

# Introduction to Statistical Machine Learning

## Homework 3

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1. A decision tree can classify linearly separable data. A boundary made by such a tree looks like stairs approximating  $\mathbf{w}^T \mathbf{x} + w_0 = 0$ . And, in the worst case, its depth is  $\lceil \log \lceil \frac{N}{2} \rceil \rceil + 1$  because we can separate a space of  $\mathbf{x}$  into  $\lceil \frac{N}{2} \rceil$  thin regions and balance the tree along  $\mathbf{x}_1$ .
2. A decision tree can classify data points which are not linearly separable by separating a space of  $\mathbf{x}$  into  $N$  thin regions along  $\mathbf{x}_1$ . And, in the worst case, its depth is  $\lceil \log N \rceil$  when the tree is balanced in the same way as in the problem 1.
- 3.

$$\begin{aligned}
 \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} &= \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} \frac{1}{Z} W_i^{(T)} e^{-\alpha_{T+1} y_i h_{T+1}(\mathbf{x}_i)} \\
 &= \frac{1}{Z} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} e^{\frac{1}{2} \log \frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \\
 &= \frac{1}{Z} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \\
 &= \frac{1}{Z} \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T)} \\
 &= \frac{\sqrt{\epsilon_{T+1}(1-\epsilon_{T+1})}}{Z}
 \end{aligned}$$

$$\begin{aligned}
 Z &= e^{-\alpha_{T+1}}(1-\epsilon_{T+1}) + e^{\alpha_T} \epsilon_{T+1} \\
 &= \sqrt{\frac{\epsilon_{T+1}}{1-\epsilon_{T+1}}} (1-\epsilon_{T+1}) + \sqrt{\frac{1-\epsilon_{T+1}}{\epsilon_{T+1}}} \epsilon_{T+1} \\
 &= 2\sqrt{\epsilon_{T+1}(1-\epsilon_{T+1})}
 \end{aligned}$$

$$\therefore \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} = \frac{1}{2}$$

Assume  $h_{T+2} = h_{T+1}$ .

$$\begin{aligned} \sum_{i \text{ s.t. } y_i \neq h_{T+1}(\mathbf{x}_i)} W_i^{(T+1)} &= \frac{1}{2} \\ \sum_{i \text{ s.t. } y_i \neq h_{T+2}(\mathbf{x}_i)} W_i^{(T+1)} &= \frac{1}{2} \\ \epsilon_{T+2} &= \frac{1}{2} \\ \epsilon_{T+2} &\geq \frac{1}{2} \text{ } \not\! \! \! \end{aligned}$$

$$\therefore h_{T+2} \neq h_{T+1}$$

4.

$$\begin{aligned} \frac{\partial}{\partial \alpha_t} L(H_t, X) &= 0 \\ \frac{\partial}{\partial \alpha_t} (e^{-\alpha_t}(1 - \epsilon_t) + e^{\alpha_t}\epsilon_t) &= 0 \\ -e^{-\alpha_t}(1 - \epsilon_t) + e^{\alpha_t}\epsilon_t &= 0 \\ e^{2\alpha_t} &= \frac{1 - \epsilon_t}{\epsilon_t} \\ \alpha_t &= \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t} \end{aligned}$$

5.

$$\begin{aligned} \min_{\mathbf{w}} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^N \max \{0, 1 - y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0)\} \\ \Leftrightarrow \max_{\mathbf{w}, \xi} -\frac{1}{2} \|\mathbf{w}\|^2 - C \sum_{i=1}^N \xi_i \\ \begin{cases} y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i \geq 0 \\ \xi_i \geq 0 \end{cases} \end{aligned}$$

Using Langrange multipliers,

$$\begin{aligned} & \min_{\alpha, \mu} \max_{\mathbf{w}, \xi} -\frac{1}{2} \|\mathbf{w}\|^2 - C \sum_{i=1}^N \xi_i + \sum_{i=1}^N \alpha_i (y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i) + \sum_{i=1}^N \mu_i \xi_i \\ \Leftrightarrow & \begin{cases} y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i \geq 0 \\ \xi_i \geq 0 \\ \alpha_i \geq 0 \\ \mu_i \geq 0 \\ \alpha_i(y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i) = 0 \\ \mu_i \xi_i = 0 \end{cases} \end{aligned}$$

Let  $L = -\frac{1}{2} \|\mathbf{w}\|^2 - C \sum_{i=1}^N \xi_i + \sum_{i=1}^N \alpha_i (y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + w_0) - 1 + \xi_i) + \sum_{i=1}^N \mu_i \xi_i$ .

$$\frac{\partial L}{\partial \mathbf{w}} = -\mathbf{w} + \sum_{i=1}^N \alpha_i y_i \phi(\mathbf{x}_i) = 0$$

$$\frac{\partial L}{\partial w_0} = \sum_{i=1}^N \alpha_i y_i = 0$$

$$\frac{\partial L}{\partial \xi_i} = -C + \alpha_i + \mu_i = 0$$

$$\begin{aligned} L &= -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \phi(\mathbf{x}_i) \phi(\mathbf{x}_j) + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \phi(\mathbf{x}_i) \phi(\mathbf{x}_j) - \sum_{i=1}^N \alpha_i \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \phi(\mathbf{x}_i) \phi(\mathbf{x}_j) - \sum_{i=1}^N \alpha_i \end{aligned}$$

$\therefore$  The resulting optimization problem is the below.

$$\begin{aligned} & \min_{\alpha} \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j \phi(\mathbf{x}_i) \phi(\mathbf{x}_j) - \sum_{i=1}^N \alpha_i \\ & \begin{cases} 0 \leq \alpha_i \leq C \\ \sum_{i=1}^N \alpha_i y_i = 0 \end{cases} \quad \because \alpha_i = C - \mu_i \wedge \mu_i \geq 0 \end{aligned}$$

$\therefore$  The parameters  $H, \mathbf{f}, A, \mathbf{a}, B, \mathbf{b}$  of an equivalent quadratic problem are

the below.

$$\begin{aligned}
 H &= \begin{bmatrix} y_1 y_1 \phi(\mathbf{x}_1) \phi(\mathbf{x}_1) & \cdots & y_1 y_N \phi(\mathbf{x}_1) \phi(\mathbf{x}_N) \\ \vdots & \ddots & \vdots \\ y_N y_1 \phi(\mathbf{x}_N) \phi(\mathbf{x}_1) & \cdots & y_N y_N \phi(\mathbf{x}_N) \phi(\mathbf{x}_N) \end{bmatrix} \\
 \mathbf{f} &= - \begin{bmatrix} 1 \\ \cdots \\ 1 \end{bmatrix} \\
 A &= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{bmatrix} \\
 \mathbf{a} &= \begin{bmatrix} C \\ \vdots \\ C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
 B &= \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \\
 \mathbf{b} &= \begin{bmatrix} 0 \end{bmatrix}
 \end{aligned}$$

6. Please, see a Jupyter notebook file submitted together.