COEN281 -- Introduction to Pattern Recognition and Data Mining

Lecture 3: Bayesian Decision Theory

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Syllabus

Week 1	Introduction; R (Ch.1)
Week 2	**Bayesian Decision Theory** (Ch.2; DHS: 2.1-2.6, 2.9) Parameter Estimation (DHS: 3.1-3.4)
Week 3	Linear Discriminant Functions (Ch.3&4; DHS: 3.8.2, 5.1-5.8) Regularization (Ch.6; SE: Ch.3)
Week 4	Neural Networks (DHS: 6.1-6.6, 6.8); Deep Learning
Week 5	Support Vector Machines (Ch.9)
Week 6	Decision Trees (Ch. 8.1; DHS: 8.3; Ch 2 SE)
Week 7	Ensemble Methods (Ch. 8.2; SE: Ch 4, 5)
Week 8	Clustering (Ch. 10; DHS: 10.6, 10.7) Clustering (DHS: 10.9); How many clusters are there? (DHS: 10.10)
Week 9	Non-metric: Association Rules Collaborative Filtering
Week 10	Text Retrieval; Other topics

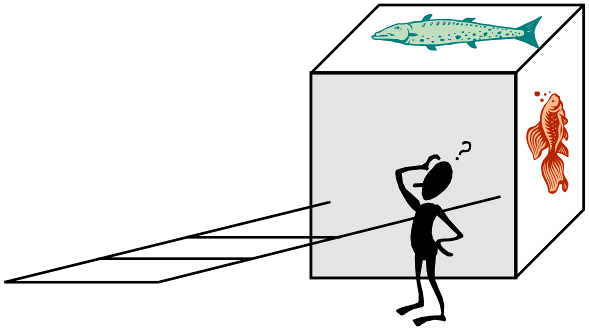
Overview

- Basic statistical concepts
  - Apriori probability, class-conditional density
  - Bayes formula & decision rule
  - Loss function & minimum-risk classifier

- Discriminant functions
- Decision regions/boundaries
- The Normal density
  - Discriminant functions (LDA)

Introduction
Statistical Approach

- A formalization of common-sense procedures…
- Quantify tradeoffs between various classification decisions using probability
- Initially assume all relevant probability values are known
- State of nature
  - What fish type (ω) will come out next?
    - ω_1 = *salmon*, ω_2 = *sea bass*
  - ω is unpredictable – i.e., a random variable



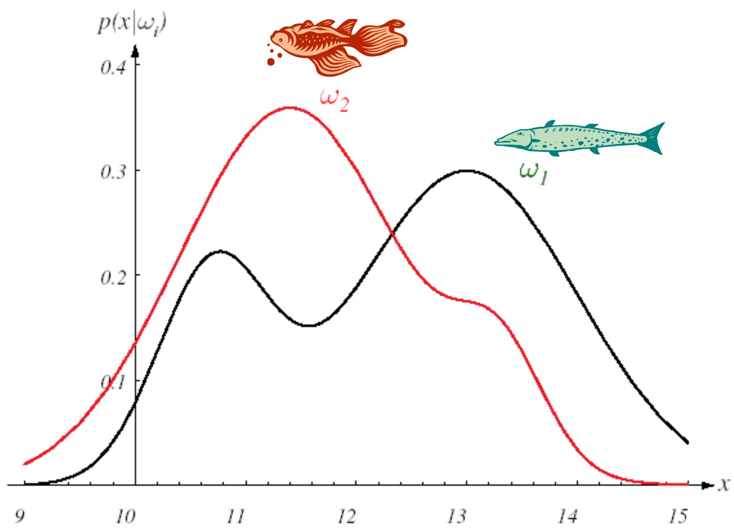
- A priori probability -- prior knowledge of how likely each fish type is -- $P(\omega_1) + P(\omega_2) = 1$

Introduction
Statistical Approach (2)

- Best decision rule about next fish type before it actually appears?
  - Decide ω_1 if $P(\omega_1) > P(\omega_2)$; otherwise decide ω_2
  - How well it works?
    - $P(error) = min\ [P(\omega_1), P(\omega_2)]$

- Incorporating lightness/length info
  - Class-conditional probability density

    $p(x|\omega_1)$ and $p(x|\omega_2)$ describe the difference in lightness between populations of sea bass and salmon



Introduction
Statistical Approach (3)

- $p(x|\omega_j)$ also called the likelihood of ω_j with respect to x
  - Other things being equal, ω_j for which $p(x|\omega_j)$ is largest is more "likely" to be true class

- Combining prior & likelihood into *posterior* – Bayes formula

$$p(\omega_j, \mathbf{x}) = p(\omega_j \mid \mathbf{x}) p(\mathbf{x}) = p(\mathbf{x} \mid \omega_j) P(\omega_j)$$

$$\boxed{p(\omega_j \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \omega_j) P(\omega_j)}{p(\mathbf{x})}}$$

where

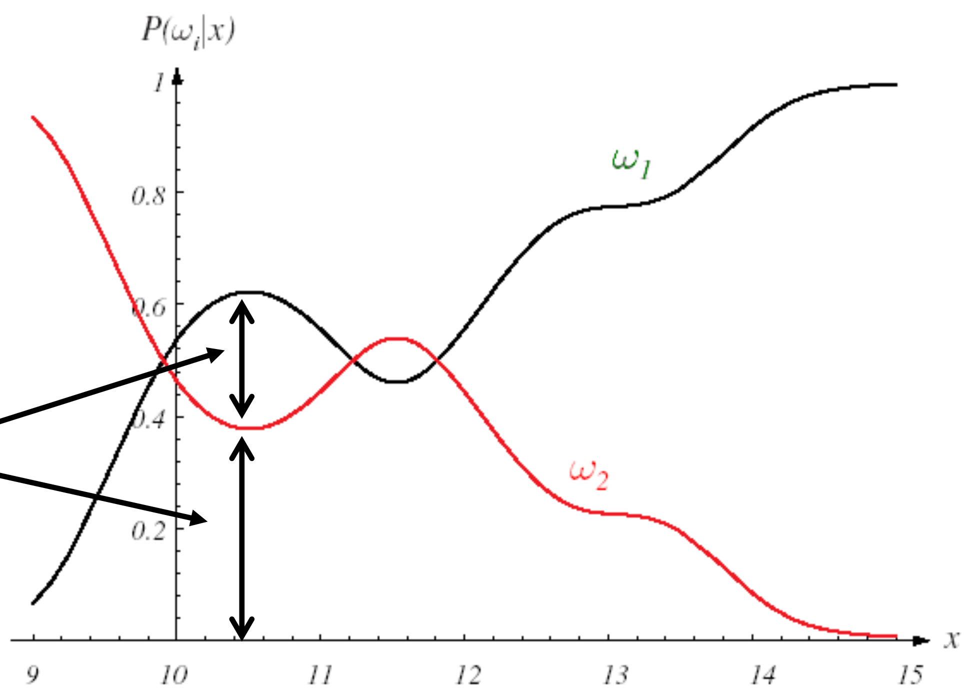
$$p(\mathbf{x}) = \sum_j p(\mathbf{x} \mid \omega_j) P(\omega_j)$$

Introduction
Statistical Approach (4)

- Bayes Decision Rule
  - Decide ω_1 if $P(\omega_1|x) > P(\omega_2|x);$ otherwise decide ω_2

    or
  - Decide ω_1 if $p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2);$ otherwise decide ω_2

Sum to 1.0 at every x



$P(\omega_1) = 2/3$

$P(\omega_2) = 1/3$

Bayesian Decision Theory
Loss Function

- $\lambda(\alpha_i | \omega_j)$: cost incurred for taking action α_i (i.e., classification or rejection) when the state of nature is ω_j

- Example
  - ***x****:* financial characteristics of firms applying for a bank loan
  - ω_0 – company did not go bankrupt
    ω_1 – company failed
  - $P(\omega_i|\mathbf{x})$ – predicted probability of bankruptcy
  - Confusion matrix:

	Algorithm: ω_0	Algorithm: ω_1
Truth: ω_0	TN	FP
Truth: ω_1	FN	TP

  - FN are 10 times as costly as FP

$$\Rightarrow \lambda(\alpha_0 | \omega_1) = \lambda_{01} = 10 \times \lambda(\alpha_1 | \omega_0) = 10 \times \lambda_{10}$$

Bayesian Decision Theory
Minimum Risk Classifier

- "Risk" (or expected loss) associated with taking action α_i

$$R(\alpha_i \mid \mathbf{x}) = \sum_{j=1}^{C} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid \mathbf{x})$$

- Decision rule: compute $R(\alpha_i|\mathbf{x})$ for $i=1,...a$ and select α_i for which $R(\alpha_i|\mathbf{x})$ is minimum

Bayesian Decision Theory
Minimum Risk Classifier (2)

- Two-category case

$$R(\alpha_0 \mid \mathbf{x}) = \lambda_{00} P(\omega_0 \mid \mathbf{x}) + \lambda_{01} P(\omega_1 \mid \mathbf{x})$$
$$R(\alpha_1 \mid \mathbf{x}) = \lambda_{10} P(\omega_0 \mid \mathbf{x}) + \lambda_{11} P(\omega_1 \mid \mathbf{x})$$

- Expressing minimum-risk rule: pick ω_o if $R(\alpha_0|\mathbf{x}) < R(\alpha_1|\mathbf{x})$, or

$$(\lambda_{10} - \lambda_{00}) P(\omega_0 \mid \mathbf{x}) > (\lambda_{01} - \lambda_{11}) P(\omega_1 \mid \mathbf{x})$$

- In our loan example: $\lambda_{00} = \lambda_{11} = 0$

$$\frac{P(\omega_0 \mid \mathbf{x})}{P(\omega_1 \mid \mathbf{x})} > \frac{\lambda_{01}}{\lambda_{10}} \quad \Longrightarrow \quad P(\omega_0 \mid \mathbf{x}) > 10 \times P(\omega_1 \mid \mathbf{x})$$

Bayesian Decision Theory
Minimum Error Rate Classifier

- Zero-one loss function leads to:

$$R(\alpha_i \mid \mathbf{x}) = \sum_{j=1}^{C} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid \mathbf{x})$$
$$= \sum_{j \neq i} P(\omega_j \mid \mathbf{x})$$
$$= 1 - P(\omega_i \mid \mathbf{x})$$

- i.e., choose ω_i for which $P(\omega_i|\mathbf{x})$ is maximum
  - same rule as in Slide 6 as expected

Bayesian Decision Theory
Discriminant Function

- A useful way of representing a classifier
  - One function $g_i(\mathbf{x})$ for each class
  - Assign $\mathbf{x}$ to ω_i if $g_i(\mathbf{x}) > g_j(\mathbf{x})$ for all $j \neq i$

- Minimum risk: $g_i(\mathbf{x}) = -R(\alpha_i|\mathbf{x})$

- Minimum error: $g_i(\mathbf{x}) = P(\omega_i|\mathbf{x})$
  - Monotonic increasing transformations are equivalent

$$g_i(\mathbf{x}) = p(\mathbf{x}|\omega_i)P(\omega_i)$$

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}|\omega_i) + \ln P(\omega_i)$$

Bayesian Decision Theory
Discriminant Function (2)

- Two-category case – dichotomizer
  - A single function suffices:

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x})$$

  - Decision rule:

Choose ω_1 if $g(\mathbf{x}) > 0$; *otherwise choose* ω_2

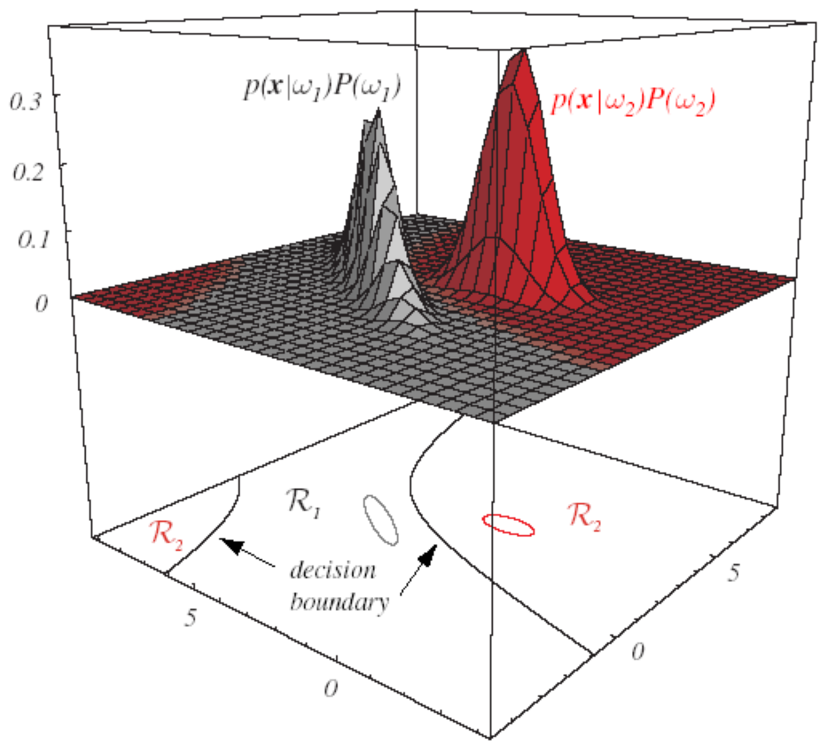
  - Convenient forms

$$g(\mathbf{x}) = P(\omega_1|\mathbf{x}) - P(\omega_2|\mathbf{x})$$

$$g(\mathbf{x}) = ln\,\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} + ln\,\frac{P(\omega_1)}{P(\omega_2)}$$

Bayesian Decision Theory

Decision Regions & Boundaries



R_i region in feature space where

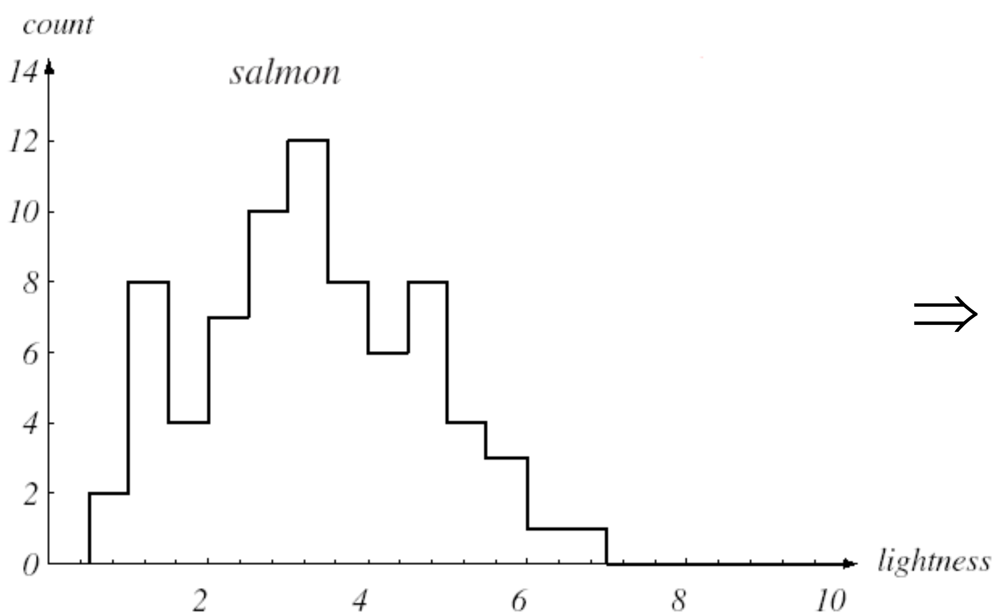
$g_i(\mathbf{x}) > g_j(\mathbf{x})$ for all $j \neq i$

– Might not be simply connected

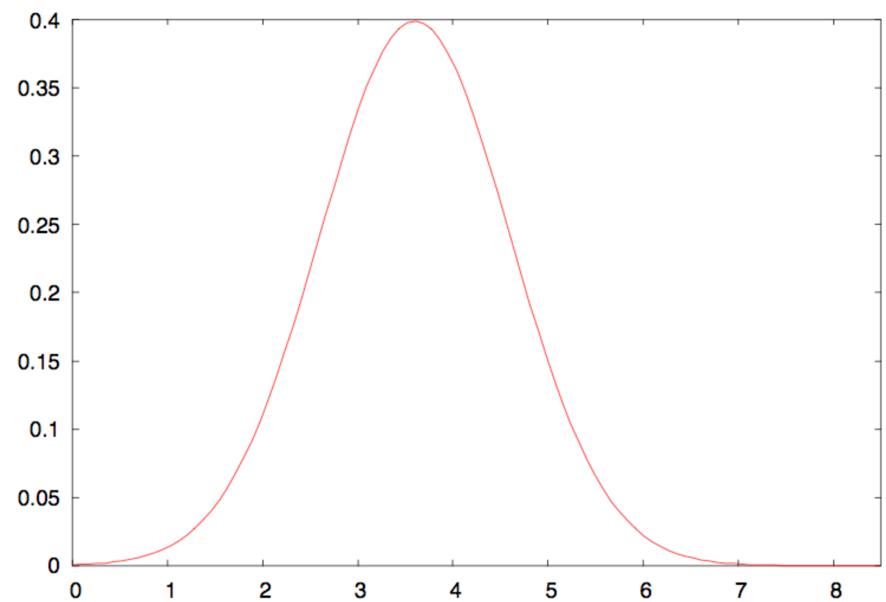
- Decision boundary: surfaces in feature space where ties occur among largest disciminant functions

Normal Density
Introduction

- Used to model $p(x|\omega_i)$



$\Rightarrow$



- Special attention due to:
  - Analytically tractable
  - A continuous-valued feature x can be seen as randomly corrupted version of a single typical μ (asymptotically Gaussian)

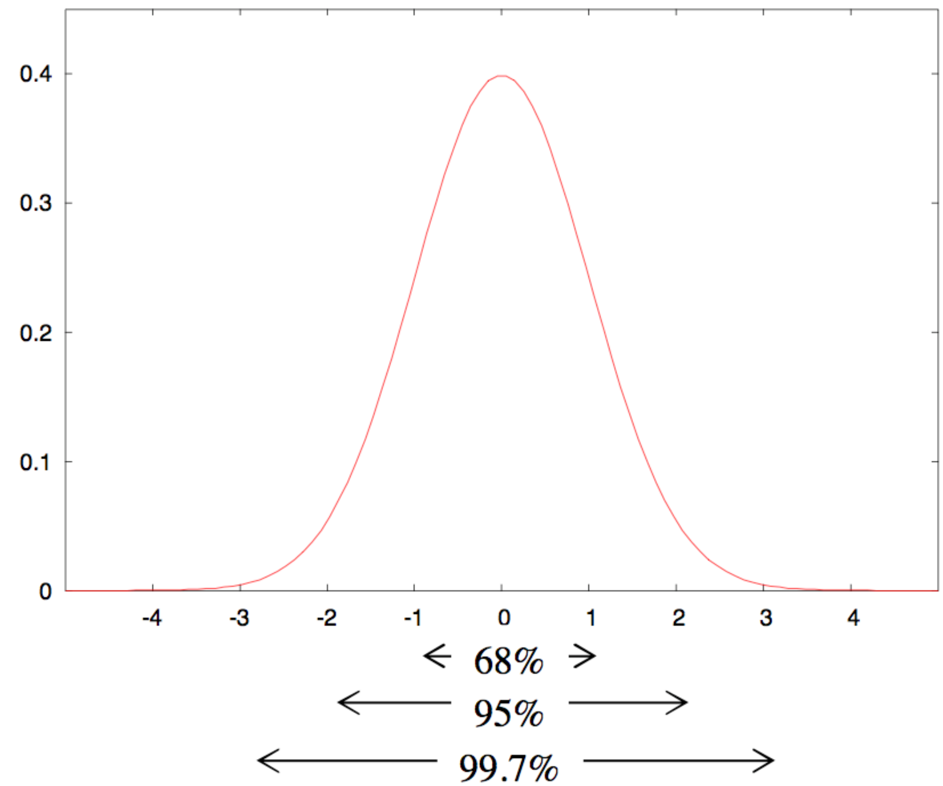
Normal Density
Univariate Case

- $x \sim N(0,\ 1)$ -- x is normally distributed with zero *mean* and unit *variance*

$$p_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$0 = \mu = \varepsilon[x]$$

$$1 = \sigma^2 = \varepsilon[(x-\mu)^2]$$



- Location-scale shift

$$z = \sigma x + \mu$$

$$\sim N(\mu,\ \sigma)$$

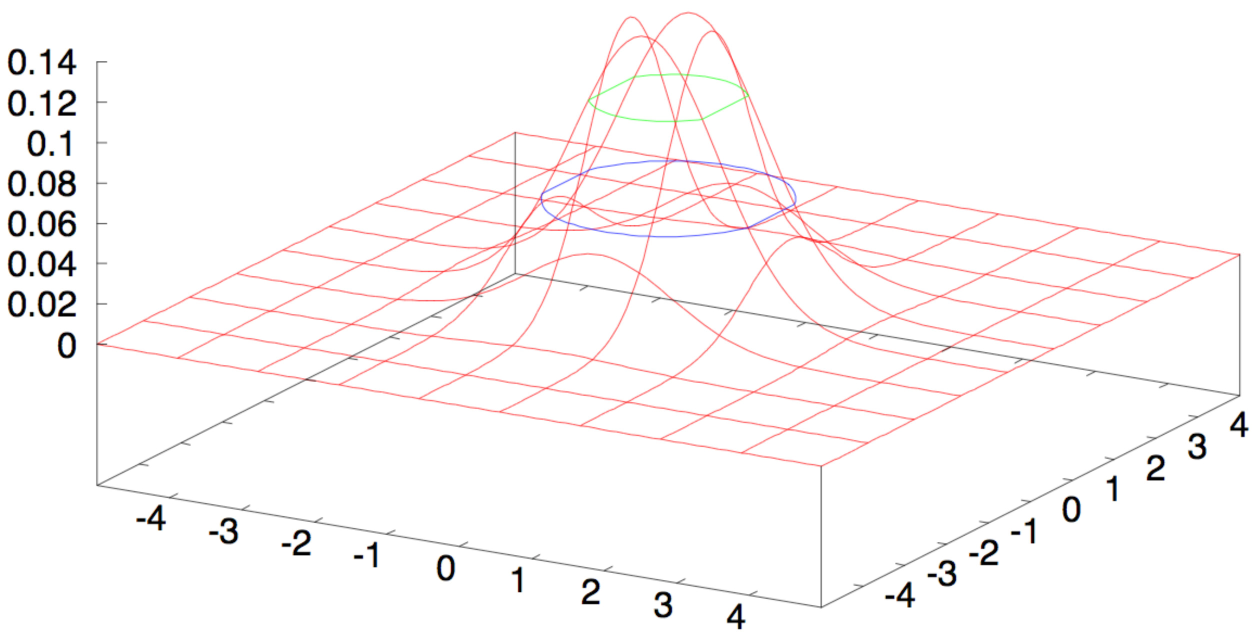
$$p_z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2} = \frac{1}{\sigma} p_x\left(\frac{z-\mu}{\sigma}\right)$$

Normal Density
Bivariate Case

- If $x \sim N(0, 1)$ and $y \sim N(0, 1)$ are independent

$$p(x,y) = p(x) \times p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+y^2)}$$

- Contours: $p(x,y) = c_1 \Rightarrow x^2 + y^2 = c_2$

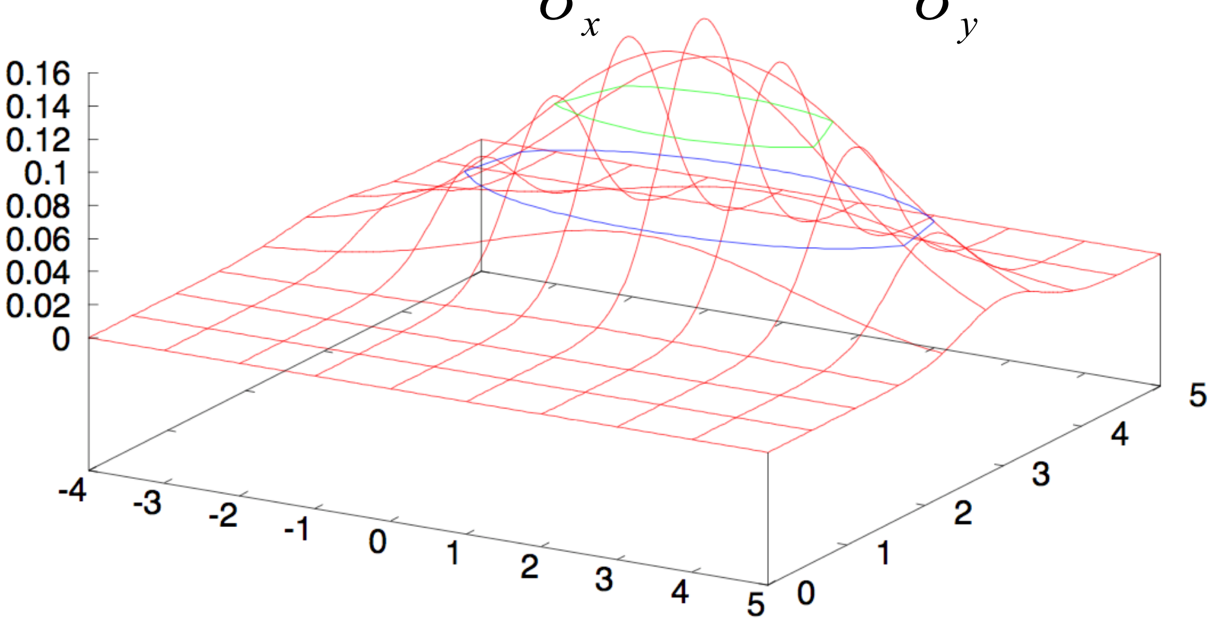


Normal Density
Bivariate Case (2)

- If $x \sim N(\mu_x, \sigma_x)$ and $y \sim N(\mu_y, \sigma_y)$ are independent

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)}$$

- Contours: $\frac{1}{\sigma_x^2}(x-\mu_x)^2 + \frac{1}{\sigma_y^2}(y-\mu_y)^2 = c$



$$p(x,y) = N\left(\begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}, \begin{bmatrix}\sigma_x^2 & 0 \\ 0 & \sigma_y^2\end{bmatrix}\right)$$

$$= N\left(\begin{bmatrix}1 \\ 8\end{bmatrix}, \underbrace{\begin{bmatrix}2^2 & 0 \\ 0 & \left(\frac{1}{2}\right)^2\end{bmatrix}}_{\text{variance-covariance matrix}}\right)$$

Normal Density
Multivariate Case

- We say $\mathbf{x} \sim N(\boldsymbol{\mu}, \Sigma)$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

where,

$\mathbf{x} = (x_1, x_2, \ldots, x_d)^t$ (t stands for the transpose vector form)

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_d)^t$ mean vector

$\boldsymbol{\Sigma}$: $d \times d$ covariance matrix

$|\Sigma|$ and Σ^{-1} are determinant and inverse respectively

$(\boldsymbol{x} - \boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})$ is (square) *Mahalanobis* distance

Normal Density
Multivariate Case

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})}$$

$$\Rightarrow \ln p(\mathbf{x}) = \ln 1 - \ln\left[(2\pi)^{d/2}|\Sigma|^{1/2}\right] + \ln\left[e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})}\right]$$

$$= 0 - \tfrac{d}{2}\ln(2\pi) - \tfrac{1}{2}\ln|\Sigma| - \tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

$$= -\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) - \tfrac{d}{2}\ln 2\pi - \tfrac{1}{2}\ln|\Sigma|$$

Bayesian Decision Theory

Discriminant Function – Normal Density

- $p(\mathbf{x} \mid \omega_i) \sim N(\boldsymbol{\mu}_i, \Sigma_i)$
- We had $g_i(\mathbf{x}) = \ln p(\mathbf{x} \mid \omega_i) + \ln P(\omega_i)$

$$\Rightarrow g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^t \Sigma_i^{-1} (\mathbf{x} - \boldsymbol{\mu}_i) - \cancel{\frac{d}{2}\ln 2\pi} - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

- Case 1: $\Sigma_i = \sigma^2 \mathbf{I}$
- Case 2: $\Sigma_i = \Sigma$

linear discriminant function

- Case 3: $\Sigma_i = arbitrary$

Bayesian Decision Theory
Discriminant Function – Normal Density (2)

- Case 1: features are statistically independent ($\sigma_{ij} = 0$) and share same variance σ^2

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\boldsymbol{\mu}_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$

$$= -\frac{1}{2\sigma^2}\left[\cancel{\mathbf{x}^t\mathbf{x}} - 2\boldsymbol{\mu}_i^t\mathbf{x} + \boldsymbol{\mu}_i^t\boldsymbol{\mu}_i\right] + \ln P(\omega_i)$$

$$= \boxed{\boldsymbol{\alpha}_i^t\mathbf{x} + \alpha_{i0}}$$

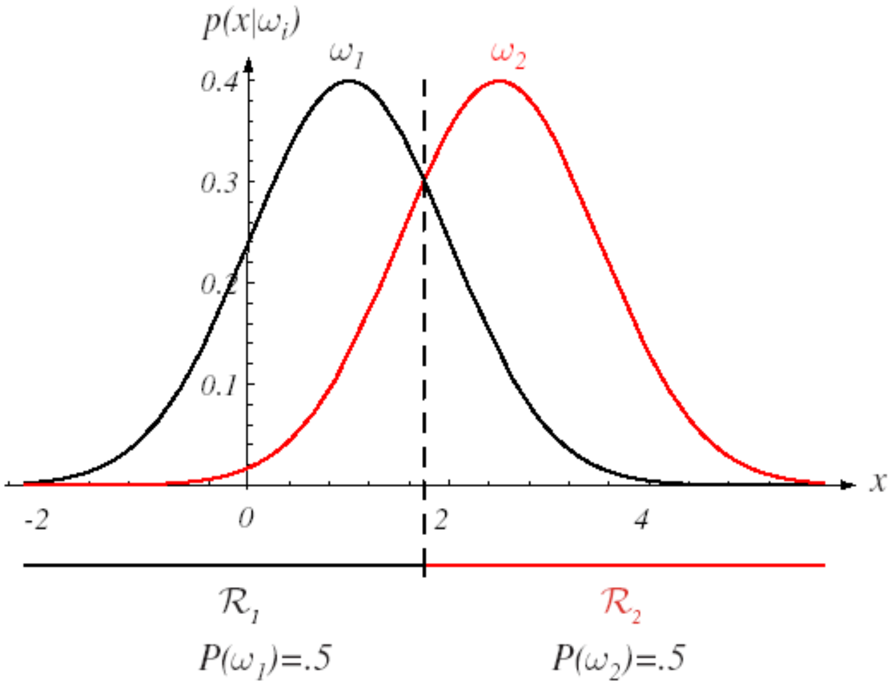
where $\boldsymbol{\alpha}_i = \frac{1}{\sigma^2}\boldsymbol{\mu}_i$

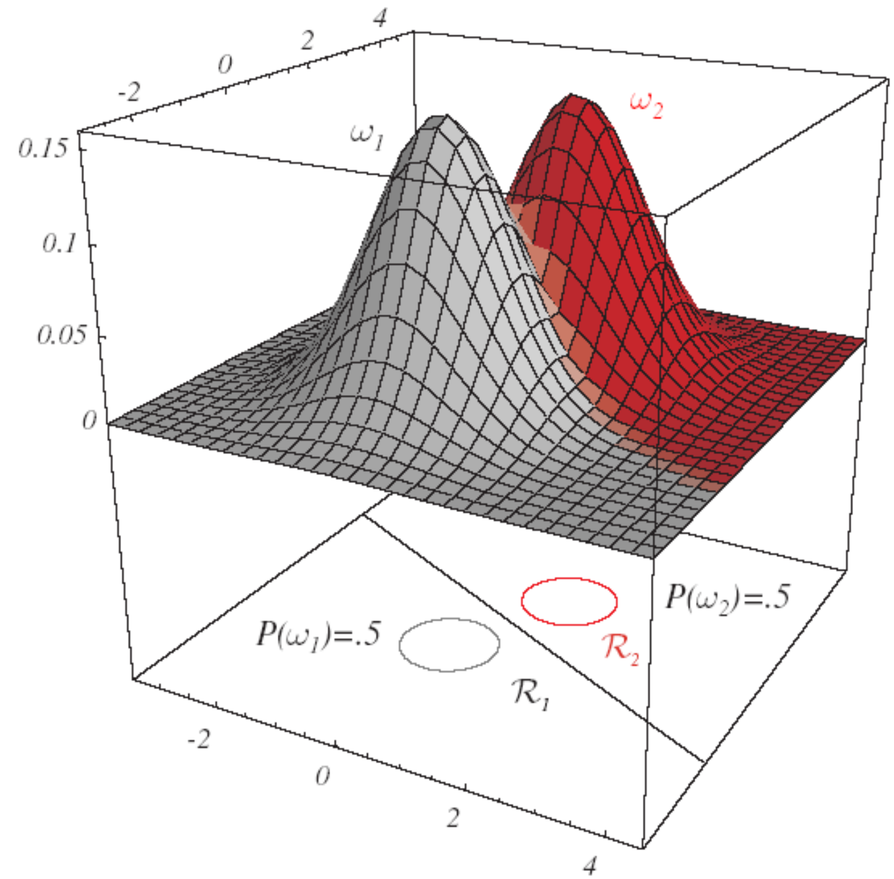
$$\alpha_{i0} = -\frac{1}{2\sigma^2}\left[\boldsymbol{\mu}_i^t\boldsymbol{\mu}_i\right] + \ln P(\omega_i)$$

- All priors equal ⇒ Minimum (Euclidean) distance classifier

Bayesian Decision Theory
Discriminant Function – Normal Density (3)

- Case 1: distributions are "spherical" in d dimensions; boundary is a *hyperplane* in d-1 dimensions perpendicular to line between means





Bayesian Decision Theory
Discriminant Function – Normal Density (4)

- Case 2: samples fall in hyperellipsoidal clusters of equal size and shape

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) + \ln P(\omega_i)$$

$$= \boldsymbol{\alpha}_i^t \mathbf{x} + \alpha_{i0} \qquad \text{as } \mathbf{x}^t \Sigma^{-1} \mathbf{x} \text{ can be dropped}$$

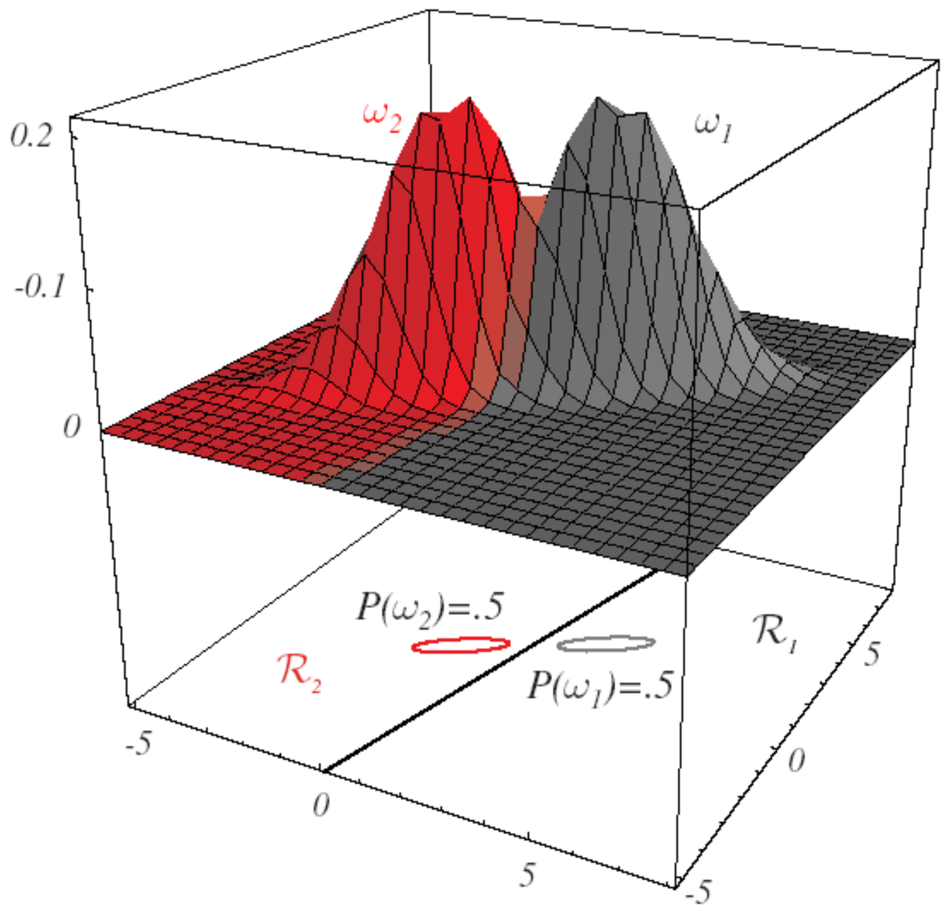
where $\boldsymbol{\alpha}_i = \Sigma^{-1} \boldsymbol{\mu}_i$

$$\alpha_{i0} = -\frac{1}{2}\boldsymbol{\mu}_i^t \Sigma^{-1} \boldsymbol{\mu}_i + \ln P(\omega_i)$$

- All priors equal $\Rightarrow$ Minimum (Mahalanobis) distance classifier

Bayesian Decision Theory
Discriminant Function – Normal Density (5)

- Case 2: hyperplane separating class regions is generally not perpendicular to line between the means



Bayesian Decision Theory
Discriminant Function – Normal Density (6)

- Case 3: decision surfaces are hyperquadratics (i.e., hyperplanes, pairs of hyperplanes, hypershpheres, hyperellipsoids, hyperhyperboloids)