

Introduction to Statistical Machine Learning

Homework 3

Yota Toyama

November 17, 2016

1. A decision tree can classify linearly separable data. A boundary made by such a tree looks like stairs approximating $\mathbf{w}^T \mathbf{x} + w_0 = 0$. And, in the worst case, its depth is $\lceil \log \lceil \frac{N}{2} \rceil \rceil + 1$ because we can separate a space of \mathbf{x} into $\lceil \frac{N}{2} \rceil$ thin regions and balance the tree along \mathbf{x}_1 .
2. A decision tree can classify data points which are not linearly separable by separating a space of \mathbf{x} into N thin regions along \mathbf{x}_1 . And, in the worst case, its depth is $\lceil \log N \rceil$ when the tree is balanced in the same way as in the problem 1.
- 3.

$$\begin{aligned}
 \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} &= \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} \frac{1}{Z} W_i^{(T)} e^{-\alpha_{T+1} y_i h_{T+1}(\mathbf{x}_i)} \\
 &= \frac{1}{Z} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} e^{\frac{1}{2} \log \frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \\
 &= \frac{1}{Z} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \\
 &= \frac{1}{Z} \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} \\
 &= \frac{\sqrt{\epsilon_{T+1}(1-\epsilon_{T+1})}}{Z}
 \end{aligned}$$

$$\begin{aligned}
 Z &= e^{-\alpha_{T+1}}(1-\epsilon_{T+1}) + e^{\alpha_T} \epsilon_{T+1} \\
 &= \sqrt{\frac{\epsilon_{T+1}}{1-\epsilon_{T+1}}} (1-\epsilon_{T+1}) + \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \epsilon_{T+1} \\
 &= 2\sqrt{\epsilon_{T+1}(1-\epsilon_{T+1})}
 \end{aligned}$$

$$\therefore \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} = \frac{1}{2}$$

Assume $h_{T+2} = h_{T+1}$.

$$\begin{aligned} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} &= \frac{1}{2} \\ \sum_{i \text{ s.t. } y_i \neq h_{T+2}(\mathbf{x}_i)} W_i^{(T+1)} &= \frac{1}{2} \\ \epsilon_{T+2} &= \frac{1}{2} \\ \epsilon_{T+2} &\geq \frac{1}{2} \text{ } \nexists \end{aligned}$$

$$\therefore h_{T+2} \neq h_{T+1}$$

4.

$$\begin{aligned} \frac{\partial}{\partial \alpha_t} L(H_t, X) &= 0 \\ \frac{\partial}{\partial \alpha_t} (e^{-\alpha_t}(1 - \epsilon_t) + e^{\alpha_t}\epsilon_t) &= 0 \\ -e^{-\alpha_t}(1 - \epsilon_t) + e^{\alpha_t}\epsilon_t &= 0 \\ e^{2\alpha_t} &= \frac{1 - \epsilon_t}{\epsilon_t} \\ \alpha_t &= \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t} \end{aligned}$$

5.

$$\begin{aligned} \operatorname{argmin}_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \max \{0, 1 - y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0)\} \\ \operatorname{argmin}_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \xi_i \\ \left\{ \begin{aligned} &y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i \geq 0 \\ &\xi_i \geq 0 \end{aligned} \right. \end{aligned}$$

6. Please, see a Jupyter notebook file submitted together.