ELEN 50 Class 19 – Natural Response of RL and RC Circuits

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You Thought I Was Joking about People in the 18^{th} Century Shocking One Another with Leyden Jar Capacitors

"Machines like these were not only made for scientific research, but a preferred toy for amusement. In the 18th century, everybody wanted to experience the electric shock. Experiments like the 'electric kiss' were a salon pastime. Although the French Abbé Nollet demonstrated in 1745 that little animals like birds and fish were killed instantaneously by the discharge of a Leyden jar, nobody was aware of the latent dangers of this type of experiments. "

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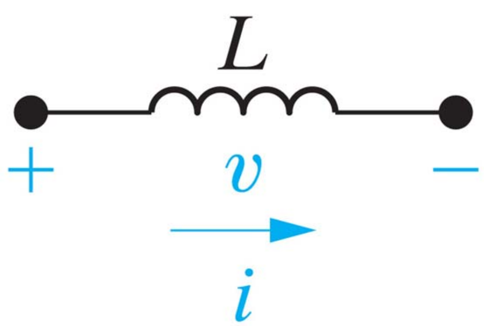
In the last class we introduced capacitors and inductors:

The Inductor

(a)

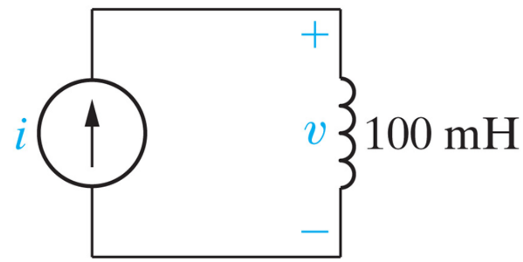


$$\boxed{v = L\frac{di}{dt}}$$

The voltage across an inductor depends on the derivative of current with time through the inductor. A time independent current (DC) produces zero voltage, therefore the inductor behaves as a DC short circuit.

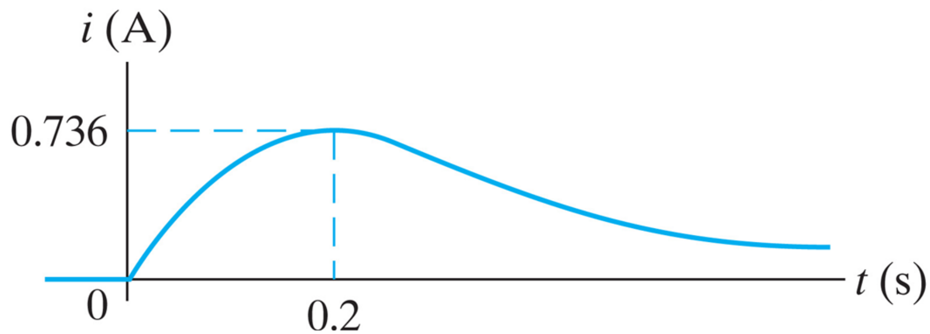
<u>Note</u>: current cannot change instantaneously through an inductor since this would result in an infinite voltage across the inductor.

As we saw before, if we apply a time varying current to an inductor:

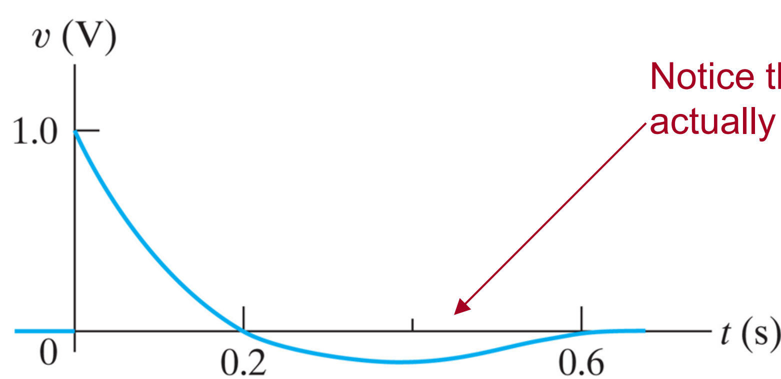


$$i = 0, \qquad t < 0$$

$$i = 10te^{-5t}\text{A}, \qquad t > 0$$

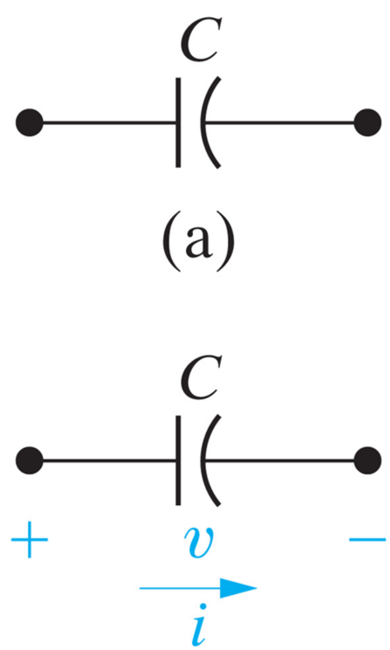


Since: $$\frac{di}{dt} = 10(-5te^{-5t} + e^{-5t}) = 10e^{-5t}(1-5t)$$



Notice the polarity of the voltage actually reverses here

The Capacitor



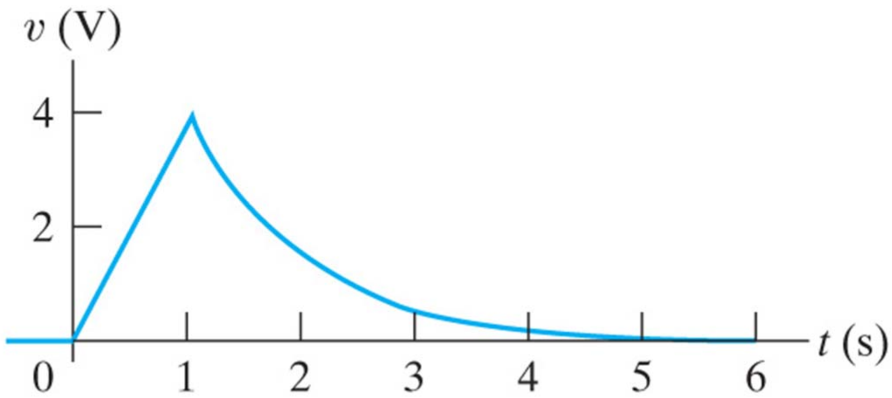
$$\boxed{i = C\frac{dv}{dt}}$$

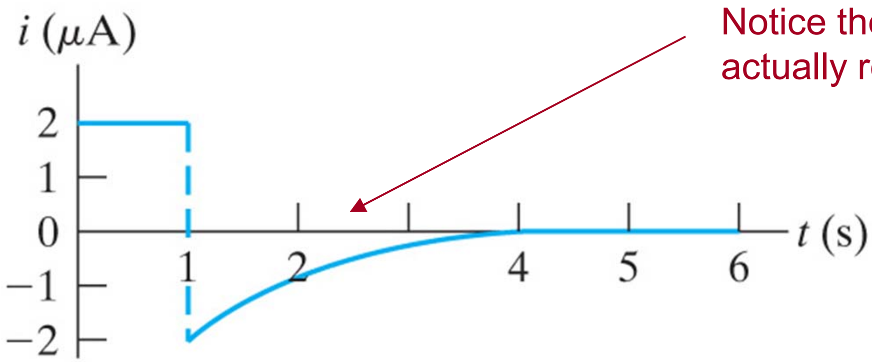
The current through a capacitor depends on the time derivative of the voltage across the capacitor. A time independent voltage (DC) results in zero current …so the capacitor acts like an open circuit to DC voltages.

Note: voltage cannot change instantaneously across a capacitor since this would result in an infinite current through the capacitor

For a voltage applied across a capacitor:

$$v(t) = 0, \qquad t \le 0$$
$$v(t) = 4tV, \qquad 0 \le t \le 1s$$
$$v(t) = 4e^{-(t-1)}V, \quad t \ge 1s$$





Notice the polarity of the current actually reverses here

Energy is stored in both capacitors and inductors. In the case of the capacitor, energy is stored in the electric field that exists between the two plates of the capacitor. In the case of the inductor, energy is stored in the magnetic field that is created by current flowing through the loops of wire that form the inductor.

For the capacitor, the stored energy is:

$$w = \frac{1}{2}Cv^2$$

For the inductor, the stored energy is:

$$w = \frac{1}{2}Li^2$$

This is different from a resistor which can dissipate power …but doesn't store any energy.

Why can a capacitor store energy?

The instantaneous power transferred into or out of a capacitor is, of course,:

$$p(t) = vi$$

$$= Cv\frac{dv}{dt} \qquad \text{since} \qquad i = C\frac{dv}{dt}$$

So the stored energy (at a time, t) will just be the time integral from neg. infinity to time, t, of the instantaneous power:

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t}\left(v\frac{dv}{dt}\right)dt = C\int_{-\infty}^{t}\left[\frac{d}{dt}\left(\frac{1}{2}v^2\right)\right]dt$$

so

$$\boxed{w = \frac{1}{2}Cv^2}$$

Similarly for an inductor:

$$p(t) = vi$$

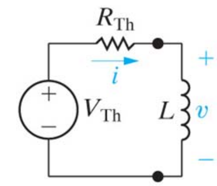
$$= Li\frac{di}{dt}$$ since $$v = L\frac{di}{dt}$$

$$w = \int_{-\infty}^{t} pdt = L\int_{-\infty}^{t}\left(i\frac{di}{dt}\right)dt = L\int_{-\infty}^{t}\left[\frac{d}{dt}\left(\frac{1}{2}i^2\right)\right]dt$$

so

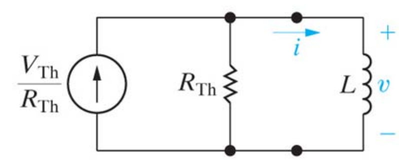
$$\boxed{w = \frac{1}{2}Li^2}$$

The natural response of an RL or RC circuit describes the transient currents and voltages that arise when the energy in an inductor or capacitor is suddenly released to a resistive network. RL and RC circuits are known as first order circuits because their voltages and currents are described by first order differential equations. These differential equations describe behavior with characteristic time constants. We can write the four possible first order circuits in terms of Thevenin or Norton equivalents:

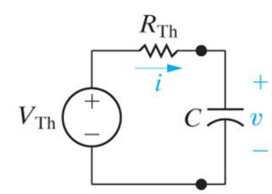


(a)

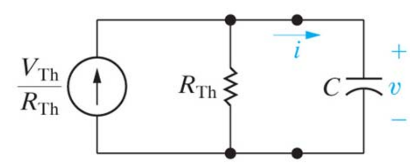
Note: no matter how complex an RL or RC circuit is, it can be reduced to a Norton or Thevenin equivalent circuit connected to the terminals of an equivalent inductor or capacitor.



(b)

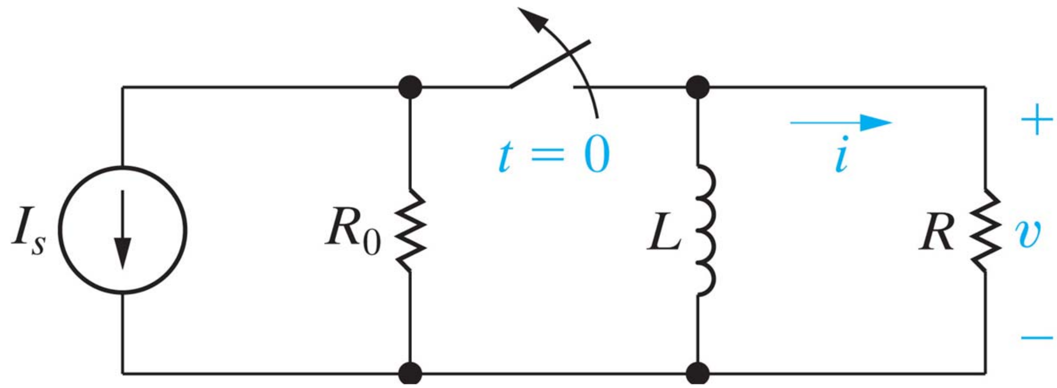


(c)



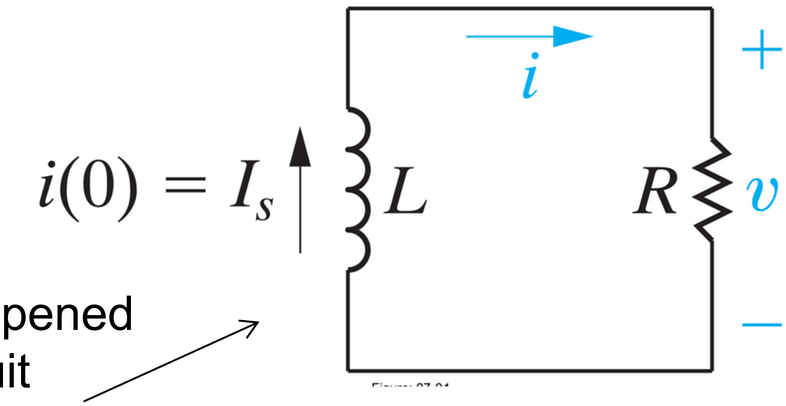
(d)

Natural response of an RL circuit



Initially, the switch is closed. After it has been closed for "a long time" (which we'll define later), all currents and voltages have reached constant values. Since the inductor is a short circuit when di/dt = 0, there will be no current in R_0 or R, therefore <u>all of the source current will flow through the inductor</u>. The natural response of the circuit involves finding the voltage and current associated with R after the switch is opened and the stored energy in the inductor is dissipated by the resistor. In other words, we want to find v(t) and i(t) for t equal to or greater than t=0.

$for\ t \geq 0$



After the switch is opened
this is the circuit

We can write <u>KVL</u> around this loop:

$$L\frac{di}{dt} + Ri = 0$$

This is a first order ordinary differential equation with constant coefficients.
We can solve it easily:

Because V = L di/dt --- remember the I-V relation
for an inductor?

What's happening here?

We spent the first seven weeks of ELEN 50 doing (fairly simple) linear algebra and now we're having to solve differential equations! Why is that?

Welcome to the world of reactive circuit elements (capacitors and inductors)! Their I-V characteristics involve derivatives so KVL and KCL produce differential equations.

$$i = C\frac{dv}{dt} \qquad v = L\frac{di}{dt}$$

Let's solve:

$$L\frac{di}{dt} + Ri = 0$$

You probably have seen differential equations like this before -- it looks like the equation for radioactive decay or Newton's Law of Cooling. To solve it, we just integrate both sides of the equation

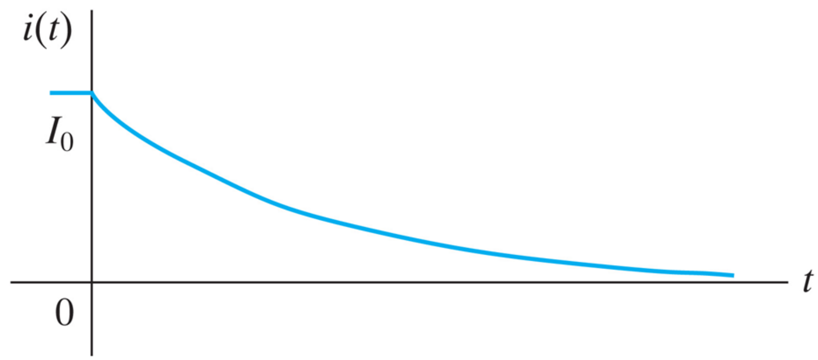
$$\frac{di}{dt}dt = -\frac{R}{L}i\ dt$$

$$\frac{di}{i} = -\frac{R}{L}dt$$

remember $\boxed{\int \frac{1}{i}\,di = \ln i + C}$

$$\ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$$

So: $$i(t) = i(0)e^{-(R/L)t}$$



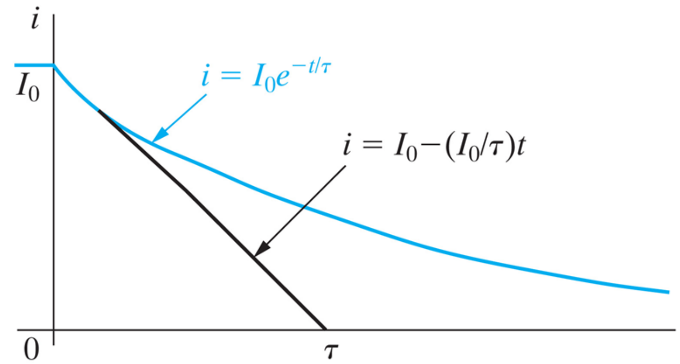
where i(0) is the initial current, I_0, flowing through the inductor before the switch was opened. We can also calculate the voltage developed across the resistor from Ohms law.

The term, (R/L), has particular significance. It is the inverse of the natural time constant for the circuit…and it's usually written as τ. Defining τ = L/R, we can rewrite the expression for current as a function of time as:

$$i(t) = i(0)e^{-t/\tau} \qquad t \geq 0$$

The <u>time constant</u> is a <u>natural time scale to describe the circuit</u>. We can talk about the time response of a first order circuit in terms of integral multiples of the time constant. After one time constant, the current in the RL circuit has been reduced to 1/e ~ 37% of its initial value. After two time constants, the current is $1/e^2$ ~ 13% of the initial value ….and after five time constants, the current is $1/e^5$ ~ 0.67% of initial value.

Also note, an extrapolation of the initial part of the decay of the current will intercept the time axis at a time equal to the time constant. Do you know why that is so?

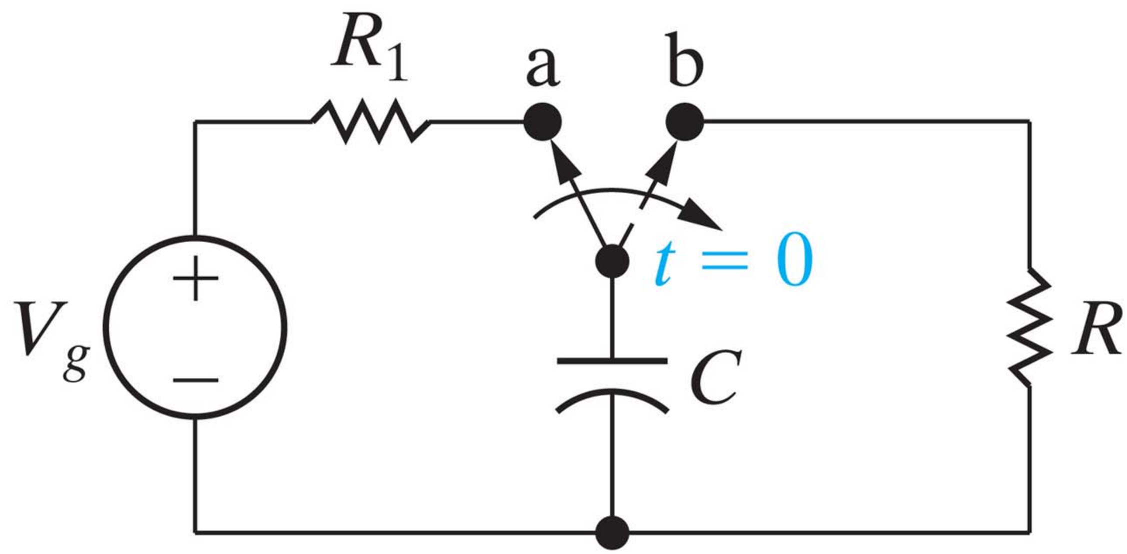


How to remember if $\tau = L/R$ or $\tau = R/L$

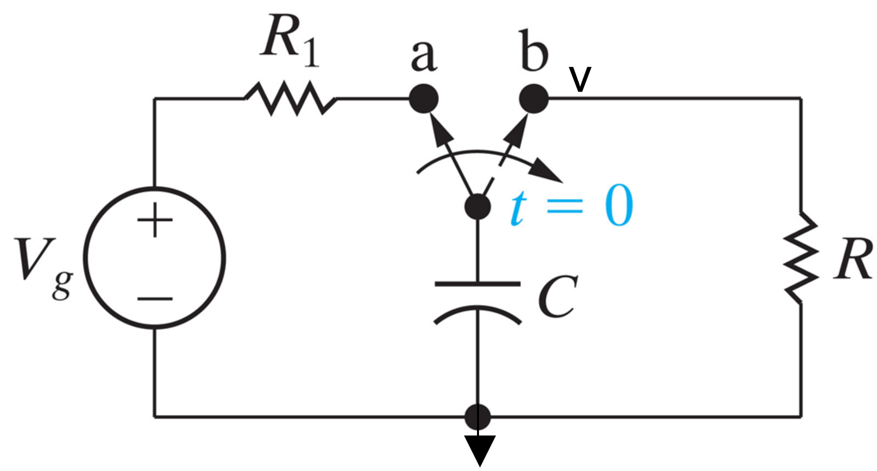
Think about the physics! The time constant is the time it takes the energy stored in the magnetic field of the inductor to dissipate in the resistor ….turning into heat because of $P = i^2R$. The larger the inductor (the larger is L) the more extensive is the magnetic field and the more total magnetic energy will be stored….causing the time constant to be longer. Similarly, the larger the resistor, the larger will be the power dissipated for a given current flowing through it and the faster the magnetic energy will be dissipated…so the time constant will be shorter.

Clearly, this requires $t = L/R$!

Natural response of an RC circuit



We can derive the natural response of this RC circuit. At time, t=0, the switch connects the fully charged capacitor to the resistor, R.



We can do a node voltage analysis for the voltage, v after the switch is thrown, as shown. Writing KCL -- summing all of the current flowing out of this node, we get:

$$C\frac{dv}{dt}+\frac{v}{R}=0$$

Because i = C dv/dt --- remember?

And, using the same approach we used for the inductor natural response:

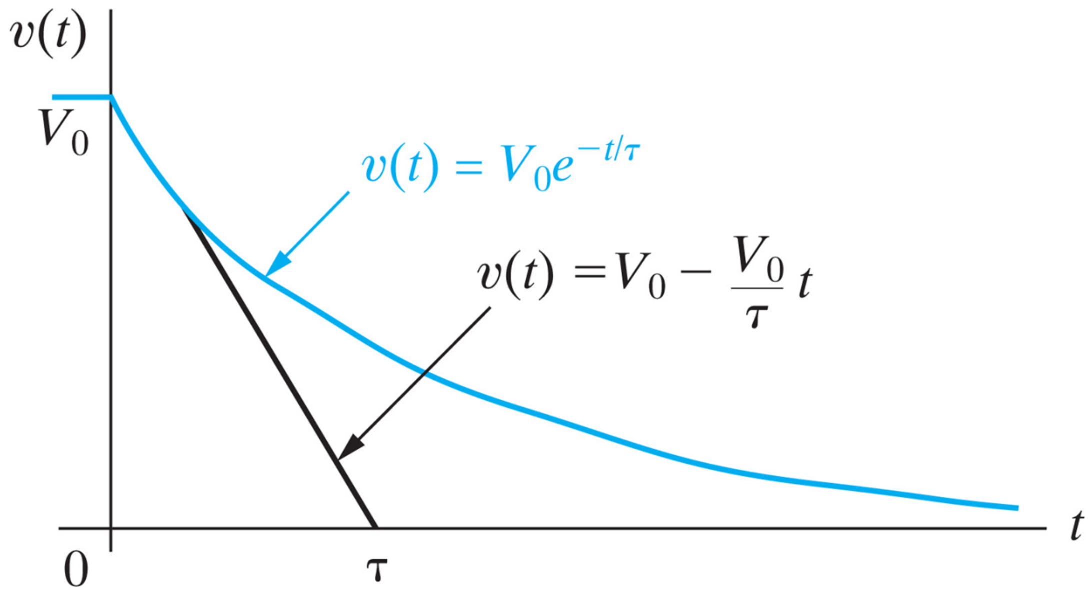
$$v(t)=V_o e^{-t/\tau}$$

$$\tau = RC$$

$$v(t) = V_o e^{-t/\tau}$$

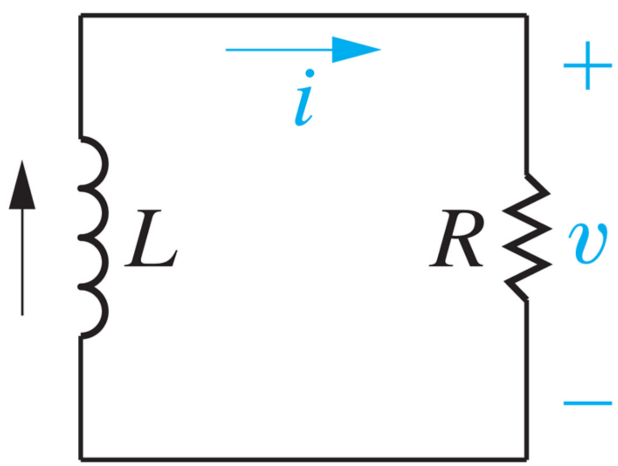
$$\tau = RC$$

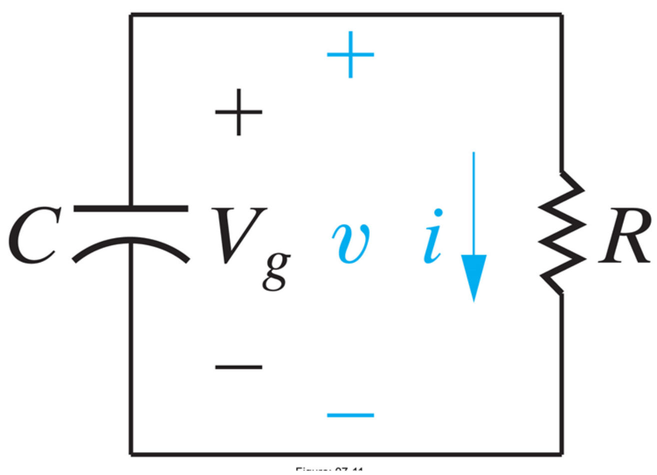
As before:



So we see that the natural response of an RC and an RL circuit are analogous. In both cases, the energy stored in the reactive element (capacitor or inductor) is dissipated in the resistor in such a way that the current (in the case of the inductor) or the voltage (in the case of the capacitor) decays exponentially with a time constant, τ.

For the RL circuit, τ = L/R, and for the RC circuit, τ = RC.





Does this circuit look familiar?

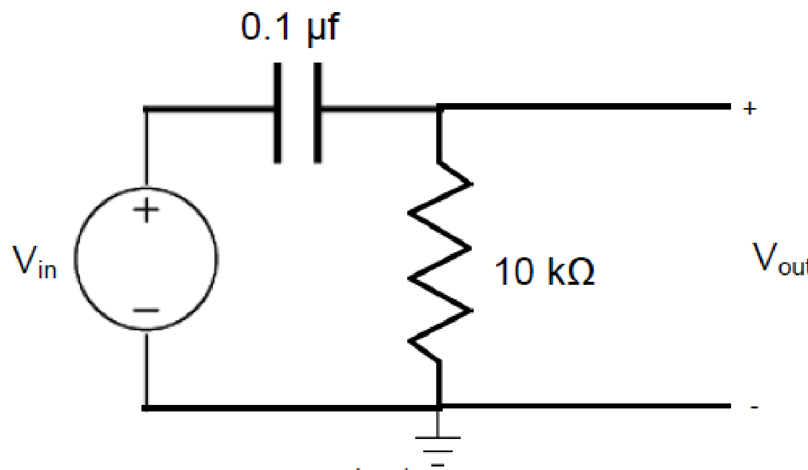


Fig 2. An RC circuit.

It should, it's the circuit you are analyzing in Lab next week! You should be able to calculate the time constant immediately in light of the previous discussion and also to be able to predict what the voltage V_{out} will look like when V_{in} is a square wave (the same thing as opening and closing the switch in the circuit we just analyzed). The response of this circuit to a sinusoidal input will be a little different …but you should be able to figure it out by going back to the basic I-V relationship for a capacitor. In the next week or so we'll introduce phasor analysis which makes solving time dependent circuits with steady state sinusoidal signals very easy.

By the way:

you saw that to solve for the natural response of an RL circuit you had to solve a KVL equation that led to:

$$L\frac{di}{dt} + Ri = 0$$

and to solve for the natural response of an RC circuit you had to solve a KCL equation that led to:

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

Both simple, first order, differential equations. What do you think will happen when there are both capacitors and inductors in a circuit?

Well…if you want to write a KVL equation for voltage drops around a loop, the resistor and inductor terms will be as before:

$$L\frac{di}{dt} + Ri$$

But the term representing the voltage drop across the capacitor will be:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$$

as we saw last time

So the KVL equation for the RLC circuit will have a first derivative of the current with time and an integral of the current over time – it will be an integrodifferential equation! We can fix this by differentiating the whole equation with respect to time ..and the integral goes away …but <u>now the equation is second order!</u> It involves both first and second derivatives with respect to time. We'll talk about what happens in this case in a future class….but, for sure, the solution is not going to be simple exponential decay.

To summarize:

We have analyzed the natural response of RC and RL circuits -- described by first order differential equations for current and voltage as a function of time:

The RC circuit produced an equation like:

$$\frac{dv}{dt} + av = b,$$

Where the constant, b, is zero in the case of the natural response (exponential decay) …and it would finite when we are considering the response of a circuit to a voltage step function.

The RL circuit produces a similar equation – except it is an equation for i(t):

$$\frac{di}{dt} + ai = b$$

We're not going to deal, explicitly, with the step response, because it's very similar to the natural response….only the end point (after many time constants have elapsed) will be a finite value of voltage or current rather than zero/ In any case we can show that the general response of these RL and RC circuits (including the step response as well as the natural response) can be written as:

$$\boxed{x(t) = x_f + \left[x(t_0) - x_f\right] e^{-(t-t_0)/\tau}}$$

where x is either the current or voltage, x_f is the final value of the current or voltage, $x(t_0)$ is the initial value of the current or voltage, and τ is the appropriate time constant for the circuit. Obviously, in the case of the natural response the final value is zero.