Problem 2.1 We note that the action has no dependence on $A_\mu$but only on its derivatives.
Using $\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}$, the E-L equation becomes

$$\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= 0 \\
\partial_\mu \Big(-\frac{1}{4}\, 2\, \frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)} F_{\rho\sigma} \Big) &= 0 \\
-\frac{1}{2}\partial_\mu \Big(\eta^{\rho\mu}\eta^{\sigma\nu} F_{\rho\sigma} - \eta^{\rho\nu}\eta^{\sigma\mu} F_{\rho\sigma} \Big) &= 0 \\
-\frac{1}{2}\partial_\mu (F^{\mu\nu} - F^{\nu\mu}) &= 0 \\
\partial_\mu F^{\nu\mu} &= 0
\end{aligned}$$

The time component of the equation gives

$$\begin{aligned}
\partial_\mu F^{0\mu} &= 0 \\
\partial_i F^{0i} &= 0 \\
\partial_i E^i &= 0 \\
\nabla \cdot \boldsymbol{E} &= 0
\end{aligned}$$

The second line is due to antisymmetry of F so $F^{00} = 0$.
The spatial component of the equation gives

$$\begin{aligned}
\partial_\mu F^{i\mu} &= 0 \\
\partial_0 F^{i0} + \partial_k F^{ik} &= 0 \\
\partial_0 E^i - \partial_k \epsilon^{ikj} B^j &= 0 \\
\partial_0 E^i - \epsilon^{ijk} \partial_j B^k &= 0 \\
\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} &= 0
\end{aligned}$$

Next we calculate the energy-momentum tensor.

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \mathcal{L}\eta^{\mu\nu} \\
&= F^{\lambda\mu}\partial^\nu A_\lambda - \mathcal{L}\eta^{\mu\nu}
\end{aligned}$$

where we used result from part **a** in the second line. This is clearly not symmetric as

$$T^{\mu\nu} - T^{\nu\mu} = F^{\lambda\mu}\partial^\nu A_\lambda - F^{\lambda\nu}\partial^\mu A_\lambda \neq 0$$

Following the prescription in the problem, we have

$$\begin{aligned}
\hat{T}^{\mu\nu} &= F^{\lambda\mu}\partial^\nu A_\lambda + \partial_\lambda (F^{\mu\lambda} A^\nu) - \mathcal{L}\eta^{\mu\nu} \\
&= F^{\lambda\mu}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu - \mathcal{L}\eta^{\mu\nu} \\
&= -\eta_{\lambda\rho} F^{\mu\lambda} F^{\nu\rho} - \mathcal{L}\eta^{\mu\nu}
\end{aligned}$$

which is manifestly symmetric.
The energy density is given by

$$
\begin{aligned}
\hat{T}^{00} &= E^i E_i + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\
&= E^2 + \frac{1}{2}F^{0i}F_{0i} + \frac{1}{4}F^{ij}F_{ij} \\
&= E^2 + \frac{1}{2}\eta_{00}\eta_{ij}E^iE^j + \frac{1}{4}\eta_{im}\eta_{jn}F^{ij}F^{mn} \\
&= \frac{1}{2}E^2 + \frac{1}{4}\eta_{im}\eta_{jn}\epsilon^{ijk}B^k\epsilon^{mnr}B^r \\
&= \frac{1}{2}E^2 + \frac{1}{4}\epsilon^{ijk}\epsilon^{ijr}B^kB^r \\
&= \frac{1}{2}E^2 + \frac{1}{4}(2\delta^{kr}B^kB^r) \\
&= \frac{1}{2}(E^2 + B^2)
\end{aligned}
$$

The momentum density is given by

$$
\begin{aligned}
\hat{T}^{0i} &= -\eta_{\lambda\rho}F^{0\lambda}F^{i\rho} \\
&= F^{0j}F^{ij} \\
&= E^j\epsilon^{ijk}B^k \\
&= (\boldsymbol{E}\times\boldsymbol{B})^i
\end{aligned}
$$

Problem 2.2 Using $\mathcal{L} = \dot{\phi}^*\dot{\phi} - \nabla\phi^*\cdot\nabla\phi - m^2\phi^*\phi$,

$$
\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*
$$

$$
\pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi}
$$

The CCR becomes

$$
\begin{aligned}
[\phi(x), \dot{\phi}^*(x')] &= i\delta^{(3)}(x-x') \\
[\phi^*(x), \dot{\phi}(x')] &= i\delta^{(3)}(x-x')
\end{aligned}
$$

These are consistent with ϕ^*being the complex conjugate of ϕ.
The Hamiltonian density is then given by

$$
\begin{aligned}
\mathcal{H} &= \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} \\
&= \dot{\phi}^*\dot{\phi} + \dot{\phi}\dot{\phi}^* - \dot{\phi}^*\dot{\phi} + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi \\
&= \pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi \\
&= \pi^*\pi + \phi^*(-\nabla^2 + m^2)\phi
\end{aligned}
$$

The last equality is obtained by integrating by parts.

Then it is easy to obtain the equation of motion:

$$\begin{aligned}
\ddot{\phi} &= \dot{\pi^*} \\
&= i[H, \pi^*] \\
&= -\int d^3x'\, \delta^{(3)}(x'-x)(-\nabla^2 + m^2)\phi(x) \\
\ddot{\phi} - \nabla^2\phi + m^2\phi &= 0 \\
\Box\phi + m^2\phi &= 0
\end{aligned}$$

Since ϕ is not identically equal to ϕ^*, we need to introduce two sets of annihilation and creation operators:

$$\begin{aligned}
\phi &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(a_k e^{-ikx} + b_k^\dagger e^{ikx}\Big) \\
\phi^* &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(a_k^\dagger e^{ikx} + b_k e^{-ikx}\Big)
\end{aligned}$$

If $a = b$ then ϕ will be identically equal to ϕ^* and the situation is reduced to that of a real scalar field. From these we also have

$$\begin{aligned}
\pi &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(i\omega_k a_k^\dagger e^{ikx} - i\omega_k b_k e^{-ikx}\Big) \\
\pi^* &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(-i\omega_k a_k e^{-ikx} + i\omega_k b_k^\dagger e^{ikx}\Big) \\
\nabla\phi &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(i\boldsymbol{k} a_k e^{-ikx} - i\boldsymbol{k} b_k^\dagger e^{ikx}\Big) \\
\nabla\phi^* &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\Big(-i\boldsymbol{k} a_k^\dagger e^{ikx} + i\boldsymbol{k} b_k e^{-ikx}\Big)
\end{aligned}$$

Note that under spatial integration the exponents will reduce to delta functions, thus we can easily evaluate H as

$$\begin{aligned}
H &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\Big(\omega_k^2 a_k^\dagger a_k + \omega_k^2 b_k b_k^\dagger - \omega_k^2 a_k^\dagger b_{-k}^\dagger - \omega_k^2 a_k b_{-k}^\dagger + k^2 a_k^\dagger a_k + k^2 b_k b_k^\dagger + k^2 a_k^\dagger b_{-k}^\dagger + k^2 a_k b_{-k}^\dagger \\
&\quad + m^2 a_k^\dagger a_k + m^2 b_k b_k^\dagger + m^2 a_k^\dagger b_{-k}^\dagger + m^2 a_k b_{-k}^\dagger\Big) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\Big(2\omega_k^2 (a_k^\dagger a_k + b_k b_k^\dagger)\Big) \\
&= \int \frac{d^3k}{(2\pi)^3}\omega_k\Big(a_k^\dagger a_k + b_k^\dagger b_k + (2\pi)^3\delta^{(3)}(0)\Big) \\
:H: &= \int \frac{d^3k}{(2\pi)^3}\omega_k\Big(n_{a,k} + n_{b,k}\Big)
\end{aligned}$$

In the last line we introduce normal ordering[1]. It can be seen that there are two sets of particles (a and b) with the same mass m.

[1] See section 2.3 of http://www.damtp.cam.ac.uk/user/tong/qft/qft.pdf

From the form of $\mathcal{L}$ it is obvious that there is a $U(1)$ (or $O(2)$, if we write the Lagrangian in terms of a two-component vector with the real and imaginary parts as components) symmetry as terms involving ϕ always come in form of products with ϕ^*. The corresponding infinitesimal transform is $\phi \to \phi + i\theta\mathbb{1}\phi$ and $\phi^* \to \phi^* - i\theta\mathbb{1}\phi^*$.. $\mathbb{1}$ is the generator of the algebra $u(1)$ which is just equal to 1 but we write it explicitly for later convenience. By Noether's theorem, the conserved current is then

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathbb{1}\phi - c.c. \\ J^0 &= \pi\mathbb{1}\phi - c.c. \end{aligned}$$

We are free to multiply this by any constant. To make the current real, we multiply by $-i$.

$$\begin{aligned} Q &= \int d^3x\, i\,(\phi^*\pi^* - \pi\phi) \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\Big(\omega_k a_k^\dagger a_k - \omega_k b_k b_k^\dagger + \omega_k(b_k a_{-k} - a_k^\dagger b_{-k}^\dagger) + \omega_k a_k^\dagger a_k - \omega_k b_k b_k^\dagger - \omega_k(b_k a_{-k} - a_k^\dagger b_{-k}^\dagger)\Big) \\ :Q: &= \int \frac{d^3k}{(2\pi)^3}\Big(n_{a,k} - n_{b,k}\Big) \end{aligned}$$

The last equality is due to normal ordering. We see that the two types of particles have opposite charges.

Now if we have two complex fields, we can arrange them in the form of a 2-dimensional complex vector $\boldsymbol{\phi}$ (or a 4-dimensional real vector). The form of $\mathcal{L}$ will be the same and everything follows through. However the $U(1)$ (or $O(2)$) symmetry will now become a $U(2)$ (or a $O(4)$) symmetry. By replacing $\mathbb{1}$ by the other generators σ^i of the algebra, we obtain three (or five) more charges.

The commutation relations of the charges are

$$[Q^i, Q^j] = -\Big(\int d^3x\, d^3x' [\boldsymbol{\phi}^*\sigma^i\boldsymbol{\pi}^*, \boldsymbol{\phi}^*\sigma^j\boldsymbol{\pi}^*] - c.c.\Big)$$

This is valid as π, ϕ commutes with π^*, ϕ^*. We focus on the commutator inside the integral. Using $[\phi_a(x), \pi_b(x')] = i\delta_{ab}\delta^{(3)}(x - x')$, and let $[\sigma^i, \sigma^j] = f^{ijk}\sigma^k$,

$$\begin{aligned} &[\phi_a^*\sigma_{ab}^i\pi_b^*, \phi_c^*\sigma_{cd}^j\pi_d^*] \\ =&\sigma_{ab}^i\sigma_{cd}^j\Big(\phi_a^*[\pi_b^*, \phi_c^*]\pi_d^* + \phi_c^*[\phi_a^*, \pi_d^*]\pi_b^*\Big) \\ =& -i\phi_a^*\sigma_{ac}^i\sigma_{cd}^j\pi_d^* + i\phi_c^*\sigma_{cd}^j\sigma_{db}^i\pi_b^* \\ =&\boldsymbol{\phi}^*(-i[\sigma^i, \sigma^j])\boldsymbol{\pi}^* \\ =& -if^{ijk}\boldsymbol{\phi}^*\sigma^k\boldsymbol{\pi}^* \end{aligned}$$

Thus,

$$\begin{aligned} [Q^i, Q^j] &= f^{ijk} i\int d^3x\,\Big(\boldsymbol{\phi}^*\sigma^k\boldsymbol{\pi}^* - \boldsymbol{\pi}\sigma^k\boldsymbol{\phi}\Big) \\ &= f^{ijk}Q^k \end{aligned}$$

In other words, the commutation relations of the charges will mirror that of the underlying algebra.

Problem 2.3 c.f. Exercise 3.5 for Brezin.

Problem 3.1 Using $g^{\mu\nu} = \eta^{\mu\nu} = diag(1,-1,-1,-1)$, and the identity $\epsilon^{imn}\epsilon^{ips} = \delta^{mp}\delta^{ns} - \delta^{np}\delta^{ms}$,

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}\epsilon^{imn}\epsilon^{jps}[J^{mn}, J^{ps}] \\
&= \frac{i}{4}\epsilon^{imn}\epsilon^{jps}(g^{np}J^{ms} - g^{mp}J^{ns} - g^{ns}J^{mp} + g^{ms}J^{np}) \\
&= iJ^{ij} \\
&= i\epsilon^{ijk}L^k
\end{aligned}$$

$$\begin{aligned}
[K^i, K^j] &= i(-g^{00}J^{ij}) \\
&= -i\epsilon^{ijk}L^k
\end{aligned}$$

$$\begin{aligned}
[L^i, K^j] &= \frac{1}{2}\epsilon^{imn}[J^{mn}, J^{0j}] \\
&= \frac{i}{2}\epsilon^{imn}(-g^{nj}J^{m0} + g^{mj}J^{n0}) \\
&= \frac{i}{2}(\epsilon^{ijm}J^{0m} + \epsilon^{ijn}J^{0n}) \\
&= i\epsilon^{ijk}K^k
\end{aligned}$$

Using these results, we have

$$\begin{aligned}
[J_+^i, J_+^j] &= \frac{1}{4}\Big([L^i, L^j] - [K^i, K^j] + i[K^i, L^j] + i[L^i, K^j]\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k + i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{ijk}K^k\Big) \\
&= \frac{1}{2}i\epsilon^{ijk}(L^k + iK^k) \\
&= i\epsilon^{ijk}J_+^k
\end{aligned}$$

$$\begin{aligned}
[J_-^i, J_-^j] &= \frac{1}{4}\Big([L^i, L^j] - [K^i, K^j] - i[K^i, L^j] - i[L^i, K^j]\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k + i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k + \epsilon^{ijk}K^k\Big) \\
&= \frac{1}{2}i\epsilon^{ijk}(L^k - iK^k) \\
&= i\epsilon^{ijk}J_-^k
\end{aligned}$$

$$\begin{aligned}
[J_+^i, J_-^j] &= \frac{1}{4}\Big([L^i, L^j] + [K^i, K^j] + i[K^i, L^j] - i[L^i, K^j]\Big) \\
&= \frac{1}{4}\Big(i\epsilon^{ijk}L^k - i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k + \epsilon^{ijk}K^k\Big) \\
&= 0
\end{aligned}$$

Hence we have shown that the Lorentz algebra can be decoupled into two independent angular momentum algebrae. For $(\frac{1}{2}, 0)$, we have $\boldsymbol{J}_+ = \frac{\boldsymbol{\sigma}}{2}$, $\boldsymbol{J}_- = 0$, thus

$$\boldsymbol{L} = \boldsymbol{J}_+ = \frac{\boldsymbol{\sigma}}{2}$$

$$\boldsymbol{K} = -i\boldsymbol{J}_+ = \frac{-i\boldsymbol{\sigma}}{2}$$

which reproduces the transformation of ψ_L. Similarly $(0, \frac{1}{2})$ reproduces the transformation for ψ_R.

If we have two column vectors ψ, ϕ it is possible to combine them into a matrix $A = \phi\psi^T$. Similarly, if we have a left-handed spinor and a right-handed spinor, we can encode the two spinors using a matrix $A = i\psi_R\psi_L^T\sigma^2$, with $A^\dagger = -i\sigma^2\psi_L^*\psi_R^\dagger$. Since $\sigma^2\psi_L^*$ transforms like a right-handed spinor and $\psi_R^\dagger$ transform like $\psi_L^T\sigma^2$, $A^\dagger$ transforms the same way as A. If we require the matrix to be Hermitian as in the problem, we need $\psi_R = -i\sigma^2\psi_L^*$ so we are really limiting ourselves to Majorana spinors when adopting such a representation.

$$\begin{aligned}
A &= V_\mu\sigma^\mu \\
&\to (1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2})V_\mu\sigma^\mu(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}) \\
&= V_\mu\sigma^\mu - i\theta_j\frac{\sigma^j}{2}V_\mu\sigma^\mu + iV_\mu\sigma^\mu\theta_j\frac{\sigma^j}{2} + \beta_j\frac{\sigma^j}{2}V_\mu\sigma^\mu + V_\mu\sigma^\mu\beta_j\frac{\sigma^j}{2} \\
&= V_\mu\sigma^\mu + \frac{i}{2}[\sigma^\mu, \sigma^j]V_\mu\theta_j + \frac{1}{2}\{\sigma^\mu, \sigma^j\}\beta_j V_\mu \\
V'_\mu\sigma^\mu &= V_0 + V_0\beta_j\sigma^j + V_i\sigma^i - V_i\theta_j\epsilon^{ijk}\sigma^k + \beta_i V_i
\end{aligned}$$

Comparing both sides, we have

$$\begin{aligned}
V'_0 &= V_0 + \beta_i V_i \\
V'_i &= V_i + \beta_i V_0 + \epsilon^{ijk}\theta_j V_k
\end{aligned}$$

which is clearly the Lorentz transform for a 4-vector.

Problem 3.2 Using $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $\not{p}u(p) = mu(p)$,

$$\begin{aligned}
\bar{u}(p')\Big[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]u(p) &= \frac{1}{4m}\bar{u}(p')\Big[2g^{\mu\nu}p'_\nu + 2g^{\mu\nu}p_\nu - [\gamma^\mu, \gamma^\nu]p'_\nu + [\gamma^\mu + \gamma^\nu]p_\nu\Big]u(p) \\
&= \frac{1}{4m}\bar{u}(p')\Big[(\{\gamma^\mu, \gamma^\nu\} - [\gamma^\mu, \gamma^\nu])p'_\nu + (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu])p_\nu\Big]u(p) \\
&= \frac{1}{2m}\bar{u}(p')\Big[\not{p}'\gamma^\mu + \gamma^\mu\not{p}\Big]u(p) \\
&= \bar{u}(p')\gamma^\mu u(p)
\end{aligned}$$

Problem 3.3[2] We first show that $\not{p}\not{q} = p \cdot q$ for any 4-vector p and q:

$$\begin{aligned}\gamma^\mu\gamma^\nu p_\mu p_\nu &= \frac{1}{2}(\gamma^\mu\gamma^\nu p_\mu p_\nu + \gamma^\nu\gamma^\mu p_\nu p_\mu)\\ &= g^{\mu\nu}p_\mu p_\nu\end{aligned}$$

Thus

$$\begin{aligned}\not{k}_0 u_{R0} &= \not{k}_0\not{k}_1 u_{L0}\\ &= k_0 \cdot k_1 u_{L0}\\ &= 0\end{aligned}$$

$$\begin{aligned}\not{p}u_L(p) &= \frac{1}{\sqrt{2p\cdot k_0}}\not{p}\not{p}u_{R0}\\ &= \frac{1}{\sqrt{2p\cdot k_0}}p^2 u_{R0}\\ &= 0\end{aligned}$$

and similarly $\not{p}u_R(p) = 0$. Therefore $u_L(p)$ and $u_R(p)$ are solutions to the massless Dirac equation $\not{p}u = 0$. To construct the spinors explicitly, we start with

$$u_{L0} = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Thus

$$\begin{aligned}u_{R0} &= -\gamma^1 u_{L0}\\ &= -\sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\end{aligned}$$

$$\begin{aligned}u_L(p) &= \frac{1}{\sqrt{p_0+p_3}}\begin{pmatrix}0 & 0 & p_0+p_3 & p_1-ip_2\\ 0 & 0 & p_1+ip_2 & p_0-p_3\\ p_0-p_3 & -p_1+ip_2 & 0 & 0\\ -p_1-ip_2 & p_0+p_3 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}\\ &= -\frac{1}{\sqrt{p_0+p_3}}\begin{pmatrix}p_0+p_3\\p_1+ip_2\\0\\0\end{pmatrix}\end{aligned}$$

[2] https://arxiv.org/pdf/1101.2414v1.pdf

$$u_R(p) = \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} 0 & 0 & p_0+p_3 & p_1-ip_2 \\ 0 & 0 & p_1+ip_2 & p_0-p_3 \\ p_0-p_3 & -p_1+ip_2 & 0 & 0 \\ -p_1-ip_2 & p_0+p_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \frac{1}{\sqrt{p_0+p_3}} \begin{pmatrix} 0 \\ 0 \\ -p_1+ip_2 \\ p_0+p_3 \end{pmatrix}$$

$$s(p,q) = \bar{u}_R(p)u_L(q) = -\frac{1}{\sqrt{(p_0+p_3)(q_0+q_3)}} \begin{pmatrix} 0 & 0 & -p_1-ip_2 & p_0+p_3 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} q_0+q_3 \\ q_1+iq_2 \\ 0 \\ 0 \end{pmatrix}$$
$$= -\frac{1}{\sqrt{(p_0+p_3)(q_0+q_3)}} \Big[(-p_1-ip_2)(q_0+q_3)+(p_0+p_3)(q_1+iq_2)\Big]$$
$$= \frac{(q_0+q_3)(p_1+ip_2)-(p_0+p_3)(q_1+iq_2)}{\sqrt{(p_0+p_3)(q_0+q_3)}}$$

It is clear from the form of the expression that $s(p,q) = -s(q,p)$.

$$t(p,q) = \bar{u}_L(p)u_R(q) = -\frac{1}{\sqrt{(p_0+p_3)(q_0+q_3)}} \begin{pmatrix} p_0+p_3 & p_1-ip_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -q_1+iq_2 \\ q_0+q_3 \end{pmatrix}$$
$$= \frac{1}{\sqrt{(p_0+p_3)(q_0+q_3)}} \Big[(p_0+p_3)(q_1-iq_2)-(q_0+q_3)(p_1-ip_2)\Big]$$

Again it is not hard to see that $t(p,q) = -s(p,q)^* = s(q,p)^*$. Finally

$$|s(p,q)|^2 = -(p_1+ip_2)(q_1-iq_2)-(p_1-ip_2)(q_1+iq_2)+\frac{p_0+p_3}{q_0+q_3}(q_1^2+q_2^2)+\frac{q_0+q_3}{p_0+p_3}(p_1^2+p_2^2)$$
$$= -2(p_1q_1+p_2q_2)+\frac{p_0+p_3}{q_0+q_3}(q_0^2-q_3^2)+\frac{q_0+q_3}{p_0+p_3}(p_0^2-p_3^2)$$
$$= 2(-p_1q_1-p_2q_2)+(p_0+p_3)(q_0-q_3)+(q_0+q_3)(p_0-p_3)$$
$$= 2(p_0q_0-p_1q_1-p_2q_2-p_3q_3)$$
$$= 2p\cdot q$$

Note that we also have $|t(p,q)|^2 = 2p\cdot q$.

Problem 3.4 We start from the Lorentz transform of γ^μ:

$$\Lambda^\mu_\nu \gamma^\nu = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}$$
$$= (1+\frac{i}{2}\omega_{\rho\lambda}S^{\rho\lambda})\gamma^\mu(1-\frac{i}{2}\omega_{\rho\lambda}S^{\rho\lambda})$$
$$\begin{pmatrix} 0 & \Lambda^\mu_\nu\sigma^\nu \\ \Lambda^\mu_\nu\bar{\sigma}^\nu & 0 \end{pmatrix} = \begin{pmatrix} 1+\beta\cdot\frac{\sigma}{2}+i\theta\cdot\frac{\sigma}{2} & 0 \\ 0 & 1-\beta\cdot\frac{\sigma}{2}+i\theta\cdot\frac{\sigma}{2} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 1-\beta\cdot\frac{\sigma}{2}-i\theta\cdot\frac{\sigma}{2} & 0 \\ 0 & 1+\beta\cdot\frac{\sigma}{2}-i\theta\cdot\frac{\sigma}{2} \end{pmatrix}$$

Thus we have

$$\Lambda^\mu_\nu \bar{\sigma}^\nu = (1 - \beta \cdot \frac{\sigma}{2} + i\theta \cdot \frac{\sigma}{2})\bar{\sigma}^\mu (1 - \beta \cdot \frac{\sigma}{2} - i\theta \cdot \frac{\sigma}{2})$$

and

$$\begin{aligned}\bar{\sigma} \cdot \partial\chi &= \bar{\sigma}^\mu \Lambda^\nu_\mu \partial'_\nu \chi \\ &= (1 - \beta \cdot \frac{\sigma}{2} + i\theta \cdot \frac{\sigma}{2})\bar{\sigma}^\mu \partial'_\mu (1 - \beta \cdot \frac{\sigma}{2} - i\theta \cdot \frac{\sigma}{2})\chi \\ &= (1 - \beta \cdot \frac{\sigma}{2} + i\theta \cdot \frac{\sigma}{2})\bar{\sigma} \cdot \partial'_\mu \chi' \\ &= U\, \bar{\sigma} \cdot \partial'_\mu \chi'\end{aligned}$$

$$\begin{aligned}\sigma^2 \chi^* &= \sigma^2 (1 + \beta \cdot \frac{\sigma^*}{2} - i\theta \cdot \frac{\sigma^*}{2})(1 - \beta \cdot \frac{\sigma^*}{2} + i\theta \cdot \frac{\sigma^*}{2})\chi^* \\ &= (1 - \beta \cdot \frac{\sigma}{2} + i\theta \cdot \frac{\sigma}{2})\chi'^* \\ &= U\, \chi'^*\end{aligned}$$

Therefore we have

$$i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = U(i\bar{\sigma} \cdot \partial'\chi' - im\sigma^2\chi'^*) = 0$$

Hence the equation is invariant.
To show that it reduces to K-G equation, we apply $-i\sigma \cdot \partial$:

$$\begin{aligned}-i\sigma \cdot \partial(i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^*) &= \partial^2\chi - m\sigma^2\bar{\sigma}^* \cdot \partial\chi^* \\ &= \partial^2\chi - m\sigma^2(m\sigma^{2*}\chi) \\ &= \partial^2\chi + m^2\chi\end{aligned}$$

Next we want to show that the action S, or equivalently $\mathcal{L}$, is real:

$$\begin{aligned}\mathcal{L}^* &= \chi^T i\bar{\sigma}^* \cdot \partial\chi^* - \frac{im}{2}(-\chi^\dagger\sigma^{2*}\chi^* + \chi^T\sigma^{2*}\chi) \\ &= (\chi^T i\bar{\sigma}^* \cdot \partial\chi^*)^T + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \\ &= \chi^\dagger i\bar{\sigma} \cdot \partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \\ &= \mathcal{L}\end{aligned}$$

Varying with respect to $\chi^\dagger$ gives

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\chi^\dagger} &= 0 \\ i\bar{\sigma} \cdot \partial\chi - \frac{im}{2}\sigma^2\chi^* - \frac{im}{2}\sigma^2\chi^* &= 0 \\ i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^* &= 0\end{aligned}$$

Using $\psi = \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix}$, we can write the Dirac Lagrangian as

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \\
&= \begin{pmatrix} \chi_1^\dagger & -i\chi_2^T\sigma^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} - m \begin{pmatrix} \chi_1^\dagger & -i\chi_2^T\sigma^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\
&= \begin{pmatrix} \chi_1^\dagger & -i\chi_2^T\sigma^2 \end{pmatrix} \begin{pmatrix} i\bar{\sigma}\cdot\partial & 0 \\ 0 & i\sigma\cdot\partial \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} - m\Big(i\chi_1^\dagger\sigma^2\chi_2^* - i\chi_2^T\sigma^2\chi_1 \Big) \\
&= \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^T i\bar{\sigma}^*\cdot\partial\chi_2^* + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) \\
&= \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + (\chi_2^T i\bar{\sigma}^*\cdot\partial\chi_2^*)^T + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) \\
&= \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2 + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*)
\end{aligned}$$

Note that the action reduces to the Majorana action if we let $\chi_1 = \chi_2$. There is a global $U(1)$ symmetry given by $\chi_1 \to e^{i\alpha}\chi_1$, $\chi_2 \to e^{-i\alpha}\chi_2$ which is inherited from the $U(1)$ symmetry of the Dirac theory. The corresponding Noether current is thus

$$\begin{aligned}
\mathcal{J}^\mu &= \bar{\psi}\gamma^\mu\psi \\
&= \chi_1^\dagger\bar{\sigma}^\mu\chi_1 + \chi_2^T\sigma^2\sigma^\mu\sigma^2\chi_2^* \\
&= \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2
\end{aligned}$$

For the Majorana action, it is obvious that the mass term is not invariant under $\chi \to e^{i\alpha}\chi$. Therefore Majorana action does not have a global $U(1)$ symmetry and Majorana fermion does not have a $U(1)$ current. This can be also be seen from the above expression by putting $\chi_1 = \chi_2$, which leads to $\mathcal{J}^\mu = 0$. If we consider massless Majorana fermion, however, then we can construct a conserved current by considering the chiral symmetry $\psi \to e^{i\alpha\gamma^5}\psi$, which then produces the axial current

$$\begin{aligned}
\mathcal{J}^\mu &= \bar{\psi}\gamma^\mu\gamma^5\psi \\
&= \begin{pmatrix} \chi^\dagger & -i\chi^T\sigma^2 \end{pmatrix} \begin{pmatrix} -\bar{\sigma} & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \chi \\ i\sigma^2\chi \end{pmatrix} \\
&= -2\chi^\dagger\bar{\sigma}^\mu\chi
\end{aligned}$$

To quantize this theory, we first need to find a basis for the solution to the field equation for χ. We can start from the mode expansion for Dirac field:

$$\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(a_p^s u_p^s e^{-ipx} + b_p^{s\dagger} v_p^s e^{ipx} \right)$$

Separating the first two components and the last two components, we have

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_s \left(a_p^s \xi^s e^{-ipx} + b_p^{s\dagger}\xi^s e^{ipx} \right)$$

$$i\sigma^2\chi^* = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}}{2E_p}}\sum_s \left(a_p^s \xi^s e^{-ipx} - b_p^{s\dagger}\xi^s e^{ipx} \right)$$

where we choose the basis to be $\xi^{1,2} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$. If we take the complex conjugate of the second equality, we have

$$\begin{aligned}
i\sigma^2\chi &= \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}^*}{2E_p}}\sum_s\Big(a_p^{s\dagger}\xi^s e^{ipx} - b_p^s\xi^s e^{-ipx}\Big)\\
\chi &= \int \frac{d^3p}{(2\pi)^3}\sigma^2\sqrt{\frac{p\cdot\bar{\sigma}^*}{2E_p}}\sum_s\Big(-ia_p^{s\dagger}\xi^s e^{ipx} + ib_p^s\xi^s e^{-ipx}\Big)\\
&= \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_s\Big(-i\sigma^2a_p^{s\dagger}\xi^s e^{ipx} + i\sigma^2 b_p^s\xi^s e^{-ipx}\Big)
\end{aligned}$$

where we invoked the relation $\sigma^2\sigma^{i*} = -\sigma^i\sigma^2$ in the last equality. Therefore, by direct comparison, we have $b_p^s = i\sigma^2 a_p^s$. Thus we can write

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_s\Big(a_p^s\xi^s e^{-ipx} - i\sigma^2 a_p^{s\dagger}\xi^s e^{ipx}\Big)$$

If we impose the anticommutation condition $\{a_p^s, a_q^t\} = \delta^{st}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})$, we can verify that[3]

$$\{\chi_a(\boldsymbol{x}), \chi_b^\dagger(\boldsymbol{y})\} = \int\frac{d^3p}{(2\pi)^3}\frac{p\cdot\sigma}{2E_p}\Big(\delta_{ab}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \delta_{ab}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Big)$$

We focus on the first term:

$$\begin{aligned}
\int\frac{d^3p}{(2\pi)^3}\frac{p\cdot\sigma}{2E_p}\delta_{ab}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} &= \frac{\delta_{ab}}{2}\Big(\int\frac{d^3p}{(2\pi)^3}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + \int\frac{d^3p}{(2\pi)^3}\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E_p}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\
&= \frac{\delta_{ab}}{2}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) + \int_\infty^{-\infty}\frac{d^3(-p)}{(2\pi)^3}\frac{(-\boldsymbol{p})\cdot\boldsymbol{\sigma}}{E_p}e^{-i(-\boldsymbol{p})\cdot(\boldsymbol{x}-\boldsymbol{y})}\\
&= \frac{\delta_{ab}}{2}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) - \int\frac{d^3p}{(2\pi)^3}\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E_p}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}
\end{aligned}$$

where we changed variable $\boldsymbol{p}\to-\boldsymbol{p}$ in the second equality. Note that the minus sign comes from reverting the integration limits.

The second term is then obvious:

$$\int\frac{d^3p}{(2\pi)^3}\frac{p\cdot\sigma}{2E_p}\delta_{ab}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \frac{\delta_{ab}}{2}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) + \int\frac{d^3p}{(2\pi)^3}\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E_p}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

Thus we finally have

$$\{\chi_a(\boldsymbol{x}), \chi_b^\dagger(\boldsymbol{y})\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

which confirms that our mode expansion is correct.

Now we diagonalize the Hamiltonian. Remembering that $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}} = i\chi^\dagger$ and $\mathcal{H} = \pi\dot{\chi} - \mathcal{L}$, we have

$$H = \int d^3x\Big(-i\chi^\dagger\boldsymbol{\sigma}\cdot\nabla\chi + \frac{im}{2}(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi)\Big)$$

[3] Remember we are considering equal-time anticommutator so the time dependence is trivial and can be dropped.

We list here the mode expansions of various terms:

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_s \left(a_p^s\xi^s e^{-ipx} - i\sigma^2 a_p^{s\dagger}\xi^s e^{ipx}\right)$$

$$\chi^\dagger = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}\sum_s \left(a_p^{s\dagger}\xi^{s\dagger} e^{ipx} + i a_p^s\xi^{s\dagger}\sigma^2 e^{-ipx}\right)\sqrt{p\cdot\sigma}$$

$$\chi^* = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar\sigma}{2E_p}}\sum_s \left(a_p^{s\dagger}\xi^s e^{ipx} - i\sigma^2 a_p^s\xi^s e^{-ipx}\right)$$

$$\chi^T = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}\sum_s \left(a_p^s(\xi^s)^T e^{-ipx} + i a_p^{s\dagger}(\xi^s)^T\sigma^2 e^{ipx}\right)\sqrt{p\cdot\bar\sigma}$$

$$\boldsymbol{\sigma}\cdot\nabla\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_s \left(i(\boldsymbol{p}\cdot\boldsymbol{\sigma})a_p^s\xi^s e^{-ipx} - (\boldsymbol{p}\cdot\boldsymbol{\sigma})\sigma^2 a_p^{s\dagger}\xi^s e^{ipx}\right)$$

where we have used $\sqrt{p\cdot\sigma^*} = \sqrt{p\cdot\sigma^T} = \sqrt{\sigma^2 p\cdot\bar\sigma\sigma^2} = \sqrt{p\cdot\bar\sigma}$ (eigenvalues of $\sigma^2 A\sigma^2$ are the same as A). We also need the following identities:

$$\begin{aligned}\sqrt{p\cdot\sigma}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{p\cdot\sigma} &= (p\cdot\sigma)(\boldsymbol{p}\cdot\boldsymbol{\sigma}) \\ &= \frac{1}{2}(p\cdot\sigma)(p\cdot\sigma - p\cdot\bar\sigma) \\ &= \frac{1}{2}(E^2 + p^2 + 2E(\boldsymbol{p}\cdot\boldsymbol{\sigma}) - E^2 + p^2 - 2E(\boldsymbol{p}\cdot\boldsymbol{\sigma})) \\ &= p^2\end{aligned}$$

$$\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} = m$$

$$\sqrt{p\cdot\sigma}(-\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{p\cdot\bar\sigma} = -m(\boldsymbol{p}\cdot\boldsymbol{\sigma})$$

where $p^2 = \boldsymbol{p}\cdot\boldsymbol{p}$. We then substitute the mode expansions and note that the exponents will reduce to delta functions when integrated over space. After some manipulation we are able to reduce H to

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big[(p^2+m^2)\sum_s a_p^{s\dagger}a_p^s + (-p^2-m^2)\sum_s a_p^s a_p^{s\dagger} + im(\boldsymbol{p}\cdot\boldsymbol{\sigma})a_p^r(\sigma^2)_{rs}a_{-p}^s + im(\boldsymbol{p}\cdot\boldsymbol{\sigma})a_p^{r\dagger}(\sigma^2)_{rs}a_{-p}^{s\dagger}\Big]$$

The last two terms will vanish because the integrands are odd functions of $\boldsymbol{p}$, e.g.

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\boldsymbol{p}\cdot\boldsymbol{\sigma})a_p^r(\sigma^2)_{rs}a_{-p}^s &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{-p}}(-\boldsymbol{p}\cdot\boldsymbol{\sigma})a_{-p}^r(\sigma^2)_{rs}a_p^s \\ &= -\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\boldsymbol{p}\cdot\boldsymbol{\sigma})a_p^r(\sigma^2)_{rs}a_{-p}^s \\ &= 0\end{aligned}$$

After normal ordering we have

$$
\begin{aligned}
:H: &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} 2(p^2+m^2) \sum_s a_p^{s\dagger} a_p^s \\
&= \int \frac{d^3p}{(2\pi)^3} E_p \sum_s a_p^{s\dagger} a_p^s
\end{aligned}
$$

which is the desired result.

Problem 3.5 Remembering that for two Grassmann numbers α, β we have $(\alpha\beta)^* = \beta^*\alpha^*$, therefore we have $(\epsilon^T\sigma^2\chi)^* = (\epsilon^1(-i)\chi^2 + \epsilon^2 i\chi^1)^* = \chi^{2*}i\epsilon^{1*} + \chi^{1*}(-i)\epsilon^{2*} = \chi^\dagger\sigma^2\epsilon^*$. We also have $(\bar{\sigma}\cdot\partial)(\sigma\cdot\partial) = \partial^2$. Then,

$$
\begin{aligned}
\delta\mathcal{L} &= \partial_\mu\delta\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger i\bar{\sigma}\cdot\partial\delta\chi + \delta F^* F + F^*\delta F \\
&= \partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi) + \epsilon^\dagger F^* i\bar{\sigma}\cdot\partial\chi + \epsilon^T\sigma^2\sigma\cdot\partial\phi^* i\bar{\sigma}\cdot\partial\chi \\
&\qquad + \chi^\dagger i\bar{\sigma}\cdot\partial(\epsilon F) + \chi^\dagger i\bar{\sigma}\cdot\partial(\sigma\cdot\partial\phi)\sigma^2\epsilon^* + i\chi^\dagger\bar{\sigma}\cdot\overleftarrow{\partial}\epsilon F - F * i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi \\
&= -(i\chi^\dagger\sigma^2\epsilon^*)\partial^2\phi + i\epsilon^T\sigma^2\chi\partial^2\phi^* + i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi F^* - i\epsilon^T\sigma^2\chi\partial^2\phi^* \\
&\qquad + \chi^\dagger i\bar{\sigma}\cdot\partial(\epsilon F) + \chi^\dagger i\sigma^2\epsilon^*\partial^2\phi - i\chi^\dagger\bar{\sigma}\cdot\partial(\epsilon F) - i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi F* \\
&= 0
\end{aligned}
$$

where we moved the partial differential operator using integration by parts in the third equality.

For the mass term, using similar techniques and the identity $\epsilon^T A\chi = -\chi^T A^T\epsilon$, we have

$$
\begin{aligned}
\delta\mathcal{L}_M &= -imF(\epsilon^T\sigma^2\chi) - im\phi(\epsilon^\dagger\bar{\sigma}\cdot\partial\chi) + \frac{1}{2}imF(\epsilon^T\sigma^2\chi) + \frac{1}{2}im\phi\epsilon^\dagger\bar{\sigma}\cdot\partial\chi + \frac{1}{2}imF\chi^T\sigma^2\epsilon + \frac{1}{2}im\chi^T\bar{\sigma}^T\cdot\partial\phi\epsilon * + c.c. \\
&= -\frac{1}{2}imF(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon) - \frac{1}{2}im\phi(\epsilon^\dagger\bar{\sigma}\cdot\partial\chi) - \frac{1}{2}im\epsilon^\dagger\bar{\sigma}\cdot\partial\phi\,\chi + c.c. \\
&= -\frac{1}{2}imF(\epsilon^T\sigma^2\chi - \epsilon^T\sigma^2\chi) + c.c. \\
&= 0
\end{aligned}
$$

The equation for F is easy to obtain:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial F} &= 0 \\
F * + m\phi &= 0
\end{aligned}
$$

Thus, by eliminating F, we have

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + m^2\phi\phi^* - m^2\phi\phi^* - m^2\phi\phi^* + \frac{1}{2}im(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \\
&= \partial_\mu\phi^*\partial^\mu\phi - m^2\phi\phi^* + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{1}{2}im(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)
\end{aligned}
$$

We can see that the Lagrangian reduces to that of a free scalar field and a free spinor field with the same mass.

Finally we consider adding the interaction term involving the superpotential:

$$\begin{aligned}
\delta\mathcal{L}_{int} &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi_i\frac{\partial W}{\partial\phi_i} + F_i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(-i\epsilon^T\sigma^2\chi_j) + i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2(\epsilon F_j + \sigma\cdot\partial\phi_j\sigma^2\epsilon^*) \\
&\quad + \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}(-i\epsilon^T\sigma^2\chi_k)(\chi_i^T\sigma^2\chi_j) + c.c. \\
&= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi_i\frac{\partial W}{\partial\phi_i} + i\frac{\partial W}{\partial\phi_i}\chi_i^T\bar{\sigma}^T\cdot\overleftarrow{\partial}\frac{\partial\phi_j}{\partial\phi_j}\epsilon^* + \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}(-i\epsilon^T\sigma^2\chi_k)(\chi_i^T\sigma^2\chi_j) + c.c. \\
&= \frac{1}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\epsilon^T\sigma^2\chi_k\chi_i^T\sigma^2\chi_j + c.c. \\
&= 0
\end{aligned}$$

The last equality is due to the fact that the partial derivative is symmetric in (i,k) but the product $(...\chi_k\chi_i^T...)$ is antisymmetric.

For $W = \frac{g\phi^3}{3}$, we have

$$\begin{aligned}
\mathcal{L}_{int} &= gF\phi^2 + ig\phi\chi^T\sigma^2\chi + c.c. \\
&= -g(\phi^*)^2\phi^2 + ig\phi\chi^T\sigma^2\chi + c.c.
\end{aligned}$$

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - g(\phi^*\phi)^2 + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + ig(\phi\chi^T\sigma^2\chi - \phi^*\chi^\dagger\sigma^2\chi^*)$$

The field equations are

$$\begin{aligned}
\partial^2\phi + 2g\phi^*\phi^2 &= 0 \\
i\bar{\sigma}\cdot\partial\chi + -2ig\phi^*\sigma^2\chi^* &= 0
\end{aligned}$$

Problem 3.6 This problem is more manageable with the help of your favorite CAS (e.g. MATLAB, Mathematica, Octave, etc.). The correct normalizations are $(A = 1, 2, ..., 16)$

$$\Gamma^A = \{1, \gamma^0, i\gamma^1, i\gamma^2, i\gamma^3, \gamma^{01}, \gamma^{02}, \gamma^{03}, i\gamma^{12}, i\gamma^{13}, i\gamma^{23}, i\gamma^{012}, i\gamma^{013}, i\gamma^{023}, \gamma^{123}, i\gamma^{0123}\}$$

Note that the higher-order matrices can be built from lower order matrices, e.g.

$$\gamma^{0123} = \frac{1}{4}\left(\gamma^0\gamma^{123} - \gamma^1\gamma^{023} + \gamma^2\gamma^{013} - \gamma^3\gamma^{012}\right)$$

The identity we want to prove is

$$(\Gamma^A)_{ij}(\Gamma^B)_{kl} = \sum_{C,D} C^{AB}_{CD}(\Gamma^C)_{il}(\Gamma^D)_{kj}$$

Such linear decomposition is possible because Γ^A's form a complete basis for all 4×4 matrices[4]. The remaining problem is to find the coefficients C^{AB}_{CD}.

[4] http://mathworld.wolfram.com/DiracMatrices.html

To do so we multiply both sides by $(\Gamma^E)_{li}(\Gamma^F)_{jk}$:

$$\begin{aligned}
(\Gamma^E)_{li}(\Gamma^F)_{jk}(\Gamma^A)_{ij}(\Gamma^B)_{kl} &= \sum_{C,D} C^{AB}_{CD}(\Gamma^E)_{li}(\Gamma^F)_{jk}(\Gamma^C)_{il}(\Gamma^D)_{kj} \\
\mathrm{tr}(\Gamma^E\Gamma^A\Gamma^F\Gamma^B) &= \sum_{C,D} C^{AB}_{CD}\mathrm{tr}(\Gamma^E\Gamma^C)\mathrm{tr}(\Gamma^F\Gamma^D) \\
\mathrm{tr}(\Gamma^E\Gamma^A\Gamma^F\Gamma^B) &= 16C^{AB}_{EF} \\
C^{AB}_{CD} &= \frac{1}{16}\mathrm{tr}(\Gamma^C\Gamma^A\Gamma^D\Gamma^B)
\end{aligned}$$

where the third equality is due to the completeness relation and the last equality results from relabelling of indices. The Fierz transformation law is then a direct application of this matrix identity:

$$\begin{aligned}
(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) &= (\bar{u}_1)_i(u_2)_j(\bar{u}_3)_k(u_4)_l(\Gamma^A)_{ij}(\Gamma^B)_{kl} \\
&= (\bar{u}_1)_i(u_2)_j(\bar{u}_3)_k(u_4)_l\sum_{C,D} C^{AB}_{CD}(\Gamma^C)_{il}(\Gamma^D)_{kj} \\
&= \sum_{C,D} C^{AB}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2)
\end{aligned}$$

Lastly we write down two concrete examples of Fierz transformation laws:

$$\begin{aligned}
(\bar{u}_1 u_2)(\bar{u}_3 u_4) &= \sum_{C,D} C^{11}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2) \\
&= \sum_{C,D}\frac{1}{16}\mathrm{tr}(\Gamma^C\Gamma^D)(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2) \\
&= \frac{1}{4}\sum_C(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^C u_2)
\end{aligned}$$

$$\begin{aligned}
(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4) &= \sum_{C,D}\frac{1}{16}\mathrm{tr}(\Gamma^C\gamma^\mu\Gamma^D\gamma_\mu)(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2) \\
&= \sum_{C',D'}\frac{1}{16}\mathrm{tr}(\Gamma^{C'}\Gamma^{D'})(\bar{u}_1\Gamma^{C'}\gamma_\mu u_4)(\bar{u}_3\Gamma^{D'}\gamma^\mu u_2) \\
&= \frac{1}{4}\sum_C(\bar{u}_1\Gamma^C\gamma^\mu u_4)(\bar{u}_3\Gamma^C\gamma_\mu u_2)
\end{aligned}$$

In the second example we used the fact that $\Gamma^C\gamma^\mu$ is just another Dirac matrix which we label as $\Gamma^{C'}$ (the factors of i cancel each other since there are 4 occurrences of Γ).

Problem 3.7 Parity transform gives (we leave out phases because they will cancel anyway)

$$\begin{aligned}
P\bar{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\psi(t,\boldsymbol{x})P &= P\bar{\psi}(t,\boldsymbol{x})P\sigma^{\mu\nu}P\psi(t,\boldsymbol{x})P \\
&= \bar{\psi}(t,-\boldsymbol{x})\gamma^0\sigma^{\mu\nu}\gamma^0\psi(t,-\boldsymbol{x})
\end{aligned}$$

We separate the anticommutation relation into cases: ($k, l \neq 0$ in all of the following equations)

$$\gamma^0\sigma^{00} = \sigma^{00}\gamma^0$$

$$\gamma^0\sigma^{0k} = -\sigma^{0k}\gamma^0$$

$$\gamma^0\sigma^{kl} = \sigma^{kl}\gamma^0$$

Therefore we get

$$\gamma^0\sigma^{\mu\nu}\gamma^0 = (-1)^\mu(-1)^\nu\sigma^{\mu\nu}$$

with $(-1)^\mu$ defined as in the last part of section 3.6.
Time reversal gives

$$\begin{aligned} T\bar{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\psi(t,\boldsymbol{x})T &= T\bar{\psi}(t,\boldsymbol{x})TT\sigma^{\mu\nu}\psi(t,\boldsymbol{x})T \\ &= T\bar{\psi}(t,\boldsymbol{x})T(\sigma^{\mu\nu})^*T\psi(t,\boldsymbol{x})T \\ &= -\bar{\psi}(-t,\boldsymbol{x})\gamma^1\gamma^3(\sigma^{\mu\nu})^*\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) \end{aligned}$$

Again we need to separate the anticommutation relation into cases. Remembering that γ^0, γ^1, γ^3 are unchanged under complex conjugation but γ^2 flips sign, and use the following definition for s:

$$\gamma^1\gamma^3(\sigma^{\mu\nu})^* = s\sigma^{\mu\nu}\gamma^1\gamma^3$$

We have, by inspection,

s	μ	ν
-1	0	1,3
-1	0	2
+1	1	3
+1	1,3	2

Thus, summarizing, we have

$$\gamma^1\gamma^3(\sigma^{\mu\nu})^*\gamma^1\gamma^3 = (-1)^\mu(-1)^\nu\sigma^{\mu\nu}$$

Finally charge conjugation gives

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C &= C\bar{\psi}C\sigma^{\mu\nu}C\psi C \\ &= (-i\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(-i\bar{\psi}\gamma^0\gamma^2)^T \\ &= -(\gamma^0)_{ab}(\gamma^2)_{bc}\psi_c(\sigma^{\mu\nu})_{af}\bar{\psi}_d(\gamma^0)_{de}(\gamma^2)_{ef} \\ &= \bar{\psi}_d(\gamma^0)_{de}(\gamma^2)_{ef}(\sigma^{\mu\nu})_{af}(\gamma^0)_{ab}(\gamma^2)_{bc}\psi_c \\ &= \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi \end{aligned}$$

Since γ^0, γ^2 are symmetric and γ^1, γ^3 are antisymmetric, we can obtain the following table by inspection: (defining s as in $\gamma^0\gamma^2(\sigma^{\mu\nu})^T = s\sigma^{\mu\nu}\gamma^0\gamma^2$)

s	μ	ν
-1	0,2	1,3
-1	0	2
-1	1,3	1,3

Thus

$$\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2 = -\sigma^{\mu\nu}$$

Thereby we have completed the table at the end of section 3.6.
Now we consider the discrete transformations for complex-valued K-G field. We start from the mode expansion

$$\phi = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_k e^{-ikx} + b_k^\dagger e^{ikx}\right)$$

and thus

$$\begin{aligned}
P\phi(t,\boldsymbol{x})P &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_{-k} e^{-ikx} + b_{-k}^\dagger e^{ikx}\right) \\
&= \int \frac{d^3\tilde{k}}{(2\pi)^3\sqrt{2\omega_{\tilde{k}}}}\left(a_{\tilde{k}} e^{-i\tilde{k}\tilde{x}} + b_{\tilde{k}}^\dagger e^{i\tilde{k}\tilde{x}}\right) \\
&= \phi(t,-\boldsymbol{x})
\end{aligned}$$

where $\tilde{k} = (k^0, -\boldsymbol{k})$ and $\tilde{x} = (t, -\boldsymbol{x})$. We also have

$$\begin{aligned}
T\phi(t,\boldsymbol{x})T &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_{-k} e^{ikx} + b_{-k}^\dagger e^{-ikx}\right) \\
&= \int \frac{d^3\tilde{k}}{(2\pi)^3\sqrt{2\omega_{\tilde{k}}}}\left(a_{\tilde{k}} e^{-i\tilde{k}x'} + b_{\tilde{k}}^\dagger e^{i\tilde{k}x'}\right) \\
&= \phi(-t,\boldsymbol{x})
\end{aligned}$$

where $x' = (-t, \boldsymbol{x})$. Lastly we have

$$\begin{aligned}
C\phi C &= \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(b_k e^{-ikx} + a_k^\dagger e^{ikx}\right) \\
&= \phi^*
\end{aligned}$$

The transformation properties of the current follows simply from the transformation property of ϕ and ∂_μ: (indices are not summed over)

$$PJ^\mu(t,\boldsymbol{x})P = (-1)^\mu J^\mu(t,\boldsymbol{x})$$
$$TJ^\mu(t,\boldsymbol{x})T = (-1)^\mu J^\mu(-t,\boldsymbol{x})$$
$$CJ^\mu C = -J^\mu$$

From the table at the end of section 3.6, we can see that any Lorentz scalar formed using various bilinear forms of ψ must have positive CPT parity. It is easy to see that any term with ϕ must be PT positive. They are also C positive because the operator has to be Hermitian, and thus any occurrences of ϕ are accompanied by complex conjugates, e.g. $\phi + \phi^*$ or $\phi\phi^*$, and vice versa. Therefore any Hermitian Lorentz scalar built from ψ and ϕ is inevitably CPT positive.

Problem 3.8 We can write the wavefunction of the positronium schematically as

$$\Psi \sim \int d^3k \left[\int d^3r e^{ikr} \Psi(r) \right] |k, -k\rangle \otimes |Spin\rangle$$
$$\sim (Spatial) \times (Operator) |0\rangle \otimes |Spin\rangle$$

The spatial part carries the orbital angular momentum. The spin part has the symmetry of the usual singlet & triplet spin states: the singlet state flips sign under spin exchange and the triplet state is symmetric under spin exchange. The operator part consists of products of creation operators of the form $a^\dagger b^\dagger$.

Now the spatial part should have the same symmetry as the spherical harmonic $Y_{lm}(r)$ which obeys $Y_{lm}(-r) = (-1)^l Y_{lm}(r)$. Therefore the spatial part of the wavefunction has spatial parity $(-1)^l$. The spin part is not affected by parity transform. Under parity transform, products in the form of $a^\dagger b^\dagger$ will become $(\eta_a \eta_b)^* a^\dagger b^\dagger = (-1)\, a^\dagger b^\dagger$. Therefore the total spatial parity is $(-1)^{l+1}$.

Exchanging the charges on the two particles is equivalent to "swapping" the two particles. The spatial part will have a parity of $(-1)^l$ since swapping the positions and momenta is equivalent to a parity transform on the spatial part. The spin part will undergo a spin exchange so the parity will be 1 for triplet states and -1 for the singlet state, i.e. the parity will be $(-1)^{s+1}$. The operator part under charge conjugation will become $a^\dagger b^\dagger \to b^\dagger a^\dagger = -a^\dagger b^\dagger$. Thus we have a combined parity of $(-1)^l(-1)^{s+1}(-1) = (-1)^{l+s}$.

Charge conjugation reverses the direction of the electric and magnetic fields, i.e. $A^\mu \to -A^\mu$. Therefore the photon has charge parity of -1. Since charge parity has to be conserved, we need $(-1)^{l+s} = (-1)^n$.

A simpler argument would be that 2 spin-1 particles cannot combine to give a total spin of one. Therefore it is impossible for a spin-1 particle to decay into 3 photons. Invoking charge parity is not necessary.

Problem 4.1 The probability that no particle is created is given by

$$P(0) = |_{out}\langle 0|0\rangle_{in}|^2$$
$$= \lim_{T\to\infty} |\langle 0|e^{-iH(2T)}|0\rangle|^2$$
$$= |\langle 0|T\Big\{ \exp[i \int d^4x\, j(x)\phi_I(x)] \Big\}|0\rangle|^2$$

We can calculate the integral using mode expansion of ϕ_I:

$$J = \int d^4x\, j(x) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_k e^{-ikx} + a_k^\dagger e^{ikx} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} a_k \tilde{j}(k) + a_k^\dagger \tilde{j}^*(k)$$

It should be clear that any odd power of J has zero vacuum expectation value. Thus, if we expand the amplitude (the expression inside the norm-square) to the first non-zero order:

$$\mathcal{M} = \langle 0|1 + \frac{(iJ)^2}{2}|0\rangle$$
$$= 1 - \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} |j(k)|^2 + O(j^4)$$
$$= 1 - \frac{1}{2}\lambda + O(j^4)$$

where for the second equality we note that the only non-vanishing term in J^2 comes from the $a_k a_{k'}^\dagger$ term (remembering $[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^{(3)}(k - k')$).

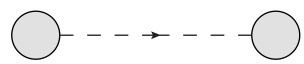
The probability is thus

$$\begin{aligned} P(0) &= |\mathcal{M}|^2 \\ &= (1 - \frac{\lambda}{2} + O(j^4))^2 \\ &= 1 - \lambda + O(j^4) \end{aligned}$$

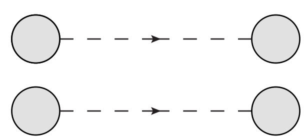
We can also write the amplitude without invoking mode expansion:

$$\begin{aligned} \mathcal{M} &= 1 - \frac{1}{2}\int d^4x d^4y \, T\{j(x)\phi(x)j(y)\phi(y)\} + O(j^4) \\ &= 1 - \frac{1}{2}\int d^4x d^4y \, j(x)j(y)\phi(x)\phi(y) + O(j^4) \\ &= 1 - \frac{1}{2}\int d^4x d^4y \, j(x)j(y)D_F(x-y) + O(j^4) \end{aligned}$$

The corresponding diagram is



By comparing with the previous expression we know that the diagram has value $-\lambda$ with combinatorial factor $\frac{1}{2}$. The circles denote the source at x and y.For higher order terms the diagram is just consisting of multiple copies of the same diagram, e.g.



which has value $(-\lambda)^2$, with combinatorial factor $\frac{1}{4!} \times 3$ since there are 3 ways to pair up the 4 ϕ's for contractions.

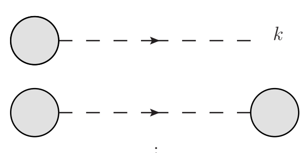
In general the n-th term has value $(-\lambda)^n$ with combinatorial factor $\frac{1}{(2n)!} \times (2n-1)!! = \frac{1}{2^n n!}$ (exponentiation factor$\times$time-ordering factor$\times$ways to contraction)[5].

Therefore the total amplitude is

$$\mathcal{M} = \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{1}{2^n (n)!} = e^{-\frac{\lambda}{2}}$$

Thus $P(0) = |\mathcal{M}|^2 = e^{-\lambda}$.

Now we turn to the problem of the source creating a single particle. We first note that the diagram will have the generic form of



[5]The number of ways to form pairs in a group of n objects is given by $(n-1)!!$. This is because the first object has $n-1$ possible partners, the first object in the remaining unpaired pool has $n-3$ possible partners, etc.

Therefore the amplitude is just the value of the top open dumbbell, which we call $\mathcal{K}$, multiplied by the zero-source amplitude $e^{-\frac{\lambda}{2}}$. We can calculate $\mathcal{K}$ by

$$\begin{aligned}\mathcal{K} &= \langle k|iJ|0\rangle \\ &= \langle k|i\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2E_{k'}}}a_{k'}\tilde{j}(k') + a_{k'}^\dagger \tilde{j}^*(k')|0\rangle \\ &= i\tilde{j}^*(k)\end{aligned}$$

Therefore the probability is just

$$P(k) = |j(k)|^2 e^{-\lambda}$$

For multiple particles with momenta $k_1, k_2, ..., k_n$, we can use the same trick and factorize out the open dumbbells, thus the amplitude is

$$\mathcal{M} = \mathcal{K}_1\mathcal{K}_2\cdots\mathcal{K}_n e^{-\frac{\lambda}{2}}$$

The factorial coming from exponentiation will cancel that from contraction so there is no additional combinatorial factor. The probability is

$$P(k_1, k_2, ..., k_n) = |j(k_1)|^2 ... |j(k_n)|^2 e^{-\lambda}$$

If we ony care about the number of particles generated but not the momenta, we can integrate over the momenta and divide by $n!$ since we do not distinguish the particles:

$$P(n) = \frac{\lambda^n}{n!}e^{-\lambda}$$

which is the Poisson distribution.

The Poisson distribution has the following properties:

$$\sum_{n=0}^{\infty} P(n) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda} = 1$$

$$\langle n\rangle = 0 + \sum_{n=1}^{\infty} nP(n) = e^{-\lambda}\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = \lambda$$

$$\begin{aligned}\langle n^2\rangle &= e^{-\lambda}\lambda\sum_{n=0}^{\infty}\frac{(n+1)\lambda^n}{n!} = \lambda^2 + \lambda \\ \langle (n-\lambda)^2\rangle &= \lambda^2 + \lambda - 2\lambda^2 + \lambda^2 = \lambda^2\end{aligned}$$

Problem 4.2 The amplitude is given by

$$\begin{aligned}-i\mu\langle k_1k_2|\int d^4x\, T\{\Phi\phi\phi\}|q\rangle &= -2i\mu \\ &= i\mathcal{M}\end{aligned}$$

where contractions between the fields and the states give exponents which after integration result in a delta function that amounts to conservation of momenta. This delta function can be dropped to get the amplitude. The factor of 2 is a combinatorial factor since there are two ways of contracting the ϕ's into the final states.

The decay rate is thus

$$\Gamma = \frac{4\mu^2}{2M}\int \frac{d^3k_1 d^3k_2}{(2\pi)^6 4E_{k_1}E_{k_2}}(2\pi)^4\delta^{(4)}(q-k_1-k_2)$$

The integration is Lorentz-invariant so we can choose to work in the rest frame of the initial particle for convenience. From kinematics it is easy to deduce

$$E_{k_1} = E_{k_2} = \frac{M}{2}$$
$$k_1^2 = k_2^2 = \frac{M^2}{4} - m^2$$

Thus, integrating out the delta function, we have

$$\begin{aligned}\Gamma &= \frac{\mu^2}{2M}\int \frac{k_1^2 d\Omega dk_1}{(2\pi)^2(\frac{M}{2})^2}\delta(M - 2\sqrt{m^2+k_1^2}) \\ &= \frac{\mu^2}{2\pi^2 M^3}\cdot\frac{4\pi}{2}\cdot\sqrt{\frac{M^2}{4}-m^2}\cdot\frac{M}{4} \\ &= \frac{\mu^2}{8\pi M}\sqrt{1-\frac{4m^2}{M^2}}\end{aligned}$$

where the solid angle integrates to $\frac{4\pi}{2}$ because the two particles are indistinguishable so antipodal points correspond to the same final state. The lifetime is just the reciprocal of the decay rate:

$$\tau = \frac{8\pi M}{\mu^2}\Big(1-\frac{4m^2}{M^2}\Big)^{-\frac{1}{2}}$$

Problem 4.3 Φ^i commutes with Φ^j for $i \neq j$ thus we have

$$\overbracket{\Phi^i(x)\Phi^j}(y) = \delta^{ij}D_F(x-y)$$

To find the value of the vertex, we first write down the interaction Hamiltonian:

$$\mathcal{H}_I = \frac{\lambda}{4}\Big[\sum_{i,j}\Phi^i\Phi^i\Phi^j\Phi^j\Big]$$

Thus there is only one type of vertex that couples four Φ's. The value of this vertex can be obtained from

$$K = -i\frac{\lambda}{4}\langle\Phi^k\Phi^l|\int d^4x\, T\{\sum_{m,n}\Phi^m\Phi^n\Phi^m\Phi^n\}|\Phi^i\Phi^j\rangle$$

We only consider contraction between the fields and incoming/outgoing states (i.e. we don't consider diagrams with loops). Such contraction will result in delta functions enforcing momenta conservation which we are not interested and can be dropped. We can write down all the combinations (the factor of 4 is due to permuatations):

$$\begin{aligned}K &= -i\frac{\lambda}{4}\times 4\times\Big[\delta^{km}\delta^{lm}\delta^{in}\delta^{jn} + \delta^{kn}\delta^{ln}\delta^{im}\delta^{jm} + \delta^{km}\delta^{ln}\delta^{in}\delta^{jm} + \delta^{kn}\delta^{lm}\delta^{in}\delta^{jm} + \delta^{km}\delta^{ln}\delta^{im}\delta^{jn} + \delta^{kn}\delta^{lm}\delta^{im}\delta^{jn}\Big] \\ &= -2i\lambda(\delta^{kl}\delta^{ij} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl})\end{aligned}$$

For $\Phi^1\Phi^2 \to \Phi^1\Phi^2$, $\mathcal{M} = 2\lambda$, the differential cross-section is $\frac{\lambda^2}{16\pi^2 E_{cm}^2}$.
For $\Phi^1\Phi^1 \to \Phi^2\Phi^2$, $\mathcal{M} = 2\lambda$, the differential cross-section is $\frac{\lambda^2}{16\pi^2 E_{cm}^2}$.
For $\Phi^1\Phi^1 \to \Phi^1\Phi^1$, $\mathcal{M} = 6\lambda$, the differential cross-section is $\frac{9\lambda^2}{16\pi^2 E_{cm}^2}$.

Now we turn to the case with $m^2 = -\mu^2$. We choose the ground state to be given by $\Phi^i(x) = 0\,(i \neq N)$ and $\Phi^N(x) = v$, where v is a constant given by

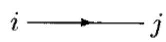
$$\partial_\Phi\left(\frac{1}{4}\lambda\Phi^4 - \frac{1}{2}\mu^2\Phi^2\right)\Big|_{\Phi=v} = 0$$
$$v = \sqrt{\frac{\mu^2}{\lambda}}$$

Under the transformation $\Phi^i(x) = \pi^i(x)$, $\Phi^N(x) = v + \sigma(x)$, the potential V becomes

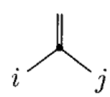
$$\begin{aligned} V &= -\frac{1}{2}\mu^2(\pi^i)^2 - \frac{1}{2}\mu^2(v+\sigma)^2 + \frac{\lambda}{4}\Big[(\pi^i)^4 + 2(\pi^i)^2(\pi^j)^2 + 2(\pi^i)^2(v+\sigma)^2 + (v+\sigma)^4\Big] \\ &= \mu^2\sigma^2 + \frac{\lambda}{4}((\pi^i)^2)^2 + \mu\sqrt{\lambda}(\pi^i)^2\sigma + \frac{\lambda}{2}(\pi^i)^2\sigma^2 + \mu\sqrt{\lambda}\sigma^3 + \frac{\lambda}{4}\sigma^4 \end{aligned}$$

The Feynman rules are as follows:

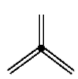
$\frac{i}{k^2 - 2\mu^2}$

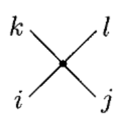


$\frac{i}{k^2}\delta^{ij}$

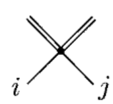


$-2i\mu\sqrt{\lambda}\delta^{ij}$

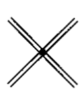

$-6i\mu\sqrt{\lambda}$



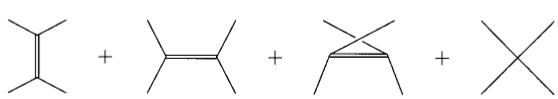
$-2i\lambda(\delta^{kl}\delta^{ij} + \delta^{jk}\delta^{il} + \delta^{ik}\delta^{jl})$



$-2i\lambda\delta^{ij}$


$-6i\lambda$

Now consider the process $\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$, the sum of the four diagrams shown gives



$$
\begin{aligned}
\mathcal{M} &= (-2i\mu\sqrt{\lambda})^2\Big[\delta^{ik}\delta^{jl}\frac{i}{t-2\mu^2} + \delta^{ij}\delta^{kl}\frac{i}{s-2\mu^2} + \delta^{il}\delta^{jk}\frac{i}{u-2\mu^2}\Big] - 2i\lambda(\delta^{kl}\delta^{ij} + \delta^{jk}\delta^{il} + \delta^{ik}\delta^{jl}) \\
&= -2i\lambda\Big(1 + \frac{2\mu^2}{s-2\mu^2}\Big)\delta^{ij}\delta^{kl} - 2i\lambda\Big(1 + \frac{2\mu^2}{t-2\mu^2}\Big)\delta^{ik}\delta^{jl} - 2i\lambda\Big(1 + \frac{2\mu^2}{u-2\mu^2}\Big)\delta^{il}\delta^{jk}
\end{aligned}
$$

If one of the initial pions (say p_1) has zero momentum, then it is a zero vector and the Mandelstam variables will be equal to p^2 of one of the other pions, which is again zero since they are massless. Therefore at threshold, $s = t = u = 0$ and $\mathcal{M} = 0$.

If $N = 2$, then

$$
\begin{aligned}
\mathcal{M} &= -2i\lambda\Big(1 + \frac{2\mu^2}{s-2\mu^2} + 1 + \frac{2\mu^2}{t-2\mu^2} + 1 + \frac{2\mu^2}{u-2\mu^2}\Big) \\
&= -2i\lambda\Big(\frac{s}{s-2\mu^2} + \frac{t}{t-2\mu^2} + \frac{u}{u-2\mu^2}\Big) \\
&\approx i\frac{\lambda}{\mu^2}\Big(s+t+u\Big) \\
&= 0
\end{aligned}
$$

where we have used the Taylor expansion $\frac{x}{x-a} \approx -\frac{x}{a} + O(x^2)$ and $s + t + u = \sum_{i=1}^4 m_i^2 = 0$.

If a symmetry breaking term is added, we have

$$
V = -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 - a\Phi^N
$$

The minimum becomes (again we have $\Phi^i = 0$ for $i \neq N$ and $\Phi^N = \Phi$)

$$
\begin{aligned}
\partial_\Phi V = -\mu^2\Phi + \lambda\Phi^3 - a &= 0 \\
\lambda\Phi^3 &= \mu^2\Phi + a
\end{aligned}
$$

We expand around $\Phi = v$ to get

$$
\begin{aligned}
3\mu^2\delta &= \mu^2\delta + a \\
\delta &= \frac{a}{2\mu^2}
\end{aligned}
$$

The mass term of pion becomes

$$
-\frac{\mu^2}{2}(\pi^i)^2 + \frac{\lambda}{2}v^2(\pi^i)^2 = \frac{\sqrt{\lambda}a}{2\mu}(\pi^i)^2 + O(a^2)
$$

Thus the pion acquires a mass of $O(a)$. The threshold amplitude to lowest order becomes

$$
\begin{aligned}
\mathcal{M} &\approx i\frac{\lambda}{\mu^2}(s+t+u) \\
&\sim a
\end{aligned}
$$

since $s, t, u = m_\pi \sim a$.

Problem 4.4 Using the mode expansion for the Dirac field, we have

$$\begin{aligned}\langle p'|iT|p\rangle &= \langle p'| - ie\int d^4x\, \bar{\psi}\gamma^\mu\psi A_\mu|p\rangle \\ &= -ie\int d^4x\, \bar{u}(p')e^{ip'x}\gamma^\mu u(p)e^{-ipx}A_\mu \\ &= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p)\end{aligned}$$

Now we calculate the cross-section. The incoming wavepacket can be written as

$$|\psi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}e^{-i\boldsymbol{b}\cdot\boldsymbol{k}}\psi(\boldsymbol{k})|\boldsymbol{k}\rangle$$

The probability that the wavepacket will be scattered into some momentum state $|\boldsymbol{p}\rangle$ is given by

$$\begin{aligned}\mathcal{P} &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|_{out}\langle\boldsymbol{p}|\psi\rangle_{in}|^2 \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k\,d^3k'}{(2\pi)^6}\frac{1}{\sqrt{4E_kE_{k'}}}e^{-i\boldsymbol{b}\cdot(\boldsymbol{k}-\boldsymbol{k'})}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})\langle p|iT|k\rangle\langle p|iT|k'\rangle^* \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k\,d^3k'}{(2\pi)^6}\frac{1}{\sqrt{4E_kE_{k'}}}e^{-i\boldsymbol{b}\cdot(\boldsymbol{k}-\boldsymbol{k'})}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})(2\pi)^2|\mathcal{M}|^2\delta(E_p-E_k)\delta(E_p-E_{k'})\end{aligned}$$

To proceed, let us denote momenta normal to the cross-sectional area by $k_\parallel$and along the cross-sectional area by $\boldsymbol{k}_\perp$. We can integrate over $\boldsymbol{b}$ to get another delta function. The double delta function also allow us to replace $\delta(E_p - E_{k'})$ by $\delta(E_k - E_{k'})$. Thus we have

$$\begin{aligned}d\sigma &= \int \mathcal{P}\,db^2 \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k\,d^3k'}{(2\pi)^6}\frac{1}{2E_k}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})|\mathcal{M}|^2(2\pi)^4\delta^{(2)}(\boldsymbol{k}_\perp-\boldsymbol{k}'_\perp)\delta(E_p-E_k)\delta(E_k-E_{k'}) \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k\,dk'_\parallel}{(2\pi)^3}\frac{1}{2E_k}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})|\mathcal{M}|^2(2\pi)\delta(E_p-E_k)\frac{E_k}{k_\parallel}\delta(k_\parallel-k'_\parallel) \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\frac{1}{v_k}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})|\mathcal{M}|^2(2\pi)\delta(E_p-E_k)\end{aligned}$$

Thanks to the delta function we can pull out terms depending on E_k and get

$$\begin{aligned}d\sigma &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\frac{1}{v_k}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'})|\mathcal{M}|^2(2\pi)\delta(E_p-E_k) \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\frac{1}{2E_k}\frac{1}{v_k}|\mathcal{M}|^2(2\pi)\delta(E_p-E_k)\int\frac{d^3k}{(2\pi)^3}\psi(\boldsymbol{k})\psi^*(\boldsymbol{k'}) \\ &= \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\frac{1}{2E_k}\frac{1}{v_k}|\mathcal{M}|^2(2\pi)\delta(E_p-E_k) \\ &= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{2E_i}\frac{1}{v_i}|\mathcal{M}|^2(2\pi)\delta(E_f-E_i)\end{aligned}$$

The last line is a mere change in notation. We can integrate over the norm p_f to get

$$\begin{aligned} d\sigma &= \int \frac{{p_f}^2 dp_f d\Omega}{(2\pi)^3} \frac{1}{2E_f} \frac{1}{2E_i} \frac{1}{v_i} |\mathcal{M}|^2 (2\pi)\delta(E_f - E_i) \\ \frac{d\sigma}{d\Omega} &= \int \frac{dp_f d\Omega}{(2\pi)^3} \frac{p_f}{2E_f} \frac{p_f}{2E_i} \frac{E_i}{p_i} |\mathcal{M}|^2 (2\pi) \frac{E_i}{p_i} \delta(p_f - p_i) \\ &= \frac{1}{(4\pi)^2} |\mathcal{M}(p,\theta)|^2 \end{aligned}$$

The amplitude is given by

$$\begin{aligned} \langle p'|iT|p\rangle &= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p) \\ &= -ie\bar{u}(p')\gamma^0 u(p)\tilde{A}_0(p'-p) \\ &= -iZe^2\bar{u}(p')\gamma^0 u(p) \int d^3r \, \frac{e^{-i(p'-p)\cdot r}}{4\pi r} \\ &= -i\frac{Ze^2}{(p'-p)^2}\bar{u}(p')\gamma^0 u(p) \end{aligned}$$

In the non-relativistic limit, the spinor product is given by

$$\begin{aligned} \frac{1}{2}\sum_s \bar{u}^s(p')\gamma^0 u^s(p) &= \frac{1}{2}\sum_s u^{s\dagger}(p')\gamma^0\gamma^0 u^s(p) \\ &= \frac{1}{2}\sum_s \begin{pmatrix} \sqrt{E}\xi^{s\dagger} & \sqrt{E}\xi^{s\dagger} \end{pmatrix} \begin{pmatrix} \sqrt{E}\xi^s \\ \sqrt{E}\xi^s \end{pmatrix} \\ &= 2E \end{aligned}$$

The energies are equal because the scattering is elastic. The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 e^4}{4\pi^2} \frac{E^2}{(p'-p)^4}$$

From simple kinematics we have

$$\begin{aligned} (p'-p)^2 &= (p\cos\theta - p)^2 + (p\sin\theta)^2 \\ &= 2p^2(1-\cos\theta) \\ &= 2p^2(1-\cos\theta) \\ &= 4p^2\sin^2\frac{\theta}{2} \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{Z^2e^4}{16\pi^2} \frac{E^2}{4p^4\sin^4(\frac{\theta}{2})} \\ &= \frac{Z^2\alpha^2}{4m^2v^4\sin^4(\frac{\theta}{2})} \end{aligned}$$

where we have used $p = mv$ and $p/E = v$.