Introduction to Performance Modelling

Solutions to Past Exams

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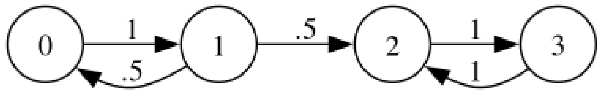
Contents

OPEN BOOK EXAM 2024

1. Poisson Process: Suppose we construct an arrival process from a Poisson process by shifting each arrival by Δ_i to the right with probability p_i, for $i = 1, \ldots, 10$. Why is the constructed arrival process a Poisson process from time $t = \max_i \Delta_i$ onwards?

2. Discrete-Time Markov Chains: Give an example of a Markov chain where all the states have period 2 with at least one transient state and at least one recurrent state.

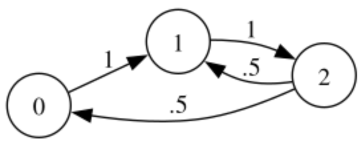
Solution:



3. Discrete-Time Markov Chains: Consider a single roll of 5 dice. Set up a DTMC that allows you to compute the probability of having a full house, meaning the result of the roll is three dice with one number and two dice with another number (5 identical numbers do not count). Try to minimize the number of states in your DTMC. There is no need to compute the actual probability.

4. Discrete-Time Markov Chains: Let $(X_n)_{n\geq 0}$ be an irreducible aperiodic DTMC. Let the $(Y_n)_{n\geq 0}$ be the DTMC obtained by censoring out state 0 of $(X_n)_{n\geq 0}$. In other words the DTMC $(Y_n)_{n\geq 0}$ is identical to $(X_n)_{n\geq 0}$ except that the points in time where state 0 is visited are skipped. Is $(Y_n)_{n\geq 0}$ necessarily aperiodic? Prove this or give a counter example.

Solution: When removing state 0, we find a periodic DTMC.



5. Continuous-Time Markov Chains: Give an example of a positive recurrent CTMC $(X_t)_{t\geq 0}$ with an infinite number of states such that $\lim_{i\to\infty} D_i = 0$, where D_i is the drift of the embedded Markov chain $(Y_n)_{n\geq 0}$ in state i, that is, $D_i = E[Y_{n+1} - Y_n | Y_n = i]$.

Solution: Consider the DTMC with $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$. We calculate the drift

$$D_i = (-1)p_{i,i-1} + 1 \cdot p_{i,i+1} = 0$$

The corresponding rate matrix has entries $q_{i,i-1} = q_{i,i+1} = 1$ and $q_{i,i} = -2$. (Except for state 0, where $q_{0,0} = -1$ and $q_{0,1} = 1$)

6. Applications: Consider an M/M/C queue. Suppose we increase the arrival rate and the rate of each of the C servers by a factor k. Does this affect the queue length

distribution? Explain. What about the mean response time? [Hint: You can answer this last question using a simple formula].

Solution: The queue length distribution in an M/M/C system depends on the traffic intensity $\rho = \frac{\lambda}{C\mu}$. If both the arrival rate λ and the service rate μ are scaled by a factor k, the new traffic intensity becomes:

$$\rho' = \frac{k\lambda}{C \cdot k\mu} = \rho.$$

Since ρ remains unchanged, the queue length distribution is unaffected.

For the mean response time W, which can be expressed as the sum of the mean waiting time in the queue and the mean service time:

$$W = W_q + \frac{1}{\mu},$$

scaling μ by a factor k reduces the mean service time to $\frac{1}{k\mu}$. Therefore, while the waiting time W_q remains unchanged (as it depends on ρ), the mean response time decreases.

Intuitively, this result makes sense: the system processes jobs faster due to the higher service rate, but the balance of arrivals and departures remains the same, preserving the queue's overall behavior.

7. Applications: Consider a Jackson network in which each of the M queues is replaced by an infinite number of servers, where the speed of a single server is μ_i for queue i. Assume the routing is such that $(I-P)^{-1}$ exists. Consider the CTMC that keeps track of the number of busy servers in each queue. Derive the global balance equations for this CTMC. Clearly explain the different terms appearing in your equations. What is the condition needed for the Markov chain to be positive recurrent? Give an expression for the stationary distribution (without actually proving it)

Solution: We can write the global balance equation as

$$\begin{aligned}\lambda_0\pi(\mathbf{n}) + \sum_{i=1}^{M} p_{i0}\pi(\mathbf{n})\mu_i n_i 1[n_i > 0] = &\sum_{i=1}^{M} \lambda_0 p_{0i}\pi(\mathbf{n}-\mathbf{e}_i) \\ &+ \sum_{i=1}^{M} \mu_i n_i p_{i0}\pi(\mathbf{n}+\mathbf{e}_i) \\ &+ \sum_{\substack{i,j=1 \\ i\neq j}}^{M} \mu_j n_j p_{ji}\pi(\mathbf{n}+\mathbf{e}_j-\mathbf{e}_i).\end{aligned}$$

- $\lambda_0\pi(\mathbf{n})$:
  Represents the rate at which external arrivals contribute to the state $\mathbf{n}$.
- $\sum_{i=1}^{M} p_{i0}\pi(\mathbf{n})\mu_i n_i 1[n_i > 0]$:
  Accounts for the departures from the system to an external sink, where a job leaves the queue at rate $\mu_i n_i$ given there is at least one job in queue i.
- $\sum_{i=1}^{M} \lambda_0 p_{0i}\pi(\mathbf{n}-\mathbf{e}_i)$:
  This term represents external arrivals into queue i.

- $\sum_{i=1}^{M} \mu_i n_i p_{i0} \pi(\mathbf{n} + \mathbf{e}_i)$:
  This term represents jobs leaving the system entirely.

- $\sum_{\substack{i,j=1 \\ i \neq j}}^{M} \mu_j n_j p_{ji} \pi(\mathbf{n} + \mathbf{e}_j - \mathbf{e}_i)$:
  This term represents jobs moving between queues.

The routing matrix P satisfies $(I - P)^{-1}$, ensuring a stable flow in the network. As long as λ_i is finite and $\mu_i > 0$, the CTMC is positive recurrent.

The stationary distribution can be described as

$$\pi(\mathbf{n}) = \prod_{i=1}^{M} \frac{\rho_i^{n_i}}{n_i!} \exp(-\rho_i)$$

Note: The whole excercise is a variant on Jackson's theorem.

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1. Poisson Process: Consider a Poisson process with rate λ. Let T_n be the time of the n-th arrival. Consider a second arrival process and denote Z_n as the time of the n-th arrival of this second arrival process. Assume $Z_n = cT_n$ for some constant c > 0. What can you say about this second arrival process? Prove your answer. What do we know about the superposition of these two arrival processes?

Solution: We have to wait c times longer on average in the new arrival process compared to the original process. Consequently, there are c times fewer arrivals in the new process than in the original process over the same time interval. This implies that the new arrival process is also Poisson, but with rate $\frac{\lambda}{c}$.

Note: *Intuitively clear, but I'm not too sure about the proof.*

Let the interarrival times of the original process be I_n, which are exponentially distributed with rate λ, i.e., $I_n \sim \exp(\lambda)$. The n-th arrival time T_n of the original process is the sum of the first n interarrival times:

$$T_n = \sum_{i}^{n} I_i$$

The new arrival process is defined by scaling the original arrival times

$$Z_n = cT_n = c\sum_{i}^{n} I_i = \sum_{i}^{n} cI_i$$

Since $I_i \sim \exp(\lambda)$, scaling I_i by c results in new interarrival times $I_i^Z = cI_i \sim \exp(\frac{\lambda}{c})$. This scaling does not affect the independence of the interarrival times I_i, which means the interarrival times of the new process remain independent.

The independence and exponential distribution of the interarrival times confirm that the new process satisfies the memoryless property, confirming our claim that it is a Poisson process with rate $\frac{\lambda}{c}$.

From Theorem 3.1 we know that the superposition of two independent Poisson processes with parameters λ_1 and λ_2 is again a Poisson process with parameter $\lambda_1 + \lambda_2$. Therefore, the rate of the superposition λ_s is equal to $\lambda + \frac{\lambda}{c}$.

2. Discrete-Time Markov Chains: Given a DTMC with state space S such that state i = 1 has period $d_1 = 3$, state i = 2 has period $d_2 = 5$ and there are at least 3 open communicating classes. What is the smallest possible value for $|S|$? Explain your answer. Give an example (by defining P) of a DTMC such that $|S|$ is minimized

Solution: The smallest size for a communicating class C_i with period $d_i = k$ has $|C_i| = k$. This means we need at least two cycles C_1 and C_2 with size 3 and 5 respectively, since a state can't be part of multiple communicating classes. Adding an edge from C_1 to C_2 (or vice versa) creates our first open communicating class. The smallest communicating class consists of a single state, so we'll have to add two additional states, resulting in:

$$\boxed{|S| = 10}$$

with

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3. Discrete-Time Markov Chains: Assume we flip a coin infinitely often. How many coin flips do we need on average until we flipped the sequence HTH (heads-tails-heads)? Explain your answer.

Solution: We can model this problem as a DTMC where each transition has a probability of $\frac{1}{2}$. Each state represents a step in the sequence. We now calculate the mean hitting time for $A = \{3\}$, where state 3 represents the successful observation of "HTH."

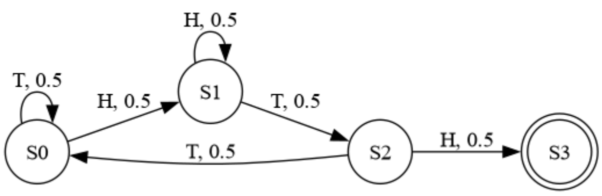

Figure 1: DTMC modelling coin flips to reach an HTH sequence

Remember: Let $\{X_n, n \geq 0\}$ be a Markov chain characterized by transition matrix P, and let $A \subseteq S$ be a subset of states. Then, the mean hitting times k_i^A of A are the smallest non-negative solution to:

$$x_i = \begin{cases} 0, & i \in A, \\ 1 + \sum_{j \notin A} p_{i,j} x_j, & i \notin A. \end{cases}$$

Define x_i as the mean hitting time (expected number of flips) to reach state 3 from state i.

- $x_3 = 0$ since $3 \in A$, which is our target state.
- For x_0 (starting from the initial state):

$$x_0 = 1 + \frac{1}{2}x_1 + \frac{1}{2}x_0$$

Rearranging, we get:

$$\frac{1}{2}x_0 = 1 + \frac{1}{2}x_1$$
$$x_0 = 2 + x_1$$

- For x_1 (starting from the state where "H" is observed):

$$x_1 = 1 + \frac{1}{2}x_2 + \frac{1}{2}x_1$$

Rearranging, we get:

$$\frac{1}{2}x_1 = 1 + \frac{1}{2}x_2$$
$$x_1 = 2 + x_2$$

- For x_2 (starting from the state where "HT" is observed):

$$x_2 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_0$$

Simplifying, since $x_3 = 0$:

$$x_2 = 1 + \frac{1}{2}x_0$$

We now solve this system of equations step-by-step:

1. Substitute $x_1 = 2 + x_2$ into $x_0 = 2 + x_1$:

$$x_0 = 4 + x_2$$

2. Substitute $x_0 = 4 + x_2$ into $x_2 = 1 + \frac{1}{2}x_0$

$$x_2 = 1 + \frac{1}{2}(4 + x_2)$$
$$\frac{1}{2}x_2 = 3$$
$$x_2 = 6$$

3. Substitute $x_2 = 6$ into $x_0 = 4 + x_2$:

$$x_0 = 4 + 6 = 10$$

Thus, the mean hitting time to reach the sequence "HTH" (state 3) from the starting state (state 0) is:

$$\boxed{x_0 = 10}$$

Therefore, on average, it takes 10 coin flips to observe the sequence "HTH" for the first time.

4. Continuous-Time Markov Chains: Consider an infinite state continuous-time Markov chain (CTMC) with state space $\{0, 1, 2, \ldots\}$ and rate matrix Q, such that for $i \neq j$ we have:

$$q_{i,j} = \begin{cases} q_{i,i+1} & \text{if } i \text{ is odd, } j = i+1, \\ q_{i,i+3} & \text{if } i \text{ is odd, } j = i+3, \\ q_{i,i-1} & \text{if } i > 0 \text{ is even, } j = i-1, \\ q_{i,i-3} & \text{if } i > 0 \text{ is even, } j = i-3, \\ q_{0,2} & \text{if } i = 0, j = 2, \\ q_{2,0} & \text{if } i = 2, j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\pi_0, \pi_1, \ldots)$ be its invariant distribution. Give an explicit expression for $\frac{\pi_{13}}{\pi_0}$. Explain your answer.

5. APPLICATIONS: Consider an M/M/1 queue with arrival rate λ and service rate $\mu > \lambda$. Assume a job is labelled type-1 upon arrival with probability p and type-2 otherwise. Jobs are labelled independent of each other. Instead of serving the jobs in FCFS order, type-1 jobs get priority over type-2 jobs. A type-2 job can only be in service if no type-1 jobs are present and is interrupted (and resumed later) if a type-1 job arrives. What is the mean response time of a type-1 job in such a system? Explain. Derive an expression for the mean response time of a type-2 job.

Solution: Consider $\lambda_1 = p\lambda$ and $\lambda_2 = (1-p)\lambda$ as the arrival rates for type-1 and type-2 jobs respectively. For a type-1 job, type-2 jobs don't exist, as they are given priority. We find the mean response time for type-1 jobs W_1 using Little's law

$$W_1 = \frac{1}{\mu - \lambda_1} = \frac{1}{\mu - p\lambda}$$

A type-2 job can only be serviced when there are no type-1 jobs present in the systems. We define $\rho_1 = \frac{\lambda_1}{\mu}$ as the fraction of the time that the server is busy servicing type-1 jobs. The actual service time for type-2 jobs μ_2 can be written as

$$\mu_2 = \mu(1-\rho_1) = \mu(1 - \frac{\lambda_1}{\mu}) = \mu - \lambda_1$$

The mean response time W_2 becomes

$$W_2 = \frac{1}{\mu_2 - \lambda_2} = \frac{1}{\mu - \lambda_1 - \lambda_2} = \frac{1}{\mu - p\lambda - (1-p)\lambda} = \frac{1}{\mu - \lambda}$$

6. APPLICATIONS: Consider a Jackson network with $M = 4$ queues and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$. All new arrivals either join queue 1 or 2 (with equal probability). The routing probabilities are as follows: $p_{1,3} = p_{1,4} = \frac{1}{2}$, $p_{2,3} = \frac{1}{8}$, $p_{2,4} = \frac{7}{8}$, $p_{4,0} = 1$, $p_{3,3} = p_{3,0} = \frac{1}{2}$. State a necessary and sufficient condition such that the Markov chain of the joint queue lengths is positive recurrent. For which of these 4 queues is the input process Poisson? Derive an expression for the probability that both queue 3 and 4 are empty at the same time.

Solution: For the Markov chain to be positive recurrent, we require $\lambda_i < \mu_i$ for all queues. We calculate each λ_i using the flow-balance equation:

$$\lambda_i = \lambda_0 p_{0,i} + \sum_{j=1}^{M} \lambda_j p_{j,i}.$$

The computations for each λ_i are as follows:

$$\begin{aligned}
\lambda_1 &= \lambda_0 p_{0,1} = \frac{\lambda_0}{2}, \\
\lambda_2 &= \lambda_0 p_{0,2} = \frac{\lambda_0}{2}, \\
\lambda_3 &= \lambda_1 p_{1,3} + \lambda_2 p_{2,3} + \lambda_3 p_{3,3}, \\
\lambda_4 &= \lambda_1 p_{1,4} + \lambda_2 p_{2,4}.
\end{aligned}$$

Substituting into the equation for λ_3:

$$\lambda_3 = \frac{\lambda_1}{2} + \frac{\lambda_2}{8} + \frac{\lambda_3}{2}.$$

Rearranging terms:

$$\lambda_3 - \frac{\lambda_3}{2} = \frac{\lambda_1}{2} + \frac{\lambda_2}{8},$$

which simplifies to:

$$\frac{\lambda_3}{2} = \frac{\lambda_1}{2} + \frac{\lambda_2}{8}.$$

Solving for λ_3:

$$\lambda_3 = \lambda_1 + \frac{\lambda_2}{4} = \frac{\lambda_0}{2} + \frac{\lambda_0}{8} = \frac{5\lambda_0}{8}.$$

Similarly, for λ_4:

$$\lambda_4 = \lambda_1 p_{1,4} + \lambda_2 p_{2,4},$$

substituting:

$$\lambda_4 = \frac{\lambda_1}{2} + \frac{7\lambda_2}{8} = \frac{\lambda_0}{4} + \frac{7\lambda_0}{16} = \frac{11\lambda_0}{16}.$$

For positive recurrence, we require $\lambda_i < \mu_i = 1$ for all i, leading to the condition:

$$\lambda_0 < 2.$$

Poisson Input Processes: The input process for all queues except queue 3 is Poisson. Queue 3 has a self-loop (as $p_{3,3} > 0$), which introduces correlation between interarrival times, violating the memoryless property of Poisson processes.

Probability of Empty Queues: The probability that a queue i is empty is given by $(1-\rho_i)$, where $\rho_i = \frac{\lambda_i}{\mu_i}$. Assuming independence between the queues, the joint probability that both queue 3 and queue 4 are empty is:

$$(1-\rho_3)(1-\rho_4).$$

Substituting for ρ_3 and ρ_4:

$$(1-\rho_3)(1-\rho_4) = \left(1 - \frac{\lambda_3}{\mu_3}\right)\left(1 - \frac{\lambda_4}{\mu_4}\right).$$

With $\mu_3 = \mu_4 = 1$, we have:

$$(1-\rho_3)(1-\rho_4) = \left(1 - \frac{5\lambda_0}{8}\right)\left(1 - \frac{11\lambda_0}{16}\right).$$

This expression is valid only for:

$$\lambda_0 \in [0, \frac{16}{11}].$$

7. Applications: In which of the following two systems is the blocking probability the largest: (1) an Erlang-B system with arrival rate λ, mean call duration $\frac{1}{\mu}$ and C lines or (2) an Engset system with $\lambda' = \frac{\lambda}{N}$, mean call duration $\frac{1}{\mu}$ and C lines? Explain.

Solution: The blocking probability is largest in the Erlang-B system. This is because the Erlang-B system assumes an infinite source pool with a constant arrival rate λ, leading to higher traffic intensity and more frequent blocking when all C lines are occupied. In contrast, the Engset system has a finite source pool, so the arrival rate decreases as more lines are occupied, reducing the likelihood of blocking. Therefore, the Engset system has a smaller blocking probability due to its finite-source nature.

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1. Poisson Process: Consider a Poisson process with arrival rate $\lambda = 2$. Assume there is one arrival in [0, 2] and there is one (possibly the same) arrival in [1, 3]. What is the probability that we have only one arrival in [0, 3]?

Solution: We are looking for the following probability:

$$P(N_{0,3} = 1 | N_{0,2} = 1, N_{1,3} = 1)$$

This is the same as looking for these three conditions:

1. $P(N_{0,1} = 0)$
2. $P(N_{1,2} = 1)$
3. $P(N_{2,3} = 0)$

For each we can use the Poisson distribution , given us the following solutions:

$$P(N_{0,1} = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-2}$$
$$P(N_{1,2} = 1) = 2.e^{-2}$$
$$P(N_{2,3} = 0) = e^{-2}$$

Since each probability (in each internal) is independent from each other , we can simplify mutliply all of them with each other giving us the final solution of $2e^{-6}$.

2. Markov Chains: Consider a DTMC on the state space $S = \{(i_1, j_1, i_2, j_2) | 1 \leq i_1, j_1, i_2, j_2 \leq 8\}$, that is, the state of the DTMC is characterized by marking two squares (i_1, j_1) and (i_2, j_2) on an 8×8 board. Assume the transition probabilities are such that the state (i'_1, j'_1, i'_2, j'_2) visited from state (i_1, j_1, i_2, j_2) is such that (i'_1, j'_1) is a random neighbor of (i_1, j_1) and (i'_2, j'_2) is a random neighbor of (i_2, j_2). For instance, from state $(1, 3, 4, 4)$ we move to $(1, 4, 4, 5)$ with probability $\frac{1}{12} = \frac{1}{3} \cdot \frac{1}{4}$ as $(1, 3)$ has 3 neighbors and square $(4, 4)$ has 4 neighbors. How many communicating classes does this DTMC have? What is the period of each class?

Solution: From each square it is possible to reach any other square in a finite number of steps (irreducible). Since we can return to a square in an arbitrary number of steps, it is also aperiodic. Thus, we have a *single* communicating class with a period $d = 1$.

3. Markov Chains: Consider an irreducible, positive recurrent DTMC. Let P be its transition probability matrix and $\pi = (\pi_0, \pi_1, \ldots)$ its unique stationary distribution. Give an expression for

$$\tilde{p}_{i,j} = \lim_{n \to \infty} P[X_n = j | X_{n+1} = i],$$

in terms of the entries of P and π. Define a DTMC with transition probability matrix $\tilde{P}$ such that entry (i, j) of $\tilde{P}$ equals $\tilde{p}_{i,j}$. Is this DTMC positive recurrent? If so, what can you say about its stationary distribution?

Solution: Using Bayes' theorem we can write $\tilde{p}_{i,j}$ as

$$\tilde{p}_{i,j} = \lim_{n \to \infty} P[X_n = j | X_{n+1} = i] = \lim_{n \to \infty} \frac{P[X_{n+1} = i | X_n = j] \cdot P[X_n = j]}{P[X_{n+1} = i]}$$

Since the process is stationary, $P[X_n = j] = \pi_j$ and $P[X_{n+1} = i] = \pi_i$. Also, $P[X_{n+1} = i|X_n = j] = P_{j,i}$, where $P_{j,i}$ is the transition probability from j to i. Thus,

$$\tilde{p}_{i,j} = \frac{P_{j,i} \cdot \pi_j}{\pi_i}$$

We can now define the transition matrix $\tilde{P}$ where

$$\tilde{P}_{i,j} = \tilde{p}_{i,j} = \frac{P_{j,i} \cdot \pi_j}{\pi_i}$$

If the original DTMC was irreducible, the reversed DTMC based on it's stationary distribution will also be irreducible, therefore positive recurrent. We know that $\tilde{\pi}$ must satisfy $\tilde{\pi}\tilde{P} = \tilde{\pi}$. For a fixed j, we find

$$\tilde{\pi}_j = \sum_i \pi_i \cdot \tilde{p}_{i,j} = \sum_i \pi_i \frac{p_{j,i} \cdot \pi_j}{\pi_i} = \sum_i \pi_j \cdot p_{j,i} = \pi_j \sum_i p_{j,i} = \pi_j \cdot 1 = \pi_j$$

Thus,

$$\boxed{\tilde{\pi} = \pi}$$

4. Markov Chains: Give an example of an irreducible, transient DTMC with period 2. Explain why your example has these properties.

Solution: A DTMC that is both transient and irreducible must have an infinte state space. A simple example is the BD Markov chain where $p_0 = 1$ and $p_i = \frac{1}{2}$.

5. PASTA: Consider a queueing system with Poisson arrivals, where the service time of customer n depends on the inter-arrival time between customer $n-1$ and n. Can we apply the PASTA property for this queueing system? Explain your answer.

Solution: The requirement on the independence in the PASTA theorem states that at time t, the system should have no information on the arrival time of future arrivals. This dependence doesn't give that information, so PASTA still holds. Local service time doesn't affect the equilibrium.

6. Applications: Consider a queueing network consisting of 2 queues (not necessarily M/M/1 queues). Type 1 jobs arrive at rate λ_1 and first visit queue 1 followed by queue 2. Type 2 jobs arrive at rate λ_2 and visit queue 1 only, while type 3 jobs arrive at rate λ_3 and only visit queue 2. Assume that the mean time that a random job spends in the queueing network equals 5, the mean time that a random type 1 or type 2 job spends in queue 1 is 3 and that the mean time that a random type 1 or 3 job spends in queue 2 is 4. Determine the arrival rates λ_1, λ_2 and λ_3 given that $\lambda_1 = \lambda_2$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 1$. Does your answer change if type 1 jobs visit both queues in the opposite order? Explain.

Solution: Let T denote the mean total time. We can write this as

$$T = P_1 T_1 + P_2 T_2 + P_3 T_3$$

Where

- T_i is the mean total time spent by a job of type i in the network.

- $P_i = \frac{\lambda_i}{\lambda}$ is the proportion of jobs of type i.

Since $\lambda = 1$, $P_i = \lambda_i$. We also know

$$\begin{aligned} T_1 &= T_{1,1} + T_{1,2} = 3 + 4 = 7 \\ T_2 &= T_{2,2} = 3 \\ T_3 &= T_{3,2} = 4 \end{aligned}$$

We find

$$T = 7\lambda_1 + 3\lambda_2 + 4\lambda_3 = 5$$

Given that $\lambda_1 = \lambda_2$, we find that $10\lambda_1 = 5 - 4\lambda_3$.
We also know $\lambda_1 + \lambda_2 + \lambda_3 = \lambda = 1$, thus $\lambda_3 = 1 - 2\lambda_1$.
Using the equation above in the mean total time equation:

$$\begin{aligned} 10\lambda_1 &= 5 - 4(1 - 2\lambda_1) = 5 - 4 + 8\lambda_1 \\ 2\lambda_1 &= 1 \\ \lambda_1 &= \frac{1}{2} \end{aligned}$$

We conclude

$$\boxed{\lambda_1 = \lambda_2 = \frac{1}{2}, \lambda_3 = 0}$$

If type 1 jobs where to visit the second queue first, this wouldn't affect the mean total time in the system, therefore our answer doesn't change.

7. Applications: Consider an M/M/1 queue (arrival rate λ, service rate μ) with the additional property that when a customer arrives it starts an exponential timer with mean $1/\theta$ and leaves the queueing system immediately if that timer expires before its service starts. How can you model this queueing system using a CTMC? Give an expression for the stationary distribution. For which value of θ is this distribution a Poisson distribution?

Solution: Just like the M/M/∞ queue, we increase the rate of a job leaving the queue according to the number of jobs in the queue so

$$\begin{aligned} q_{i,i-1} &= \mu + \frac{i}{\theta} \\ q_{i,i} &= -(\lambda + \mu + \frac{i}{\theta}) \\ q_{i,i+1} &= \lambda \end{aligned}$$

with $q_{0,0} = -\lambda$. The stationary distribution π satisfies $\pi Q = 0$ and $\sum_{i=0}^{\infty} \pi = 1$.
Using the balance equations, for $i = 0$

$$\lambda \pi_0 = \left(\mu + \frac{1}{\theta}\right) \pi_1$$

For $i \geq 1$

$$\lambda \pi_i = \left(\mu + \frac{i+1}{\theta}\right) \pi_{i+1}$$

From these terms we can express π_{i+1} in terms of π_i. Starting from π_0, we can recursively compute

$$\pi_i = \pi_0 \prod_{k=1}^{i} \frac{\lambda}{\mu + \frac{k}{\theta}}$$

We can easily see that when $\mu = 0$ and $\theta = \lambda$

$$\pi_i = \pi_0 \prod_{k=1}^{i} \frac{\lambda}{\frac{k}{\lambda}} = \pi_0 \prod_{k=1}^{i} \frac{\lambda^2}{k} = \pi_0 \frac{\lambda^{2i}}{i!}$$

and $\pi_0 = \exp(-\lambda^2)$ as $\sum_{i=0}^{\infty} \pi_i = 1$ must be satisfied.

$$\sum_{i=0}^{\infty} \pi_i = \sum_{i=0}^{\infty} \pi_0 \frac{\lambda^{2i}}{i!} = 1$$

$\sum_{i=0}^{\infty} \frac{\lambda^{2i}}{i!}$ is the Taylor expasion of $\exp(\lambda^2)$. Thus

$$\pi_0 \exp(\lambda^2) = 1$$

Solving for π_0 gives us $\pi_0 \exp(-\lambda^2)$

Like Theorem 1.4, this means the stationary distribution is Poisson with parameter λ^2.

OPEN BOOK EXAM 2021

1. Poisson process: Let P_1 and P_2 be two independent Poisson processes with rates λ_1 and λ_2 . Let $N_{t,i}$ be the number of arrivals of process P_i in [0, t], for i = 1, 2. Show that:

$$\lambda_2^k P(N_{t,1} = N_{t,2} + k) = \lambda_1^k P(N_{t,2} = N_{t,1} + k)$$

Solution:

$$P(N_{t,1} = N_{t,2} + k) = \sum_{n=0}^{\infty} P(N_{t,1} = n + k and N_{t,2} = n)$$

Since P_1 and P_2 are independent we can write this probability as a product

$$P(N_{t,1} = N_{t,2} + k) = \sum_{n=0}^{\infty} P(N_{t,1} = n + k)P(N_{t,2} = n)$$

We substitute with the Poisson probabilities

$$\begin{aligned} P(N_{t,1} = N_{t,2} + k) &= \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^{(n+k)}}{(n+k)!} \exp(-\lambda_1 t) \frac{(\lambda_2 t)^n}{n!} \exp(-\lambda_2 t) \\ &= \exp(-(\lambda_1 + \lambda_2)t)(\lambda_1 t)^k \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{(n+k)!} \frac{(\lambda_2 t)^n}{n!} \\ &= \exp(-(\lambda_1 + \lambda_2)t)(\lambda_1 t)^k \sum_{n=0}^{\infty} \frac{(\lambda_1 \lambda_2 t^2)^n}{n!(n+k)!} \end{aligned}$$

For the reversed case we find

$$P(N_{t,1} = N_{t,2} + k) = \exp(-(\lambda_1 + \lambda_2)t)(\lambda_2 t)^k \sum_{n=0}^{\infty} \frac{(\lambda_1 \lambda_2 t^2)^n}{n!(n+k)!}$$

By multiplying the probabilities with λ_2^k and λ_1^k respectively, we end up with the given equation.

2. Markov Chains: Give an example of a DTMC with 6 states, where some states have period 2 and others have period 3, while not all states are recurrent. Explain your answer. What is the maximum number of non-zero entries in P for such a chain? Explain.

Solution: States 1-2 communicate with period of 2. States 3-5 communicate with period of 3. State 6 does not go anywhere to ensure that it is not recurrent. Writing out the matrix P gives us six non-zero entries.

NOTE: Amount of non-zero entries is incorrect. Example is correct, but states 1 and 2 are also transient.

A better example:

$$\begin{bmatrix} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

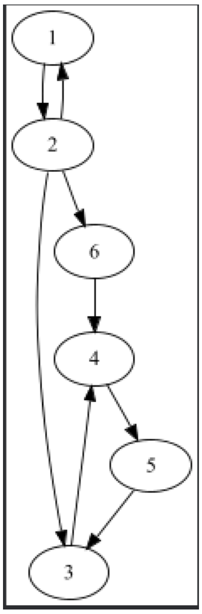

Figure 2: DTMC with 6 states. 1,2 have period 2, 3-5 period 3 and 6 not recurrent

- Yonah

3. Markov Chains: Let Y_i, for $i \geq -1$, be an infinite set of independent random variables with $P[Y_i = -1] = P[Y_i = 0] = P[Y_i = 1] = \frac{1}{3}$. Define $X_n = Y_{n-1} + Y_n$ for $n \geq 0$. Is $(X_n)_{n\geq 0}$ a DTMC? If so, give its transition matrix P; if not, explain why.

Solution:
We have $X_0 = Y_{-1} + Y_0$ and $X_1 = Y_1 + X_0 - Y_{-1}$. The state X_{n+1} should be fully determined by the state of our system at time n, but as we see we also need to know what Y_{n-2} was. Therefore the Markov property does not hold and this is not a DTMC.

4. Markov Chains: Consider two independent Poisson processes P_1 and P_2 with rates λ_1 and λ_2. Assume we have an infinite bag, and whenever an arrival occurs of process P_1 we add a ball to the bag. If an arrival occurs of process P_2, we remove half of the balls from the bag, that is, we remove $\lceil k/2 \rceil$ balls if the bag contains k balls ($\lceil x \rceil$ rounds x to the smallest integer larger than or equal to x). Let X_t be the number of balls in the bag at time t. Argue that $(X_t)_{t\geq 0}$ is an irreducible CTMC. Given λ_1, how large must λ_2 be such that this CTMC is positive recurrent? Explain your answer.

Solution: Since the two Poisson processes are independent from each other, it is intuitive that the arrival rate of P_1 or P_2 is also a Poisson process. We also have the following:

$$k \longrightarrow k + 1 \text{with rate} \lambda_1$$
$$k \longrightarrow \lceil k/2 \rceil \text{with rate} \lambda_2$$

Irreducible:
Since it is possible to increase k to k+1, k+2,... from any k by consecutive arrivals of P_1. And it is possible to decrease k all the way to 0 by consecutive arrivals of P_2. We find that all states can be reached by all states by a sequence of events, proving the irreducibility

of the CTMC.
Positive recurrence:
The expected return time to any state must be finite. This is equivalent to ensuring that X_t does not "drift to infinity". The key is to balance λ_1 and λ_2. We have:

- Upward drift: Balls are added at a rate λ_1, causing the chain to move upward.
- Downward drift: The balls are removed at rate λ_2, and the amount removed increases with k (since $\lceil k/2 \rceil$ is larger for larger k).

The expected rate of change in the number of balls ($E[\Delta X]$) depends on the current state k:

$$E[\Delta X] = \lambda_1 - \lambda_2 E[\lceil k/2 \rceil]$$

For large k, $[\lceil k/2 \rceil]$ approximates $E[\lceil k/2 \rceil]$. To ensure the process is positive recurrent, the downward drift must counteract the upward drift, requiring $\lambda_2$to be sufficiently large such that:

$$\lambda_2 > \frac{2\lambda_1}{k} \text{ for large k}$$

Since k grows, we consider the expected average drift over all states. This requires:

$$\lambda_2 > 2\lambda_1$$

5. Applications: Consider 2 queueing systems. The first is an M/M/1 queue with arrival rate $\lambda = 0.3$ and mean service time equal to 2. The second is a Jackson network with $M = 1$, $\mu_1 = 4$, and $p_{1,1} = \frac{1}{3}$. How should we set λ_0 such that the queue length distribution is the same in both queueing systems? Explain.

Solution: The mean queue length of an M/M/1 queue L is computed as

$$L = \frac{\rho}{1-\rho}$$

For a Jackson network with a single M/M/1 queue, this is equivalent. We now need to find λ_1 in function of λ_0 to solve the problem.

In a Jackson network, λ_i is calculated as

$$\lambda_i = \lambda_0 p_{0i} + \sum_{j=1}^{M} p_{ji}\lambda_j$$

For our problem, we have $p_{01} = 1$, $p_{11} = \frac{1}{3}$, and $M = 1$. The above equation simplifies to

$$\lambda_1 = \lambda_0 + \frac{\lambda_1}{3}$$

Rearranging, we find

$$\lambda_1 = \frac{3\lambda_0}{2}.$$

We now look for a λ_0 such that $\rho = \frac{\lambda}{\mu} = \frac{0.3}{2} = 0.15$ equals the system load $\rho_1 = \frac{\lambda_1}{\mu_1}$ of the queue in our Jackson network.

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{\frac{3\lambda_0}{2}}{4}.$$

Equating ρ_1 to 0.15, we get

$$\frac{\frac{3\lambda_0}{2}}{4} = 0.15.$$

Simplify to solve for λ_0:

$$\frac{3\lambda_0}{8} = 0.15,$$

$$\lambda_0 = \frac{8 \cdot 0.15}{3} = \frac{1.2}{3} = 0.4.$$

Thus, $\lambda_0 = 0.4$.

6. Applications: Consider an Erlang-C system with C servers, i.e. an M/M/C/C+Q queue with $Q = \infty$. Assume we know that on average k jobs are waiting in the waiting room to enter a server. Does the mean time that a customer spends in the system depend on C, k, or both C and k? Explain.

Solution: Let $L_q = k$ be the mean waiting room length. Little's law tells us that

$$L_q = \lambda W_q$$

where W_q is the mean waiting time in the queue. The mean total time W is the sum of the mean waiting time W_q and the mean service time $S = \frac{1}{\mu}$.

$$W = W_q + S$$

From L_q we calculate $W_q = \frac{L_q}{\lambda}$. We notice that none of the formulas depend on C, so the mean total time W depends only on k.

OPEN BOOK EXAM 2020

1. Spam Mail: Suppose spam mail arrives in your mailbox as a Poisson process with rate λ_s, regular mail arrives as a Poisson process with rate λ_r and both processes are independent. (1) What is the mean time between any two mails? (2) What is the probability that exactly 5 out of 10 consecutive mails are spam mails?

Solution:

(1) The superposition of two independent Poisson processes with parameters λ_1 and λ_2 is again a Poisson process with parameter $\lambda_1 + \lambda_2$. We find a Poisson process with paramter $\lambda = \lambda_s + \lambda_r$. The mean time between any two mails is therefore $\frac{1}{\lambda}$.

(2) If a Poisson process with parameter λ is randomly split into two subprocesses with probability p, then the resulting processes are indenependent Poisson processes with parameters λp and $\lambda(1-p)$. We know the original processes have rates λ_s and λ_r, so p_s must be $\frac{\lambda_s}{\lambda}$. For p_r we see,

$$p_r = 1 - p_s = 1 - \frac{\lambda_s}{\lambda_s + \lambda_r} = \frac{\lambda_s + \lambda_r}{\lambda_s + \lambda_r} - \frac{\lambda_s}{\lambda_s + \lambda_r} = \frac{\lambda_r}{\lambda_s + \lambda_r}$$

Intuitively, this makes sense. The probability for 5 spam mails is p_s^5, for 5 regular mails p_r^5. There are 6 positions where the consecutive spam mails could arrive. The probability becomes $6p_s^5 p_r^5$.

2. Discrete Time Markov Chains: Let d_i be the period of state i in a DTMC. Give an example of a DTMC with 7 states (by defining P) such that $(d_1, d_2, \ldots, d_7) = (3, 3, 2, 1, 3, 2, 3)$ and exactly 2 states are recurrent. Explain your answer.

3. Discrete Time Markov Chains: Consider a DTMC with state space $\mathbb{Z}$. Let $p_{0,1} = \frac{1}{2}$, $p_{i,i+1} = e^{-i/100}$ for $i > 0$, and $p_{i,i-1} = e^{i/100}$ for $i < 0$. Further, let $p_{i,i+1} = 1 - p_{i,i-1}$ for $i \in \mathbb{Z}$. Is this DTMC positive recurrent, null recurrent, or transient? Prove your answer.

Solution: The DTMC is irreducible, each state communicates with every other state. We find the drift for state i

$$D_i = \sum_{k=-i}^{\infty} k p_{i,i+k}$$

$$D_i = 1 \cdot e^{-i/100} + (-1)(1 - e^{-i/100}) = 2e^{-i/100} - 1 \text{ for } i > 0$$
$$D_i = 1 \cdot (1 - e^{i/100}) + (-1)(e^{i/100}) = 1 - 2e^{i/100} \text{ for } i < 0$$

- $i \to \infty$, we see that the drift is negative and goes to -1.
- $i \to -\infty$, the drift goes to 1. But since we are working with negative states, this is a drift back to the positives (Reverse case as above).

The chain is therefore positive recurrent with $\delta = 1$.

Note: It might be that I abused Pakes' lemma here.

4. Repairing Servers: Consider two machines maintained by a single repairman. Machine i, for $i = 1, 2$, operates for an exponentially distributed amount of time with mean $1/\gamma_i$ before breaking down. The repair time for machine i is also exponential with mean $1/\beta_i$, but the repairman can only work on one machine at a time. Assume that machines are repaired in the order in which they fail. Set up a CTMC to find the long-run proportions of time that each machine is working (no need to compute these values). How does your answer change if the repairman always works on machine 1 first if it fails?

Solution: We define state 0 as OK, state 1 as 1 broken, state 2 as 2 broken, state 3 as both broken with 1 first and state 4 as both broken with 2 first. The transitions are intuitive, i.e 1 can transition to 3 and 0, etc.

We define the rate matrix Q as

$$\begin{bmatrix} -(\gamma_1+\gamma_2) & \gamma_1 & \gamma_2 & 0 & 0 \\ \beta_1 & -(\beta_1+\gamma_2) & 0 & \gamma_2 & 0 \\ \beta_2 & 0 & -(\beta_2+\gamma_1) & 0 & \gamma_1 \\ 0 & 0 & \beta_1 & -\beta_1 & 0 \\ 0 & \beta_2 & 0 & 0 & -\beta_2 \end{bmatrix}$$

If the repairman always repairs 1 first we don't need the extra state:

$$\begin{bmatrix} -(\gamma_1+\gamma_2) & \gamma_1 & \gamma_2 & 0 \\ \beta_1 & -(\beta_1+\gamma_2) & 0 & \gamma_2 \\ \beta_2 & 0 & -(\beta_2+\gamma_1) & \gamma_1 \\ 0 & 0 & \beta_1 & -\beta_1 \end{bmatrix}$$

5. Server Order: Assume jobs arrive in a system with 2 servers as a Poisson process with rate λ. All jobs must be served by both servers, and the service time of a job in server i is exponential with mean $1/\mu_i$, for $i = 1, 2$. Assume a fraction p of the incoming jobs first joins server 1 and then proceeds to server 2, while the remaining fraction $1 - p$ visits both servers in the reversed order. Can you use a Jackson network to determine the optimal choice of p such that the time that a job spends in the system is minimized?

Solution: While a Jackson network does not directly allow conditional routing like "visit server 1 first, then server 2" or "server 2 first, then server 1," this routing can be approximated on average by treating the system as if jobs are sent to servers probabilistically at the entry point with probabilities p and $1-p$, and then routed between servers in the same proportion. We find

$$\lambda_1 = \lambda_0 p_{0,1} + \lambda_2 p_{2,1}$$
$$\lambda_2 = \lambda_0 p_{0,2} + \lambda_1 p_{1,2}$$

Define $p = p_{0,1}$, then $p_{0,2} = 1 - p$ and $p_{1,2} = p_{0,1} = p$. Therefore

$$\lambda_1 = \lambda_0 p + \lambda_2(1-p)$$
$$\lambda_2 = \lambda_0(1-p) + \lambda_1 p$$

From the last equation we find

$$\lambda_1 = \frac{\lambda_2 - \lambda_0(1-p)}{p}$$

Substituting λ_1 in the first equation gives us

$$\begin{aligned}
\frac{\lambda_2 - \lambda_0(1-p)}{p} &= \lambda_0 p + \lambda_2(1-p) \\
\Leftrightarrow \lambda_2 - \lambda_0(1-p) &= \lambda_0 p^2 + \lambda_2(1-p)p \\
\Leftrightarrow \lambda_2 - \lambda_2(1-p)p &= \lambda_0 p^2 + \lambda_0(1-p) \\
\Leftrightarrow \lambda_2(1-(p-p^2)) &= \lambda_0(p^2+1-p) \\
\Leftrightarrow \lambda_2 &= \lambda_0
\end{aligned}$$

We find

$$\lambda_1 = \lambda_0 p + \lambda_2(1-p) = \lambda_0 p + \lambda_0(1-p) = \lambda_0$$

The total average response time W for an entire network is

$$W = \frac{1}{\lambda_0}\sum_{i=1}^{M}\frac{\rho_i}{1-\rho_i}$$

$\rho_1 = \frac{\lambda_0}{\mu_1}$ and $\rho_2 = \frac{\lambda_0}{\mu_2}$. We conclude that the average response time doesn't depend on p.

6. M/M/1 with Group Arrivals: Consider an M/M/1 queue with arrival rate λ and service rate μ, but instead of having one arrival at a time, arrivals occur in groups of 2. Set up a CTMC to study this queueing model. Write down the global balance equations and use them to prove that

$$\lambda\pi_{i-2} + \lambda\pi_{i-1} = \mu\pi_i,$$

holds for $i \geq 2$, where π_i represents the stationary probability that the queue contains i jobs.

Solution: We define the rate matrix Q as:

$$q_{i,j} = \begin{cases} \lambda & \text{if } j = i+2, \\ \mu & \text{if } j = i-1, \\ -(\mu+\lambda) & \text{if } j = i \neq 0, \\ -\lambda & \text{if } j = i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The global balance equations are defined as

$$\sum_{j \neq i} \pi_j q_{j,i} = \pi_i \sum_{j \neq i} q_{i,j}$$

For $i \geq 2$, we find

$$\pi_{i+1}q_{i+1,i} + \pi_{i-2}q_{i-2,i} = \pi_i(q_{i,i+2} + q_{i,i-1})$$

$$\pi_{i+1}\mu + \pi_{i-2}\lambda = \pi_i(\lambda + \mu)$$

For $i = 0, 1, 2$, we have

$$\begin{aligned}\pi_0\lambda &= \pi_1\mu \\ \pi_1(\lambda+\mu) &= \pi_2\mu \\ \pi_2(\lambda+\mu) &= \pi_0\lambda + \lambda_3\mu\end{aligned}$$

We see

$$\begin{aligned}\pi_2\mu &= \pi_1(\lambda+\mu) \\ &= \pi_1\mu + \pi_1\lambda \\ &= \pi_0\lambda + \pi_1\lambda\end{aligned}$$

Thus the equation holds for i.
We now try to prove for $i+1$. From the global balance equation we know

$$\begin{aligned}\pi_{i+1}\mu + \pi_{i-2}\lambda &= \pi_i(\lambda+\mu) \\ \pi_{i+1}\mu &= \pi_i\lambda + \pi_i\mu - \pi_{i-2}\lambda\end{aligned}$$

Using our induction hypothesis for i, we get

$$\begin{aligned}\pi_{i+1}\mu &= \pi_i\lambda + \pi_{i-1}\lambda + \pi_{i-2}\lambda - \pi_{i-2}\lambda \\ &= \pi_i\lambda + \pi_{i-1}\lambda\end{aligned}$$

Thus we have proven that the equation holds for all $i \geq 2$

7. Blocking Time: Consider an Erlang-B system (i.e., an M/M/C/C queue). Such a system alternates between periods where incoming calls are blocked and periods where calls are accepted. What is the mean duration of a period during which calls are blocked? How can you determine the mean duration of a period during which calls are accepted?

Solution: A system doesn't accept calls when all lines are full, i.e. state C. State C is visited for an exponential amount of time with mean $\frac{1}{C\mu}$. A system goes into the blocked state with a rate of arrival to state C, equal to $\lambda\pi_{C-1}$. The mean duration of this period is $\frac{1}{\lambda\pi_{C-1}}$.

OPEN BOOK EXAM 2019

1. Poisson Process: Consider a Poisson process with rate λ. Assume n arrivals occurred in between time 0 and t. Give an expression for the probability that no arrivals occur between time $t/3$ and $2t$? Explain how you obtained your expression.

Solution: Arrivals are independent. Even though there were n arrivals between 0 and t, this doesn't influence the probability of having no arrivals between $t/3$ and $2t$.

$$P[I_n \leq t] = 1 - \exp(-\lambda t) \Rightarrow P[I_n > t] = 1 - (1 - \exp(-\lambda t)) = \exp(-\lambda t)$$

We find our t as $2t - t/3 = \frac{5t}{3}$. Our final expression is $\exp(-\lambda \frac{5t}{3})$.

2. Discrete Time Markov Chains: What is the least number of non-zero entries in the transition probability matrix P of a discrete time irreducible aperiodic Markov chain with $n > 2$ states? Does your answer change if we demand that the chain has period 2? Explain your answers.

Solution: For an irreducible aperiodic Markov chain each state we need a single SCC. We need one transition per node to make cycle, and one more for a self loop to ensure aperiodicity, so $n + 1$.

For an irreducible chain with period 2, it differs for even or odd states. For an even amount of states we create a cycle and then a single transition in the reverse direction, since the period is defined as the greatest common divisor of all the time epochs at which the system can be in state i, a cycle with even nodes and a possibility to increase the cycle with any multiple of two gives us a period of 2. So $n + 1$ non-zero entries. For an odd amount of states, we need to make a chain with two transitions per state, $2(n - 1)$ non-zero entries. (Think of a BD chain)

3. Continuous Time Markov Chains: Consider a continuous time Markov chain on the state space $S = \{(i, j) | i \geq 0, j \geq 0\}$ and assume the transition rates are such that

$$q_{(i,j),(i+1,j-1)} = 2, \text{ for } j > 0, \quad q_{(i,0),(0,i-1)} = 2, \text{ for } i > 0,$$

$$q_{(i,j),(i-1,j+1)} = 1, \text{ for } i > 0, \quad q_{(0,j),(j+1,0)} = 1.$$

Prove that this Markov chain is positive recurrent.

4. You've Got Mail: Bob receives both regular mail and spam mail. An incoming spam mail is detected as spam with probability p and moved to Bob's spam folder. Undetected spam mails are collected in Bob's regular mail folder together with all of his regular mails. Bob has on average 4 unopened mails in his regular mail folder and 80 mails in his spam folder. Bob cleans his spam folder every 5 days and opens all of his unopened regular mail every 3 hours (on average). Which fraction p of Bob's incoming spam mail is certainly detected as spam?

5. Queueing Theory: Consider a queue with Poisson arrivals with rate 1 and exponential service times with mean 1/2. Assume jobs are served first-come-first-serve and at most N jobs can wait in the waiting room. Incoming jobs that find the waiting room full are lost (i.e., immediately leave). Show that the loss probability in this system equals $\frac{1}{2N+1-1}$.

6. Queueing Networks: Consider a system consisting of 3 queues (labeled 1, 2, and 3) and assume we have N jobs in the system. A job requires an exponential amount of service with mean $1/\mu_i$ in queue i and moves to queue $(i \mod 3)+1$ afterwards. Thus we have a fixed number of N jobs in the system. The service discipline is first-come-first-serve in each queue. Show that the probability to have i jobs in queue 1 and j jobs in queue 2 is given by the following product form

$$\frac{\frac{1}{\mu_1^i \mu_2^j \mu_3^{N-i-j}}}{\sum_{i=0}^{N}\sum_{j=0}^{N-i}\frac{1}{\mu_1^i \mu_2^j \mu_3^{N-i-j}}}.$$

7. Bianchi Model: Discuss the changes required to the 802.11 Bianchi model if we replace the uniform backoff time by some other distribution on $\{0,\ldots,W_i-1\}$, for $i=1,\ldots,m$. More specifically, indicate the changes required in the Markov chain used to determine the throughput. Why does the uniform distribution seem like a sensible choice?

OPEN BOOK EXAM 2018

1. Poisson Process: Consider a Poisson process with rate $\lambda = 2$. Assume 4 arrivals occurred in between time 0 and 1. What is the probability that you observe another 4 arrivals between time 1 and 2? Explain your answer.

Solution: Arrivals are independent. The probability of k arrivals in time t is given by

$$P[N_t = k] = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$$

Thus for $t = 2 - 1 = 1$ and $k = 4$ we have

$$P[N_1 = 4] = \frac{(\lambda \cdot 1)^4}{4!} \exp(-\lambda \cdot 1) = \frac{\lambda^4}{24} \exp(-\lambda)$$

2. Discrete Time Markov chains: Give an example of an irreducible Markov chain with 5 states that has period 2. Explain your answer.

3. Discrete Time Markov Chains: Consider a discrete-time positive recurrent Markov chain and let $m_{i,j}$ be the mean number of steps to reach state j from state i. Prove or disprove that $m_{j,j} \leq m_{i,j} + m_{j,i}$.

4. Continuous Time Markov Chains: Give an example of an infinite state, transient continuous time Markov chain such that its uniformized and embedded Markov chain are characterized by the same transition probability matrix. Explain your answer.

Solution: Consider the CTMC with $q_{i,i+1} = \lambda$, $q_{i,i} = -\lambda$ and $q_{i,j} = 0$ for all other j.

- **Uniformization** transforms Q into $P = Q/\lambda + I$. Then $p_{i,i} = \frac{-\lambda}{\lambda} + 1 = 0$ and $p_{i,i+1} = \frac{\lambda}{\lambda} = 1$.
- The **embedded** Markov chain characterized by P with $p_{i,j} = \frac{q_{i,j}}{-q_{i,i}}$ is obviously the same P as before.

The chain is transient, since once we transition out of a state, we can never get back to it.

5. Server Farm: Consider a server farm consisting of 2 servers. Jobs arrive according to a Poisson process with rate λ and are probabilistically split among the two servers with a fraction p of the jobs going to server 1. The service time of a job on server i is exponential with mean $1/\mu_i$, for $i = 1, 2$. Give an expression for the mean response time of a job in the server farm. How does your result change if the service times are not exponential but have the same means?

6. Little's Law and PASTA: Consider a system that allows at most 120 jobs in the system. Jobs arrive as a Poisson process with rate 10 and are dropped when there are already 120 jobs in the system. If you know that the steady state probability of having 120 jobs in this system is 0.05 and the mean number of jobs in the system is 76, then what is the mean response time of a job that enters the system (that is, of a job that is not dropped)?

7. Wireless Networks: Consider a cell in a wireless network that can support up to C simultaneous calls. Assume the mean time until a call ends or leaves the cell is exponential with parameter μ, while new calls arrive at rate λ_n and handover calls at rate λ_h. Assume we have $g \geq 1$ guard channels, meaning handover calls are blocked when all the C channels are busy, while new calls are also blocked when there are $C - g$ or more busy channels. Indicate how to adapt the Markov chain of the Erlang-B loss system. Derive an expression for the steady state probabilities.

OPEN BOOK EXAM 2017

1. Poisson Process: Assume requests arrive at a web server farm according to a Poisson process with a rate of 106 requests per hour. Assume 1.3 million requests arrived in the last hour. What is the probability that at least 650 thousand requests arrived in the last half hour?

2. Discrete Time Markov Chains: Assume we have m white and m black balls that are randomly distributed over 2 bins such that each bin contains exactly m balls. Next, assume we repeatedly pick one ball from each bin at random and exchange these two balls. Explain how you can model the content of these bins using a discrete time Markov chain (give the state space and transition probabilities). Is your chain irreducible and aperiodic? Derive an expression for the steady state probabilities of this Markov chain (if they exist).

3. Discrete Time Markov Chains: Give an example of an irreducible DTMC such that its invariant distribution equals $(1/10, 1/10, 1/10, 1/10, 1/5, 1/5, 1/5)$.

4. Continuous Time Markov Chains: Give an example of an irreducible transient CTMC with period 2. Explain why the chain is transient and has period 2.

5. Continuous Time Markov Chains: Consider an $M/D/1$ queue, that is, a queue with Poisson arrivals with rate λ, an infinite waiting room, one server, and the service time of a customer equals one. Let X_t denote the number of customers in the system at time $t \geq 0$. Is $(X_t)_{t\geq 0}$ a continuous time Markov chain? (Explain your answer.)

6. Applications: Assume consultant Frank receives on average 120 assignments per year and on average he has 4 unfinished assignments. How many days does it on average take before Frank completes an assignment if he processes his assignments in first-come-first-served order? Does your answer change if Frank uses a different order to process his assignments, which also results in an average of 4 unfinished assignments (explain your answer)? What is the optimal strategy that minimizes the average completion time of an assignment (no proof needed)?

7. Applications: Consider a Jackson network with $M = 2$ queues. Let $p_{0,1} = 1$, $p_{1,1} = 1/4$, $p_{1,2} = 3/4$, $p_{2,1} = 1/2$, and $p_{2,0} = 1/2$. Determine the arrival rates λ_1 and λ_2. What is the average response time of a job in the network if $\mu_1 = 3$ and $\mu_2 = 4$? What is the largest value of λ_0 for which the system is stable, i.e., the average response time is finite, when $\mu_1 = 3$ and $\mu_2 = 4$?

8. Applications: Is it possible to adapt Bianchi's model for the 802.11 network in case we only double the backoff window after every 2 failed transmissions? If so, briefly describe the changes necessary without going into too much detail.

OPEN BOOK EXAM 2016

1. Bernoulli and Poisson Process: Indicate whether the following statements are true or false. Explain your answer.

(a) If some number was not part of the 6 lottery numbers during the last 10 weeks, it is more likely to be part of the six numbers of this week as all numbers appear equally often in the long run (due to the law of the large numbers),

(b) Points in time at which queries arrive at a DNS server, which translates domain names in IP addresses, is fairly well approximated by a Poisson process,

(c) the time epochs at which students log on to Blackboard for a specific course.

2. Branching Processes: Determine the extinction probabilities q_1, q_2, and q_3 for the following multi-type branching process, where $p^{(s)}_{i,j,k}$ is the probability that a type s individual has i children of type 1, j of type 2, and k of type 3:

- $p^{(1)}_{1,0,4} = 0.5$, $p^{(1)}_{0,1,3} = 0.25$, and $p^{(1)}_{0,2,1} = 0.25$.
- $p^{(2)}_{0,0,3} = 0.2$, $p^{(2)}_{0,1,2} = 0.4$, and $p^{(2)}_{0,2,17645} = 0.4$.
- $p^{(3)}_{0,0,k} = \binom{16}{k} \left(\frac{1}{20}\right)^k \left(1 - \frac{1}{20}\right)^{16-k}$, for $0 \leq k \leq 16$.

3. Markov Chains: Consider an irreducible, finite Markov chain with transition matrix $P = P_w + P_b$, with P_w and P_b substochastic as in Exercise 17 in the course notes. If we only observe this Markov chain after the white transitions, we obtain a new Markov chain.

(a) Show by means of an example that the new Markov chain is not necessarily irreducible.

(b) Is it possible that this new Markov chain contains two (or more) closed communicating classes?

4. Bianchi Model: How would you proceed to determine the saturation throughput of the 802.11 DCF function if we replace the binary exponential back-off algorithm with a simple ALOHA scheme (i.e., uniform back-off between 0 and W)? What are the required changes to the Bianchi model? Give an explicit expression for the failure probability p.

5. Markov Chains: Suppose customers arrive at an ATM machine according to a Poisson process with rate λ, and their transactions have a constant duration (e.g., 1 minute). Set up a Markov chain to determine the probability that a customer finds exactly i customers queued at the ATM machine at the time of their arrival. [Hint: the probability that there are i customers queued at an arrival time is identical to the probability that there are i customers queued at a departure time (i.e., when a customer completes their transaction).]

6. Bernoulli and Poisson Processes: Suppose we split a Poisson process in a probabilistic manner into process A and B and that process A is again split in a

probabilistic manner into processes A_1 and A_2. What is the mean arrival rate of the superposition of processes A_2 and B? Is this superposition also a Poisson process?

7. Erlang B Formula: Suppose that a telecom operator has both premium and regular customers, and whenever C_p or more of the available C lines are occupied (with $C_p < C$), only premium customer calls are accepted. Adapt the Markov chain in the course notes to incorporate this policy (it is not necessary to determine an expression for the steady state probabilities). How would you determine the blocking probabilities from the steady state probability vector (detailed formulas are not required)?

OPEN BOOK EXAM 2015a

1. Bernoulli and Poisson process: Indicate whether the following two processes are approximately Poisson. Explain your answer.

(a) The times at which people arrive at the US airport immigration.

(b) The points in time at which an item is ordered at `amazon.com`.

2. Discrete Time Markov Chains: Give an example of a discrete time Markov chain that is (explain your answer):

(a) Positive recurrent and its unique steady state distribution π is equal to $\pi = \frac{(1,2,4,8,16)}{31}$.

(b) Reducible with both transient and recurrent states.

3. Discrete Time Markov Chains: You toss a die repeatedly until the product of the last two outcomes is equal to 12. What is the average number of times you toss your die?

4. Continuous Time Markov Chains: Consider the continuous time Markov chain of the queue length process of an M/M/1 queue. Construct the transition probability matrix of both the uniformized and embedded Markov chain. Do both these chains have the same steady state distribution whenever $\rho < 1$?

5. Applications: Assume Sue receives on average 50 emails per day and on average she has 12 unread emails in her mailbox. How many hours does it on average take before Sue has read an incoming mail? An email is marked as read as soon as Sue opens it.

6. Applications: Consider a tandem queueing network, that is, a Jackson network with $M = 2$ queues with $p_{1,2} = 1$, $p_{2,0} = 1$, $p_{0,1} = 1$, and $p_{0,2} = 0$. Explain how the speed μ_1 of the first server affects the mean queue length and mean waiting time at the second queue.

7. Bianchi Model: Discuss the changes required to the 802.11 Bianchi model if we replace the uniform backoff time with a geometric backoff time. More specifically, indicate the changes required in the state space of the Markov chain used to determine the throughput.

OPEN BOOK EXAM 2015b

1. Bernoulli and Poisson process: Indicate whether the following two processes are Poisson. Explain your answer.

(a) Points in time at which cars pass by on a road if the time between two cars has a uniform distribution with a mean of 27 seconds.

(b) Points in time at which spam mail arrives in your mailbox.

OPEN BOOK EXAM 2014

1. Poisson Process: Consider a set of n hard disk drives and assume the lifetime of a drive is exponential with parameter μ. When a disk crashes it is not replaced. How long will it take on average before all of the n disks have crashed?

OPEN BOOK EXAM 2013

1. Bernoulli and Poisson process: Indicate whether the Poisson process is a suitable process to model the time epochs at which:

(a) orders are placed at `amazon.com`;

(b) people buy online tickets for a rock concert;

(c) queries arrive at a DNS server;

(d) handovers occur in a wireless network.

2. Bernoulli and Poisson process: Assume that the time it takes before a new computer gets infected by a virus is exponentially distributed and that about 20% get infected within the first 6 months. What is the probability that a new computer remains uninfected during the first 3 years (assuming all computers are equally vulnerable)?

OPEN BOOK EXAM 2012a

1. Bernoulli and Poisson process: Indicate whether the Poisson process is a suitable process to model:

(a) The time epochs at which hand-overs take place in a wireless network.

(b) The time epochs at which a packet is sent by a single TCP source.

(c) The time epochs at which packets are sent by a large number of TCP sources sharing a single link.

(d) The time epochs at which emails arrive in your mailbox.

2. Branching Processes: Give an example of a single-type branching process such that the extinction probability is equal to p, for any $0 < p < 1$.

3. Branching Processes: Construct a branching process to model the coordinated splitting tree algorithm (CSTA). Whenever n users collide in the CSTA, they are allowed to exchange information such that they split into n groups, each consisting of a single user. Collisions still occur on the channel as colliding users get no information with respect to the arrival times of the new users. Explain how we can determine the maximum stable throughput by means of this branching process.

4. Markov Chains: Given an example of a Markov chain that is:

(a) Irreducible, positive recurrent with period $d = 3$.

(b) Irreducible, transient, with period $d = 2$.

Explain in detail why your examples meet the above requirements.

5. Markov Chains: Assume we have 2 machines to process jobs. Machine i needs to process k_i jobs, and the processing time of a job on machine i is exponential with mean $1/\mu_i$, respectively, for $i = 1$ and 2. How can we determine the probability that machine 1 finishes its k_1 jobs before machine 2 finishes its k_2 jobs (use a Markov chain)? Assume that as soon as one machine finishes its work, it also starts to process pending jobs from the other machine (if any). How can we determine the mean time until all the $k_1 + k_2$ jobs are completed?

6. Processor Sharing Queue: Assume jobs arrive according to a Poisson process with rate λ at a single server. A job requires an exponential amount of work with mean $1/\mu$. All the jobs in the queue share the single server, meaning a job is served at rate μ/n whenever there are n jobs in the queue. Set up a Markov chain to determine the number of jobs in the queue (at arrival or departure times). Determine the condition for positive recurrence and give an explicit expression for the steady state probabilities. Give an expression for the queue length distribution at arrival times only. Do you recognize the latter expression from an earlier course?

7. Bianchi Model: Assume we use the simple ALOHA scheme to retransmit a packet instead of the binary exponential back-off algorithm, with the additional requirement

that a packet can be retransmitted at most $m > 0$ times. Discuss the changes required to the Bianchi model to determine the saturation throughput. How does m influence the saturation throughput?

OPEN BOOK EXAM 2012b

1. Bernoulli and Poisson process: Indicate whether either the Bernoulli or Poisson process is a suitable process to model:

(a) The time epochs at which search queries are performed at `www.google.be`.

(b) The days of the year during which it rains in Antwerp city

2. Bernoulli and Poisson process: Consider a (RAID) system consisting of 80 hard disk drives. Assume that the life time of a disk has an exponential distribution with a mean of 1200 days. If a disk fails it is immediately replaced by another new disk. On average how many disks need replacement per month? Explain your answer using the properties of the exponential distribution and the Poisson process.

OPEN BOOK EXAM 2010a

1. Bernoulli and Poisson Processes: Indicate whether the following processes can be well modelled by a Bernoulli or Poisson process:

(a) The process that indicates when a read or write request is generated when modelling a hard disk.

(b) The time points at which buses arrive at a bus stop.

(c) The time points at which students log in to Blackboard for a specific course.

Solution:

(a) This process can be described as a Poisson process as reading and writing from a disc is continuous and can occur at any point in time.

(b) Assuming the bus follows a regular schedule (This scenario is obviously not realistic in Belgium), this process can be described as a Bernoulli process. Each time interval we have a possibility p that the bus will arrive and the arrival of the bus in one interval is independent of the arrival of the bus in the next interval.

(c) We may observe that more students will login before the exam of a course or a specific time where the course is given in class. However, students can login at any point in time. This is a continuos process and can therefore be described as a Poisson process.

2. Branching Processes: Determine the extinction probabilities q_1, q_2, and q_3 for the following multi-type branching process, where $p(s)_{i,j,k}$ is the probability that a type s individual has i children of type 1, j of type 2, and k of type 3:

- $p(1)_{1,0,4} = 0.5$, $p(1)_{0,1,3} = 0.25$, and $p(1)_{0,2,1} = 0.25$.
- $p(2)_{0,0,3} = 0.2$, $p(2)_{0,1,2} = 0.4$, and $p(2)_{0,2,17645} = 0.4$.
- $p(3)_{0,0,k} = \binom{16}{k} \left(\frac{1}{20}\right)^k \left(1 - \frac{1}{20}\right)^{16-k}$ for $0 \leq k \leq 16$.

3. Markov Chains: Consider an irreducible, finite Markov chain with transition matrix $P = P_w + P_b$, where P_w and P_b are sub-stochastic as in exercise 17 of the course. When we observe this Markov chain only after a white transition, we obtain a new Markov chain.

(a) Show with an example that this new chain is not necessarily irreducible.

(b) Can this Markov chain contain 2 or more closed (communicating) classes?

4. Bianchi Model: Assume we want to determine the saturation throughput of the 802.11 DCF function in the case where we replace the binary exponential back-off algorithm with simple ALOHA (i.e., uniform back-off between 0 and W). How do we

proceed (what do we need to adjust)? Provide an explicit expression for p, the probability that the transmission fails.

5. Markov Chains: Assume that customers arrive at a payment machine according to a Poisson process with rate λ, and that they perform their payment in a constant time (e.g., 1 minute). Set up a Markov chain that allows us to determine the probability that a customer i has customers in front of them when they arrive at the payment machine. [Hint: The probability that a customer i has other customers in front of them upon arrival is equal to the probability that there are i customers waiting when a customer has finished their payment (or, the queue length distribution at arrival times is equal to that at departure times).]

6. Bernoulli and Poisson Processes: Suppose I probabilistically split a Poisson process into processes A and B, and then again probabilistically split A into A_1 and A_2. What is the average arrival rate of the superposition of A_2 and B, and is this also a Poisson process?

7. Erlang B Formula: Suppose a telecom operator divides its customers into premium and regular customers and applies the rule that when there are C_p or more of the C lines occupied when initiating a phone call, only calls from premium customers are accepted. Adjust the Markov chain to model this behaviour; you do not need to derive the steady state expression. How do you now calculate the loss probabilities (formulas are not needed)?

OPEN BOOK EXAM 2010b

1. Bernoulli and Poisson Processes: Argue that the minimum spanning tree (MST) of the basic binary tree algorithm does not change when we implement the following adjustment. A user who creates a new packet will send it in the next slot with probability a and in the subsequent slot with probability $1-a$.

2. Branching Processes: Suppose we use the basic binary tree algorithm on a channel with multiple reception capabilities. On such a channel, we can also correctly receive packets involved in a collision consisting of k or fewer packets. How should we adjust the branching process from the course to determine the MST (as a function of k)? What should we do with d when we increase k? How can we immediately see that $p = \frac{1}{2}$ is still an optimal choice?

3. Markov Chains: Consider an irreducible Markov chain with a finite number of states n. For which values of n can the Markov chain be periodic with a period equal to 5? Explain your answer.

4. Markov Chains: Assume that people arrive at an attraction in an amusement park according to a Poisson process with rate λ people per minute. Assume that a cart arrives every 90 seconds, accommodating (and requiring) 6 people. At which time points should we observe this system to obtain a Markov chain? What is the state space of the Markov chain, and formulate the transition matrix. For which values of λ is your chain positive recurrent? What do you expect to happen to the average waiting time when a cart arrives every minute that accommodates only 4 people?

5. Erlang C Formula: Suppose we wish to generalize the Erlang C formula by also accounting for the impatience of waiting customers. We assume that the amount of patience of a waiting customer is exponentially distributed with a mean of $1/\theta$. We construct a Markov chain with the same state space as in the Erlang C formula, but we now observe the chain not only at arrival and service completion time points but also at the times when waiting customers lose their patience and leave. In short, when there is one or more waiting customers, there are three possibilities for the next event: an arrival (λ), a service completion (μ), or a customer who loses their patience (θ).

(a) Provide the transition matrix for this system; you do not need to determine the invariant vector.

(b) When is this Markov chain positive recurrent (use Pakes)?

6. Bianchi Model: Assume that the maximum window size is 8 times greater than the minimum window size. In the course, a packet can be retransmitted an unlimited number of times; suppose we want to limit this number to 16. How should we adjust the Markov chain (without going into too much detail)?