Note For some reason only the first 4 exercises are numbered in the book. For later exercises we label them by the section number in which they appear. We also skip exercises that are followed by the author's solution.

Exercise 1 The de Broglie wavelength is given by

$$\lambda = \frac{h}{\sqrt{2mkT}}$$

The intermolecular distance is given by

$$d = (\frac{1}{n})^{\frac{1}{3}}$$

The condition $\lambda \approx d$ thus implies

$$T = \frac{h^2 n^{\frac{2}{3}}}{2mk} \approx 0.03K$$

where we used $n \approx 2.7 \times 10^{25}\, m^{-3}$, $m \approx 4.7 \times 10^{-26}\, kg$.

Exercise 2 From thermodynamics, the free energy satisfies

$$dF = -SdT - PdV + \mu dN$$

Thus S, P and μ are all obtainable from derivatives of F, and thus are related to Z.

The partition function for a classical gas of non-interacting particles is (assuming they all have the same mass for simplicity)

$$\begin{aligned} Z_0 &= \frac{1}{N! h^{3N}} \int d^{3N}q d^{3N}p \, \exp(-\beta \sum_i \frac{p_i^2}{2m}) \\ &= \frac{V^N}{N!} \Big(\frac{1}{h} \int dp \, e^{-\frac{\beta}{2m} p_i^2} \Big)^{3N} \\ &= \frac{V^N}{N!} (2mh^2 kT)^{\frac{3N}{2}} \end{aligned}$$

Exercise 3 From $Z_0 = \frac{V^N}{N!}(2mh^2kT)^{\frac{3N}{2}}$ we have

$$\begin{aligned} \log Z_0 &= N \log V + \frac{3N}{2} \log(2mh^2kT) - \log N! \\ &= N \log V + \frac{3N}{2} \log(2mh^2kT) - N \log N + N + O(\log N) \end{aligned}$$

$$\begin{aligned} \frac{\log Z_0}{N} &= \log \frac{V}{N} + \frac{3}{2} \log(2mh^2kT) + 1 + O(\frac{\log N}{N}) \\ &\xrightarrow{N,V \to \infty} -\log v + \frac{3}{2} \log(2mh^2kT) + 1 \end{aligned}$$

Exercise 4 We use U to denote the potential energy to avoid confusion with the symbol for volume. We consider the spatial part of the partition function:

$$\begin{aligned} Z_q &= \frac{1}{N!}\int d^{3N}q\ \exp\Big(-\beta U(q_1,...,q_{3N})\Big) \\ &= \frac{V^N}{N!}\int d^{3N}x\ \exp\Big(-\beta V^{s/3}U(x_1,...,x_{3N})\Big) \end{aligned}$$

where we remove the volume dependence from the integral by introducing $x_i = q_i/V^{\frac{1}{3}}$. The pressure is then given by

$$\begin{aligned} p &= kT\frac{\partial}{\partial V}\log Z_q \\ &= kT\partial_V\Big(N\log V + \log\int d^{3N}x\ \exp(-\beta V^{s/3}U)\Big) \\ &= kTv - \frac{\frac{s}{3}V^{s/3-1}\int d^{3N}x\, U\exp(-\beta V^{s/3}U)}{\int d^{3N}x\ \exp(-\beta V^{s/3}U)} \\ &= T^{1-3/s}\Big(kvT^{3/s} - \frac{s}{3}v^{1-s/3}T^{3/s-1}F(\beta V^{s/3})\Big) \end{aligned}$$

where F is some function. Now we note the following:

$$\begin{aligned} F(\beta V^{s/3}) &= F\Big((\frac{k}{N}vT^{3/s})^{-s/3}\Big) = f(vT^{3/s}) \\ v^{1-s/3}T^{3/s-1} &= (vT^{(3/s-1)/(1-s/3)})^{1-s/3} = (vT^{3/s})^{1-s/3} \end{aligned}$$

Thus we have shown that

$$p = T^{1-3/s}\varphi(vT^{3/s})$$

Suppose at T_0 there is a phase transition between states of two different densities, then there will not be a critical point due to the functional form of p - varying T (e.g. $T\to\alpha T$) is equivalent to a rescaling of the $p-V$ graph ($p\to\alpha^{1-3/s}p$, $v\to\alpha^{3/s}p$) without changing the shape of the curve. Hence there will always be such phase transition no matter the temperature.

Exercise 2.2 Let $N = a^d$. Each point has a maximum degree of $2d$. For each surface of the hypercube, we need to cut off degree of points on the surface by 1, so the total degree is $2Nd - (2d)\cdot(a^{d-1})$. Each linkage consumes 2 degrees. Thus we have

$$\begin{aligned} L/N &= \frac{2Nd - 2N^{(d-1)/d}d}{2N} \\ &= \Big(1 - \frac{1}{N^{1/d}}\Big)d \\ &\xrightarrow{N\to\infty} d \end{aligned}$$

Another method is to count the number of all linkage along a direction, say the x-direction. There is a total of $(a-1)\cdot a^{d-1}$ of such linkages. By symmetry $L = d\cdot(a-1)\cdot a^{d-1} = Nd - a^{d-1}d \to Nd$.

Exercise 2.5 Denote $\sigma(\vec{r}_i)$ by σ_i, $P(\sigma_2 = i, \sigma_1 = j)$ by P_{ij}, $P(\sigma_2 = i \,|\, \sigma_1 = j)$ by $P_{i|j}$. Note that the probilibities observe Z_2 symmetry, in particular $P(\sigma_i = 1) = P(\sigma_i = -1) = \frac{1}{2}$, $P_{ij} = P_{-i-j}$.

$$\begin{aligned}
\langle \sigma_1 \sigma_2 \rangle &= P_{11} + P_{-1-1} - P_{1-1} - P_{-11} \\
&= 2(P_{11} - P_{-11}) \\
&= (P_{1|1} - (1 - P_{1|1})) \\
P_{1|1} &= \frac{1}{2}(1 + \langle \sigma_1 \sigma_2 \rangle)
\end{aligned}$$

Exercise 3.1 The partition function is integrated over momentum, thus a constant shift in momentum can be absorbed by a change of variable. If we let $p_i' = p_i - eA$, then $d^3 p_i' = d^3 p_i$ and thus

$$\begin{aligned}
Z' &= \frac{1}{h^{3N} N!} \int d^{3N} q d^{3N} p \, \rho(p - eA, q) \\
&= \frac{1}{h^{3N} N!} \int d^{3N} q d^{3N} p' \, \rho(p', q) \\
&= Z
\end{aligned}$$

Therefore the shift in momentum will not be reflected in the partition function at all.

Even if we allow for A to depend on q the result will still be the same, since

$$\frac{\partial(p', q')}{\partial(p, q)} = \begin{pmatrix} 1 & -e\frac{\partial A}{\partial q} \\ 0 & 1 \end{pmatrix}$$

so the Jacobian will be 1 and our argument remains valid.

Exercise 3.2 We can form spin operators as tensor products:

$$S_1^i = \begin{pmatrix} \sigma^i & 0 & 0 & 0 \\ 0 & \sigma^i & 0 & 0 \\ 0 & 0 & \sigma^i & 0 \\ 0 & 0 & 0 & \sigma^i \end{pmatrix}, \; S_2^i = \begin{pmatrix} (\sigma^i)_{11} I_{2x2} & (\sigma^i)_{12} I_{2x2} & 0 & 0 \\ (\sigma^i)_{21} I_{2x2} & (\sigma^i)_{22} I_{2x2} & 0 & 0 \\ 0 & 0 & (\sigma^i)_{11} I_{2x2} & (\sigma^i)_{12} I_{2x2} \\ 0 & 0 & (\sigma^i)_{21} I_{2x2} & (\sigma^i)_{22} I_{2x2} \end{pmatrix}, \; S_3^i = \begin{pmatrix} (\sigma^i)_{11} I_{4x4} & (\sigma^i)_{12} I_{4x4} \\ (\sigma^i)_{21} I_{4x4} & (\sigma^i)_{22} I_{4x4} \end{pmatrix}$$

Using your favorite CAS, the eigenvalues can be found: (the number in bracket denotes multiplicity)

$$E/J = -0.5\,(4),\ 0\,(2),\ 1\,(2)$$

For the case of circle, they are

$$E/J = 0.75\,(4),\ -0.75\,(4)$$

Exercise 3.3

$$
\begin{aligned}
Z &= \sum_N \frac{z^N}{N!} Q_N \\
&= \sum_{N,\mathcal{C}_N} z^N e^{\beta\epsilon \sum_{\langle i,j\rangle} n_i n_j} \\
&= \sum_{\mathcal{C}} e^{(n_1+\cdots+n_{\mathcal{N}})+\beta\epsilon \sum_{\langle i,j\rangle} n_i n_j}
\end{aligned}
$$

where $\mathcal{C}_N$ denotes that we are summing over configurations with total occupation number equal to N, while the sum over $\mathcal{C}$ doesn't have such requirement.

Exercise 3.5

$$
\begin{aligned}
\langle \varphi(x)\varphi(y)\rangle &= \int d^4p\, \frac{e^{ip\cdot(x-y)}}{p^2+m^2} \\
&= \int d^3p\, dp_0\, \frac{e^{-ip_o(x_0-y_0)}}{-p_0^2+\omega^2} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= i\pi \int d^3p\, \frac{1}{\omega} e^{i\omega|t|} e^{i\boldsymbol{p}\cdot\boldsymbol{r}} \\
&= 2\pi^2 \int_0^\infty dk\, k^2 \frac{e^{ikr}-e^{-ikr}}{ikr} \frac{e^{i\omega|t|}}{\omega} \\
&= 4\pi^2 i \int dp\, p \frac{e^{i\sqrt{p^2+m^2}|t|}}{\sqrt{p^2+m^2}\, r} \sin pr
\end{aligned}
$$

Since the integral is Lorentz-invariant, we can choose $t = 0$ for spacelike intervals.

$$
\begin{aligned}
\langle \varphi(x)\varphi(y)\rangle &\sim \int dp\, \frac{p}{\sqrt{p^2+m^2}\, r} \sin pr \\
&\sim \frac{1}{r} K_1(mr)
\end{aligned}
$$

where K_1 is the modified Bessel function. For $r \to \infty$, $K_1(mr) \sim \frac{1}{\sqrt{mr}} e^{-mr}$, so we have

$$
\langle \varphi(x)\varphi(y)\rangle \sim \frac{1}{r^{3/2}} e^{-mr}
$$

Exercise 4.1