FIRST AND SECOND-ORDER TRANSIENT CIRCUITS

IN CIRCUITS WITH INDUCTORS AND CAPACITORS VOLTAGES AND CURRENTS CANNOT CHANGE INSTANTANEOUSLY.
EVEN THE APPLICATION, OR REMOVAL, OF CONSTANT SOURCES CREATES A TRANSIENT BEHAVIOR

LEARNING GOALS

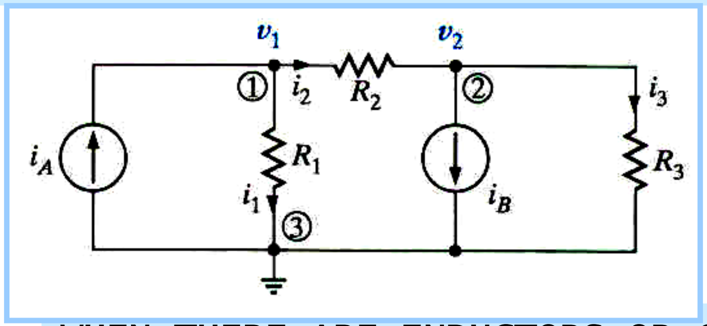
FIRST ORDER CIRCUITS
Circuits that contain a single energy storing elements.
Either a capacitor or an inductor

SECOND ORDER CIRCUITS
Circuits with two energy storing elements in any combination

ANALYSIS OF LINEAR CIRCUITS WITH INDUCTORS AND/OR CAPACITORS

THE CONVENTIONAL ANALYSIS USING MATHEMATICAL MODELS REQUIRES THE DETERMINATION OF (A SET OF) EQUATIONS THAT REPRESENT THE CIRCUIT.
ONCE THE MODEL IS OBTAINED ANALYSIS REQUIRES THE SOLUTION OF THE EQUATIONS FOR THE CASES REQUIRED.

FOR EXAMPLE IN NODE OR LOOP ANALYSIS OF RESISTIVE CIRCUITS ONE REPRESENTS THE CIRCUIT BY A SET OF ALGEBRAIC EQUATIONS



THE MODEL

$$(G_1 + G_2)v_1 - G_2 v_2 = i_A$$
$$-G_2 v_1 + (G_2 + G_3)v_2 = -i_B$$

WHEN THERE ARE INDUCTORS OR CAPACITORS THE MODELS BECOME LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODEs). HENCE, IN GENERAL, ONE NEEDS ALL THOSE TOOLS IN ORDER TO BE ABLE TO ANALYZE CIRCUITS WITH ENERGY STORING ELEMENTS.

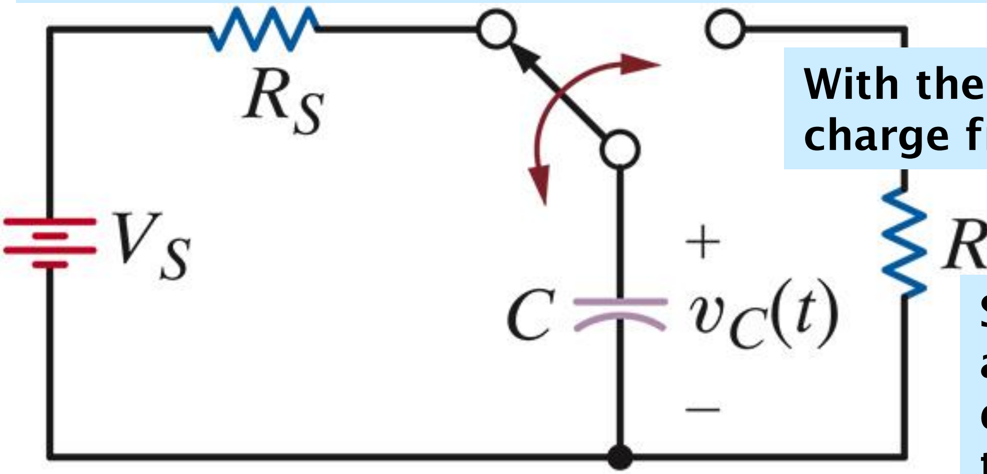
A METHOD BASED ON THEVENIN WILL BE DEVELOPED TO DERIVE MATHEMATICAL MODELS FOR ANY ARBITRARY LINEAR CIRCUIT WITH ONE ENERGY STORING ELEMENT.

THE GENERAL APPROACH CAN BE SIMPLIFIED IN SOME SPECIAL CASES WHEN THE FORM OF THE SOLUTION CAN BE KNOWN BEFOREHAND.
THE ANALYSIS IN THESE CASES BECOMES A SIMPLE MATTER OF DETERMINING SOME PARAMETERS.
TWO SUCH CASES WILL BE DISCUSSED IN DETAIL FOR THE CASE OF CONSTANT SOURCES.
ONE THAT ASSUMES THE AVAILABILITY OF THE DIFFERENTIAL EQUATION AND A SECOND THAT IS ENTIRELY BASED ON ELEMENTARY CIRCUIT ANALYSIS… BUT IT IS NORMALLY LONGER

WE WILL ALSO DISCUSS THE PERFORMANCE OF LINEAR CIRCUITS TO OTHER SIMPLE INPUTS

AN INTRODUCTION

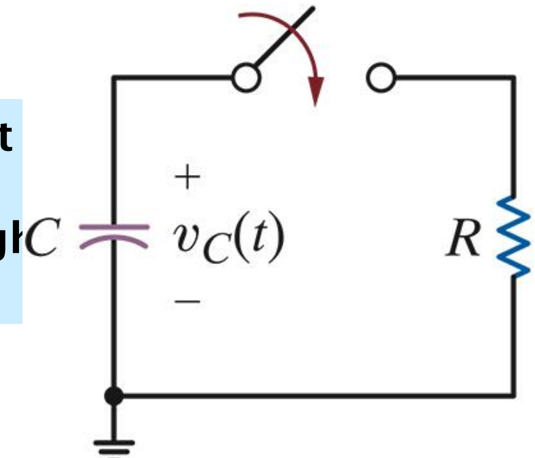
INDUCTORS AND CAPACITORS CAN STORE ENERGY. UNDER SUITABLE CONDITIONS THIS ENERGY CAN BE RELEASED. THE RATE AT WHICH IT IS RELEASED WILL DEPEND ON THE PARAMETERS OF THE CIRCUIT CONNECTED TO THE TERMINALS OF THE ENERGY STORING ELEMENT

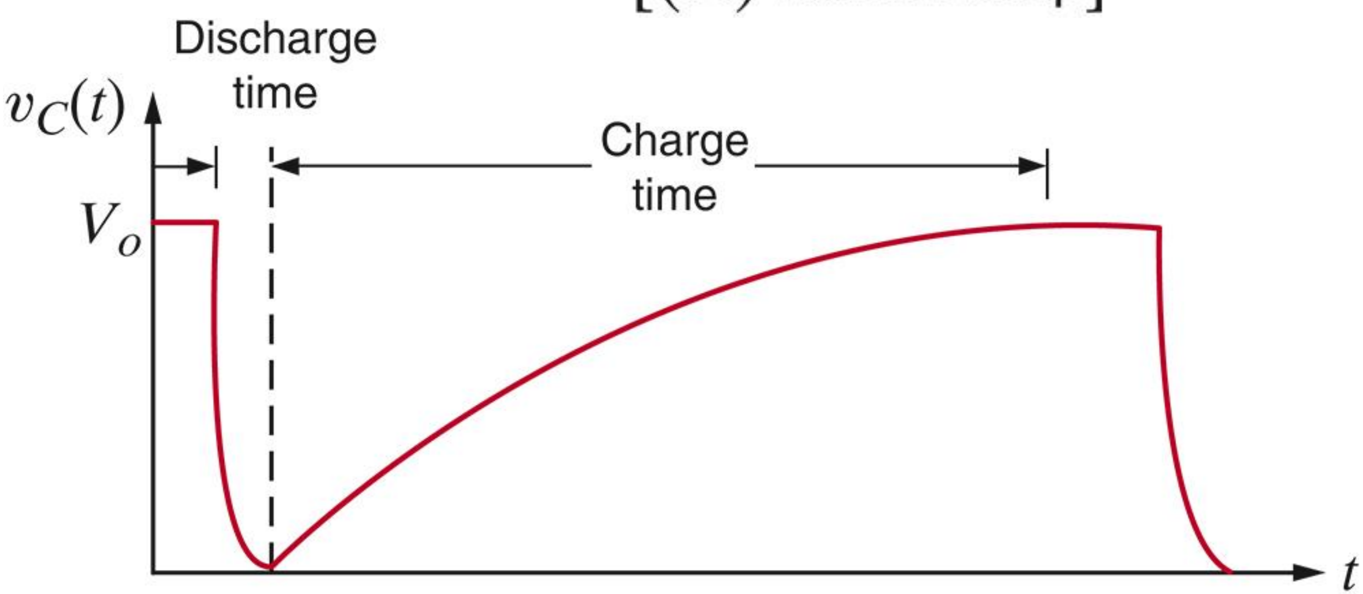


With the switch on the left the capacitor receives charge from the battery.

Switch to the right and the capacitor discharges through the lamp

[(R) Xenon lamp]





GENERAL RESPONSE: FIRST ORDER CIRCUITS

Including the initial conditions the model for the capacitor voltage or the inductor current will be shown to be of the form

$$\frac{dx}{dt}(t) + ax(t) = f(t);\ x(0+) = x_0$$

$$\boxed{\tau}\frac{dx}{dt} + x = f_{TH};\ x(0+) = x_0$$

Solving the differential equation using integrating factors, one tries to convert the LHS into an exact derivative

$$\tau\frac{dx}{dt} + x = f_{TH}\quad /*\frac{1}{\tau}e^{\frac{t}{\tau}}$$

$$e^{\frac{t}{\tau}}\frac{dx}{dt} + \frac{1}{\tau}e^{\frac{t}{\tau}}x = \frac{1}{\tau}e^{\frac{t}{\tau}}f_{TH}$$

$$\int_{t_0}^{t}\qquad \frac{d}{dt}\left(e^{\frac{t}{\tau}}x\right) = \frac{1}{\tau}e^{\frac{t}{\tau}}f_{TH}$$

$$e^{\frac{t}{\tau}}x(t) - e^{\frac{t_0}{\tau}}x(t_0) = \int_{t_0}^{t}\frac{1}{\tau}e^{\frac{x}{\tau}}f_{TH}(x)dx \quad */e^{-\frac{t}{\tau}}$$

$$x(t) = e^{-\frac{t-t_0}{\boxed{\tau}}}x(t_0) + \frac{1}{\tau}\int_{t_0}^{t}e^{-\frac{t-x}{\tau}}f_{TH}(x)dx$$

THIS EXPRESSION ALLOWS THE COMPUTATION OF THE RESPONSE FOR ANY FORCING FUNCTION. WE WILL CONCENTRATE IN THE SPECIAL CASE WHEN THE RIGHT HAND SIDE IS CONSTANT

τ is called the "time constant."
it will be shown to provide significant information on the reaction speed of the circuit

The initial time, t_o, is arbitrary. The general expression can be used to study sequential switchings.

FIRST ORDER CIRCUITS WITH CONSTANT SOURCES

$$\tau \frac{dx}{dt} + x = f_{TH}; \quad x(0+) = x_0$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{1}{\tau} \int_{t_0}^{t} e^{-\frac{t-x}{\tau}} f_{TH}(x)dx$$

If the RHS is constant

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau} \int_{t_0}^{t} e^{-\frac{t-x}{\tau}} dx$$

$$e^{-\frac{t-x}{\tau}} = e^{-\frac{t}{\tau}} e^{\frac{x}{\tau}} \Rightarrow$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau} e^{-\frac{t}{\tau}} \int_{t_0}^{t} e^{\frac{x}{\tau}} dx$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{f_{TH}}{\tau} e^{-\frac{t}{\tau}} \left(\tau e^{\frac{x}{\tau}} \right)\Big|_{t_0}^{t}$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + f_{TH} e^{-\frac{t}{\tau}} \left(e^{\frac{t}{\tau}} - e^{\frac{t_0}{\tau}} \right)$$

$$x(t) = f_{TH} + \left(x(t_0) - f_{TH} \right) e^{-\frac{t-t_0}{\tau}}$$

$$t \geq t_0$$

The form of the solution is

$$x(t) = K_1 + \underbrace{K_2 e^{-\frac{t-t_0}{\boxed{\tau}}}}_{\text{TRANSIENT}}; \; t \geq t_0$$

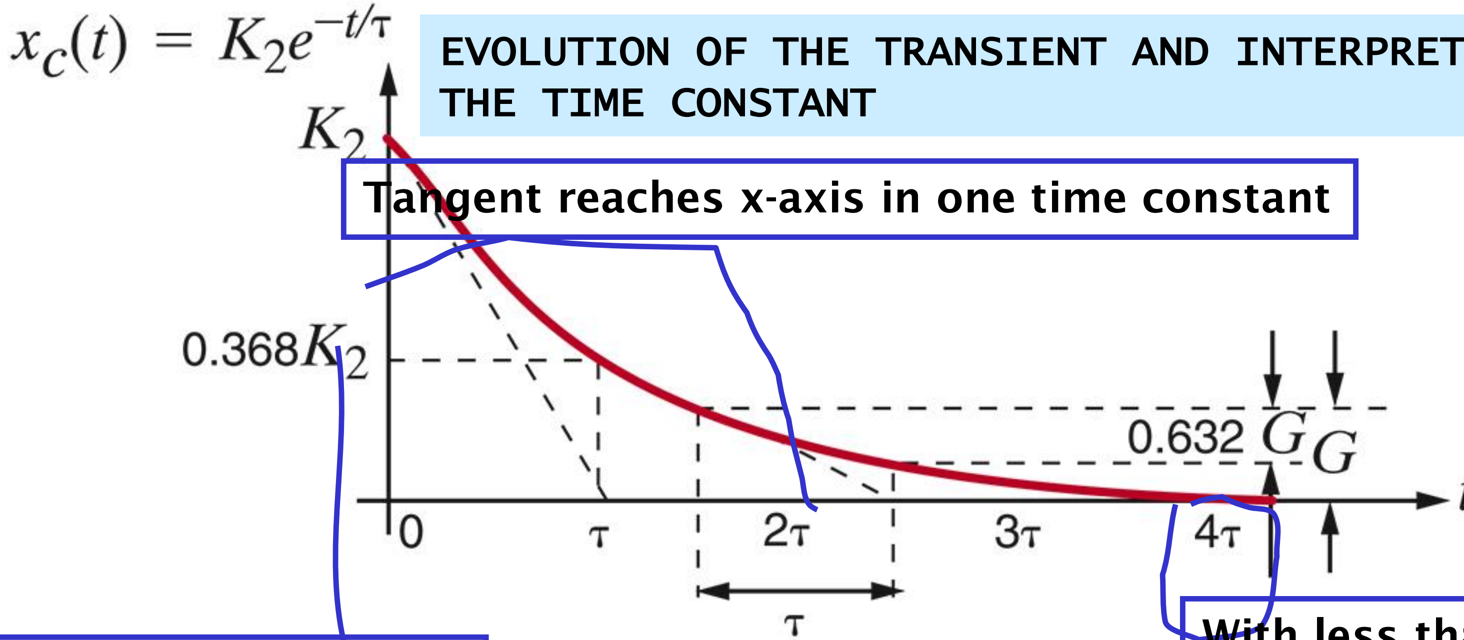
TIME CONSTANT

Any variable in the circuit is of the form

$$y(t) = K_1 + K_2 e^{-\frac{t-t_0}{\tau}}; \; t \geq t_0$$

Only the values of the constants K_1, K_2 will change

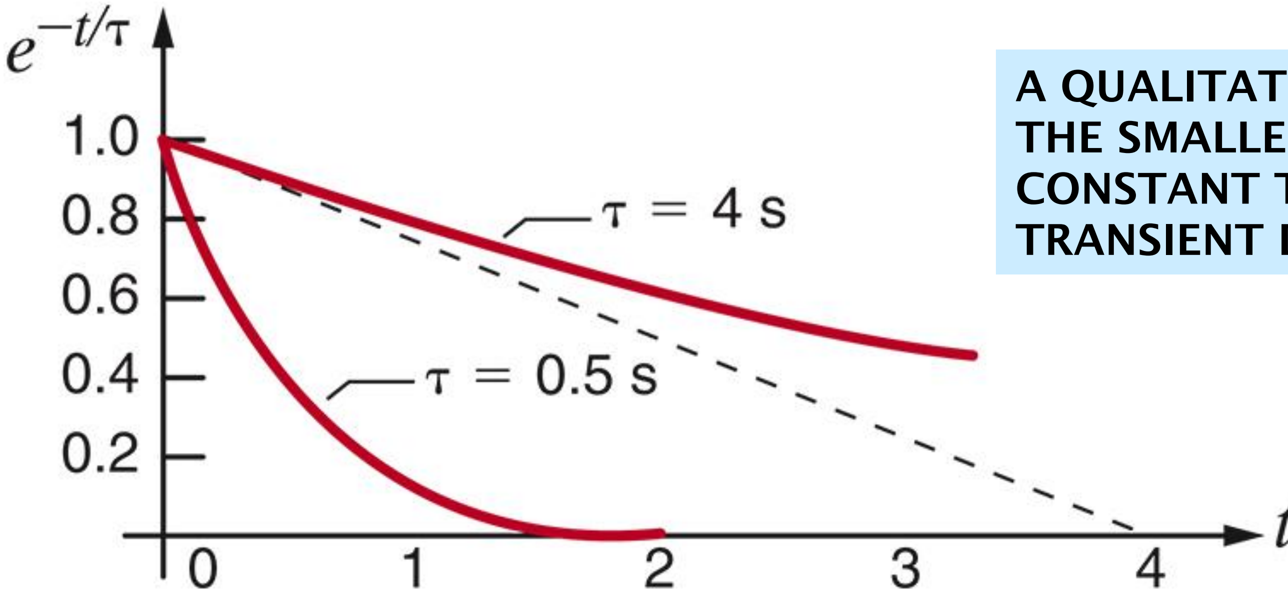
EVOLUTION OF THE TRANSIENT AND INTERPRETATION OF THE TIME CONSTANT



Tangent reaches x-axis in one time constant

Drops 0.632 of initial value in one time constant

With less than 2% error transient is zero beyond this point

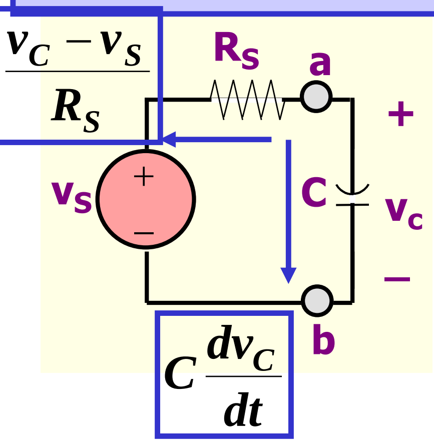


A QUALITATIVE VIEW:
THE SMALLER THE THE TIME CONSTANT THE FASTER THE TRANSIENT DISAPPEARS

THE TIME CONSTANT

The following example illustrates the physical meaning of time constant

Charging a capacitor



KCL@a:

$$C\frac{dv_c}{dt} + \frac{v_C - v_S}{R_S} = 0$$

The model

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$

$$\boxed{\tau = R_{TH}C}$$

Assume

$$v_S = V_S, v_C(0) = 0$$

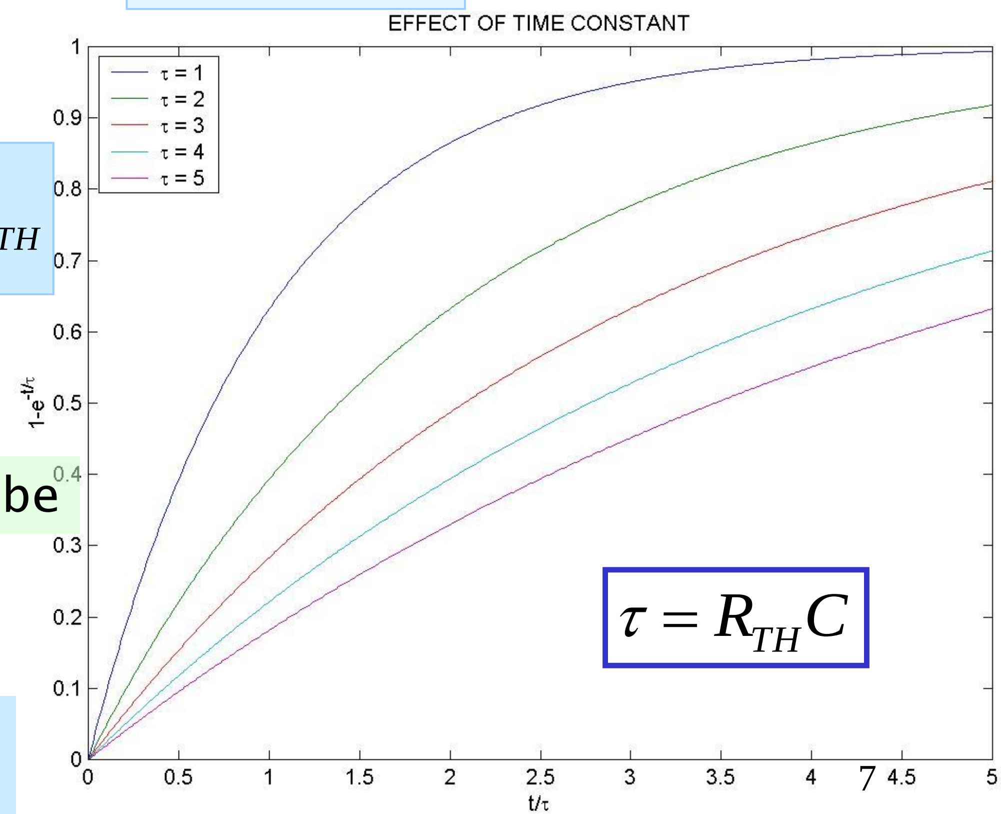
The solution can be shown to be

$$v_C(t) = V_S - \underbrace{V_S e^{-\frac{t}{\tau}}}_{\text{transient}}$$

For practical purposes the capacitor is charged when the transient is negligible

t	$e^{-\frac{t}{\tau}}$
τ	0.368
2τ	0.135
3τ	0.0498
4τ	0.0183
5τ	0.0067

With less than 1% error the transient is negligible after five time constants



CIRCUITS WITH ONE ENERGY STORING ELEMENT

THE DIFFERENTIAL EQUATION APPROACH

CONDITIONS

1. THE CIRCUIT HAS ONLY CONSTANT INDEPENDENT SOURCES
2. THE DIFFERENTIAL EQUATION FOR THE VARIABLE OF INTEREST IS SIMPLE TO OBTAIN. NORMALLY USING BASIC ANALYSIS TOOLS; e.g., KCL, KVL. . . OR THEVENIN
3. THE INITIAL CONDITION FOR THE DIFFERENTIAL EQUATION IS KNOWN, OR CAN BE OBTAINED USING STEADY STATE ANALYSIS

FACT: WHEN ALL INDEPENDENT SOURCES ARE CONSTANT FOR ANY VARIABLE, $y(t)$, IN THE CIRCUIT THE SOLUTION IS OF THE FORM

$$y(t) = K_1 + K_2 e^{-\frac{(t-t_o)}{\tau}}, t > t_o$$

<u>SOLUTION STRATEGY</u>: USE THE DIFFERENTIAL EQUATION AND THE INITIAL CONDITIONS TO FIND THE PARAMETERS K_1, K_2, τ

If the diff eq for y is known in the form

$$a_1 \frac{dy}{dt} + a_0 y = f$$

$$y(0+) = y_0$$

We can use this info to find the unknowns

Use the diff eq to find two more equations by replacing the form of solution into the differential equation

$$y(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0 \Rightarrow \frac{dy}{dt} = -\frac{K_2}{\tau} e^{-\frac{t}{\tau}}$$

$$a_1\left(-\frac{K_2}{\tau} e^{-\frac{t}{\tau}}\right) + a_0\left(K_1 + K_2 e^{-\frac{t}{\tau}}\right) = f$$

$$a_0 K_1 = f \Rightarrow \boxed{K_1 = \frac{f}{a_0}}$$

$$\left(-\frac{a_1}{\tau} + a_0\right) K_2 e^{-\frac{t}{\tau}} = 0 \Rightarrow \boxed{\tau = \frac{a_1}{a_0}}$$

Use the initial condition to get one more equation

$$y(0+) = K_1 + K_2$$

$$\boxed{K_2 = y(0+) - K_1}$$

SHORTCUT: WRITE DIFFERENTIAL EQ. IN NORMALIZED FORM WITH COEFFICIENT OF VARIABLE = 1.

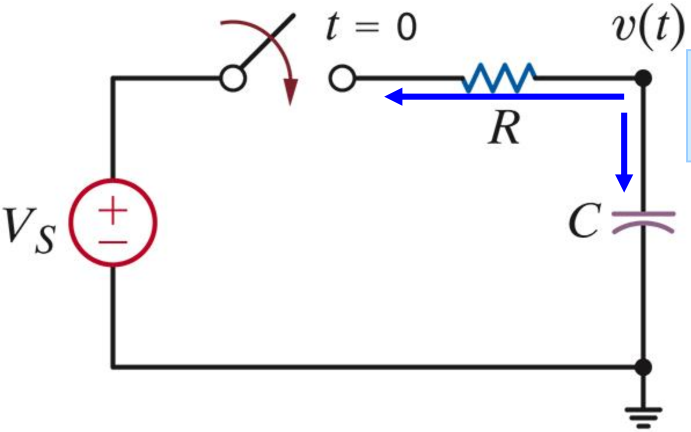
$$a_1 \frac{dy}{dt} + a_0 y = f \Rightarrow \underbrace{\boxed{\frac{a_1}{a_0}}}_{\tau} \frac{dy}{dt} + y = \underbrace{\boxed{\frac{f}{a_0}}}_{K_1}$$

EXAMPLE 1

FIND $v(t), t > 0$. ASSUME $v(0) = V_S/2$

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$$K_1 = x(\infty);\ K_1 + K_2 = x(0+)$$



ANSWER : $v(t) = V_S - (V_S/2)e^{-\frac{t}{RC}}, t > 0$

MODEL FOR $t > 0$. USE KCL @ $v(t)$

$$\frac{v(t) - V_S}{R} + C\frac{dv}{dt}(t) = 0 \quad \boxed{*/R}$$

initial condition $v(0) = V_S/2$

(DIFF. EQ. KNOWN, INITIAL CONDITION KNOWN)

STEP 1 TIME CONSTANT

$$\boxed{\tau}\frac{dy}{dt} + y = f$$

$$\boxed{RC}\frac{dv}{dt}(t) + v(t) = V_s$$

Get time constant as coefficient of derivative

STEP 2 STEADY STATE ANALYSIS

SOLUTION IS $v(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

for $\tau > 0$ and $t \to \infty$, $v(t) \to K_1$ (steady state value)

IN STEADY STATE THE SOLUTION IS A CONSTANT. HENCE ITS DERIVATIVE IS ZERO. FROM DIFF EQ.

$$\frac{dv}{dt} = 0 \Rightarrow v = V_S$$

Steady state value from diff. eq.

$\therefore$ (equating steady state values)

$K_1 = V_S$

IF THE MODEL IS $\tau\frac{dy}{dt} + y = f$ THEN $K_1 = f$

STEP 3 USE OF INITIAL CONDITION

AT $t = 0$

$$v(0) = K_1 + K_2 \Rightarrow K_2 = v(0) - K_1$$

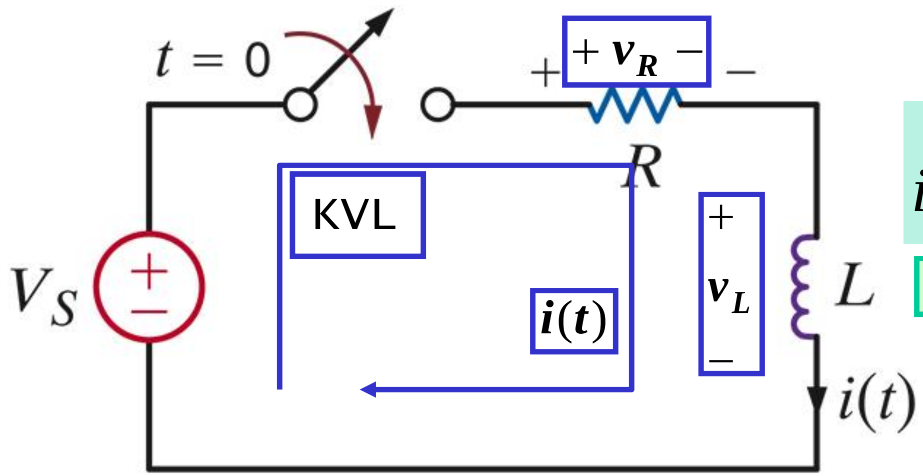
$$K_2 = v(0) - f$$

$$v(0) = V_S/2 \Rightarrow K_2 = -V_S/2$$

EXAMPLE 2

FIND $i(t), t > 0$

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$
$$K_1 = x(\infty);\ K_1 + K_2 = x(0+)$$



$$i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

MODEL. USE KVL FOR $t > 0$

$$V_S = v_R + v_L = Ri(t) + L\frac{di}{dt}(t)$$

INITIAL CONDITION

$$\left.\begin{array}{l} t < 0 \Rightarrow i(0-) = 0 \\ \text{inductor} \Rightarrow i(0-) = i(0+) \end{array}\right\} i(0+) = 0$$

STEP 1 $\quad \frac{L}{R}\frac{di}{dt}(t) + i(t) = \frac{V_S}{R} \qquad \tau = \frac{L}{R}$

STEP 2 STEADY STATE $\quad i(\infty) = K_1 = \frac{V_S}{R}$

STEP 3 INITIAL CONDITION

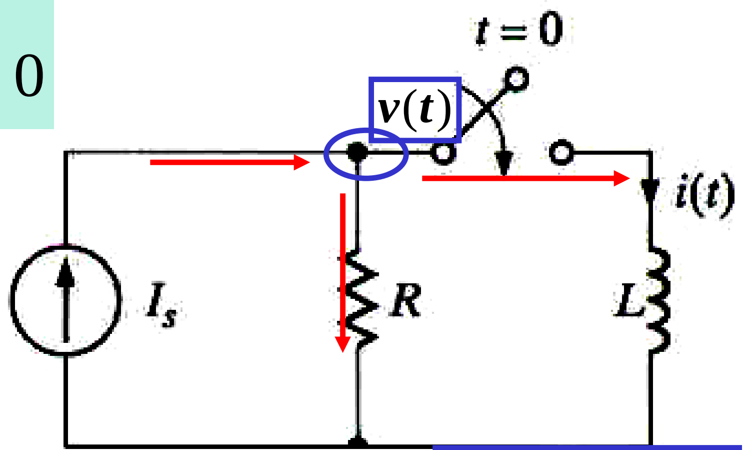
$$i(0+) = K_1 + K_2$$

ANS: $i(t) = \frac{V_S}{R}\left(1 - e^{-\frac{t}{L/R}}\right)$

EXAMPLE 3

Find $i(t)$ for $t > 0$ in the following network:

$$i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$



MODEL. KCL FOR $t > 0$

$$I_S = \frac{v(t)}{R} + i(t)$$

$$v(t) = L\frac{di}{dt}(t) \Rightarrow$$ $$I_S = \boxed{\frac{L}{R}}\frac{di}{dt}(t) + i(t)$$

INITIAL CONDITION: $i(0+) = 0$

STEP 1 $\boxed{\tau = \frac{L}{R}}$

STEP 2 $i(\infty) = I_S \Rightarrow K_1 = I_S$

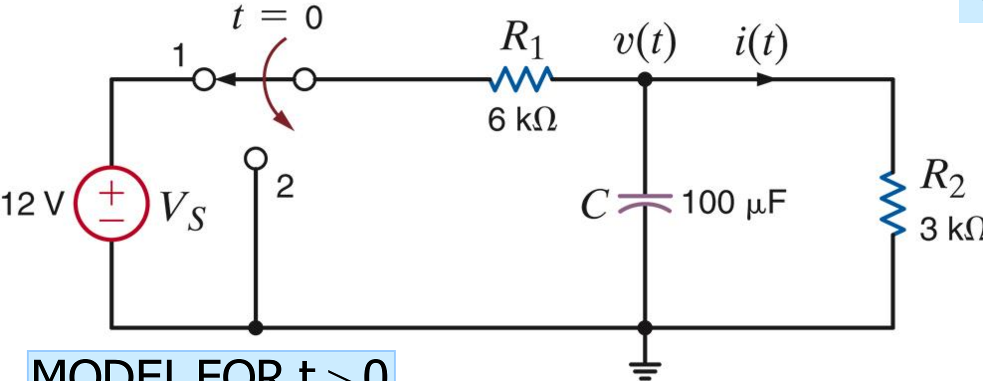
STEP 3 $i(0+) = 0 = K_1 + K_2$

ANS: $i(t) = I_S\left(1 - e^{-\frac{t}{L/R}}\right)$

EXAMPLE 4

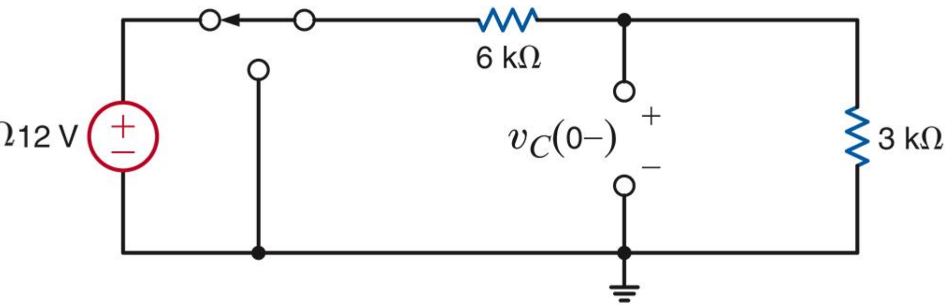
$$i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

Assuming that the switch has been in position 1 for a long time, at time $t = 0$ the switch is moved to position 2. We wish to calculate the current $i(t)$ for $t > 0$.

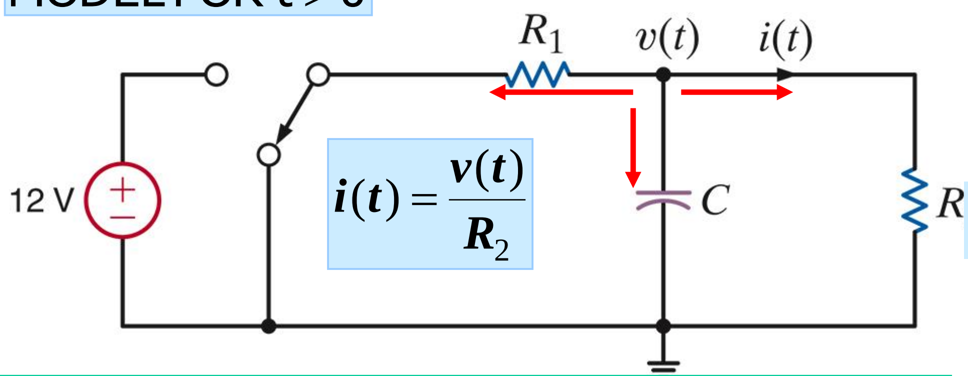


INITIAL CONDITIONS

CIRCUIT IN STEADY STATE FOR $t < 0$



MODEL FOR $t > 0$



$$i(t) = \frac{v(t)}{R_2}$$

$$v_C(0-) = \frac{3k}{3k + 6k}(12) = 4V \Rightarrow v(0+) = 4V$$

STEP 1

$$\tau = R_P C = (2\times10^3\Omega)(100\times10^{-6}F) = 0.2s$$

STEP 2 $v(\infty) = K_1 = 0$

STEP 3 $v(0+) = K_1 + K_2 = 4V \Rightarrow K_2 = 4V$

IT IS SIMPLER TO DETERMINE MODEL FOR CAPACITOR VOLTAGE

$$\frac{v(t)}{R_1} + C\frac{dv}{dt}(t) + \frac{v(t)}{R_2} = 0; \; R_P = R_1 \| R_2$$

$$R_P = 3k \| 6k = 2k\Omega$$

$$C\frac{dv}{dt}(t) + \frac{v(t)}{R_P} = 0$$

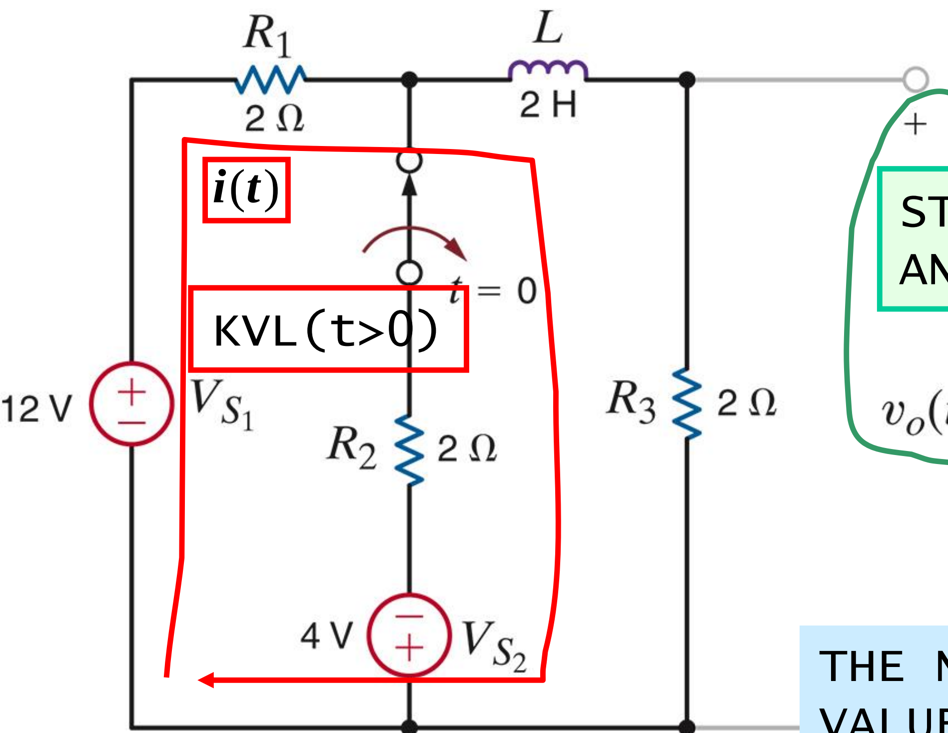
$$v(t) = 4e^{-\frac{t}{0.2}}[V], t > 0$$

$$\text{ANS}: i(t) = \frac{4}{3}e^{-\frac{t}{0.2}}[mA], t > 0$$

EXAMPLE 5

FIND $v_O(t), t>0$

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$$
$$K_1 = x(\infty);\ K_1 + K_2 = x(0+)$$



$$v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$$

STEP 2: FIND K1 USING STEADY STATE ANALYSIS

$$0.5\frac{dv_O}{dt}(t) + v_O(t) = 6 \Rightarrow v_O(\infty) = 6V$$

$$v_O(\infty) = K_1$$

$$\therefore K_1 = 6V$$

THE NEXT STEP REQUIRES THE INITIAL VALUE OF THE VARIABLE, $v_O(0+)$

MODEL FOR $t>0$. USE KVL

$$-V_{S1} + R_1 i(t) + L\frac{di}{dt}(t) + R_3 i(t) = 0$$

$$2\frac{di}{dt}(t) + 4i(t) = 12$$

$$v_O(t) = 2i(t)[V]$$

$$0.5\frac{di}{dt}(t) + i(t) = 3[A]$$

$$0.5\frac{dv_O}{dt}(t) + v_O(t) = 6V$$

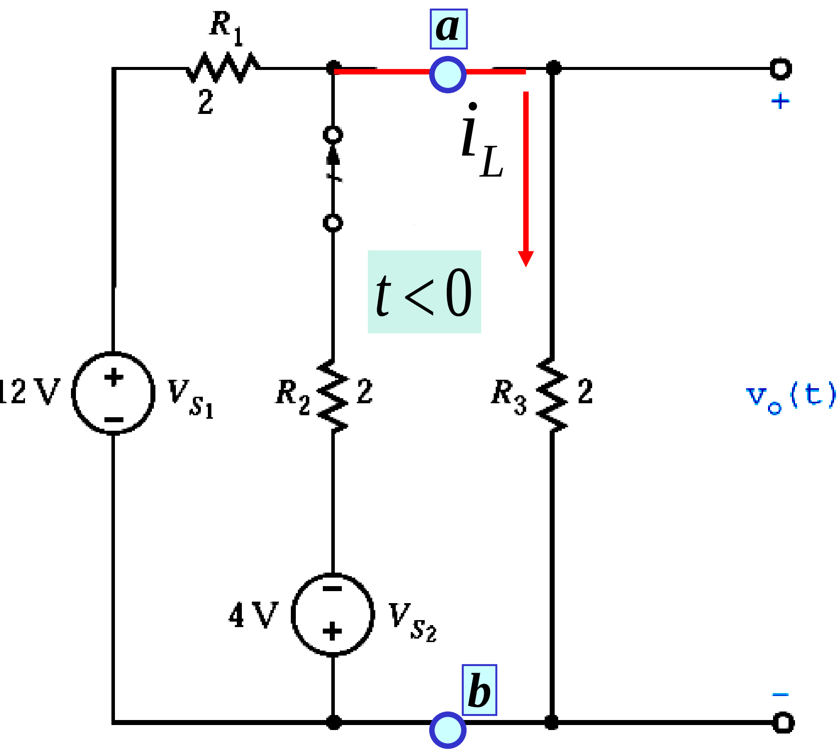
$$\tau = 0.5$$

STEP 1

FOR THE INITIAL CONDITION ONE NEEDS THE INDUCTOR CURRENT FOR t<0 AND USES THE CONTINUITY OF THE INDUCTOR CURRENT DURING THE SWITCHING .

THE STEADY STATE ASSUMPTION FOR t<0 SIMPLIFIES THE ANALYSIS

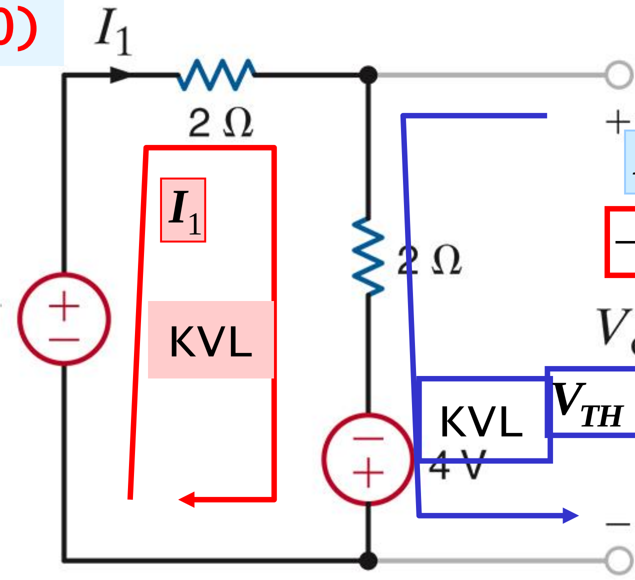
CIRCUIT IN STEADY STATE (t<0)



MUST FIND $i_L(t)$

FOR EXAMPLE USE THEVENIN ASSUMING INDUCTOR IN STEADY STATE

$$v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

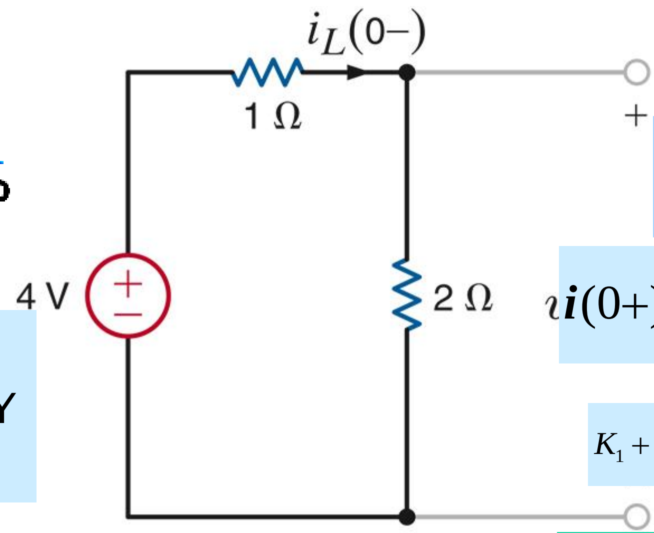


$$R_{TH} = 2 \,\|\, 2 = 1\Omega$$

$$-12 + 4I_1 - 4 = 0$$

$$I_1 = 4[A]$$

$$V_{TH} = V_{OC} = 2I_1 - 4 = 4[V]$$



$$i_L(0-) = i(0+) = \frac{4}{3}[A]$$

$$i(0+) = \frac{4}{3} \Rightarrow v_O(0+) = \frac{8}{3}[V]$$

$$K_1 + K_2 = \frac{8}{3} = 6 - K_2 \Rightarrow K_2 = -\frac{10}{3}$$

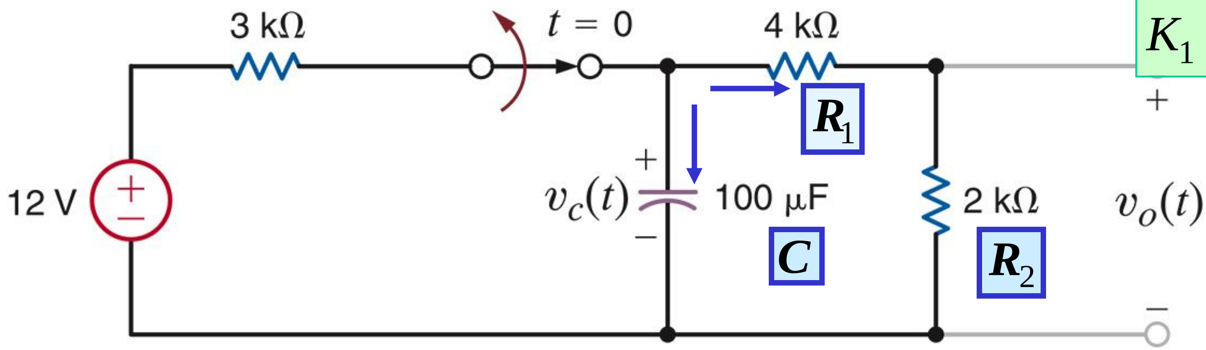
$$i(t) = 3 - \frac{5}{3} e^{-\frac{t}{0.5}}, t > 0$$

$$v_O(t) = 6 - \frac{10}{3} e^{-\frac{t}{0.5}}[V], t > 0$$

EXAMPLE 6

FIND $v_O(t), t>0$

$$v_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$$
$$K_1 = v_C(\infty);\ K_1 + K_2 = v_C(0+)$$



MODEL FOR $t>0$. USE KCL

DETERMINE $v_c(t)$

$$C\frac{dv_C}{dt}(t) + \frac{v_C}{R_1+R_2} = 0 \Rightarrow (R_1+R_2)C\frac{dv_C}{dt}(t) + v_c = \boxed{0}$$

$$v_O(t) = \frac{2}{2+4}v_C(t) = \frac{1}{3}v_C(t)$$

STEP 1

$$\tau = (R_1+R_2)C = (6\times10^3\Omega)(100\times10^{-6}F) = 0.6s$$

$$v_O(t) = \frac{8}{3}e^{-\frac{t}{0.6}}[V], t>0$$

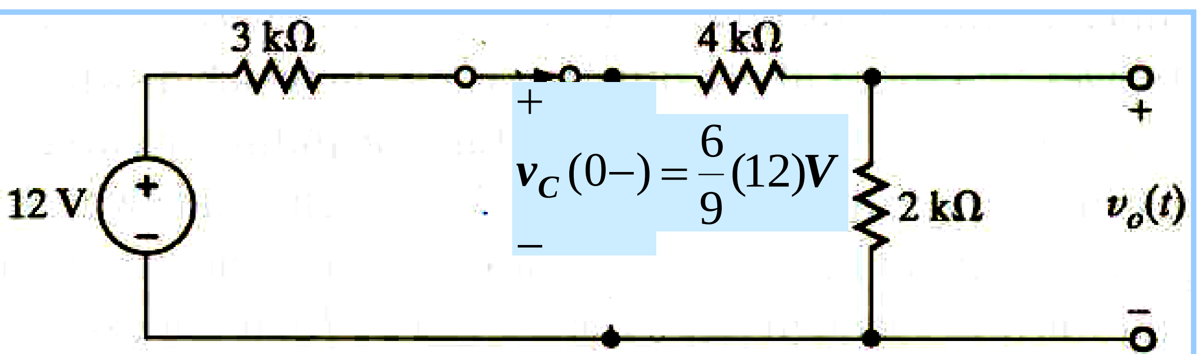
STEP 2

$$v_C(t) = \boxed{K_1} + K_2 e^{-\frac{t}{\tau}}, t>0$$

$$K_1 = 0$$

INITIAL CONDITIONS. CIRCUIT IN STEADY STATE t<0

STEP 3



$$v_C(0-) = \frac{6}{9}(12)V$$

$$v_C(0+) = 8 = K_1 + K_2 \Rightarrow K_2 = 8[V]$$

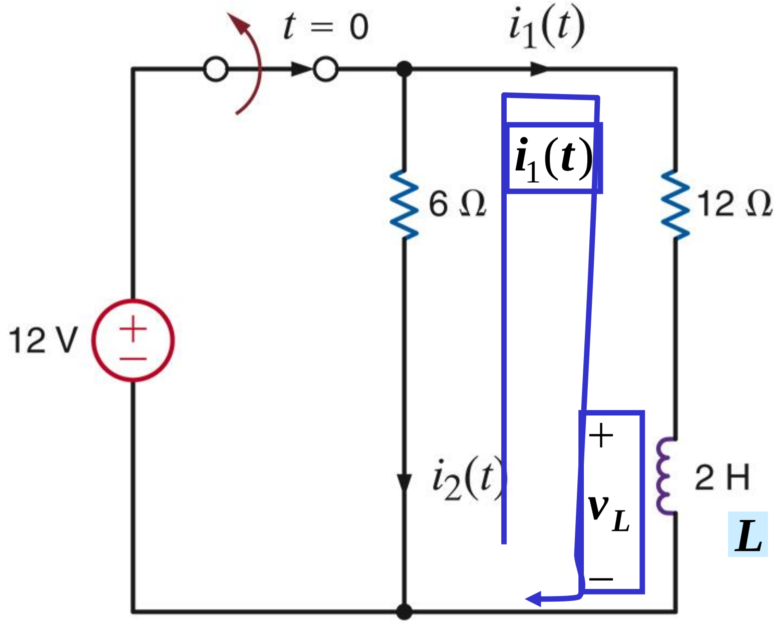
$$v_C(t) = 8e^{-\frac{t}{0.6}}[V], t>0$$

EXAMPLE 7

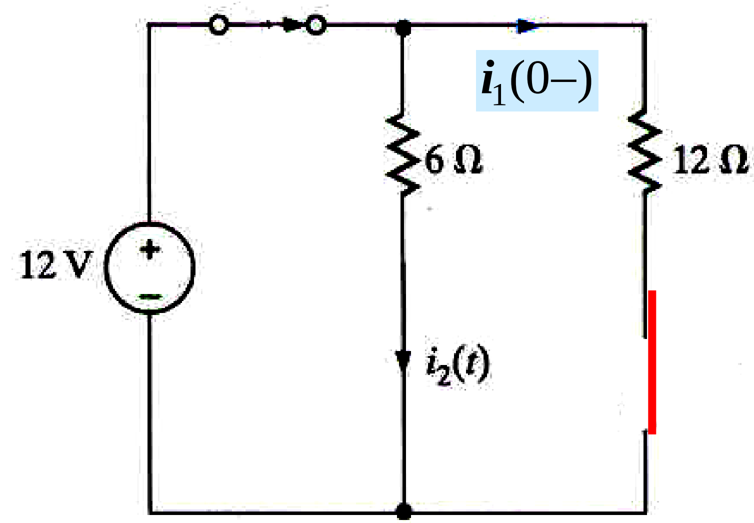
FIND $i_1(t), t>0$

$$i_1(t)=K_1+K_2 e^{-\frac{t}{\tau}}, t>0$$

$$K_1=i_1(\infty);\ K_1+K_2=i_1(0+)$$



CIRCUIT IN STEADY STATE PRIOR TO SWITCHING



$$i_1(0-)=\frac{12V}{12\Omega}=1A$$

MODEL FOR t > 0. USE KVL

$$L\frac{di_1}{dt}+18i_1(t)=0\Rightarrow\boxed{\frac{1}{9}}\frac{di_1}{dt}(t)+i_1(t)=\boxed{0}$$

STEP 1 $\boxed{\tau=\frac{1}{9}s}$

STEP 2 $\boxed{K_1=0}$

STEP 3

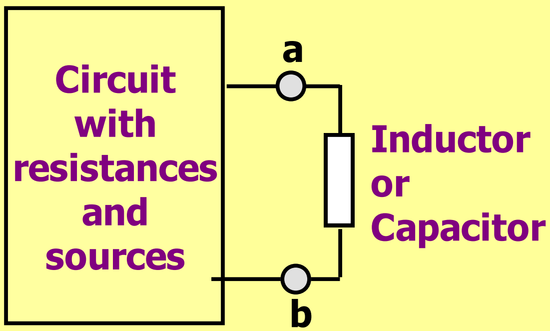
$$i_1(0-)=i_1(0+)=K_1+K_2\Rightarrow K_2=1[A]$$

FOR INITIAL CONDITIONS ONE NEEDS INDUCTOR CURRENT FOR t<0

$$\text{ANS: } i_1(t)=e^{-\frac{t}{1/9}}[A]=e^{-9t}[A], t>0$$

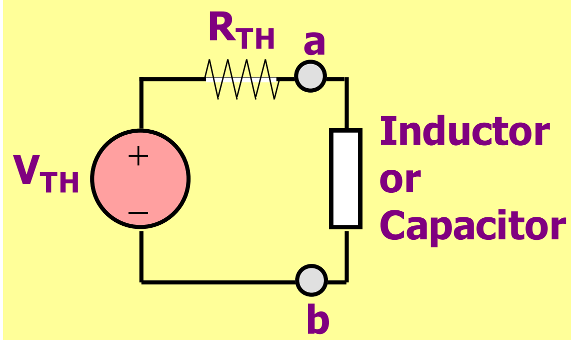
USING THEVENIN TO OBTAIN MODELS

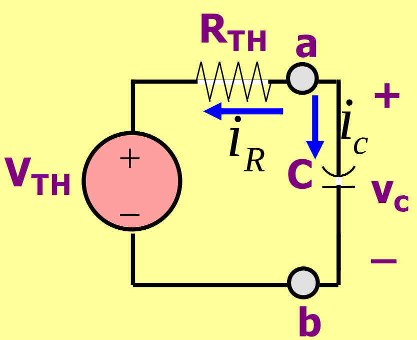
Obtain the voltage across the capacitor or the current through the inductor



Representation of an arbitrary circuit with one storage element

Thevenin





Case 1.1
Voltage across capacitor

KCL@ node a

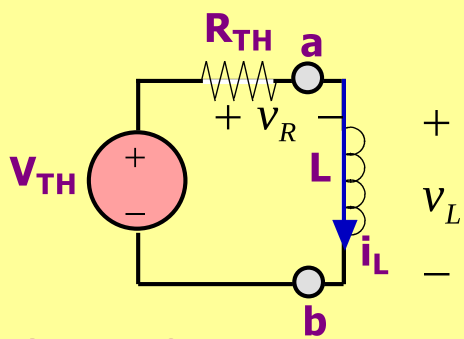
$$i_c + i_R = 0$$

$$i_c = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_C - v_{TH}}{R_{TH}}$$

$$C\frac{dv_C}{dt} + \frac{v_C - v_{TH}}{R_{TH}} = 0$$

$$\boxed{R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}}$$



Case 1.2
Current through inductor

Use KVL

$$v_R + v_L = v_{TH}$$

$$v_R = R_{TH}i_L$$

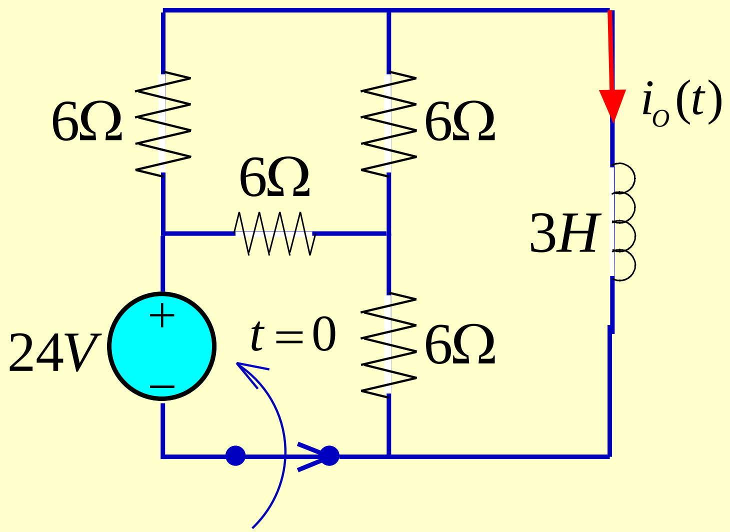
$$v_L = L\frac{di_L}{dt}$$

$$L\frac{di_L}{dt} + R_{TH}i_L = v_{TH}$$

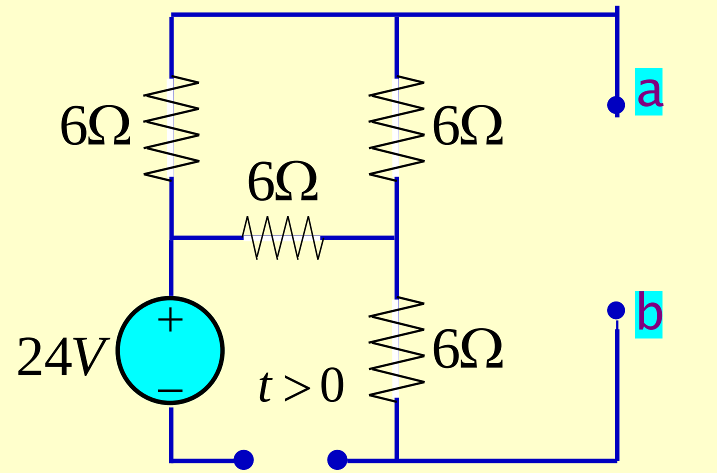
$$\boxed{\left(\frac{L}{R_{TH}}\right)\frac{di_L}{dt} + i_L = \frac{v_{TH}}{R_{TH}}} = i_{SC}$$

EXAMPLE 8

Find $i_O(t); t > 0$



Thevenin for t>0
at inductor terminals



$$v_{TH} = 0 \qquad R_{TH} = 6 + (6 \| (6+6))$$

The variable of interest is the inductor current. The model is

$$\frac{L}{R_{TH}}\frac{di_O}{dt} + i_O = \frac{v_{TH}}{R_{TH}}$$

$$\tau = \frac{L}{R_{TH}} = \frac{3H}{10\Omega} = 0.3s$$

$$0.3\frac{di_O}{dt} + i_O = \boxed{0}; \; t > 0$$

And the solution is of the form

$$i_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}; t > 0$$

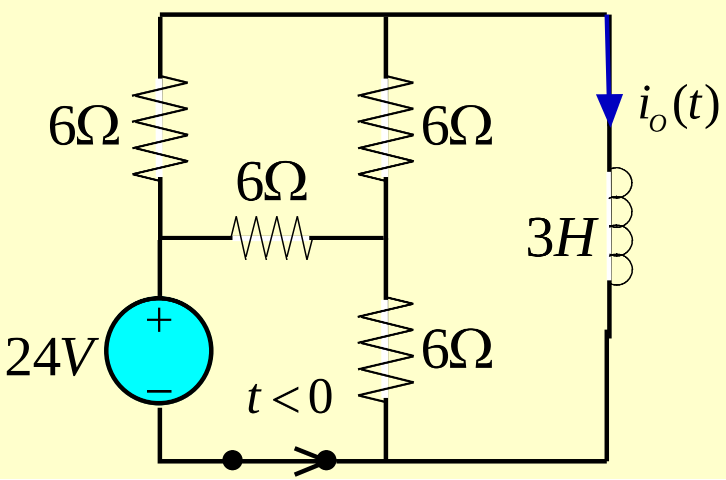
$$0.3\left(-\frac{K_2}{0.3}e^{-\frac{t}{0.3}}\right) + K_1 + K_2 e^{-\frac{t}{0.3}} = 0$$

$$\boxed{K_1 = 0 \Rightarrow} \quad i_O(t) = K_2 e^{-\frac{t}{0.3}}; t > 0$$

Next: Initial Condition

Determine $i_O(0+)$. Use steady state assumption and continuity of inductor current

Circuit for t<0



$$6i_1 + 6(i_1 - i_3) + 6(i_1 - i_2) = 0$$

$$-24 + 6(i_2 - i_1) + 6(i_2 - i_3) = 0$$

$$6(i_3 - i_1) + 6(i_3 - i_2) = 0$$

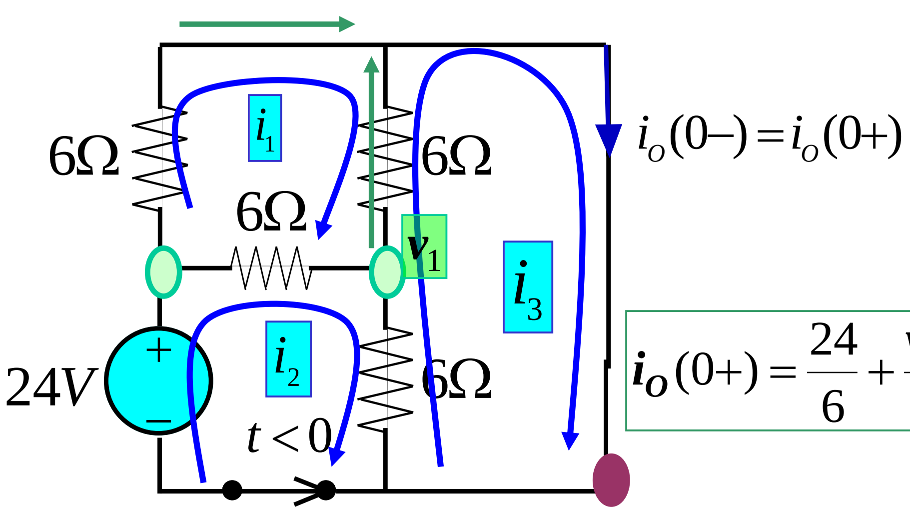
Loop analysis

$i_C(0+) = i_3$

$$\frac{v_1}{6} + \frac{v_1}{6} + \frac{v_1 - 24}{6} = 0 \Rightarrow v_1 = 8$$

Node analysis

solution: $\mathrm{i_C}(0+) = \dfrac{32}{6}A$



$$i_O(0+) = \frac{24}{6} + \frac{v_1}{6}$$

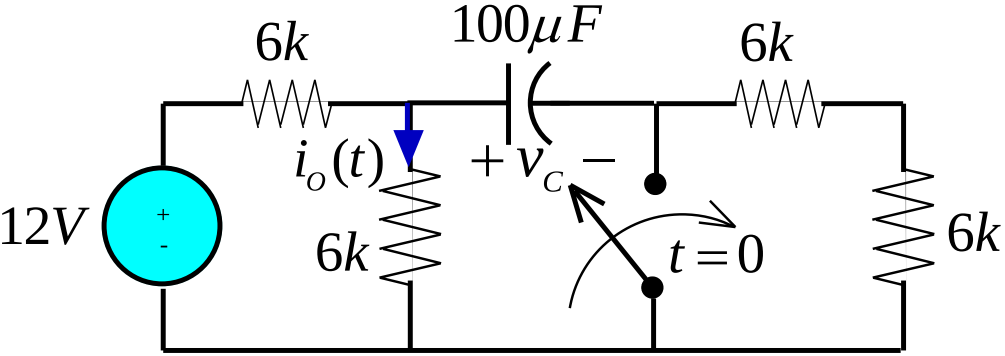
Since K1=0 the solution is

$$i_O(t) = K_2 e^{-\frac{t}{0.3}};\ t > 0$$

Evaluating at 0+ $\quad K_2 = \dfrac{32}{6}$

$$\boldsymbol{i_O(t) = \frac{32}{6} e^{-\frac{t}{0.3}};\ t > 0}$$

EXAMPLE 9 Find $i_o(t), t>0$



$v_{TH} = 6V$

$R_{TH} = 6k \| 6k = 3k$

$\tau = 3*10^3\Omega*100*10^{-6}F = 0.3s$

For $t>0$ $i_o = v_C/6k$

Model for v_C

$$0.3\frac{dv_C}{dt} + v_C = \boxed{6}$$

$$v_C = \boxed{K_1} + K_2 e^{-\frac{t}{0.3}}$$

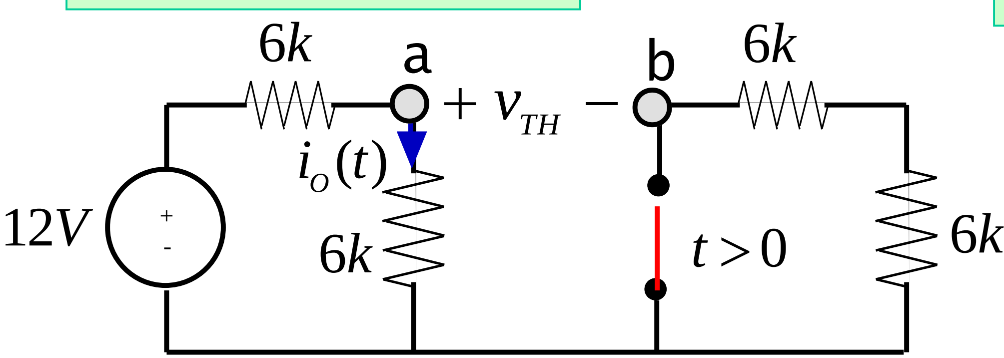
Hence, if the capacitor voltage is known the problem is solved

Model for v_c

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$

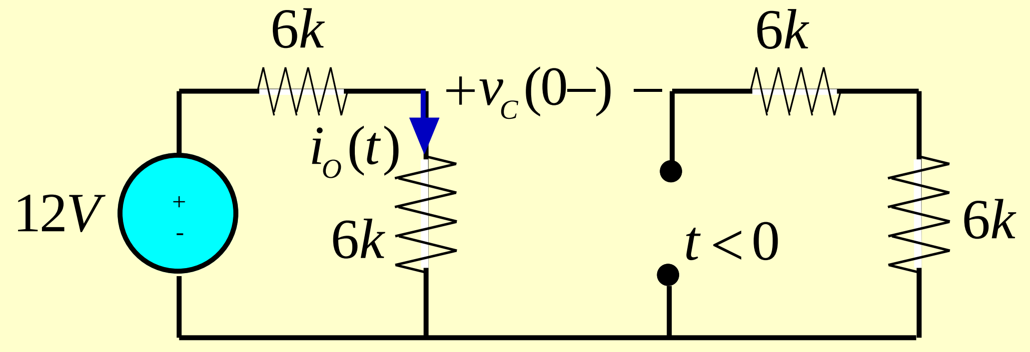
$$1.5\left(-\frac{K_2}{1.5}e^{-\frac{t}{1.5}}\right) + K_1 + K_2 e^{-\frac{t}{0.3}} = 6$$

$K_1 = 6$



Now we need to determine the initial value v_c(0+) using continuity and the steady state assumption

circuit in steady state
before the switching



$$v_C(0-)=6V$$

Continuity of capacitor voltage

$$v_C(0+)=6V$$

$$K_1+K_2=v_C(0+)$$

$$\boxed{K_1=6 \Rightarrow K_2=0}$$

$$v_C(t)=6V\,;\,t>0 \Rightarrow$$

$$\boxed{i_O(t)=\frac{v_C}{6k}=1mA;\,t>0}$$

ANALYSIS OF CIRCUITS WITH ONE ENERGY STORING ELEMENT

CONSTANT INDEPENDENT SOURCES

A STEP-BY-STEP APPROACH

THIS APPROACH RELIES ON THE KNOWN FORM OF THE SOLUTION BUT FINDS THE CONSTANTS K_1, K_2, τ USING BASIC CIRCUIT ANALYSIS TOOLS AND FORGOES THE DETERMINATION OF THE DIFFERENTIAL EQUATION MODEL

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

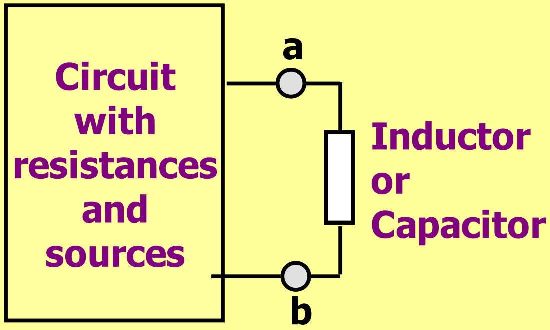
K_1 is the steady state value of the variable and can be determined analyzing the circuit in steady state

$x(0+)$ is the initial condition and provides the second equation to compute the constants K_1, K_2

τ is the time constant and can be determined using Thevenin across the energy storing element

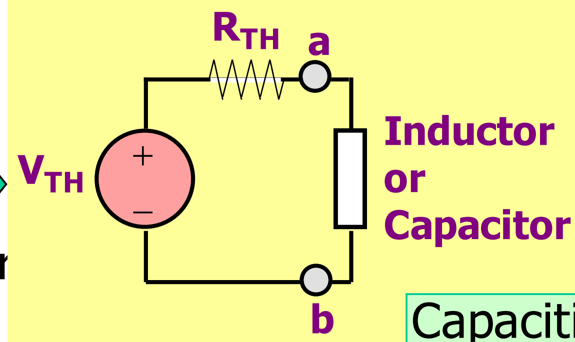
CIRCUITS WITH ONE ENERGY STORING ELEMENT

Obtaining the time constant: A General Approach



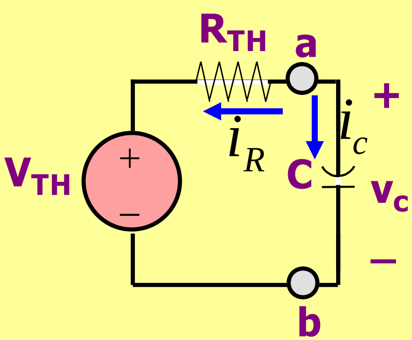
Representation of an arbitrary circuit with one storage element

Thever



Capacitive Circuit $\tau = R_{TH}C$

Inductive Circuit $\tau = \dfrac{L}{R_{TH}}$



Case 1.1
Voltage across capacitor

KCL@ node a

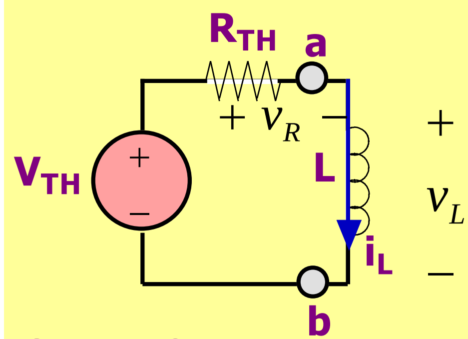
$$i_c + i_R = 0$$

$$i_c = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_C - v_{TH}}{R_{TH}}$$

$$C\frac{dv_C}{dt} + \frac{v_C - v_{TH}}{R_{TH}} = 0$$

$$R_{TH}C\frac{dv_C}{dt} + v_C = v_{TH}$$



Case 1.2
Current through inductor

Use KVL

$$v_R + v_L = v_{TH}$$

$$v_R = R_{TH}i_L$$

$$v_L = L\frac{di_L}{dt}$$

$$L\frac{di_L}{dt} + R_{TH}i_L = v_{TH}$$

$$\left(\frac{L}{R_{TH}}\right)\frac{di_L}{dt} + i_L = \frac{v_{TH}}{R_{TH}} = i_{SC}$$

THE STEPS

STEP 1. THE FORM OF THE SOLUTION

$$x(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$$

$$K_1 = x(\infty);\ K_1 + K_2 = x(0+)$$

DETERMINE $x(0+)$

STEP 2: DRAW THE CIRCUIT IN STEADY STATE PRIOR TO THE SWITCHING AND DETERMINE CAPACITOR VOLTAGE OR INDUCTOR CURRENT

STEP 3: DRAW THE CIRCUIT AT 0+ THE CAPACITOR ACTS AS A VOLTAGE SOURCE. THE INDUCTOR ACTS AS A CURRENT SOURCE.
DETERMINE THE VARIABLE AT t=0+

DETERMINE $x(\infty)$

STEP 4: DRAW THE CIRCUIT IN STEADY STATE AFTER THE SWITCHING AND DETERMINE THE VARIABLE IN STEADY STATE.

STEP 5: DETERMINE THE TIME CONSTANT

$\tau = R_{TH} C$ circuit with one capacitor

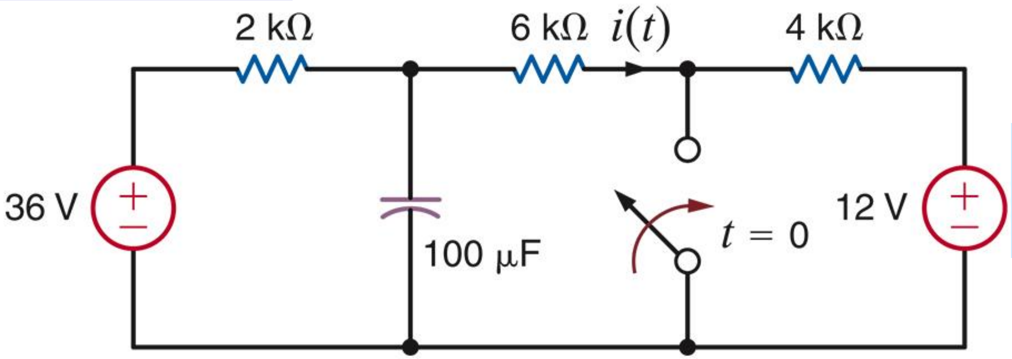
$\tau = \frac{L}{R_{TH}}$ circuit with one inductor

STEP 6: DETERMINE THE CONSTANTS K1, K2

$$K_1 = x(\infty),\quad K_1 + K_2 = x(0+)$$

EXAMPLE 1

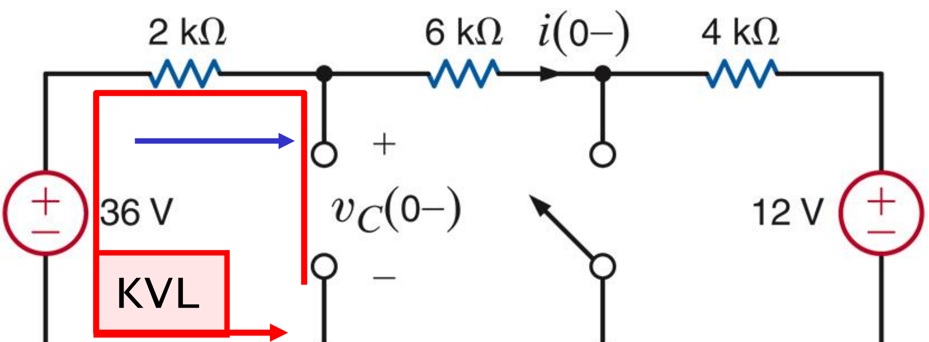
FIND $i(t), t > 0$



STEP 1: $i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: Initial voltage across capacitor

USE CIRCUIT IN STEADY STATE PRIOR TO THE SWITCHING



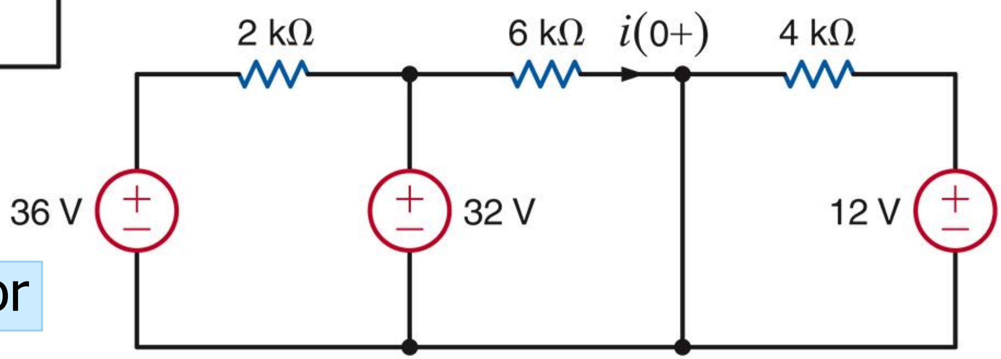
$i = \dfrac{24V}{12k\Omega} = 2mA$ $= i(0-)$

$v_C(0+) = v_c(0-)$

$v_c(0-) = 36V - (2mA)(2k) = 32[V]$

STEP 3: Determine $i(0+)$

USE A CIRCUIT VALID FOR t=0+. THE CAPACITOR ACTS AS SOURCE



$$i(0+) = \frac{32V}{6k} = \frac{16}{3} mA$$

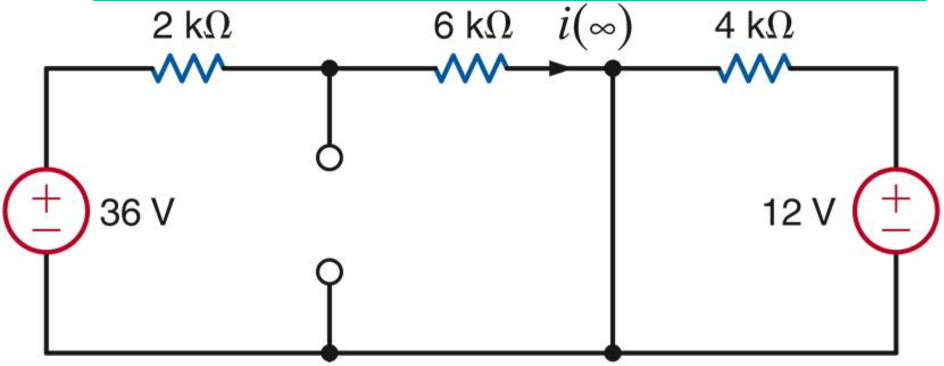
NOTES FOR INDUCTIVE CIRCUIT

(1) DETERMINE INITIAL INDUCTOR CURRENT IN STEP 2

(2) FOR THE t=0+ CIRCUIT REPLACE INDUCTOR BY A CURRENT SOURCE

STEP 4: Determine $i(\infty)$

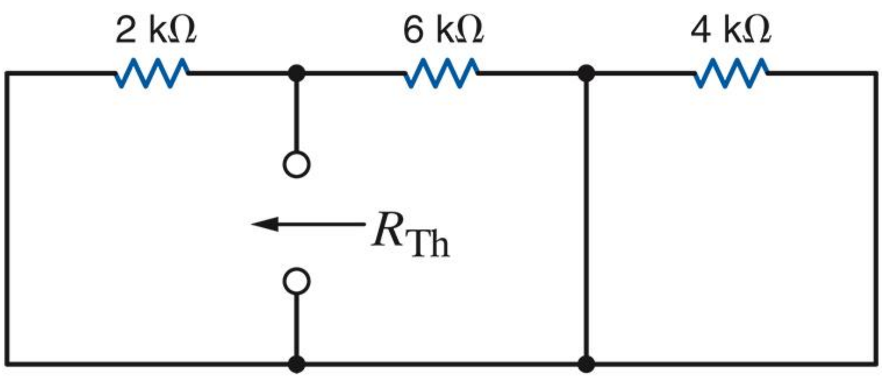
USE CIRCUIT IN STEADY STATE AFTER SWITCHING



$$i(\infty) = \frac{36}{8} mA$$

STEP 5: Determine time constant

Capacitive circuit: $\tau = R_{TH} C$



$R_{TH} = 2k \,||\, 6k = 1.5 k\Omega$ $\quad C = 100\,\mu F$

$$\tau = (1.5 \times 10^3 \Omega)(100 \times 10^{-6} F) = 0.15 s$$

STEP 6: Determine K_1, K_2

(STEP 1) $\quad i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

(STEP 3) $\quad i(0+) = \frac{16}{3} mA$ $\boxed{= K_1 + K_2}$

(STEP 4) $\quad i(\infty) = \frac{36}{8} mA$ $\boxed{= K_1}$

$$\therefore K_2 = \frac{16}{3} - \frac{36}{8} = \frac{5}{6}$$

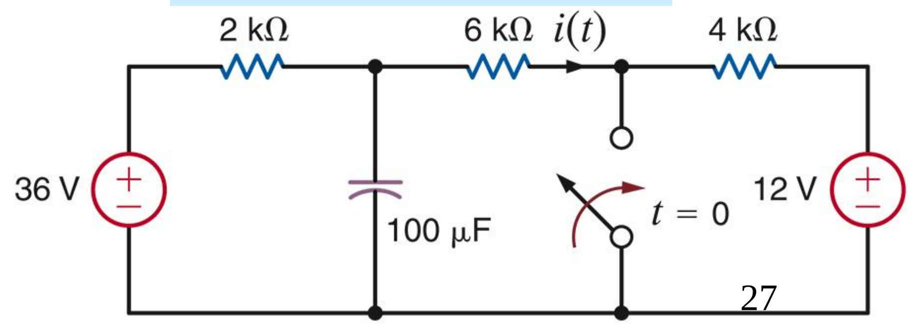
FINAL ANSWER

$$i(t) = \frac{36}{8} + \frac{5}{6} e^{-\frac{t}{0.15}}, t > 0$$

NOTE: FOR INDUCTIVE CIRCUIT

$$\tau = \frac{L}{R_{TH}}$$

ORIGINAL CIRCUIT



USING MATLAB TO DISPLAY FINAL ANSWER

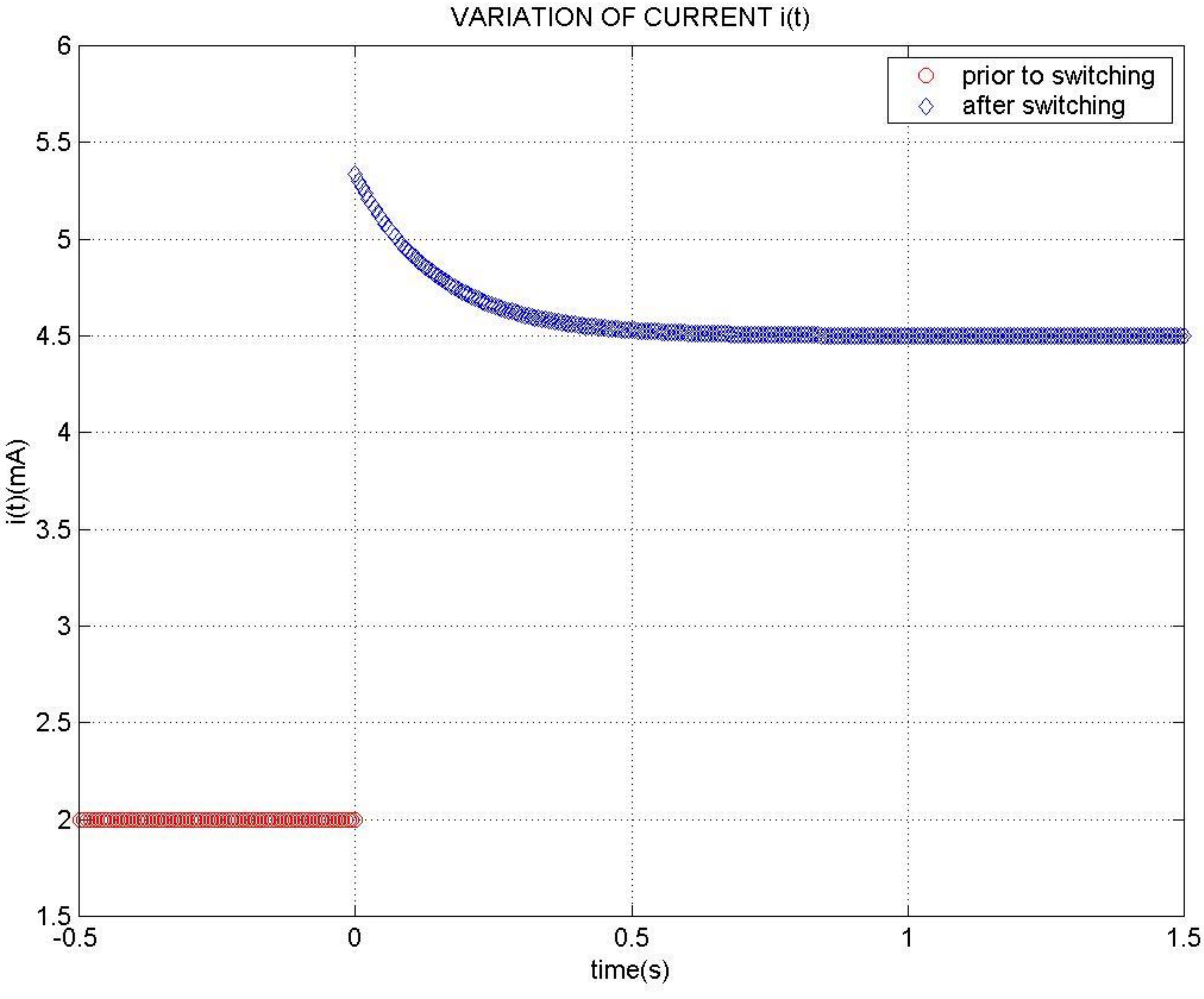
$$i(t) = \begin{cases} 2mA & t \le 0 \\ \dfrac{36}{8} + \dfrac{5}{6} e^{-\frac{t}{0.15}}, & t > 0 \end{cases}$$

Command used to define linearly spaced arrays

» help linspace
LINSPACE Linearly spaced vector.
LINSPACE(x1, x2) generates a row vector of 100 linearly equally spaced points between x1 and x2.
LINSPACE(x1, x2, N) generates N points between x1 and x2.
See also LOGSPACE, :.

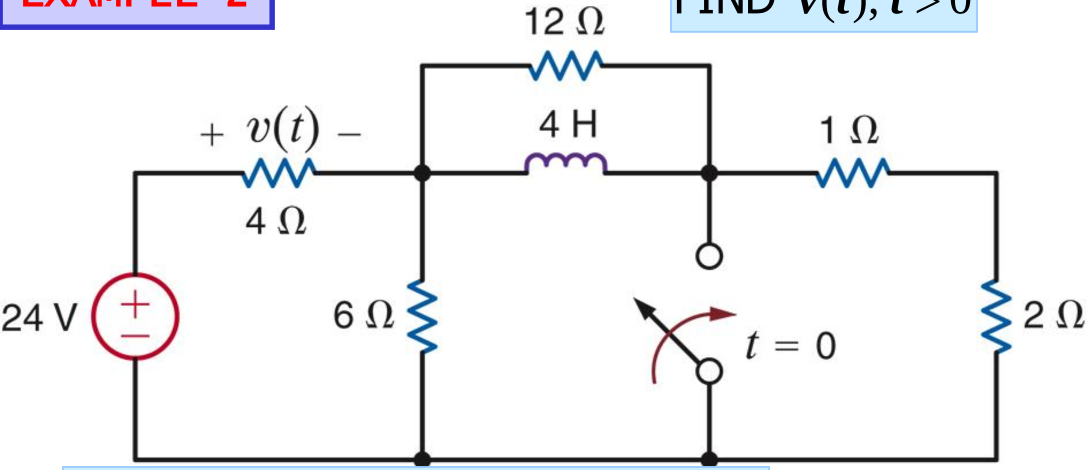
Script (m-file) with commands used. Prepared with the MATLAB Editor

```
%example6p3.m
%commands used to display funtion i(t)
%this is an example of MATLAB script or M-file
%must be stored in a text file with extension ".m"
%the commands are executed when the name of the M-file is typed at the
%MATLAB prompt (without the extension)
tau=0.15; %define time constant
tini=-4*tau; %select left starting point
tend=10*tau; %define right end point
tminus=linspace(tini,0,100); %use 100 points for t<=0
tplus=linspace(0,tend, 250); % and 250 for t>=0
iminus=2*ones(size(tminus)); %define i for t<=0
iplus=36/8+5/6*exp(-tplus/tau); %define i for t>=0
plot(tminus,iminus,'ro',tplus,iplus,'bd'), grid; %basic plot command specifying
                                                 %color and marker
title('VARIATION OF CURRENT i(t)'), xlabel('time(s)'), ylabel('i(t)(mA)')
legend('prior to switching', 'after switching')
axis([-0.5,1.5,1.5,6]);%define scales for axis [xmin,xmax,ymin,ymax]
```


VARIATION OF CURRENT i(t)
prior to switching
after switching
i(t)(mA)
time(s)
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
-0.5
0
0.5
1
1.5

EXAMPLE 2

FIND $v(t), t > 0$

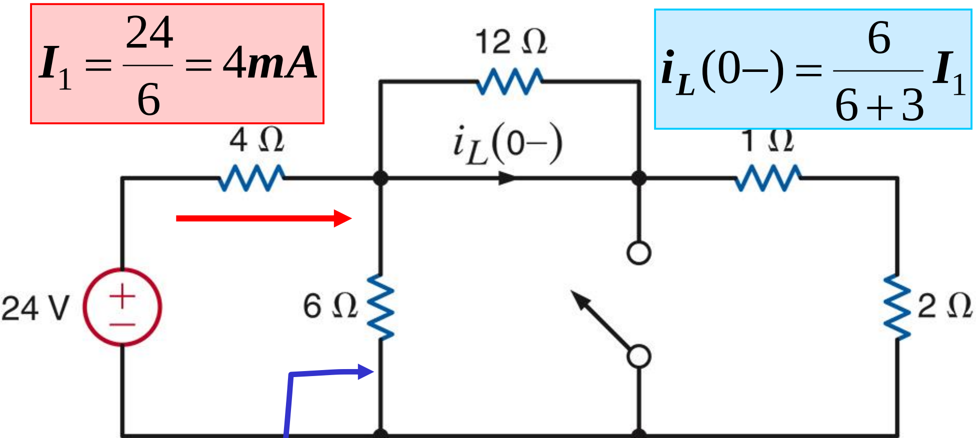


STEP 1: $v(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: Initial inductor current

Use circuit in steady state prior to switching

$$I_1 = \frac{24}{6} = 4mA$$



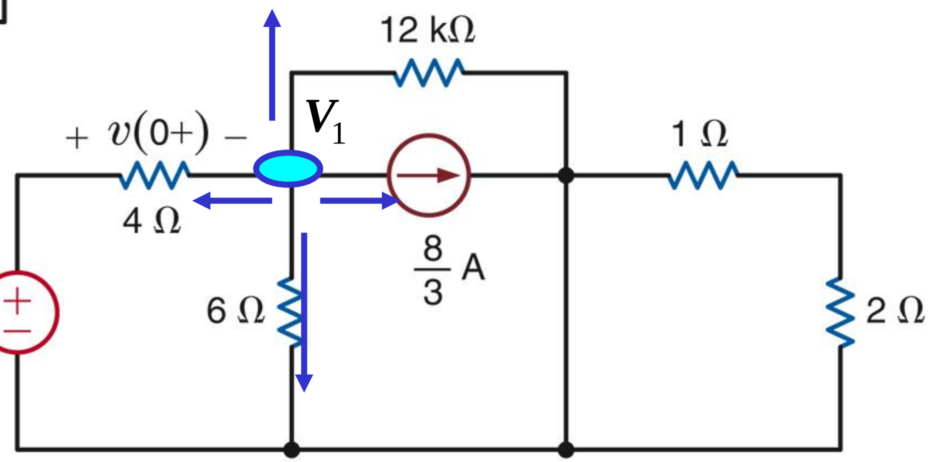
$$i_L(0-) = \frac{6}{6+3} I_1$$

$6k \| 3k$

$$i_L(0-) = \frac{8}{3} mA$$

STEP 3: Determine $v(0+)$

Use circuit at t=0+. Inductor is replaced by current source



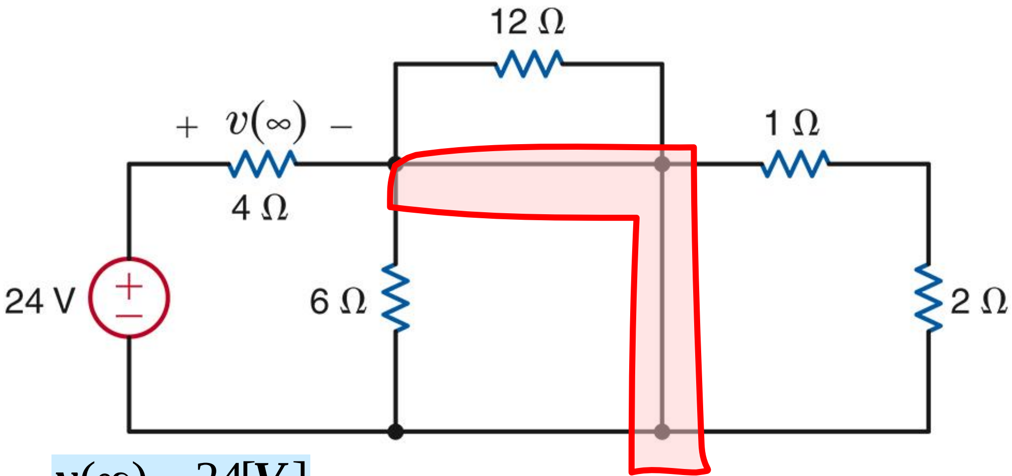
$$\frac{V_1 - 24}{4} + \frac{V_1}{6} + \frac{V_1}{12} + \frac{8}{3} = 0$$

$$V_1 = \frac{20}{3}[V]$$

$$v(0+) = 24[V] - V_1 = \frac{52}{3}[V]$$

STEP 4: DETERMINE $v(\infty)$

USE CIRCUIT IN STEADY STATE AFTER SWITCHING



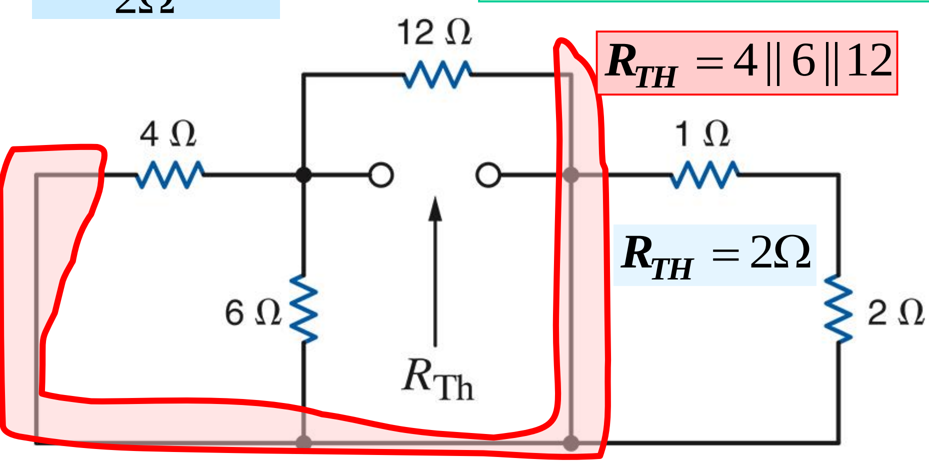
$v(\infty) = 24[V]$

STEP 5: DETERMINE TIME CONSTANT

$\tau = \frac{4H}{2\Omega} = 2s$

Inductive Circuit: $\tau = \frac{L}{R_{TH}}$

$R_{TH} = 4 \| 6 \| 12$



$R_{TH} = 2\Omega$

STEP 6: DETERMINE K_1, K_2

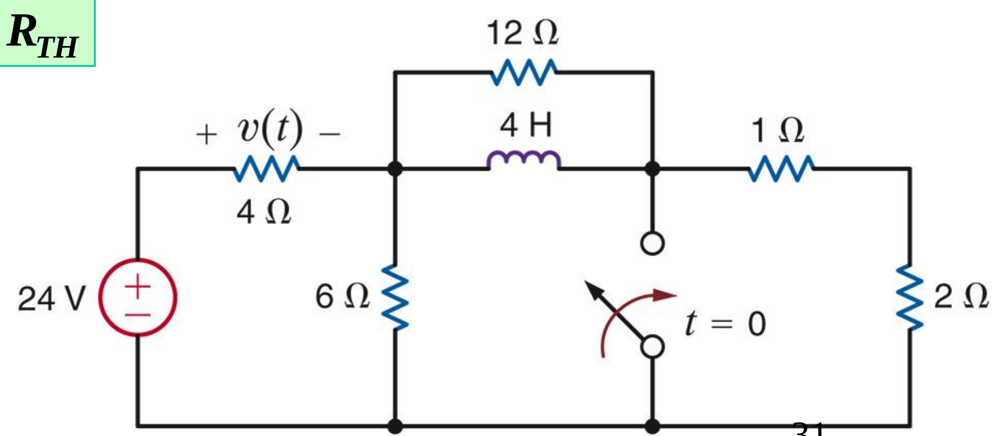
$K_1 = v(\infty) = 24[V]$ (step 4)

$v(0+) = \frac{52}{3} = K_1 + K_2$ (step 3)

$K_2 = \frac{52}{3} - 24 = -\frac{20}{3}[V]$

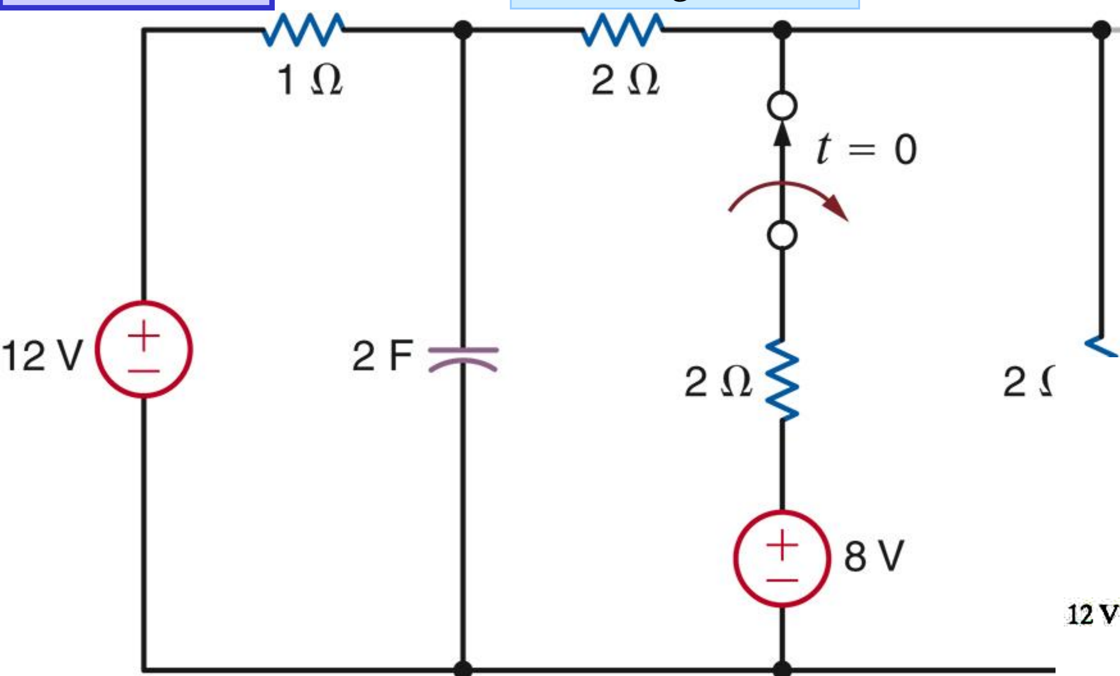
ANS: $v(t) = 24 - \frac{20}{3} e^{-\frac{t}{2}}, t > 0$

ORIGINAL CIRCUIT



EXAMPLE 3

FIND $v_O(t), t>0$

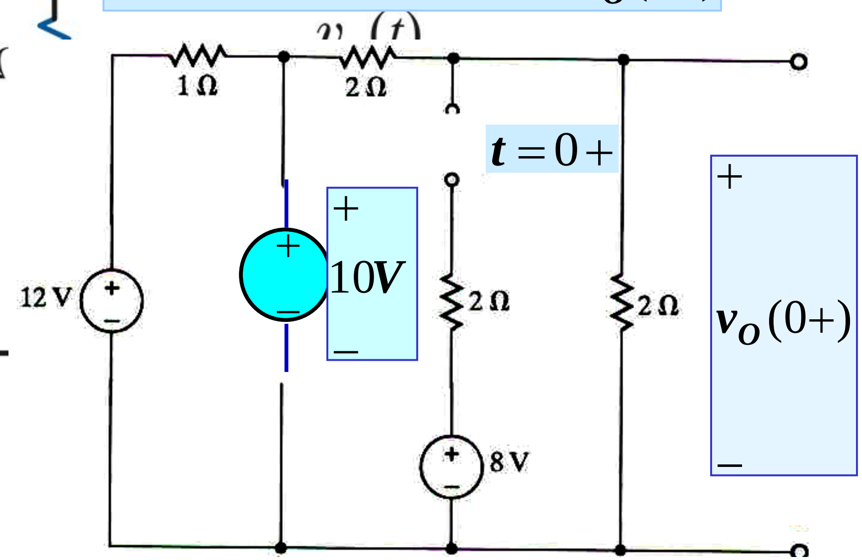


KCL @ v_1: $\frac{v_1-12}{3}+\frac{v_1-8}{2}+\frac{v_1}{2}=0$ $*/6$

$8v_1-48=0 \Rightarrow v_1=6[V]$

$v_2=2[V] \Rightarrow v_C(0-)=v_C(0+)=10[V]$

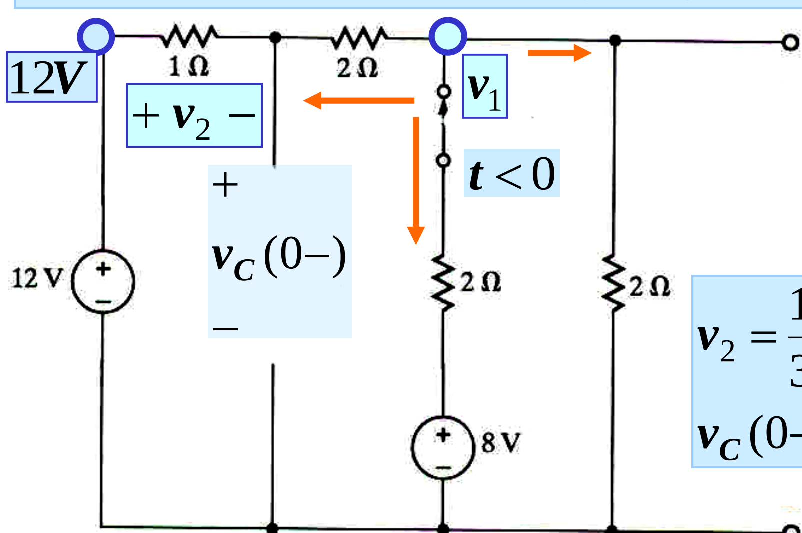
STEP 3: DETERMINE $v_O(0+)$



$$v_O(0+)=\frac{2}{2+2}(10)=5[V]$$

STEP 1: $v_o(t)=K_1+K_2e^{-\frac{t}{\tau}}, t>0$

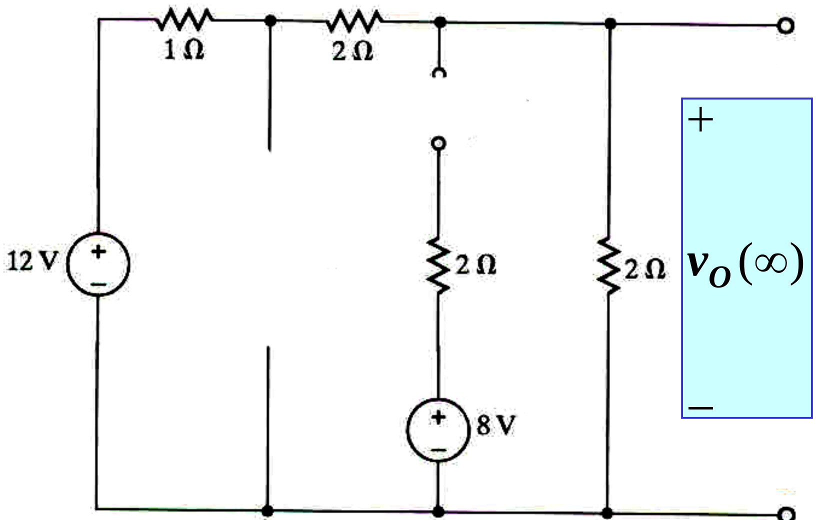
STEP 2: INITIAL CAPACITOR VOLTAGE



$$v_2=\frac{1}{3}(12-v_1)$$

$$v_C(0-)=12-v_2$$

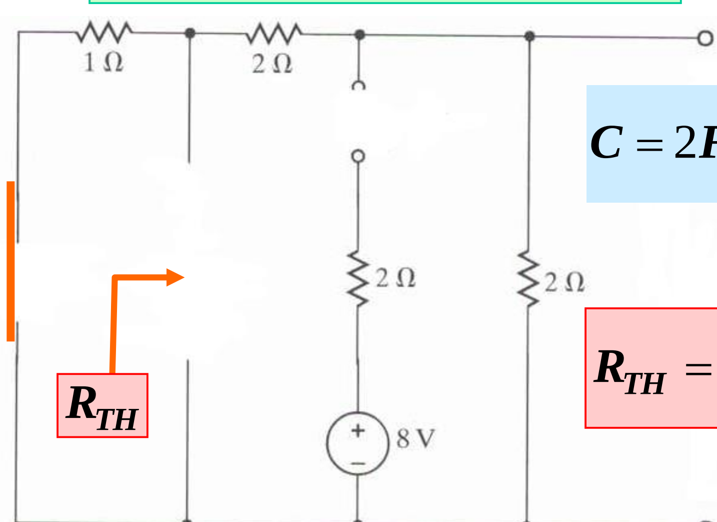
STEP 4: DETERMINE $v_O(\infty)$



$$v_O(\infty) = \frac{2}{5}(12) = \frac{24}{5}[V]$$

STEP 5: DETERMINE TIME CONSTANT

Capacitive Circuit: $\tau = R_{TH}C$



$$C = 2F \Rightarrow \tau = \frac{8}{5}s$$

$$R_{TH} = 1 \| 4 = \frac{4}{5}\Omega$$

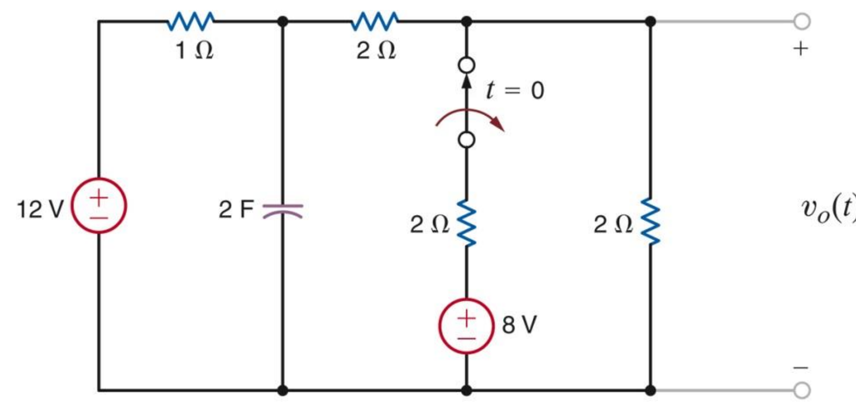
STEP 6: DETERMINE K_1, K_2

$$K_1 = v_O(\infty) = \frac{24}{5}[V]$$

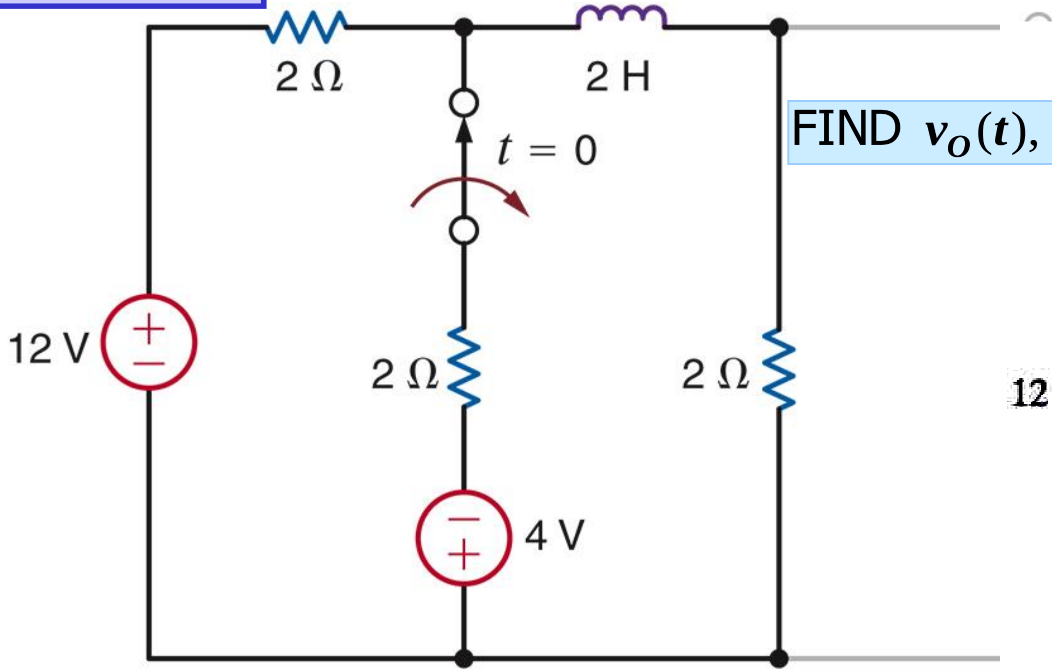
$$v_O(0+) = 5[V] = K_1 + K_2 \Rightarrow K_2 = \frac{1}{5}[V]$$

$$\text{ANS:} \quad v_O(t) = \frac{24}{5} + \frac{1}{5}e^{-\frac{t}{8/5}}[V];\ t > 0$$

ORIGINAL CIRCUIT



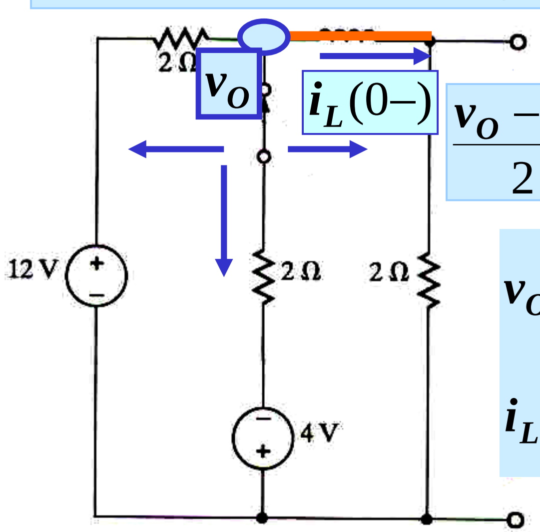
EXAMPLE 4



FIND $v_O(t), t > 0$

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$

STEP 2: INITIAL INDUCTOR CURRENT

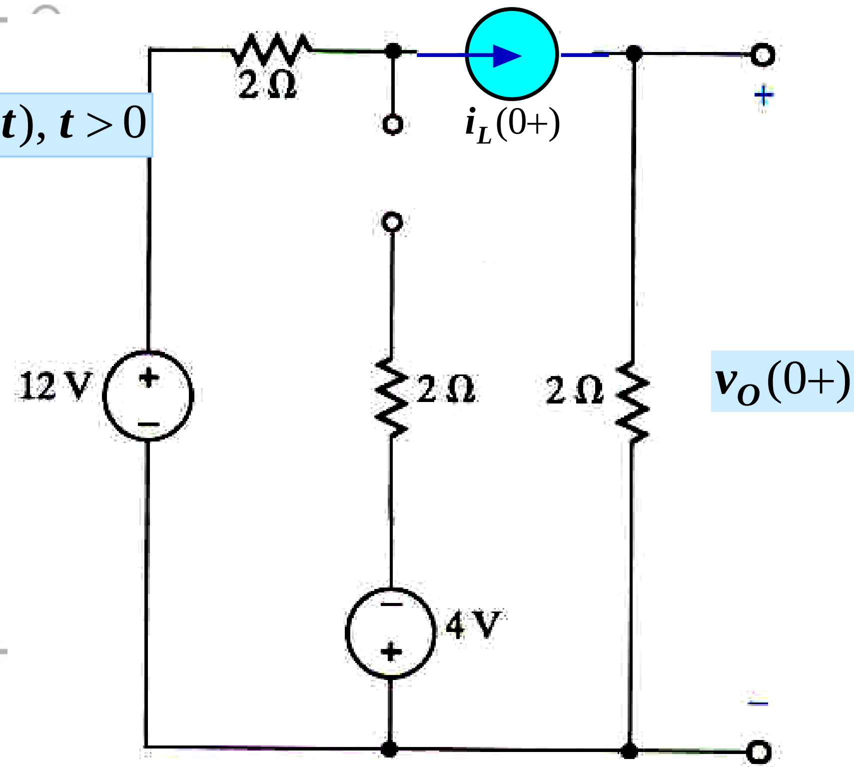


$$\frac{v_O - 12}{2} + \frac{v_O - (-4)}{2} + \frac{v_O}{2} = 0$$

$$v_O = \frac{8}{3}[V]$$

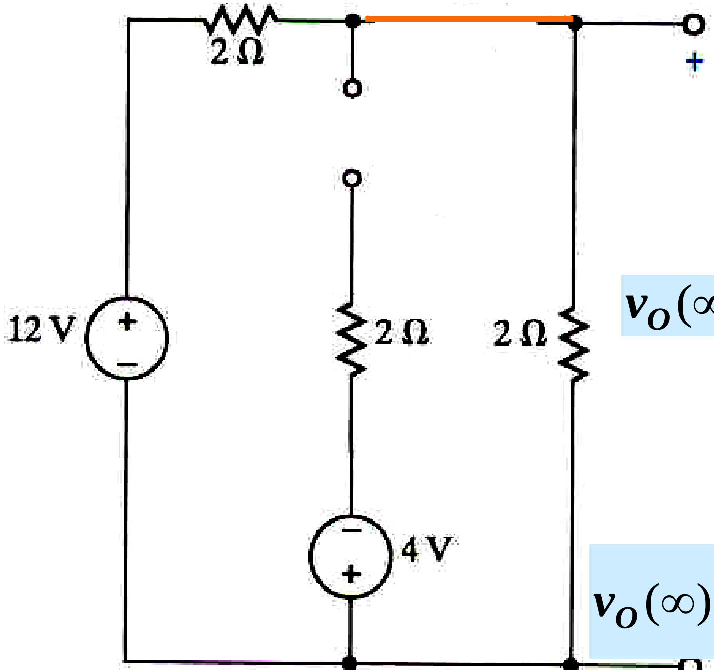
$$i_L(0-) = i_L(0+) = \frac{4}{3}[A]$$

STEP 3: DETERMINE $v_O(0+)$



$$v_O(0+) = 2 i_L(0+) = \frac{8}{3}[V]$$

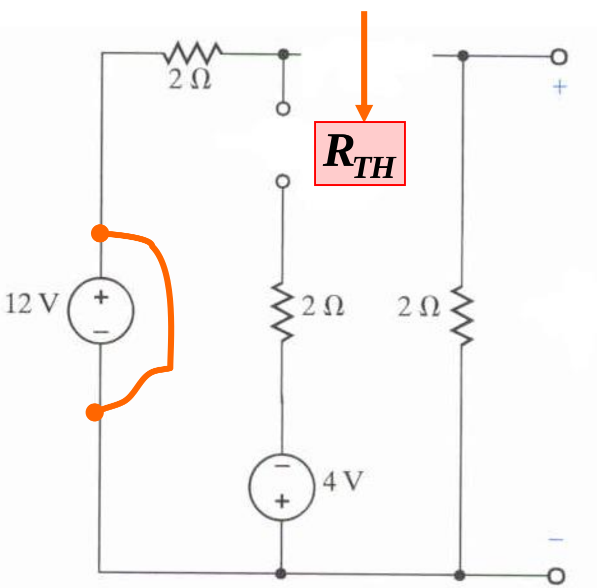
STEP 4: DETERMINE $v_O(\infty)$



$v_O(\infty)$

$$v_O(\infty) = \frac{2}{2+2}(12) = 6[V]$$

STEP 5: DETERMINE TIME CONSTANT



R_{TH}

Inductive Circuit

$$\tau = \frac{L}{R_{TH}}$$

$$R_{TH} = 4\Omega$$

$$\tau = \frac{2}{4} = 0.5s$$

STEP 6: DETERMINE K_1, K_2

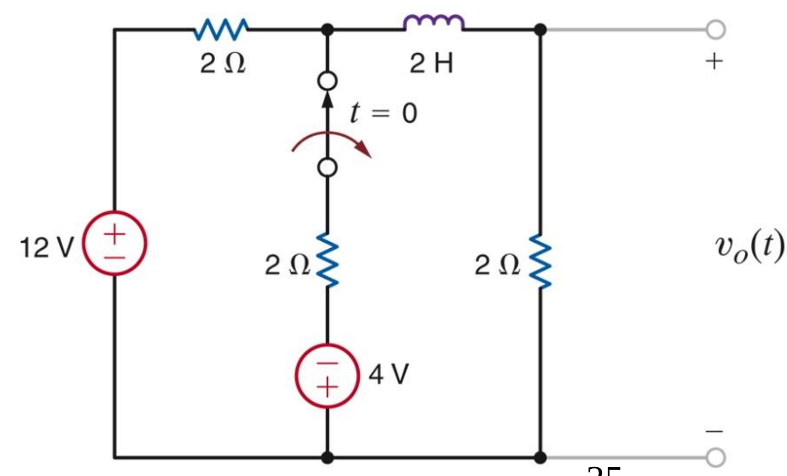
$K_1 = v_O(\infty) = 6[V]$ (step 4)

$v_O(+) = \frac{8}{3} = K_1 + K_2$ (step 3)

$$K_2 = \frac{8}{3} - 6 = -\frac{10}{3}[V]$$

ANS: $v_O(t) = 6 - \frac{10}{3}e^{-\frac{t}{0.5}}, t > 0$

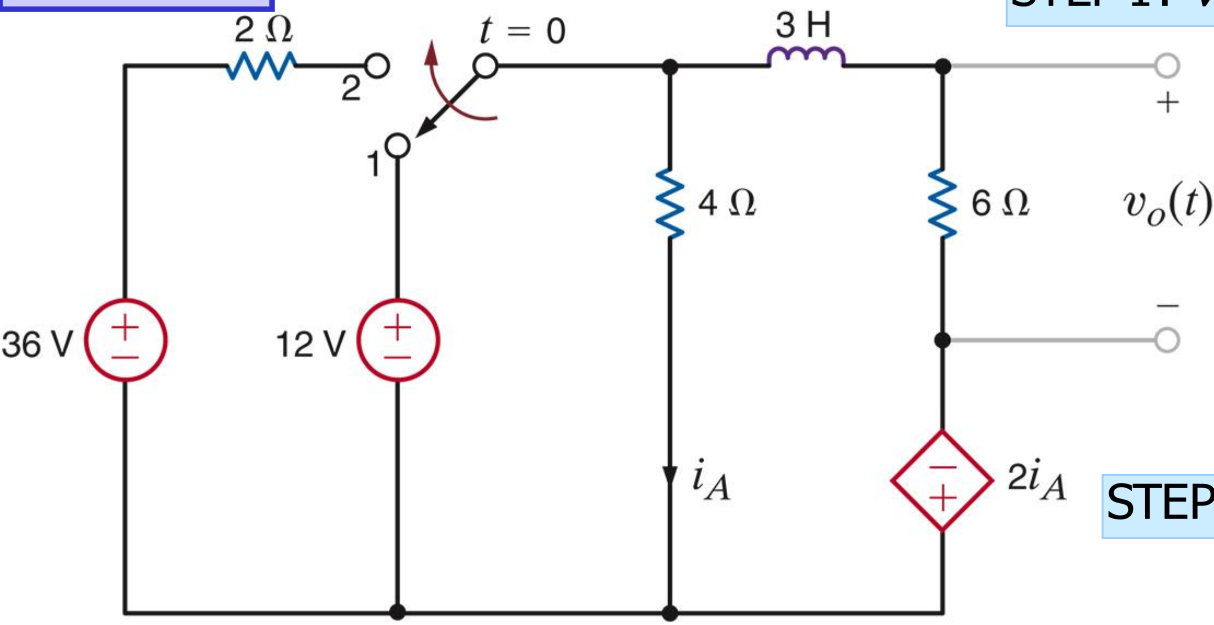
ORIGINAL CIRCUIT



EXAMPLE 5

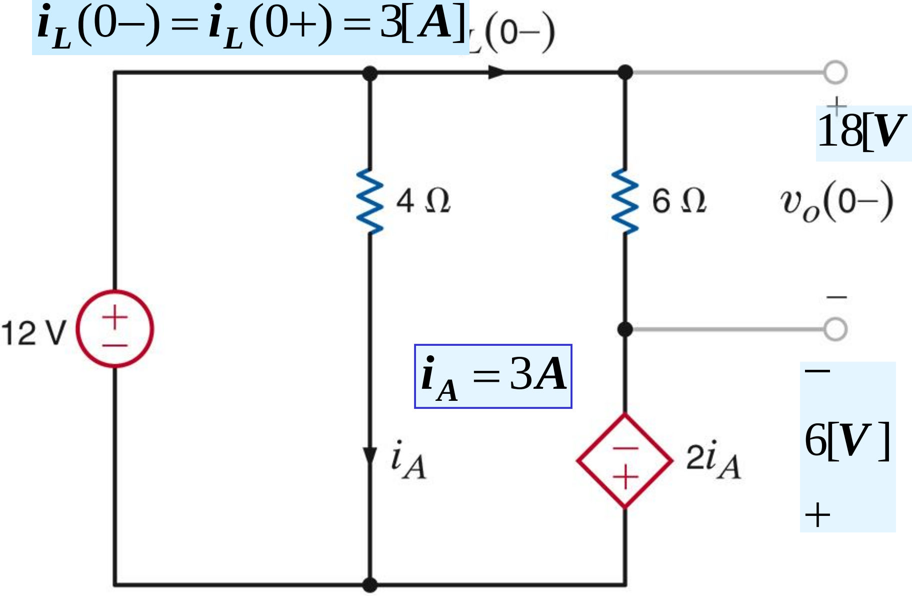
FIND $v_O(t), t > 0$

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t > 0$



STEP 2: DETERMINE $i_L(0+)$

$i_L(0-) = i_L(0+) = 3[A]$



$18[V]$

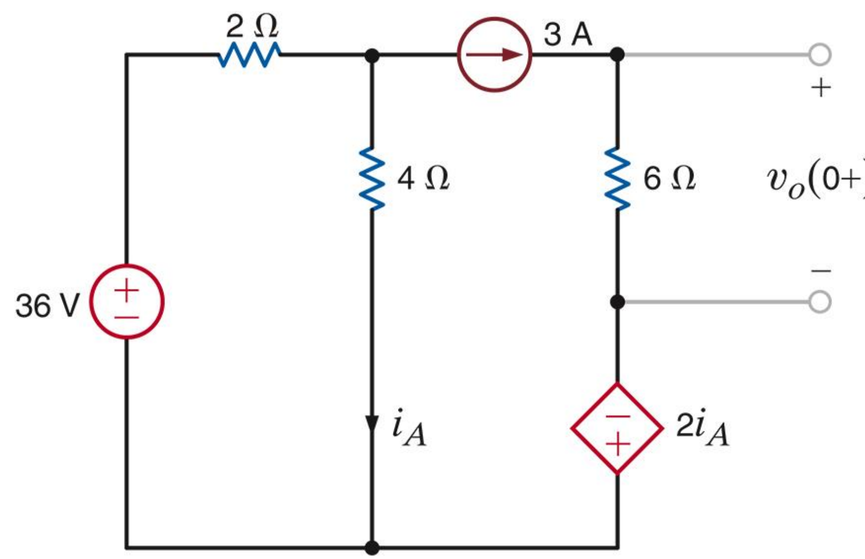
$i_A = 3A$

$-$

$6[V]$

$+$

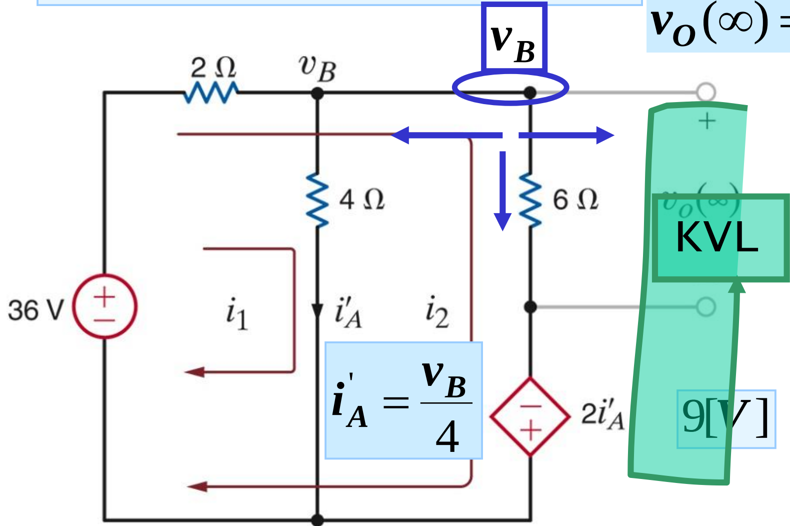
STEP 3: DETERMINE $v_O(0+)$



$v_O(0+) = 18[V]$

STEP 4: DETERMINE $v_O(\infty)$

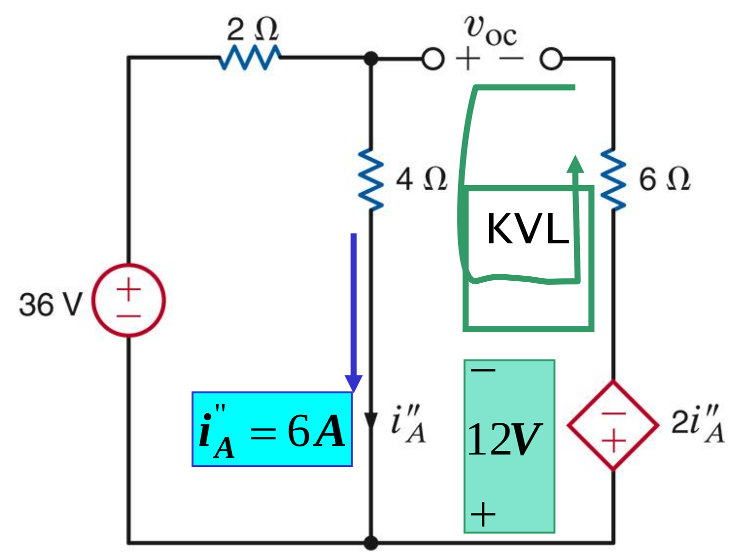
$$v_O(\infty)=27[V]$$



$$\frac{v_B-36}{2}+\frac{v_B}{4}+\frac{v_B-(-2i'_A)}{6}=0 \quad * /12$$

$$11v_B+4i'_A=36\times 6 \qquad v_B=18[V],\ i'_A=4.5[A]$$

OPEN CIRCUIT VOLTAGE



$$v_{OC}=24+12=36[V]$$

STEP 5: DETERMINE TIME CONSTANT

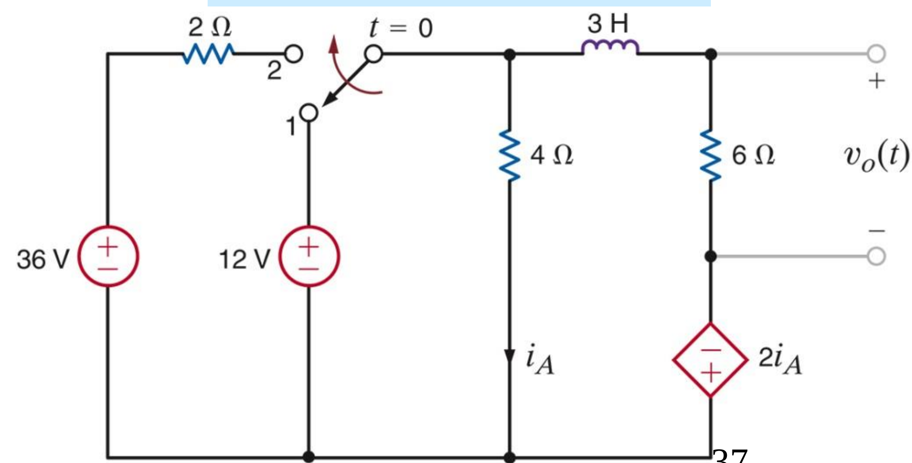
inductive circuit

$$\tau=\frac{L}{R_{TH}}$$

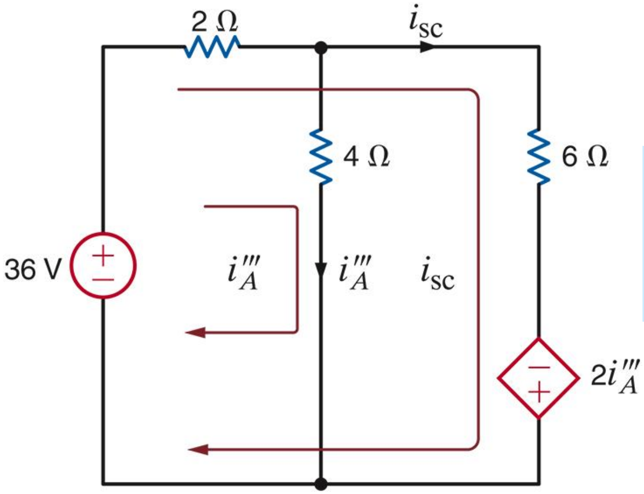
Circuit with dependent sources

$$R_{TH}=\frac{v_{OC}}{i_{SC}}$$

ORIGINAL CIRCUIT



SHORT CIRCUIT CURRENT



$$i_A''' = i_1$$
$$i_{SC} = i_2$$

NOTE: FOR THE INDUCTIVE CASE THE CIRCUIT USED TO COMPUTE THE SHORT CIRCUIT CURRENT IS THE SAME USE TO DETERMINE $\boxed{v_O(\infty)}$

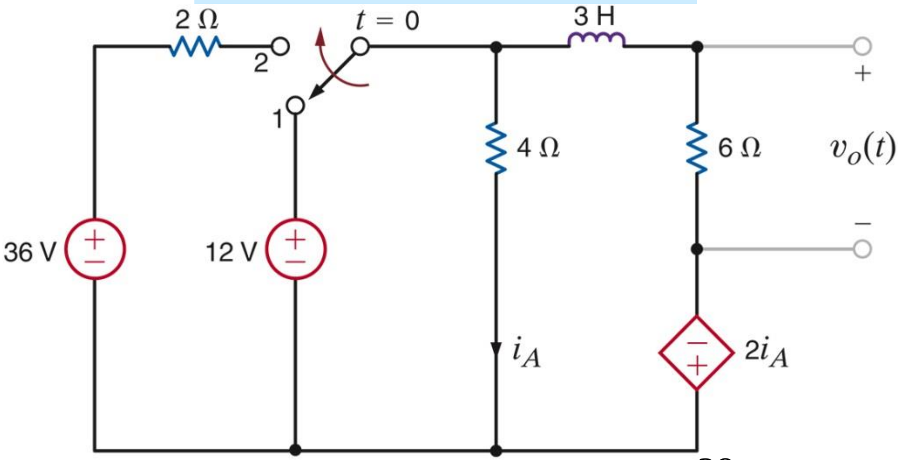
$$36 = 2(i_1 + i_2) + 4i_1$$
$$36 = 2(i_1 + i_2) + 6i_2 - 2i_A'''$$

$$i_{SC} = \frac{36}{8}[A]$$

$$\left.\begin{matrix} v_{OC} = 36[V] \\ i_{SC} = 36/8[A] \end{matrix}\right\} \Rightarrow R_{TH} = 8\Omega$$

$$L = 3H \Rightarrow \tau = \frac{3}{8}s$$

ORIGINAL CIRCUIT



STEP 6: DETERMINE K_1, K_2

$v_O(\infty) = 27 = K_1$ (step 4)

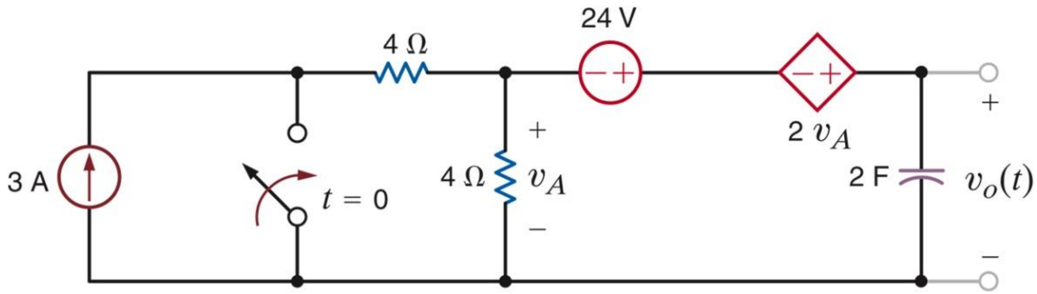
$v_O(0+) = 18 = K_1 + K_2 \Rightarrow K_2 = -9[V]$ (step 3)

ANS: $v_O(t) = 27 - 9e^{-\frac{t}{3/8}}, \ t > 0$

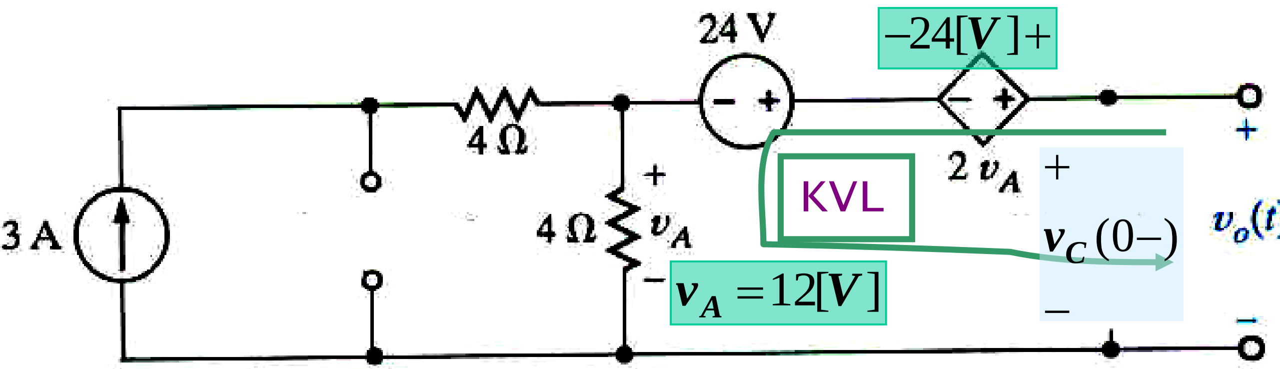
EXAMPLE 6

FIND $v_o(t), t>0$

STEP 1: $v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, t>0$



STEP 2: DETERMINE CAPACITOR VOLTAGE AT $t = 0+$



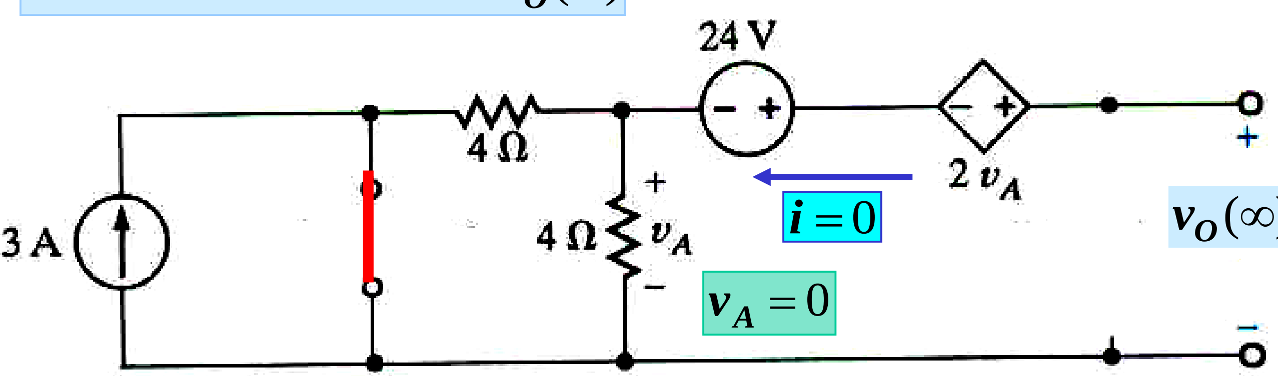
$v_C(0-) = 24 + 24 + 12 = 60[V]$

$v_C(0+) = v_C(0-)$

STEP 3: DETERMINE $v_O(0+)$

$v_O = v_C \Rightarrow v_O(0+) = 60[V]$

STEP 4: DETERMINE $v_O(\infty)$



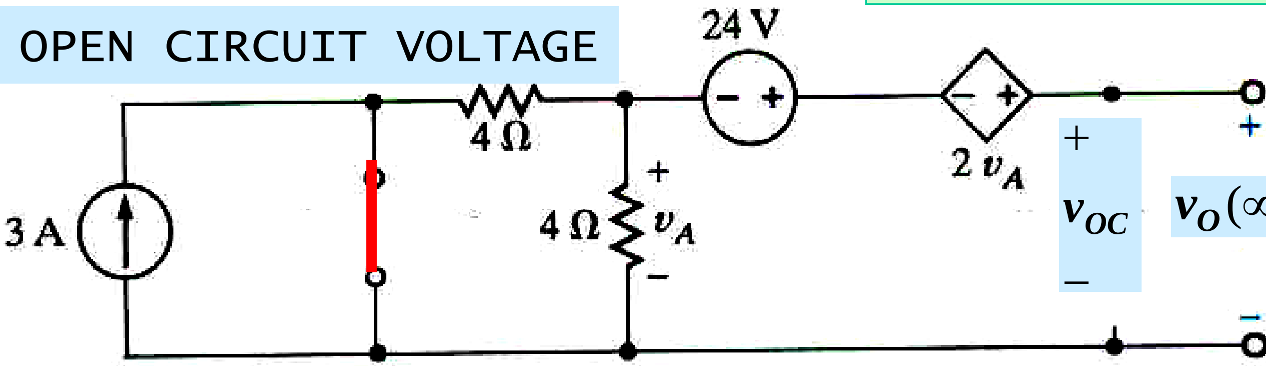
$v_O(\infty) = 24[V]$

STEP 5: DETERMINE TIME CONSTANT

capacitive circuit $\Rightarrow \tau = R_{TH}C$

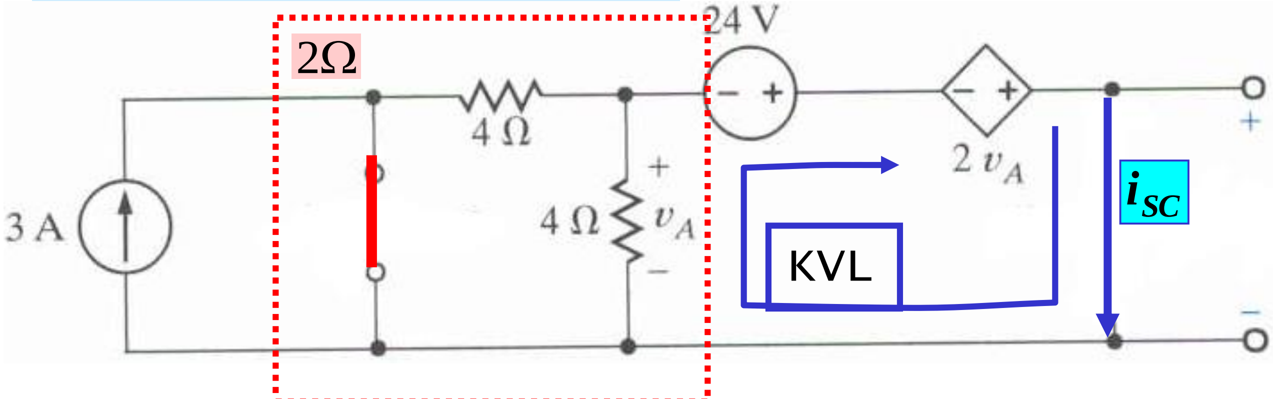
$$R_{TH} = \frac{v_{OC}}{i_{SC}}$$

OPEN CIRCUIT VOLTAGE



$$v_{OC} = v_O(\infty) = 24[V]$$

SHORT CIRCUIT CURRENT



$$2i_{SC} - 24 - 2v_A = 0$$

$$v_A = -2i_{SC}$$

$$i_{SC} = 4[A]$$

$$R_{TH} = \frac{24}{4} = 6\Omega$$

$$\tau = 6\Omega \times 2F = 12s$$

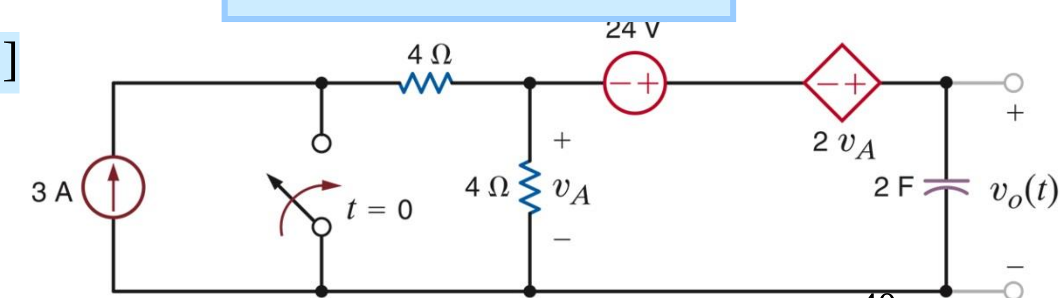
STEP 6: DETERMINE K_1, K_2

$K_1 = v_O(\infty) = 24$ (step 4)

$v_O(0+) = 60 = K_1 + K_2$ (step 3) $\quad K_2 = 36[V]$

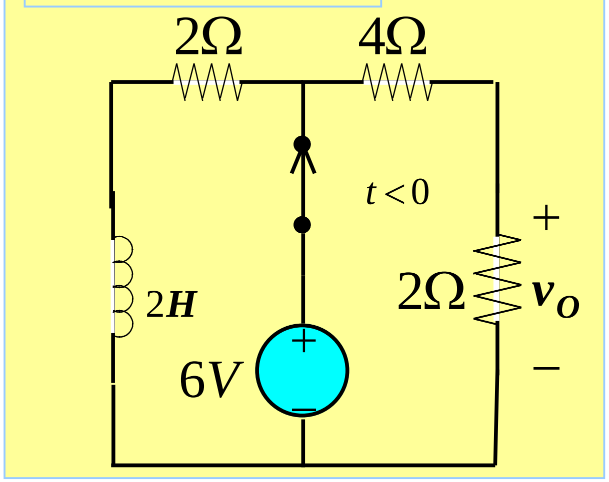
ANS: $v_O(t) = 24 + 36e^{-\frac{t}{12}}, \; t > 0$

ORIGINAL CIRCUIT



EXAMPLE 7

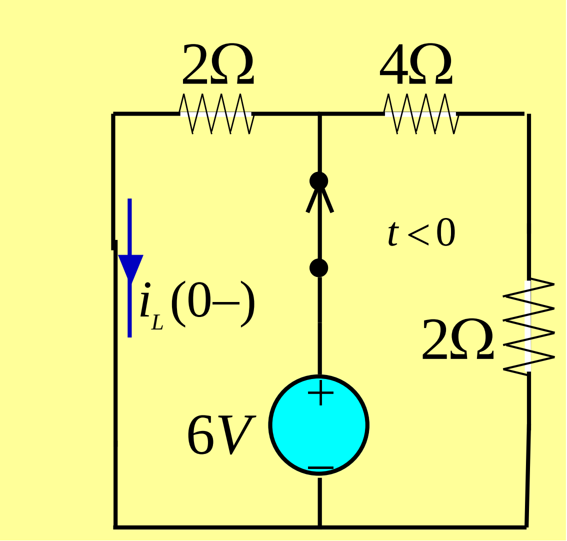
FIND $v_O(t), t > 0$



STEP 1: Form of the solution

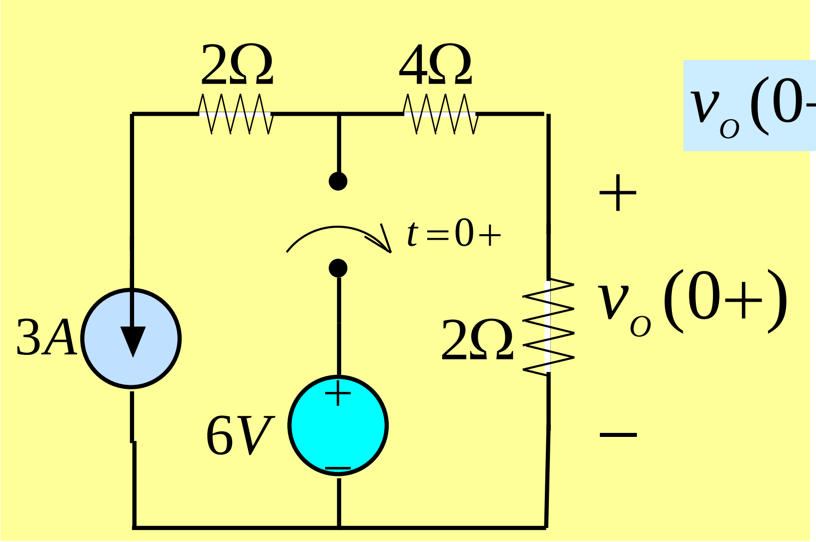
$$v_O(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$$

STEP 2: Initial inductor current



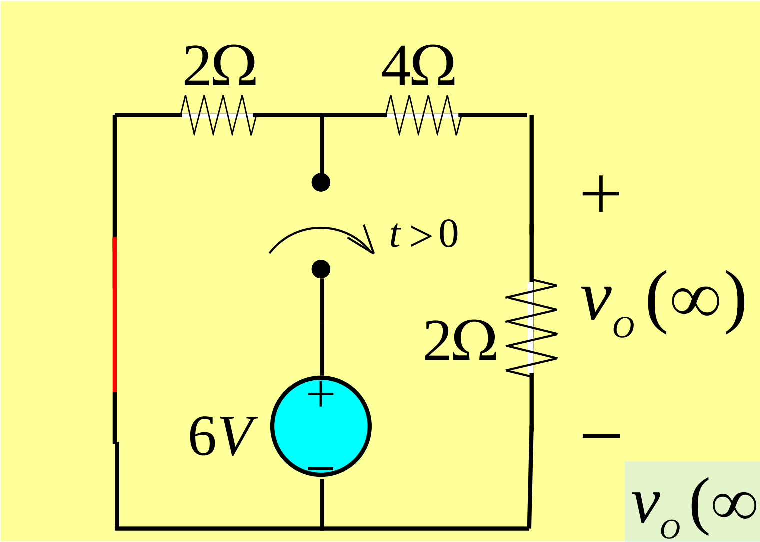
$$i_L(0-) = 3A$$

STEP 3: Determine output at 0+
(inductor current is constant)

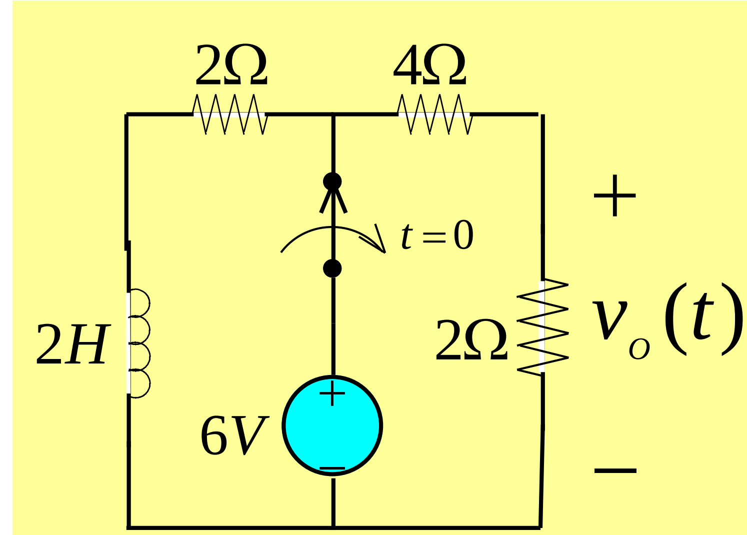


$$v_O(0+) = -6V$$

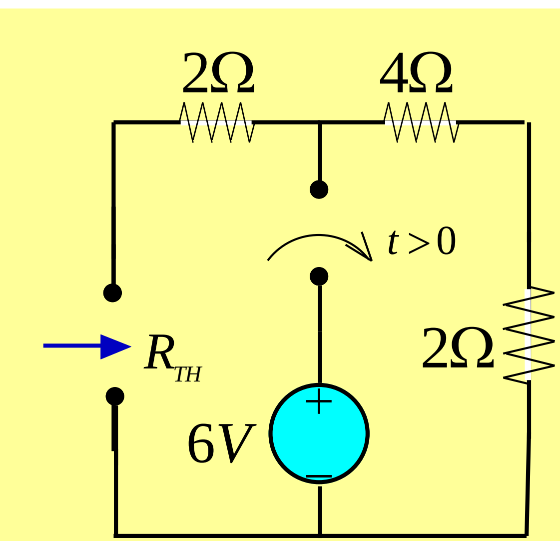
STEP 4: Find output in steady state after the switching



$$v_o(\infty) = 0$$



STEP 5: Find time constant after switch



$$\tau = \frac{L}{R_{TH}}$$

$$R_{TH} = 8\Omega$$

$$\tau = 0.25\,s$$

STEP 6: Find the solution

$$K_1 + K_2 = v_O(0+) = -6V$$

$$K_1 = v_O(\infty) = 0$$

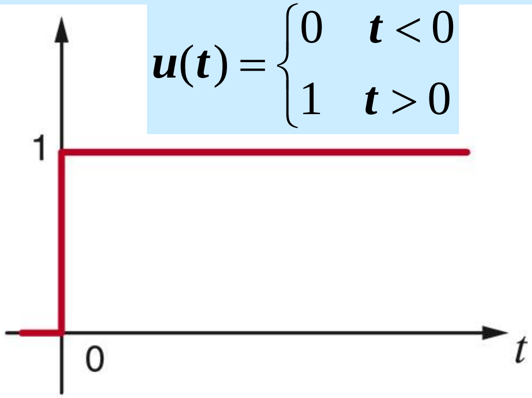
$$v_O(t) = -6e^{-\frac{t}{0.25}}\,;\, t > 0$$

$$\boldsymbol{v_O(t) = -6e^{-4t}\,;\, t > 0}$$

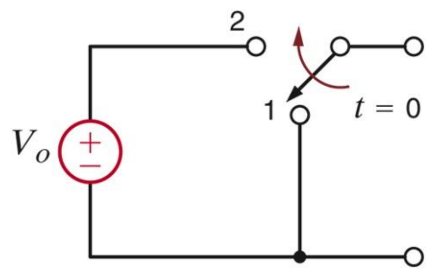
PULSE RESPONSE

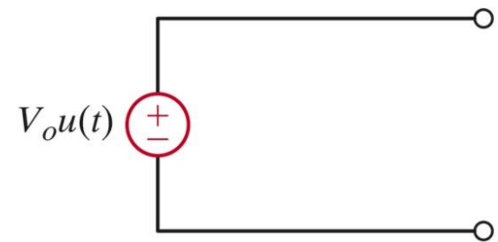
WE STUDY THE RESPONSE OF CIRCUITS TO A SPECIAL CLASS OF *SINGULARITY FUNCTIONS*

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

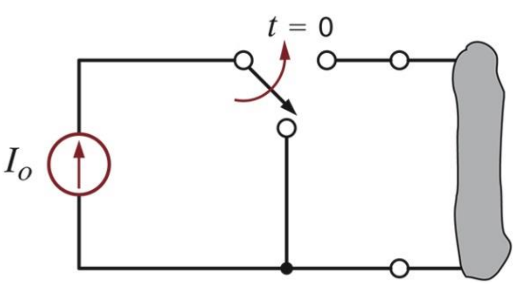


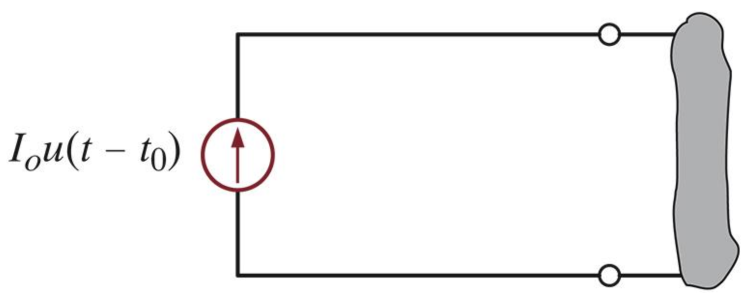
VOLTAGE STEP



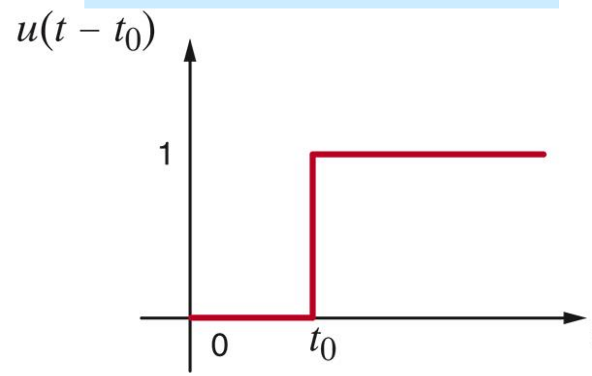


CURRENT STEP



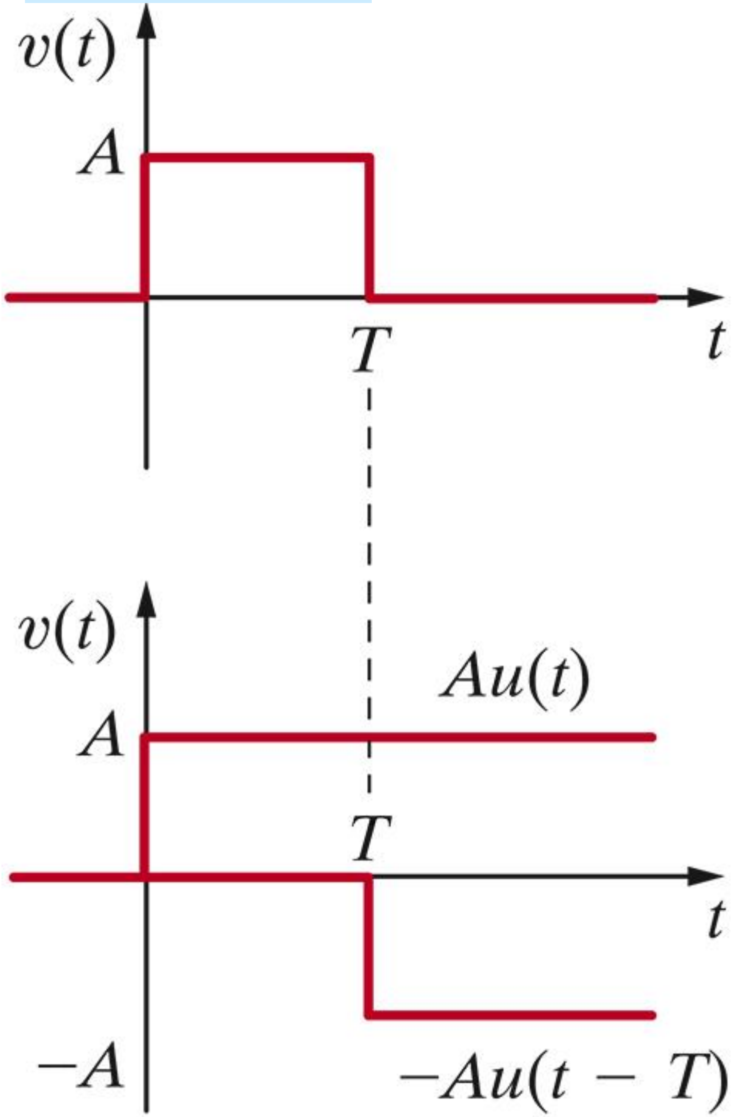


TIME SHIFTED STEP



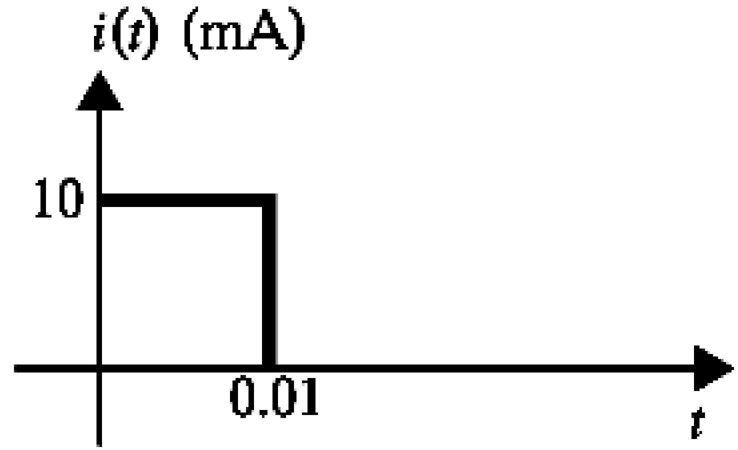
LEARN BY DOING

PULSE SIGNAL

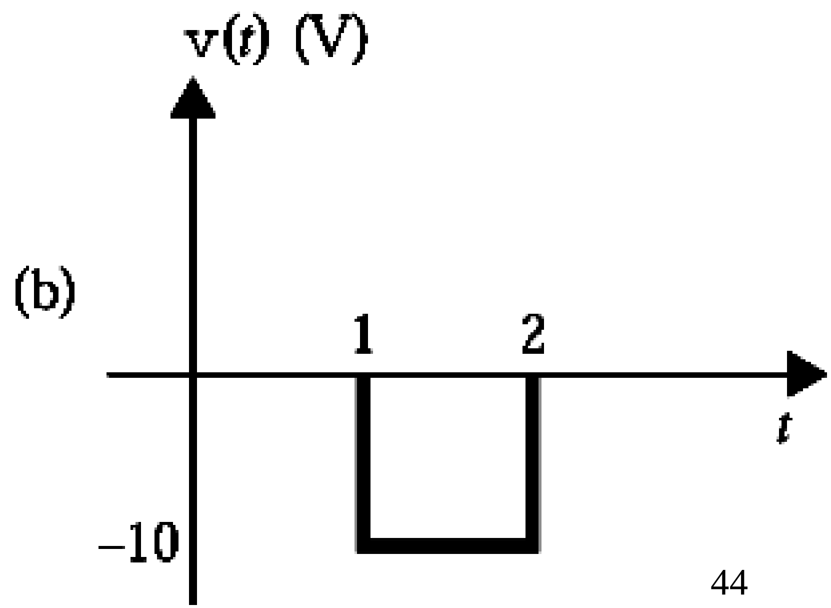


PULSE AS SUM OF STEPS

$$i(t) = 10[u(t) - u(t - 0.01)](mA)$$



$$v(t) = -10[u(t-1) - u(t-2)](V)$$



NONZERO INITIAL TIME AND REPEATED SWITCHING

$$\tau \frac{dx}{dt} + x = f_{TH}; \; x(t_0+) = x_0$$

$$x(t) = e^{-\frac{t-t_0}{\tau}} x(t_0) + \frac{1}{\tau} \int_{t_0}^{t} e^{-\frac{t-x}{\tau}} f_{TH}(x) dx$$

$$x(t) = K_1 + K_2 e^{-\frac{t-t_0}{\tau}}; \; t \geq t_0$$

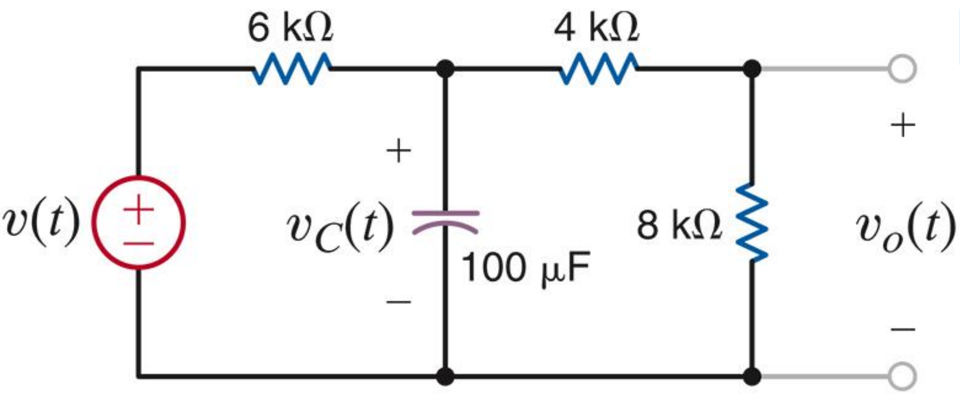
RESPONSE FOR CONSTANT SOURCES

This expression will hold on ANY interval where the sources are constant. The values of the constants may be different and must be evaluated for each interval

The values at the end of one interval will serve as initial conditions for the next interval

EXAMPLE 8

FIND THE OUTPUT VOLTAGE $v_O(t);\ t>0$



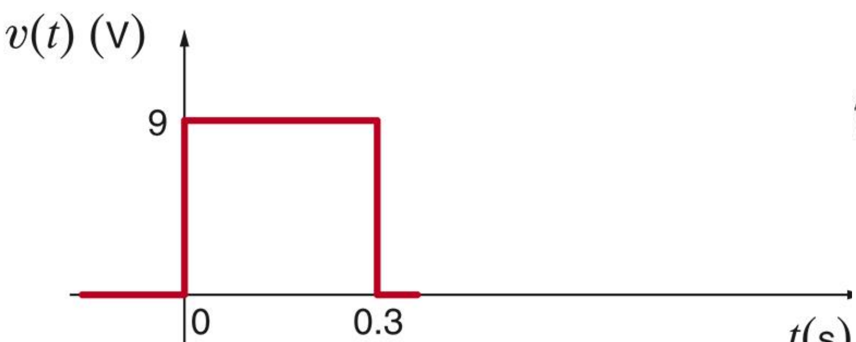
$t>0.3 \Rightarrow v(t)=0$ $\quad t_o=0.3$ $\quad v_o(0.3+)=4(1-e^{-\frac{0.3}{0.4}})$

$$v_O(t)=K_1''+K_2''e^{-\frac{(t-0.3)}{\tau'}}$$

$\tau'=0.4$

$v_o(\infty)=0 \Rightarrow K_1''=0$ $\quad K_2''=v_o(0.3+)=2.11(V)$

$$v_o(t)=2.11e^{-\frac{t-0.3}{0.4}};\ t>0.3$$



$t<0 \Rightarrow v(t)=0 \Rightarrow v_O(t)=0$ $\quad v_O(0+)=0$

$t>0 \Rightarrow v(t)=9V$

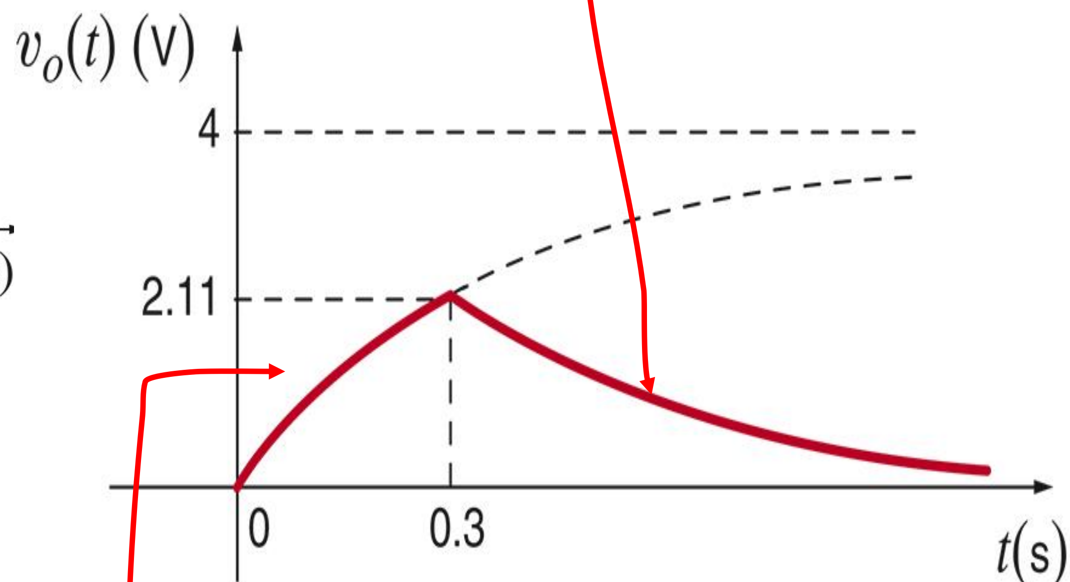
$$v_o(t)=K_1'+K_2'e^{-\frac{t}{\tau}}$$

$$\tau=R_{TH}C=(6k\,\|\,12k)\times 100\mu F=0.4s$$

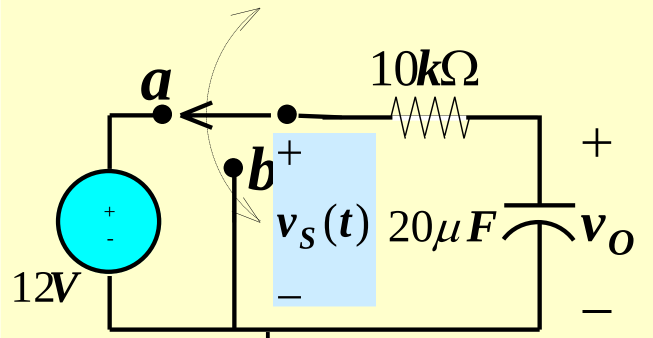
$$v_o(\infty)=\frac{8}{10+8}(9)=K_1'$$

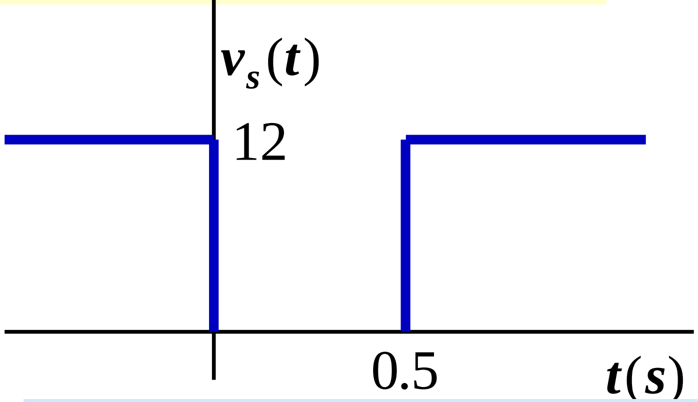
$$v_o(0+)=K_1'+K_2'=0$$

$$v_o(t)=4\left(1-e^{-\frac{t}{0.4}}\right)$$



EXAMPLE 9 THE SWITCH IS INITIALLY AT a. AT TIME t=0 IT MOVES TO b AND AT t=0.5 IT MOVES BACK TO a. FIND $v_O(t), t > 0$





Piecewise constant source

ON EACH INTERVAL WHERE THE SOURCE IS CONSTANT THE OUTPUT IS OF THE FORM

$$v_O(t) = K_1 + K_2 e^{-\frac{t-t_O}{\tau}}$$

FOR $0 < t < 0.5$ (switch at b) $t_o = 0$

$$v_O(t) = K_1^{'} + K_2^{'} e^{-\frac{t}{\tau}}$$

$$v(0+) = 12[V] = K_1^{'} + K_2^{'}$$

$$v_O(\infty) = 0 = K_1^{'}$$

$$\tau = (10k\Omega)(20\mu F) = 0.2s$$

$$v_O(t) = 12e^{-\frac{t}{0.2}}, 0 < t < 0.5$$

FOR $t > 0.5$ (switch at a) $t_o = 0.5$

$$v_O(0.5+) = v_O(0.5-) = 12e^{-\frac{0.5}{0.2}} = 0.985$$

$$v_O(t) = K_1^{''} + K_2^{''} e^{-\frac{(t-0.5)}{\tau'}}$$

$$v_O(0.5+) = 0.985 = K_1^{''} + K_2{''}$$

$$v_O(\infty) = 12 = K_1^{''}$$

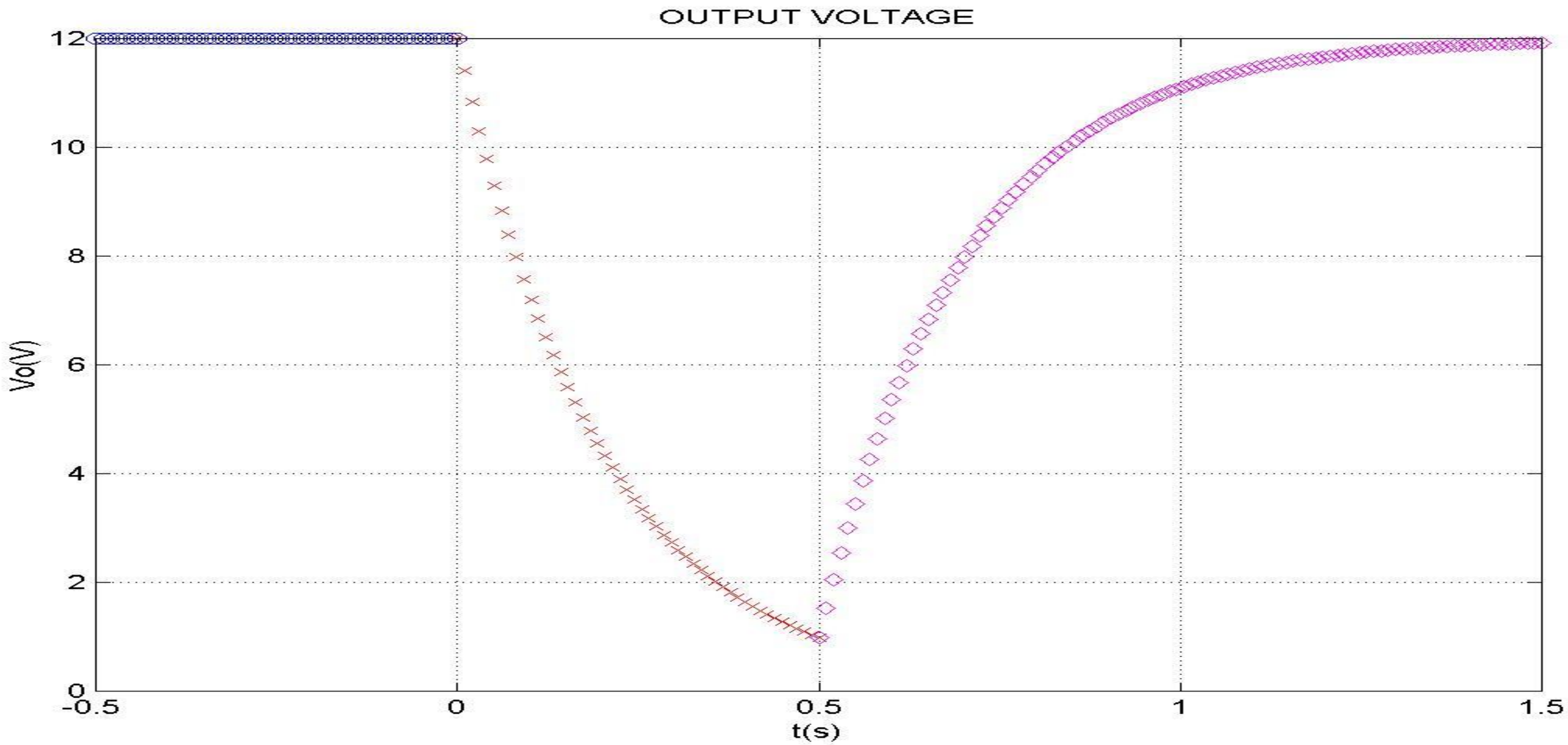
$$K_2^{''} = 0.985 - 12 = -11.015$$

$$v_O(t) = 12 - 11.015e^{-\frac{t-0.5}{0.2}}, t > 0.5$$

The constants are determined in the usual manner

USING MATLAB TO DISPLAY OUTPUT VOLTAGE

```
%pulse1.m
% displays the response to a pulse response
tmin=linspace(-0.5,0,50); %negative time segment
t1=linspace(0,0.5,50); %first segment
t2=linspace(0.5, 1.5,100); %second segment
vomin=12*ones(size(tmin));
vo1=12*exp(-t1/0.2); %after first switching
vo2=12-11.015*exp(-(t2-0.5)/0.2); %after second switching
plot(tmin,vomin,'bo',t1,vo1,'rx',t2,vo2,'md'),grid
title('OUTPUT VOLTAGE'), xlabel('t(s)'),ylabel('Vo(V)')
```



EXERCISE PROBLEMS

7.1 Use the differential equation approach to find $i(t)$ for $t > 0$ in the network in Fig. P7.1.

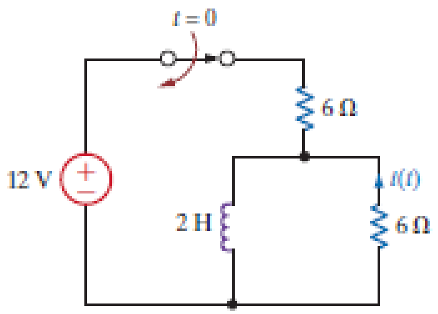


Figure P7.1

7.3 Use the differential equation approach to find $v_o(t)$ for $t > 0$ in the circuit in Fig. P7.3 and plot the response including the time interval just prior to switch action.

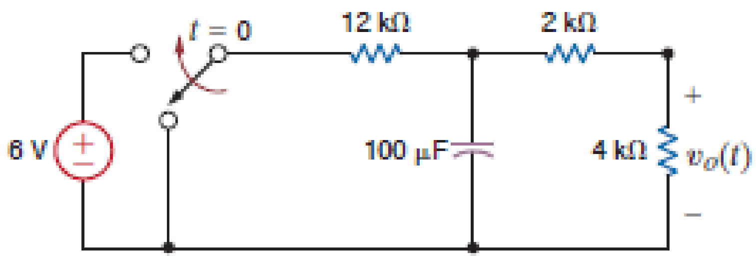


Figure P7.3

7.5 Use the differential equation approach to find $v_C(t)$ for $t > 0$ in the circuit in Fig. P7.5.

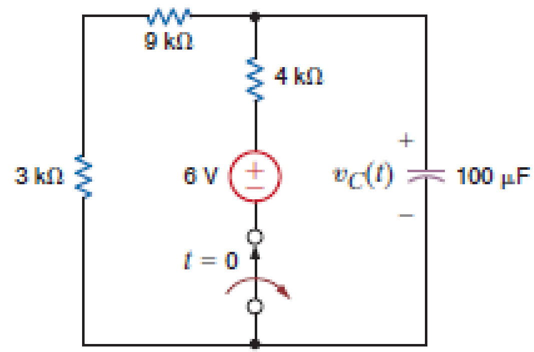
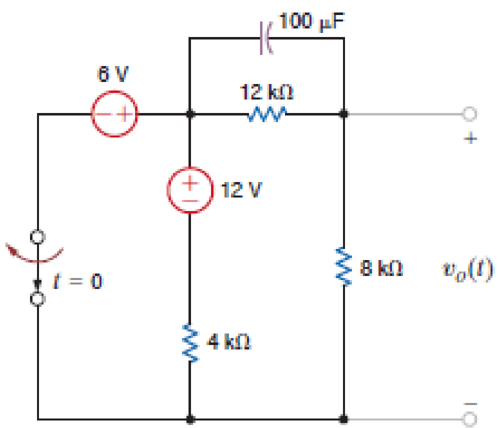


Figure P7.5

7.12 Use the differential equation approach to find $v_o(t)$ for $t > 0$ in the circuit in Fig. P7.12 and plot the response, including the time interval just prior to opening the switch.



EXERCISE PROBLEMS

7.15 Use the step-by-step technique to find $i_o(t)$ for $t > 0$ in the network in Fig. P7.15.

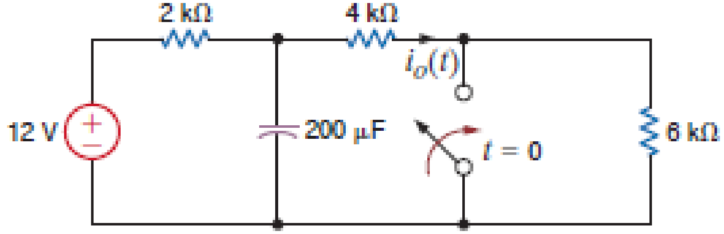


Figure P7.15

7.19 Use the step-by-step method to find $i_o(t)$ for $t > 0$ in the circuit in Fig. P7.19.

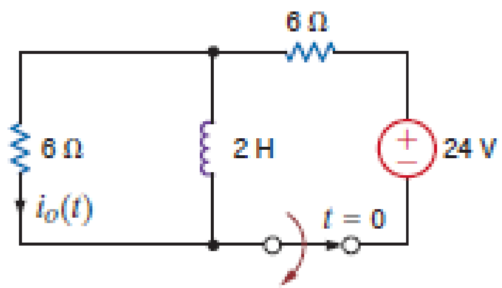


Figure P7.19

7.25 Use the step-by-step technique to find $i_o(t)$ for $t > 0$ in the network in Fig. P7.25.

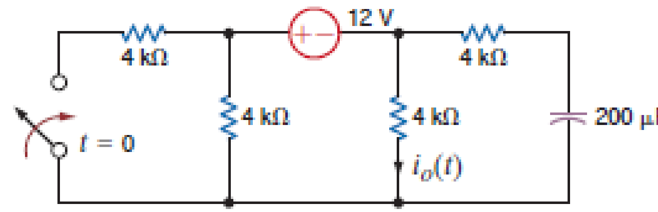


Figure P7.25

7.28 Find $i_o(t)$ for $t > 0$ in the circuit in Fig. P7.28.

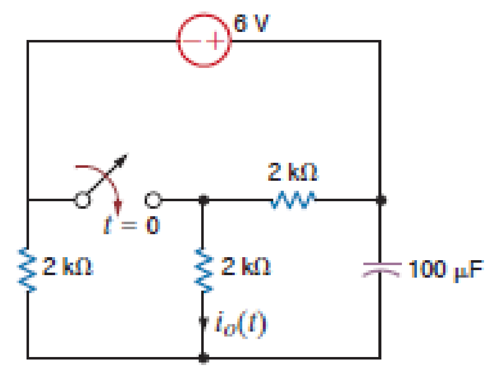


Figure P7.28

EXERCISE PROBLEMS

7.32 Use the step-by-step method to find $v_o(t)$ for $t > 0$ in the network in Fig. P7.32.

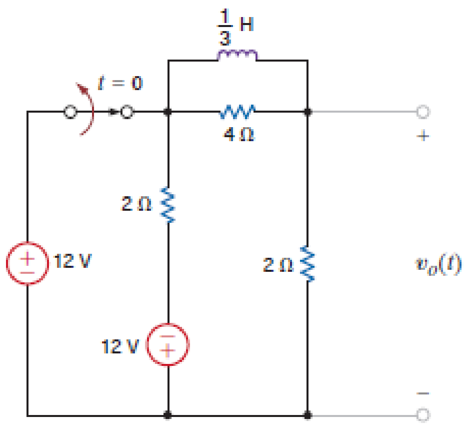


Figure P7.32

7.39 The switch in the circuit in Fig. P7.39 is opened at $t = 0$. Find $i(t)$ for $t > 0$.

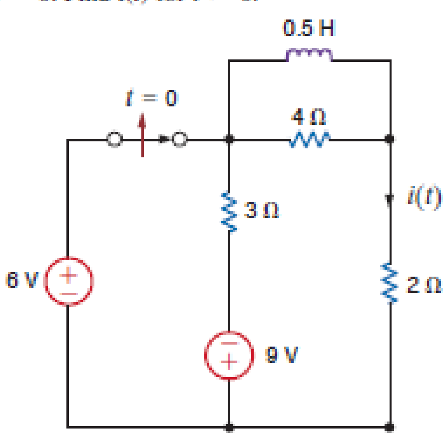


Figure P7.39

7.66 Find $i_o(t)$ for $t > 0$ in the network in Fig. P7.66.

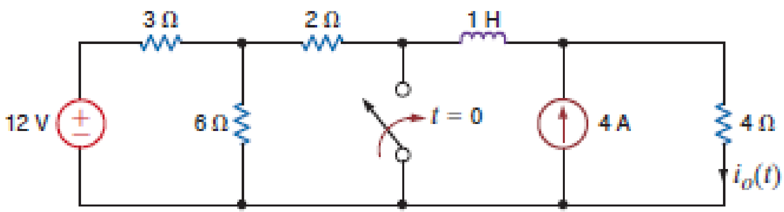


Figure P7.66

7.73 Find $i_o(t)$ for $t > 0$ in the network in Fig. P7.73 using the step-by-step method.

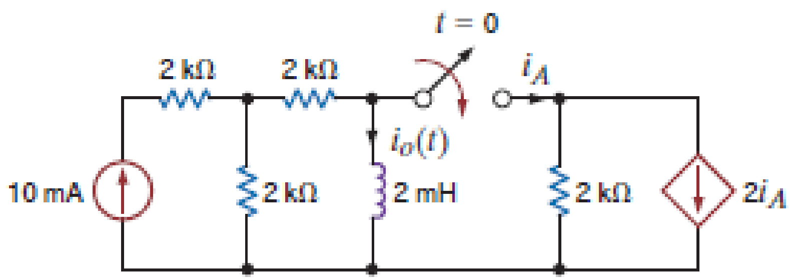


Figure P7.73

EXERCISE PROBLEMS

7.74 Use the step-by-step technique to find $v_o(t)$ for $t > 0$ in the network in Fig. P7.74.

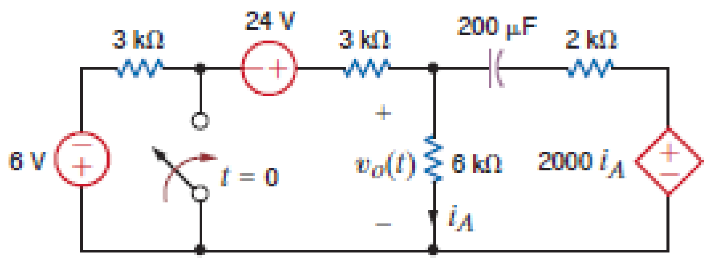
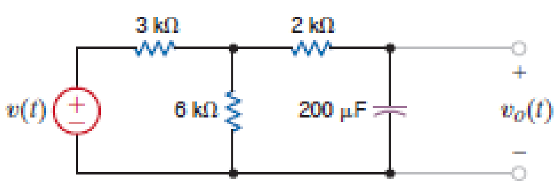
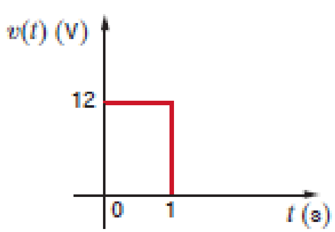


Figure P7.74

7.75 Determine the equation for the voltage $v_o(t)$ for $t > 0$ in Fig. P7.75a when subjected to the input pulse shown in Fig. P7.75b.



(a)

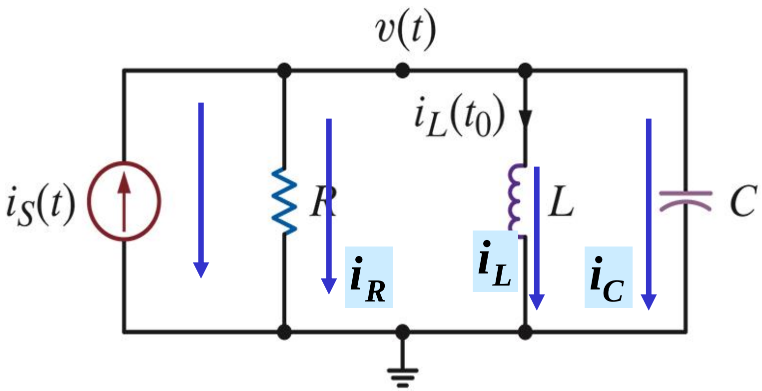


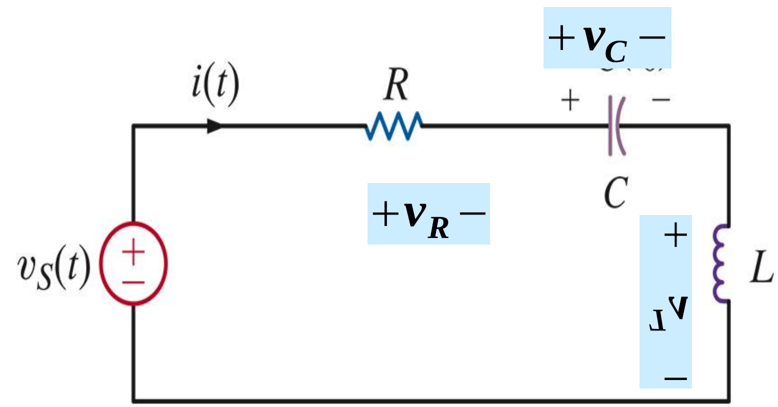
(b)

Figure P7.75

SECOND-ORDER CIRCUITS

THE BASIC CIRCUIT EQUATION





Single Node-pair: Use KCL

$$-i_S + i_R + i_L + i_C = 0$$

$$i_R = \frac{v(t)}{R}; \quad i_L = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i_L(t_0); \quad i_C = C\frac{dv}{dt}(t)$$

$$\frac{v}{R} + \frac{1}{L}\int_{t_0}^{t} v(t)dt + i_L(t_0) + C\frac{dv}{dt}(t) = i_S$$

Differentiating

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = \frac{di_S}{dt}$$

Single Loop: Use KVL

$$-v_S + v_R + v_C + v_L = 0$$

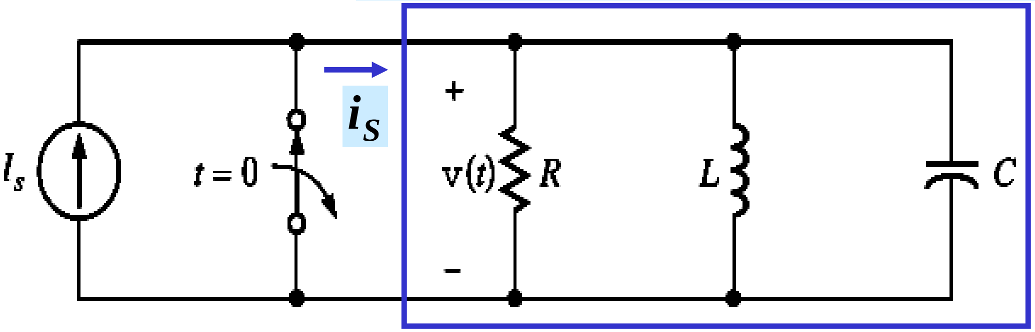
$$v_R = Ri; \; v_C = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v_C(t_0); \; v_L = L\frac{di}{dt}(t)$$

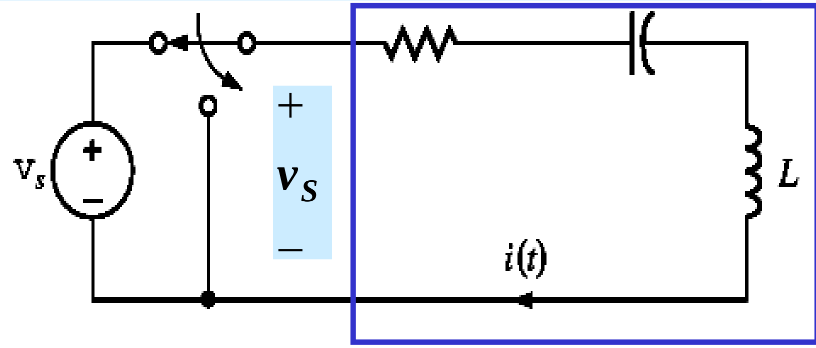
$$Ri + \frac{1}{C}\int_{t_0}^{t} i(t)dt + v_C(t_0) + L\frac{di}{dt}(t) = v_S$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv_S}{dt}$$

EXAMPLE 1

WRITE THE DIFFERENTIAL EQUATION FOR $v(t), i(t)$, RESPECTIVELY





$$i_S(t)=\begin{cases}0 & t<0\\ I_S & t>0\end{cases}$$

$$\frac{di_S}{dt}(t)=0;\ t>0$$

$$v_S(t)=\begin{cases}V_S & t<0\\ 0 & t>0\end{cases}$$

$$\frac{dv_S}{dt}(t)=0;\ t>0$$

MODEL FOR RLC PARALLEL

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}=\frac{di_S}{dt}$$

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}=0$$

MODEL FOR RLC SERIES

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=\frac{dv_S}{dt}$$

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=0$$

THE RESPONSE EQUATION

WE STUDY THE SOLUTIONS FOR THE EQUATION

$$\frac{d^2x}{dt^2}(t) + a_1\frac{dx}{dt}(t) + a_2x(t) = f(t)$$

KNOWN: $x(t) = x_p(t) + x_c(t)$

x_p particular solution

x_c complementary solution

THE COMPLEMENTARY SOLUTION SATISIFES

$$\frac{d^2x_c}{dt^2}(t) + a_1\frac{dx_c}{dt}(t) + a_2x_c(t) = 0$$

IF THE FORCING FUNCTION IS A CONSTANT

$f(t) = A \Rightarrow x_p = \dfrac{A}{a_2}$ is a particular solution

PROOF: $x_p = \dfrac{A}{a_2} \Rightarrow \dfrac{dx_p}{dt} = \dfrac{d^2x_p}{dt^2} = 0 \Rightarrow a_2x_p = A$

FOR ANY FORCING FUNCTION $f(t) = A$

$$x(t) = \frac{A}{a_2} + x_c(t)$$

THE HOMOGENEOUS EQUATION

$$\frac{d^2x}{dt^2}(t)+a_1\frac{dx}{dt}(t)+a_2x(t)=0$$

NORMALIZED FORM

$$\frac{d^2x}{dt^2}(t)+2\varsigma\omega_n\frac{dx}{dt}(t)+\omega_n^2x(t)=0$$

ω_n (undamped) natural frequency

ς damping ratio

CHARACTERISTIC EQUATION

$$s^2+2\varsigma\omega_n s+\omega_n^2=0$$

$$a_2=\omega_n^2\Rightarrow\omega_n=\sqrt{a_2}$$

$$a_1=2\varsigma\omega_n\Rightarrow\varsigma=\frac{a_1}{2\sqrt{a_2}}$$

LEARNING BY DOING

DETERMINE THE CHARACTERISTIC EQUATION, DAMPING RATIO AND NATURAL FREQUENCY

$$4\frac{d^2x}{dt^2}(t)+8\frac{dx}{dt}(t)+16x(t)=0$$

COEFFICIENT OF SECOND DERIVATIVE MUST BE ONE

$$\frac{d^2x}{dt^2}(t)+\boxed{2\frac{dx}{dt}(t)}+\boxed{4x(t)}=0$$

CHARACTERISTIC EQUATION

$$s^2+\boxed{2s}+\boxed{4}=0$$

DAMPING RATIO, NATURAL FREQUENCY

$$\frac{d^2x}{dt^2}(t)+\boxed{2}\frac{dx}{dt}(t)+\boxed{4}x(t)=0$$

$$\boxed{2\varsigma\omega_n}\qquad\boxed{\omega_n^2}\Rightarrow\omega_n=2$$

$$\Downarrow$$

$$\varsigma=0.5$$

ANALYSIS OF THE HOMOGENEOUS EQUATION

NORMALIZED FORM

$$\frac{d^2x}{dt^2}(t) + 2\varsigma\omega_n \frac{dx}{dt}(t) + \omega_n^2 x(t) = 0$$

$x(t) = Ke^{st}$ is a solution iff

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

Iff s is solution of the *characteristic equation*

PROOF: $\frac{dx}{dt}(t) = sKe^{st}$; $\frac{d^2x}{dt^2} = s^2Ke^{st}$

$$\frac{d^2x}{dt^2}(t) + 2\varsigma\omega_n \frac{dx}{dt}(t) + \omega_n^2 x(t) = (s^2 + 2\varsigma\omega_n s + \omega_n^2)Ke^{st}$$

CHARACTERISTIC EQUATION

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

$$(s + \varsigma\omega_n)^2 + (\omega_n^2 - \varsigma^2\omega_n^2) = 0$$

$$s = -\varsigma\omega_n \pm \sqrt{\varsigma^2\omega_n^2 - \omega_n^2}$$

$$s = -\varsigma\omega_n \pm \omega_n\sqrt{\varsigma^2 - 1}$$

(modes of the system)

CASE 1: $\varsigma > 1$ (real and distinct roots)

$$x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

CASE 2: $\varsigma < 1$ (complex conjugate roots)

$$x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$s = -\varsigma\omega_n \pm j\omega_n\sqrt{1 - \varsigma^2}$$

$$s = -\sigma \pm j\omega_d$$

$x(t)$ real $\Rightarrow K_2 = K_1^*$

$\omega_d =$ damped oscillation frequency

$\sigma =$ damping factor

$$x(t) = e^{-\sigma t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)$$

HINT: $e^{st} = e^{-(\varsigma\omega_n \pm j\omega_d)t} = e^{-\varsigma\omega_n t} e^{\mp j\omega_d t}$

$$e^{\mp j\omega_d t} = \cos\omega_d t \mp j\sin\omega_d t$$

ASSUME $K_1 = (A_1 + jA_2)/2$

$$\left.\begin{array}{l} K_2 = K_1^* \\ s = -\sigma \pm j\omega_d \end{array}\right\} \Rightarrow x(t) = 2\,\mathrm{Re}\left[K_1 e^{-(\sigma + j\omega_d)t}\right]$$

CASE 3: $\varsigma = 1$ (real and equal roots)

$$s = -\varsigma\omega_n$$

$$x(t) = \left(B_1 + B_2 t\right)e^{-\varsigma\omega_n t}$$

HINT: te^{st} is solution iff

$$(s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0)\ AND\ (2s + 2\varsigma\omega_n = 0)$$

EXAMPLE 2

DETERMINE THE GENERAL FORM OF THE SOLUTION

$$\frac{d^2x}{dt^2}(t)+4\frac{dx}{dt}(t)+4x(t)=0$$

CHARACTERISTIC EQUATION

$$s^2+4s+4=0$$

$$\omega_n^2=4\Rightarrow\omega_n=2 \qquad 2\varsigma\omega_n=4\Rightarrow\varsigma=1$$

$$s^2+4s+4=0\Rightarrow(s+2)^2=0$$

Roots are real and equal

this is a critically damped (case 3) system

$$x(t)=(B_1+B_2t)e^{st}$$

$$x(t)=(B_1+B_2t)e^{-2t}$$

$$4\frac{d^2x}{dt^2}(t)+8\frac{dx}{dt}(t)+16x(t)=0$$

Divide by coefficient of second derivative

$$\frac{d^2x}{dt^2}(t)+2\frac{dx}{dt}(t)+4x(t)=0$$

$$\omega_n^2=4\Rightarrow\omega_n=2 \qquad 2\varsigma\omega_n=2\Rightarrow\varsigma=0.5$$

$$s^2+2s+4=(s+1)^2+3=0\Rightarrow s=-1\pm j\sqrt{3}$$

Roots are complex conjugate

ω_d

underdamped (case 2) system

$$\sigma=\varsigma\omega_n=1;\quad \omega_d=\omega_n\sqrt{1-\varsigma^2}=2\sqrt{1-0.25}=\sqrt{3}$$

$$x(t)=e^{-\sigma t}\left(A_1\cos\omega_d t+A_2\sin\omega_d t\right)$$

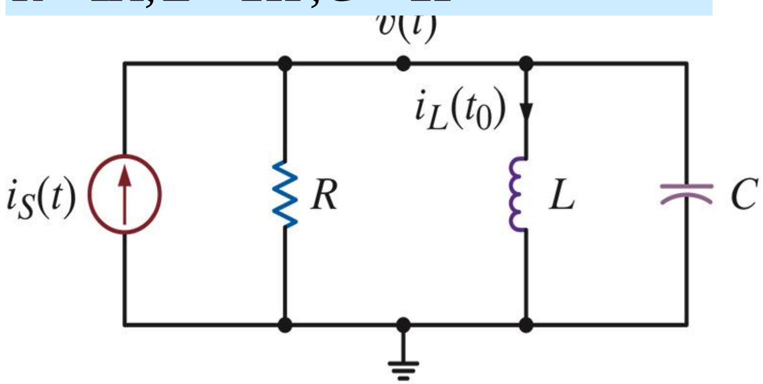
$$x(t)=e^{-t}\left(A_1\cos\sqrt{3}t+A_2\sin\sqrt{3}t\right)$$

Form of the solution

EXAMPLE 3

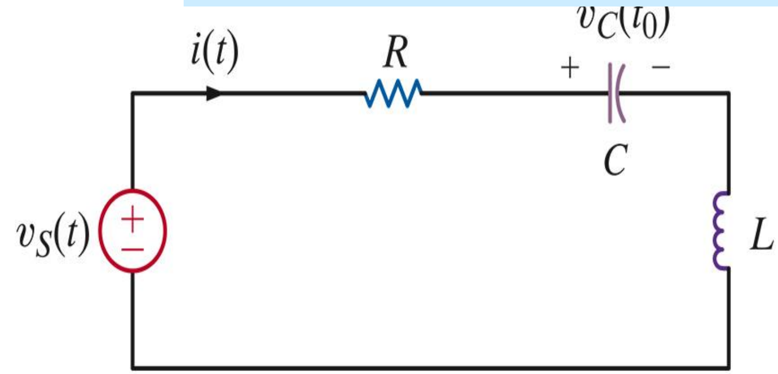
RLC PARALLEL CIRCUIT WITH

$R = 1\Omega, L = 2H, C = 2F$



RLC SERIES CIRCUIT WITH

$R = 2\Omega; L = 1H, C = 0.5F, 1F, 2F$



HOMOGENEOUS EQUATION

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = 0$$

$$2\frac{d^2v}{dt^2} + \frac{dv}{dt} + \frac{v}{2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{2}\frac{dv}{dt} + \frac{v}{4} = 0$$

$$s^2 + \frac{s}{2} + \frac{1}{4} = (s + \frac{1}{4})^2 + \frac{3}{16} = 0$$

$$\omega_n = \frac{1}{2}; \varsigma\omega_n = \frac{1}{4} \Rightarrow \varsigma = \frac{1}{2}$$

$$\sigma = \frac{1}{4}$$

$$\omega_d = \omega_n\sqrt{1-\varsigma^2} = \frac{1}{2}\sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{4}$$

$$v_c(t) = e^{-\frac{t}{4}}\left(A_1\cos\frac{\sqrt{3}}{4}t + A_2\sin\frac{\sqrt{3}}{4}t\right)$$

Classify the responses for the given values of C

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$ $:/L$ & replace values

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + \frac{i}{C} = 0$$

$$\omega_n = \frac{1}{\sqrt{C}}; \; 2\varsigma\omega_n = 2 \Rightarrow \varsigma = \sqrt{C}$$

C=0.5	underdamped
C=1.0	critically damped
C=2.0	overdamped

$$\text{discriminant} = 4 - \frac{4}{C}$$

DETERMINING THE CONSTANTS

NORMALIZED FORM

$$\frac{d^2x}{dt^2}(t) + 2\varsigma\omega_n \frac{dx}{dt}(t) + \omega_n^2 x(t) = A$$

$$x(t) = \frac{A}{\omega_n^2} + K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$x(0+) - \frac{A}{\omega_n^2} = K_1 + K_2$$

$$\frac{dx}{dt}(0+) = s_1 K_1 + s_2 K_2$$

$$x(t) = \frac{A}{\omega_n^2} + e^{-\varsigma\omega_n t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)$$

$$x(0+) - \frac{A}{\omega_n^2} = A_1$$

$$\frac{dx}{dt}(0+) = -\varsigma\omega_n A_1 + \omega_d A_2$$

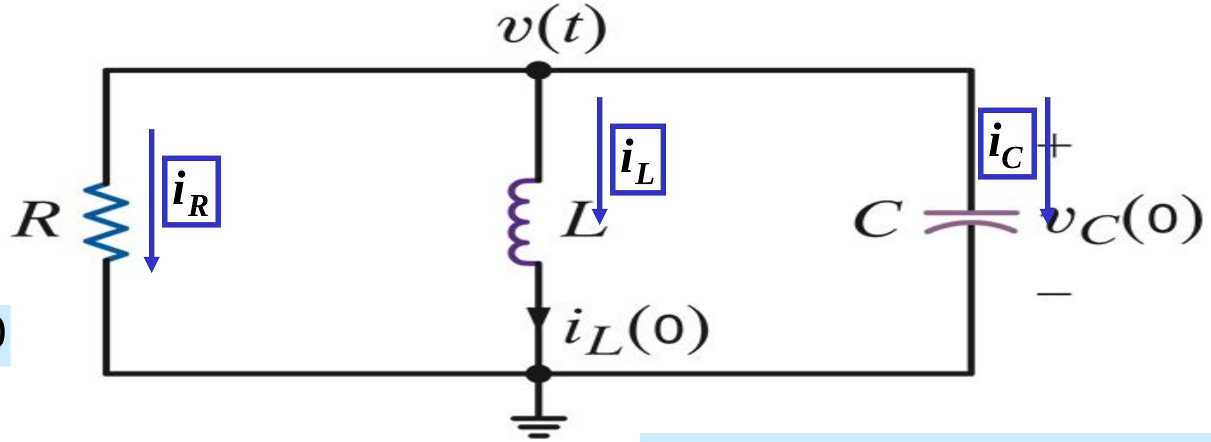
$$x(t) = \frac{A}{\omega_n^2} + \left(B_1 + B_2 t\right) e^{-\varsigma\omega_n t}$$

$$x(0+) - \frac{A}{\omega_n^2} = B_1$$

$$\frac{dx}{dt}(0+) = -\varsigma\omega_n B_1 + B_2$$

EXAMPLE 4

$R = 2\Omega, L = 5H, C = \frac{1}{5}F$ $i_L(0) = -1A,\ v_C(0) = 4V$



$i_R + i_L + i_C = 0$

$$\frac{v}{R} + \frac{1}{L}\int_0^t v(x)dx + i_L(0) + C\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

STEP 1
MODEL

CHARACTERISTIC EQUATION

STEP 2

$$s^2 + 2.5s + 1 = 0 \Rightarrow \omega_n = 1;\ \varsigma = 1.5$$

$$s = \frac{-2.5 \pm \sqrt{(2.5)^2 - 4}}{2} = \frac{-2.5 \pm 1.5}{2}$$

STEP 3
ROOTS

$$v(t) = K_1 e^{-2t} + K_2 e^{-0.5t}$$

STEP 4
FORM OF
SOLUTION

STEP 5: FIND CONSTANTS

To determine the constants we need

$$v(0+);\ \frac{dv}{dt}(0+)$$

IF NOT GIVEN FIND $v_C(0), i_L(0)$

$$v(0+) = v_C(0+) = v_C(0) = 4V$$

KCL AT $t = 0+$

$$\frac{v_C(0+)}{R} + i_L(0+) + C\frac{dv}{dt}(0+) = 0$$

ANALYZE
CIRCUIT AT
t=0+

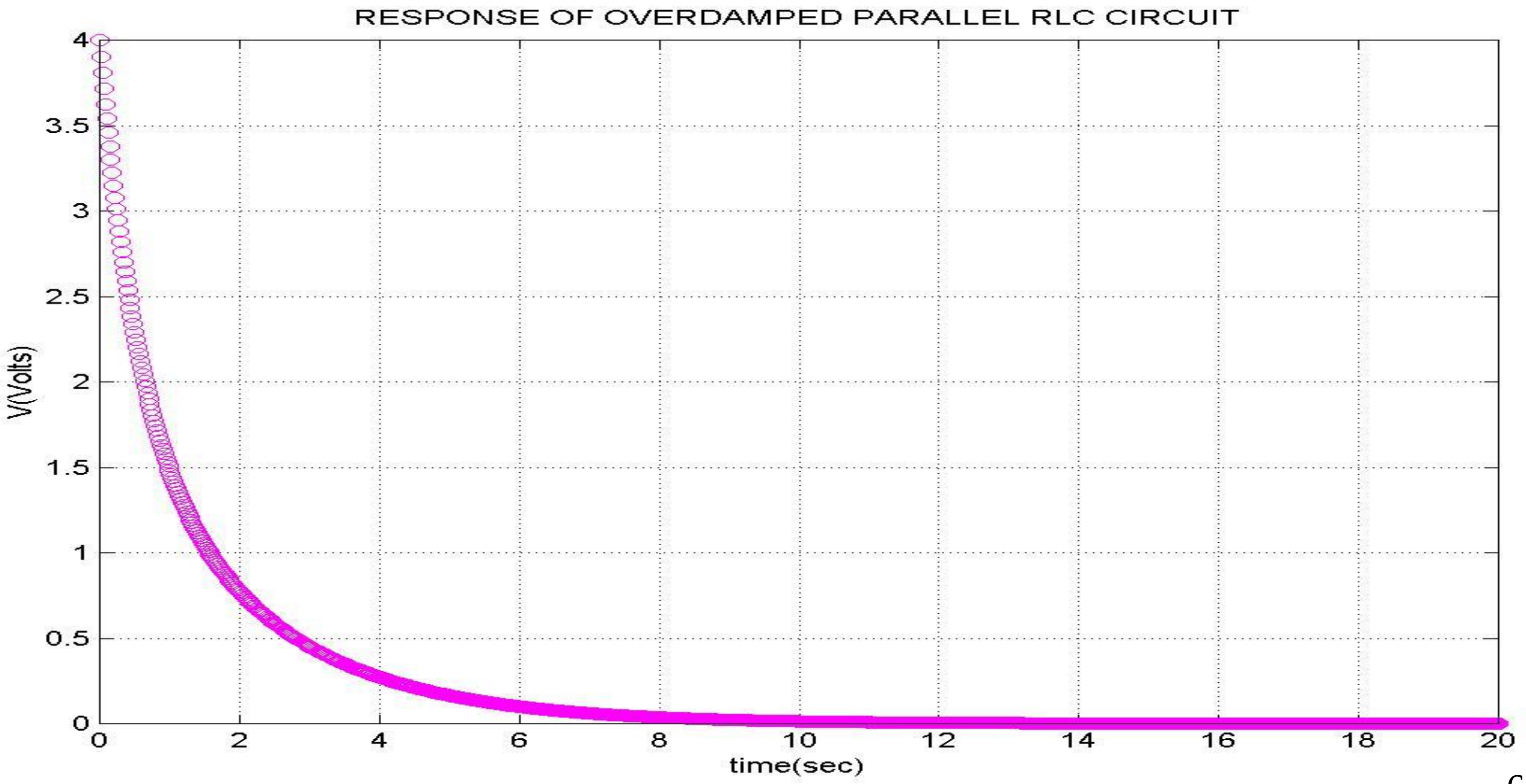
$$\frac{dv}{dt}(0+) = -\frac{4}{2(1/5)} - \frac{(-1)}{(1/5)} = -5$$

$$\left.\begin{aligned} K_1 + K_2 &= 4 \\ -2K_1 - 0.5K_2 &= -5 \end{aligned}\right\} \Rightarrow K_1 = 2;\ K_2 = 2$$

$$v(t) = 2e^{-2t} + 2e^{-0.5t};\ t > 0$$

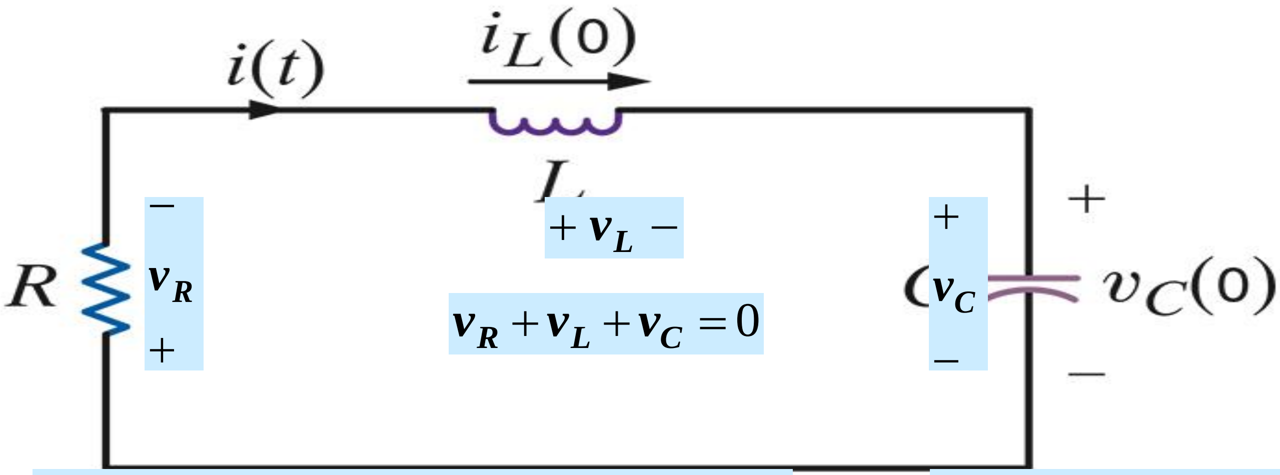
USING MATLAB TO VISUALIZE THE RESPONSE

```
%script6p7.m
%plots the response in Example 6.7
%v(t)=2exp(-2t)+2exp(-0.5t); t>0
t=linspace(0,20,1000);
v=2*exp(-2*t)+2*exp(-0.5*t);
plot(t,v,'mo'), grid, xlabel('time(sec)'), ylabel('V(Volts)')
title('RESPONSE OF OVERDAMPED PARALLEL RLC CIRCUIT')
```



EXAMPLE 5

$R = 6\Omega, L = 1H, C = 0.04F$ $\quad i_L(0) = 4A;\ v_C(0) = -4V$



NO SWITCHING OR DISCONTINUITY AT t=0. USE t=0 OR t=0+

$$Ri(t) + L\frac{di}{dt}(t) + \frac{1}{C}\int_0^t i(x)dx + v_C(0) = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt}(t) + \frac{1}{LC}i(t) = 0 \quad \text{model}$$

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt}(t) + 25i(t) = 0$$

Ch.Eq.: $s^2 + 6s + 25 = 0$

$$\omega_n^2 = 25 \Rightarrow \omega_n = 5$$

$$2\varsigma\omega_n = 6 \Rightarrow \varsigma = 0.6$$

roots: $s = \dfrac{-6 \pm \sqrt{36-100}}{2} = -3 \pm j\boxed{4}\ \omega_d$

Form: $i(t) = e^{-3t}(A_1 \cos 4t + A_2 \sin 4t)$

$$i(0) = i_L(0) = 4A \Rightarrow A_1 = 4$$

TO COMPUTE $\dfrac{di}{dt}(0+)$ $\quad v_L(t) = L\dfrac{di}{dt}(t)$

$$L\frac{di}{dt}(0) = -Ri(0) - v_C(0) \Rightarrow \frac{di}{dt}(0+) = -20$$

$$\frac{di}{dt}(t) = -3i(t) + e^{-3t}(-4A_1 \sin 4t + 4A_2 \cos 4t)$$

$$@t = 0:\ -20 = -3\times(4) + 4A_2 \Rightarrow A_2 = -2$$

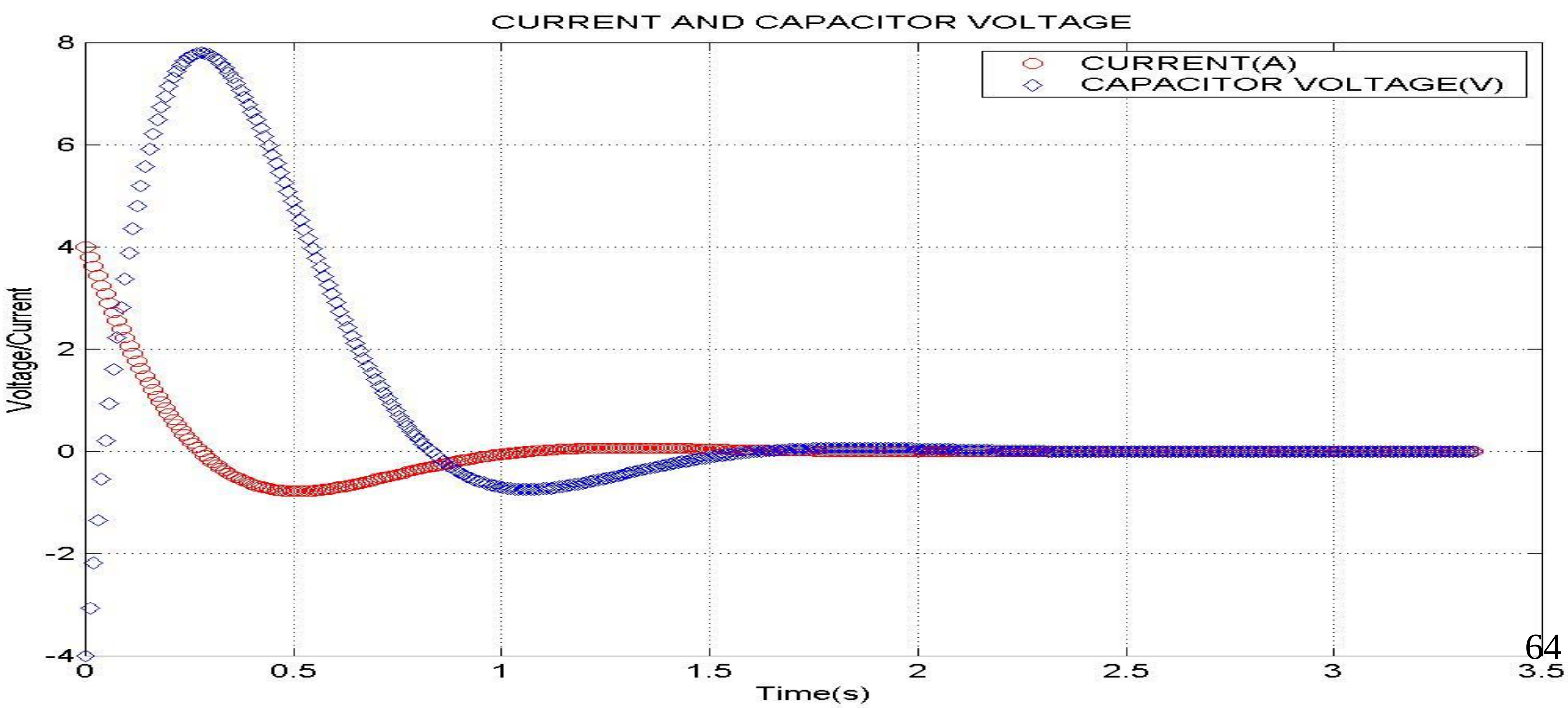
$$i(t) = e^{-3t}(4\cos 4t - 2\sin 4t)[A];\ t > 0$$

$$v_C(t) = -Ri(t) - L\frac{di}{dt}(t) = v_C(0) + \frac{1}{C}\int_0^t i(x)dx$$

$$v_C(t) = e^{-3t}(-4\cos 4t + 22\sin 4t)[V];\ t > 0$$

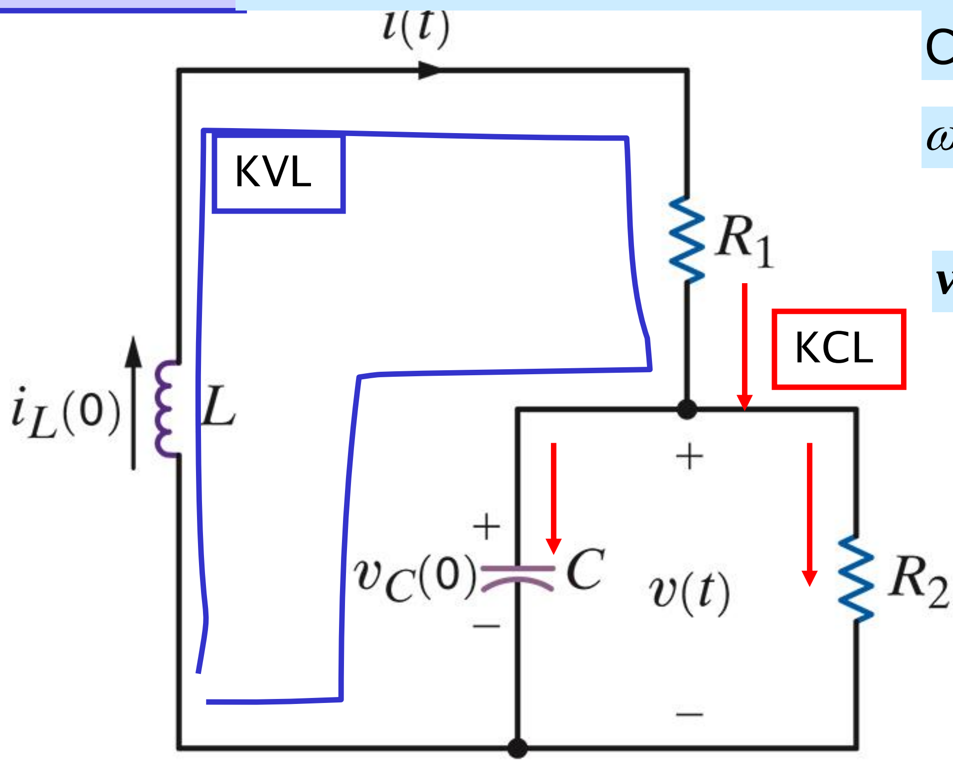
USING MATLAB TO VISUALIZE THE RESPONSE

```
%script6p8.m
%displays the function i(t)=exp(-3t)(4cos(4t)-2sin(4t))
% and the function vc(t)=exp(-3t)(-4cos(4t)+22sin(4t))
% use a simle algorithm to estimate display time
tau=1/3;
tend=10*tau;
t=linspace(0,tend,350);
it=exp(-3*t).*(4*cos(4*t)-2*sin(4*t));
vc=exp(-3*t).*(-4*cos(4*t)+22*sin(4*t));
plot(t,it,'ro',t,vc,'bd'),grid,xlabel('Time(s)'),ylabel('Voltage/Current')
title('CURRENT AND CAPACITOR VOLTAGE')
legend('CURRENT(A)','CAPACITOR VOLTAGE(V)')
```



EXAMPLE 6 $R_1 = 10\Omega,\ R_2 = 8\Omega,\ C = 1/8F,\ L = 2H$

$v_C(0) = 1V,\ i_L(0) = 0.5A$



$$L\frac{di}{dt}(t) + R_1 i(t) + v(t) = 0 \qquad i(t) = \frac{v(t)}{R_2} + C\frac{dv}{dt}(t)$$

$$L\left(\frac{1}{R_2}\frac{dv}{dt}(t) + C\frac{d^2v}{dt^2}\right) + R_1\left(\frac{v(t)}{R_2} + C\frac{dv}{dt}(t)\right) + v(t) = 0$$

$$\frac{d^2v}{dt^2}(t) + \left(\frac{1}{R_2 C} + \frac{R_1}{L}\right)\frac{dv}{dt}(t) + \frac{R_1 + R_2}{R_2 LC}v(t) = 0$$

$$\frac{d^2v}{dt^2}(t) + 6\frac{dv}{dt}(t) + 9v(t) = 0$$

Ch.Eq.: $s^2 + 6s + 9 = 0$

Ch.Eq.: $s^2 + 6s + 9 = 0 = (s+3)^2$

$\omega_n = 3,\ 2\varsigma\omega_n = 6 \Rightarrow \varsigma = 1$ $\quad v(t) = e^{-3t}(B_1 + B_2 t)$

$v(0+) = v_c(0+) = 1V$

NO SWITCHING OR DISCONTINUITY AT t=0. USE t=0 OR t=0+

KCL AT $t = 0+$

$$i(0) = i_L(0) = \frac{v(0)}{R_2} + C\frac{dv}{dt}(0) \Rightarrow \frac{dv}{dt}(0) = 3$$

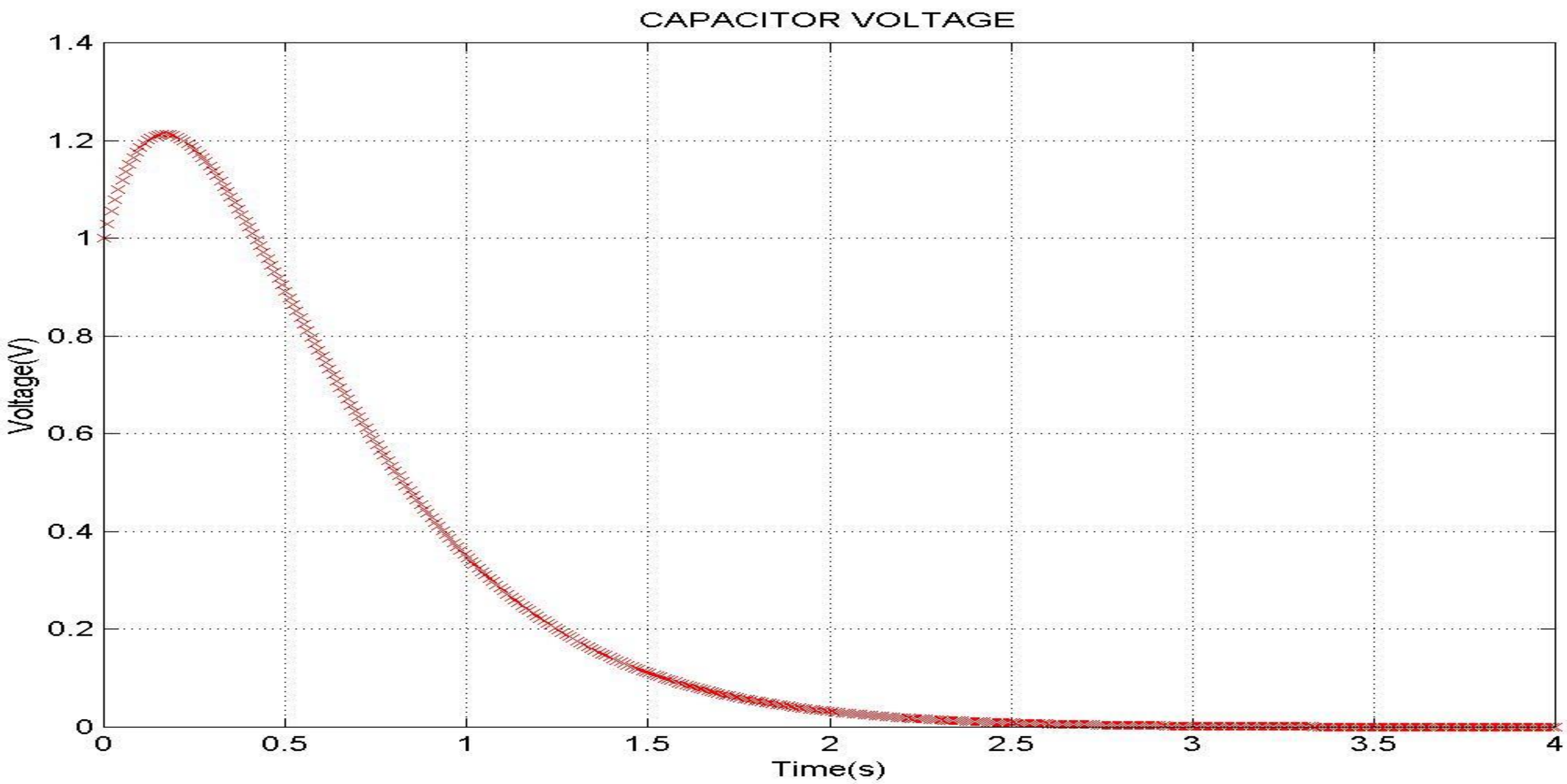
$v(0) = 1 = B_1$

$$\frac{dv}{dt}(0) = -3v(0) + B_2 = 3 \Rightarrow B_2 = 6$$

$$v(t) = e^{-3t}(1 + 6t);\ t > 0$$

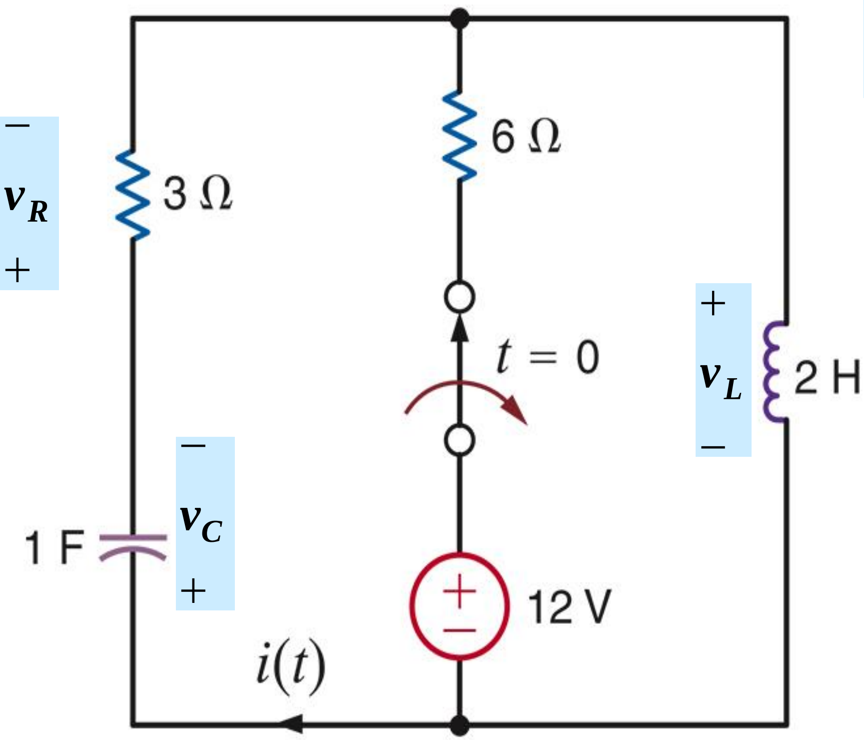
USING MATLAB TO VISUALIZE RESPONSE

```
%script6p9.m
%displays the function v(t)=exp(-3t)(1+6t)
tau=1/3;
tend=ceil(10*tau);
t=linspace(0,tend,400);
vt=exp(-3*t).*(1+6*t);
plot(t,vt,'rx'),grid, xlabel('Time(s)'), ylabel('Voltage(V)')
title('CAPACITOR VOLTAGE')
```



EXAMPLE 7

FIND $i(t), t>0$



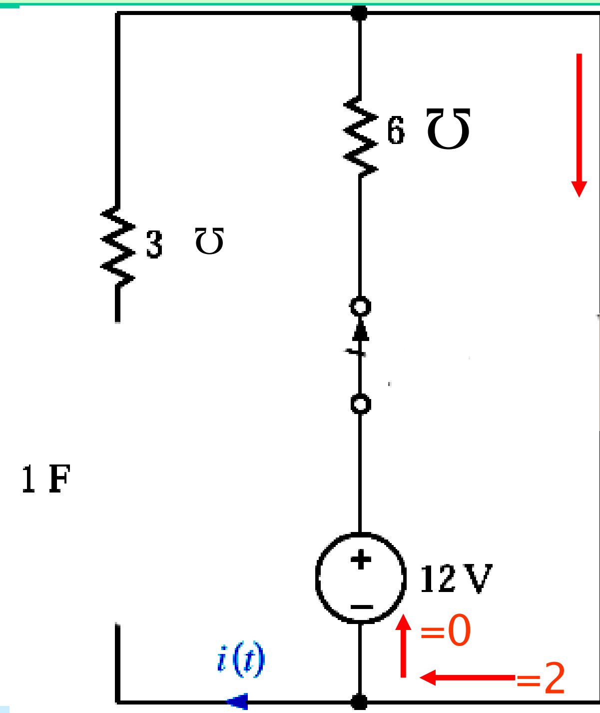
$$\frac{d^2i}{dt^2}(t)+\frac{3}{2}\frac{di}{dt}(t)+\frac{1}{2}i(t)=0$$

Ch.Eq.: $s^2+1.5s+0.5=0$

roots: $s=-1,-0.5$

$$i(t)=K_1e^{-t}+K_2e^{-\frac{t}{2}};\ t>0$$

To find initial conditions use steady state analysis for t<0



$i_L(0)=2A$

$v_C(0)=0V$

$$\frac{di}{dt}(0+)=-3$$

And analyze circuit at t=0+

Once the switch opens the circuit is RLC series

$$3i(t)+2\frac{di}{dt}(t)+v_C(0)+\int_0^t i(x)dx=0$$

$i(0+)=2A$

KCL at $t=0+$

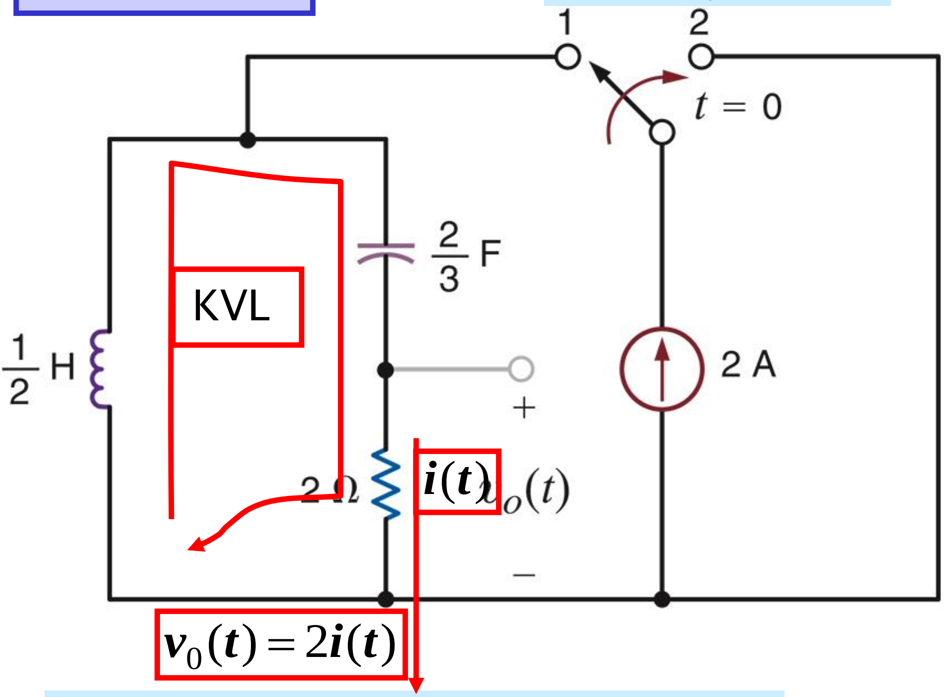
$$2=K_1+K_2$$

$$-3=-K_1-\frac{1}{2}K_2$$

$$i(t)=4e^{-t}-2e^{-\frac{t}{2}};\ t>0$$

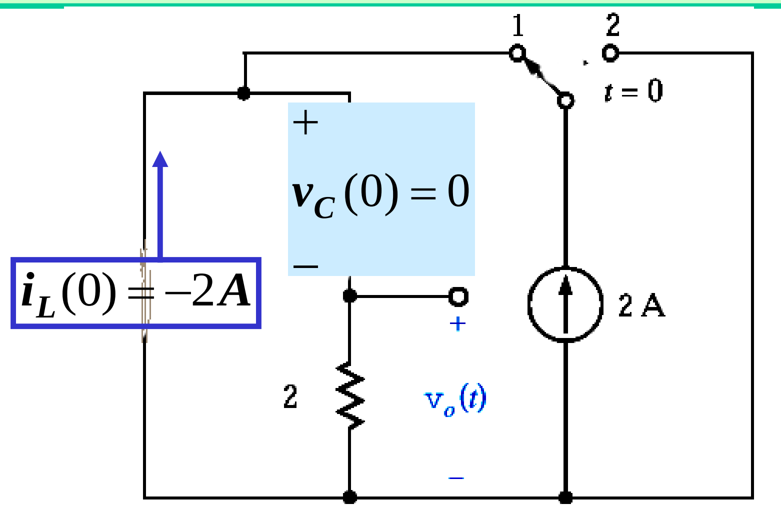
EXAMPLE 8

FIND $v_0(t), t>0$



$v_0(t) = 2i(t)$

To find initial conditions we use steady state analysis for t<0



For t>0 the circuit is RLC series

$$\frac{1}{2}\frac{di}{dt}(t) + \frac{1}{2/3}\int_0^t i(x)dx + v_C(0) + 2i(t) = 0$$

$$\frac{d^2i}{dt^2}(t) + 4\frac{di}{dt}(t) + 3i(t) = 0$$

Ch. Eq. : $s^2 + 4s + 3 = 0$

roots : $s = -1, -3$

$i(t) = K_1e^{-t} + K_2e^{-3t};\ t>0$

And analyze circuit at t=0+

$i(0+) = -2A$

$$v_L(0+) = L\frac{di}{dt}(0+) = 4\ \text{V}$$

$i(0+) = 0 \Rightarrow K_1 + K_2 = -2$

$\frac{di}{dt}(0+) = 8 \Rightarrow -K_1 - 3K_2 = 8$

$K_2 = 1$

$K_1 = -3$

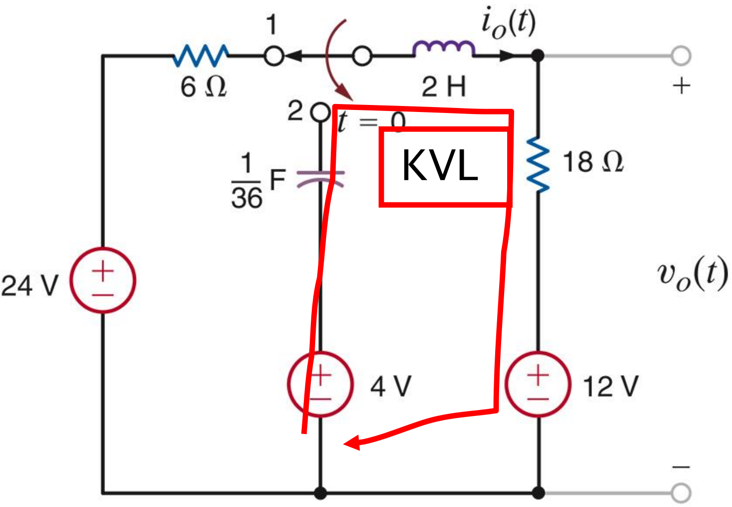
$$\therefore i(t) = e^{-t} - 3e^{-3t};\ t>0$$

$$v_0(t) = 2\left(e^{-t} - 3e^{-3t}\right);\ t>0$$

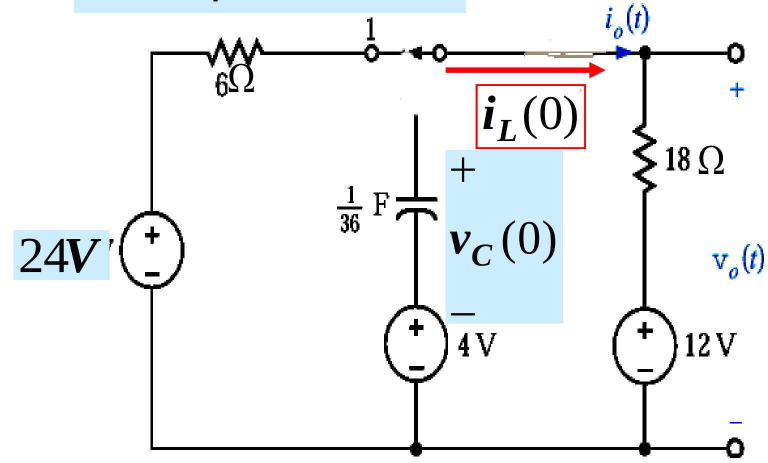
EXAMPLE 9

DETERMINE $i_0(t), v_0(t);\ t>0$

$v_0(t) = 18 i_0(t) + 12(V)$



Steady state t<0



$$v_C(0) = 0 \qquad i_L(0) = 0.5A$$

$$-4 + \frac{1}{1/36}\int_0^t i(x)dx + v_C(0) + 2\frac{di}{dt}(t) + 18i(t) + 12 = 0$$

$$\frac{d^2 i}{dt^2}(t) + 9\frac{di}{dt}(t) + 18i(t) = 0$$

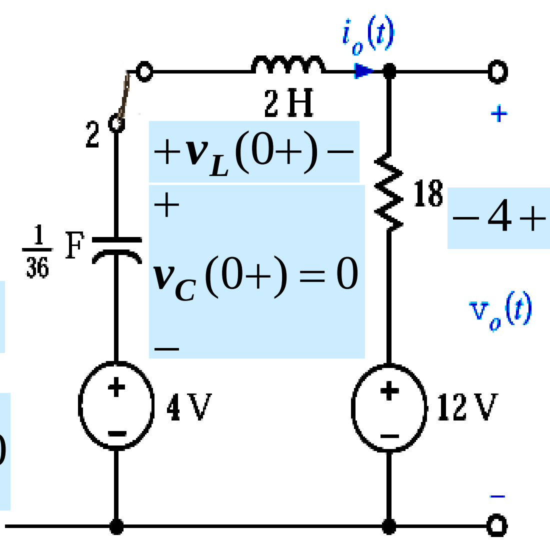
Ch. Eq.: $s^2 + 9s + 18 = 0$

roots: $s = -3, -6$

$$i_0(t) = K_1 e^{-3t} + K_2 e^{-6t};\ t>0$$

$$i_0(t) = -\frac{11}{6}e^{-3t} + \frac{14}{6}e^{-6t};\ t>0$$

Analysis at t=0+



$$i_0(0+) = i_L(0+) = 0.5(A)$$

$$v_L(0+) = L\frac{di_L}{dt}(0+) = L\frac{di_0}{dt}(0+)$$

$$-4 + v_L(0+) + 18 i_L(0+) + 12 = 0 \qquad v_L(0+) = -17$$

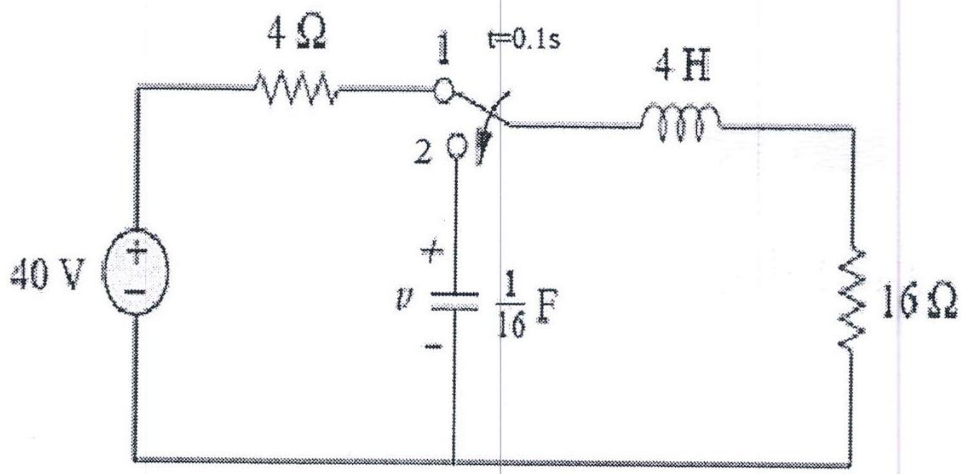
$$\frac{di_0}{dt}(0+) = -17/2 = -3K_1 - 6K_2$$

$$i_0(0+) = 0.5 = K_1 + K_2$$

$$K_1 = -\frac{11}{6};\quad K_2 = \frac{14}{6}$$

EXAMPLE 10 (PREVIOUS YEAR FINAL EXAM)

2) (30 **points**) The switch in the figure was in position 1 for $0 < t < 0.1$ s. At $t = 0.1$ s, it is switched to position 2. Find and plot v(t) for $t > 0$.



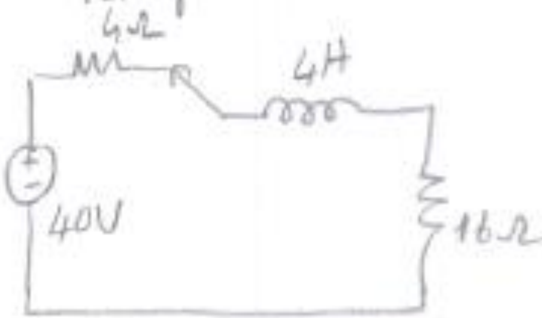
Answer:
v(t)= ________

$i_L(0) = 0$

$V_C(0) = 0$

EXAMPLE 10 (PREVIOUS YEAR FINAL EXAM)

- for position 1



$$i_L(t) = i_L(\infty) - (i_L(\infty) - i_L(0))e^{-\frac{t}{\tau}}$$

$$i_L(\infty) = \frac{40}{4+16} = 2\,A \qquad i_L(0) = 0$$

4 H sc

$$\tau = \frac{L}{R_{TH}} = \frac{4}{4+16} = \frac{1}{5} = 0.2s$$

$$i_L(t) = 2 - 2e^{-\frac{t}{0.2}}\ A$$

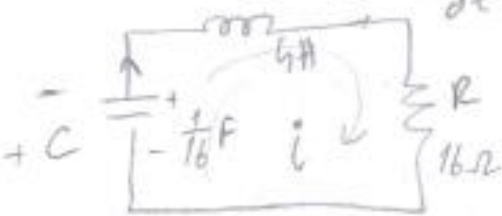
10

at $t = 0.1s$ $\quad i_L(t) = 2 - 2e^{-\frac{0.1}{0.2}} \Rightarrow i_L(t=0.1s) = 0.79\ A$

EXAMPLE 10 (PREVIOUS YEAR FINAL EXAM)

- For position 2

$i = C\frac{dV_c}{dt}$



$$V_c + LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} = 0$$

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{(Characteristic equation)}$$

$16 - 4$ $\quad 4 \cdot \frac{1}{16}$

$$s^2 + 4s + 4 = 0$$

$$s_1 = s_2 = -2 \quad \text{[Critically damped response]}$$ 10

$$V_c(t \cancel{-t_0}) = K_1 e^{-2(t-t_0)} + K_2(t-t_0)e^{-2(t-t_0)}$$

$t_0 = 0.1s$

At $t = t_0$ $\quad V_c(t_0) = 0 = K_1 e^{-0} + 0 \Rightarrow K_1 = 0$

$$C\frac{dV_c}{dt}(t_0) = i_L(t_0) = 0.79A \Rightarrow \frac{dV_c}{dt}(t_0) = 12.64$$

$$\frac{dV_c}{dt}(t_0) = K_2 e^{-2(t_0-t_0)} - \cancel{2K_2(t_0-t_0)e^{-2(t_0-t_0)}} = 12.64 = K_2$$

$$V_c(t-t_0) = 12.64(t-t_0) \cdot e^{-2(t-t_0)} \, V$$

10

EXERCISE PROBLEMS

7.87 The voltage $v_1(t)$ in a network is defined by the equation

$$\frac{d^2v_1(t)}{dt^2} + 4\frac{dv_1(t)}{dt} + 5v_1(t) = 0$$

Find

(a) the characteristic equation of the network

(b) the circuit's natural frequencies

(c) the expression for $v_1(t)$.

7.93 Find $v_C(t)$ for $t > 0$ in the circuit in Fig. P7.93.

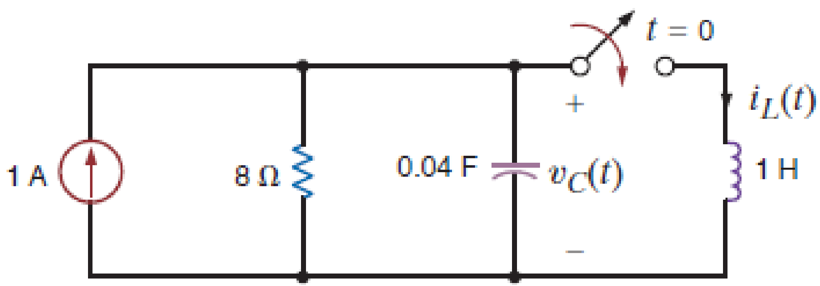


Figure P7.93

7.95 Find $v_o(t)$ for $t > 0$ in the circuit in Fig. P7.95 and plot the response, including the time interval just prior to closing the switch.

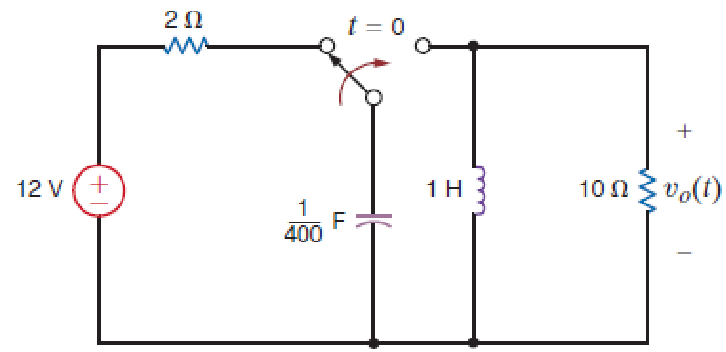


Figure P7.95

7.99 In the circuit shown in Fig. P7.99, find $v(t) > 0$.

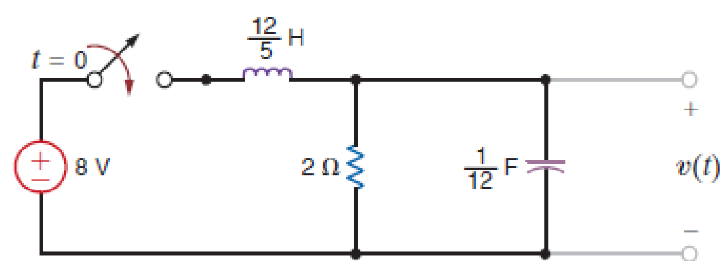


Figure P7.99

EXERCISE PROBLEMS

7.98 Find $v_o(t)$ for $t > 0$ in the circuit in Fig. P7.98 and plot the response, including the time interval just prior to closing the switch.

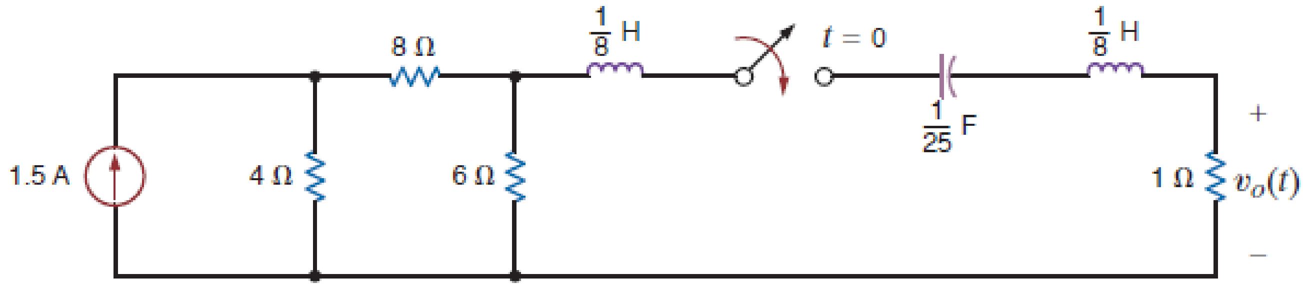


Figure P7.98

7.100 Find $v_o(t)$ for $t > 0$ in the circuit in Fig. P7.100 and plot the response, including the time interval just prior to moving the switch.

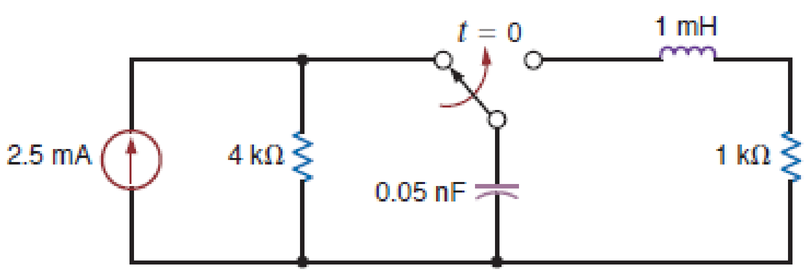


Figure P7.100

7.102 Find $v_o(t)$ for $t > 0$ in the network in Fig. P7.102 and plot the response, including the time interval just prior to moving the switch.

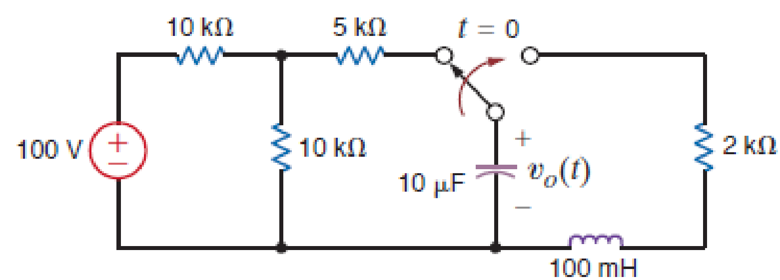


Figure P7.102