Orbital and Spin Motion of Mercury

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1 Introduction

It is well-known that Mercury is in 3:2 spin-orbit resonance, meaning that $P_{\rm rot} = (2/3)P_{\rm orb}$, where $P_{\rm rot}$ is the spin period and $P_{\rm orb}$ is the orbital period. Furthermore, Mercury has a fairly high eccentricity: $e = 0.20563069$ (see `http://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html`). It is easy to show that the orbital angular velocity $\dot{\theta}$ is given by

$$\dot{\theta} = \frac{a^2\sqrt{1-e^2}}{r^2}\bar{\Omega}_{\rm orb},$$

where a is the orbital semi-major axis, r is the distance from the sun and $\bar{\Omega}_{\rm orb} = 2\pi/P_{\rm orb}$ is the mean orbital angular velocity. At perihelion, $r = a(1-e)$ and hence the orbital angular velocity at perihelion is

$$\dot{\theta} = \frac{\sqrt{1-e^2}}{(1-e)^2}\bar{\Omega}_{\rm orb} = 1.55086\bar{\Omega}_{\rm orb} > \frac{2\pi}{P_{\rm rot}} \qquad (1)$$

for Mercury. As a result, the sun moves eastward relative to horizon near perihelion as seen in Mercury's sky. More interestingly, Mercury is approximately tidally locked over a significant fraction of its orbit close to the perihelion, as mentioned in *Foundations of Astrophysics* by Ryden and Peterson.

The purpose of this calculation is to visualize this approximate tidal locking. Consider two points P and Q at Mercury's equator. As shown in Figure 1 where Mercury is at perihelion, P faces the sun directly ("subsolar point") and Q is at an angle $\pi/2$ to the west of P. We will calculate the position of P and Q as a function of time as Mercury spins and revolves around the sun, and see how their relative angular positions with respect to the sun change with time. Since Q is simply $\pi/2$ to the west of P, it suffices to compute the position of P. Mercury's spin axis tilts about 7° with respect to its orbital angular momentum. However, the calculation ignores this small tilt.

Animations are generated to show how the positions of P and Q relative to the sun change with time.

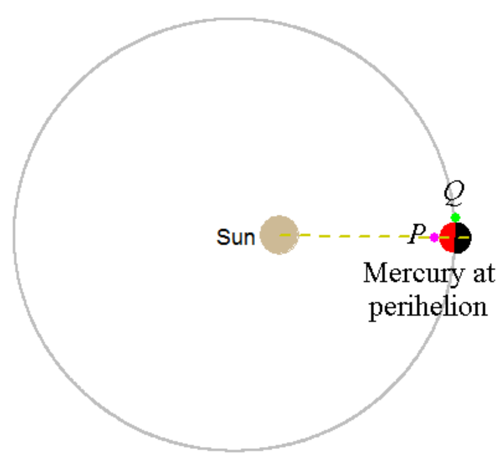


Figure 1: Mercur's orbit. The red region denotes the hemisphere facing the sun and is the day side. The black hemisphere is the night side.

2 Computation Method

Set up a coordinate system when the sun is at the origin and $\hat{x}$ points to the perihelion point. Let θ be the true anomaly of Mercury, ϕ be the phase angle of point P measured from the $\hat{x}$ direction and H be the hour angle of the sun as seen from point P (see Figure 2). It follows that $H = \pi + \phi - \theta$. Integer factors of 2π may be added to H and so the expression

$$H = \phi - \theta - \pi \tag{2}$$

is also valid. This expression is adopted so that $H = 0$ at the configuration shown in Figure 1 with $\theta = 0$. If Mercury were tidally locked, H would be time independent. The rate of change of H is therefore a measure of the degree of tidal locking, and it is simply the difference of the rotational angular velocity and orbital angular velocity:

$$\dot{H} = \dot{\phi} - \dot{\theta} = \frac{2\pi}{P_{\rm rot}} - \frac{2\pi a^2\sqrt{1-e^2}}{r^2 P_{\rm orb}} = \frac{2\pi}{P_{\rm orb}}\left(\frac{3}{2} - \frac{a^2\sqrt{1-e^2}}{r^2}\right) \tag{3}$$

Assume that $t = 0$ is the configuration shown in Figure 1. The phase angle is simply given by

$$\phi(t) = \pi + \Omega_{\rm rot} t = \pi + \frac{2\pi t}{P_{\rm rot}} = \pi + \frac{3\pi t}{P_{\rm orb}}. \tag{4}$$

The true anomaly can be determined by solving Kepler's equation

$$E - e\sin E = M = \frac{2\pi t}{P_{\rm orb}}, \tag{5}$$

where E is the eccentric anomaly and M is the mean anomaly. Mercury's center position (x, y) is given by

$$x = a(\cos E - e) \quad , \quad y = a\sqrt{1-e^2}\,\sin E \tag{6}$$

and θ is the argument of the complex number $x + yi$. Alternatively, θ can be computed by

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}. \tag{7}$$

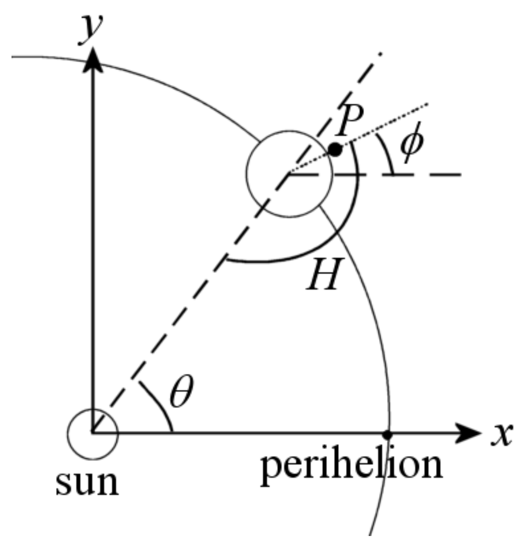


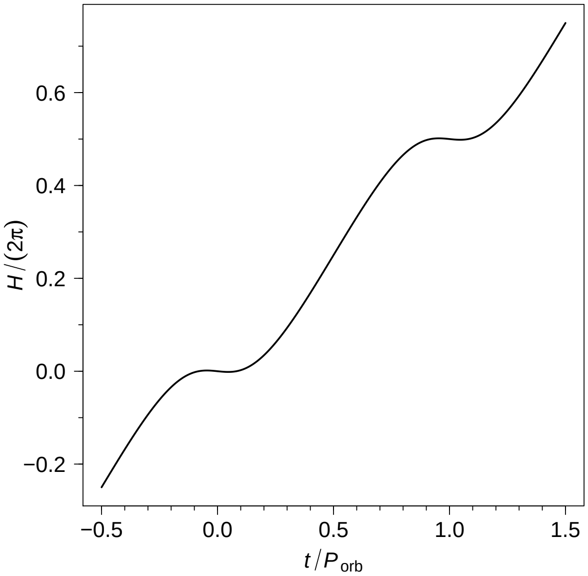
Figure 2: Hour angle H, true anomaly θ and phase angle ϕ. The sizes of Mercury, sun and Mercury's orbit are not drawn to scale.

The distance from the sun is $r = \sqrt{x^2+y^2} = a(1 - e\cos E)$. Substituting this equation to (3) gives

$$\dot{H} = \frac{2\pi}{P_{\rm orb}}\left[\frac{3}{2} - \frac{\sqrt{1-e^2}}{(1-e\cos E)^2}\right]. \tag{8}$$

So here is the recipe: for any given time t, compute the phase angle ϕ using (4) and solve the Kepler equation (5) for E. Next compute Mercury's position using (6) and calculate the true anomaly θ. The position of P, (x_P, y_P), is given by

$$x_P = x + R\cos\phi \quad , \quad y_P = y + R\sin\phi, \tag{9}$$



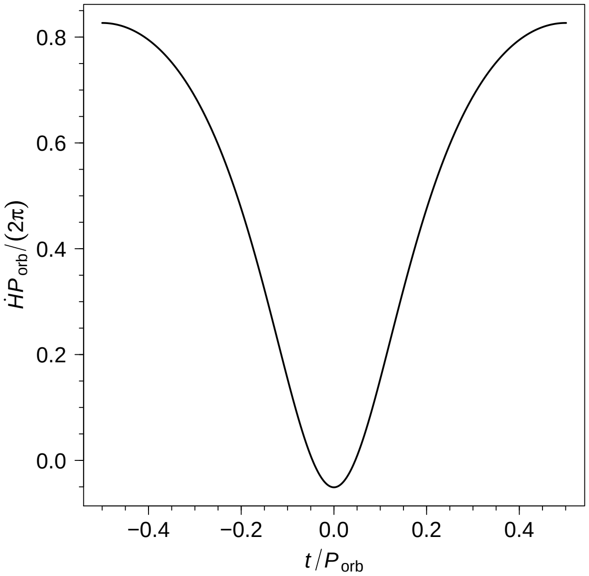


Figure 3: Left: Hour angle H of the sun as seen at point P (the magenta point in Figure 1) as a function of time. Right: The time derivative of H as a function of time.

where R is Mercury's radius. These quantities are sufficient to draw the configuration at time t. The hour angle H and its time derivative can also be computed using (2) and (3) [or (8)].

It should be noted that it is actually not necessary to solve the transcendental equation (5), as x, y, and t can be parametrized by the eccentric anomaly E, i.e. $x = x(E)$, $y = y(E)$, and $t = t(E)$. One may choose to use E as the independent variable in the calculation instead. However, if one wants to make t uniformly spaced, which is useful if one wishes to make movies, the spacing in E needs to be chosen appropriately. This is not too difficult for small time spacing Δt since it follows from (5) that

$$\Delta E \approx \frac{2\pi}{(1 - e\cos E)P_{\rm orb}}\Delta t.$$

A uniform time spacing can be created approximately by using this particular ΔE. However, solving the Kepler equation numerically is rather straightforward and was adopted in our calculation.

3 Result

Choose $t = 0$ to be the configuration in Figure 1. Figure 3 shows H and $\dot{H}$ as a function of time as seen from the point P (the magenta point in Figure 1). We see that there is a significant amount of time when H remains nearly constant, an indication of approximate tidal locking. The $\dot{H}$ curve also shows the amount of time around the perihelion where $\dot{H} < 0$. Recall that $\dot{H} < 0$ means the sun's motion is eastward relative to horizon as seen in Mercury's sky.

Data from the calculation show that $\dot{H} < 0$ when $-0.046 < t/P_{\rm orb} < 0.046$. Since $P_{\rm orb} = 87.969$ (Earth) days, the retrograde motion lasts about 8 days. During this period, the hour angle has changed by $\Delta H = -0.0194$ radians. For comparison, the angular radius of the sun as seen on Mercury at $|t| < 0.046P_{\rm orb}$ is about 0.015 radians. At point P, the sun is high near the zenith.

However, at point Q which is $\pi/2$ to the west of P (see Figure 1), the sun hovers about the horizon during this period.

It is interesting to analyze the sun's positions as seen from Q. The hour angle of the sun at Q, H_Q, is related to H by $H_Q = H - \pi/2$. Sunrise is defined as the moment when the sun's upper edge touches the horizon. Since Mercury doesn't have an atmosphere, we don't need to correct for atmospheric refraction as on Earth. Sunrise occurs when $H_Q = -\pi/2 - \alpha_\odot$ or $H = -\alpha_\odot$, where $\alpha_\odot = R_\odot/r$ is the angular radius of the sun and $R_\odot = 6.955 \times 10^8$ m is the solar radius. Calculation shows that sunrise occurs around $t = -0.1P_{\rm orb}$ and the angular radius of the sun is $\alpha_\odot = 0.014\text{rad} = 0.8°$. At $t = -0.046P_{\rm orb}$, the sun's center is 0.0097 radians (0.56°) above the horizon and the lower edge of the sun is still below the horizon. The sun then moves backward in the sky and is now setting. At $t = 0.046P_{\rm orb}$, the sun's center is 0.0097 radians (0.56°) below the horizon but the upper edge of the sun is still above the horizon. The sun rises again as $\dot{H} > 0$. The lower edge of the sun touches the horizon at $t = 0.1P_{\rm orb}$. Hence the whole sunrise lasts $0.2P_{\rm orb}$ or 17.6 days as seen at Q. Although an observer at Q sees the upper edge of the sun above the horizon during this period, observers at some points further west of Q will first see the sun rise, but before the whole sun rises it sets and rises again later when Mercury moves further away from the perihelion.

Figure 3 suggests that the period when H is nearly constant lasts longer than the period when $\dot{H} < 0$. Exactly how long it is depends on the criterion of "nearly constant." For $-0.114 < t/P_{\rm orb} < 0.114$, $-0.03 < H < 0.03$. For $-0.125 < t/P_{\rm orb} < 0.125$, $-0.045 < H < 0.045$. Hence for a quarter of orbital period around perihelion, or 22 (Earth) days, the sun remains less than 2.6° from the zenith as seen from P. This gives us an idea of the degree of tidal locking near perihelion.

Finally, Figure 4 shows the configurations of Mercury at various times of the orbit. Two animations have been created to show the spin and orbit motions. One animation shows the configurations displayed in Figure 4 as a function of time. The other one shows the change of day-night boundary as a function of time as seen in a frame corotating with Mercury. In the animations, however, the time is shifted by $0.5P_{\rm orb}$, i.e. $t(\text{animation}) = t - 0.5P_{\rm orb}$. So $t(\text{animation}) = 0$ corresponds to Mercury at aphelion. The shift is made in order to see the occurrence of approximate tidal locking when Mercury moves close to the perihelion. If the animations start at Mercury at perihelion, we will miss the beginning of the approximate tidal locking and will have to wait until Mercury completes one orbit.

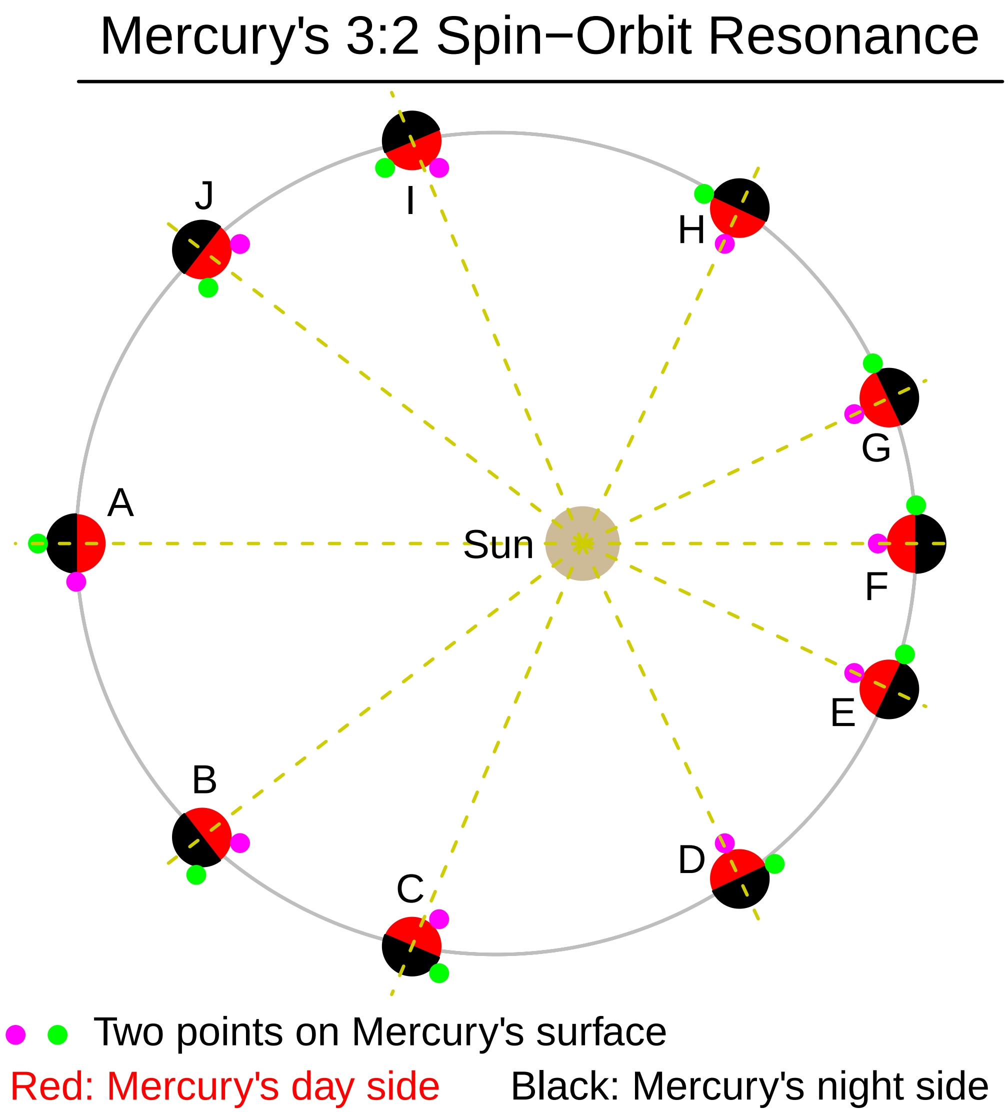


Figure 4: Mercury's spin-orbit configuration at various times in its orbit. The times from points A–J are $t/P_{\rm orb}$ =-0.5, -0.35, -0.25, -0.125, -0.046, 0, 0.046, 0.125, 0.25, and 0.35. The magenta and green points are the same points P and Q shown in Figure 1. Red hemisphere is Mercury's day side and black hemisphere is the night side. The sun's motion relative to horizon is eastward from points E to G. Approximate tidal locking occurs from points D to H, where the hour angle changes by $\Delta H = 0.06$ radians (about 5°) only. The configurations are periodic with the synodic period equal to $2P_{\rm orb}$. The configurations for $0.5 < t/P_{\rm orb} < 1.5$ can be obtained from the configurations at time $t - P_{\rm orb}$ by simply putting the magenta and green points to the opposite side of Mercury.