Introduction to Performance Modelling

Solutions to Past Exams

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Contents

OPEN BOOK EXAM 2023

1. Poisson Process: Consider a Poisson process with rate λ. Let T_n be the time of the n-th arrival. Consider a second arrival process and denote Z_n as the time of the n-th arrival of this second arrival process. Assume $Z_n = cT_n$ for some constant c > 0. What can you say about this second arrival process? Prove your answer. What do we know about the superposition of these two arrival processes?

Solution: The inter-arrival time of the first process $(T_n - T_{n-1})$ with mean $\frac{1}{\lambda}$ is independent of the second process $c.(T_n - T_{n-1})$, with mean $\frac{c}{\lambda}$. We can deduce that the second process is a Poisson process with a scaled rate of $\frac{\lambda}{c}$.

Furthermore, looking at the superposition of the two arrival processes, we know that the combination of two Poisson processes is just another Poisson process. In this case a Poisson process with arrival rate:

$$\lambda + \frac{\lambda}{c} = \lambda\left(1 + \frac{1}{c}\right)$$

2. Discrete-Time Markov Chains: Given a DTMC with state space S such that state i = 1 has period $d_1 = 3$, state i = 2 has period $d_2 = 5$ and there are at least 3 open communicating classes. What is the smallest possible value for $|S|$? Explain your answer. Give an example (by defining P) of a DTMC such that $|S|$ is minimized

Solution: The smallest size for a communicating class C_i with period $d_i = k$ has $|C_i| = k$. This means we need at least two cycles C_1 and C_2 with size 3 and 5 respectively, since a state can't be part of multiple communicating classes. Adding an edge from C_1 to C_2 (or vice versa) creates our first open communicating class. The smallest communicating class consists of a single state, so we'll have to add two additional states, resulting in:

$$\boxed{|S| = 10}$$

with

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3. Discrete-Time Markov Chains: Assume we flip a coin infinitely often. How many coin flips do we need on average until we flipped the sequence HTH (heads-tails-heads)? Explain your answer.

Solution: We can model this problem as a DTMC where each transition has a probability of $\frac{1}{2}$. Each state represents a step in the sequence. We now calculate the mean hitting time for $A = \{3\}$, where state 3 represents the successful observation of "HTH."

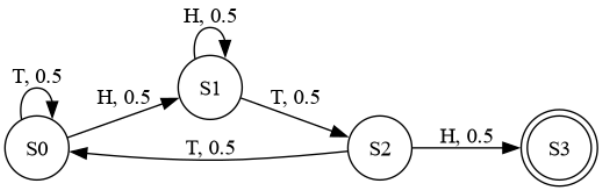

Figure 1: DTMC modelling coin flips to reach an HTH sequence

Remember: Let $\{X_n, n \geq 0\}$ be a Markov chain characterized by transition matrix P, and let $A \subseteq S$ be a subset of states. Then, the mean hitting times k_i^A of A are the smallest non-negative solution to:

$$x_i = \begin{cases} 0, & i \in A, \\ 1 + \sum_{j \notin A} p_{i,j} x_j, & i \notin A. \end{cases}$$

Define x_i as the mean hitting time (expected number of flips) to reach state 3 from state i.

- $x_3 = 0$ since $3 \in A$, which is our target state.
- For x_0 (starting from the initial state):

$$x_0 = 1 + \frac{1}{2}x_1 + \frac{1}{2}x_0$$

Rearranging, we get:

$$\frac{1}{2}x_0 = 1 + \frac{1}{2}x_1$$
$$x_0 = 2 + x_1$$

- For x_1 (starting from the state where "H" is observed):

$$x_1 = 1 + \frac{1}{2}x_2 + \frac{1}{2}x_1$$

Rearranging, we get:

$$\frac{1}{2}x_1 = 1 + \frac{1}{2}x_2$$
$$x_1 = 2 + x_2$$

- For x_2 (starting from the state where "HT" is observed):

$$x_2 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_0$$

Simplifying, since $x_3 = 0$:

$$x_2 = 1 + \frac{1}{2}x_0$$

We now solve this system of equations step-by-step:

1. Substitute $x_1 = 2 + x_2$ into $x_0 = 2 + x_1$:

$$x_0 = 4 + x_2$$

2. Substitute $x_0 = 4 + x_2$ into $x_2 = 1 + \frac{1}{2}x_0$

$$x_2 = 1 + \frac{1}{2}(4 + x_2)$$

$$\frac{1}{2}x_2 = 3$$

$$x_2 = 6$$

3. Substitute $x_2 = 6$ into $x_0 = 4 + x_2$:

$$x_0 = 4 + 6 = 10$$

Thus, the mean hitting time to reach the sequence "HTH" (state 3) from the starting state (state 0) is:

$$\boxed{x_0 = 10}$$

Therefore, on average, it takes 10 coin flips to observe the sequence "HTH" for the first time.

4. Continuous-Time Markov Chains: Consider an infinite state continuous-time Markov chain (CTMC) with state space $\{0, 1, 2, \ldots\}$ and rate matrix Q, such that for $i \neq j$ we have:

$$q_{i,j} = \begin{cases} q_{i,i+1} & \text{if } i \text{ is odd, } j = i + 1, \\ q_{i,i+3} & \text{if } i \text{ is odd, } j = i + 3, \\ q_{i,i-1} & \text{if } i > 0 \text{ is even, } j = i - 1, \\ q_{i,i-3} & \text{if } i > 0 \text{ is even, } j = i - 3, \\ q_{0,2} & \text{if } i = 0, j = 2, \\ q_{2,0} & \text{if } i = 2, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\pi_0, \pi_1, \ldots)$ be its invariant distribution. Give an explicit expression for $\frac{\pi_{13}}{\pi_0}$. Explain your answer.

5. Applications: Consider an M/M/1 queue with arrival rate λ and service rate $\mu > \lambda$. Assume a job is labelled type-1 upon arrival with probability p and type-2 otherwise. Jobs are labelled independent of each other. Instead of serving the jobs in FCFS order, type-1 jobs get priority over type-2 jobs. A type-2 job can only be in service if no type-1 jobs are present and is interrupted (and resumed later) if a type-1 job arrives. What is the mean response time of a type-1 job in such a system? Explain. Derive an expression for the mean response time of a type-2 job.

6. Applications: Consider a Jackson network with $M = 4$ queues and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$. All new arrivals either join queue 1 or 2 (with equal probability). The routing probabilities are as follows: $p_{1,3} = p_{1,4} = \frac{1}{2}$, $p_{2,3} = \frac{1}{8}$, $p_{2,4} = \frac{7}{8}$, $p_{4,0} = 1$, $p_{3,3} = p_{3,0} = \frac{1}{2}$.

State a necessary and sufficient condition such that the Markov chain of the joint queue lengths is positive recurrent. For which of these 4 queues is the input process Poisson? Derive an expression for the probability that both queue 3 and 4 are empty at the same time.

7. APPLICATIONS: In which of the following two systems is the blocking probability the largest: (1) an Erlang-B system with arrival rate λ, mean call duration $\frac{1}{\mu}$ and C lines or (2) an Engset system with $\lambda' = \frac{\lambda}{N}$, mean call duration $\frac{1}{\mu}$ and C lines? Explain.

OPEN BOOK EXAM 2022

1. Poisson Process: Consider a Poisson process with arrival rate $\lambda = 2$. Assume there is one arrival in [0, 2] and there is one (possibly the same) arrival in [1, 3]. What is the probability that we have only one arrival in [0, 3]?

Solution: We are looking for the following probability:

$$P(N_{0,3} = 1 | N_{0,2} = 1, N_{1,3} = 1)$$

This is the same as looking for these three conditions:

1. $P(N_{0,1} = 0)$
2. $P(N_{1,2} = 1)$
3. $P(N_{2,3} = 0)$

For each we can use the Poisson distribution , given us the following solutions:

$$\begin{aligned} P(N_{0,1} = 0) &= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-2} \\ P(N_{1,2} = 1) &= 2.e^{-2} \\ P(N_{2,3} = 0) &= e^{-2} \end{aligned}$$

Since each probability (in each internal) is independent from each other , we can simplify mutliply all of them with each other giving us the final solution of $2e^{-6}$.

2. Markov Chains: Consider a DTMC on the state space $S = \{(i_1, j_1, i_2, j_2) | 1 \leq i_1, j_1, i_2, j_2 \leq 8\}$, that is, the state of the DTMC is characterized by marking two squares (i_1, j_1) and (i_2, j_2) on an 8×8 board. Assume the transition probabilities are such that the state (i'_1, j'_1, i'_2, j'_2) visited from state (i_1, j_1, i_2, j_2) is such that (i'_1, j'_1) is a random neighbor of (i_1, j_1) and (i'_2, j'_2) is a random neighbor of (i_2, j_2). For instance, from state $(1, 3, 4, 4)$ we move to $(1, 4, 4, 5)$ with probability $\frac{1}{12} = \frac{1}{3} \cdot \frac{1}{4}$ as $(1, 3)$ has 3 neighbors and square $(4, 4)$ has 4 neighbors. How many communicating classes does this DTMC have? What is the period of each class?

Solution: From each square it is possible to reach any other square in a finite number of steps (irreducible). Since we can return to a square in an arbitrary number of steps, it is also aperiodic. Thus, we have a *single* communicating class with a period $d = 1$.

3. Markov Chains: Consider an irreducible, positive recurrent DTMC. Let P be its transition probability matrix and $\pi = (\pi_0, \pi_1, \ldots)$ its unique stationary distribution. Give an expression for

$$\tilde{p}_{i,j} = \lim_{n \to \infty} P[X_n = j | X_{n+1} = i],$$

in terms of the entries of P and π. Define a DTMC with transition probability matrix $\tilde{P}$ such that entry (i, j) of $\tilde{P}$ equals $\tilde{p}_{i,j}$. Is this DTMC positive recurrent? If so, what can you say about its stationary distribution?

Solution: Using Bayes' theorem we can write $\tilde{p}_{i,j}$ as

$$\tilde{p}_{i,j} = \lim_{n \to \infty} P[X_n = j | X_{n+1} = i] = \lim_{n \to \infty} \frac{P[X_{n+1} = i | X_n = j] \cdot P[X_n = j]}{P[X_{n+1} = i]}$$

Since the process is stationary, $P[X_n = j] = \pi_j$ and $P[X_{n+1} = i] = \pi_i$. Also, $P[X_{n+1} = i | X_n = j] = P_{j,i}$, where $P_{j,i}$ is the transition probability from j to i. Thus,

$$\tilde{p}_{i,j} = \frac{P_{j,i} \cdot \pi_j}{\pi_i}$$

We can now define the transition matrix $\tilde{P}$ where

$$\tilde{P}_{i,j} = \tilde{p}_{i,j} = \frac{P_{j,i} \cdot \pi_j}{\pi_i}$$

If the original DTMC was irreducible, the reversed DTMC based on it's stationary distribution will also be irreducible, therefore positive recurrent. We know that $\tilde{\pi}$ must satisfy $\tilde{\pi}\tilde{P} = \tilde{\pi}$. For a fixed j, we find

$$\tilde{\pi}_j = \sum_i \pi_i \cdot \tilde{p}_{i,j} = \sum_i \pi_i \frac{p_{j,i} \cdot \pi_j}{\pi_i} = \sum_i \pi_j \cdot p_{j,i} = \pi_j \sum_i p_{j,i} = \pi_j \cdot 1 = \pi_j$$

Thus,

$$\boxed{\tilde{\pi} = \pi}$$

4. Markov Chains: Give an example of an irreducible, transient DTMC with period 2. Explain why your example has these properties.

Solution: Define the Markov chain with $S = \{1, 2\}$ and a transition matrix P:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- **Irreducible**: Each state can be reached from the other. Single communicating class.
- **Period of 2**: For each state, any (simple) path returning to it involves exactly two steps. (Every timestep we visit state i after starting in i is divisible by two).
- **Transience**: The chain never returns to the same state within one step.

5. PASTA: Consider a queueing system with Poisson arrivals, where the service time of customer n depends on the inter-arrival time between customer $n-1$ and n. Can we apply the PASTA property for this queueing system? Explain your answer.

6. Applications: Consider a queueing network consisting of 2 queues (not necessarily M/M/1 queues). Type 1 jobs arrive at rate λ_1 and first visit queue 1 followed by queue 2. Type 2 jobs arrive at rate λ_2 and visit queue 1 only, while type 3 jobs arrive at rate λ_3 and only visit queue 2. Assume that the mean time that a random job spends in the queueing network equals 5, the mean time that a random type 1 or type 2 job spends in queue 1 is 3 and that the mean time that a random type 1 or 3 job spends in queue 2 is 4. Determine the arrival rates λ_1, λ_2 and λ_3 given that $\lambda_1 = \lambda_2$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 1$. Does your answer change if type 1 jobs visit both queues in the opposite order? Explain.

7. Applications: Consider an M/M/1 queue (arrival rate λ, service rate μ) with the additional property that when a customer arrives it starts an exponential timer with

mean $1/\theta$ and leaves the queueing system immediately if that timer expires before its service starts. How can you model this queueing system using a CTMC? Give an expression for the stationary distribution. For which value of θ is this distribution a Poisson distribution?

OPEN BOOK EXAM 2021

1. Poisson process: Let P_1 and P_2 be two independent Poisson processes with rates λ_1 and λ_2 . Let $N_{t,i}$ be the number of arrivals of process P_i in [0, t], for i = 1, 2. Show that:

$$\lambda_2^k P(N_{t,1} = N_{t,2} + k) = \lambda_1^k P(N_{t,2} = N_{t,1} + k)$$

Solution: We look at the first probability:

$$\begin{aligned} &P(N_{t,1} = N_{t,2} + k) \\ &\quad = \sum_{n=0}^{\infty} P(N_{t,1} = n + k, N_{t,2} = n) \end{aligned}$$

Since we have that $N_{t,1}$ and $N_{t,2}$ are independent , we can do the following:

$$\begin{aligned} &P(N_{t,1} = N_{t,2} + k) \\ &\quad = \sum_{n=0}^{\infty} P(N_{t,1} = n + k).P(N_{t,2} = n) \\ &\quad = \sum_{n=0}^{\infty} \left(\frac{(\lambda_1 t)^{n+k}}{(n+k)!} e^{-\lambda_1 t}\right) \cdot \left(\frac{(\lambda_2 t)^n}{n!} e^{-\lambda_2 t}\right) \end{aligned}$$

Factoring out $e^{(\lambda_1+\lambda_2)t}$, we get:

$$\begin{aligned} &P(N_{t,1} = N_{t,2} + k) \\ &\quad = e^{-(\lambda_1+\lambda_2)t} \sum_{n=0}^{\infty} \left(\frac{(\lambda_1 t)^{n+k}}{(n+k)!}\right) \cdot \left(\frac{(\lambda_2 t)^n}{n!}\right) \end{aligned}$$

For clarity , we can introduce the variable m to make the equality more clear with:

$$\begin{aligned} m &= n + k \\ n &= m - k \end{aligned}$$

This transforms the equation to:

$$\begin{aligned} &P(N_{t,1} = N_{t,2} + k) \\ &\quad = e^{-(\lambda_1+\lambda_2)t} \sum_{m=k}^{\infty} \left(\frac{(\lambda_1 t)^{m}}{(m)!}\right) \cdot \left(\frac{(\lambda_2 t)^{m-k}}{(m-k)!}\right) \end{aligned}$$

This makes it clearer to see the initial equality of :

$$\lambda_2^k P(N_{t,1} = N_{t,2} + k) = \lambda_1^k P(N_{t,2} = N_{t,1} + k)$$

2. Markov Chains: Give an example of a DTMC with 6 states, where some states have period 2 and others have period 3, while not all states are recurrent. Explain your answer. What is the maximum number of non-zero entries in P for such a chain? Explain.

3. Markov Chains: Let Y_i, for $i \geq -1$, be an infinite set of independent random variables with $P[Y_i = -1] = P[Y_i = 0] = P[Y_i = 1] = \frac{1}{3}$. Define $X_n = Y_{n-1} + Y_n$ for $n \geq 0$. Is $(X_n)_{n\geq 0}$ a DTMC? If so, give its transition matrix P; if not, explain why.

4. Markov Chains: Consider two independent Poisson processes P_1 and P_2 with rates λ_1 and λ_2. Assume we have an infinite bag, and whenever an arrival occurs of process P_1 we add a ball to the bag. If an arrival occurs of process P_2, we remove half of the balls from the bag, that is, we remove $\lceil k/2 \rceil$ balls if the bag contains k balls ($\lceil x \rceil$ rounds x to the smallest integer larger than or equal to x). Let X_t be the number of balls in the bag at time t. Argue that $(X_t)_{t\geq 0}$ is an irreducible CTMC. Given λ_1, how large must λ_2 be such that this CTMC is positive recurrent? Explain your answer.

5. Applications: Consider 2 queueing systems. The first is an M/M/1 queue with arrival rate $\lambda = 0.3$ and mean service time equal to 2. The second is a Jackson network with $M = 1$, $\mu_1 = 4$, and $p_{1,1} = \frac{1}{3}$. How should we set λ_0 such that the queue length distribution is the same in both queueing systems? Explain.

6. Applications: Consider an Erlang-C system with C servers, i.e. an M/M/C/C+Q queue with $Q = \infty$. Assume we know that on average k jobs are waiting in the waiting room to enter a server. Does the mean time that a customer spends in the system depend on C, k, or both C and k? Explain.

OPEN BOOK EXAM 2020

1. SPAM MAIL: Suppose spam mail arrives in your mailbox as a Poisson process with rate λ_s, regular mail arrives as a Poisson process with rate λ_r and both processes are independent. (1) What is the mean time between any two mails? (2) What is the probability that exactly 5 out of 10 consecutive mails are spam mails?

Solution: (1) Since spam and regular mails arrive as two independent Poisson processes, with rates λ_s and λ_r respectively, the combined arrival process (any type of mail) is also a Poisson process with rate:

$$\lambda = \lambda_s + \lambda_r$$

This gives us a mean inter-arrival time of:

$$\frac{1}{\lambda} = \frac{1}{\lambda_s + \lambda_r}$$

(2) The probability that any given mail is spam is given with the following distribution:

$$p_{spam} = \frac{\lambda_s}{\lambda_s + \lambda_r}$$

Using binomial distribution we can deduce given=10 (total mails) and a probability p_spam of being spam, the probability that exactly k=5 of these mails are spam is:

$$p(5, 10) = \binom{10}{5} \left(\frac{\lambda_s}{\lambda_s + \lambda_5}\right)^5 \left(1 - \frac{\lambda_s}{\lambda_s + \lambda_5}\right)^{10}$$

2. DISCRETE TIME MARKOV CHAINS: Let d_i be the period of state i in a DTMC. Give an example of a DTMC with 7 states (by defining P) such that $(d_1, d_2, \ldots, d_7) = (3, 3, 2, 1, 3, 2, 3)$ and exactly 2 states are recurrent. Explain your answer.

3. DISCRETE TIME MARKOV CHAINS: Consider a DTMC with state space $\mathbb{Z}$. Let $p_{0,1} = \frac{1}{2}$, $p_{i,i+1} = e^{-i/100}$ for $i > 0$, and $p_{i,i-1} = e^{i/100}$ for $i < 0$. Further, let $p_{i,i+1} = 1 - p_{i,i-1}$ for $i \in \mathbb{Z}$. Is this DTMC positive recurrent, null recurrent, or transient? Prove your answer.

4. REPAIRING SERVERS: Consider two machines maintained by a single repairman. Machine i, for $i = 1, 2$, operates for an exponentially distributed amount of time with mean $1/\gamma_i$ before breaking down. The repair time for machine i is also exponential with mean $1/\beta_i$, but the repairman can only work on one machine at a time. Assume that machines are repaired in the order in which they fail. Set up a CTMC to find the long-run proportions of time that each machine is working (no need to compute these values). How does your answer change if the repairman always works on machine 1 first if it fails?

5. SERVER ORDER: Assume jobs arrive in a system with 2 servers as a Poisson process with rate λ. All jobs must be served by both servers, and the service time of a job in server i is exponential with mean $1/\mu_i$, for $i = 1, 2$. Assume a fraction p of the incoming jobs first joins server 1 and then proceeds to server 2, while the remaining fraction $1 - p$

visits both servers in the reversed order. Can you use a Jackson network to determine the optimal choice of p such that the time that a job spends in the system is minimized?

6. M/M/1 WITH GROUP ARRIVALS: Consider an M/M/1 queue with arrival rate λ and service rate μ, but instead of having one arrival at a time, arrivals occur in groups of 2. Set up a CTMC to study this queueing model. Write down the global balance equations and use them to prove that

$$\lambda\pi_{i-2} + \lambda\pi_{i-1} = \mu\pi_i,$$

holds for $i \geq 2$, where π_i represents the stationary probability that the queue contains i jobs.

7. BLOCKING TIME: Consider an Erlang-B system (i.e., an M/M/C/C queue). Such a system alternates between periods where incoming calls are blocked and periods where calls are accepted. What is the mean duration of a period during which calls are blocked? How can you determine the mean duration of a period during which calls are accepted?

OPEN BOOK EXAM 2019

1. Poisson Process: Consider a Poisson process with rate λ. Assume n arrivals occurred in between time 0 and t. Give an expression for the probability that no arrivals occur between time $t/3$ and $2t$? Explain how you obtained your expression.

Solution: We first calculate the interval,we are interested in:

$$\begin{aligned} < t > &= 2t - \frac{1}{3}t \\ &= \frac{5}{3}t \end{aligned}$$

The probability of having no arrivals in a specific interval can be written as:

$$P(N(t) = k) = \frac{\lambda < t > e^{-\lambda < t >}}{k!}$$

In this case we have k=0 and $< t >= \frac{5}{3}$. This gives us the following expression:

$$\begin{aligned} P(N(t) = 0) &= \left(\frac{\lambda \frac{5}{3} t}{0!} e^{-\lambda 5/3t} \right) \\ &= e^{5/3t\lambda} \end{aligned}$$

2. Discrete Time Markov Chains: What is the least number of non-zero entries in the transition probability matrix P of a discrete time irreducible aperiodic Markov chain with $n > 2$ states? Does your answer change if we demand that the chain has period 2? Explain your answers.

3. Continuous Time Markov Chains: Consider a continuous time Markov chain on the state space $S = \{(i, j) | i \geq 0, j \geq 0\}$ and assume the transition rates are such that

$$q_{(i,j),(i+1,j-1)} = 2, \text{ for } j > 0, \quad q_{(i,0),(0,i-1)} = 2, \text{ for } i > 0,$$

$$q_{(i,j),(i-1,j+1)} = 1, \text{ for } i > 0, \quad q_{(0,j),(j+1,0)} = 1.$$

Prove that this Markov chain is positive recurrent.

4. You've Got Mail: Bob receives both regular mail and spam mail. An incoming spam mail is detected as spam with probability p and moved to Bob's spam folder. Undetected spam mails are collected in Bob's regular mail folder together with all of his regular mails. Bob has on average 4 unopened mails in his regular mail folder and 80 mails in his spam folder. Bob cleans his spam folder every 5 days and opens all of his unopened regular mail every 3 hours (on average). Which fraction p of Bob's incoming spam mail is certainly detected as spam?

5. Queueing Theory: Consider a queue with Poisson arrivals with rate 1 and exponential service times with mean 1/2. Assume jobs are served first-come-first-serve and at most N jobs can wait in the waiting room. Incoming jobs that find the waiting room full are lost (i.e., immediately leave). Show that the loss probability in this system equals $\frac{1}{2N+1-1}$.

6. Queueing Networks: Consider a system consisting of 3 queues (labeled 1, 2, and 3) and assume we have N jobs in the system. A job requires an exponential amount of service with mean $1/\mu_i$ in queue i and moves to queue $(i \mod 3)+1$ afterwards. Thus we have a fixed number of N jobs in the system. The service discipline is first-come-first-serve in each queue. Show that the probability to have i jobs in queue 1 and j jobs in queue 2 is given by the following product form

$$\frac{\frac{1}{\mu_1^i \mu_2^j \mu_3^{N-i-j}}}{\sum_{i=0}^{N} \sum_{j=0}^{N-i} \frac{1}{\mu_1^i \mu_2^j \mu_3^{N-i-j}}}.$$

7. Bianchi Model: Discuss the changes required to the 802.11 Bianchi model if we replace the uniform backoff time by some other distribution on $\{0, \ldots, W_i - 1\}$, for $i = 1, \ldots, m$. More specifically, indicate the changes required in the Markov chain used to determine the throughput. Why does the uniform distribution seem like a sensible choice?

OPEN BOOK EXAM 2018

1. Poisson Process: Consider a Poisson process with rate $\lambda = 2$. Assume 4 arrivals occurred in between time 0 and 1. What is the probability that you observe another 4 arrivals between time 1 and 2? Explain your answer.

Solution: We have: $\lambda = 2$. Using the following formula for Poisson distribution:

$$\begin{aligned} P(N = 4) &= \lambda^4 \frac{e^{-\lambda}}{k!} \\ &\approx 0.0902 \end{aligned}$$

This gives a 0.0902 chance of observing another 4 arrivals between 1 and 2.

2. Discrete Time Markov chains: Give an example of an irreducible Markov chain with 5 states that has period 2. Explain your answer.

3. Discrete Time Markov Chains: Consider a discrete-time positive recurrent Markov chain and let $m_{i,j}$ be the mean number of steps to reach state j from state i. Prove or disprove that $m_{j,j} \leq m_{i,j} + m_{j,i}$.

4. Continuous Time Markov Chains: Give an example of an infinite state, transient continuous time Markov chain such that its uniformized and embedded Markov chain are characterized by the same transition probability matrix. Explain your answer.

5. Server Farm: Consider a server farm consisting of 2 servers. Jobs arrive according to a Poisson process with rate λ and are probabilistically split among the two servers with a fraction p of the jobs going to server 1. The service time of a job on server i is exponential with mean $1/\mu_i$, for $i = 1, 2$. Give an expression for the mean response time of a job in the server farm. How does your result change if the service times are not exponential but have the same means?

6. Little's Law and PASTA: Consider a system that allows at most 120 jobs in the system. Jobs arrive as a Poisson process with rate 10 and are dropped when there are already 120 jobs in the system. If you know that the steady state probability of having 120 jobs in this system is 0.05 and the mean number of jobs in the system is 76, then what is the mean response time of a job that enters the system (that is, of a job that is not dropped)?

7. Wireless Networks: Consider a cell in a wireless network that can support up to C simultaneous calls. Assume the mean time until a call ends or leaves the cell is exponential with parameter μ, while new calls arrive at rate λ_n and handover calls at rate λ_h. Assume we have $g \geq 1$ guard channels, meaning handover calls are blocked when all the C channels are busy, while new calls are also blocked when there are $C - g$ or more busy channels. Indicate how to adapt the Markov chain of the Erlang-B loss system. Derive an expression for the steady state probabilities.

OPEN BOOK EXAM 2017

1. Poisson Process: Assume requests arrive at a web server farm according to a Poisson process with a rate of 106 requests per hour. Assume 1.3 million requests arrived in the last hour. What is the probability that at least 650 thousand requests arrived in the last half hour?

Solution: We calculate that arrival rate per half hour:

$$\lambda = \frac{10^6}{2}$$
$$= 500000$$

We are looking for $P(X \geq 650000)$.We can write this as:

$$P(X \geq 650000) = 1 - P(X < 650000)$$
$$= 1 - P(X \leq 649999)$$

Looking fist for $P(X \leq 649999)$, we have:

$$P(X \leq 649999) = \sum_i P(X = i)$$
$$= \frac{\lambda^i e^{-\lambda}}{i!}$$

Since we are summing up to a very large number (i: $1 \longrightarrow 649999$), we can approximate this as 0, leaving us with a 100% change the probability that least 650 thousand requests arrived in the last half hour.

2. Discrete Time Markov Chains: Assume we have m white and m black balls that are randomly distributed over 2 bins such that each bin contains exactly m balls. Next, assume we repeatedly pick one ball from each bin at random and exchange these two balls. Explain how you can model the content of these bins using a discrete time Markov chain (give the state space and transition probabilities). Is your chain irreducible and aperiodic? Derive an expression for the steady state probabilities of this Markov chain (if they exist).

3. Discrete Time Markov Chains: Give an example of an irreducible DTMC such that its invariant distribution equals $(1/10, 1/10, 1/10, 1/10, 1/5, 1/5, 1/5)$.

4. Continuous Time Markov Chains: Give an example of an irreducible transient CTMC with period 2. Explain why the chain is transient and has period 2.

5. Continuous Time Markov Chains: Consider an $M/D/1$ queue, that is, a queue with Poisson arrivals with rate λ, an infinite waiting room, one server, and the service time of a customer equals one. Let X_t denote the number of customers in the system at time $t \geq 0$. Is $(X_t)_{t\geq 0}$ a continuous time Markov chain? (Explain your answer.)

6. Applications: Assume consultant Frank receives on average 120 assignments per year and on average he has 4 unfinished assignments. How many days does it on average

take before Frank completes an assignment if he processes his assignments in first-come-first-served order? Does your answer change if Frank uses a different order to process his assignments, which also results in an average of 4 unfinished assignments (explain your answer)? What is the optimal strategy that minimizes the average completion time of an assignment (no proof needed)?

7. Applications: Consider a Jackson network with $M = 2$ queues. Let $p_{0,1} = 1$, $p_{1,1} = 1/4$, $p_{1,2} = 3/4$, $p_{2,1} = 1/2$, and $p_{2,0} = 1/2$. Determine the arrival rates λ_1 and λ_2. What is the average response time of a job in the network if $\mu_1 = 3$ and $\mu_2 = 4$? What is the largest value of λ_0 for which the system is stable, i.e., the average response time is finite, when $\mu_1 = 3$ and $\mu_2 = 4$?

8. Applications: Is it possible to adapt Bianchi's model for the 802.11 network in case we only double the backoff window after every 2 failed transmissions? If so, briefly describe the changes necessary without going into too much detail.

OPEN BOOK EXAM 2016

1. Bernoulli and Poisson Process: Indicate whether the following statements are true or false. Explain your answer. (a) If some number was not part of the 6 lottery numbers during the last 10 weeks, it is more likely to be part of the six numbers of this week as all numbers appear equally often in the long run (due to the law of the large numbers), (b) Points in time at which queries arrive at a DNS server, which translates domain names in IP addresses, is fairly well approximated by a Poisson process, (c) the time epochs at which students log on to Blackboard for a specific course.

Solution: (a) False. In lottery process the events are **independent** of each other. This means the results of the last 10 weeks don't affect the probability of a number being drawn in the next lotteries)
(b) True.

1. Events (such as DNS queries) occur independently
2. There is a constant average rate of occurrences over time.
3. The inter-arrival times between events are exponentially distributed.

2. Branching Processes: Determine the extinction probabilities q_1, q_2, and q_3 for the following multi-type branching process, where $p^{(s)}_{i,j,k}$ is the probability that a type s individual has i children of type 1, j of type 2, and k of type 3:

- $p^{(1)}_{1,0,4} = 0.5$, $p^{(1)}_{0,1,3} = 0.25$, and $p^{(1)}_{0,2,1} = 0.25$.
- $p^{(2)}_{0,0,3} = 0.2$, $p^{(2)}_{0,1,2} = 0.4$, and $p^{(2)}_{0,2,17645} = 0.4$.
- $p^{(3)}_{0,0,k} = \binom{16}{k}\left(\frac{1}{20}\right)^k\left(1-\frac{1}{20}\right)^{16-k}$, for $0 \le k \le 16$.

3. Markov Chains: Consider an irreducible, finite Markov chain with transition matrix $P = P_w + P_b$, with P_w and P_b substochastic as in Exercise 17 in the course notes. If we only observe this Markov chain after the white transitions, we obtain a new Markov chain.

(a) Show by means of an example that the new Markov chain is not necessarily irreducible.

(b) Is it possible that this new Markov chain contains two (or more) closed communicating classes?

4. Bianchi Model: How would you proceed to determine the saturation throughput of the 802.11 DCF function if we replace the binary exponential back-off algorithm with a simple ALOHA scheme (i.e., uniform back-off between 0 and W)? What are the required changes to the Bianchi model? Give an explicit expression for the failure probability p.

5. Markov Chains: Suppose customers arrive at an ATM machine according to a Poisson process with rate λ, and their transactions have a constant duration (e.g., 1 minute). Set up a Markov chain to determine the probability that a customer finds exactly

i customers queued at the ATM machine at the time of their arrival. [Hint: the probability that there are i customers queued at an arrival time is identical to the probability that there are i customers queued at a departure time (i.e., when a customer completes their transaction).]

6. Bernoulli and Poisson Processes: Suppose we split a Poisson process in a probabilistic manner into process A and B and that process A is again split in a probabilistic manner into processes A_1 and A_2. What is the mean arrival rate of the superposition of processes A_2 and B? Is this superposition also a Poisson process?

7. Erlang B Formula: Suppose that a telecom operator has both premium and regular customers, and whenever C_p or more of the available C lines are occupied (with $C_p < C$), only premium customer calls are accepted. Adapt the Markov chain in the course notes to incorporate this policy (it is not necessary to determine an expression for the steady state probabilities). How would you determine the blocking probabilities from the steady state probability vector (detailed formulas are not required)?

OPEN BOOK EXAM 2015a

1. Bernoulli and Poisson process: Indicate whether the following two processes are approximately Poisson. Explain your answer.

(a) The times at which people arrive at the US airport immigration.

(b) The points in time at which an item is ordered at `amazon.com`.

Solution:

(a) Not Poisson: The number of people arriving at the US airport immigrations are clustered and dependent on the flight schedule. This makes them **dependent** and therefore not suitable for Poisson distribution.

(b) This scenario may approximate a Poisson process under certain conditions. That is assuming a high volume and independence rate of orders. These assumptions could work to approximate this process as Amazon experiences a large volume of independent customer orders. While certain times (like holidays or sales events) might see spikes, during normal hours, individual orders from different users might be close to independent.

2. Discrete Time Markov Chains: Give an example of a discrete time Markov chain that is (explain your answer):

(a) Positive recurrent and its unique steady state distribution π is equal to $\pi = \frac{(1,2,4,8,16)}{31}$.

(b) Reducible with both transient and recurrent states.

3. Discrete Time Markov Chains: You toss a die repeatedly until the product of the last two outcomes is equal to 12. What is the average number of times you toss your die?

4. Continuous Time Markov Chains: Consider the continuous time Markov chain of the queue length process of an M/M/1 queue. Construct the transition probability matrix of both the uniformized and embedded Markov chain. Do both these chains have the same steady state distribution whenever $\rho < 1$?

5. Applications: Assume Sue receives on average 50 emails per day and on average she has 12 unread emails in her mailbox. How many hours does it on average take before Sue has read an incoming mail? An email is marked as read as soon as Sue opens it.

6. Applications: Consider a tandem queueing network, that is, a Jackson network with $M = 2$ queues with $p_{1,2} = 1$, $p_{2,0} = 1$, $p_{0,1} = 1$, and $p_{0,2} = 0$. Explain how the speed μ_1 of the first server affects the mean queue length and mean waiting time at the second queue.

7. Bianchi Model: Discuss the changes required to the 802.11 Bianchi model if we replace the uniform backoff time with a geometric backoff time. More specifically, indicate the changes required in the state space of the Markov chain used to determine the throughput.

OPEN BOOK EXAM 2015b

1. Bernoulli and Poisson process: Indicate whether the following two processes are Poisson. Explain your answer.

(a) Points in time at which cars pass by on a road if the time between two cars has a uniform distribution with a mean of 27 seconds.

(b) Points in time at which spam mail arrives in your mailbox.

Solution:

(a) This cannot be Poisson since a Poisson process requires the inter-arrival times to follow an exponential distribution and the problem indicates a uniform distribution.

(b) Assuming the spam emails arrive independently and at a relatively constant rate, then this process could be modelled as an approximate Poisson process.

OPEN BOOK EXAM 2014

1. Poisson Process: Consider a set of n hard disk drives and assume the lifetime of a drive is exponential with parameter μ. When a disk crashes it is not replaced. How long will it take on average before all of the n disks have crashed?

Solution: The expectant time (T) for all n disks to fail can be written as follows:

$$\begin{aligned} T &= \frac{1}{n\mu} + \frac{1}{(n-1)\mu} + ... + \frac{1}{\mu} \\ &= \frac{1}{\mu}(\frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{n}) \\ &= \frac{1}{\mu}\sum_{k=1}^{n}\frac{1}{k} \end{aligned}$$

Where we recognize the sum as the n-th harmonic number, where for n going to infinity (see Wikipedia) is equal to :

$$\gamma + \ln n$$

This gives us:

$$T = \frac{1}{\mu}(\gamma + \ln n)$$

OPEN BOOK EXAM 2013

1. Bernoulli and Poisson process: Indicate whether the Poisson process is a suitable process to model the time epochs at which:

(a) orders are placed at `amazon.com`;

(b) people buy online tickets for a rock concert;

(c) queries arrive at a DNS server;

(d) handovers occur in a wireless network.

Solution:

(a) Suitable. The arrival of orders can be modelled with a Poisson process for many practical applications, especially when looking at a sufficiently large time interval.

(b) Not suitable. The purchase of tickets can vary significantly based on several factors such as the popularity of the concert, timing of sales, and marketing activities. This process will not have a constant average rate of ticket buying making it unsuitable for a Poisson process.

(c) Suitable. DNS queries typically arrive randomly and independently, especially during regular operation when users access websites at varying times. Under normal conditions, they can be assumed to occur at a constant average rate.

(d) Not suitable. Handovers can depend heavily on user mobility patterns, signal conditions, and network configurations. These factors often make handover events correlated rather than independent, with variations in rate under different conditions.

2. Bernoulli and Poisson process: Assume that the time it takes before a new computer gets infected by a virus is exponentially distributed and that about 20% get infected within the first 6 months. What is the probability that a new computer remains uninfected during the first 3 years (assuming all computers are equally vulnerable)?

Solution: The exponential distribution function is:

$$f(t) = \lambda e^{-\lambda t}$$

The probability that a computer is infected by time (t), is given by:

$$P(T \leq t) = 1 - e^{-\lambda t}$$

The probability that a computer remains uninfected until time (t) is:

$$P(T > t) = e^{-\lambda t}$$

We know from the problem statement that about 20% of computers get infected within the first 6 months, implying:

$$P(T \leq 0.5) = 0.20$$

This can be written as:

$$\begin{aligned} 1 - e^{-\lambda \cdot 0.5} &= 0.20 \\ \iff e^{-\lambda \cdot 0.5} &= 0.80 \\ \iff -\lambda \cdot 0.5 &= \ln(0.80) \\ \iff \lambda &= -\frac{2\ln(0.80)}{1} \end{aligned}$$

with $\ln(0.80) \approx -0.2231$ This gives:

$$\lambda \approx -\frac{2 \cdot (-0.2231)}{1} \approx 0.4462$$

Now that we have the value of (λ), we can find the probability that a new computer remains uninfected during the first 3 years (which is equal to (t = 3) years): Using the survival function:

$$P(T > 3) = e^{-\lambda \cdot 3}$$

Substituting the value of (λ):

$$P(T > 3) = e^{-0.4462 \cdot 3} = e^{-1.3386}$$

Calculating ($e^{-1.3386}$):

$$e^{-1.3386} \approx 0.2610$$

This gives us a probability that a new computer remains uninfected during the first 3 years is approximately 26.1%.

OPEN BOOK EXAM 2012a

1. Bernoulli and Poisson process: Indicate whether the Poisson process is a suitable process to model:

(a) The time epochs at which hand-overs take place in a wireless network.

(b) The time epochs at which a packet is sent by a single TCP source.

(c) The time epochs at which packets are sent by a large number of TCP sources sharing a single link.

(d) The time epochs at which emails arrive in your mailbox.

Solution:

(a) Not suitable. The handover times are often dependent on different factors such as a user going from one place to the other or network load. This process can therefore not be modelled with a Poisson process.

(b) Not suitable. A single TCP source transmits packets according to its congestion control and flow control mechanisms. This means that the packet transmission could be affected by network conditions, previous congestion, and other factors, which can make the time between packet transmissions variable and dependent on these conditions

(c) Suitable. In this case, while the individual packet transmissions of TCP sources may not follow a Poisson process, when considering a large number of sources working under varying loads, the aggregated packet arrivals can often approximate a Poisson process, especially if the number of sources is large and they are independently sending packets.

(d) Suitable. Email arrivals can often be viewed as independent events, and while they may vary throughout the day, the overall arrival rate can be relatively constant over long periods (in a general sense). Factors like promotions or all at once notifications can cause bursts, but on average, it can often be approximated as arriving at a constant average rate.

2. Branching Processes: Give an example of a single-type branching process such that the extinction probability is equal to p, for any $0 < p < 1$.

3. Branching Processes: Construct a branching process to model the coordinated splitting tree algorithm (CSTA). Whenever n users collide in the CSTA, they are allowed to exchange information such that they split into n groups, each consisting of a single user. Collisions still occur on the channel as colliding users get no information with respect to the arrival times of the new users. Explain how we can determine the maximum stable throughput by means of this branching process.

4. Markov Chains: Given an example of a Markov chain that is:

(a) Irreducible, positive recurrent with period $d = 3$.

(b) Irreducible, transient, with period $d = 2$.

Explain in detail why your examples meet the above requirements.

5. Markov Chains: Assume we have 2 machines to process jobs. Machine i needs to process k_i jobs, and the processing time of a job on machine i is exponential with mean $1/\mu_i$, respectively, for $i = 1$ and 2. How can we determine the probability that machine 1 finishes its k_1 jobs before machine 2 finishes its k_2 jobs (use a Markov chain)? Assume that as soon as one machine finishes its work, it also starts to process pending jobs from the other machine (if any). How can we determine the mean time until all the $k_1 + k_2$ jobs are completed?

6. Processor Sharing Queue: Assume jobs arrive according to a Poisson process with rate λ at a single server. A job requires an exponential amount of work with mean $1/\mu$. All the jobs in the queue share the single server, meaning a job is served at rate μ/n whenever there are n jobs in the queue. Set up a Markov chain to determine the number of jobs in the queue (at arrival or departure times). Determine the condition for positive recurrence and give an explicit expression for the steady state probabilities. Give an expression for the queue length distribution at arrival times only. Do you recognize the latter expression from an earlier course?

7. Bianchi Model: Assume we use the simple ALOHA scheme to retransmit a packet instead of the binary exponential back-off algorithm, with the additional requirement that a packet can be retransmitted at most $m > 0$ times. Discuss the changes required to the Bianchi model to determine the saturation throughput. How does m influence the saturation throughput?

OPEN BOOK EXAM 2012b

1. Bernoulli and Poisson process: Indicate whether either the Bernoulli or Poisson process is a suitable process to model:

(a) The time epochs at which search queries are performed at `www.google.be`.

(b) The days of the year during which it rains in Antwerp city

Solution:

(a) Search queries happen randomly and frequently over time. They are generally independent of one another, without a predictable interval, which makes a Poisson process suitable here. The Poisson process effectively models the random occurrences of events (search queries) over continuous time intervals.

(b) Rain on a particular day is more accurately a binary event (rain: yes or no) and can be observed at fixed daily intervals. This makes a Bernoulli process suitable if we consider each day independently and assign a probability p to the occurrence of rain. We can view each day as an independent trial with a probability of rain, which aligns with the structure of a Bernoulli process.

2. Bernoulli and Poisson process: Consider a (RAID) system consisting of 80 hard disk drives. Assume that the life time of a disk has an exponential distribution with a mean of 1200 days. If a disk fails it is immediately replaced by another new disk. On average how many disks need replacement per month? Explain your answer using the properties of the exponential distribution and the Poisson process.

Solution: Disk failures can be modelled as a Poisson process, where failures are independent events occurring at a constant average rate. The Poisson's mean rate can be calculated as follows:

$$\begin{aligned} \lambda &= \frac{1}{mean} \\ &= \frac{1}{1200} \end{aligned}$$

The total failure rate (for 80 disks) per month (30) is:

$$80 \cdot 1200 \cdot 30 \approx 2$$

OPEN BOOK EXAM 2010a

1. Bernoulli and Poisson Processes: Indicate whether the following processes can be well modelled by a Bernoulli or Poisson process:

(a) The process that indicates when a read or write request is generated when modelling a hard disk.

(b) The time points at which buses arrive at a bus stop.

(c) The time points at which students log in to Blackboard for a specific course.

Solution:

(a) This process can be described as a Poisson process as reading and writing from a disc is continuous and can occur at any point in time.

(b) Assuming the bus follows a regular schedule (This scenario is obviously not realistic in Belgium), this process can be described as a Bernoulli process. Each time interval we have a possibility p that the bus will arrive and the arrival of the bus in one interval is independent of the arrival of the bus in the next interval.

(c) We may observe that more students will login before the exam of a course or a specific time where the course is given in class. However, students can login at any point in time. This is a continuos process and can therefore be described as a Poisson process.

2. Branching Processes: Determine the extinction probabilities q_1, q_2, and q_3 for the following multi-type branching process, where $p(s)_{i,j,k}$ is the probability that a type s individual has i children of type 1, j of type 2, and k of type 3:

- $p(1)_{1,0,4} = 0.5$, $p(1)_{0,1,3} = 0.25$, and $p(1)_{0,2,1} = 0.25$.
- $p(2)_{0,0,3} = 0.2$, $p(2)_{0,1,2} = 0.4$, and $p(2)_{0,2,17645} = 0.4$.
- $p(3)_{0,0,k} = \binom{16}{k}\left(\frac{1}{20}\right)^k\left(1-\frac{1}{20}\right)^{16-k}$ for $0 \leq k \leq 16$.

3. Markov Chains: Consider an irreducible, finite Markov chain with transition matrix $P = P_w + P_b$, where P_w and P_b are sub-stochastic as in exercise 17 of the course. When we observe this Markov chain only after a white transition, we obtain a new Markov chain.

(a) Show with an example that this new chain is not necessarily irreducible.

(b) Can this Markov chain contain 2 or more closed (communicating) classes?

4. Bianchi Model: Assume we want to determine the saturation throughput of the 802.11 DCF function in the case where we replace the binary exponential back-off algorithm with simple ALOHA (i.e., uniform back-off between 0 and W). How do we

proceed (what do we need to adjust)? Provide an explicit expression for p, the probability that the transmission fails.

5. Markov Chains: Assume that customers arrive at a payment machine according to a Poisson process with rate λ, and that they perform their payment in a constant time (e.g., 1 minute). Set up a Markov chain that allows us to determine the probability that a customer i has customers in front of them when they arrive at the payment machine. [Hint: The probability that a customer i has other customers in front of them upon arrival is equal to the probability that there are i customers waiting when a customer has finished their payment (or, the queue length distribution at arrival times is equal to that at departure times).]

6. Bernoulli and Poisson Processes: Suppose I probabilistically split a Poisson process into processes A and B, and then again probabilistically split A into A_1 and A_2. What is the average arrival rate of the superposition of A_2 and B, and is this also a Poisson process?

7. Erlang B Formula: Suppose a telecom operator divides its customers into premium and regular customers and applies the rule that when there are C_p or more of the C lines occupied when initiating a phone call, only calls from premium customers are accepted. Adjust the Markov chain to model this behaviour; you do not need to derive the steady state expression. How do you now calculate the loss probabilities (formulas are not needed)?

OPEN BOOK EXAM 2010b

1. Bernoulli and Poisson Processes: Argue that the minimum spanning tree (MST) of the basic binary tree algorithm does not change when we implement the following adjustment. A user who creates a new packet will send it in the next slot with probability a and in the subsequent slot with probability $1-a$.

2. Branching Processes: Suppose we use the basic binary tree algorithm on a channel with multiple reception capabilities. On such a channel, we can also correctly receive packets involved in a collision consisting of k or fewer packets. How should we adjust the branching process from the course to determine the MST (as a function of k)? What should we do with d when we increase k? How can we immediately see that $p = \frac{1}{2}$ is still an optimal choice?

3. Markov Chains: Consider an irreducible Markov chain with a finite number of states n. For which values of n can the Markov chain be periodic with a period equal to 5? Explain your answer.

4. Markov Chains: Assume that people arrive at an attraction in an amusement park according to a Poisson process with rate λ people per minute. Assume that a cart arrives every 90 seconds, accommodating (and requiring) 6 people. At which time points should we observe this system to obtain a Markov chain? What is the state space of the Markov chain, and formulate the transition matrix. For which values of λ is your chain positive recurrent? What do you expect to happen to the average waiting time when a cart arrives every minute that accommodates only 4 people?

5. Erlang C Formula: Suppose we wish to generalize the Erlang C formula by also accounting for the impatience of waiting customers. We assume that the amount of patience of a waiting customer is exponentially distributed with a mean of $1/\theta$. We construct a Markov chain with the same state space as in the Erlang C formula, but we now observe the chain not only at arrival and service completion time points but also at the times when waiting customers lose their patience and leave. In short, when there is one or more waiting customers, there are three possibilities for the next event: an arrival (λ), a service completion (μ), or a customer who loses their patience (θ).

(a) Provide the transition matrix for this system; you do not need to determine the invariant vector.

(b) When is this Markov chain positive recurrent (use Pakes)?

6. Bianchi Model: Assume that the maximum window size is 8 times greater than the minimum window size. In the course, a packet can be retransmitted an unlimited number of times; suppose we want to limit this number to 16. How should we adjust the Markov chain (without going into too much detail)?