathbf{I}_{eff}\right|^2 Z = \frac{1}{2}\left|\mathbf{I}\right|^2 Z$$

Next class we'll have a review of the whole course but, now, just for fun …..

We can't have a course on electrical circuits without talking a little about lightning

Lightening was probably the first (dramatic) example of the operation of a kind of electrical circuit (the Eaarth and the sky) that humans ever witnessed. It was such a big deal that early religions had specific god for lightening and thunder.

If you are interested, there is a list on Wikipedia:

https://en.wikipedia.org/wiki/List_of_thunder_gods

There are 57 thunder/lightening gods on this page …not including the 130 different gods, lords, goddesses, kings, marshalls, and generals from Chinese mythology.

Although most people don't blame lightening gods for misfortune today, lightening still causes a lot of property damage and even injury and loss of life (about 2000 people each year!). Damage to the electrical grid is particularly important and people have worked to develop ways of testing power grid components in the case of lightening strikes.

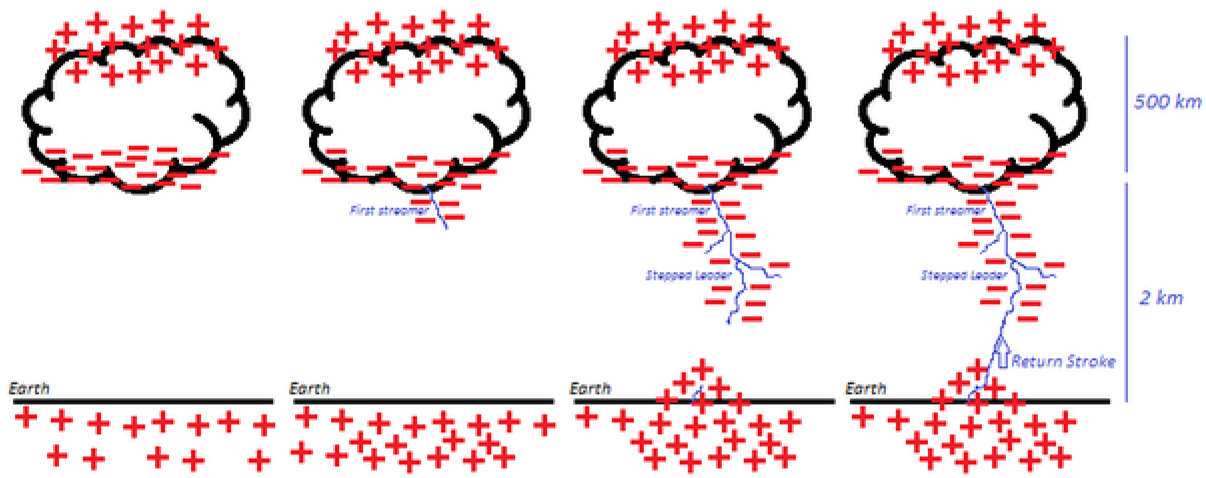
What do you think a good simulation for a lightening strike should be?

You might have guessed a Tesla Coil:

But these dramatic sparks are high frequency alternating current …and usually very low current discharges – not like lightening at all

"musical" Tesla coil

Lightening is actually a kind of capacitor discharge process:



So the best simulation of lightening strikes to earth (and power lines/ transformers/ substations/ etc.) is a capacitor discharging a very high current

remember

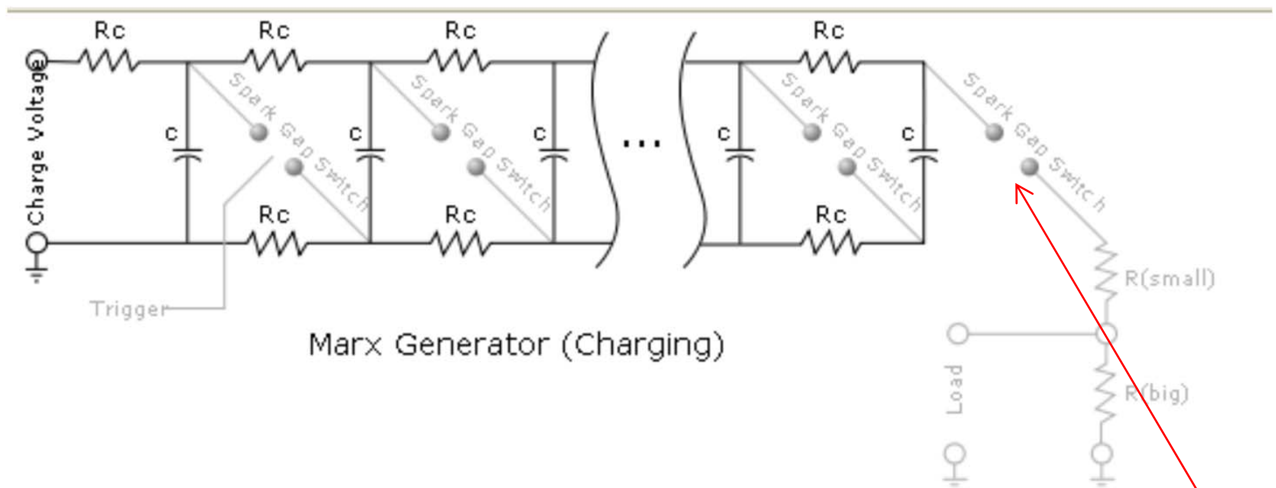
$$i(t) = I_o e^{-t/\tau}$$

$$\tau = RC$$

Once the lightening strike begins, R is small, and C and V_o can be quite large

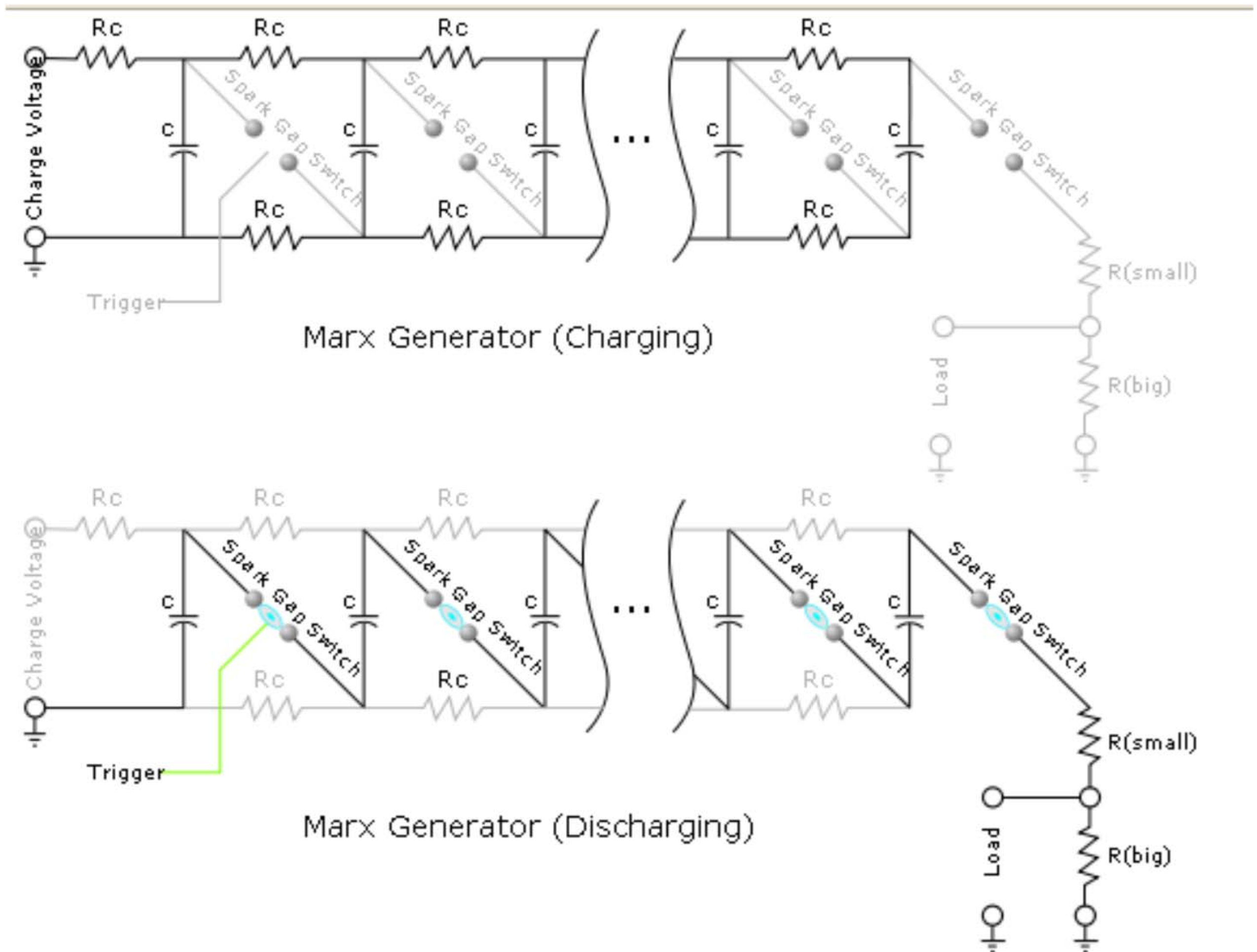
Here's really fascinating capacitor circuit that is used to simulate lightning:

The Marx Generator (invented by Otto Marx in 1924)



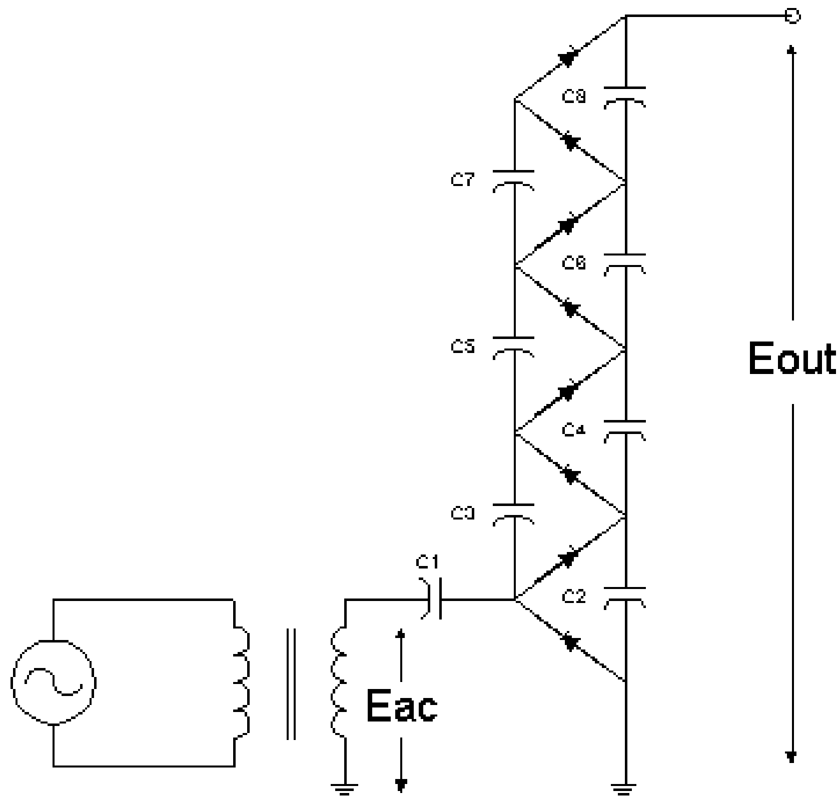
Notice that the Marx generator uses a funny circuit element – a spark gap – but otherwise it's just capacitors and resistors and a DC voltage source.

The spark gap is a breakdown device that is either an open circuit at low voltages or a short circuit at voltages in excess of the threshold voltage. When the Marx generator is charging, all of the spark gaps are in the "open circuit" state …so the circuit looks like a bunch of capacitors and resistors in parallel – when fully charged, all of the capacitors will have the same voltage across them …and current through the resistors will be zero…and the bottom of all the capacitors will be at ground potential.



Typical Marx generators have many capacitors in the stack to achieve high voltages. Here is one http://www.youtube.com/watch?v=IrQsghadA8A&feature=related with 16 capacitors operating at 2 MV total. It can deliver a 1000 A spark in 12 μs….simulating a lightening strike!

Here’s a similar idea – this circuit (called a Cockroft – Walton multiplier) produces a steady high voltage output from a low voltage AC input. The output voltage is approximately equal to twice the peak input voltage ..multiplied by the number of diode capacitor stages.



It was used extensively in old (CRT type) TV sets to provide the 30 kV high voltage for the picture tube.