Mathematical Toolkit Assignment 1

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1. (a)
$$\dim(A) = \operatorname{rank}(A) + \operatorname{null}(A)$$
$$n = m + \operatorname{null}(A)$$
$$\operatorname{null}(A) = n - m$$

   (b)
$$\operatorname{null}(A) = \dim(\ker(A))$$

   Then, $\ker(A)$ can have a basis B.

   i.e. $\forall \mathbf{v} \in \ker(A), \exists a_1, ..., a_{n-m} \in \mathbb{F}_2, \mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_{n-m}\mathbf{b}_{n-m}$ ($\mathbf{b}_i \in B$)

   $\therefore$ The answer is 2^{n-m}.

   (c)
$$\forall \mathbf{x} \text{ s.t. } A\mathbf{x} = \mathbf{b}, A(\mathbf{x} - \mathbf{x}_0) = 0 \because A\mathbf{x}_0 = \mathbf{b}$$
$$\therefore \ \mathbf{x} - \mathbf{x}_0 \in \ker(A)$$

   Then, choosing each element of $\mathbf{x}$ carefully (1 or 0), $\mathbf{x} - \mathbf{x}_0$ can be any element of $\mathbb{F}_2^n$.

$$\therefore \ \{\mathbf{x} - \mathbf{x}_0 | A\mathbf{x} = \mathbf{b}\} = \ker(A)$$

   $\therefore \mathbf{x} - \mathbf{x}_0$ has 2^{n-m} solutions.
   $\therefore \mathbf{x}$ has 2^{n-m} solutions.

2. (a)
$$\forall \mathbf{v}, f(c\mathbf{v} + (-c)\mathbf{v}) \geq \min\{f(\mathbf{v}), f(\mathbf{v})\}$$
$$f(\mathbf{0}_V) \geq f(\mathbf{v})$$

   (b) Because every element $\mathbf{v}_t \in V_t$ is in V by definition.
$$V_t \subseteq V$$

3.

$$p(x) = x^2 + bx + c$$
$$= (x - r_1)(x - r_2)$$
$$= x^2 - (r_1 + r_2)x + r_1 r_2$$
$$\therefore \; b = -r_1 - r_2, c = r_1 r_2$$

I'm lost.

4.

$$\mu(P, Q) = \text{degree}(PQ)$$
$$= \text{degree}(QP)$$
$$= \mu(Q, P)$$
$$\mu(0, 0) = \text{degree}(0)$$
$$= 0$$
$$\forall P \neq 0, \mu(P, P) = \text{degree}(P^2)$$
$$= 2\text{degree}(P)$$
$$> 0$$
$$\mu(P + Q, R) = \text{degree}((P + Q)R)$$
$$= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R)$$
$$\mu(P, R) + \mu(Q, R) = \max\{\text{degree}(P) + \text{degree}(R), \text{degree}(Q) + \text{degree}(R)\}$$
$$= \max\{\text{degree}(P), \text{degree}(Q)\} + \text{degree}(R)$$
$$\therefore \; \mu(P + Q, R) = \mu(P, R) + \mu(P, Q)$$
$$c \in \mathbb{R}, \mu(cP, R) = \text{degree}(cPR)$$
$$= \text{degree}(PR)$$
$$\neq c \cdot \text{degree}(P, R)$$

$\therefore \mu(\cdot, R)$ is not a LT.
$\therefore \mu$ is not a IP.

5.

$$\alpha\beta\mathbf{x} = \lambda\mathbf{x}$$
$$\beta\alpha\beta\mathbf{x} = \beta(\lambda\mathbf{x})$$
$$\beta\alpha(\beta\mathbf{x}) = \lambda(\beta\mathbf{x})$$

$\therefore \lambda$ is an eigenvalue of $\beta\alpha$.

6. (a)

$$\varphi(\mathbf{v}) = \lambda \mathbf{v}$$
$$\varphi(\mathbf{v}) = \lambda \varphi(\mathbf{v})$$
$$(\lambda - 1)\varphi(\mathbf{v}) = 0$$
$$\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) = 0$$
$$\lambda = 1_{\mathbb{F}} \vee \varphi(\mathbf{v}) = 0_{\mathbb{F}} \mathbf{v}$$
$$\therefore \ \lambda \in \{0_{\mathbb{F}}, 1_{\mathbb{F}}\}$$

(b) Let $\forall \mathbf{v}, \varphi(\mathbf{v}) = \mathbf{v}_0$ ($\mathbf{v}_0$ is fixed.) and assume $\varphi = \varphi^*$.

$$\forall \mathbf{v}, \mathbf{w} \in V \text{ s.t. } \mathbf{v} \neq \mathbf{w},$$
$$\langle \varphi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle$$
$$\langle \mathbf{v}_0, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v}_0 \rangle$$
$$\langle \mathbf{w}, \mathbf{v}_0 \rangle = \langle \mathbf{v}, \mathbf{v}_0 \rangle$$
$$\mathbf{v} = \mathbf{w} \ \lightning$$

$\therefore$ not always $\varphi = \varphi^*$

7. (a)

$$\langle \varphi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle$$
$$\langle \varphi^*(\mathbf{w}), \mathbf{v} \rangle = \langle \mathbf{w}, \varphi(\mathbf{v}) \rangle$$
$$\therefore \ (\varphi^*)^* = \varphi$$

(b)

$$\forall \mathbf{v} \in \ker(\varphi), \varphi(\mathbf{v}) = 0$$
$$\langle \varphi(\mathbf{v}), \mathbf{w} \rangle = 0$$
$$\langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle = 0$$
$$\mathbf{v} \in (\operatorname{im}(\varphi^*))^{\perp}$$
$$\forall \mathbf{v} \in (\operatorname{im}(\varphi^*))^{\perp}, \forall \mathbf{w} \in W, \langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle = 0$$
$$\langle \varphi(\mathbf{v}), \mathbf{w} \rangle = 0$$
$$\varphi(\mathbf{v}) = 0$$
$$\mathbf{v} \in \ker(\varphi)$$
$$\therefore \ \ker(\varphi) = (\operatorname{im}(\varphi^*))^{\perp}$$

(c)

$$\forall \mathbf{w} \in \text{im}(\varphi), \exists \mathbf{v} \in V \text{ s.t. } \mathbf{w} = \varphi(\mathbf{v}), \forall \mathbf{w}' \in \ker(\varphi^*), \langle \mathbf{w}, \mathbf{w}' \rangle = \langle \varphi(\mathbf{v}), \mathbf{w}' \rangle$$
$$= \langle \mathbf{v}, \varphi^*(\mathbf{w}') \rangle$$
$$= 0$$
$$\mathbf{w} \in (\ker(\varphi^*))^\perp$$

$$\forall \mathbf{v} \in V, \forall \mathbf{w} \in \ker(\varphi^*), \varphi^*(\mathbf{w}) = 0$$
$$\langle \mathbf{v}, \varphi^*(\mathbf{w}) \rangle = 0$$
$$\therefore \text{ im}(\varphi) = (\ker(\varphi^*))^\perp$$

(d)

$$\text{rank}(\varphi) = \dim(\text{im}(\varphi))$$
$$= \dim((\ker(\varphi^*))^\perp)$$
$$= \dim(W) - \dim(\ker(\varphi^*))$$
$$= \dim(im(\varphi^*))$$
$$= \text{rank}(\varphi^*)$$

(e)

$$\text{Let} \quad A = BC, B \in \mathbb{C}^{m \times r}, C \in \mathbb{C}^{r \times n}$$
$$\text{then} \quad A_{i,:} = \sum_{j=1}^{r} B_{i,j} C_{i,:}$$
$$A_{:,i} = \sum_{j=1}^{r} C_{j,i} B_{:,i}$$
$$\therefore \begin{cases} rank_{row}(A) \leq rank_{row}(C) \leq r \\ rank_{column}(A) \leq rank_{column}(B) \leq r \end{cases}$$

Choose a minimal r.
Then rows of C form a minimal spanning set of rows of A.
And, columns of C form a minimal spanning set of columns of A.
$\therefore r$ is the rank of both row and column spaces of A.

$$\therefore \ rank_{row}(A) = rank_{column}(A)$$