

Introduction to Statistical Machine Learning

Homework 1

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1.

$$\begin{aligned}
 \frac{\partial}{\partial \mathbf{w}} R(\mathbf{w}) &= 0 \\
 \frac{\partial}{\partial \mathbf{w}} E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})^2] &= 0 \\
 E_{p(\mathbf{x}, y)} [2(y - \mathbf{w}^T \mathbf{x})(-\mathbf{x})] &= 0 \\
 E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})\mathbf{x}] &= 0 \\
 \mathbf{a}^T E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})\mathbf{x}] &= 0 \\
 E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})\mathbf{a}^T \mathbf{x}] &= 0
 \end{aligned}$$

2.

$$\begin{aligned}
 E_{p(\mathbf{x}, y)} [y - \mathbf{w}^T \mathbf{x}] &= 0 \\
 E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})E_{p(\mathbf{x})} [\mathbf{x}]] &= 0 \\
 E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})\mathbf{x}] + E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})E_{p(\mathbf{x})} [\mathbf{x}]] &= 0 \\
 E_{p(\mathbf{x}, y)} [(y - \mathbf{w}^T \mathbf{x})(\mathbf{x} - E_{p(\mathbf{x})} [\mathbf{x}])] &= 0
 \end{aligned}$$

\therefore the correlation between data and prediction errors is 0.

3.

$$\begin{aligned}\hat{\mathbf{w}} &= \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i=1}^N (y_i - \mathbf{w}^T \mathbf{x}_i)^2 \\ &= (X^T X)^{-1} X^T y\end{aligned}$$

Let $C \in \mathbb{R}^{(d+1) \times (d+1)}$ be a diagonal matrix s.t. $\tilde{X} = XC$

$$\begin{aligned}\hat{\tilde{\mathbf{w}}} &= \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i=1}^N (y_i - \mathbf{w}^T \tilde{\mathbf{x}}_i)^2 \\ &= (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T y \\ &= ((XC)^T XC)^{-1} (XC)^T y \\ &= (CX^T XC)^{-1} CX^T y \\ &= C^{-1} (X^T X)^{-1} C^{-1} CX^T y \\ &= C^{-1} (X^T X)^{-1} X^T y \\ \tilde{X} \hat{\tilde{\mathbf{w}}} &= XCC^{-1} (X^T X)^{-1} X^T y \\ &= X(X^T X)^{-1} X^T y \\ &= X\hat{\mathbf{w}} \text{ as required}\end{aligned}$$

4.

$$\begin{aligned}
\hat{\sigma}^2 &= \operatorname{argmax}_{\sigma^2} \sum_{i=1}^N \log p(y_i | \mathbf{x}_i; \mathbf{w}, \sigma) \\
&= \operatorname{argmax}_{\sigma^2} -\frac{1}{2\sigma^2} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 - N \log \sigma \sqrt{2\pi} \\
&= \operatorname{argmin}_{\sigma^2} \frac{1}{\sigma^2} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 + N \log 2\pi \sigma^2 \\
&= \operatorname{argmin}_{\sigma^2} \frac{1}{\sigma^2} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 + N \log \sigma^2 \\
\frac{\partial}{\partial \sigma^2} \frac{1}{\sigma^2} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 + N \log \sigma^2 &= 0 \\
-\frac{1}{\sigma^4} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 + \frac{N}{\sigma^2} &= 0 \\
\sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 - N\sigma^2 &= 0 \\
\sigma^2 &= \frac{1}{N} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2 \\
\therefore \hat{\sigma}^2 &= \frac{1}{N} \sum_{i=1}^N (y_i - f(\mathbf{x}_i; \mathbf{w}))^2
\end{aligned}$$

5.

$$\begin{aligned}
\text{Let } \beta_{\hat{y},y} &= \begin{cases} 1 & \text{if } \hat{y} \leq y \\ \alpha & \text{otherwise} \end{cases} \\
l_{\alpha}(\hat{y}, y) &= \begin{cases} (\hat{y} - y)^2 & \text{if } \hat{y} \leq y \\ \alpha(\hat{y} - y)^2 & \text{otherwise} \end{cases} \\
&= \beta_{\hat{y},y}(\hat{y} - y)^2 \\
\frac{\partial}{\partial \mathbf{w}} L_{\alpha} &= \frac{\partial}{\partial \mathbf{w}} \sum_{i=1}^N l_{\alpha}(\hat{y}_i, y_i) \\
&= \frac{\partial}{\partial \mathbf{w}} \frac{1}{N} \sum_{i=1}^N \beta_{\hat{y}_i, y_i} (\hat{y}_i - y_i)^2 \\
&= \frac{\partial}{\partial \mathbf{w}} \frac{1}{N} \sum_{i=1}^N \beta_{\hat{y}_i, y_i} (\mathbf{w}^T \mathbf{x}_i - y_i)^2 \\
&= \frac{1}{N} \sum_{i=1}^N 2\beta_{\hat{y}_i, y_i} (\mathbf{w}^T \mathbf{x}_i - y_i) \mathbf{x}_i
\end{aligned}$$