SPRINGER TEXTS IN STATISTICS

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PROBABILITY
Instructor's Manual

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

Section 1.1

1. a) 2/3 b) 66.67% c) 0.6667 d) 4/7 e) 57.14% f) 0.5714

2. a) 7/10 = 0.7 b) 4/10 = 0.4 c) 4/10 = 0.4

3. a) If the tickets are drawn with replacement, then, as in Example 3, there are n^2 equally likely outcomes. There is just one pair in which the first number is 1 and the second number is 2, so $P(\text{first ticket is 1 and second ticket is 2}) = 1/n^2$.
   b) The event (the numbers on the two tickets are consecutive integers) consists of $n-1$ outcomes: $(1,2), (2,3), \ldots (n-1, n)$. So its probability is $(n-1)/n^2$.
   c) Same as Problem 3 of Example 3. Answer: $(1 - 1/n)/2$
   d) If the draws are made without replacement, then there are only $n^2 - n$ equally likely possible outcomes, since we have to exclude the outcomes $(1,1), (2,2), \ldots, (n,n)$. So replace the denominators in a) through c) by $n(n-1)$.

4. a) 2/38
   b) $1 - (2/38) = 36/38$
   c) 1, since $P(\text{both win}) = 0$.

5. a) #(total) $= 52 \times 51 = 2652$ possibilities.
   b) #(first card ace) $= 4 \times 51 = 204$. Thus $P(\text{first card ace}) = 4/52 = 1/13$.
   c) This will be exactly the same calculation as in part b) – just substitute "second" for "first". Thus, $P(\text{second card ace}) = 1/13$. Because you have all possible ordered pairs of cards, any probability statement concerning the first card by itself must also be true for the second card by itself.
   d) $P(\text{both aces}) = \frac{\#(\text{both aces})}{\#(\text{total})} = \frac{4\times 3}{2652} = \frac{1}{221}$.
   e) $P(\text{at least one ace}) = P(\text{first card ace}) + P(\text{second card ace}) - P(\text{both cards aces}) = \frac{1}{13} + \frac{1}{13} - \frac{1}{221} = \frac{33}{221}$.

6. a) $52 \times 52 = 2704$
   b) $(52 \times 4)/(52 \times 52) = 1/13$
   c) Same as b)
   d) $(4 \times 4)/(52 \times 52) = 1/169$
   e) $P(\text{at least one ace}) = P(\text{first card ace}) + P(\text{second card ace}) - P(\text{both cards aces}) = \frac{1}{13} + \frac{1}{13} - \frac{1}{169} = \frac{25}{169}$.

7. a) $P(\text{maximum} \le 2) = P(\text{both dice} \le 2) = 4/36 = 1/9$
   b) $P(\text{maximum} \le 3) = P(\text{both dice} \le 3) = 9/36 = 1/4$
   c) $P(\text{maximum} = 3) = P(\text{maximum} \le 3) - P(\text{maximum} \le 2) = 5/36$
   d)

Outcome	1	2	3	4	5	6
Probability	1/36	3/36	5/36	7/36	9/36	11/36

   e) Since this covers all the possible outcomes of the experiment, and these events are mutually exclusive, you should expect $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$.

8. a-d) As in Example 3, the outcome space consists of n^2 equally likely pairs of numbers, each number between 1 and n. The event (the maximum of the two numbers is less than or equal to x) is represented by the set of pairs having both entries less than or equal to x. There are x^2 possible pairs of this type, so for $x = 0$ to n: $P(\text{maximum } \le x) = x^2/n^2$ and for $x = 1$ to n

$$\begin{aligned} P(x) &= P(\text{maximum is exactly } x) \\ &= P(\text{maximum } \le x) - P(\text{maximum } \le x - 1) \\ &= (2x-1)/n^2. \end{aligned}$$

e) As in the previous exercise, $\sum_{x=1}^{n} P(x) = 1$.

Remark. It follows that $\sum_{x=1}^{n}(2x-1) = n^2$. In other words, the sum of the first n odd numbers is n^2, a fact which you can check in other ways.

9. 1/11, 1/6

10.

Play	Chance of win	Payoff Odds = r_{pay} to 1	House Percentage
A	18/38	1 to 1	5.26%
B	12/38	2 to 1	5.26%
C	12/38	2 to 1	5.26%
D	6/38	5 to 1	5.26%
E	5/38	6 to 1	7.89%
F	4/38	8 to 1	5.26%
G	3/38	11 to 1	5.26%
H	2/38	17 to 1	5.26%
I	1/38	35 to 1	5.26%

Use the formula: House percentage $= (1 - P(\text{win}) \times (r_{pay} + 1)) \times 100\%$

11. Call the event A. By definition of fair odds, we have $r_{fair} = (1 - P(A))/P(A)$. Solve for $P(A) = 1/(r_{fair}+1)$ and substitute in the formula for the house percentage: House percentage $= (1 - P(A) \cdot (r_{pay} + 1)) \times 100\%$.

Section 1.2

1. The relative frequency interpretation makes no sense here. Presumably, what is meant is "more likely than not in the subjective opinion of the judge".

2. Less than 1/100. Since $r_{pay} < r_{fair}$ (we assume the bookmaker wants to make a profit), it follows that
$$P(\text{win}) = 1/(r_{fair} + 1) < 1/(r_{pay} + 1).$$

3. a) Let r_i^* to 1 be the fair odds against horse i winning, say in the opinion of the bookmaker, and let p_i^* be the probability of horse i winning. Then (presuming the bookmaker wants a profit) $r_i < r_i^*$, so
$$\Sigma = \sum_{i=1}^{10} p_i = \sum_{i=1}^{10} \frac{1}{r_i + 1} > \sum_{i=1}^{10} \frac{1}{r_i^* + 1} = \sum_{i=1}^{10} p_i^* = 1.$$

   b) Yes. Bet a total of $\$B$ on the horses, with proportion p_i/Σ of this total, i.e., $\$p_iB/\Sigma$ on horse i. If horse i wins, you get back an amount $\$(r_i + 1)p_iB/\Sigma = \$B/\Sigma > \$B$. So, no matter which horse wins (and regardless of any probabilities), you get back more than you bet. A golden opportunity rarely found at the races!

4. a) Overall gain $= (\$8) \times 10 - (\$1) \times 90 = -\$10$.

   b) Average gain per bet = (overall gain) / (# bets) $= \frac{-\$10}{100} = -10$ cents.

   c) Gambler's average gain per bet = (overall gain) / (# bets)
$$= \frac{r_{pay} \times (\#\text{wins}) - 1 \times (\#\text{losses})}{\#\text{wins}+\#\text{losses}} = \frac{r_{pay} - r_\#}{1 + r_\#}.$$

   The house's average financial gain per bet is therefore $(r_\# - r_{pay})/(r_\# + 1)$ over this sequence of bets. Recall that if the fair (chance) odds against the gambler winning are r_{fair} to 1, then the house percentage is $(r_{fair} - r_{pay})/(r_{fair} + 1) \times 100\%$. As the number of bets made increases, we should expect (according to the frequency interpretation of probability) $r_\#$ to approach r_{fair}, and therefore the house's average financial gain per bet should approach the house percentage. So we can interpret the house percentage as the long run average financial gain on a one dollar bet.

Section 1.3

1. The cake is $(2 \times 2) + 2 + 1 = 7$ times as large as the piece your neighbor gets, so you get 4/7 of the cake.

2. a) $(AB^c) \cup (A^cB)$
   b) $A^cB^cC^c$
   c) Exactly one : $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC)$
   Exactly two : $(ABC^c) \cup (AB^cC) \cup (A^cBC)$
   Three : ABC

3. Take $\Omega = \{1, 2, ..., 500\}$.
   a) $\{17, 93, 202\}$
   b) $\{17, 93, 202, 4, 101, 102, 398\}^c$
   c) $\{16, 18, 92, 94, 201, 203\}$

4. a) Yes: $\{0, 1\}$
   b) Yes: $\{1\}$
   c) No. This is a subset of the event $\{1\}$ but it is not identical to $\{1\}$ because the event (first toss tails, second toss heads) also is a subset of $\{1\}$.
   d) Yes: $\{1, 2\}$

5. a) first coin lands heads
   b) second coin lands tails
   c) same as a)
   d) at least two heads
   e) exactly two tails
   f) first two coins land the same way.

6. a)

outcome	1	2	4	6	7	8
probability	1/10	1/5	3/10	1/5	1/10	1/10

   b)

outcome	1	2	3
probability	3/5	1/5	1/5

7. a) As in Example 3, using the addition rule and the symmetry assumption, $P(1) = P(6) = p/2$ and $P(2) = P(3) = P(4) = P(5) = (1 - p)/4$
   b) Use the additivity property: $P(3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = \frac{3(1-p)}{4} + \frac{p}{2} = \frac{3-p}{4}$

8. a) $P(A \cup B) = P(A) + P(B) - P(AB) = 0.6 + 0.4 - 0.2 = 0.8$
   b) $P(A^c) = 1 - P(A) = 0.4$
   c) Similarly $P(B^c) = 0.6$
   d) $P(A^cB) = P(B) - P(AB) = 0.4 - 0.2 = 0.2$
   e) $P(A \cup B^c) = P[(A^cB)^c] = 1 - 0.2 = 0.8$
   f) $P(A^cB^c) = P[(A \cup B)^c] = 1 - 0.8 = 0.2$

9. a) 0.9 b) 1 c) 0.1

10. a) $P(\text{exactly 2 of } A, B, C) = P(ABC^c) + P(AB^cC) + P(A^cBC)$

$$= (P(AB) - P(ABC)) + (P(AC) - P(ABC)) + (P(BC) - P(ABC))$$
$$= P(AB) + P(AC) + P(BC) - 3P(ABC)$$

b) $P(\text{exactly 1 of } A, B, C) = P(AB^cC^c) + P(A^cB^cC) + P(A^cB^cC)$

$$= P(A) - P(A\cap(B\cup C)) + P(B) - P(B\cap(A\cup C)) + P(C) - P(C\cap(A\cup B))$$
$$= \dots$$
$$= P(A) + P(B) + P(C) - 2\,(P(AB) + P(AC) + P(BC)) + 3P(ABC)$$

c) $P(\text{exactly none}) = 1 - P(A\cup B\cup C)$

$$= 1 - (P(A) + P(B) + P(C)) + (P(AB) + P(AC) + P(BC)) - P(ABC)$$

11. By inclusion-exclusion for $n = 2$,

$$P(A\cup B\cup C) = P[(A\cup B)\cup C] = P(A\cup B) + P(C) - P[(A\cup B)C]$$
$$= P(A) + P(B) - P(AB) + P(C) - P[(A\cup B)C].$$

Now $(A\cup B)C = (AC)\cup(BC)$, which is clear from a Venn diagram. So by another application of inclusion-exclusion for $n = 2$,

$$P[(A\cup B)C] = P[(AC)\cup(BC)] = P(AC) + P(BC) - P[(AC)(BC)]$$
$$= P(AC) + P(BC) - P(ABC)$$

Substitute this into the previous expression.

12. Formula has been proved for $n = 1, 2, 3$. Proceed by mathematical induction. We should first prove

$$(\cup_{i=1}^n A_i)\cap A_{n+1} = \cup_{i=1}^n (A_iA_{n+1}),$$

which is easily checked, e.g. by another mathematical induction using the $n = 2$ case: $(A\cup B)C = (AC)\cup(BC)$. Next, assume that our hypothesis is true for n. Then

$$P(\cup_{i=1}^{n+1} A_i) = P[(\cup_{i=1}^n A_i)\cup A_{n+1}] = P(\cup_{i=1}^n A_i) + P(A_{n+1}) - P[\cup_{i=1}^n (A_iA_{n+1})]$$

(by inclusion-exclusion for $n = 2$)

$$= \sum_{i=1}^n P(A_i) - \sum_{i<j\le n} P(A_iA_j) + \sum_{i<j<k\le n} P(A_iA_jA_k) - \dots$$
$$+P(A_{n+1}) - \sum_{i\le n} P(A_iA_{n+1}) + \sum_{i<j\le n} P(A_iA_jA_{n+1}) - \dots$$

(by applying the induction hypothesis to the first and third term)

$$= \sum_{i=1}^{n+1} P(A_i) - \sum_{i<j\le n+1} P(A_iA_j) + \sum_{i<j<k\le n+1} P(A_iA_jA_k) - \dots$$

(by regrouping terms.) So the claim holds for $n+1$.

13. For $n = 2$, $P(A_1\cup A_2) = P(A_1) + P(A_2) - P(A_1\cap A_2) \le P(A_1) + P(A_2)$ Now use induction by assuming true for n.

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\left[\bigcup_{i=1}^{n} A_i\right]\cup A_{n+1}\right) \le P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) \le \sum_{i=1}^n P(A_i) + P(A_{n+1})$$

14. Use the identity $P(A\cup B) = P(A) + P(B) - P(AB)$ and the fact that $P(A\cup B)\le 1$.

15. Let $A_i = B_i^c$. Then

$$P(B_1B_2\cdots B_n) = P(\bigcap_{i=1}^n A_i^c) = P((\bigcup_{i=1}^n A_i)^c) = 1 - P(\bigcup_{i=1}^n A_i)$$
$$\ge 1 - \sum_{i=1}^n P(A_i) = 1 - \sum_{i=1}^n (1 - P(B_i)) = \sum_{i=1}^n P(B_i) - (n-1)$$

16. a) Use mathematical induction. The claim holds for $n = 1$ and 2. Suppose the claim holds for unions of n sets. To show the claim for $n + 1$ sets, use inclusion-exclusion for 2 sets to write

$$P(\cup_{i=1}^{n+1} A_i) = P\left((\cup_{i=1}^{n} A_i) \cup A_{n+1}\right) = P(\cup_{i=1}^{n} A_i) + P(A_{n+1}) - P\left((\cup_{i=1}^{n} A_i)A_{n+1}\right)$$

Use the induction hypothesis to bound the first term:

$$P(\cup_{i=1}^{n} A_i) \geq \sum_{i \leq n} P(A_i) - \sum_{i<j\leq n} P(A_iA_j)$$

and use Boole's inequality to bound the last term:

$$P\left((\cup_{i=1}^{n} A_i)A_{n+1}\right) = P(\cup_{i=1}^{n} A_iA_{n+1}) \leq \sum_{i \leq n} P(A_iA_{n+1})$$

Regroup to see the claim holds for $n + 1$.

b) Define for $m \geq 1$ and sets $A_1, \dots, A_n$

$$S(m; A_1, \dots, A_n) = \sum_{i \leq n} P(A_i) - \sum_{i<j\leq n} P(A_iA_j) + \cdots + (-1)^{m+1} \sum_{k_1 < \dots < k_m \leq n} P(A_{k_1} \cdots A_{k_m})$$

The claim is that

Claim: For each $m \geq 1$ and each $n \geq 1$: If $A_1, \dots, A_n$ are n sets then

$$(-1)^m \left(P(\cup_1^n A_i) - S(m; A_1, \dots, A_n)\right) \geq 0.$$

Proof: Use mathematical induction. The claim holds for $m = 1$ (by a)) (and also for $m = 2$, by b)). Say the claim holds for m. Now show that this implies the claim holds for $m + 1$, i.e., that:

Subclaim: (m is fixed) For each $n \geq 1$: If $A_1, \dots, A_n$ are n sets then

$$(-1)^{m+1} \left(P(\cup_1^n A_i) - S(m+1; A_1, \dots, A_n)\right) \geq 0.$$

Proof of subclaim: Again use mathematical induction. This subclaim holds for $n = 1, \dots, m$ because the inclusion-exclusion formula (Exercise 12) shows that the left side equals zero. Suppose the claim holds for n. Now show that this implies the subclaim holds for $n + 1$, i.e., that

$$(-1)^{m+1} \left(P(\cup_1^{n+1} A_i) - S(m+1; A_1, \dots, A_{n+1})\right) \geq 0. \qquad (*)$$

To do this, first observe that

$$S(m+1; A_1, \dots, A_n) + P(A_{n+1}) - S(m; A_1A_{n+1}, \dots, A_nA_{n+1}) = S(m+1; A_1, \dots, A_{n+1}). \qquad (**)$$

Therefore

$$(-1)^{m+1} \left(P(\cup_1^{n+1} A_i) - S(m+1; A_1, \dots, A_{n+1})\right)$$
$$= (-1)^{m+1} \left(P(\cup_1^n A_i) + P(A_{n+1}) - P(\cup_1^n A_iA_{n+1}) - S(m+1; A_1, \dots, A_{n+1})\right)$$

(by inclusion-exclusion for 2 sets)

$$= (-1)^{m+1} \{P(\cup_1^n A_i)$$
$$- P(\cup_1^n A_iA_{n+1}) + S(m; A_1A_{n+1}, \dots, A_nA_{n+1}) - S(m+1; A_1, \dots, A_n)\}$$

(by identity (**))

$$= (-1)^{m+1} \{P(\cup_1^n A_i) - S(m+1; A_1, \dots, A_n)\}$$
$$+ (-1)^m \{P(\cup_1^n A_iA_{n+1}) - S(m; A_1A_{n+1}, \dots, A_nA_{n+1})\}$$

The first term is nonnegative by the induction hypothesis on n; the second term is nonnegative by the induction hypothesis on m. So (*) holds, and we're done!

Section 1.4

1. Let π denote the proportion of women in the population. Then the proportion r of righthanders is given by $r = .92 \times \pi + .88 \times (1 - \pi) = .88 + .04\pi$.

   a) (iii) ; since π is unknown.

   b) (i) ; since $0 \leq \pi \leq 1$ implies $.88 \leq r \leq .92$.

   c) (i) ; $.88 + .04 \times .5 = .9$.

   d) (i) ; solve $.88 + .04 \times \pi = .9$, you'll get $\pi = .5$.

   e) (i) ; if $\pi \geq .75$, then $r \geq .91$.

2. Pick a light bulb at random. Let D be the event (the light bulb is defective), and B be the event (the light bulb is made in city B). Then $P(B) = 1/3$, $P(D|B) = .01$, and

$$P(D^cB) = P(D^c|B)P(B) = [1 - P(D|B)]P(B) = 0.99 \times 1/3 = 0.33$$

3. $P(\text{rain tomorrow} \mid \text{rain today}) = \dfrac{P(\text{rain today and tomorrow})}{P(\text{rain today})} = \dfrac{30\%}{40\%} = \frac{3}{4}$

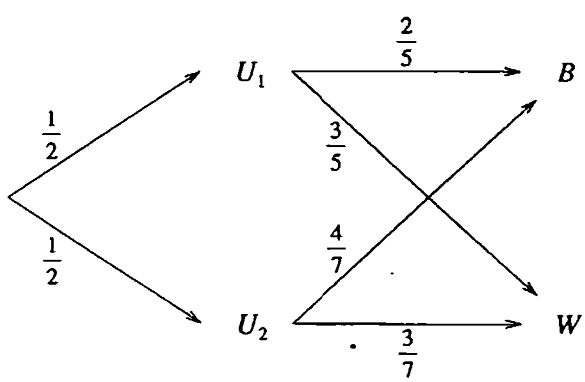
4. Call the events A (having probability .1) and B (having probability .3).

   a) $P(A^cB^c) = P(A^c)P(B^c) = 0.9 \times 0.7 = 0.63$

   b) $1 - P(A^cB^c) = 0.37$

   c) $P(AB^c \cup A^cB) = P(AB^c) + P(A^cB) = 0.1 \times 0.7 + 0.9 \times 0.3 = 0.34$

5. a) Let U_1 = (urn 1 chosen), U_2 = (urn 2 chosen), B = (black ball chosen), W = (white ball chosen).



   b) $P(U_1) = \frac{1}{2} = P(U_2); P(W|U_1) = \frac{3}{5}; P(B|U_1) = \frac{2}{5}; P(W|U_2) = \frac{3}{7}; P(B|U_2) = \frac{4}{7}$

   c) $P(B) = P(BU_1) + P(BU_2) = P(B|U_1)P(U_1) + P(B|U_2)P(U_2) = \frac{4}{7} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{2} = \frac{17}{35}$

6. $P(\text{second spade} \mid \text{first black})$

$$= \frac{P(\text{first black and second spade})}{P(\text{first black})}$$

$$= \frac{P(\text{first spade and second spade}) + P(\text{first club and second spade})}{P(\text{first black})}$$

$$= \frac{(13/52)(12/51) + (13/52)(13/51)}{(26/32)} = \frac{25}{102}.$$

Or you may use a symmetry argument as follows:
$P(\text{second black} \mid \text{first black}) = 25/51$,
and $P(\text{second spade} \mid \text{first black}) = P(\text{second club} \mid \text{first black})$
by symmetry. Therefore $P(\text{second spade} \mid \text{first black}) = 25/102$.

Discussion. The frequency interpretation is that over the long run, out of every 102 deals yielding a black card first, about 25 will yield a spade second.

7. a) $P(B) = 0.3$

   b) $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$
   $\Longrightarrow P(B) = \frac{P(A \cup B) - P(A)}{1 - P(A)} = \frac{0.3}{0.5} = 0.6$

8. Assume n cards and all $2n$ faces are equally likely to show on top.
   P(white on bottom | black on top)

   $$= \frac{P(\text{white on bottom and black on top})}{P(\text{black on top})} = \frac{50\% \times \frac{1}{2}}{50\% \times \frac{1}{2} + 20\%} = 5/9$$

9. By scheme A, P(student selected is from School 1) $= 100/1000 = 1/10$, while by scheme B, that chance is just $1/3$, which is the chance that school 1 is selected. So these two schemes are not probabilistically equivalent.
   Consider a particular student x, and suppose she is from school i. The chance that x will be selected by scheme A is $1/1000$. The chance that x will be selected by scheme C is

   $$P(x \text{ is selected}) = P(\text{School } i \text{ is selected, and then } x) = p_i \cdot \frac{1}{(\text{size of class } i)}.$$

   For scheme C to be equivalent to scheme A, this chance should be $1/1000$. Therefore,
   $p_i = (\text{size of class } i)/1000$. So $p_1 = .1$, $p_2 = .4$, $p_3 = .5$.

10. a) Let A_i be the event that the ith source works.
    $P(\text{zero work}) = P(A_1^c A_2^c) = 0.6 \times 0.5 = 0.3$
    $P(\text{exactly one works}) = P(A_1 A_2^c) + P(A_1^c A_2) = 0.4 \times 0.5 + 0.6 \times 0.5 = 0.5$
    $P(\text{both work}) = P(A_1 A_2) = 0.4 \times 0.5 = 0.2$

    b) $P(\text{enough power}) = 0.6 \times 0.5 + 1 \times 0.2 = 0.5$

11. Let B_1 = (firstborn is a boy), G_1 = (firstborn is a girl), similarly for B_2 and G_2.

    a) $P(B_1 B_2) = P(B_1 B_2 \mid \text{identical})P(\text{identical}) + P(B_1 B_2 \mid \text{fraternal})P(\text{fraternal})$

    $$= \frac{1}{2} \cdot p + \frac{1}{4} \cdot (1 - p) = \frac{1+p}{4}.$$

    b) $P(B_1 G_2) = 0 \cdot p + \frac{1}{4} \cdot (1 - p) = \frac{1-p}{4}$.

    c) Note that the chance that the firstborn is a boy is $1/2$ whether identical or fraternal, so $P(B_1) = 1/2$. Similarly $P(B_2) = P(G_1) = P(G_2) = 1/2$. So

    $$P(G_2|B_1) = \frac{P(B_1 G_2)}{P(B_1)} = \frac{1/4 \cdot (1-p)}{1/2} = \frac{1}{2}(1 - p).$$

    d)

    $$P(G_2|G_1) = \frac{P(G_1 G_2)}{P(G_1)} = \frac{1/4 \cdot (1+p)}{1/2} = \frac{1}{2}(1 + p).$$

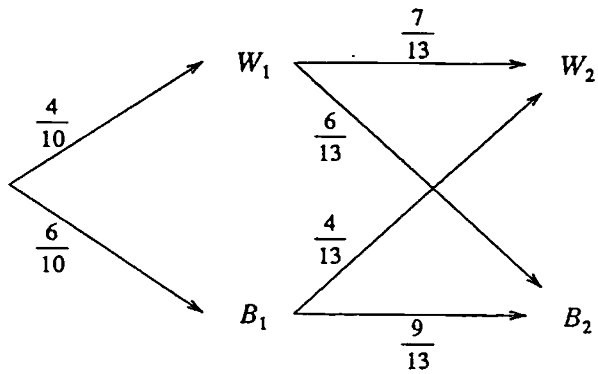
12. $\frac{P(F) - P(FG)}{1 - P(G)}$

Section 1.5

1. a) $P(\text{black}) = P(\text{black} \mid \text{odd})P(\text{odd}) + P(\text{black} \mid \text{even})P(\text{even}) = \frac{1}{4} \times \frac{1}{2} + \frac{2}{6} \times \frac{1}{2} = \frac{7}{24}$

   b) $P(\text{even} \mid \text{white}) = \dfrac{P(\text{white} \mid \text{even})P(\text{even})}{P(\text{white})} = \frac{(2/3)\times(1/2)}{17/24} = \frac{8}{17}$

2.

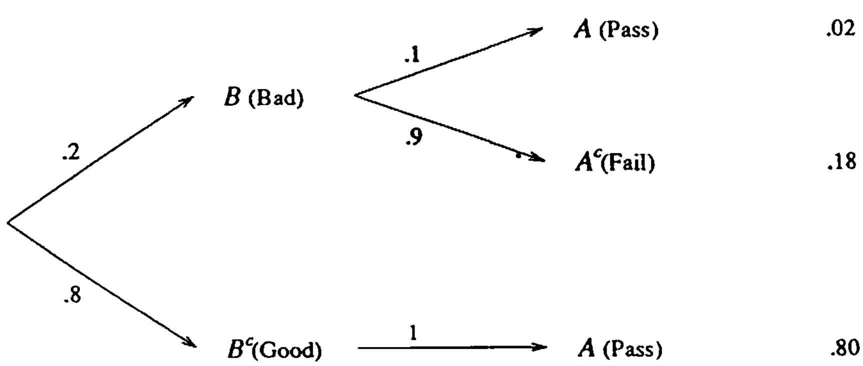


   a) $P(W_2) = P(W_1W_2) + P(B_1W_2) = \frac{4}{10} \cdot \frac{7}{13} + \frac{6}{10} \cdot \frac{4}{13} = \frac{2}{5}$

   b) $P(B_1|W_2) = \frac{P(B_1W_2)}{P(W_2)} = \frac{\frac{6}{10}\cdot\frac{4}{13}}{\frac{4}{10}} = \frac{6}{13}$.

   c) Repeat a), with symbols:
   $P(W_2) = \frac{w}{w+b} \cdot \frac{w+d}{w+b+d} + \frac{b}{w+b} \cdot \frac{w}{w+b+d} = \frac{w}{w+b}$

3. Pick at chip at random. Let $B =$ (chip is bad), let $A =$ (chip passes the cheap test). Then $P(B) = .2$, $P(A|B) = .1$, and $P(A|B^c) = 1$, which imply $P(B^c) = .8$, and $P(A^c|B) = .9$.



   a) $P(B^c|A) = \frac{P(A|B^c)P(B^c)}{P(A|B^c)P(B^c)+P(A|B)P(B)} = \frac{0.80}{0.02+0.80} = \frac{40}{41}$. Or use Bayes' rule for odds.

   b) $P(B|A) = 1 - P(B^c|A) = 1/41$.

4. a) $P(T_0|R_1) = \frac{P(R_1|T_0)P(T_0)}{P(R_1|T_0)P(T_0)+P(R_1|T_1)P(T_1)} = \frac{(.01)(.5)}{(.01)(.5)+(.98)(.5)} = \frac{1}{99}$

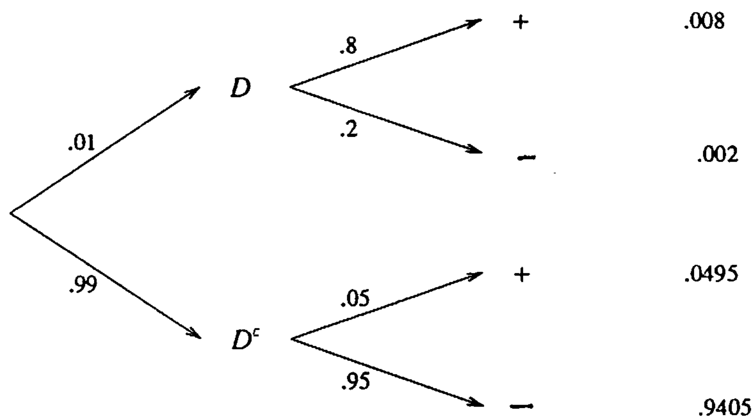
   b) $\frac{(.01)(.2)}{(.01)(.2)+(.98)(.8)} = \frac{1}{393}$

c) $P(\text{error in transmission}) = P(R_0T_1 \cup R_1T_0)$

$$= P(R_0T_1) + P(R_1T_0)$$
$$= P(R_0|T_1)P(T_1) + P(R_1|T_0)P(T_0)$$
$$= (.02)(.5) + (.01)(.5) = 3/200$$

d) $(.02)(.8) + (.01)(.2) = 9/500$

5. Let D denote the event (person has the disease), $+$ denote the event (person is diagnosed as having the disease), and $-$ denote the event (person is diagnosed as healthy). Then $P(D) = .01$, $P(+|D^c) = .05$, $P(-|D) = .2$, which in turn imply that $P(D^c) = .99$, $P(-|D^c) = .95$, $P(+|D) = .8$.



a) $P(+) = P(+|D)P(D) + P(+|D^c)P(D^c) = .0575$.

b) $P(- \cap D) = P(-|D)P(D) = .002$

c) $P(- \cap D^c) = P(-|D^c)P(D^c) = .9405$

d) $P(D|+) = \frac{P(+|D)P(D)}{P(+)} = \frac{16}{115}$.
Or we may argue using odds ratios, as follows:
$\frac{P(D)}{P(D^c)} = \frac{1}{99}$, $\frac{P(+|D)}{P(+|D^c)} = \frac{0.8}{0.05} = 16 \Longrightarrow \frac{P(D|+)}{P(D^c|+)} = \frac{1}{99} \times 16 = \frac{16}{99}$
Hence $P(D|+) = \frac{16}{99+16} = \frac{16}{115}$.

e) Yes. See the explanation after Example 3.

6. a) The experimenter is assuming that before the experiment, H_1, H_2, and H_3 are equally likely. That is, prior probabilities are given by $P(H_1) = P(H_2) = P(H_3) = 1/3$.

b) No, because the above assumption is being made. Since the prior probabilities are subjective, so are the posterior probabilities.

c) Prior probabilities: $P(H_1) = .5$, $P(H_2) = .45$, $P(H_3) = .05$.
Likelihoods of A: $P(A|H_1) = .1$, $P(A|H_2) = .01$, $P(A|H_3) = .39$.
[Now all these probabilities have a long run frequency interpretation, so the posterior probabilities will as well.]
Posterior probabilities are proportional to: .05, .0045, .0195. So H_3 is no longer the most likely hypothesis; H_1 is, and $P(H_3|A) = \frac{.0195}{.05+.0045+.0195} = .263$.

7. a) As in Example 1,

i	1	2	3
$P(\text{Box } i \text{ and white})$	1/6	2/9	1/4
$P(\text{Box } i \text{ and black})$	1/6	1/9	1/12
$P(\text{Box } i \mid \text{white})$	6/23	8/23	9/23
$P(\text{Box } i \mid \text{black})$	6/13	4/13	3/13

Since $P(\text{Box } i \mid \text{white})$ is largest when $i = 3$ and $P(\text{Box } i \mid \text{black})$ is largest when $i = 1$, the strategy is to guess Box 3 when a white ball is drawn and guess Box 1 when a black ball is drawn. The probability of guessing correctly is

$$P(\text{Box 3 and white}) + P(\text{Box 1 and black}) = 5/12.$$

b) Suppose your strategy is to guess box i with probability p_i ($i = 1, 2, 3$; $p_1 + p_2 + p_3 = 1$) whenever a white ball is drawn, and suppose I am picking each box with probability 1/3. Then in cases where a white ball is drawn, the probability that you guess correctly is

$$p_1\frac{6}{23} + p_2\frac{8}{23} + p_3\frac{9}{23} = \frac{9}{23} - p_1\frac{3}{23} - p_2\frac{1}{23}.$$

Clearly the probability of your guessing correctly is greatest when $p_3 = 1$; that is, when you guess Box 3 every time that you see a white ball. A similar argument for the case where a black ball is drawn shows that the probability of your guessing correctly is greatest when you guess Box 1 every time that you see a black ball.

c) Here $P(\text{Box 1}) = 1/2$, $P(\text{Box 2}) = 1/4$, $P(\text{Box 3}) = 1/4$, so that

i	1	2	3
$P(\text{Box } i \text{ and white})$	1/4	1/6	3/16
$P(\text{Box } i \text{ and black})$	1/4	1/12	1/16
$P(\text{Box } i \mid \text{white})$	12/29	8/29	9/29
$P(\text{Box } i \mid \text{black})$	12/19	4/19	3/19

If you continue to use the strategy found in a), i.e. guess Box 3 if a white ball is seen, and guess Box 1 if a black ball is seen, then the probability of guessing correctly is

$$P(\text{Box 3 and white}) + P(\text{Box 1 and black}) = \frac{3}{16} + \frac{1}{4} = \frac{7}{16}.$$

d) If you are convinced that I am using either the (1/3, 1/3, 1/3) strategy or the (1/2, 1/4, 1/4) strategy, then you can decide which one it is by observing the long-run proportion of trials that your guesses are correct. If I am using the (1/3, 1/3, 1/3) strategy then, by the frequency interpretation of probability, the proportion of trials that you guess correctly should be approximately 5/12, while if I am using the (1/2, 1/4, 1/4) strategy then this proportion should be closer to 7/16. If I am in fact using the (1/2, 1/4, 1/4) picking strategy, you can improve upon the guessing strategy found in a) as follows: Since $P(\text{Box } i \mid \text{white})$ is largest when $i = 1$ and $P(\text{Box } i \mid \text{black})$ is largest when $i = 1$, the strategy is to always guess Box 1. The probability of guessing correctly is then

$$P(\text{Box 1}) = \frac{1}{2}.$$

Remark: If you're not sure at all what strategy I am using, you can determine it approximately, as follows: Suppose I am using a (p_1, p_2, p_3) strategy to pick the boxes, and suppose you decide that you will always pick (say) Box 1 when you see a white ball, Box 2 when you see a black ball. It is possible for you to determine my strategy, provided you can keep track of the proportion of trials where a white ball was chosen from Box 1 and the proportion of trials where a black ball was chosen from Box 2. (This you can do, because you can see the color of the ball drawn, and you are told whether your guess is correct.) By the frequency interpretation of probability, these two proportions should in the long run approximate the probabilities

$$P(\text{Box 1 and white}) = \frac{1}{2}p_1 \text{ and } P(\text{Box 2 and black}) = \frac{1}{3}p_2$$

respectively. Thus you should be able to determine (approximately) the values p_1 and p_2 (and hence p_3).

8. a) The prior probabilities (of the boxes that I pick) are as follows:

i	1	2	3
π_i	6/23	9/23	8/23

Use Bayes' rule $P(\text{Box } i \mid \text{black}) = \dfrac{P(\text{Box } i \text{ and black})}{P(\text{black})} = \dfrac{\pi_i \cdot \frac{1}{i+1}}{\pi_1 \cdot \frac{1}{2} + \pi_2 \cdot \frac{1}{3} + \pi_3 \cdot \frac{1}{4}}$ to determine the posterior probabilities:

i	1	2	3
$P(\text{Box } i \mid \text{black})$	3/8	3/8	1/4

So you should guess Box 1 or 2. For either choice, $P(\text{you guess correctly} \mid \text{black}) = 3/8$.

b) Use Bayes' rule $P(\text{Box } i \mid \text{white}) = \dfrac{P(\text{Box } i \text{ and white})}{P(\text{white})} = \dfrac{\pi_i \cdot \frac{i}{i+1}}{\pi_1 \cdot \frac{1}{2} + \pi_2 \cdot \frac{2}{3} + \pi_3 \cdot \frac{3}{4}}$ to determine the posterior probabilities:

i	1	2	3
$P(\text{Box } i \mid \text{white})$	1/5	2/5	2/5

[You can use $P(\text{white}) = 1 - P(\text{black}) = 15/23$.] So you should guess Box 2. For either choice, $P(\text{you guess correctly} \mid \text{white}) = 2/5$.

c) $P(\text{guess correctly})$

$= P(\text{guess correctly} \mid \text{black})P(\text{black}) + P(\text{guess correctly} \mid \text{white})P(\text{white})$

$= \frac{3}{8} \times \frac{8}{23} + \frac{2}{5} \times \frac{15}{23} = \frac{9}{23}$.

This is your chance of winning, no matter which of the two best choices you make in each case. For instance, you could simply always guess Box 2, in which case the event of a win for you is the same as the event that I pick Box 2.

To see why the probability of your guessing correctly is at most 9/23 whatever your strategy may be, suppose that your strategy whenever a black ball is drawn is to guess Box i with probability p_i. Then

$P(\text{guess correctly} \mid \text{black})$

$= p_1 P(\text{Box 1} \mid \text{black}) + p_2 P(\text{Box 2} \mid \text{black}) + p_3 P(\text{Box 3} \mid \text{black})$

$= p_1 \frac{3}{8} + p_2 \frac{3}{8} + p_3 \frac{1}{4} = \frac{3}{8} - \frac{1}{8} b_3 \le \frac{3}{8}$.

Similarly, whatever your strategy whenever a white ball is drawn,

$P(\text{guess correctly} \mid \text{white}) \le \frac{2}{5}$. Therefore

$P(\text{guess correctly})$

$= P(\text{guess correctly} \mid \text{black})P(\text{black}) + P(\text{guess correctly} \mid \text{white})P(\text{white})$

$\le \frac{3}{8} \times \frac{8}{23} + \frac{2}{5} \times \frac{15}{23} = \frac{9}{23}$.

d) $P(\text{guess correctly} \mid \text{Box 1})$

$= P(\text{guess 1} \mid \text{black \& Box 1})P(\text{black \& Box 1} \mid \text{Box 1})$

$+ P(\text{guess 1} \mid \text{white \& Box 1})P(\text{white \& Box 1} \mid \text{Box 1})$

$= \frac{18}{23} \times \frac{1}{2} + 0 \times \frac{1}{2} = \frac{9}{23}$.

Similarly $P(\text{guess correctly} \mid \text{Box 2}) = \frac{5}{23} \times \frac{1}{3} + \frac{11}{23} \times \frac{2}{3} = \frac{9}{23}$ and

$P(\text{guess correctly} \mid \text{Box 3}) = 0 \times \frac{1}{4} + \frac{12}{23} \times \frac{3}{4} = \frac{9}{23}$.

Section 1.6

1. This is just like the birthday problem:

   $P(2 \text{ or more under same sign}) = 1 - P(\text{all different signs})$.

$$n = 2 : \text{chance} = 1 - \frac{11}{12} = \frac{1}{12} < \frac{1}{2}$$
$$n = 3 : \text{chance} = 1 - \frac{11}{12} \times \frac{10}{12} = \frac{17}{72} < \frac{1}{2}$$
$$n = 4 : \text{chance} = 1 - \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} = \frac{123}{288} < \frac{1}{2}$$
$$n = 5 : \text{chance} = 1 - \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{8}{12} = \frac{178}{288} > \frac{1}{2}$$

   So, the answer is 5.

2. Assume the successive hits are independent, each occurring with chance 0.3.

   a) $1 - (0.7)^2 = 0.51$

   b) $1 - (0.7)^3 = 0.657$

   c) $1 - (0.7)^n$

3. a) $P(\text{at least two heads} \mid \text{at least one head}) = \dfrac{P(\text{at least two H and at least one H})}{P(\text{at least one H})}$

   $= \dfrac{P(\text{at least two H})}{P(\text{at least one H})} = \frac{3(2/3)^2(1/3)+(2/3)^3}{1-(1/3)^3} = \frac{.741}{.963} = .7695.$

   b) $P(\text{exactly one H} \mid \text{at least one H}) + P(\text{at least two H} \mid \text{at least one H}) = 1$,
   because (at least one H) = (exactly one H) $\cup$ (at least two H), and the two events are disjoint.
   So $P(\text{exactly one H} \mid \text{at least one H}) = 1 - .7695 = .2305$.

4. a) $\frac{1}{20} \times \frac{9}{20} \times \frac{1}{20} = 9/8000$

   b) $\left(\frac{1}{20} \times \frac{9}{20} \times \frac{19}{20}\right) + \left(\frac{1}{20} \times \frac{11}{20} \times \frac{1}{20}\right) + \left(\frac{19}{20} \times \frac{9}{20} \times \frac{1}{20}\right) = 353/8000$

   c) $P(\text{jackpot}) = \frac{3}{20} \times \frac{1}{20} \times \frac{3}{20} = 9/8000$, same as before;
   $P(\text{two bells}) = \left(\frac{3}{20} \times \frac{1}{20} \times \frac{17}{20}\right) + \left(\frac{3}{20} \times \frac{19}{20} \times \frac{3}{20}\right) + \left(\frac{17}{20} \times \frac{1}{20} \times \frac{3}{20}\right) = 273/8000$
   The chance of the jackpot is the same on both machines, but the 1-9-1 machine encourages you to play, because you have a better chance of two bells. It will seem that you are "close" to a jackpot more frequently.

5. a) $P(\text{at least one student has the same birthday as mine})$
   $= 1 - P(\text{all } n-1 \text{ other students have different birthdays from mine})$
   $= 1 - (364/365)^{n-1}$
   [My birthday is one particular day: in order for the event to occur, each of the other students must have been born on one of the remaining 364 days .]

   b) Use the argument in Example 3. The desired event has probability at least 1/2 if and only if its complement has probability at most 1/2. This occurs if and only if $(364/365)^{n-1} \le 1/2 \iff$
   $n - 1 \ge \frac{\log(1/2)}{\log(364/365)} \approx 253.6$. So $n = 254$ will do.

   c) The difference between this and the standard birthday problem is that a *particular* birthday (say January 1st) must occur twice in the class. This is a much stricter requirement than in the usual birthday problem.

6. a) $p_8 = p_9 = \cdots = 0$. (Never need to roll more than seven times)
   $p_1 = 0$
   $p_2 = 1/6$
   $p_3 = (5/6) \cdot (2/6)$ (2nd different from first, third the same as first or second)

$p_4 = (5/6) \cdot (4/6) \cdot (3/6)$
$p_5 = (5/6) \cdot (4/6) \cdot (3/6) \cdot (4/6)$
$p_6 = (5/6) \cdot (4/6) \cdot (3/6) \cdot (2/6) \cdot (5/6)$
$p_7 = (5/6) \cdot (4/6) \cdot (3/6) \cdot (2/6) \cdot (1/6) \cdot 1$

b) $p_1 + \cdots + p_{10} = 1$: you must stop before the tenth roll, and the events determining p_1, p_2, etc., are mutually exclusive.

c) Of course you can compute them and add them up. Here's another way. In general, let A_i be the event that the first i rolls are different, then $p_i = P(A_{i-1}) - P(A_i)$ for $i = 2, \ldots, 7$, with $P(A_1) = 1$, and $P(A_7) = 0$. Adding them up, you can easily check that the sum is 1.

7. a) $p_3(p_1 + p_2 - p_1 p_2) = p_3 p_1 q_2 + p_3 q_1 p_2 + p_3 p_1 p_2$.

b) $p_4 + P(\text{flows along top}) - p_4 \cdot P(\text{flows along top})$ where $P(\text{flows along top})$ was calculated in a).

8. a) $P(B_{12} \text{ and } B_{23}) = P(\text{all three have the same birthdates}) = \frac{365}{365} \cdot \frac{1}{365} \cdot \frac{1}{365} = \frac{1}{(365)^2}$.
$P(B_{12}) = \frac{365}{365} \cdot \frac{1}{365} = \frac{1}{365} = P(B_{23})$.
So $P(B_{12} \text{ and } B_{23}) = \frac{1}{(365)^2} = \frac{1}{365} \cdot \frac{1}{365} = P(B_{12})P(B_{23})$.
Therefore, B_{12} and B_{23} are independent!

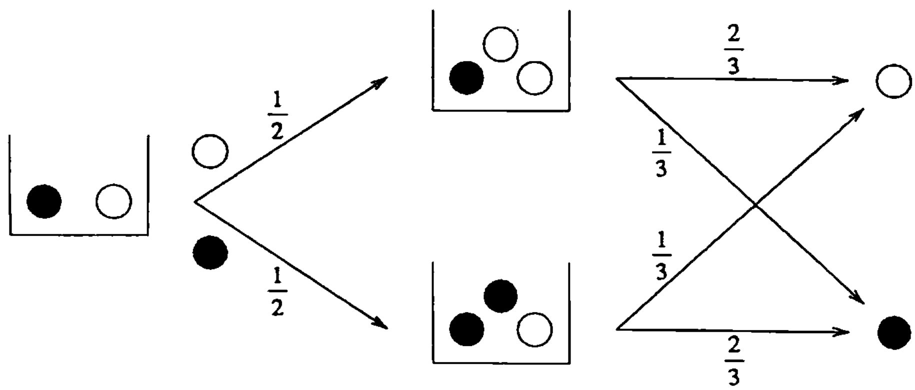
b) No. If you tell me B_{12} and B_{23} have occurred, then all three have the same birthday, so B_{13} also has occurred. That is, $P(B_{12}B_{23}B_{31}) = \frac{1}{365^2} \neq \frac{1}{365^3} = P(B_{12})P(B_{23})P(B_{31})$.

c) Yes, each pair is independent by the same reason as a).

Chapter 1: Review

1. $P(\text{both defective} \mid \text{item picked at random defective})$
$= \dfrac{P(\text{both defective})}{P(\text{item picked at random defective})} = \frac{3\%}{3\%+5\%\times 1/2} = \frac{6}{11}$

2.



From the above tree diagram,
$P(\text{black}) = P(\text{white}) = 1/2$,
$P(\text{black, black}) = P(\text{white, white}) = 1/3$,
$P(\text{black, white}) = P(\text{white, black}) = 1/6$.
So $P(\text{white} \mid \text{at least one of two balls drawn was white}) = \frac{1/6+1/3}{1/6+1/6+1/3} = 3/4$.

3. False. Of course $P(HHH \text{ or } TTT) = 1/4$. The problem with the reasoning is that while two of the coins at least must land the same way, which two is not known in advance. Thus given say two or more H's, i.e. HHH, HHT, HTH, or THH, these 4 outcomes are equally likely, so $P(HHH \text{ or } TTT \mid \text{at least two H's}) = 1/4$, not $1/2$. Similarly given at least two T's.

4. a) $P(\text{black} \mid \text{Box } 1) = 3/5 = P(\text{black} \mid \text{Box } 2)$
and $P(\text{red} \mid \text{Box } 1) = 2/5 = P(\text{red} \mid \text{Box } 2)$.
Hence $P(\text{black}) = P(\text{black} \mid \text{Box } i)$ and $P(\text{red}) = P(\text{red} \mid \text{Box } i)$ for $i = 1, 2$, and so the color of the ball is independent of which box is chosen. Or you can check that $P(\text{black, Box } 1) = P(\text{black})P(\text{Box } 1)$, etc, from the following table.

	Box 1	Box 2	
black	3/10	3/10	3/5
red	1/5	1/5	2/5
	1/2	1/2	

b)

	Box 1	Box 2	
black	3/10	5/18	26/45
red	1/5	2/9	19/45
	1/2	1/2	

Observe that $P(\text{black, Box } 1) \neq P(\text{black})P(\text{Box } 1)$, so color of ball is not independent of which box is chosen.

5. For either of the permissible orders in which to attempt the tasks, we have

$$\{\text{pass test}\} = S_1S_2 \cup S_1^cS_2S_3$$

where S_i denotes the event {task i performed successfully}; you don't have to worry abo [illegible] ie third task if you already have performed the first two tasks successfully. For the first order [illegible] y, hard, easy) the probability of passing the test is therefore

$$P(\text{pass test}) = P(S_1S_2) + P(S_1^cS_2S_3) = zh + (1-z)hz = zh(2-z).$$

By symmetry, the probability for the second order (hard, easy, hard) is $hz(2-h)$. S [illegible] $-h > 2-z$, the second order maximizes the probability of passing the test.

6. a) Since $P(B) = P(AB) + P(A^cB)$,

$$P(A^cB) = P(B) - P(AB) = P(B) - P(A)P(B) = P(B)(1-P(A)) = P(B)P(A^c)$$

b) Just reverse the roles of A and B in part a).

c) $P(A^cB^c) = P((A \cup B)^c) = 1 - P(A \cup B)$
$= 1 - P(A) - P(B) + P(AB)$
$= 1 - P(A) - P(B) + P(A)P(B)$ (by independence of A and B)
$= (1-P(A))(1-P(B)) = P(A^c)P(B^c)$

7. a) The probability that there will be no one in favor of 134 is the probability that the 1st person doesn't favor 134 and the 2nd person doesn't favor 134 and the 3rd person doesn't favor 134 and the 4th person doesn't favor 134. This is just $\frac{20}{50} \times \frac{19}{49} \times \frac{18}{48} \times \frac{17}{47} = .021$

b) The probability that at least one person favors 134 is 1− the probability that no one favors 134; but the probability that no one favors 134 was done in part a). Thus the answer is $1 - .021 = .979$.

c) The probability that exactly one person favors 134 is $\binom{4}{1}$ times the probability that the 1st person favors and the 2nd person doesn't and the 3rd person doesn't and the 4th person doesn't. This is $\binom{4}{1} \times \frac{30}{50} \times \frac{19}{49} \times \frac{18}{48} \times \frac{17}{47} = 0.126$.

d) The probability that a majority favor 134 is the probability that three people favor 134 plus the probability that four people favor 134. This probability is $\binom{4}{1} \times \frac{30}{50} \times \frac{29}{49} \times \frac{28}{48} \times \frac{20}{47} + \frac{30}{50} \times \frac{29}{49} \times \frac{28}{48} \times \frac{27}{47} = 0.472$

8. a) 0.08228 b) 0.7785 c) 0.1866

9. a) By independence, $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{60} = .01666...$

b) $P(A \text{ or } B \text{ or } C) = 1 - P(A^cB^cC^c)$
$= 1 - P(A^c) \cdot P(B^c) \cdot P(C^c) = 1 - \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{3}{5} = .6.$
Or use inclusion-exclusion.

c) $P(\text{exactly one of the three events occurs}) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)$
$= \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{26}{60} = \frac{13}{30} = .433...$

10. a)

$$\begin{aligned} P(\text{same type}) &= P(\text{both A or both B or both AB or both O}) \\ &= P(\text{both A}) + P(\text{both B}) + P(\text{both AB}) + P(\text{both O}) \\ &= (.42)^2 + (.10)^2 + (.04)^2 + (.44)^2 = .3816; \\ P(\text{different types}) &= 1 - P(\text{same type}) = 1 - .3816 = .6184 \end{aligned}$$

b) Since $P(1) + P(2) + P(3) + P(4) = 1$, we need only evaluate three of these; the fourth can be obtained by subtraction.

$$\begin{aligned} P(1) &= P(\text{all A or all B or all AB or all O}) \\ &= P(\text{all A}) + P(\text{all B}) + P(\text{all AB}) + P(\text{all O}) \\ &= (.42)^4 + (.10)^4 + (.04)^4 + (.44)^4 = .0687 \end{aligned}$$

$$
\begin{aligned}
P(4) &= (\#\text{ arrangements})(.42)(.10)(.04)(.44) = (4!)(.0007392) = .01774 \\
P(2) &= \big[(.42)^2(.10)^2 + (.42)^2(.04)^2 + (.42)^2(.44)^2 \\
&\qquad + (.10)^2(.04)^2 + (.10)^2(.44)^2 + (.04)^2(.44)^2\big] \times 6 \\
&\quad + \big[(.42)^3(1-.42) + (.10)^3(1-.10) \\
&\qquad + (.04)^3(1-.04) + (.44)^3(1-.44)\big] \times 4 \\
&= .5973
\end{aligned}
$$

So $P(3) = 1 - .5973 - .0687 - .01774 = .3163$.

11.

$$
\begin{aligned}
P(\text{fair} \mid \text{HT}) &= \frac{P(\text{fair and HT})}{P(\text{HT})} \\
&= \frac{P(\text{HT} \mid \text{fair})P(\text{fair})}{P(\text{HT} \mid \text{fair})P(\text{fair}) + P(\text{HT} \mid \text{biased})P(\text{biased})} \\
&= \frac{(1/4) \times (f/n)}{(1/4) \times (f/n) + (2/9) \times (b/n)} = \frac{9f}{9f + 8b}
\end{aligned}
$$

12. a) If $n \geq 7$, the chance is 0: the die has only six different faces! If $n = 1$, the chance is 1.

$$
\begin{aligned}
n = 2 &: \text{chance} = \frac{5}{6} \\
3 &: \text{chance} = \frac{5}{6} \cdot \frac{4}{6} \\
4 &: \text{chance} = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \\
5 &: \text{chance} = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \\
6 &: \text{chance} = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6}
\end{aligned}
$$

b) $1-$ above : the events in a) and b) are complementary.

13.

$$
\begin{aligned}
P(A|B) = P(AB|B) &= \sum_{i=1}^{n} P(AB_i|B) = \sum_{i=1}^{n} P(AB_i)/P(B) \\
&= \sum_{i=1}^{n} P(A|B_i)P(B_i)/P(B) = \sum_{i=1}^{n} P(A|B_i)P(B_i|B) \quad \text{since } B_iB = B_i
\end{aligned}
$$

14. a) Since the prior probabilities of the various boxes are the same, choose the one with the greatest likelihood: Box 100.

b) Here are the prior probabilities and likelihoods given a gold coin is drawn:

	Box 1	Box 2	Box 3	...	Box 98	Box 99	Box 100
prior:	2/150	1/150	2/150	...	1/150	2/150	1/150
likelihoods:	1/100	2/100	3/100	...	98/100	99/100	100/100

Clearly, the product (prior odds $\times$ likelihoods) is greatest for Box 99. So Box 99 is the choice with the greatest posterior probability given a gold coin is drawn.

15. $P(\text{Box } 1 \mid \text{gold}) = \frac{1}{1+0+1/2} = \frac{2}{3}$

16. a) Student 1 can choose any of the remaining $n-1$, student 2 can choose any of the eligible $n-2$, student 3 can choose any of the eligible $n-2$ except student 1, student 4 can choose any of the eligible $n-2$ other than students 1 and 2, and so on. So

$$p_r = \frac{(n-1)}{(n-1)} \times \frac{(n-2)}{(n-2)} \times \frac{(n-3)}{(n-2)} \times \frac{(n-4)}{(n-2)} \times \cdots \times \frac{(n-r)}{(n-2)}.$$

b) Use the exponential approximation:

$$\begin{aligned}\log(p_r) &= \log\left(1 - \frac{1}{n-2}\right) + \log\left(1 - \frac{2}{n-2}\right) + \cdots + \log\left(1 - \frac{r-2}{n-2}\right) \\ &\approx \left(\frac{-1}{n-2}\right) + \left(\frac{-2}{n-2}\right) + \cdots + \left(\frac{-(r-2)}{n-2}\right) \quad \text{since } \log(1+z) \approx z \text{ for small } z \\ &= -\frac{1}{n-2} \times \frac{(r-2)(r-1)}{2}.\end{aligned}$$

So for $n = 300$ and $r = 30$,

$$\log(p_{30}) \approx -\frac{28 \times 29}{298 \times 2} = -1.362416$$

so $p_{30} \approx e^{-1.362416} = .256041$.

17. False. They presumably send the same letter to 10,000 people, including the name of the winner, the name of the person to whom the letter is sent, and the name of one other person. In that case, I learn nothing from the information given, so my chances of winning the Datsun are still 1 in 10,000.

Section 2.1

1. a) $\binom{7}{4}$; b) $\binom{7}{4}(1/6)^4(5/6)^3$

2. $P(\text{2 boys and 2 girls}) = \binom{4}{2}(1/2)^4 = 6/2^4 = 0.375 < 0.5$. So families with different numbers of boys and girls are more likely than those having an equal number of boys and girls, and the relative frequencies are (respectively): 0.625, 0.375.

3. a) $P(\text{2 sixes in 5 rolls}) = \binom{5}{2}(1/6)^2(5/6)^3 = 0.160751$
   b) $P(\text{at least 2 sixes in 5 rolls}) = 1 - P(\text{at most 1 six})$
   $= 1 - [P(\text{0 sixes}) + P(\text{1 six})]$
   $= 1 - (5/6)^5 - \binom{5}{1}(1/6)(5/6)^4 = 0.196245$
   (This is shorter than adding the chances of 2, 3, 4, 5 sixes.)
   c) $P(\text{at most 2 sixes}) = P(\text{0 sixes}) + P(\text{1 six}).P(\text{2 sixes})$
   $= (5/6)^5 + \binom{5}{1}(1/6)(5/6)^4 + \binom{5}{2}(1/6)^2(5/6)^3 = 0.964506$
   d) The probability that a single die shows 4 or greater is $3/6 = 1/2$.
   $P(\text{exactly 3 show 4 or greater}) = \binom{5}{3}(1/2)^5 = 0.3125$
   e) $P(\text{at least 3 show 4 or greater}) = \left[\binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right](1/2)^5 = 0.5$
   (The binomial $(5, 1/2)$ distribution is symmetric about 2.5)

4.
$$\begin{aligned}
&P(\text{2 sixes in first five rolls}\mid\text{3 sixes in all eight rolls})\\
&= \frac{P(\text{2 sixes in first 5, and 3 sixes in all eight})}{P(\text{3 sixes in all eight})}\\
&= \frac{P(\text{2 sixes in first five, and 1 six in next three})}{P(\text{3 sixes in all eight})}\\
&= \frac{\binom{5}{2}(1/6)^2(5/6)^3 \cdot \binom{3}{1}(1/6)(5/6)^2}{\binom{8}{3}(1/6)^3(5/6)^5} = \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}} = \frac{10 \times 3}{56} = 0.535714
\end{aligned}$$

5. a) $\dfrac{\binom{19}{11}}{\binom{20}{12}}$ b) $\dfrac{\binom{18}{10}}{\binom{20}{12}}$ c) $1 - \left\{ \dfrac{\binom{15}{12}}{\binom{20}{12}} + 5 \times \dfrac{\binom{15}{11}}{\binom{20}{12}} \right\}$

6. a) $P(\text{exactly 4 hits}) = \binom{8}{4}(0.7)^4(0.3)^4 = 0.1361367$
   b) $P(\text{exactly 4 hits}\mid\text{at least 2 hits}) = \frac{P(\text{exactly 4 hits \& at least 2 hits})}{P(\text{at least 2 hits})}$
   $= \frac{P(\text{exactly 4 hits})}{1-P(\text{exactly 0 hits})-P(\text{exactly 1 hit})} = 0.1363126$
   c) $P(\text{exactly 4 hits}\mid\text{first 2 shots hit})$
   $= \frac{P(\text{exactly 4 hits \& first 2 shots hit})}{P(\text{first 2 shots hit})}$
   $= \frac{P(\text{first 2 shots hit \& exactly 2 hits in last 6 shots})}{P(\text{first 2 shots hit})}$
   $= P(\text{exactly 2 hits in last 6 shots})$ (since shots are independent)
   $= \binom{6}{2}(0.7)^2(0.3)^4 = 0.059535$

7. The chance that you win in this game is $15/36 = 5/12$ (list all 36 outcomes!), so the chance you win at least four times in five plays is
$$\binom{5}{4}(5/12)^4(7/12) + \binom{5}{5}(5/12)^5 = 0.100469$$

8. Let n be a positive integer. Using the formula for the mode of the binomial (n, p) distribution:

   If $np + p < 1$ then the most likely number of successes is $int\,(np + p) = 0$.

If $np+p=1$ then the most likely number of successes is 0 or 1 (both equally likely).

- If $np+p>1$ then the most likely number of successes is at least 1.

Hence the most likely number of successes is zero if and only if $np+p \le 1$, and the largest p for which zero is the most likely number of successes is $p = \frac{1}{n+1}$.

9. a) Most likely number $= int\ (326 \times 1/38) = 8$.

$$P(8) = P(6) \cdot \frac{P(7)}{P(6)} \cdot \frac{P(8)}{P(7)}$$

$$= .104840 \cdot \frac{325-7+1}{7} \cdot \frac{1}{37} \cdot \frac{325-8+1}{8} \cdot \frac{1}{37} = .138724$$

b) $P(10) = P(8) \cdot \frac{P(9)}{P(8)} \cdot \frac{P(10)}{P(9)}$
$= .138724 \times \frac{325-9+1}{9} \cdot \frac{1}{37} \cdot \frac{325-10+1}{10} \cdot \frac{1}{37} = .1127847$

c) $P(10 \text{ wins in } 326 \text{ bets})$

$$= P(9 \text{ wins in } 325 \text{ bets}) \times \frac{1}{38} + P(10 \text{ wins in } 325 \text{ bets}) \times \frac{37}{38}$$

$$= .132058 \times \frac{1}{38} + .1127847 \times \frac{37}{38}$$
$$= .1132919$$

10. a) $k/(n+1)$ b) $(n-k+1)/(n+1)$.

11. a) The tallest bar in the binomial $(15, 0.7)$ histogram is at int $((n+1)p) = \text{int}(16 \times 0.7) = \text{int}(11.2) = 11$

b) $P(11 \text{ adults in sample of } 15) = \binom{15}{11}(0.7)^{11}(0.3)^4$

$$= \binom{15}{4}(0.7)^{11}(0.3)^4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = (0.7)^{11}(0.3)^4 = 0.218623$$

12. a) $P(\text{makes exactly 8 bets before stopping})$
$= P(\text{wins at 8th game, and has won 4 out of the previous seven})$
$= \binom{7}{4}(18/38)^4(20/38)^3 \cdot (18/38) = 0.1216891$

b) $P(\text{plays at least 9 times})$
$= P(\text{wins at most four bets out of the first eight})$
$= (20/38)^8 + \binom{8}{1}(18/38)(20/38)^7 + \binom{8}{2}(18/38)^2(20/38)^6 + \binom{8}{3}(18/38)^3(20/38)^5 + \binom{8}{4}(18/38)^4(20/38)^4$

$= 0.6926167$

13. a) No! Since the child must get one allele from his/her mother, which will be the dominant B, he/she must have brown eyes.

b) Taking one allele from each parent, we see that there are four (equally likely) allele pairs: Bb, Bb, bb, bb. Thus the probability of the child having brown eyes is 50% = 0.5.

c) Once again there are four possibilities: Bb, Bb, Bb, bb. Thus there ia a 75% = 0.75 chance that the child will have brown eyes.

d) Given that the mother is brown eyed and her parents were both Bb, the probability she is Bb is 2/3 and BB 1/3. Thus

$$P(\text{child brown-eyed}) = P(\text{child brown-eyed}|\text{mother Bb}) \cdot P(\text{mother Bb})$$
$$+P(\text{child brown-eyed}|\text{mother BB}) \cdot P(\text{mother BB})$$
$$= \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$$

So

$$P(\text{woman Bb}|\text{child Bb}) = \frac{P(\text{child Bb}|\text{woman Bb})P(\text{woman Bb})}{P(\text{child Bb})} = \frac{(1/2)(2/3)}{2/3} = \frac{1}{2}$$

14. a) (Ts, Pw); tall and purple.

b)

Genetic Combinations	Probability	
(TT, PP)	1/16	tall and purple
(TT, Pw)	1/8	tall and purple
(TT, ww)	1/16	tall and white
(Ts, PP)	1/8	tall and purple
(Ts, Pw)	1/4	tall and purple
(Ts, ww)	1/8	tall and white
(ss, PP)	1/16	short and purple
(ss, Pw)	1/8	short and purple
(ss, ww)	1/16	short and white

	tall and purple	tall and white	short and purple	short and white
Probability	9/16	3/16	3/16	1/16

c) $1 - (7/16)^{10} - 10(9/16)(7/16)^9$.

15. a) If $0 < p < 1$, then $int\ (np + p) = np$, since np is an integer.

b) Note that $np = int\ (np) + [np - int\ (np)]$.
If $[np - int\ (np)] + p \geq 1$, then $int\ (np + p) = int\ (np) + 1$, which is the integer above np.
If $0 < [np - int\ (np)] + p < 1$, then $int\ (np + p) = int\ (np)$, which is the integer below np.

c) Consider the case $n = 2, p = 1/3$, where 1 is the closest integer to np and the integer above np, but 0 is also a mode. Also 0 is the integer below np, but 1 is also a mode.

Section 2.2

1. The number of heads in 400 tosses has binomial (400, 0.5) distribution. Use the normal approximation with $\mu = 400 \times (1/2) = 200$ and $\sigma = \sqrt{400 \times \frac{1}{2} \times \frac{1}{2}} = 10$.

   a) $P(190 \le H \le 210) \approx \Phi(1.05) - \Phi(-1.05) = 0.7062$
   b) $P(210 \le H \le 220) \approx \Phi(2.05) - \Phi(0.95) = 0.1509$
   c) $P(H = 200) \approx \Phi(0.05) - \Phi(-0.05) = 0.0398$
   d) $P(H = 210) \approx \Phi(1.05) - \Phi(0.95) = 0.0242$

2. Now $\mu = 204$ and $\sigma = 9.998$.

   a) $P(190 \le H \le 210) \approx \Phi(0.65) - \Phi(-1.45) = 0.6686$
   b) $P(210 \le H \le 220) \approx \Phi(1.65) - \Phi(-0.55) = 0.2417$
   c) $P(H = 200) \approx \Phi(-0.35) - \Phi(-0.45) = 0.0368$
   d) $P(H = 210) \approx \Phi(0.65) - \Phi(0.55) = 0.0333$

3. a) Law of large numbers: the first one.
   b) Binomial (100, .5) mean 50, SD 5:

$$1 - \Phi\left(\frac{54.5 - 50}{5}\right) = 1 - \Phi(.9) = 1 - .8159 = .1841$$

   Binomial (400, .5) mean 200, SD 10:

$$1 - \Phi\left(\frac{219.5 - 200}{10}\right) = 1 - \Phi(1.95) = 1 - .9744 = .0256$$

4. Let X be the number of patients helped by the treatment. Then $E(X) = 100$, $SD(X) = 8.16$ and $P(X > 120) = P(X \ge 120.5) \approx 1 - \Phi(2.51) = .006$.

5. Want the chance that you win at least 13 times. The number of times that you win has binomial (25, 18/38) distribution. Use the normal approximation with $\mu = 11.84, \sigma = 2.50$:
   $P(13 \text{ or more wins}) = 1 - P(12 \text{ or fewer wins}) \approx 1 - \Phi\left(\frac{12.5-11.84}{2.50}\right) = 1 - \Phi(0.26) = 0.3974$

6. The number of opposing voters in the sample has the binomial (200, .45) distribution. This gives $\mu = 90$ and $\sigma = \sqrt{200 \times .45 \times .55} = 7.035$. Use the normal approximation:

   a) The required chance is approximately

$$\Phi\left(\frac{90.5 - 90}{7.035}\right) - \Phi\left(\frac{89.5 - 90}{7.035}\right) = .5283 - .4717 = .0567 \text{ (about 6\%)}$$

   b) Now the required chance is approximately

$$1 - \Phi\left(\frac{100.5 - 90}{7.035}\right) = 1 - \Phi(1.49) = 1 - .9319 = 0.0681 \text{ (about 7\%)}$$

7. a) The city A sample has 400 people, the city B sample has 600 people, so city B accuracy is $\sqrt{6/4} = 1.22$ times greater.
   b) Both have the same accuracy, since the absolute sizes of the two samples are equal.
   c) The city A sample has 4000 people, the city B sample has 4500 people, so the city B sample is $\sqrt{4500/4000} = 1.06$ times more accurate, even though the percent of population sampled in city B is smaller than the percent sampled in city A.

8. Use the normal approximation: $\sigma = \sqrt{npq} = 9.1287$. Want relative area between 99.5 and 100.5 under the normal curve with mean $600 \times (1/6) = 100$, and $\sigma = 9.1287$. That is, want area between $-.055$ and $.055$ under the standard normal curve. Required area $= \Phi(.055) - [1 - \Phi(.055)] = .0438$.

9. a) Think of 324 independent trials, each a "success" (the person shows up) with probability 0.9. So the number of arrivals has binomial (324, 0.9) distribution. We want P(more than 300 successes in 324 trial: Use the normal approximation with $\mu = 324 \times 0.9 = 291.6$ and $\sigma = \sqrt{324 \times 0.9 \times 0.1} = 5.4$ to compute the required probability:
$1 - \Phi\left(\frac{300.5-291.6}{5.4}\right) = 1 - \Phi(1.65) \approx 0.0495$.

b) Increase: in the long run, each group must show up with probability .9. So effectively, traveling in groups reduces the number of trials, keeping p the same. So the histogram for the proportion of successes has more mass in the tails, since n is smaller.

c) Repeat a), with the 300 seats replaced by 150 pairs, and the 324 people replaced by 162 pairs. So the number of pairs that arrive has binomial (162, 0.9) distribution. Use the normal approximation with $\mu = 162 \times 0.9 = 145.8$ and $\sigma = \sqrt{162 \times 0.9 \times 0.1} = 3.82$. We want $1 - \Phi\left(\frac{150.5-145.8}{3.82}\right) = 1 - \Phi(1.23) = 0.1093$.

10. We have 30 independent repetitions of a binomial $(200, \frac{1}{2})$. For each of these repetitions, the probability of getting exactly 100 heads can be gotten by normal approximation where $\mu = (200 \times \frac{1}{2}) = 100$ and $\sigma = \sqrt{200 \times \frac{1}{2} \times \frac{1}{2}} = 7.07$.

The chance of exactly 100 heads can be approximated by

$$P(100 \text{ heads }) = \Phi\left(\frac{(100+.5)-100}{7.07}\right) - \Phi\left(\frac{(100-.5)-100}{7.07}\right) \approx .5282 - .4718 = .0564$$

Since the students are independent, the probability that all 30 students do not get exactly 100 heads is $(1 - P(100 \text{ heads }))^{30}$ and the normal approximation gives us

$$(1 - P(100 \text{ heads }))^{30} \approx (1 - .0564)^{30} = 0.175$$

11. The number of hits in the next 100 times at bat has binomial (100, 0.3) distribution. Use the normal approximation with $\mu = 30$ and $\sigma = \sqrt{100 \times 0.3 \times 0.7} = 4.58$.

a) $P(\geq 31 \text{ hits}) \approx 1 - \Phi\left(\frac{30.5-30}{4.58}\right) = 1 - \Phi(0.11) = 0.4562$

b) $P(\geq 33 \text{ hits}) \approx 1 - \Phi\left(\frac{32.5-30}{4.58}\right) = 1 - \Phi(0.545) = 0.2929$

c) $P(\leq 27 \text{ hits}) \approx \Phi\left(\frac{27.5-30}{4.58}\right) = \Phi(-0.545) = 0.2929$

d) No, independence would be lost, because if the player has been doing well the last few times at bat, he's more likely to do well the next time. Similarly, if he's been doing badly, then he's more likely to continue doing badly. This will increase all the chances above.

e) From part b), we see that the player has about a 29% chance of such a performance, if his form stayed the same (long run average .300) and if the hits were independent. So it is reasonable to conclude that this performance is "due to chance," since 29% is quite a large probability.

12. For $n = 10,000$ independent trials with success probability $p = 1/2$,

$$\mu = np = 5000 \text{ and } \sigma = \sqrt{npq} = 50$$

Hence, by the normal approximation, the probability of between $5000 - m$ and $5000 + m$ successes is approximately

$$\Phi\left(\frac{m+1/2}{50}\right) - \Phi\left(\frac{-m-1/2}{50}\right) = 2\Phi\left(\frac{m+1/2}{50}\right) - 1$$

This equals 2/3 if and only if $\Phi((m + 1/2)/50) = 5/6$ which implies that $(m + 1/2)/50 = 0.97$ and $m = 48$. In other words, there is about a 2/3 chance that the number of heads in 10,000 tosses of a fair coin is within about one standard deviation of the mean.

13. First find z such that $\Phi(-z, z) = 95\%$. This means $\Phi(z) = 0.9750$ and by the normal table, $z = 1.96$.

$$95\% = P\left(\hat{p} \text{ in } p \pm 1.96\sqrt{\frac{pq}{n}}\right) \leq P\left(\hat{p} \text{ in } p \pm 1.96\frac{.5}{\sqrt{n}}\right)$$

We want $1.96(.5/\sqrt{n}) \leq 1\%$, so

$$n \geq \left(\frac{1.96(.5)}{.01}\right)^2 = 9604$$

14. a) # working devices in box has binomial $(400, .95)$ distribution. This is approximately normal with $\mu = 380$, $\sigma \approx 4.36$. Required percent $\approx 1 - \Phi(\frac{389.5-380}{4.36}) = 1 - \Phi(2.18) = 0.0146$ (This normal approximation is pretty rough due to the skewness of the distribution. The exact probability is 0.0094 correct to 4 decimal places. The skew-normal approximation is 0.0099 which is much better.)

b) Using the normal approximation, want largest k so that $1 - \Phi(\frac{k-0.5-380}{4.36}) \geq 0.95$ so $\frac{k-0.5-380}{4.36} = -1.65$, so $k = 373$.

15. a) $\phi'(z) = \frac{1}{\sqrt{2\pi}}(-z)e^{-z^2/2} = -z\phi(z)$

b) $\phi''(z) = -\phi(z) - z(-z\phi(z)) = (z^2 - 1)\phi(z)$

c) Sketch: Outside $(-4, 4)$, they are close to zero.

d) Let $f(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})$. Want $f''(x)$

$$f''(x) = \frac{1}{\sigma}\phi''(\frac{x-\mu}{\sigma}) = \frac{1}{\sigma}((\frac{x-\mu}{\sigma})^2 - 1)\phi(\frac{x-\mu}{\sigma})\frac{1}{\sigma^2}$$

e) For $\mu - \sigma < x < \mu + \sigma$, the second derivative in d) is negative. So the curve is concave in this region. For $x > \mu + \sigma$ or $x < \mu - \sigma$. The second derivative is positive, so curve is convex.

16. a) Using the results of Exercise 15,

$$\phi'''(z) = 2z\phi(z) - (z^2 - 1)z\phi(z) = (-z^2 + 3z)\phi(z)$$

b) Fundamental theorem of calculus; ϕ'' vanishes at $-\infty$. The first equality of integrals is because ϕ''' is an odd function.

$$2\phi(\sqrt{3}) = ((\sqrt{3})^2 - 1)\phi(-\sqrt{3}) = \phi''(\sqrt{3}).$$

$$\int_0^{\sqrt{3}} \phi'''(z)dz) = \int_{-\infty}^{\sqrt{3}} \phi'''(z)dz - \int_{-\infty}^{0} \phi'''(z)dz) = 2\phi(\sqrt{3}) - (-\phi(0))$$

c) ϕ''' switches signs at $-\sqrt{3}$ and $\sqrt{3}$. $\phi(0) + 2\phi\sqrt{3}$ is the biggest area possible over an interval because areas are negative when the curve is negative.

17. a) $1 - \Phi(z) = 1 - \int_{-\infty}^{z} \phi(x)dx = \int_{-\infty}^{\infty} \phi(x)dx - \int_{-\infty}^{z} \phi(x)dx = \int_{z}^{\infty} \phi(x)dx.$

b) Over the range of the integral $z < x$, so x/z is greater than 1. Multiplying a positive integrand by a number greater than 1 gives an integral with a larger value.

c)

$$\begin{aligned}
1 - \Phi(z) &< \int_z^\infty \frac{1}{\sqrt{2\pi}z} e^{-\frac{1}{2}x^2} x\,dx \\
&= \int_{\frac{1}{2}z^2}^\infty \frac{1}{\sqrt{2\pi}z} e^{-u} du \qquad [u = \frac{1}{2}x^2, du = x\,dx] \\
&= -\frac{1}{\sqrt{2\pi}z} e^{-u} \Big|_{\frac{1}{2}z^2}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}z} e^{-\frac{1}{2}z^2} = \frac{\phi(z)}{z}
\end{aligned}$$

Section 2.3

1. Let $p = P(0)$, then $P(k) = R(k) \cdot R(k-1) \cdots R(1)p$.
Use $\sum_{i=1}^{n} P(k) = 1 - p$ to get the value of p.

2. Put $n = 10,000$, $p = 0.5$, $k = 5000$ in Formula (3). Then $\mu = 5000$, $\sigma = 50$, and the desired probability is approximately
$$\frac{1}{\sqrt{2\pi} \times 50} = 0.008 \approx 0.01$$

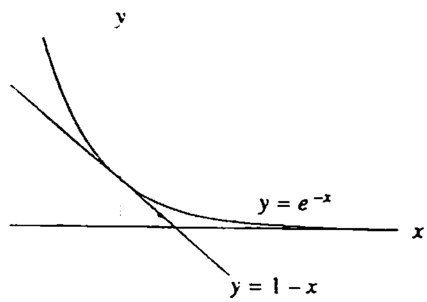
3. a) Use $P(m+1 \text{ in } 2m) = P(m \text{ in } 2m) \cdot R(m+1)$ where $R(m+1) = \frac{2m-(m+1)+1}{m+1} = \left(1 - \frac{1}{m+1}\right)$
Similarly, $P(m-1 \text{ in } 2m) = P(m \text{ in } 2m)/R(m)$ where $R(m) = \frac{2m-m+1}{m}$. (This could also have been deduced by symmetry.)

b)
$$\begin{aligned} &P(m+1 \text{ in } 2m+2) \\ &= P(m-1 \text{ in } 2m, \text{ then HH}) + P(m \text{ in } 2m, \text{ then HT or TH}) + P(m+1 \text{ in } 2m, \text{ then TT}) \\ &= \frac{1}{4}P(m-1 \text{ in } 2m) + \frac{1}{2}P(m \text{ in } 2m) + \frac{1}{4}P(m+1 \text{ in } 2m) \end{aligned}$$

c) Substitute $P(m-1 \text{ in } 2m) = P(m \text{ in } 2m)\left(1 - \frac{1}{m+1}\right)$ and $P(m+1 \text{ in } 2m) = P(m \text{ in } 2m)\left(1 - \frac{1}{m+1}\right)$ into (b), and simplify. This can also be checked by cancelling factorials.

d) Write
$$\begin{aligned} P(m \text{ in } 2m) &= \frac{P(m \text{ in } 2m)}{P(m-1 \text{ in } 2m-2)} \cdot \frac{P(m-1 \text{ in } 2m-2)}{P(m-2 \text{ in } 2m-4)} \cdots\cdots \frac{P(2 \text{ in } 4)}{P(1 \text{ in } 2)} \cdot P(1 \text{ in } 2) \\ &= \cdot\left(1 - \frac{1}{2m}\right) \cdot \left(1 - \frac{1}{2(m-1)}\right) \cdots\cdots \left(1 - \frac{1}{2 \times 2}\right) \cdot \left(1 - \frac{1}{2 \times 1}\right). \end{aligned}$$

e) Use the fact that $1 - x \le e^{-x}$ with equality iff $x = 0$ (see diagram).



Then the inequalities follow easily:
$$0 < P(m \text{ in } 2m) < e^{-\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}\right)} < e^{-\frac{1}{2}\log(m)} = \frac{1}{\sqrt{m}}$$

(For the last inequality, draw a graph of $1/x$ and remember that the area under this graph from 1 to m is $\log(m)$).

f) By part (c),
$$\frac{\alpha_m}{\alpha_{m-1}} = 1 - \frac{1}{2m}.$$

Square this, substitute, and simplify:
$$\frac{m+\frac{1}{2}}{m-1+\frac{1}{2}}\left(\frac{\alpha_m}{\alpha_{m-1}}\right)^2 = \frac{2m+1}{2m-1}\left(\frac{2m-1}{2m}\right)^2 = \frac{(2m-1)(2m+1)}{(2m)^2} = 1 - \frac{1}{4m^2}.$$

g) As in (d), write

$$2\left(m+\frac{1}{2}\right)\alpha_m^2 = 2\frac{\left(m+\frac{1}{2}\right)\alpha_m^2}{\left(m-1+\frac{1}{2}\right)\alpha_{m-1}^2}\cdot\frac{\left(m-1+\frac{1}{2}\right)\alpha_{m-1}^2}{\left(m-2+\frac{1}{2}\right)\alpha_{m-2}^2}\cdots\cdots\frac{\left(2+\frac{1}{2}\right)\alpha_2^2}{\left(1+\frac{1}{2}\right)\alpha_1^2}\cdot\frac{\left(1+\frac{1}{2}\right)\alpha_1^2}{\left(0+\frac{1}{2}\right)\alpha_0^2}\left(0+\frac{1}{2}\right)\alpha_0^2$$

$$= \left(1-\frac{1}{4m^2}\right)\left(1-\frac{1}{4(m-1)^2}\right)\cdots\left(1-\frac{1}{4(1)^2}\right)$$

(since $\left(0+\frac{1}{2}\right)\alpha_0^2 = 1/2$). The result follows from factoring

$$\left(1-\frac{1}{4k^2}\right) = \left(1-\frac{1}{2k}\right)\left(1+\frac{1}{2k}\right).$$

h)

$$\alpha_m = P(\text{m in 2m}) = P(\text{mode}) \sim \frac{1}{\sqrt{2\pi}\sigma}$$

with $\sigma = \frac{1}{2}\sqrt{2m}$. So $\alpha_m \sim \frac{K}{\sqrt{m}}$ where $K = \frac{1}{\sqrt{\pi}}$. But this means

$$\frac{2}{\pi} = 2K^2 = 2\lim_{m\to\infty}(m)\alpha_m^2 = 2\lim_{m\to\infty}\left(m+\frac{1}{2}\right)\alpha_m^2 = \lim_{m\to\infty}\frac{2m+1}{2m}\cdot\frac{2m-1}{2m}\cdot\frac{2m-1}{2(m-1)}\cdot\frac{2m-3}{2(m-1)}\cdots\cdots\frac{3}{2}\cdot\frac{1}{2}.$$

Section 2.4

1. a) Approximately Poisson(1).

   b) Approximately Poisson(2).

   c) Approximately Poisson(0.3284).

   d) This is the distribution of the number of successes in 1000 independent trials, where the success probability is $p = .998$. The distribution of the number of failures has binomial (1000, .002) distribution, which is approximately Poisson(2). Since #successes + #failures = 1000, it follows that the histogram for the number of successes (i.e., the desired histogram) looks like the left-to-right mirror image of a Poisson(2) histogram, with the block at 0 being sent to 1000, the block at 1 being sent to 999, etc.

2. a) The number of successes in 500 independent trials with success probability .02 has binomial (500, .02) distribution with mean $\mu = 10$. By the Poisson approximation,

$$P(1 \text{ success}) \approx e^{-\mu}\frac{\mu^1}{1!} = \mu e^{-\mu} = .000454.$$

   b) $P(2 \text{ or fewer successes}) = P(0 \text{ or } 1 \text{ or } 2 \text{ successes})$

$$\approx e^{-\mu}\frac{\mu^0}{0!} + e^{-\mu}\frac{\mu^1}{1!} + e^{-\mu}\frac{\mu^2}{2!}$$

$$= e^{-\mu}(1 + \mu + \frac{\mu^2}{2}) = .002769$$

   c) $P(4 \text{ or more successes}) = 1 - P(3 \text{ or fewer successes}) = .989664.$

3. The number of times you see 25 or more sixes has binomial distribution with $\mu = 365 \times .022 = 8.03$.

   a) $P(\text{at least once}) = 1 - P(\text{exactly 0 times}) \approx 1 - e^{-\mu} = .999674.$

   b) $P(\text{at least twice}) = 1 - P(\text{exactly 0 times}) - P(\text{exactly 1 time}) \approx 1 - e^{-\mu} - \mu e^{-\mu} = .997060.$

4. Here $\mu = 365 \times 0.00068 = 0.2482$, and

   a) $1 - e^{-\mu} = .219796;$

   b) $1 - e^{-\mu} - \mu e^{-\mu} = 0.026150.$

5. The number of wins has binomial (52, 1/100) distribution with mean $\mu = 52/100$, and by the Poisson approximation $P(k \text{ wins}) \approx e^{-\mu}\mu^k/k!$.
$k = 0 : P(0 \text{ wins}) \approx e^{-\mu} = .594521;$
$k = 1 : P(1 \text{ win}) \approx \mu e^{-\mu} = .309151;$
$k = 2 : P(2 \text{ wins}) \approx \frac{\mu^2}{2}e^{-\mu} = .080379.$

6. a) The number of black balls seen in a series of 100 draws with replacement has binomial (1000, 2/1000) distribution with mean $\mu = 2$. By the Poisson approximation,

$$P(\text{fewer than 2 black balls}) \approx e^{-\mu}\frac{\mu^0}{0!} + e^{-\mu}\frac{\mu^1}{1!} = e^{-\mu}(1 + \mu) = .406006.$$

$$P(\text{exactly 2 black balls}) = e^{-\mu}\frac{\mu^2}{2!} = .270671.$$

   Calculate the probability of more than 2 black balls by subtraction, conclude that getting fewer than 2 black balls is most likely.

b) $P(\text{both series see same number of black balls}) = \sum_{k=0}^{\infty} P(\text{both series see k black balls})$

$$\approx \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}e^{-\mu}\frac{\mu^k}{k!} \text{ by independence}$$

$$= e^{-4}\sum_{k=0}^{\infty}\frac{4^k}{(k!)^2} = 0.207002$$

7. a) 2 b) .2659 c) .2475 d) .2565 e) $m = 250$. Normal approximation .0266 f) $m = 2$. Poisson approximation: .2565

8.

$$P(k) = e^{-\mu}\frac{\mu^k}{k!}$$

$$P(k-1) = e^{-\mu}\frac{\mu^{k-1}}{(k-1)!}$$

so $R(k) = P(k)/P(k-1)$ is given by

$$R(k) = \frac{\mu}{k}$$

which is decreasing as k increases. The maximimum probability will occur at the largest value of k for which $R(k) \geq 1$; after this $P(k)$ will decrease. Thus the maximum occurs at $int(\mu)$. There is a double maximum if and only if $R(k) = 1$ for some k. This can only occur if μ is an integer. The two values of k that maximize are then μ and $\mu - 1$. There can never be a triple maximum since this would imply that $R(k) = 1$ and $R(k-1) = 1$ for some k.

9. Assume that each box of cereal has a prize with chance .95, independently of all others. The number of prizes collected by the family in 52 weeks has the binomial distribution with parameters 52 and .95. This gives $\mu = 52 \times .95 = 49.4$, and $\sigma = \sqrt{52 \times .95 \times .05} = 1.57$. Since σ is very small (less than 3), the normal approximation is not good. Use the Poisson instead. The number of "dud" boxes has the Poisson distribution with paramter $52 \times .05 = 2.6$. We want the chance of 46 or more prizes, that is, 6 or less duds. This is

$$e^{-2.6}\{1 + 2.6 + 2.6^2/2 + 2.6^3/3! + 2.6^4/4! + 2.6^5/5! + 2.6^6/6!\} = .982830.$$

[For comparison, the exact binomial probability is .985515. The normal approximation gives .993459.]

10. Distribution of the number of successes is

$$\text{binomial } (n, 1/N) \approx \text{ Poisson } (n/N) \approx \text{ Poisson } (5/3).$$

$$P(\text{at least two}) = 1 - P(0) - P(1) \approx 1 - e^{-5/3}(1 + 5/3) = 1 - e^{-5/3}\cdot\frac{8}{3} \approx 0.49633 \approx 0.5$$

Section 2.5

1. a) $\frac{\binom{20}{4}\binom{30}{6}}{\binom{50}{10}}$ b) $\binom{10}{4}(2/5)^4(3/5)^6$

2. a) $\frac{26}{52}\cdot\frac{26}{51}\cdot\frac{25}{50}$ b) $3\times$ a) c) $1-\frac{26}{52}\cdot\frac{25}{51}\cdot\frac{24}{50}$

3. a) $\frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}$ b) $\frac{\binom{3}{3}\binom{48}{9}}{\binom{51}{12}}$ c) $\frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}-\binom{48}{13}}$ d) 0

4. The exact chance is $\dfrac{\sum_{k=45}^{100}\binom{40{,}000}{k}\binom{60{,}000}{100-k}}{\binom{100{,}000}{100}}$

   For the approximation, use the normal curve with $\mu = 40$, $\sigma = 4.9$. Chance is approximately $1-\Phi(\frac{44.5-40}{4.9}) = 0.1788$

5. Solve $\frac{-.05}{\sqrt{.55\times.45/n}} \le -2.326$, then $n \ge 537$.

6. a) $\frac{\binom{36}{13}}{\binom{52}{13}}$ b) $\frac{\binom{40}{13}}{\binom{52}{13}}-\frac{\binom{36}{13}}{\binom{52}{13}}$ c) (a) + 4(b).

7. Denote by B_i the event (ith ball is black), similarly for R_i.
   a) $P(B_1B_2B_3B_4) = P(B_1)P(B_2|B_1)P(B_3|B_1B_2)P(B_4|B_1B_2B_3) = \frac{50}{80}\frac{49}{79}\frac{48}{78}\frac{47}{77} = .1456$.
   b) This is four times $P(B_1B_2B_3R_4)$, so by d) equals .3716.
   c) $P(B_1B_2B_3R_4) = P(B_1)P(B_2|B_1)P(B_3|B_1B_2)P(R_4|B_1B_2B_3) = \frac{50}{80}\frac{49}{79}\frac{48}{78}\frac{30}{77} = .0929$.

8. Let the outcome space be the set of all 3-element subsets from the set $\{1, ..., 100\}$, so that, e.g., the 3-set $\{23, 21, 1\}$ means that the winning tickets were tickets #1, #21, and #23. Each of the $\binom{100}{3}$ such 3-sets is equally likely.

   a) The event (one person gets all three winning tickets) is the disjoint union of the 10 equally likely events (person i gets all three tickets). The event (person i gets all three tickets) corresponds to all 3-sets consisting entirely of tickets bought by person i. There are $\binom{10}{3}$ such 3-sets, so the desired probability is
   $$10\times\frac{\binom{10}{3}}{\binom{100}{3}} = .007421.$$

   b) The event (there are three different winners) is the disjoint union of the $\binom{10}{3}$ equally likely events (the winning tickets were bought by persons i, j, and k) (i, j, k all different). The event (the winning tickets were bought by persons i, j, and k) consists of $\binom{10}{1}\times\binom{10}{1}\times\binom{10}{1}$ 3-sets, so the desired probability is
   $$\binom{10}{3}\times\frac{\binom{10}{1}\binom{10}{1}\binom{10}{1}}{\binom{100}{3}} = .742115.$$

   c) By subtraction, .250464. Just to be sure, use the above technique to obtain the desired probability:
   $$10\times 9\times\frac{\binom{10}{2}\binom{10}{1}}{\binom{100}{3}} = .250464.$$

9. a) Let A_i be the event that the 1^{st} sample contains exactly i bad items, $i = 0, 1, 2, 3, 4, 5$. And let B_j be the event that the 2^{nd} sample contains exactly j bad items, $j = 0, 1, ..., 10$. Then P(2nd sample drawn and contains more than one bad item)
   $$= P\left[A_1 \cap \{\cup_{j=2}^{10} B_j\}\right]$$
   $$= P\left[\cup_{j=2}^{10} B_j \mid A_1\right] P(A_1)$$

$$= P(A_1) \cdot \{1 - P[B_0 \cup B_1 \mid A_1]\}$$
$$= P(A_1) \cdot \{1 - P(B_0|A_1) - P(B_1|A_1)\}$$
$$= \frac{\binom{10}{1}\binom{40}{4}}{\binom{50}{5}} \cdot \left\{1 - \frac{\binom{9}{0}\binom{36}{10}}{\binom{45}{10}} - \frac{\binom{9}{1}\binom{36}{9}}{\binom{45}{10}}\right\}$$
$$= 0.431337 \cdot (1 - 0.079678 - 0.265592) = 0.282409$$

b) $P(\text{lot accepted})$

$$= P[A_0 \cup \{A_1 \cap (B_0 \cup B_1)\}]$$
$$= P(A_0) + P(A_1) \cdot P[B_0 \cup B_1 \mid A_1]$$
$$= \frac{\binom{10}{0}\binom{40}{5}}{\binom{50}{5}} + \frac{\binom{10}{1}\binom{40}{4}}{\binom{50}{5}} \cdot \left\{\frac{\binom{9}{0}\binom{36}{10}}{\binom{45}{10}} + \frac{\binom{9}{1}\binom{36}{9}}{\binom{45}{10}}\right\}$$
$$= 0.310563 + 0.431337 \cdot (0.079678 + 0.265592) = 0.459491.$$

10. Every sequence of k_1 good, k_2 bad and k_3 indifferent elements has the same probability:

$$(G/N)^{k_1}(B/N)^{k_2}(I/N)^{k_3}$$

This is clear if you think of the sample as n independent trials. The probability of each pattern of k_1 good, k_2 bad and k_3 indifferent elements is a product of k_1 factors of G/N, k_2 factors of B/N, and k_3 factors of I/N. The number of different patterns of k_1 good, k_2 bad and k_3 indifferent elements is $n!/(k_1!k_2!k_3!)$, which proves the formula.

11. The set is $\{g : \max\{0, n - N + G\} \le g \le \min\{n, G\}\}$. This is because the maximum possible number of good elements in sample is $\min\{n, G\}$. And the minimum possible number of good elements in the sample is n minus the max number of bad elements in sample, i.e.

$= n - \min\{n, N - G\} = \max\{0, n - N + G\}$

Formula is correct for all g because $\binom{a}{b} = 0$ for $b < 0$ or $b > a$.

12. There are $\binom{52}{5} = 2,598,960$ possible *distinct* poker hands.

a) There are 52 cards in a pack: $A, 2, 3, ..., 10, J, Q, K$ in each of the four suits: spades, clubs, diamonds and hearts. A straight is five cards in sequence. An A can be at the begining or the end of a sequence, but never in the middle. i.e. $A, 2, 3, 4, 5$ and $10, J, Q, K, A$ are legitimate straights, but $K, A, 2, 3, 4$ (a "round-the-corner" straight) is not. So there are are 10 possible starting points for the straight flush, (A through 10) and 4 suits for it to be in, giving a total of 40 hands.

Thus $P(\text{straight flush}) = \frac{40}{\binom{52}{5}} = 0.0000154.$

b) $P(\text{four of a kind}) = \frac{13 \times 48}{\binom{52}{5}} = \frac{624}{2598960} = 0.000240$

c) $P(\text{full house}) = \frac{13 \times 12 \times \binom{4}{3} \times \binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.00144$

d) $P(\text{flush}) = P(\text{all same suit}) - P(\text{straight flush})$
$= \frac{4 \times \binom{13}{5} - 4 \times 10}{\binom{52}{5}} = \frac{5108}{2598960} = 0.00197$

e) $P(\text{straight}) = P(5 \text{ consec. ranks}) - P(\text{straight flush}) = \frac{10 \times \binom{4}{1}^5 - 10 \times 4}{\binom{52}{5}} = \frac{10200}{2598960} = 0.00392$

f) $P(\text{three of a kind}) = \frac{13 \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1} \times \binom{4}{1}}{\binom{52}{5}} = \frac{54912}{2598960} = 0.0211$

g) $P(\text{two pairs}) = \frac{\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 11 \times \binom{4}{1}}{\binom{52}{5}} = \frac{123552}{2598960} = 0.0475$

h) $P(\text{one pair}) = \frac{13 \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}}{\binom{52}{5}} = \frac{1098240}{2598960} = 0.423$

i) Since the events are mutually exclusive, the probability of none of the above is
$1 - \text{sum of (a) through (h)} = 0.501.$

13.
$$P(\text{pass}) = 1 - P(\text{fail}) = 1 - [P(> \ 10 \text{ defectives})]^2$$
$$P(> \ 10 \text{ defectives}) \approx 1 - \Phi\left(\frac{10.5 - 500(.05)}{\sqrt{500(.05)(.95)}}\right) \approx 1 - \Phi(-2.98) = .9986$$
$$P(\text{pass}) = 1 - (.9986)^2 = .0028$$

Chapter 2: Review

1. a) $\binom{10}{4}(1/6)^4(5/6)^6$

   b) $\binom{10}{4}(1/5)^4(4/5)^6$

   c) $\frac{10!}{4!3!2!}/6^{10}$

   d) $\frac{\binom{6}{3}}{\binom{10}{3}} = \frac{\binom{7}{4}}{\binom{10}{4}}$

2. 0.007

3. a) $P(3H) = P(3H|3 \text{ spots})P(3 \text{ spots}) + \ldots + P(3H|6 \text{ spots})P(6 \text{ spots})$
   $= \left\{\binom{3}{3}(1/2)^3 + \binom{4}{3}(1/2)^4 + \binom{5}{5}(1/2)^5 + \binom{6}{3}(1/2)^6\right\} \times \frac{1}{6}$
   $= \left(\frac{1}{2^3} + \frac{4}{2^4} + \frac{10}{2^5} + \frac{20}{2^6}\right) \times \frac{1}{6} = \frac{1}{6}.$

   b)
$$P(4 \text{ spots}|3H) = \frac{P(3H|4 \text{ spots})P(4 \text{ spots})}{P(3H)} = \frac{\binom{4}{3}(1/2)^4(1/6)}{(1/6)} = \frac{1}{4}$$

4. $P(\text{exactly 9 tails} \mid \text{at least 9 tails})$
   $= P(\text{exactly 9 tails and at least 9 tails} \mid \text{at least 9 tails})$
   $= P(\text{exactly 9 tails} \mid \text{at least 9 tails})$
   $= \frac{10(1/2)^{10}}{10(1/2)^{10}+(1/2)^{10}} = 10/11$

5. $\frac{\binom{97}{57}}{\binom{100}{60}}$

6. a) $(7/10)^4 - (6/10)^4$
   b) The six must be one of the four numbers drawn. The remaining three numbers must be selected from $\{0, \ldots, 5\}$. The desired probability is therefore $\binom{6}{3}/\binom{10}{4}$.

7. $k \approx 1025$

8. $\sum_{k=0}^{10} \left[\binom{10}{k}(1/6)^k(5/6)^{10-k}\right]^2$

9. a) 80%

   b) $P(\# \text{ of kids} = x| \text{ at least 2 girls})$

$$= \frac{0.4 \times \frac{1}{4}}{P(\geq 2 \text{ girls})} \quad \text{for } x = 2$$

$$= \frac{0.3 \times \frac{4}{8}}{P(\geq 2 \text{ girls})} \quad \text{for } x = 3$$

$$= \frac{0.1 \times \frac{11}{16}}{P(\geq 2 \text{ girls})} \quad \text{for } x = 4$$

   So $x = 3$ is most likely.

   c) $P(1\text{G},3\text{B} \ \& \text{ pick G}) + P(2\text{G},2\text{B} \ \& \text{ pick G}) + P(3\text{G},1\text{B} \ \& \text{ pick G})$

$$= \frac{4}{16} \cdot \frac{1}{4} + \frac{6}{16} \cdot \frac{2}{4} + \frac{4}{16} \cdot \frac{3}{4} = 0.4375$$

10. a) We can use the binomial approximation to the hypergeometric distribution with $n = 10$, $p = .15$. So $1-(1-.15)^{10} = .8031$. And the histogram will be that of the binomial $(10, 0.15)$ distribution.

    b) No. Presumably some machines are more reliable than others. Then results of successive tests on a machine picked at random are not independent. So the independence assumption required for the binomial distribution is not satisfied.

11.

$$P(\text{bad}) = \frac{2}{3} \times \frac{1}{100} + \frac{1}{3} \times \frac{2}{100} = 0.0133$$

$$P(2 \text{ bad out of } 12) = \binom{12}{2}(.0133)^2(1 - .0133)^{10} = 0.0102$$

Assume items are bad independent of each other. Reasonable since both A and B produce a large number of items each day.

12. a) 0.423 (See solution to Exercise 2.5.12).

b) Want chance of at most 149 "one pair"s in the first 399 deals

$$= \sum_{k=0}^{149} \binom{399}{k} (0.423)^k (0.577)^{399-k}$$

c) Use normal approximation: $\mu = 168.78$, $\sigma = 9.87$.
Want $\Phi(\frac{149.5 - 168.78}{9.87}) = \Phi(-1.95) = .0256$

13. The number of "dud" seeds in each packet has the binomial (50, .01) distribution, which is very well approximated by the Poisson (.5) distribution. The chance that a single packet has to be replaced is therefore

$$1 - e^{-0.5}\{1 + .5 + .5^2/2\} = .0144.$$

Assuming that packets are independent of each other, the number of replaceable packets out of the next 4000 has the binomial distribution with parameters 4000 and .0144. This gives $\mu = 57.6$ and $\sigma = 7.535$, which is much bigger than 3. So the normal approximation will work well. The chance that more than 40 out of the next 4000 packets have to be replaced is very close to

$$1 - \Phi\left(\frac{40.5 - 57.6}{7.535}\right) = 1 - \Phi(-2.33) = \Phi(2.33) = .9901.$$

14. a) 1/5 b) 2/9
c) line: $\frac{n-k+1}{\binom{n}{k}} = \frac{(n-k+1)!k!}{n!}$; circle: $\frac{n}{\binom{n}{k}} = \frac{(n-k)!k!}{(n-1)!}$.

15. a) $\binom{20}{5}(0.4)^5(0.6)^{15}$

b) $\frac{20!}{2!4!6!8!}(0.1)^2(0.2)^4(0.3)^6(0.4)^8$

c) $P(25th$ ball is red, and there are 2 red balls in first 24 draws)

$$= 0.1 \times \binom{24}{2}(0.1)^2(0.9)^{22}$$

16. a) $\frac{\binom{48}{4}}{\binom{52}{8}}$ b) $\frac{1}{\binom{52}{8}}$ c) $13 \times \frac{\binom{48}{4}}{\binom{52}{8}} - \binom{13}{2} \times \frac{1}{\binom{52}{8}}$

17. a) $\frac{\binom{6}{1}}{6^4}$ b) $\binom{6}{1} \times \binom{5}{1} \times \frac{\binom{4}{1}}{6^4}$ c) $\binom{6}{2} \times \frac{\binom{4}{2}}{6^4}$

18. a) $\binom{7}{3}(1/6)^3(5/6)^4$ b) $6 \times 5 \times \frac{\binom{7}{4}}{6^7}$ c) $\frac{\binom{7}{2}\binom{5}{2}}{6^7}$ d) $6 \times \frac{\binom{7}{2} \times 5!}{6^7}$
e) $P(\text{sum} \geq 9) = 1 - P(\text{sum} < 9)$
$= 1 - P(\text{seven 1's}) - P(\text{six 1's and a 2})$
$= 1 - (1/6)^7 - 7(1/6)^7$.

19. a) $(2/3)^4$; b) $\binom{4}{1}(2/3)^4(1/3) + (2/3)^4$

20. a) $\sum_{x=0}^{k} \binom{n}{x} q^x p^{n-x}$

b) $(0.99)^8 + 8 \cdot 0.01(0.99)^7 + \binom{8}{2}(0.01)^2(0.99)^6$.

21. For n odd, $\sum_{x=0}^{(n-1)/2} \binom{n}{x} q^x p^{n-x}$.

22. a) Assume each person who buys a ticket shows up independently of all others with probability 0.97. If n tickets are sold, then the number of people who show up has binomial $(n, 0.97)$ distribution. Find n so that

$$.95 \leq P(0 \text{ to } 400) \approx \Phi\left(\frac{400.5 - .97n}{\sqrt{(.97)(.03)n}}\right).$$

Therefore $\frac{400.5-.97n}{\sqrt{(.97)(.03)n}} \geq 1.645$, which implies $n \leq 407$.

23.

$$P(\text{at least one girl} \mid \text{at least one boy}) = \frac{P(\text{at least one girl and at least one boy})}{P(\text{at least one boy})}$$
$$= \frac{P(\text{not all girls or all boys})}{(.2 \times 1/2) + (.4 \times 3/4) + (.2 \times 7/8) + (.1 \times 15/16)}$$
$$= \frac{(.4 \times 1/2) + (.2 \times 6/8) + (.1 \times 14/16)}{(.2 \times 1/2) + (.4 \times 3/4) + (.2 \times 7/8) + (.1 \times 15/16)}.$$

24. Denote by $\pi(n)$, $n = 0$ to 5, the proportion of families having n children.

a) We assume that in every family, every child is equally likely to be a boy or a girl. In this case, if a family is chosen at random then

$$P(\text{family has } n \text{ children and exactly 2 girls})$$
$$= P(n \text{ children})P(\text{exactly 2 girls} \mid n \text{ children})$$
$$= \pi(n) \times \binom{n}{2}\frac{1}{2^n}$$

(interpret $\binom{n}{2}$ as 0 if $n < k$). Hence

$$P(\text{family has exactly 2 girls}) = \sum_{n=0}^{5} \pi(n)\binom{n}{2}\frac{1}{2^n} = .203125.$$

b) Since each child is equally likely to be chosen, we have

$$P(\text{child comes from a family having exactly 2 girls}) = \frac{\#\text{such children}}{\#\text{children in population}}.$$

Suppose the population has N families, N assumed large. Denominator: There are $N\pi(n)$ *families* having exactly n children, so there are $nN\pi(n)$ *children* belonging to n-child families; hence there are $\sum_{n=0}^{5} nN\pi(n)$ children in the population.

Numerator: In part (a) we showed that the proportion of families having n children and exactly 2 girls is $\pi(n)\binom{n}{2}\frac{1}{2^n}$. Thus there are $nN\pi(n)\binom{n}{2}\frac{1}{2^n}$ children who come from families having n children and exactly 2 girls, and $\sum_{n=0}^{5} nN\pi(n)\binom{n}{2}\frac{1}{2^n}$ children who come from families having exactly 2 girls.

Therefore the chance that a child chosen at random from the children in this population comes from a family having exactly 2 girls is

$$\frac{\sum_{n=0}^{5} n\pi(n)\binom{n}{2}\frac{1}{2^n}}{\sum_{n=0}^{5} n\pi(n)} = .294207.$$

25. a) $P(\text{win in 3 sets}) = P(\text{win all three}) = p^3$
$P(\text{win in exactly 4 sets}) = P(\text{win 2 out of first three, then win fourth}) = \binom{3}{2}p^2q \cdot p = 3p^3q$
$P(\text{win in exactly 5 sets}) = P(\text{win 2 out of first four, then win fifth}) = \binom{4}{2}p^2q^2 \cdot p = 6p^3q^2$

b) $P(\text{player A wins the match})$
$= P(\text{wins in exactly 3 sets}) + P(\text{wins in exactly 4 sets}) + P(\text{wins in exactly 5})$
$= p^3 + 3p^3q + 6p^3q^2$

c)

$$P(\text{match lasts only 3 sets}|\text{A won}) = \frac{P(\text{A won in only 3 sets})}{P(\text{A won})}$$

$$= \frac{p^3}{p^3 + 3p^3q + 6p^3q^2} = \frac{1}{1 + 3q + 6q^2}$$

d) If $p = 2/3$, the answer in c) is 0.375.

e) No: if player A has lost the first two sets, he may be nervous about saving the match, and that could affect his performance. In general, the assumption of independence of a player's performance in successive games is rather suspect.

26. a) $8/\binom{10}{3}$

b) Using the inclusion-exclusion formula, the probability in question is

$$P(A\&B \cup B\&C \cup \cdots \cup J\&A)$$

$$= P(A\&B) + P(B\&C) + \cdots + P(J\&A) - \sum P(\cap\text{'s of two pairs}) + \sum P(\cap\text{'s of three pairs}) - \ldots$$

$$= 10 \times 8/\binom{10}{3} - 10/\binom{10}{3} = 7/12$$

because the probability of each intersection of 3 or more pairs is 0.

27. Assume that every quadruplet of 4 exam groups appears equally often in the population of students, so that the proportion of students having a specific set of 4 groups is $1/\binom{18}{4}$.

By equally likely outcomes: There are $\binom{6}{4}3^4$ quadruplets which correspond to different exam days (count the ways to pick the 4 days, then the ways to pick an exam time from each of those 4 days). Hence the desired proportion is $\binom{6}{4}3^4/\binom{18}{4} = .3971$.

By conditioning: Pick a student at random. Let D_i denote the day of the ith exam. Then

$$P(D_1, D_2, D_3, D_4 \text{ different}) = P(D_1, D_2 \text{ different}) \times$$
$$P(D_1, D_2, D_3 \text{ different}|D_1, D_2 \text{ different}) \times$$
$$P(D_1, D_2, D_3, D_4 \text{ different}|D_1, D_2, D_3 \text{ different})$$
$$= \frac{15}{17} \times \frac{12}{16} \times \frac{9}{15} = .3971.$$

28. a) $1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n-1}}{n!}$

b) $1 - e^{-1}$

29. What kinds of throws are wimpouts? Firstly, they cannot have any numbers in them. Of the the throws with only letters the only ones that contain a W either have four different other letters or 2 or more letters of the same type. Both of these are scoring throws. Thus the only throws that do not score consist entirely of the letters A, B, C, D with no more than two of any letter. Condition on the dice with the W because it has only A, B, C. Suppose it is A. Then the outcomes of the other dice that lead to a wimpout are: ABCD, ABCC, ABBC, ABDD, ABBD, ACDD, ACCD, BBCC, BBCD, BBDD, BCDD, BCCD, CCDD. Taking into account the different possible orderings and multiplying by 3 we get a total of 450 outcomes. Since there are $6^5 = 7776$ equally likely outcomes possible when you roll 5 dice,

$$P(\text{wimpout}) = \frac{450}{7776} = \frac{25}{432} = 0.05787$$

30. Draw the curve $y = \log x$ from 1 to n. Between x and $x + 1$, draw a box with height $\log x$. (The first "box" has 0 height.) Connect the upper left corners of the boxes by straight lines. The area under the curve is the sum of three parts; the area of the boxes, the area of the triangles above the boxes, and the area of the slivers above the triangles and below the curve. The area under the curve is

$$\int_1^n \log x\,dx = n \log n - n + 1 = \log\left[\left(\frac{n}{e}\right)^n\right] + 1$$

The area of the boxes is

$$\sum_{x=1}^{n-1} \log x = \log (n-1)!$$

The area of the triangles is

$$\sum_{x=1}^{n-1} \frac{1}{2}(\log(x+1) - \log x) = \frac{1}{2}\log n$$

By moving the slivers so that they all have their right ends on the point $(2, \log 2)$, you can see that none of the slivers overlap and all of the slivers fit in a box between 1 and 2 with height $\log 2$. So the area of the slivers is some number $c_n < \log 2$ for all n. We see that the area under the curve is also equal to

$$\log (n-1)! + \frac{1}{2}\log n + c_n = \log n! - \frac{1}{2}\log n + c_n$$

By setting the two expressions for the area under the curve to be equal, and solving for $\log n!$, we get

$$\log n! = \log \left[\left(\frac{n}{e}\right)^n\right] + 1 + \frac{1}{2}\log n - c_n = \log \left[\left(\frac{n}{e}\right)^n \sqrt{n} e^{1-c_n}\right]$$

where c_n increases to a limit c with $c < \log 2$. So

$$n! = \left(\frac{n}{e}\right)^n \sqrt{n} e^{1-c_n} \sim C\left(\frac{n}{e}\right)^n \sqrt{n}$$

where $C = e^{1-c}$.

The exact probability of getting m heads in $2m$ coin tosses is

$$\begin{aligned}\binom{2m}{m}\left(\frac{1}{2}\right)^{2m} &= \frac{(2m)!}{(m!)^2}\left(\frac{1}{2}\right)^{2m} \\ &\sim \frac{C(2m/e)^{2m}\sqrt{2m}}{C^2(m/e)^{2m}m}\left(\frac{1}{2}\right)^{2m} \\ &= \frac{\sqrt{2}}{C\sqrt{m}}\end{aligned}$$

By the normal approximation, this is approximately

$$\frac{1}{\sqrt{2m(1/2)^2}\sqrt{2\pi}} = \frac{1}{\sqrt{m\pi}}$$

Setting these expressions equal gives $C = \sqrt{2\pi}$.

31. No Solution

32. Those who have computed the probabilities of all the 5 card poker hands and know that a straight is about twice as likely as a flush may be surprised to learn that the answer to this question depends on the size of the hand, h. There are $\binom{52}{h}$ distinct h card hands possible. Of these, $4\binom{13}{h}$ are *flushes*: 4 suits (hearts, diamonds, clubs, spades) and we want to choose h cards from any one of these suits. There are $(13-h+1)$ cards that a *straight* of length h could start on, and for each card in the sequence $d, d+1, \ldots, d+h$ there are four possible suits to choose from, giving a total of $(13-h+1)4^h$ possible *straights*. For $h = 1, 2, 3, 4$, $4\binom{13}{h} > (13-h+1)4^h$ and for $5 \leq h \leq 13$, $4\binom{13}{h} < (13-h+1)4^h$. The case $h = 1$ is trivial because any single card is both a *straight* and a *flush*. For $h > 13$ neither a *straight* nor a *flush* is possible, so they both have zero probability.

33. a) Note that $P(HH|HH \text{ or } HT \text{ or } TH) = 1/3$. So toss the fair coin twice. Report 1 if the outcome is HH, and 0 if the outcome is HT or TH, keep trying otherwise.

    b) Note that $P(HT|HT \text{ or } TH) = P(TH|HT \text{ or } TH) = 1/2$ independent of p. So toss the biased coin twice. Report 1 if the outcome is HT, and 0 if the outcome is TH, keep trying otherwise.

34. a) Note that the number of heads and the number of tails have the same distribution, therefore
$P(\#$ of heads from my toss $= \#$ of heads from your toss$)$
$= P(\#$ of heads from my toss $- \#$ of heads from your toss $= 0)$
$= P(\#$ of heads from my toss $+ \#$ of tails from your toss $- m = 0)$
$= P(\#$ of heads from my toss $+ \#$ of tails from your toss $= m)$

And the distribution of # of heads from my toss + # of tails from your toss is binomial $(2m, 1/2)$ by symmetry and the independence of tosses.

b) Let M_n be the number of heads I get on n tosses, and Y_n be the number of heads you get on n tosses. Then

$P(Y_{m+1} > M_m) = P(Y_{m+1} > M_m \mid$ your $m+1$st toss is head$) \times 1/2$
$+ P(Y_{m+1} > M_m \mid$ your $m+1$st toss is tail$) \times 1/2$
$= P(Y_m \geq M_m) \times 1/2 + P(Y_m > M_m) \times 1/2$
$= P(Y_m \geq M_m) \times 1/2 + P(Y_m < M_m) \times 1/2$ by symmetry
$= 1/2.$

35. a) $\sum_{k=20}^{35} \binom{1000}{k} (\frac{1}{38})^k (\frac{37}{38})^{1000-k}$

b) The SD is $\sigma = 5.06$. Use normal approximation.

$$\Phi(\frac{35.5 - 26.316}{5.06}) - \Phi(\frac{19.5 - 26.316}{5.06})$$
$$= \Phi(1.815) - \Phi(-1.347)$$
$$\approx .965 - (1 - .9115) = 0.8765$$

36. c) As $n \to \infty$, $H(k)$ is asymptotic to $e^{-\frac{1}{2}(k-np)^2/npq}$. Hence if $H(k) < \epsilon$, then k satisfies approximately

$$e^{-\frac{1}{2}(k-np)^2/npq} < \epsilon$$
$$-\frac{1}{2}(k-np)^2/npq < \log \epsilon$$
$$(k-np)^2 > 2npq \log \frac{1}{\epsilon}$$
$$k < np - \sqrt{2npq \log \frac{1}{\epsilon}} \text{ or } k > np + \sqrt{2npq \log \frac{1}{\epsilon}}.$$

So the a, b such that $a < m < b$ and both $H(a)$ and $H(b)$ are less than ϵ are approximately

$$a \approx np - \sqrt{2npq \log \frac{1}{\epsilon}}$$
$$b \approx np + \sqrt{2npq \log \frac{1}{\epsilon}},$$

and

$$b - a \approx 2\sqrt{2npq \log \frac{1}{\epsilon}}.$$

The run time to perform the calculation is then approximately $2K \times \sqrt{2npq \log \frac{1}{\epsilon}}$.

d) Given p and ϵ, the run time to compute every probability in the binomial (n, p) distribution to within ϵ is proportional to $\sqrt{n}$. Thus it should take $\sqrt{10}$ times as long to compute when n is multiplied by 10. Hence if it takes 2 seconds to compute the binomial (100, 18/38) distribution correct to 3 decimal places, it should take $2\sqrt{10} \approx 6.3$ seconds to compute the binomial (1000, 18/38) distribution correct to 3 decimal places.

37. No Solution

Section 3.1

1. X has binomial (3, 1/2) distribution. So $P(X=k) = \binom{3}{k}(1/2)^3$ for $k = 0, 1, 2, 3$.

a)

x	0	1	2	3
$P(X=x)$	1/8	3/8	3/8	1/8

b) Write $Y = \lvert X-1 \rvert$.

y	0	1	2
$P(Y=y)$	3/8	1/2	1/8

2. a) *Joint distribution table for (X, Y) (with replacement)*

		possible values x for X				*distn*
		1	2	3	4	*of Y*
possible	1	1/16	1/16	1/16	1/16	1/4
values	2	1/16	1/16	1/16	1/16	1/4
y	3	1/16	1/16	1/16	1/16	1/4
for Y	4	1/16	1/16	1/16	1/16	1/4
distn of X		1/4	1/4	1/4	1/4	1

$P(X \le Y) = 10 \times \frac{1}{16} = \frac{5}{8}$

b) *Joint distribution table for (X, Y) (without replacement)*

		possible values x for X				distn
		1	2	3	4	*of Y*
possible	1	0	1/12	1/12	1/12	1/4
values	2	1/12	0	1/12	1/12	1/4
y	3	1/12	1/12	0	1/12	1/4
for Y	4	1/12	1/12	1/12	0	1/4
distn of X		1/4	1/4	1/4	1/4	1

$P(X \le Y) = 6 \times \frac{1}{12} = \frac{1}{2}$

3. a) { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }

b) Distribution Table for S:

s	2	3	4	5	6	7	8	9	10	11	12
$P(S=s)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

4. a) *Joint distribution table for* (X_1, X_2)

		possible values x_1 for X_1						distn
		1	2	3	4	5	6	of X_2
possible	1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
values	2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
x_2	3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
for X_2	4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
	6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
distn of X_1		1/6	1/6	1/6	1/6	1/6	1/6	1

b) *Joint distribution table for* (Y_1, Y_2)

		possible values y_1 for Y_1						distn
		1	2	3	4	5	6	of Y_2
possible	1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
values	2	0	1/36	2/36	2/36	2/36	2/36	9/36
y_2	3	0	0	1/36	2/36	2/36	2/36	7/36
for Y_2	4	0	0	0	1/36	2/36	2/36	5/36
	5	0	0	0	0	1/36	2/36	3/36
	6	0	0	0	0	0	1/36	1/36
distn of Y_1		1/36	3/36	5/36	7/36	9/36	11/36	1

5. Distribution of X_1X_2:

z	1	2	3	4	5	6	8	9	10
$P(X_1X_2 = z)$	1/36	2/36	2/36	3/36	2/36	4/36	2/36	1/36	2/36

z	12	15	16	18	20	24	25	30	36
$P(X_1X_2 = z)$	4/36	2/36	1/36	2/36	2/36	2/36	1/36	2/36	1/36

6. There are 8 equally likely outcomes for three fair coin tosses:

outcome	probability	X	Y	$X+Y$
HHH	1/8	2	2	4
HHT	1/8	2	1	3
HTH	1/8	1	1	2
HTT	1/8	1	0	1
THH	1/8	1	2	3
THT	1/8	1	1	2
TTH	1/8	0	1	1
TTT	1/8	0	0	0

a) Joint distribution table for (X, Y)

Y \ X	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

b) X and Y are not independent, since, for instance, $P(X = 2, Y = 0) = 0$, while $P(X = 2)P(Y = 0) = (1/4)(1/4)$.

c)

z	0	1	2	3	4
$P(X+Y = z)$	1/8	2/8	2/8	2/8	1/8

7. a) $(N = 2) = (ABC^c) \cup (AB^cC) \cup (A^cBC)$

b) $P(N = 2) = ab(1-c) + a(1-b)c + (1-a)bc$.

8. a)

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

b)

y	2	3	4	5
$P(Y = y)$	0.1	0.2	0.3	0.4

c) Let $\hat{X}$ be the number of cards until the first ace when dealing from the bottom of the deck. Then $\hat{X}$ has the same distribution as X, and

$$Y = 5 - (\hat{X} - 1) = 6 - \hat{X}$$

So

$$P(Y=y)=P(6-\hat{X}=y)=P(\hat{X}=6-y)=P(X=6-y)$$

9.

x	2	3	4	5
$P(X=x)$	5/35	10/35	12/35	8/35

10. a) binomial (n,p) b) binomial (m,p) c) binomial $(n+m,p)$

d) yes; functions of disjoint blocks of independent variables are independent.

11. a) By the change of variable principle, U_n+V_m has the same distribution as S_n+T_m in Exercise 10, and this is binomial $(n+m,p)$

b) By (a), $P(U_n+V_m=k)=\binom{n+m}{k}p^k(1-p)^{n+m-k}$. But also,

$$P(U_n+V_m=k)=\sum_{j=0}^{k}P(U_n=j,V_m=k-j)=\sum_{j=0}^{k}P(U_n=j)P(V_m=k-j)$$
$$=\sum_{j=0}^{k}\binom{n}{j}p^j(1-p)^{n-j}\binom{m}{k-j}p^{k-j}(1-p)^{m-(k-j)}$$
$$=[\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}]p^k(1-p)^{n+m-k}.$$

Now equate the two expressions for $P(U_n+V_m=k)$. [To get the sum from 0 to n, note $\binom{x}{y}=0$ if $y<0$ or $y>x$]

c) Suppose you have $n+m$ objects, n of which are red, and m blue. You want to choose k out of the $n+m$ objects. Among the k selected objects, some j will be red, where j could be $0,1,2,\dots,n$. So you could just as well pick j red objects first, then $(k-j)$ blue objects.

d) This is similar to b).

e) $\binom{2n}{n}$

12. a) If we think of N_i as just counting the number of times we get category i in n trials, we don't care what happens when we don't get this category. We get category i with probability p_i, and don't get it with probability $1-p_i$, so the distribution of N_i is just binomial (n,p_i).

b) Similarly, N_i+N_j counts the number of times we get either category i or j, which happens with probability p_i+p_j, so the distribution of N_i+N_j is binomial (n,p_i+p_j).

c) Now we just consider the three categories i,j, and everything else, which gives a joint distribution which is multinomial $(n,p_i,p_j,1-p_i-p_j)$.

13. a)

$$P(X>k)=P(\text{first } k \text{ balls are different color})=1\cdot\frac{2n-2}{2n}\cdot\frac{2n-4}{2n}\cdot\dots\cdot\frac{2n-2(k-1)}{2n}$$

b)

$$\log P(X>k)=\sum_{j=0}^{k-1}\log(\frac{n-j}{n})=\sum_{j=1}^{k-1}\log(1-j/n)\approx\sum_{j=1}^{k-1}-j/n=-\frac{k(k-1)}{2n}$$

So $k(k-1)=[-\log(\frac{1}{2})]2n$. Roughly $k^2=n\log 4$, so $k=\sqrt{n\log 4}$. And $k=1177$ for $n=10^6$.

14. a) $\binom{g-1}{3}p^3q^{g-1-3}\cdot p=\binom{g-1}{3}p^4q^{g-4}$ for $g=4,5,6,7$.

b) $\sum_{g=4}^{7}\binom{g-1}{3}p^4q^{g-4}$

c) $P(\text{A wins})=1808\ /\ 2187$

d) The outcome of the World series would be the same if the teams played all 7 games. Let X be the number of times that A wins if all 7 games are played. If $X \geq 4$, then A has won the World series, since B can have won at most 3 games. And if A won the World Series, then A won four games before B did, so $X \geq 4$. So $P(\text{A wins}) = P(X \geq 4)$. Of course X has binomial $(7, p)$ distribution!

e) G has range $\{4, 5, 6, 7\}$.
$P(G = g) = P(\text{A wins in } g \text{ games}) + P(\text{B wins in } g \text{ games})$
$= \binom{g-1}{3} p^4 q^{g-4} + \binom{g-1}{3} q^4 p^{g-4}$.
$p = 1/2$ makes G and the winner independent.

15. a) $P(X = Y) = P(X = 1, Y = 1) + P(X = 2, Y = 2) + \cdots + P(X = n, Y = n)$
$= P(X = 1)P(Y = 1) + \ldots + P(X = n)P(Y = n)$ (by independence)
$= n \cdot \frac{1}{n^2} = \frac{1}{n}$.

b&c) Notice that by symmetry, $P(X > Y) = P(X < Y)$. Moreover,
$1 = P(X > Y) + P(X < Y) + P(X = Y) = 2P(X > Y) + 1/n$.
So $P(X > Y) = \frac{n-1}{2n} = P(X < Y)$.

d) $P(\max(X, Y) = k)$
$= P(X = k, Y < k) + P(X < k, Y = k) + P(X = k, Y = k)$
$= \frac{1}{n} \cdot \frac{k-1}{n} + \frac{k-1}{n} \cdot \frac{1}{n} + \frac{1}{n^2}$
$= \frac{2k-1}{n^2}$

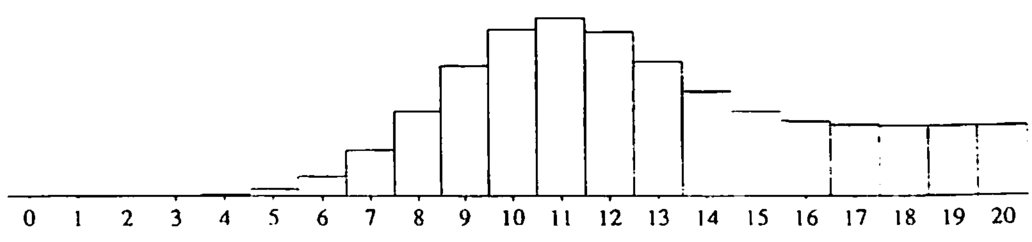
e) $P(\min(X, Y) = k) = P(\max(n + 1 - X, n + 1 - Y) = n + 1 - k)$
$= P(\max(X^*, Y^*) = n + 1 - k)$
$= \frac{2(n+1-k)-1}{n^2}$.
where X^* and Y^* are copies of X and Y respectively.

f) Note that the distribution of $X + Y$ is symmetric about $n + 1$ (recall the sum of two dice). For $2 \leq k \leq n + 1$,
$P(X + Y = k) = \sum_{j=1}^{k-1} P(X = j)P(Y = k - j) = \frac{k-1}{n^2}$.
On the other hand, for $n + 1 < k \leq 2n$
$P(X + Y = k) = P[2(n + 1) - (X + Y) = 2(n + 1) - k] = \frac{[2(n+1)-k]-1}{n^2} = \frac{2n-k+1}{n^2}$.

16. a) $P(X + Y = n) = \sum_{k=0}^{n} P(X = k, X + Y = n)$
$= \sum_{k=0}^{n} P(X = k, Y = n - k)$
$= \sum_{k=0}^{n} P(X = k)P(Y = n - k)$

b) $P(X + Y = 8) = \sum_{k=0}^{8} P(X = k)P(Y = 8 - k) = \sum_{k=0}^{6} P(X = k)P(Y = 8 - k)$
$= \frac{1}{36} \times \frac{5}{36} + \frac{2}{36} \times \frac{4}{36} + \frac{3}{36} \times \frac{3}{36} + \frac{4}{36} \times \frac{2}{36} + \frac{5}{36} \times \frac{1}{36} = \frac{35}{1296} = 0.027$

17. a) $P(Z = k) = P(Y < k, X = k) + P(Y = k, X < k) + P(Y = k, X = k)$
$= (k/21)\binom{20}{k}(1/2)^{20} + (1/21)\sum_{i=0}^{k} \binom{20}{i}(1/2)^{20}$



b) Left tail comes from binomial: very thin. Right tail comes from uniform: thick and flat.

18. a) The number of spots has a distribution symmetric about $E(\text{number of spots}) = 3 \times 3.5 = 10.5$, so $P(11 \text{ or more spots}) = P(10 \text{ or fewer spots}) = 1/2$.

b) The number of spots has a distribution symmetric about $5 \times 3.5 = 17.5$, so $P(18 \text{ or more spots}) = P(17 \text{ or fewer spots}) = 1/2$.

19. a) $P(S = k) = \sum_{i+j=k} p_i r_j$ for all $k = 2$ to 12, so in particular
$P(S = 2) = p_1 r_1$;
$P(S = 7) = p_1 r_6 + p_2 r_5 + p_3 r_4 + p_4 r_3 + p_5 r_2 + p_6 r_1$;
$P(S = 12) = p_6 r_6$.

b) $P(S = 7) > p_1 r_6 + p_6 r_1 = P(S = 2)\frac{r_6}{r_1} + P(S = 12)\frac{r_1}{r_6}$.

c) Suppose the values are equally likely. Then from (b) and calculus, $1 > \frac{r_6}{r_1} + \frac{r_1}{r_6} \geq 2$: contradiction!

d) Yes; for example, let X take values 1, 2 with probability 1/2 each, and Y take values 1, 3 with probability 1/2 each. Then $X + Y$ has uniform distribution over 2, 3, 4, 5.

20. No. Counterexample: Let X_1, X_2, X_3 be the indicators of the events H_1, H_2, S in Example 1.6.8.

21. Yes. Proof by mathematical induction.

22. a) $P(X = x) = \sum_y P(X = x, Y = y) = \sum_y f(x)g(y) = f(x)\sum_y g(y)$
Similarly $P(Y = y) = g(y)\sum_x f(x)$.

b) If $P(X = x, Y = y) = f(x)g(y)$, then
$P(X = x)P(Y = y) = f(x)\sum_j g(j) \cdot g(y)\sum_i f(i) = f(x)g(y)\left(\sum_j g(j)\right)\left(\sum_i f(i)\right)$.
So we just need to show $\left(\sum_j g(j)\right)\left(\sum_i f(i)\right) = 1$:

$$1 = \sum_i \sum_j P(X = i, Y = j) = \sum_i \sum_j f(i)g(j) = \left(\sum_i f(i)\right)\left(\sum_j g(j)\right)$$

Note: In this calculation x and y are fixed. The sums are over dummy variables which are not called x and y, to avoid confusion with the fixed values.

23. If $X \leq T$, then $Y \leq T$, since $Y \leq X$. Hence

$$(X \leq T) \subset (Y \leq T) \text{ and } P(X \leq T) \leq P(Y \leq T).$$

This argument still works if T is a random variable.

24. a) Let $p_i = P(X = i \bmod 2)$, $i = 0, 1$. Then $p_0 + p_1 = 1$,

$$P(X + Y \text{ is even}) = p_0^2 + p_1^2 = 1 - 2p_0(1 - p_0) \geq 1 - 2 \cdot \frac{1}{4}.$$

b) Let $p_i = P(X = i \bmod 3)$, $i = 0, 1, 2$. Then $p_0 + p_1 + p_2 = 1$,

$$P(\text{X + Y + Z is a multiple of 3}) = p_0^3 + p_1^3 + p_2^3 + 6p_0p_1p_2.$$

Now write simply p, q, r for p_0, p_1, p_2. So

$$p + q + r = 1, 0 \leq p, q, r \leq 1.$$

To show: $p^3 + q^3 + r^3 + 6pqr \geq \frac{1}{4}$. Consider

$$1 = (p + q + r)^3 = p^3 + q^3 + r^3 + 6pqr + 3[p^2q + p^2r + q^2p + q^2r + r^2p + r^2q].$$

Notice that $p(q + r) = pq + pr \leq 1/4$, $q(p + r) = qp + qr \leq 1/4$, and $r(p + q) = rp + rq \leq 1/4$. The probability in question is thus

$$1 - 3[p[pq + pr] + q[qp + qr] + r[rp + rq]]$$

$$\geq 1 - 3[(p + q + r) \cdot 1/4] \geq 1 - 3/4 = 1/4.$$

Section 3.2

1. $15 \times 1 + 25 \times .2 + 50 \times .7 = 41.5$

2. Average of list 1 $= 1 \times .2 + 2 \times .8 = 1.8$; Average of list 2 $= 3 \times .5 + 5 \times .5 =$

   a) $1.8 + 4 = 5.8$ (distributive property of addition)

   b) $1.8 - 4 = 2.2$

   c&d) Can't do it: need to know the order of the numbers in the two lists.

3. $E(\#$ sixes in 3 rolls$) = 3 \times \frac{1}{6} = \frac{1}{2}$.

   $E(\#$ odd numbers in 3 rolls$) = 3 \times \frac{1}{2} = \frac{3}{2}$.

4. If 25 of the numbers are 8, then all the others must be 0, since the average must be 2. If 26 or more of the numbers are 8 or more, there is no way the average can be 2, since some of the other numbers would have to be negative.

5. Suppose he bets on 6, and let N be his net gain.

k	$P(\text{get } k \text{ 6's})$	N	$n \times P(N = n)$
3	$(1/6)^3$	3	$3 \times (1$ [illegible] $)^3$
2	$3(5/6)(1/6)^2$	2	$2 \cdot 3(5/$ [illegible] $/6)^2$
1	$3(5/6)^3(1/6)$	1	$3(5/6)$ [illegible] $(1/6)$
0	$(5/6)^3$	-1	$-(5/6)^3$

   Therefore $E(N) = -.078705$, and the gambler expects to lose about 8 cents per game in the long run.

6. $X = I_1 + I_2 + \cdots + I_7$ where I_i is an indicator random variable indicating whether the ith card is a spade. Then $E(X) = 7P(\text{first card is a spade}) = \frac{7}{4}$

7. $E(X) = \sum_1^n p_i$, by linearity of E. No more assumptions required.

8. $E\left[(X + Y)^2\right] = E(X^2) + 2E(XY) + E(Y^2) = 17$.

9. $E(X - Y)^2 = E(X^2) - 2E(XY) + E(Y^2) = p - 2pr + r$

10. a) Write $Y = (I_A + I_B)^2$.
    If $I_A = 0$ and $I_B = 0$ (this occurs with prob $(1 - P(A)) \cdot (1 - P(B))$), then $Y = (0 + 0)^2$.
    If $I_A = 1$ and $I_B = 0$ (this occurs with prob $P(A) \cdot (1 - P(B))$), then $Y = (1 + 0)^2$.
    If $I_A = 0$ and $I_B = 1$ (this occurs with prob $(1 - P(A)) \cdot P(B)$), then $Y = (0 + 1)^2$.
    If $I_A = 1$ and $I_B = 1$ (this occurs with prob $P(A) \cdot P(B)$), then $Y = (1 + 1)^2$.
    So

y	$P(Y = y)$
0	$(1 - P(A)) \cdot (1 - P(B))$
1	$P(A) \cdot (1 - P(B)) + (1 - P(A)) \cdot (P(B))$
4	$P(A) \cdot (P(B))$

    b) Use properties of expectation described in this section: Note that $Y = (I_A + I_B)^2 = I_A + 2I_{AB} + I_B$ (using $x^2 = x$ for $x = 0$ and $x = 1$, so $I_A^2 = I_A$ and $I_B^2 = I_B$) so

$$E(Y) = P(A) + 2P(AB) + P(B) = P(A) + 2P(A)P(B) + P(B).$$

    Alternatively (and this is more cumbersome), use part (a) and the basic definition of expectation.

11. Use Markov's inequality:

$$P(\text{at least one win}) \le \text{expected \# of wins} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = .3$$

Or you can use Boole's inequality.
Actually, $P(\text{at least one win}) = 1 - P(\text{no wins}) = 1 - \frac{900}{1000} \cdot \frac{899}{999} \cdot \frac{898}{998} = .271$. The bound is close because the bound pretends the events (ith ticket wins) are mutually exclusive. Well, they almost are, because $P(\text{more than 1 win})$ is tiny.

12.

$$\begin{aligned} E(X) &= \sum_x xP(x) \le \sum_x bP(x) \quad (\text{since } P(X \le b) = 1) \\ &= b\sum_x P(x) = b. \end{aligned}$$

Similarly you can show the other inequality.

13. a) By linearity of expectation, $10 \times 3.5 = 35$.

b) Let X_1, X_2, X_3 be the three numbers, and let Min denote the minimum of the first three numbers.
We want $E(X_1 + X_2 + X_3 - Min) = E(X_1 + X_2 + X_3) - E(Min) = 3 \times 3.5 - E(Min)$. Exactly as in Example 9,

$$\begin{aligned} E(Min) &= q_1 + q_2 + q_3 + q_4 + q_5 + q_6 \\ &= 1 + (\frac{5}{6})^3 + (\frac{4}{6})^3 + (\frac{3}{6})^3 + (\frac{2}{6})^3 + (\frac{1}{6})^3 \\ &= \frac{441}{216} = 2.042 \end{aligned}$$

where $q_m = P(Min \ge m)$. So the required expectation is $3 \times 3.5 - 2.042 = 8.458$.

c) Let Max be the maximum of the numbers on the first five rolls. For each m between 1 and 6,

$$P(Max = m) = P(Max \le m) - P(Max \le m-1) = (\frac{m}{6})^5 - \left(\frac{m-1}{6}\right)^5.$$

So

$$E(Max) = 1 \cdot (\frac{1}{6})^5 + 2\left((\frac{2}{6})^5 - (\frac{1}{6})^5\right) + \cdots + 6\left((\frac{6}{6})^5 - (\frac{5}{6})^5\right) = 5.43.$$

Or: by symmetry, $E(Max) = 7 - E(min(X_1, \ldots, X_5))$.

d) The number of multiples of 3 in the first ten rolls has binomial (10, 1/3) distribution, so its expectation is $10/3 = 3.3333$.

e) The number of faces which fail to appear in the first ten rolls is is $I_1 + I_2 + \cdots + I_6$, where I_i is the indicator of the event (face i fails to appear in the first 10 rolls). Now for each i,

$$E(I_i) = P(\text{face } i \text{ fails to appear in the first ten rolls}) = (5/6)^{10}.$$

So the required expectation is $6 \times E(I_1) = 6 \times (5/6)^{10} = 0.969024$.

f) The number of different faces in the first ten rolls equals 6 minus the number of faces which fail to appear. So the required expectation is $6 - 6(5/6)^{10} = 5.030976$.

14. We want $E(N)$, where N is the number of floors at which the elevator makes a stop to let out one or more of the people. N is a counting variable. It's the sum of the ten indicators

$$I(\text{at least one person chooses floor } i), i = 1, \ldots, 10.$$

So by linearity,

$$E(N) = \sum_{i=1}^{10} P(\text{at least one person chooses floor } i).$$

Now for each i

$$P(\text{at least one person chooses floor } i) = 1 - P(\text{nobody chooses floor } i) = 1 - (9/10)^{12}$$

by the independence of the people's choices. Hence

$$E(N) = 10 \times \left[1 - (9/10)^{12}\right] \approx 7.18.$$

15. a) If $y \leq b$, then πy will be the profit, and $\lambda(b-y)$ will be the loss incurred from the items stocked but unsold. So $-\pi y + \lambda(b-y)$ will be the loss in this case. And, if $y > b$, then πb will be the profit, since there are only b items to be sold.

b) Write $p(y) = P(Y = y)$. Then

$$\begin{aligned} r(b) = E[L(Y,b)] &= \sum_{y \leq b} [-\pi y + \lambda \cdot (b-y)]\, p(y) - \sum_{y > b} \pi b p(y) \\ &= \sum_{y \leq b} [-\pi y + \lambda \cdot (b-y)]\, p(y) + \sum_{y \leq b} \pi b p(y) - \pi b \\ &= (\lambda + \pi) \sum_{y \leq b} (b-y) p(y) - \pi b. \end{aligned}$$

How to minimize over all integers b?
Method 1: Argue that if we buy b items, then the expected loss will be

$$r(b) = (\lambda + \pi) \sum_{y \leq b-1} (b-y)p(y) - \pi b.$$

while if we buy $b-1$ items, then the expected loss will be

$$r(b-1) = (\lambda + \pi) \sum_{y \leq b-1} (b-1-y)p(y) - \pi \cdot (b-1)$$

Therefore

$$r(b) - r(b-1) = (\lambda + \pi) \sum_{y \leq b-1} p(y) - \pi = (\lambda + \pi) P(Y \leq b-1) - \pi.$$

So buying b items will be better than buying $b-1$ items whenever

$$P(Y \leq b-1) < \frac{\pi}{\lambda + \pi}. \qquad (*)$$

Let $b\star$ be the largest value of b for which $(*)$ holds. (Such a one exists, since the left-hand side tends to 1 as b increases). Note that $b\star$ is also the smallest integer for which $P(Y \leq b) \geq \pi/(\lambda + \pi)$. Argue that

$$r(0) > \cdots > r(b\star - 1) > r(b\star) \leq r(b\star + 1) \leq \cdots$$

so $r(y)$ is minimized at $y = b\star$ (and possibly elsewhere).
Method 2: View $r(b)$ as a function of a real variable b. Argue that $r(b)$ is continuous, and that if b is not an integer, then r is differentiable at b with derivative

$$r'(b) = (\lambda + \pi) \sum_{y \leq b} p(y) - \pi = (\lambda + \pi) P(Y \leq b) - \pi.$$

The right-hand side is a nondecreasing function of b. Let $b\star$ denote the smallest integer for which the right-hand side is nonnegative. Argue that if $b < b\star$, then $r(b) > r(b\star)$, and if $b \geq b\star$, then $r(b) \geq r(b\star)$. Hence r attains its minimum over real b at $b\star$ (and possibly elsewhere).

16. a) $P(X_i = k) = P(X_1 = k) = P(\text{first } k \text{ cards are non-aces, next is ace}) = \frac{(48)_k \cdot 4}{(52)_{k+1}}$.

b) $X_1 + X_2 + X_3 + X_4 + X_5 + 4 = 52 \Longrightarrow 5E(X_1) = 48 \Longrightarrow E(X_i) = E(X_1) = 9.6.$

c) No. For instance, $P(X_1 = 30, X_2 = 30) = 0$, but $P(X_1 = 30) = P(X_2 = 30) > 0$.

17. a) $P(D \le 9) = P(3 \text{ red balls are among the first 9 draws}) = \frac{\binom{10}{9-3}}{\binom{13}{9}}$.

b) $P(D = 9) = P(D \le 9) - P(D \le 8) = \frac{\binom{10}{9-3}}{\binom{13}{9}} - \frac{\binom{10}{8-3}}{\binom{13}{8}}$.

c) Label the blue balls and the green balls, say, $b_1, \ldots, b_{10}$. Then
$D = 3 + \sum_{i=1}^{10} I(b_i \text{ is drawn before the third red ball})$, so
$E(D) = 3 + \sum_{i=1}^{10} P(b_i \text{ is drawn before the third red ball}) = 3 + 10 \times \frac{3}{4} = 10.5$.

18. Solve the system of equations

$$p(a) + p(b) = 1$$
$$ap(a) + bp(b) = \mu$$

for $p(a) = \frac{\mu - b}{a - b}$ and $p(b) = \frac{a - \mu}{a - b}$.

19. Let $x = \#$ blues. Then $\#$ reds $= 2x$, $\#w = x$, $\#g = 3x$. So

$$p_b = p_w = \frac{1}{7}, \qquad p_n = \frac{2}{7}, \qquad p_g = \frac{3}{7}$$

a) $P(X \ge 4) = P(X = 4) = (\frac{5!}{1!1!1!2!}(\frac{1}{7})^3(\frac{2}{7})(\frac{3}{7})) \times 2 + \frac{5!}{1!1!1!2!}(\frac{1}{7})^2(\frac{2}{7})^2(\frac{3}{7}) + \frac{5!}{1!1!1!2!}(\frac{1}{7})^2(\frac{2}{7})(\frac{3}{7})^2$.

b) $E(X) = E(I_b) + E(I_w) + E(I_r) + E(I_g)$ where e.g. $I_g =$ indicator "green appears"

$$= (1 - (\frac{6}{7})^5) + (1 - (\frac{6}{7})^5) + (1 - (\frac{5}{7})^5) + (1 - (\frac{4}{7})^5)$$

20. Write $p(x) = P(X = x)$. Solve the system of equations

$$p(0) + p(1) + p(2) = 1$$
$$p(1) + 2p(2) = \mu_1$$
$$p(1) + 4p(2) = \mu_2$$

for $p(2) = \frac{\mu_2 - \mu_1}{2}$, $p(1) = 2\mu_1 - \mu_2$, $p(0) = 1 - (\frac{\mu_2 - \mu_1}{2}) - (2\mu_1 - \mu_2)$.

21. a) The indicator of A^c must be 1 when A^c occurs and 0 otherwise; clearly this is true for $1 - I_A$: $A^c = 1 \Leftrightarrow I_A = 0 \Leftrightarrow 1 - I_A = 1$.

b) The indicator of AB must be 1 when A and B both occur and 0 otherwise. Since $I_A I_B = 1 \Leftrightarrow (I_A = 1 \text{ and } I_B = 1) \Leftrightarrow I_A I_B = 1$ we have $I_{AB} = I_A I_B$.

c) We wish to show that the indicator of the union can be found by the given formula, which can be shown by the following application of the rules in a) and b).

$$\begin{aligned} I_{A_1 \cup A_2 \cup \cdots \cup A_n} &= I_{(A_1^c A_2^c \cdots A_n^c)^c} \\ &= 1 - I_{(A_1^c A_2^c \cdots A_n^c)} \\ &= 1 - (I_{A_1^c} I_{A_2^c} \cdots I_{A_n^c}) \\ &= 1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n}) \end{aligned}$$

d)

$$\begin{aligned} P(\bigcup_{i=1}^n A_i) &= E(I_{A_1 \cup A_2 \cup \cdots \cup A_n}) \\ &= E(1 - (1 - I_{A_1})(1 - I_{A_2}) \cdots (1 - I_{A_n})) \\ &= E(\sum_i I_{A_i} - (\sum_{i<j} I_{A_i} I_{A_j}) + \cdots + ((-1)^{n+1} I_{A_1} I_{A_2} \cdots I_{I_n})) \\ &= \sum_i E(I_{A_i}) - (\sum_{i<j} E(I_{A_i} I_{A_j})) + \cdots + ((-1)^{n+1} E(I_{A_1} I_{A_2} \cdots I_{I_n}))) \\ &= \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n) \end{aligned}$$

22. a) Let $I_{3,i}$ be the indicator of a run of length 3 starting at the ith trial. Let $P(3,i) = E(I_{3,i})$ be the probability of this event. Then for $n > 3$

$$\begin{aligned}
E(R_{3,n}) &= E(\sum_{i=1}^{n-2} I_{3,i}) = \sum_{i=1}^{n-2} E(I_{3,i}) = \sum_{i=1}^{n-2} P(3,i) \\
&= P(3,1) + \sum_{i=2}^{n-3} P(3,i) + P(3,n-2) \\
&= p^3(1-p) + (n-4)(1-p)(p^3)(1-p) + (1-p)p^3 \\
&= 2p^3(1-p) + (n-4)(p^3)(1-p)^2
\end{aligned}$$

b) Similarly, let $I_{m,i}$ be the indicator of a run of length 3 starting at the ith trial. The for $m < n$

$$\begin{aligned}
E(R_{m,n}) &= \sum_{i=1}^{n-m+1} P(m,i) \\
&= 2p^m(1-p) + (n-m-1)(p^m)(1-p)^2
\end{aligned}$$

For $m = n$, $E(R_{n,n}) = p^n$.

c)

$$R_n = \sum_{m=1}^{n} R_{m,n}$$

so

$$\begin{aligned}
E(R_n) &= \sum_{m=1}^{n} E(R_{m,n}) \\
&= p^n + \sum_{m=1}^{n-1} 2p^m(1-p) + (n-m-1)(p^m)(1-p)^2 \\
&= p^n + 2(p-p^n) + (n-1)(1-p)^2 \sum_{m=1}^{n-1}(p^m) - (1-p)^2 \sum_{m=1}^{n-1} mp^m \\
&= 2p - p^n + \sum_{m=1}^{n-1}(n-m-1)(p^m)(1-p)^2
\end{aligned}$$

Let $\Sigma_1 = \sum_{m=1}^{n-1}(n-m-1)(p^m)(1-p)^2$. Then

$$\Sigma_1 - p\Sigma_1 = (n-1)p(1-p)^2 - \sum_{m=1}^{n-1} p^m(1-p)^2 = (n-1)p(1-p)^2 - (p-p^n)(1-p)$$

so

$$\Sigma_1 = (n-1)p(1-p) - (p-p^n)$$

and finally

$$E(R_n) = 2p - p^n + \Sigma_1 = p + (n-1)p(1-p)$$

d) Let I_j be the indicator that a run of some length starts on the jth trial. Then $R_n = \sum_{j=1}^{n} I_j$ and if we let $P(j)$ be the probability that a run of some length starts on the jth trial, we have

$$E(R_n) = E(\sum_{j=1}^{n} I_j) = \sum_{j=1}^{n} E(I_j) = \sum_{j=1}^{n} P(j)$$

where $P(1) = p$ and $P(j) = (1-p)p$ for $j > 1$. Thus

$$E(R_n) = p + (n-1)p(1-p)$$